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The Constituent-oriented Age and Residence time Theory (CART) and some of its applications to marine systems

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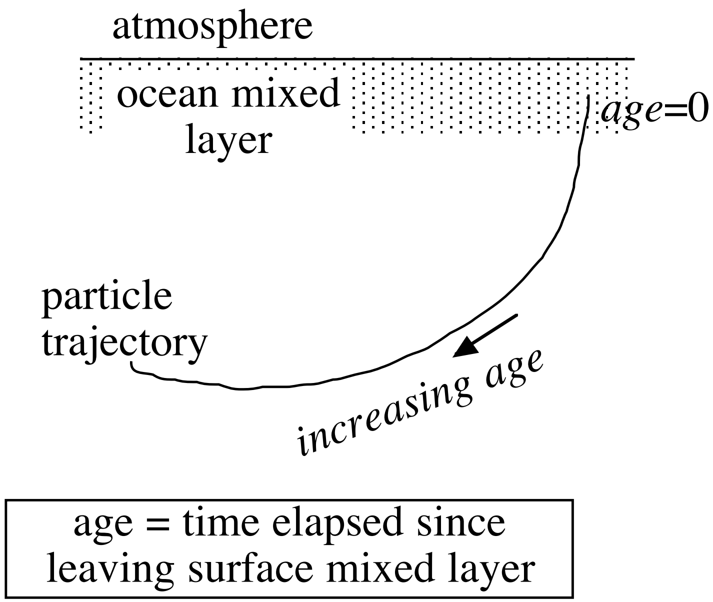
(with many thanks to Jean-Marie Beckers, Jean-Michel Campin and Anne Mouchet)

Organisation of the talk

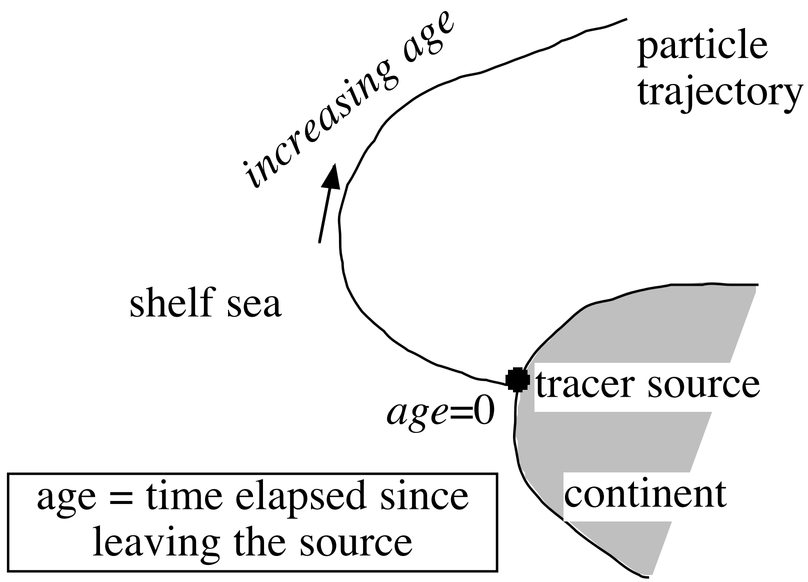
- **Motivation for developing a general theory of the age**
- **Some aspects of the age theory**
- **Applications:**
  - Diagnosing matter fluxes in ecological models
  - Age of tracers released by a point-source
- **A glimpse into the concept of residence time**
- **Concluding remarks**

Typical applications

estimating ocean ventilation rate



inferring shelf sea circulation



see for instance:

England, 1995, *Journal of Physical Oceanography*, 25, 2756-2777

Salomon et al. 1995, *Journal of Marine Systems*, 6, 515-527

Diffusion paradox of some dating techniques

- Using the radioactive decay as a clock:

$$\text{“radio-age”} \;=\; r(t,\mathbf{x}) \;=\; \gamma^{-1} \log \frac{C_p(t,\mathbf{x})}{C_r(t,\mathbf{x})}$$

C_p = concentration of a passive — i.e. inert — tracer

C_r = concentration of a radioactive tracer (half-life = $\gamma^{-1} \log 2$)

- The main assumption underlying the concept of radio-age is that diffusion is negligible. However, diffusion cannot be neglected to compute C_p and C_r. This is somewhat paradoxical...

- Similar problems in other “pragmatic” dating techniques.

- Are we really estimating an elapsed time? Is a single age really relevant for every water parcel?

A general theory of the age

A theory for evaluating the age in **numerical models** such that:

- advection, mixing, and production/destruction processes are properly accounted for;
- age is a time- and position-dependent variable;
- the age of every constituent can be evaluated separately, hence the name CAT, i.e. Constituent-oriented Age Theory;
- the age may be calculated in the Eulerian formalism;
- CAT is intended for numerical models, but some of its aspects may be of use to applications relying on field data only.

See Delhez et al. (1999, *Ocean Modelling*, 1, 17-27) and Deleersnijder et al. (2001, *Journal of Marine Systems*, 28, 229-267).

Only one arbitrary assumption

- Particles "A" and "B", mass m^{A} and m^{B}, age a^{A} and a^{B}.
- System "A+B", consisting of particles "A" and "B".
- Mass of system "A+B": $m^{\mathrm{A+B}} = m^{\mathrm{A}} + m^{\mathrm{B}}$.
  (mass is an additive quantity, i.e. basic physical principle)
- Age of system "A+B": no underlying physical principle!

Age-averaging hypothesis: mass-weighted arithmetic mean:

$$\boxed{a^{\mathrm{A+B}} = \frac{m^{\mathrm{A}} a^{\mathrm{A}} + m^{\mathrm{B}} a^{\mathrm{B}}}{m^{\mathrm{A}} + m^{\mathrm{B}}}}$$

- Age content: $m^{\mathrm{A+B}} a^{A+B} = m^{\mathrm{A}} a^{\mathrm{A}} + m^{\mathrm{B}} a^{\mathrm{B}}$ (additive quantity).

Basic variables

- $\rho c_i(t,\mathbf{x},\tau)\delta V\delta\tau$: mass of the i-th constituent in δV, whose age lies in the interval $[\tau-\delta\tau/2,\ \tau+\delta\tau/2]$ $(\delta\tau\to 0)$, where $c_i(t,\mathbf{x},\tau)$ is the **concentration distribution function**.

