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Abstract

This paper analyzes the efficiency of hedging strategies for stock options, in presence of jump clustering. In the proposed model, the asset is ruled by a jump diffusion process wherein the arrival of jumps is correlated to the amplitude of past shocks. This feature adds feedback effects and time heterogeneity to the initial jump diffusion. After a presentation of main properties of the process, a numerical method for options pricing is proposed. Next, we develop four hedging policies minimizing the variance of the final wealth. These strategies are based on first and second order approximations of option prices. The hedging instrument is either the underlying asset or either another option. The performance of these hedges is measured by simulations for put and call options, with a model fitted to the S&P 500.

Keywords: self-excitation, Hawkes process, minimum variance hedging, options pricing

1 Introduction

The sudden arrival of some grouped and unexpected information may trigger a phenomenon of considerable importance for financial markets which is the clustering of jumps in asset values (Rangel (2011))¹. This phenomenon has some important consequences on financial derivatives. Firstly, the value of financial derivatives should reflect the consequences of such a possible self-excitation. Secondly, the associated dynamic hedging strategy should be tailored to the suspected data-generating process. Ignoring this empirical feature may alter the efficiency of hedging strategies. Despite this, an inspection of the literature reveals that these pricing and hedging problems/issues have not been frequently discussed and rarely addressed from an operational point of view. Our investigation fills this gap by studying the influence of jump clustering on the design of the suitable hedging strategy.

A way to deal with the clustering of jumps is to consider a jump process equipped with a stochastic jump arrival intensity as self-excited jump processes², also called Hawkes processes. The distinctive characteristic of these processes is that their jump arrival intensity immediately responds to the arrival of

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¹The credit crunch of 2008 is among the most speaking examples of this mechanism. At that time, the waterfall of bad news and domino effects in bankruptcies shacked violently stocks markets.

²Before the more recent literature on Hawkes process, various authors have investigated the importance of introducing a state-dependent jump arrival intensity for equity price modelling. Among the first authors, Bates (2000) argues that the frequency of shocks in stock markets can increase in response to stress. He proposes a rich jump diffusion model where the volatility is stochastic and where the jump arrival intensity in the stock price is a linear function of the volatility factors. He provides evidence that the considered specification can fit the S&P500 option prices better than other nested models. But the implicit risk-neutral distributions inferred from the jump-diffusion models assign typically "a 90% risk-neutral probability of observing at least 1 weekly move of 10% [... while] none was observed". Pan (2002) essentially reconsiders the same process, except that the jump arrival intensity is assumed proportional to the stochastic volatility. His empirical investigations then show that the jump-risk premium, which is indeed correlated with the market volatility, plays an important role in explaining the joint time-series behavior of spot index returns and option prices and the cross-sectional behavior of option prices. This result has also been discussed by Eraker (2004) who introduces a model with discontinuous correlated jumps in stock prices and stock price volatility. Here the state-dependent jump arrival intensity also depends on the stochastic volatility and hence

jumps and this influence lasts for a certain time. As such, they are not Lévy processes³.

Different motivations have emerged in the literature for using self-excited jump processes in financial markets and quantitative finance. Errais, Giesecke and Goldberg (2010) are among the first to use this class of processes for modelling the default risk of a credit portfolio. Here, the intensity of defaults among the considered portfolio of loans increases temporarily following each default⁴. Some authors rely on these processes for solving some microstructural challenges (see Bacry and Muzy (2014) for a recent discussion). Maneesoonthorn et al. (2016) show how a self-excited Heston jump diffusion process with stochastic volatility can fit high-frequency data. Aït-Sahalia et al. (2014) introduce a multivariate self-exciting jump process for simultaneously modelling many asset returns and they provide empirical evidences in favor of such self- and cross-excitations. McClelland (2012)⁵, Chen and Poon (2013); Boswijk et al. (2015), Fulop et al. (2015) and Carr and Wu (2016) investigate some self-excited jump-diffusion processes for modelling stock index returns and/or stock index options. Typically, McClelland (2012) extends the jump diffusion model of Bates (1996) by relaxing the assumption that the jump arrival intensity remains constant through time. He then applies his extended specification to both the S&P500 index returns and a panel of S&P500 index option prices and he finds evidences of self-excitation. None of these authors however questions the hedging issue.

The main purpose of this article is to explore quadratic hedging strategies useful to manage the options whose underlying asset may experience some clustering of jumps. To this end, we extend the jump-diffusion model of Kou (2002) and let the intensity of jumps be ruled by a self-excited process. Our model differs from previous contributions in two main directions. Firstly, our specification postulates a linear relationship between the increase of the jump arrival intensity and the absolute value of the just realized jump, so that both positive and negative shocks matter⁶. Secondly, we assume that the return jump size is distributed according to a double exponential distribution. For comparison, the self-exciting model studied by Boswijk et al. (2015) posits that the increase of the jump arrival intensity is constant at each arrival of a new shock (whatever its size) and that the return jump size is normally distributed. McClelland (2012) discusses various specifications where the impact on the intensity is constant or exponential distributed and independent or correlated (and therefore not equal) to the impact of the unexpected news on the price return. Carr and Wu (2016) put a special emphasis on negative jumps (to insist on the leverage effect) by assuming that the compensated (negative) realized jump impacts the jump arrival intensity proportionally. In our setting, the impact on the jump arrival intensity of large shocks (in absolute value) is larger than the one of small realized jumps (in absolute value). Past realized jumps (whatever they size) can partly explain the contemporaneous level of the jump intensity but large and small shocks are differently treated qualitatively. By design, larger shocks will have a longer influence on the intensity than the small ones. This is obviously not the case in the model of Boswijk et al. (2015) where every new shock is treated the same and in the one of Carr and Wu (2016) where only jumps with negative signs are taken on account⁷. Our specification has important differences with previous models.

indirectly on associated jumps. However results suggest that "*complex jump specifications add little explanatory power in fitting options data*". This limited improvement for a significant computational cost may explain why some authors recently favor to investigate other stochastic processes and the class of Hawkes processes. E.g., Santa-Clara and Yan (2010) consider some diffusive quadratic processes for both the volatility and the jump intensity. They find that the correlation between the relative increments of these latent factors is quite low (0.17). This suggests that there is no need to systematically connect the intensity to the level of the volatility. The intensity may have a separate process.

³The very first process, developed by Hawkes (1971), has been used in seismology to model the frequency of earthquakes and aftershocks.

⁴In this sense, the overall self-excitation of the portfolio is a rough way to capture the hidden contagion within the portfolio.

⁵We thank the referee for pointing toward this unpublished PhD thesis.

⁶This specification has been exploited by Hainaut (2016a) for interest rates modeling.

⁷Because our approach does not consider any bivariate Hawkes processes such as proposed in Aït-Sahalia et al. (2014) or in Hainaut (2016 b), it is also more parsimonious and easier to calibrate.

Our study aims at designing quadratic hedging strategies in presence of jump clustering. So we do not consider any stochastic volatility as in McClelland (2012), Boswijk et al. (2015), Carr and Wu (2016), Chen and Poon (2013) or Fulop et al. (2015). These articles mainly question if the jump size matters and, for some of them, if the role of negative jumps (associated to bad news) is key to evaluate the asset risk, option prices and the risk premium of variance swaps. The contribution of our research is very different. We propose some quadratic hedging strategies and derive closed form expressions useful to manage options when the underlying asset may experience a clustering of jumps. None of the cited articles study or even mention hedging, which is of first importance for practitioners. To investigate this issue appropriately, a number of results must be provided. E.g., we characterize a class of changes of measure that preserves the price dynamics of the underlying asset under the risk neutral measure and we modify the parameters of self-excitation accordingly. We also provide some formulae to compute the option prices and the hedging parameters.

Jumps makes the market incomplete and prevents a perfect replication of contingent claims. In this case, Föllmer and Sondermann (1986) and Cont and Tankov (2004) approximate the target payoff by a self-financed trading strategy that minimizes the quadratic hedging error. Unlike approaches based on other loss functions, quadratic hedging yields linear hedging rules that are very convenient to implement as mentioned in Schweizer (2001). This motivates us to follow the same approach. Two hedging instruments will be considered: the underlying asset and another derivative. We obtain in our setting closed form expressions for the hedge ratios with first and second order approximations of option prices. These ratios explicitly depends upon the intensity of the self-exciting jump process.

The efficiency of hedges is assessed by simulations with a model fitted to the S&P500 time series. We draw several interesting conclusions from this numerical exercise. Firstly, the spreads between the delta hedge and minimum variance ratios are not significant if the option is hedged with the underlying asset. Whatever the order of the approximation, taking into account the sensitivity of the hedge to the intensity of the jump process does not reduce significantly the exposure to the risk of jump clustering. The volatility and the Value at Risk of hedging errors remain significant and close to these obtained with a classic delta neutral hedge. Secondly, the only efficient hedging instrument is another option. In this case, the minimum variance ratios are notably different from the delta hedge ratio and the policy using a second order approximation for option prices is the best choice to mitigate the risk.

The paper is organized as follows. The next section introduces the main specifications of the model. The third section presents an econometric calibration method to legitimate empirically the proposed dynamics. This part is followed by a presentation of a category of changes of measure preserving the features of the studied process. In section 5, methods to evaluate European options and their Greeks are developed. The sensitivity of the surface of implied volatility to parameters is next discussed. Section 6 proposes minimum variance hedges, using the underlying asset or another option and based on a first and second order development of option prices. The performances of these strategies are measured and compared in section 7.

2 The framework

We consider a probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on which is defined the price process $S = (S_t)_t$ of a financial asset. This asset serves us later as underlying for European derivatives. The instantaneous return of this asset is the sum of a deterministic drift, a Brownian motion

$W = (W_t)_t$ and a jump process $N = (N_t)_t$:

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= \mu dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} (e^{J_j} - 1)\right) - \lambda_t \mathbb{E}(e^J - 1) dt, \\ &= \mu dt + \sigma dW_t + (e^J - 1) dN_t - \lambda_t \mathbb{E}(e^J - 1) dt.\end{aligned}\tag{1}$$

The drift rate and the Brownian volatility are constant and positive, i.e. $\mu \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$. $N = (N_t)_t$ is a point process and the random jumps $(J_j)_j$ are independent copies of J that has a probability density function $\nu(z)$ defined on $\mathbb{R}$. So the jumps can be positive or negative. The last term of the equation (1) is the compensator of the jump process. Hence, μ is the expected return of the asset. Its presence ensures that on average, the asset price grows at a constant rate, $\mathbb{E}\left(\frac{dS_t}{S_{t-}}\right) = \mu dt$. The dynamics of S_t may also be rewritten as follows

$$d \ln S_t = \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}(e^J - 1)\right) dt + \sigma dW_t + J dN_t,$$

and we infer from this last relation the expression for S_t :

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2}\right) t - \mathbb{E}(e^J - 1) \int_0^t \lambda_s ds + \sigma W_t + \sum_{j=1}^{N_t} J_j \right).$$

We assume, from now on, that jumps are double-exponential random variables (denoted by DEJ). Note however most of the results developed in this paper are applicable to any other statistical distributions. The probability density function (pdf) $\nu(z)$ of J is defined by the three parameters $\rho^+ \in \mathbb{R}^+$, $\rho^- \in \mathbb{R}^-$ and $p \in (0, 1)$:

$$\nu(z) = p\rho^+ e^{-\rho^+ z} 1_{\{z \geq 0\}} - (1-p)\rho^- e^{-\rho^- z} 1_{\{z < 0\}},$$

where p and $(1-p)$ are respectively the probabilities of observing upward and downward (exponential) jumps. Average sizes of positive and negative shocks are equal to $\frac{1}{\rho^+}$ and $\frac{1}{\rho^-}$. The expectation of J_j is the weighted sum of mean jump sizes:

$$\mathbb{E}(J_j) = p \frac{1}{\rho^+} + (1-p) \frac{1}{\rho^-}.$$

The moment-generating function of the sum of the jump size J and its absolute value $|J|$ is required for later developments. It is give by:

$$\begin{aligned}\psi(z_1, z_2) &:= \mathbb{E}\left(e^{z_1 J_i + z_2 |J_i|}\right) \\ &:= p \frac{\rho^+}{\rho^+ - (z_1 + z_2)} + (1-p) \frac{\rho^-}{\rho^- - (z_1 - z_2)}.\end{aligned}$$

$\psi(z_1, z_2)$ exists under the condition that $(z_1 + z_2) < \rho^+$ and $(z_1 - z_2) > \rho^-$. When λ_t is held constant over time, the model corresponds to the Double Exponential Jump Diffusion (DEJD) proposed by Kou (2002). However, the DEJD has constant moments and fails to replicate the clustering of jumps displayed by financial markets. To capture such features in the DEJD, we assume that the jump arrival intensity $\lambda = (\lambda_t)_t$ is itself a stochastic process and that it depends on the sum of absolute values of past jumps. This sum, up to time t , is denoted by L_t and is equal to

$$L_t := \sum_{i=1}^{N_t} |J_i|.$$

The intensity of jump arrival $\lambda = (\lambda_t)_t$ reverts to a level θ at a speed α and it increases by $\eta|J|$ ($\eta \in \mathbb{R}^+$) when a jump occurs. One therefore has:

$$d\lambda_t = \alpha(\theta - \lambda_t)dt + \eta dL_t.$$

It is well known that $(\lambda_t, N_t)_t$ is a Markov process and by direct integration, we can show that the influence of past jumps on λ_t decays exponentially:

$$\lambda_t = \theta + e^{-\alpha t}(\lambda_0 - \theta) + \int_0^t \eta e^{\alpha(u-t)} dL_u.$$