- Concentration: $C_i(t,\mathbf{x}) = \int_0^\infty c_i(t,\mathbf{x},\tau)\ d\tau$

- Age concentration: $\alpha_i(t,\mathbf{x}) = \int_0^\infty \tau c_i(t,\mathbf{x},\tau)\ d\tau$

- Mean age: $a_i(t,\mathbf{x}) = \dfrac{\alpha_i(t,\mathbf{x})}{C_i(t,\mathbf{x})}$

Basic equations

- Simple mass budget considerations yield:

$$\frac{\partial c_i}{\partial t} = \underbrace{p_i - d_i}_{\text{source - sink}} - \nabla \bullet \underbrace{(\mathbf{u}c_i - \mathbf{K} \bullet \nabla c_i)}_{\text{advection + diffusion}} - \underbrace{\frac{\partial c_i}{\partial \tau}}_{\text{ageing}}$$

$$\frac{\partial C_i}{\partial t} = \underbrace{P_i - D_i}_{\text{source - sink}} - \nabla \bullet \underbrace{(\mathbf{u}C_i - \mathbf{K} \bullet \nabla C_i)}_{\text{advection + diffusion}}$$

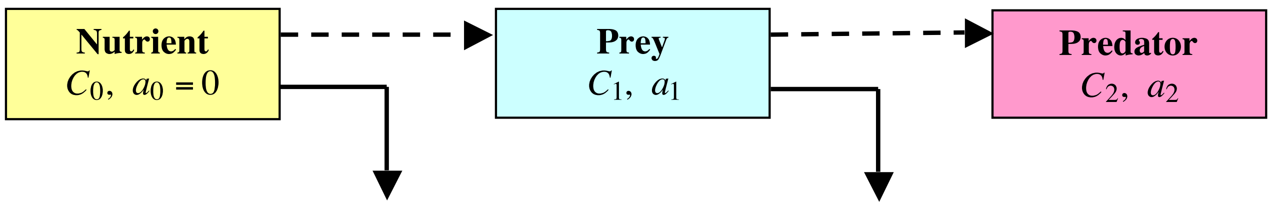
$$\frac{\partial \alpha_i}{\partial t} = \underbrace{C_i}_{\text{ageing}} + \underbrace{\pi_i - \delta_i}_{\text{source - sink}} - \nabla \bullet \underbrace{(\mathbf{u}\alpha_i - \mathbf{K} \bullet \nabla \alpha_i)}_{\text{advection + diffusion}}$$

- All advection-diffusion operators are of the same form.

Diagnosing matter fluxes in ecological models (I)

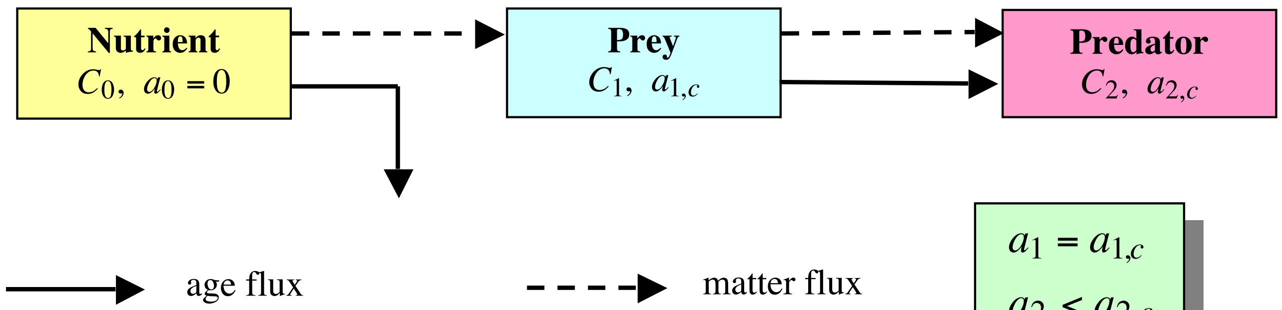
- At least two options, illustrated here in a simple/generic model:

Estimating the age of every compartment



The matter entering each compartment has zero age

Estimating cumulative ages



The prey takes zero-age nutrients.

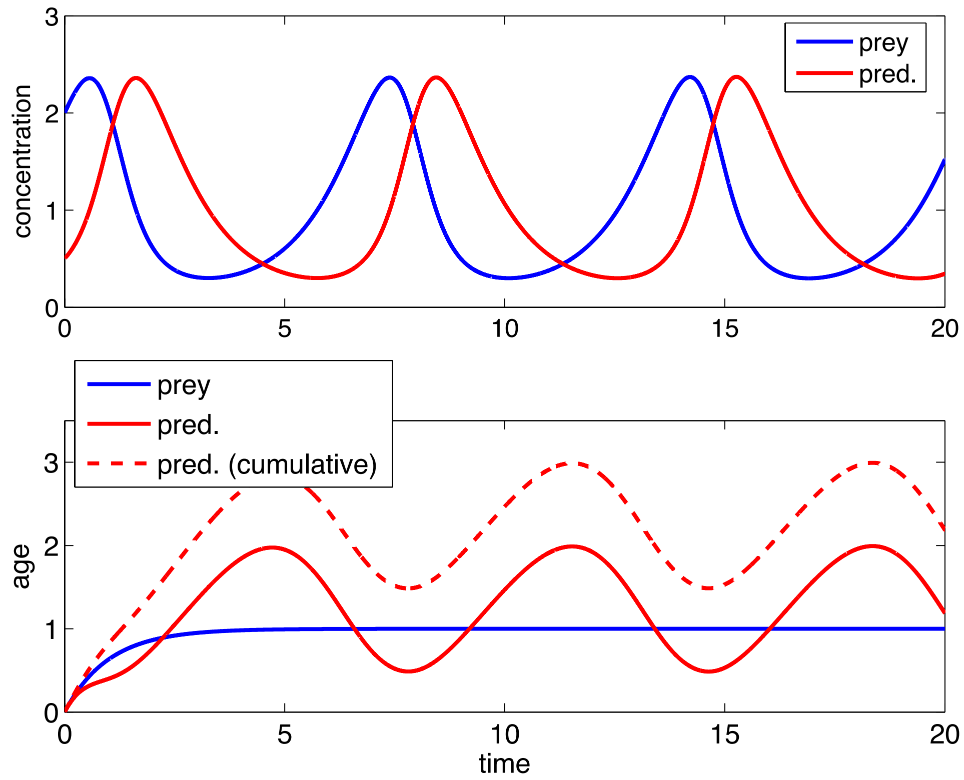
The predator takes the prey "along with its age".

$$a_1 = a_{1,c}$$

$$a_2 < a_{2,c}$$

Diagnosing matter fluxes in ecological models (II)

Assuming an infinite stock of nutrients (no nutrient-related limitation to growth), the solutions of a classical two-equation Lotka-Volterra model are (in dimensionless variables):



- The prey and predator concentrations exhibit periodic oscillations and, yet, the age of the prey tends to a constant!

Diagnosing matter fluxes in ecological models (III)

- The equations for the prey concentration and age are:

$$\frac{dC_1}{dt} = \underbrace{\frac{C_1}{\theta}}_{nutrient\ uptake} - \underbrace{\frac{C_2}{\theta^*}C_1}_{predation} \quad , \quad \frac{d\alpha_1}{dt} = \underbrace{0}_{nutrient\ uptake} - \underbrace{\left(\frac{C_2}{\theta^*}C_1\right)a_1}_{predation} + \underbrace{C_1}_{ageing}$$

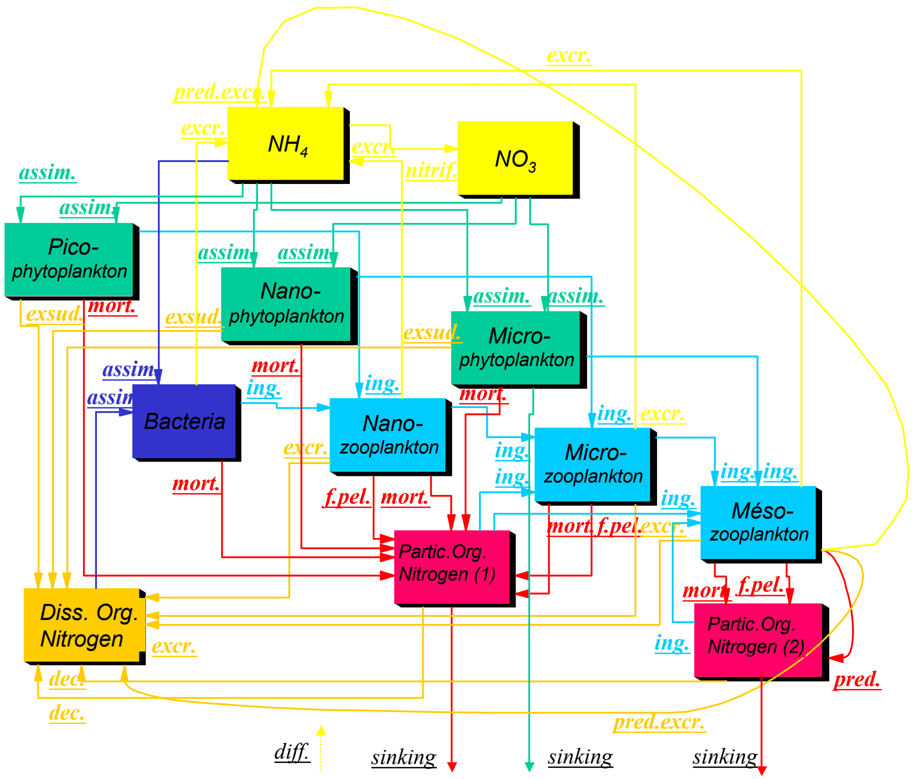
$$\Rightarrow \quad \frac{da_1}{dt} = \underbrace{-\frac{a_1}{\theta}}_{nutrient\ uptake} + \underbrace{1}_{ageing} \quad \Rightarrow \quad a_1(t) = \underbrace{[a_1(0) - \theta]e^{-t/\theta}}_{\to 0\ as\ t/\theta \to \infty} + \theta$$

The prey's age tends to the nutrient uptake timescale, θ, because the predation term is age-independent.

- Valid for any predation term of the form $f(t, C_1, C_2)C_1$.

Diagnosing matter fluxes in ecological models (IV)

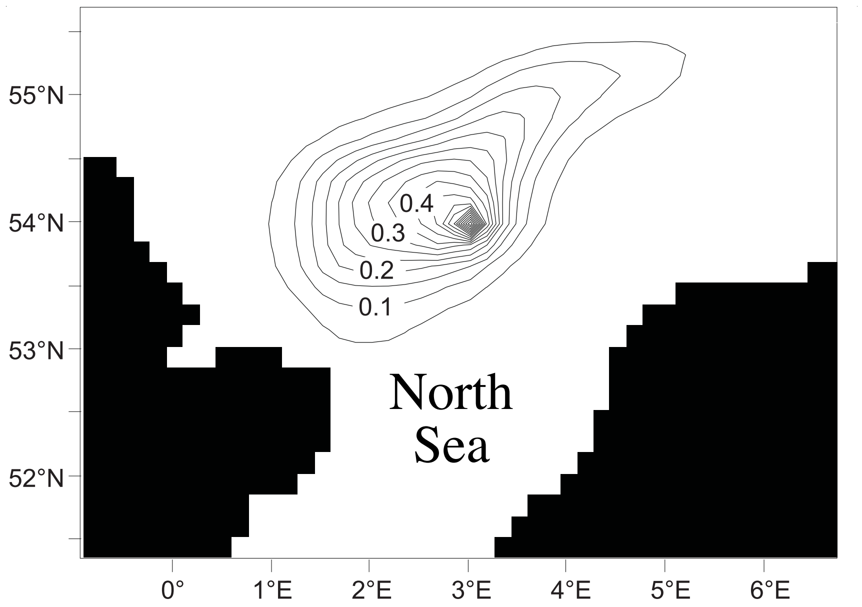
- Delhez et al. applied similar ideas to a sophisticated 1D eco-hydrodynamic model (2004, *Ocean Dynamics*, 54, 221-231).



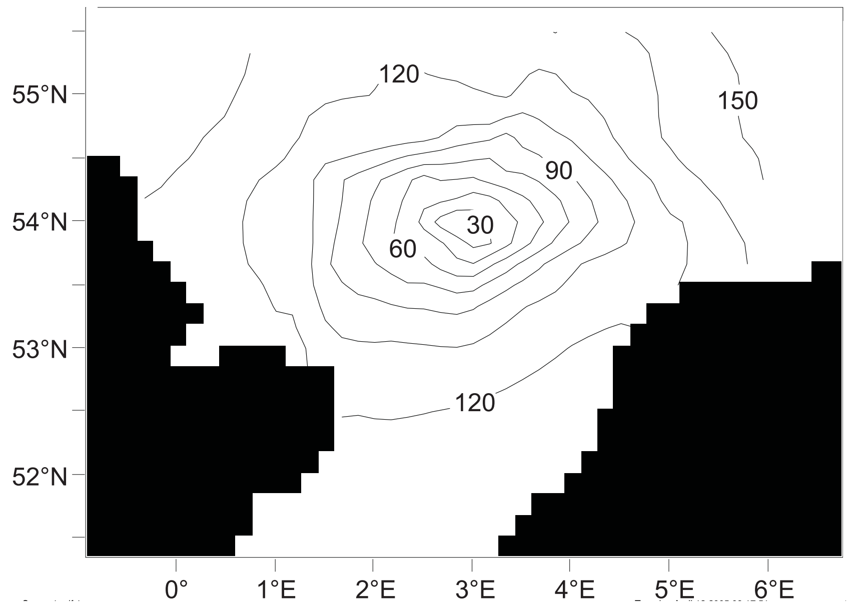
Ages of tracers released by a point-source (I)

- Modelling tracers released by a point source in the North Sea:

passive tracer concentration



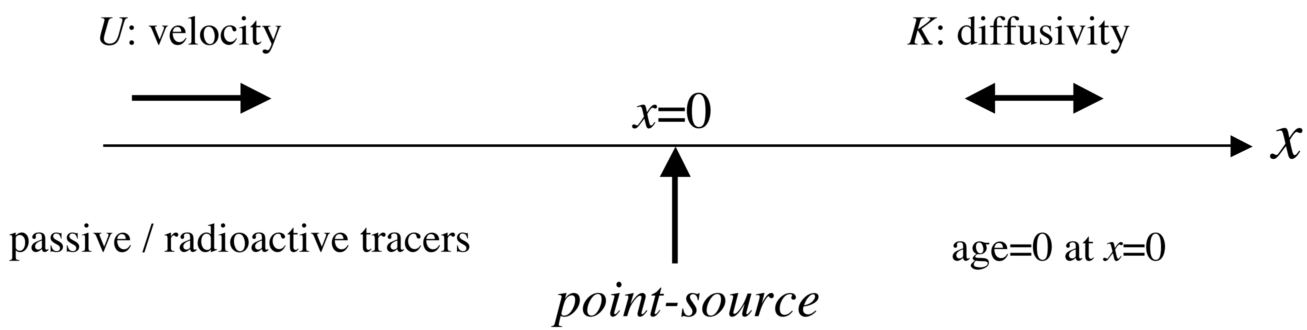
radio-age



- The tracer concentration reflects the direction of the advection, but the age does not! A numerical artefact?