The integrand in this last expression is called the kernel function. The expected intensity at horizon t is in this case equal to (for a proof see e.g. Errais et al., 2010),

$$\mathbb{E}(\lambda_t | \mathcal{F}_0) = \left(\frac{\alpha\theta}{\eta\mathbb{E}(|J_i|) - \alpha} + \lambda_0 \right) e^{(\eta\mathbb{E}(|J_i|) - \alpha)t} - \frac{\alpha\theta}{\eta\mathbb{E}(|J_i|) - \alpha}. \quad (2)$$

From this last relation, we infer that the process is stable only if $\eta\mathbb{E}(|J_i|) - \alpha \leq 0$. In this case, the asymptotic value to which $\lambda = (\lambda_t)_t$ converges when t tends to infinity is finite and equal to

$$\lambda_\infty := \lim_{t \rightarrow \infty} \mathbb{E}(\lambda_t | \mathcal{F}_0) = \frac{\alpha\theta}{\alpha - \eta\mathbb{E}(|J_i|)}.$$

Notice that this asymptotic value is above θ , except if $\eta = 0$ i.e. when there is no self-excitation.

Proposition 2.1. *The variance of λ_t is equal to the next integral:*

$$\mathbb{V}(\lambda_t | \mathcal{F}_0) = \int_0^t \eta^2 e^{-2\alpha(t-u)} \mathbb{E}(|J|^2) \mathbb{E}(\lambda_u | \mathcal{F}_0) du, \quad (3)$$

where $\mathbb{E}(\lambda_u | \mathcal{F}_0)$ is provided by equation (2) and $\mathbb{E}(|J|^2) = p \frac{2}{(\rho^+)^2} + (1-p) \frac{2}{(\rho^-)^2}$.

In the rest of the paper, one consider the process $\lambda = (\lambda_t)_t$ as an observable quantity. In practice, several techniques exist to filter this process from observations. One of these methods is the particle filtering, as used in Hainaut (2017), but is computationally intensive. The “peaks over threshold” procedure is another way to determine $\lambda = (\lambda_t)_t$ and we detail this approach in the next section. This method is simple to implement and provides a sufficiently accurate estimate of the jump arrival intensity.

The next proposition derives the moment generating function (mgf) for the log-return of S_t . Throughout the paper, the log-return of S_t is denoted by $X_t := \ln \frac{S_t}{S_0}$ and the dynamics of $X = (X_t)_t$ obeys the following stochastic differential equation:

$$dX_t = \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}(e^J - 1) \right) dt + \sigma dW_t + J dN_t.$$

This equation is of course similar to the stochastic differential equation prevailing for $d \ln S_t$. The mgf of the log-return X_t is needed in the next section to establish the dynamics of the asset under the risk neutral measure.

Proposition 2.2. *The moment generating function (mgf) of $\omega_1 X_s + \omega_2 \lambda_s$ for $s \geq t$, is given by*

$$\mathbb{E} \left(e^{\omega_1 X_s + \omega_2 \lambda_s} | \mathcal{F}_t \right) = \left(\frac{S_t}{S_0} \right)^{\omega_1} \exp(A(t, s) + B(t, s)\lambda_t),$$

where $A(t, s)$, $B(t, s)$ are solutions to the system of ODE's

$$\begin{cases} \frac{\partial}{\partial t} A &= -\omega_1 \left(\mu - \frac{\sigma^2}{2} \right) - \omega_1^2 \frac{\sigma^2}{2} - \alpha \theta B \\ \frac{\partial}{\partial t} B &= \alpha B + \omega_1 (\psi(1, 0) - 1) - [\psi(\omega_1, B\eta) - 1], \end{cases} \quad (4)$$

with the terminal conditions $A(s, s) = 0$, $B(s, s) = \omega_2$.

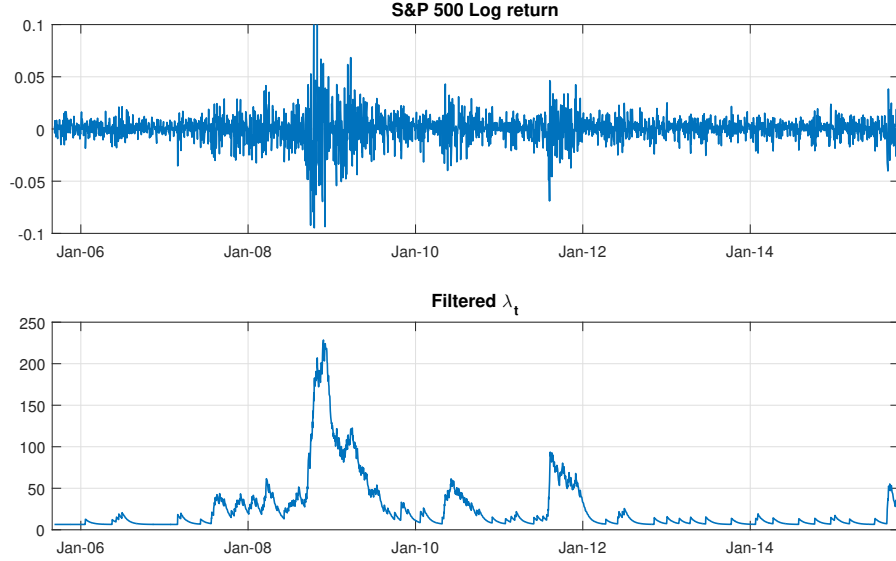


Figure 1: S&P 500 daily log-returns from the 7/9/2005 to the 13/10/2015. Clustering of jumps is clearly visible around the credit crunch and the second period of the double-dip recession. The second graph presents the sample path of $(\lambda_t)_t$ filtered by the POT procedure.

3 Data description and calibration

Before continuing with the exploration of model features, we present the data set that we use in numerical applications and to which the estimation procedure is applied. The sample of data consists of S&P 500 daily values from September 2005 to October 2015 (2543 observations). Table 1 provides summary statistics of daily returns. The yearly volatility reaches 20.64% and the very high kurtosis indicates that the distribution of returns has fat-tails. Jarque Bera and Lillie tests both reject the assumption of normality whereas the Durbin Watson statistic reveals a serial dependence. The non-normality is also confirmed by the first QQ plot of figure 2.

	Value
Mean daily return	0.02%
Standard daily deviation	1.30%
Skewness	-0.33
Kurtosis	13.51
Jarque Bera p-value	1e-3
Lillie test p-value	1e-3
Durbin Watson p-value	0e-3

Table 1: This table reports the mean, the standard deviation, the skewness, the kurtosis, statistics of normality and serial dependence, for the continuously compounded daily returns of the S&P 500 from September 2005 to October 2015.