Ages of tracers released by a point-source (II)

- Steady-state, one-dimensional idealised model:



- If K=0:

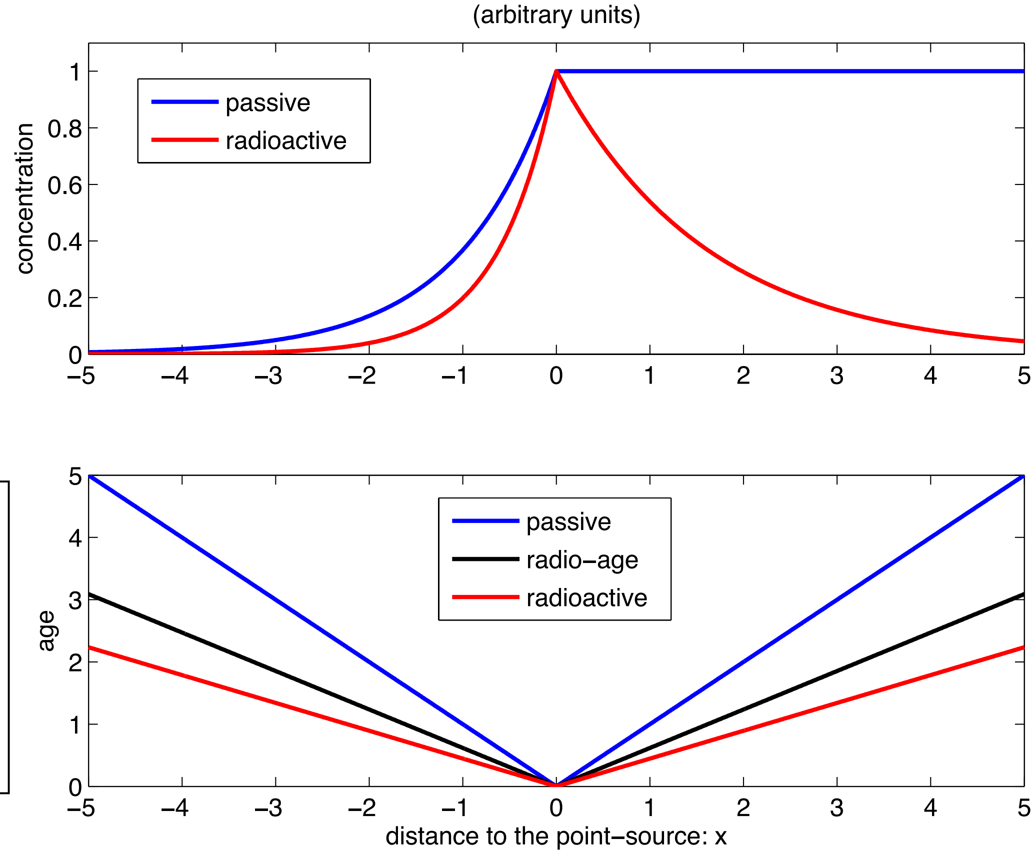
$$\begin{cases} x < 0: & \text{no tracer} \\ x > 0: & \begin{cases} C_p = 1,\ C_r = \exp\left[-\dfrac{x}{U\gamma^{-1}}\right] \\ a_p = a_r = r = \dfrac{x}{U}, \quad \text{with} \ \ r = \gamma^{-1}\log\dfrac{C_p}{C_r} \end{cases} \end{cases}$$

All ages are equal!

Ages of tracers released by a point-source (III)

• If $K \neq 0$, diffusion causes tracer to be present upstream of the source and ages to be all different, but symmetric!

The radioactive tracer age is smaller than that of the passive tracer because of decay+diffusion, rather than decay alone.



See also the Lagrangian approach to this problem by Hall and Haine (2004, *Journal of Marine Systems*, 48, 51-59)

Ages of tracers released by a point-source (IV)

• Scaling of the ages:

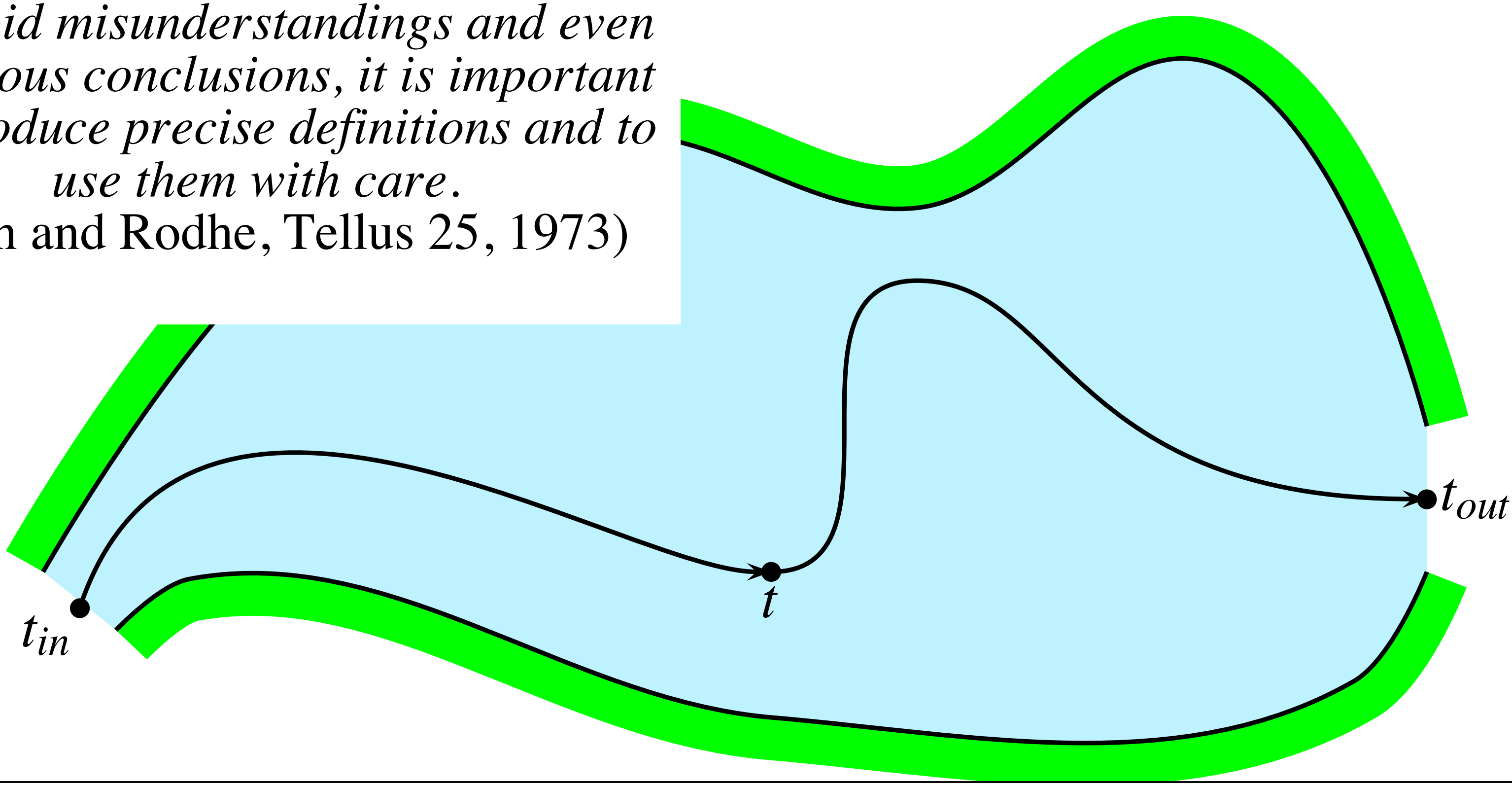
$$\begin{cases} a_p = \dfrac{|x|}{U} & \text{(passive tracer age)} \\ r = \dfrac{\sqrt{1+4Je^{-1}}-1}{2Je^{-1}} \dfrac{|x|}{U} & \text{(radio-age)} \\ a_r = \dfrac{1}{\sqrt{1+4Je^{-1}}} \dfrac{|x|}{U} & \text{(radioactive tracer age)} \end{cases}$$

with $Je = \dfrac{U(U/\gamma)}{K} = \dfrac{U^2}{K\gamma}$ (Jenkins number, i.e. Peclet with $L = U/\gamma$)

• Symmetry also arises in transient, multi-dimensional problems by prescribing that either the ages are zero at the source point or the age of the tracers released by the source is zero (Beckers et al., *SIAM Journal on Applied Mathematics*, 61, 1526-1544, 2001).

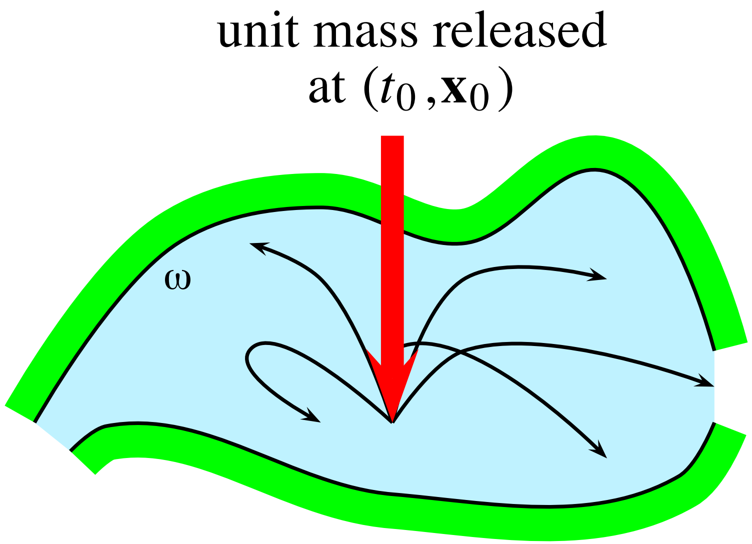
Definition of the residence time

To avoid misunderstandings and even erroneous conclusions, it is important to introduce precise definitions and to use them with care.
(Bolin and Rodhe, Tellus 25, 1973)

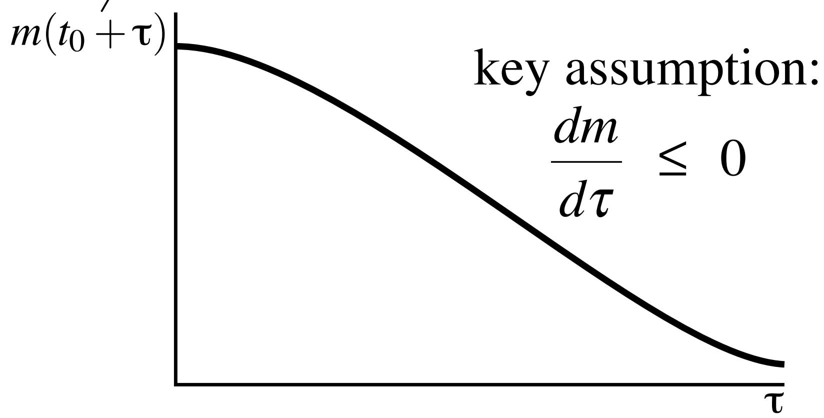


$age = t - t_{in}$ $residence\ time = t_{out} - t$ $transit\ time = t_{out} - t_{in}$

Residence time: the forward/direct procedure



$$m(t_0+\tau) = \int_\omega C(t_0+\tau, \mathbf{x})\, d\mathbf{x}$$



1. Introduce unit mass of passive tracer at time t_0 and location $\mathbf{x}_0$;
2. Calculate the mass $m(t_0+\tau)$ of the tracer in the domain ω;
3. Residence time: $\theta(t_0, \mathbf{x}_0) = -\int_1^0 \tau\, dm = \int_0^\infty m(t_0+\tau)\, d\tau$.

Residence time: the backward/adjoint procedure (I)

- Using the direct procedure, the number of models runs that are needed is equal to the number of t_0 and $\mathbf{x}_0$ at which the residence time is to be estimated.

  $\Rightarrow$ CPU cost can be prohibitive!

- Delhez et al. (2004, *Estuarine, Costal and Shelf Science*, 61, 691-702) developed an adjoint model that is potentially much more efficient, but requires backward integration in time.

- The residence time $\theta(t,\mathbf{x})$ is the solution of

$$\frac{\partial \theta}{\partial t} = -1 - \nabla \bullet (\mathbf{u}\theta + \mathbf{K} \bullet \nabla \theta)$$

Residence time: the backward/adjoint procedure (II)