The first graph of figure 1 plots returns of the index on the sampling period. The clustering of jumps is visible from September 2008 to the end 2009 (the US credit crunch period) and from September 2011 to February 2012 (the second period of the double-dip recession). Shocks during these periods do not display

any clear trend: negative abrupt movements alternate regularly with large positive jumps. This observation corroborates a link between the frequency of jumps and their absolute values, as assumed in our model.

Among the available estimation methods, we choose the asymmetric “peaks over threshold” (POT) procedure that is an enhanced version of the procedure of Embrechts et al. (2011). This approach is robust, easy to implement and computationally efficient. The discrete record of T observations of log returns, equally spaced by a lag Δ of one day of trading is denoted $\{x_1, x_2, \dots, x_T\}$. A jump is believed to occur if the return is above or below some thresholds. These thresholds, denoted $g(\alpha_1)$ and $g(\alpha_2)$, depend on the lag between observations and on two confidence levels, α_1 α_2 . To determine thresholds, we fit by log-likelihood maximization, a pure Gaussian process : $x_i \sim \left(\mu - \frac{\sigma^2}{2}\right) \Delta + \sigma W_\Delta$. If $\Phi(\cdot)$ denotes the pdf of a standard normal, $g(\alpha_1)$, $g(\alpha_2)$ are set to the α_1 and α_2 percentiles of the Brownian motion: $g(\alpha_i) = \sigma\sqrt{\Delta}\Phi^{-1}(\alpha_i)$. When a jump is detected, the variation of prices is assumed equal to the jump size:

$$\left\{ \left(x_i - \left(\mu - \frac{\sigma^2}{2}\right) \Delta\right) \sim J_i \quad \left(x_i - \left(\mu - \frac{\sigma^2}{2}\right) \Delta\right) > g(\alpha_1) \text{ or } < g(\alpha_2) \right\}.$$

Finally, levels of confidence, α_1 and α_2 are optimized such that the skewness and the kurtosis of x_i for periods without jump are close to these of a normal distribution. For the S&P 500, we find that α_1 and α_2 are respectively equal to 94% and 91%. The skewness and kurtosis of returns for days without detected jumps are equal to 0.047 and 3.28. The volatilities of the sample from which we eliminate positive, negative and both type of jumps are 18%, 16% and 12%. Once that jumps are detected, the sample path of $(\lambda_t)_t$ for a given set of parameters is approached by:

$$\Delta\lambda_i = \alpha(\theta - \lambda_{i-1})\Delta + \eta J_i I_{jump:at t_i}.$$

When Δ is small, the probability of observing a jump in the i^{th} interval of time is equal to $\lambda_i\Delta$. Jumps and intensities can then be calibrated by maximizing the log-likelihood of jumps distribution and of λ_t as follows:

$$\begin{cases} (\rho^-, \rho^+, p) &= \arg \max \sum_{i=1}^n \log \nu(x_i | \rho^-, \rho^+, p) I_{jump:at t_i} \\ (\alpha, \eta, \theta, \lambda_0) &= \arg \max \sum_{i=1}^n \log ((\lambda_i\Delta) I_{jump:at t_i} + (1 - \lambda_i\Delta) I_{no jump:at t_i}) \end{cases},$$

where $\nu(\cdot)$ is the pdf of double exponential jumps. The second graph of figure 1 shows the sample path of λ_t filtered by the POT procedure. This intensity is an excellent indicator of market stress and it reaches its highest level during the credit crunch of 2008 or during the second period of the double dip crisis. Outside these periods, the intensity converges to its asymptotic level, $\lambda_\infty = 5.62$. The parameters obtained by the calibration procedure are reported in table 2. The average return is close to 5% whereas the volatility of the Brownian part is around 12%. A pure diffusion fitted to the same data set has a standard deviation that climbs to 21%. This means that the jump process in our model, generates marginally 75% more volatility i.e. an added value of 9% to the volatility. The quality of the fit is assessed with the QQ plots in figure 2 and seems excellent. The parameters obtained in this section are used in section 7, in numerical illustrations.

Parameters			
μ	0.05	θ	6.44
σ	0.12	p	0.37
α	14.71	ρ^+	30.47
η	337.08	ρ^-	-33.90

Table 2: Parameters fitted by the “peak over threshold” procedure.

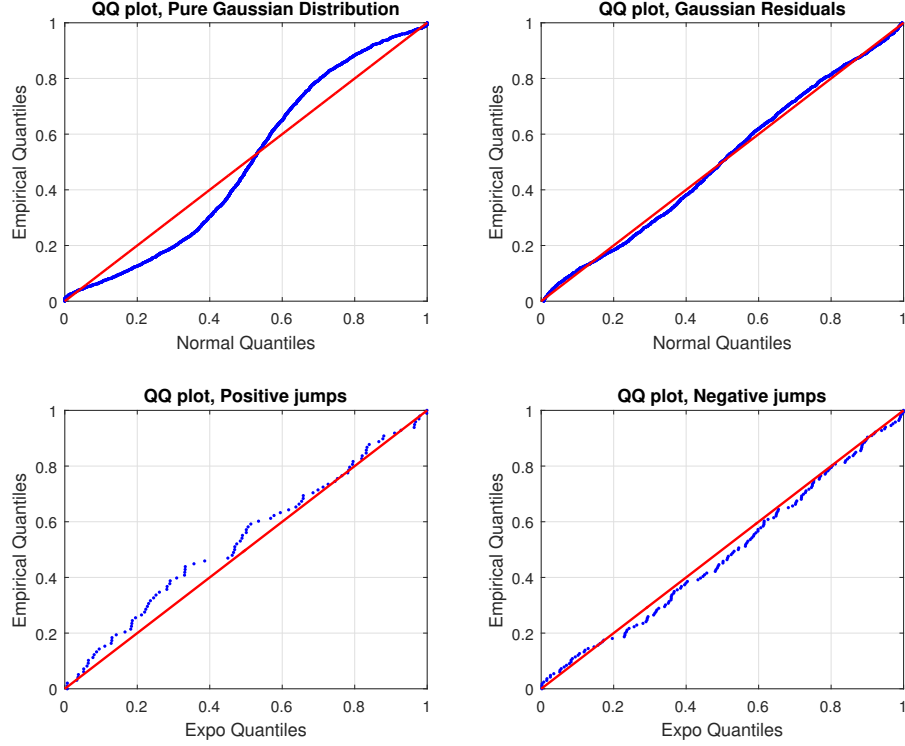


Figure 2: The upper left graph shows the QQ plot of a pure diffusion model, fitted to the S&P time series, versus the empirical percentiles. The next figure presents the QQ plots of the Gaussian part in the jump diffusion model versus the empirical distribution of returns, on days when no jump is detected by the POT method. The two lower graphs show the QQ plots of truncated exponential jumps (identified by the POT procedure) versus the corresponding exponential distribution.