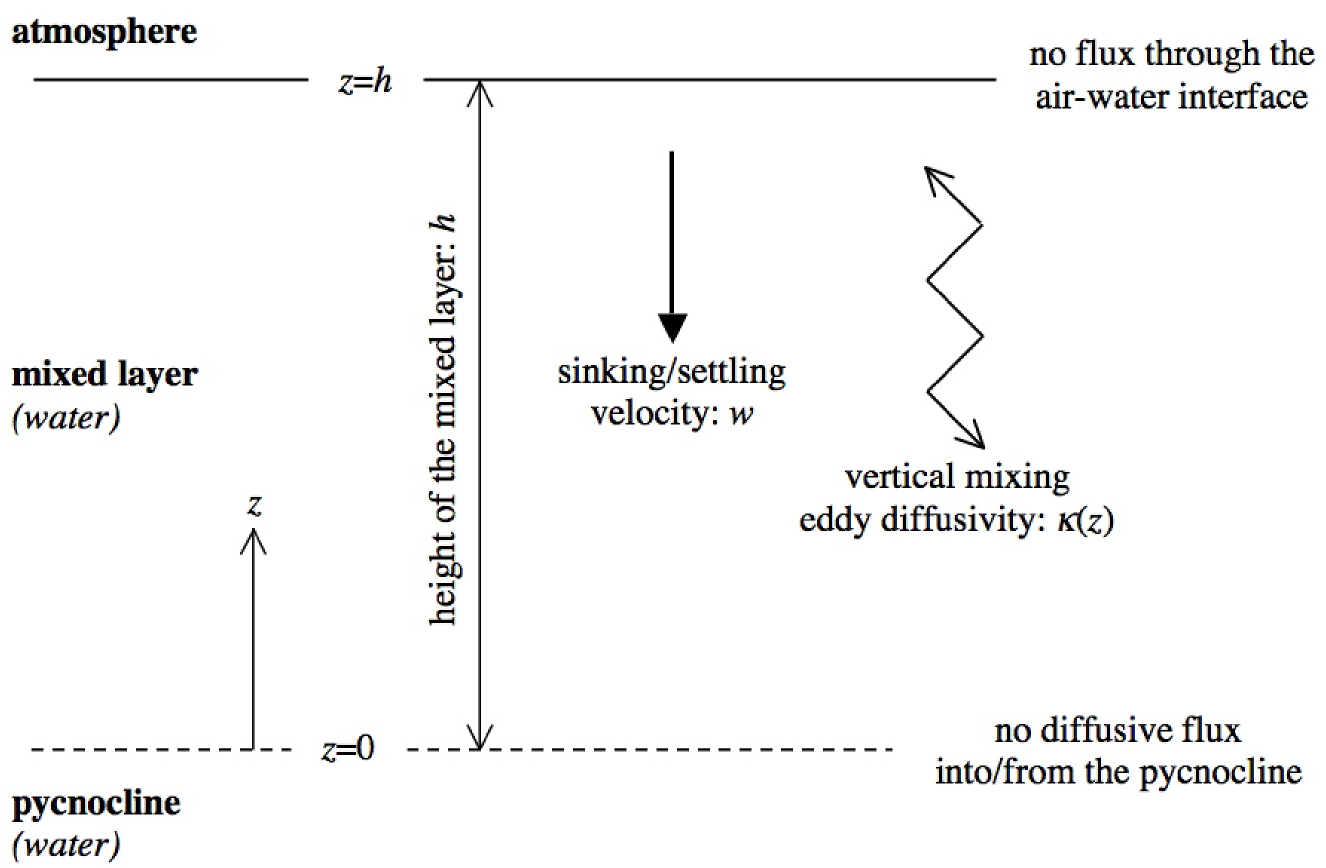
- The equation governing $\theta(t,\mathbf{x})$ is to be integrated backward in time from $t = T$, with $\theta(T,\mathbf{x}) = 0$ and $T \to \infty$.

  In practice, T is taken to be sufficiently large, so that the residence time is hopefully accurate for $t << T - O(\theta)$. For more details, see Delhez (2005, *Ocean Science Discussions*, 2, 247-265, available on the web).

- Some examples of boundary conditions:

direct problem	adjoint problem
$C = 0$	$\theta = 0$
$(\mathbf{K}\bullet\nabla C)\bullet\mathbf{n} = 0$	$(\mathbf{u}\theta + \mathbf{K}\bullet\nabla\theta)\bullet\mathbf{n} = 0$
$(\mathbf{u}C - \mathbf{K}\bullet\nabla C)\bullet\mathbf{n} = 0$	$(\mathbf{K}\bullet\nabla\theta)\bullet\mathbf{n} = 0$

Residence time in the upper mixed layer (I)



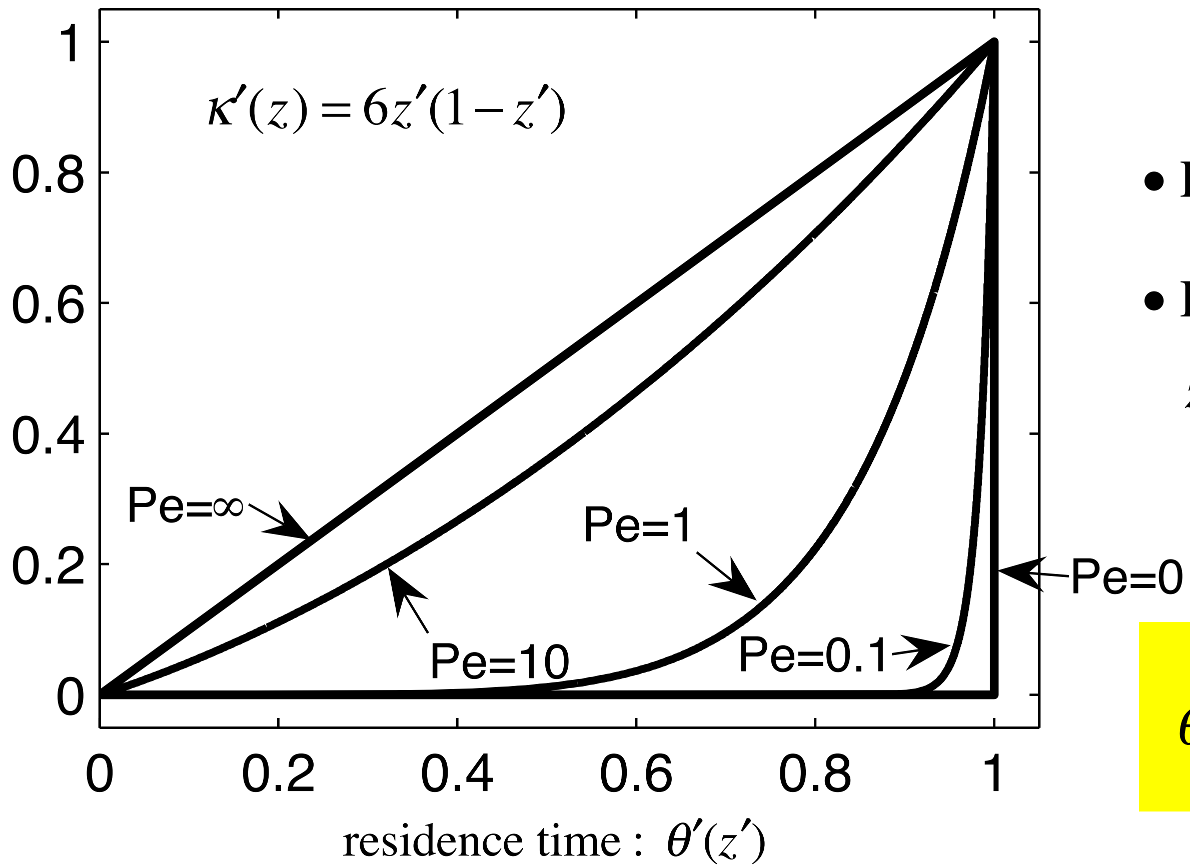
Key assumptions: horizontal homogeneity and hydrodynamics at a steady state.

Residence time in the upper mixed layer (II)

	direct/forward problem	adjoint problem
unknown	concentration: $C(t,z)$	residence time: $\theta(z)$
governing equation	$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(wC + \kappa\frac{\partial C}{\partial z}\right)$	$\frac{d}{dz}\left(w\theta - \kappa\frac{d\theta}{dz}\right) = 1$
initial condition	$C(0,z) = \delta(z - z_0)$	not applicable
boundary conditions	$\left[wC + \kappa\frac{\partial C}{\partial z}\right]_{z=h} = 0$ $\left[\kappa\frac{\partial C}{\partial z}\right]_{z=0} = 0$	$\left[\kappa\frac{d\theta}{dz}\right]_{z=h} = 0$ $\left[w\theta - \kappa\frac{d\theta}{dz}\right]_{z=0} = 0$
solution	$C(t,z) = ?$	$\theta(z) = \frac{z}{w} + \frac{1}{w}\int_z^h \exp\left[-w\int_z^\xi \frac{d\zeta}{\kappa(\zeta)}\right]d\xi$

$\theta\uparrow$ if $h\uparrow,\ \kappa\uparrow,\ w\downarrow$

Residence time in the upper mixed layer (III)



- Peclet number: $Pe = \dfrac{wh}{\overline{\kappa}}$
- Dimensionless variables:

$$z' = \frac{z}{h} \ , \quad \kappa' = \frac{\kappa}{\overline{\kappa}} \ , \quad \theta' = \frac{\theta}{h/w}$$

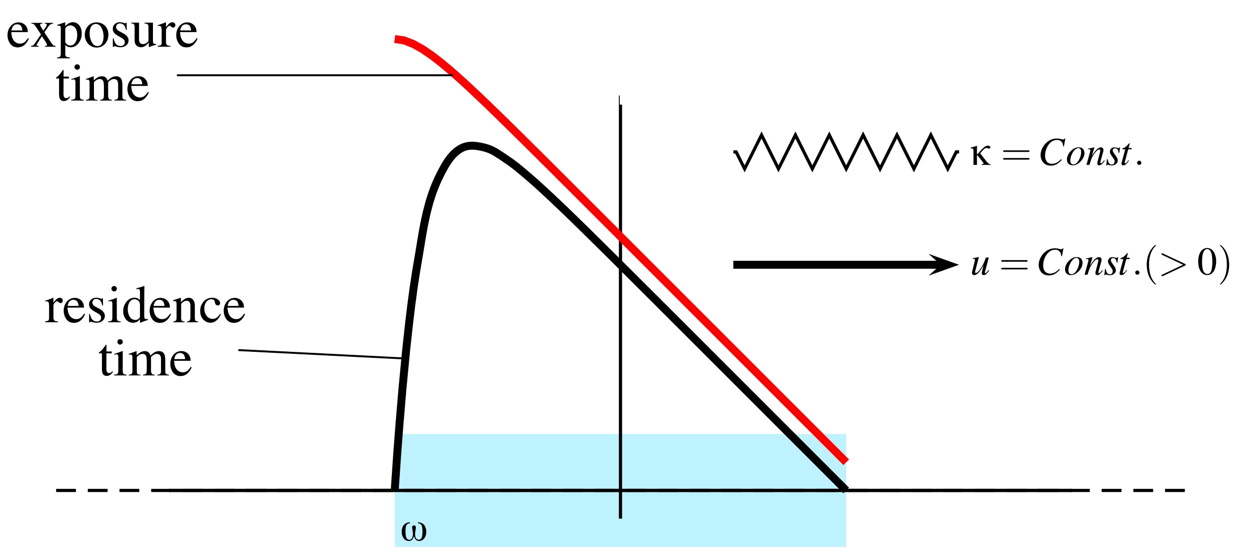
$$z' \ \leq \ \theta'(z') \ \leq \ 1$$

$\theta'(1)$ independent of $\kappa'(z')$!

(Deleersnijder et al., 2005, *Environmental Fluid Mechanics*, in press)

Residence time *vs* exposure time

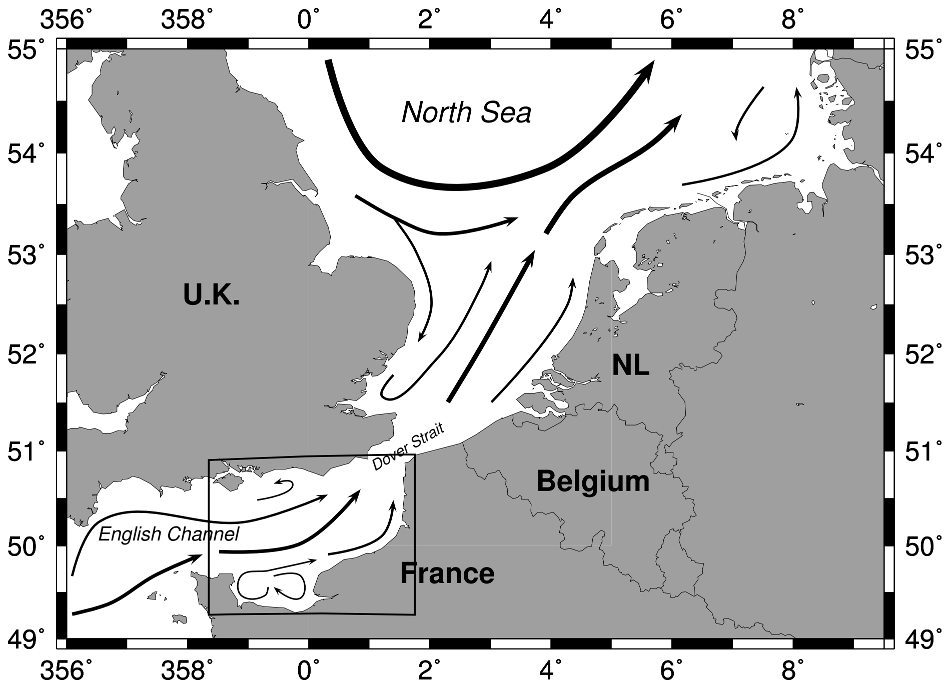
- Particles that left the domain can enter it again at some later time. This can be taken into account by means of the *exposure time*, i.e. *the time spent in the domain of interest* — whereas the residence time is the time needed to leave it for the first time.