Boswijk et al. (2015) do a similar exercise and fit a jump-diffusion with self-excitation to the S&P 500. Their model differs from our approach on several points. Firstly, we assume that jumps are distributed according to a double exponential law instead of being normal random variables. Secondly, the shock of intensity caused by a jump of the stock price is proportional to its amplitude, whereas it is constant in Boswijk et al. (2015). Finally, as our objective it to analyze the impact of self-excitation on hedging ratios, we don't consider stochastic volatility. However, a comparison of parameters estimates with these reported in Boswijk et al. (2015) allows us to understand the origin of the volatility in each model. As underlined by figures of table 3, the speeds of mean reversion α are comparable (14.71 in our model versus 18.16) but the the mean reversion is much higher in our model. In Boswijk et al. (2015), self-excited jumps correspond to rare and violent economic shocks whereas other variations are explained by the stochastic volatility. As we don't include this feature in the dynamics of stock prices, jumps are more frequent in our setting and explain a larger variety of events. The increase of intensity caused by a jump in the Boswijk et al. model is constant and around 16.62. In our framework, positive and negative jumps raise the intensity respectively on average by 11.24 and 9.92. On average, the jump intensity is then slightly less impacted by shocks of stock prices in our model than in Boswijk et al. (2015).

	Self-exciting jump diffusion without stoch. volatility		Self-exciting jump diffusion with stoch. volatility
α	14.71	α	18.16
θ	6.44	θ	0.32
$\frac{\eta}{\rho^+}$	11.24	η	16.62
$\frac{\eta}{ \rho^- }$	9.92		

Table 3: Comparison of parameter estimates with these of Boswijk et al. (2015).

4 Changes of measure

We continue to explore the properties of the self-excited process with a discussion about the choice of a risk neutral measure. The market, such as modeled, is incomplete. A consequence of this incompleteness is the existence of several equivalent measures that are all potential candidates for the definition of a risk neutral one. In this paper, we focus on a family of changes of measure that are induced by exponential martingales of the form:

$$M_t(\xi, \varphi) := \exp(\kappa_1(\xi)\lambda_t + \xi L_t + \kappa_2(\xi)t) \times \exp\left(-\frac{1}{2} \int_0^t \varphi(s)^2 ds - \int_0^t \varphi(s) dW_s\right). \quad (5)$$

where $\varphi(s)$ is a $\mathcal{F}_s$ adapted process and ξ is constant. $\kappa_1(\xi)$ and $\kappa_2(\xi)$ are functions of ξ that corresponds to the price of jump risk. Zhang et al. (2009) use a similar change of measure to simulate rare events, of a one-dimension Hawkes process, without Brownian component and only with constant jumps. In our framework, jumps are random and the affine change of measure modifies both frequencies and distribution. We focus on processes $\varphi(s)$ that are linear functions of the intensity:

$$\varphi(s) = \varphi_1 + \varphi_2 \lambda_s,$$

The next proposition details the conditions under which the process $M = (M_t)_t$ is a local martingale:

Proposition 4.1. *If for any parameter ξ , there exist solutions $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ of the system*

$$\begin{aligned} \kappa_1 \alpha - (\psi(0, \kappa_1 \eta + \xi) - 1) &= 0 \\ \kappa_2 + \kappa_1 \alpha \theta &= 0, \end{aligned} \quad (6)$$

then $(M_t(\xi))_t$ is a local martingale.

Using similar arguments to these of proposition 2.2, it is possible to show that $(M_t(\xi))_t$ is a martingale if conditions (6) are fulfilled. An equivalent measure $Q^{\xi, \varphi}$ can then be defined by the ratio:

$$\frac{dQ^{\xi, \varphi}}{dP} \Big|_{\mathcal{F}_t} = \frac{M_t(\xi, \varphi)}{M_0(\xi, \varphi)}. \quad (7)$$

This new measure is particularly interesting because it preserves the structure of the jump process as demonstrated by next proposition.

Proposition 4.2. *Let us denote by $N^Q = (N_t^Q)_t$, the counting process with the following intensity*

$$\lambda_t^Q = \psi(0, \kappa_1 \eta + \xi) \lambda_t,$$

under $Q^{\xi, \varphi}$. We also define random variables J^Q through its moment generating function:

$$\psi^Q(z_1, z_2) = \frac{\psi(z_1, z_2 + (\kappa_1 \eta + \xi))}{\psi(0, \kappa_1 \eta + \xi)},$$

and the process $L_t^Q = \sum_{j=1}^{N_t^Q} |J_j^Q|$. Then the dynamics of $\lambda = (\lambda_t)_t$ under $Q^{\xi, \varphi}$ is ruled by the following SDE

$$d\lambda_t^Q = \alpha \left(\theta^Q - \lambda_t^Q \right) dt + \eta^Q dL_t^Q,$$

where

$$\begin{aligned} \theta^Q &= \theta \psi(0, \kappa_1 \eta + \xi), \\ \eta^Q &= \eta \psi(0, \kappa_1 \eta + \xi). \end{aligned}$$

On the other hand, we show that under the risk neutral measure, jumps still have a double exponential distribution as stated in the following result:

Proposition 4.3. *Under $Q^{\xi, \varphi}$ jumps, J_i^Q are double-exponential random variables with a density equal to*

$$\nu^Q(z) = p^Q \rho^{+Q} e^{-\rho^{+Q} z} 1_{\{z \geq 0\}} - (1 - p^Q) \rho^{-Q} e^{-\rho^{-Q} z} 1_{\{z < 0\}}, \quad (8)$$

and where the parameters are adjusted as follows:

$$\begin{aligned} \rho^{+Q} &= \rho^+ - (\kappa_1 \eta + \xi), \\ \rho^{-Q} &= \rho^- + (\kappa_1 \eta + \xi), \\ p^Q &= \frac{p \rho^+ \rho^{-Q}}{(p \rho^+ \rho^{-Q} + (1 - p) \rho^- \rho^{+Q})}. \end{aligned}$$

It remains to determine the $\varphi(t)$, the $\mathcal{F}_t$ -adapted process involved in the definition of $\frac{dQ^{\xi, \varphi}}{dP}$ such that the discounted asset price is a martingale under the risk neutral measure.