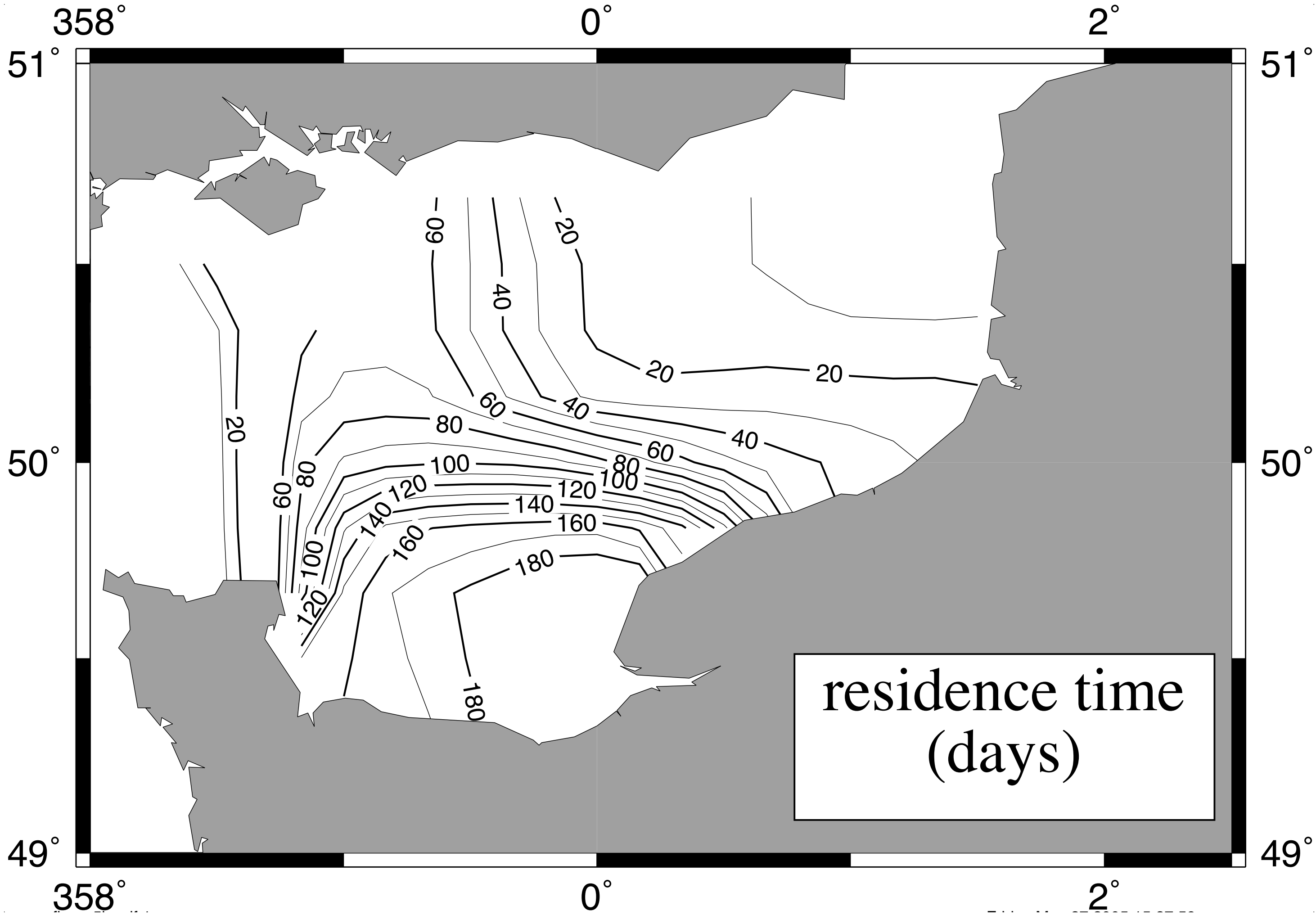
To obtain the exposure time, the same adjoint model equations are to be solved, but in a different domain and with different boundary conditions.

Residence/exposure time in the English Channel (I)

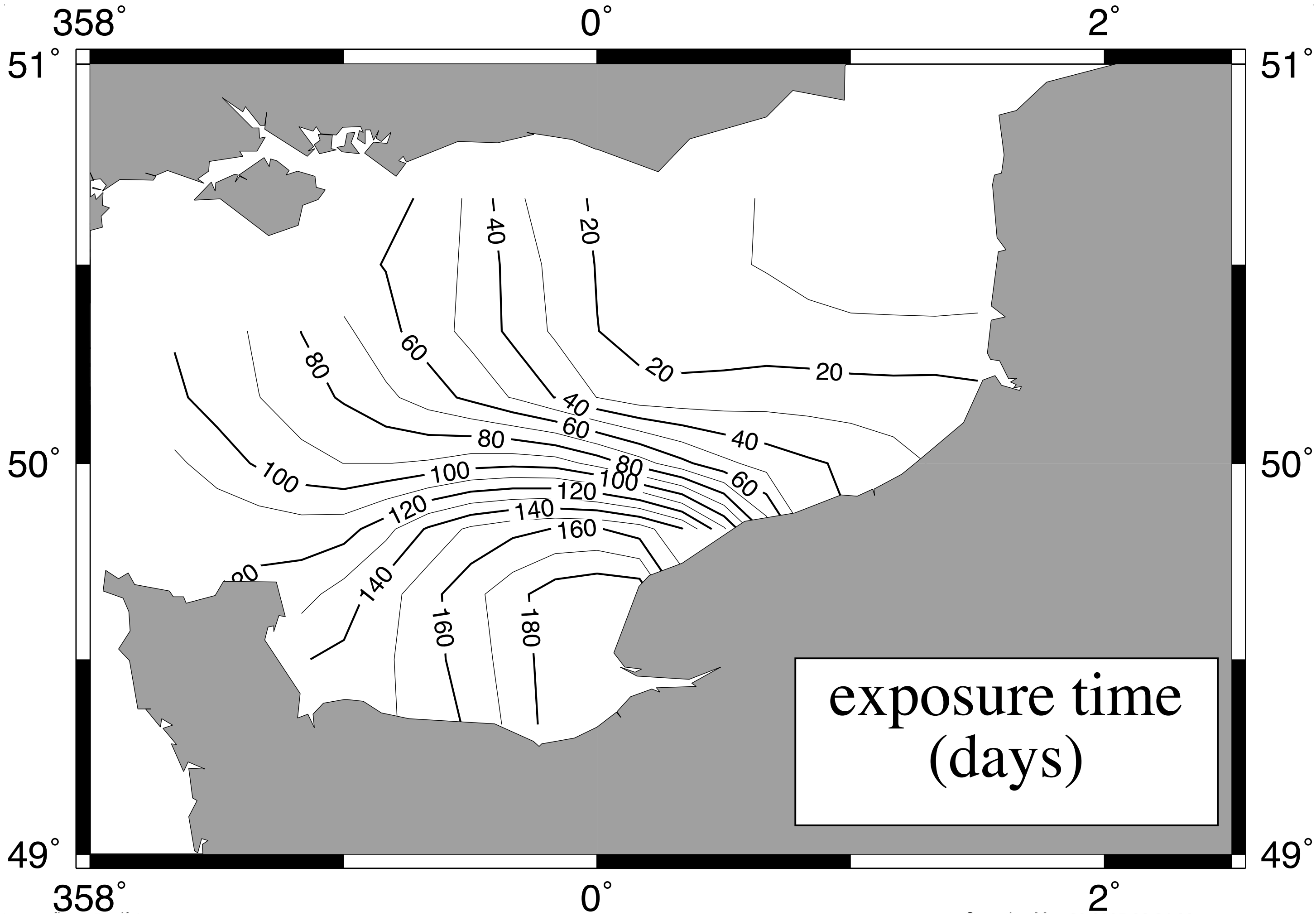
- Horizontal resolution: 10'; 10 σ-levels.
- Free-surface; baroclinic; *k* turbulence closure model.
- Forcings: 10 tidal constituents and NCEP reanalysis met. data.



Residence/exposure time in the English Channel (II)



Residence/exposure time in the English Channel (III)



Concluding remarks

- A general theory of the age has been developed, from which the age of any seawater constituent or group of constituents, passive or not, can be estimated.
- A general theory of the residence/exposure time is being developed. Open questions pertain to boundary conditions and tracers with non-linear source/sink terms.
- Age and residence/exposure times are easy to estimate numerically, are of use in a number of applications, and there is no shortage of surprising results.
- CART can be applied outside the realm of oceanography as long as the Boussinesq approximation remains valid (e.g. industrial chemistry paper by Jongen, 2004, *AICHE*, 50).

For more information about CART, see

http://www.climate.be/CART

(under development)

The journal *Estuarine, Coastal and Shelf Science* will publish a special issue on the use of timescales and tracer methods — in the Eulerian or Lagrangian formalism — that can help understand the results of complex model results. Manuscript should be submitted between December 1, 2005 and March 31, 2006.

For more information about this, see

http://www.climate.be/ECSS