Proposition 4.4. *If the $\mathcal{F}_t$ adapted process defining the martingale (5), is equal to*

$$\varphi(t) = \frac{\mu + \lambda_t [\psi(0, \kappa_1 \eta + \xi) (\psi^Q(1, 0) - 1) - (\psi(1, 0) - 1)] - r}{\sigma}, \quad (9)$$

then the equivalent measure is risk neutral and the log-return, $X_t := \ln \frac{S_t}{S_0}$, is driven by the next dynamics under the measure $Q^{\xi, \varphi}$

$$dX_t = \left(r - \frac{1}{2} \sigma^2 - \mathbb{E}^Q \left((e^{J^Q} - 1) \mid \mathcal{F}_t \right) \lambda_t^Q \right) dt + \sigma dW_t^Q + J^Q dN_t.$$

The asset price is in this case ruled by the following SDE

$$\begin{aligned} dS_t &= r S_t dt + \sigma S_t dW_t^Q \\ &+ S_t \left[(e^{J_t^Q} - 1) dN_t^Q - \mathbb{E}^Q \left((e^{J^Q} - 1) \mid \mathcal{F}_t \right) \lambda_t^Q dt \right]. \end{aligned} \quad (10)$$

Note that the function φ can be splitted into two parts to highlight that the risk premium is the sum of two components:

$$\varphi(t) = \frac{\mu - r}{\sigma} + \lambda_t \frac{\psi(0, \kappa_1 \eta + \xi) [\psi^Q(1, 0) - 1] - [\psi(1, 0) - 1]}{\sigma}.$$

The terms of this sum are respectively the risk premiums for the Brownian motion and for the jump risk. The next corollary is a direct sequel of this last proposition and is used in the next section for options pricing. It proves that the moment generating function (mgf) of the log-return is the exponential of an affine function of X_t and λ_t^Q .

ξ	θ^Q	η^Q	p^Q	ρ^{+Q}	ρ^{-Q}	$\psi(0, \kappa_1 \eta + \xi)$	λ_∞^Q
-6.00	4.68	244.82	0.36	42.74	-46.17	0.73	4.23
-5.00	4.85	254.03	0.37	41.12	-44.55	0.75	4.37
-4.00	5.05	264.50	0.37	39.40	-42.83	0.78	4.53
-3.00	5.29	276.69	0.37	37.58	-41.01	0.82	4.72
-2.00	5.57	291.36	0.37	35.58	-39.01	0.86	4.94
-1.00	5.92	310.05	0.37	33.31	-36.74	0.92	5.22
0.00	6.44	337.08	0.37	30.47	-33.90	1.00	5.62
1.00	7.67	401.45	0.38	25.25	-28.68	1.19	6.53

Table 4: This table illustrates the relationship between ξ and the jump process parameters, under $Q^{\xi, \varphi}$. The other parameters used to produce this table are these fitted by the POT procedure, and reported in table 2. Notice that when $\xi = 0$, dynamics under P and $Q^{\xi, \varphi}$ are exactly the same.

Corollary 4.5. *The moment generating function of X_s under $Q^{\xi, \varphi}$ for $s \geq t$, is given by*

$$\Upsilon_{t,s}(\omega) := \mathbb{E}^Q (e^{\omega X_s} | \mathcal{F}_t) = \exp \left(\omega X_t + A(t, s) + B(t, s) \lambda_t^Q \right).$$

where $A(t, s)$, $B(t, s)$ are solutions to the system of ODE's

$$\begin{cases} \frac{\partial}{\partial t} A &= -\omega \left(r - \frac{1}{2} \sigma^2 \right) - \omega^2 \frac{\sigma^2}{2} - \alpha \theta^Q B \\ \frac{\partial}{\partial t} B &= \alpha B + \omega \left(\psi^Q(1, 0) - 1 \right) - [\psi^Q(\omega_1, B \eta^Q) - 1] \end{cases}, \quad (11)$$

with the terminal conditions $A(s, s) = 0$, $B(s, s) = 0$.

The constant that serves us to define the new measure, ξ is the cost of the risk for the jump component in the price process. As illustrated in table 4, it significantly influences the parameters defining the price process under the risk neutral measure and the asymptotic level of the intensity. The numerical analysis reveals that for values of ξ above one, the system of equations (4.1) does not admit any real solutions for κ_1 and κ_2 . If $\xi = 0$, the parameters of the jump process are identical under the real and risk neutral measures. For $\xi \in [0, 1]$, parameters defining λ_t under $Q^{\xi, \varphi}$ are higher than these under P . Whereas for negative values of ξ , they are lower than under P . The last column of the table 4 emphasizes that ξ is directly proportional to the asymptotic value of the jump arrival intensity. Note that, because values of ξ above one are not allowed by design, this asymptotic value is bounded from above.

5 Call-Put pricing and Greeks

Let us consider European call and put options of maturity T , written on S_t . Their payoff and their strike are expressed as functions of the log-return $\ln(\frac{S_T}{S_0})$ and of k , the log-strike (such that the strike is equal to $K = S_0 e^k$). The prices of call and put options are functions of the log-strike k denoted by $C(k)$ and $P(k)$. If the risk neutral density at time $t \leq T$ of the log return $\ln \frac{S_T}{S_0} | \mathcal{F}_t$ is noted $f_{t,T}(x)$, these prices are equal to their expected discounted payoffs

$$\begin{aligned} C(k) &= S_0 \int_k^{+\infty} e^{-r(T-t)} (e^x - e^k) f_{t,T}(x) dx, \\ P(k) &= S_0 \int_{-\infty}^k e^{-r(T-t)} (e^k - e^x) f_{t,T}(x) dx. \end{aligned}$$

As $C(k)$ (resp. $P(k)$) tends to S_t (resp. $-S_t$) when $k \rightarrow -\infty$ (resp. $k \rightarrow +\infty$), $C(k)$ and $P(k)$ are not square integrable with respect to k and their Fourier transforms are not defined. For this reason, we consider the modified call and put prices denoted by $c(k) = e^{\epsilon k} C(k)$, $p(k) = e^{\epsilon k} P(k)$, for which the Fourier transform exists for some ϵ ($\epsilon > 1$ for the call and $\epsilon < -1$ for the put). The Fourier transforms of $c(k)$ and $p(k)$ are defined as follows:

$$\begin{aligned}\mathcal{FC}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega k} c(k) dk, \\ \mathcal{FP}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega k} p(k) dk.\end{aligned}$$

Recalling that $\Upsilon_{t,s}(\omega) = \mathbb{E}^Q(e^{\omega X_s} | \mathcal{F}_t)$ is given by corollary (4.5), a direct calculation leads to the same expressions of $\mathcal{FC}(\omega)$ and $\mathcal{FP}(\omega)$:

$$\mathcal{FC}(\omega) = \mathcal{FP}(\omega) = \frac{S_0 e^{-r(T-t)}}{(i\omega + \epsilon)^2 + (i\omega + \epsilon)} \Upsilon_{t,T}(i\omega + \epsilon + 1),$$

except that ϵ is positive (resp. negative) for the call (resp. put). The values of call options are then obtained by inverting the Fourier transform:

$$C(k) = \frac{S_0 e^{-\epsilon k - r(T-t)}}{\pi} \int_0^{\infty} e^{-i\omega k} \frac{\Upsilon_{t,T}(i\omega + \epsilon + 1)}{(i\omega + \epsilon)^2 + (i\omega + \epsilon)} d\omega. \quad (12)$$

As same expressions hold for puts, except that $\epsilon < 0$, we exclusively focus on call options in the remainder of this section. The naive approach consists of calculating numerically the integral present in the equation (12). Setting $\omega_j = \Delta_\omega(j-1)$, an approximation of the call price is in this case given by:

$$C(k) \approx \frac{S_0 e^{-\epsilon k - r(T-t)}}{\pi} \sum_{m=1}^M e^{-i\omega_m k} \delta_m \left[\frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} \right] \Delta_\omega, \quad (13)$$

where $\delta_j = \frac{1}{2}1_{\{j=1\}} + 1_{\{j \neq 1\}}$. An judicious choice for the discretization steps in the equation (13), allows us to use a Fast Fourier Transform algorithm to speed up calculations. This point is detailed in the next proposition.

Proposition 5.1. *Let M be the number of steps used in the Discrete Fourier Transform (DFT) and $\Delta_k = \frac{2k_{max}}{M-1}$, be the step of discretization. Let us denote $\delta_j = \frac{1}{2}1_{\{j=1\}} + 1_{\{j \neq 1\}}$, $\Delta_\omega = \frac{2\pi}{M\Delta_k}$ and $\omega_j = (j-1)\Delta_\omega$. The values of $C(k)$ at points $k_j = -\frac{M}{2}\Delta_k + (j-1)\Delta_k$ are approached by*

$$C(k_j) \approx \frac{2S_0 e^{-\epsilon k_j - r(T-t)}}{M\Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(\frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} (-1)^{m-1} \right) e^{-i\frac{2\pi}{M}(m-1)(j-1)} \right). \quad (14)$$

This last relation can be computed with a fast Fourier transform algorithm.

The next section focuses on the minimum variance strategy to hedge European derivatives. The implementation of this strategy requires to estimate the sensitivities of option prices to variations of underlying state variables. These sensitivities are measured by the first order differential of call ($\epsilon > 0$) or put ($\epsilon < 0$) prices with respect to S_t and λ_t that are evaluated by a DFT procedure:

Corollary 5.2. *The first order sensitivities of call prices with respect to state variables are given by expressions:*

$$\frac{\partial C(k_j)}{\partial S_t} = \frac{2S_0 e^{-\epsilon k_j - rT}}{M\Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(\frac{(i\omega_m + \epsilon + 1)}{S_t} \frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} (-1)^{m-1} \right) e^{-i\frac{2\pi}{M}(m-1)(j-1)} \right),$$

$$\frac{\partial C(k_j)}{\partial \lambda_t} = \frac{2S_0 e^{-\epsilon k_j - rT}}{M \Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(B(t, T) \frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} (-1)^{m-1} \right) e^{-i \frac{2\pi}{M} (m-1)(j-1)} \right),$$

where $B(t, T)$ is the function defined in corollary 4.5.

Corollary 5.3. *The second order sensitivities of call prices with respect to state variables are given by expressions:*

$$\frac{\partial^2 C(k_j)}{\partial S_t^2} = \frac{2S_0 e^{-\epsilon k_j - rT}}{M \Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(\frac{(i\omega_m + \epsilon + 1)}{S_t^2} \frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon) + 1} (-1)^{m-1} \right) e^{-i \frac{2\pi}{M} (m-1)(j-1)} \right),$$

$$\frac{\partial^2 C(k_j)}{\partial \lambda_t^2} = \frac{2S_0 e^{-\epsilon k_j - rT}}{M \Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(B(t, T)^2 \frac{\Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} (-1)^{m-1} \right) e^{-i \frac{2\pi}{M} (m-1)(j-1)} \right),$$

$$\frac{\partial^2 C(k_j)}{\partial \lambda_t \partial S_t} = \frac{2S_0 e^{-\epsilon k_j - rT}}{M \Delta_k} \operatorname{Re} \left(\sum_{m=1}^M \delta_m \left(\frac{(i\omega_m + \epsilon + 1)}{S_t} \frac{B(t, T) \Upsilon_{t,T}(i\omega_m + \epsilon + 1)}{(i\omega_m + \epsilon)^2 + (i\omega_m + \epsilon)} (-1)^{m-1} \right) e^{-i \frac{2\pi}{M} (m-1)(j-1)} \right),$$

where $B(t, T)$ is the function defined in corollary 4.5.

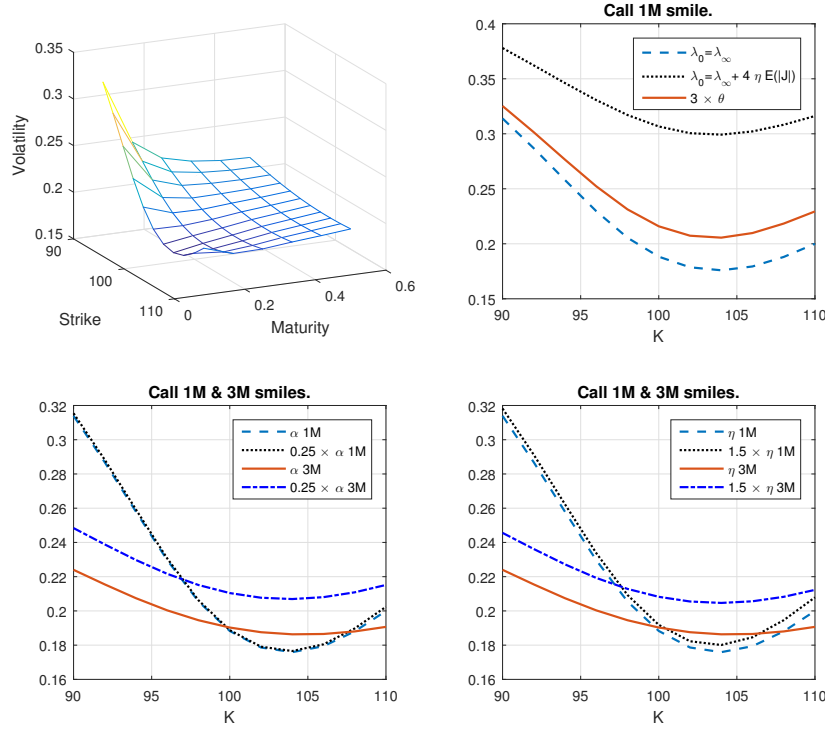


Figure 3: The upper left graph shows the surface of implied volatilities for call options, priced with parameters of table 2. The other graphs illustrate the impact of a modification of these parameters on shapes of 1 month and 3 months volatility smiles.

To figure out how self-excited jumps influence options prices, we evaluate call options with parameters of table 2 and invert the Black & Scholes formula to retrieve the smile of volatilities. The surface of implied volatilities for different maturities and moneyness is plotted in the first graph of figure 3. The

curvature of the smile is more pronounced for short term maturities than for long term ones. The right upper graph shows that in period during which jumps are more frequent, the implied volatilities raise. It also emphasizes that a higher level of mean reversion for λ_t shifts up the surface. The left lower graph illustrates the impact of a reduction of the speed of mean reversion of λ_t on the smile. At short term, its influence is limited, even null. This reduction mainly affects the long-term part of the smile of volatilities. The last graph reveals that increasing η has a similar effect.

6 Minimum variance hedging of options

A natural question that arises now is how to hedge these options? A first answer could be to “Delta” hedge the position. This approach consists of buying $\frac{\partial C}{\partial S_t}$ shares of the underlying asset to build a replicating portfolio with the same first order sensitivity as the option. However, the presence of jumps in the asset dynamics reduces the efficiency of this method. To take into account the jump risk in the hedging strategy, we opt for a minimum variance approach, as developed by Föllmer and Sondermann (1986). This consists of minimizing the variance under the risk neutral measure, of the spread between a self-financed portfolio and the option payoff.

The choice of the risk neutral measure Q to perform the optimization is debatable. However, quadratic hedging with discontinuous processes under other measures does not admit a solution in general. On the other hand, the presence of jumps makes the market incomplete and the risk neutral measure is then not unique. The parameters defining the dynamics of the underlying asset under Q may then be adjusted to reflect the uncertainty over the evolution of prices and the risk aversion of traders. In this case, the hedge is based on a riskier dynamics for the underlying asset under Q and is more conservative than any other strategy built under P . Such a robust approach was pioneered in economics by Hansen and Sargent (1995) or (2001) and it justifies minimizing the variance under a risk neutral measure.

In following sections, we consider two hedging portfolios: one composed of cash and of the underlying asset and one with cash and another option. In both cases, our analysis reveals that the presence of the self-exciting mechanism modifies the optimal hedging portfolio.

6.1 Hedging with the underlying asset

The remainder of this section focuses on the optimal minimum variance strategy to hedge a derivative, with a self-financed portfolio of cash and of the underlying asset. Under the risk neutral measure, the dynamics of the asset is driven by

$$\begin{aligned} dS_t = & rS_t dt + \sigma S_t dW_t^Q \\ & + S_t \left[\left(e^{J_t^Q} - 1 \right) dN_t^Q - \mathbb{E}^Q \left(\left(e^{J_t^Q} - 1 \right) \mid \mathcal{F}_t \right) \lambda_t^Q dt \right]. \end{aligned}$$

If $\chi(dz, dt)$ is the Poisson random measure of the the jump process under Q such that $L_t^Q = \int_0^t \int_{\mathbb{R}} z \chi(dz, dt)$, the dynamics of S_t can be rewritten as follows:

$$\begin{aligned} dS_t = & rS_t dt + \sigma S_t dW_t^Q \\ & + S_t \left[\int_{\mathbb{R}} (e^z - 1) \left(\chi(dz, dt) - \nu^Q(dz) \lambda_t^Q dt \right) \right]. \end{aligned}$$

On the other hand, we denote a self-financed strategy by ϕ_t . ϕ_t points out here the number of shares of the underlying asset. This is a predictable process $\phi : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that $\int_0^T \phi_t dS_t$ is a square integrable martingale. As previously, the risk free rate is constant and noted r . The discounted value of the self-financed portfolio, that is denoted by $\tilde{P} = e^{-rt} P_t$, is then

$$d\tilde{P}_t = \phi_t d\tilde{S}_t,$$

where $\tilde{S}_t$ is the discounted stock price. If the strategy is self-financed, $\tilde{P}_t$ satisfies the following relation:

$$\tilde{P}_t = P_0 + \int_0^t \phi_s d\tilde{S}_s.$$

The discounted stock value, $\tilde{S}_t = \frac{S_t}{B_t}$ is ruled by the following dynamics,

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t^Q + \tilde{S}_t \left[\int_{\mathbb{R}} (e^z - 1) \left(\chi(dz, dt) - \nu^Q(dz) \lambda_t^Q dt \right) \right]. \quad (15)$$

From this last relation, we infer that its quadratic bracket is given by:

$$d[\tilde{S}_t, \tilde{S}_t] = \sigma^2 \tilde{S}_t^2 dt + \tilde{S}_t^2 \int_{\mathbb{R}} (e^z - 1)^2 \chi(dz, dt).$$

Let us denote by Y the European payoff of an option expiring at date T . The minimum variance hedging strategy, due to Föllmer and Sondermann (1986) consists of determining the initial amount P_0 and the self-financed strategy ϕ_t that minimizes the quadratic hedging error:

$$\min \mathbb{E}^Q \left(\left(P_0 + \int_0^T \phi_s d\tilde{S}_s - B_T^{-1} Y \right)^2 \mid \mathcal{F}_0 \right).$$

By construction, the expected discounted payoff $\tilde{Y}_t := \mathbb{E}^Q \left(\frac{1}{B_T} Y \mid \mathcal{F}_t \right)$ is a martingale and then a stochastic integral with respect to the driving risk factors according to the martingale representation theorem. It satisfies the relation:

$$\tilde{Y}_t = \tilde{Y}_0 + \int_0^t \sigma_s^Y dW_s^Q + \int_0^t \int_{\mathbb{R}} \gamma_s^Y(z) \left(\chi(dz, ds) - \nu^Q(z) \lambda_s^Q ds \right), \quad (16)$$

where $\sigma^Y : [0, \infty) \rightarrow \mathbb{R}$ is a càdlàg $\mathcal{F}_t$ adapted process and $\gamma^Y : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ is a predictable random function. From its infinitesimal form, one has:

$$d\tilde{Y}_t = \sigma_t^Y dW_t^Q + \int_{\mathbb{R}} \gamma_t^Y(z) \chi(dz, dt) - \int_{\mathbb{R}} \gamma_t^Y(z) \nu^Q(dz) \lambda_t^Q dt,$$

we deduce that its quadratic bracket satisfies the following relation:

$$d[\tilde{Y}_t, \tilde{Y}_t] = (\sigma_t^Y)^2 dt + \int_{\mathbb{R}} (\gamma_t^Y(z))^2 \chi(dz, dt).$$

This last observation allows us to build the minimum variance price and strategy which are similar to these obtained by Cont and Tankov (2004), except the presence of self-excitation in the dynamics of S_t .

Proposition 6.1. *Let $\tilde{Y}_t$ and $\tilde{S}_t$ be defined by equations (16) and (15), then the minimum variance price is*

$$P_0^* = \tilde{Y}_0 = \mathbb{E}^Q (B_T^{-1} Y \mid \mathcal{F}_0), \quad (17)$$

and the minimum variance hedging strategy is given by:

$$\phi_t^* = \frac{1}{\tilde{S}_t} \frac{\sigma_t^Y \sigma + \int_{\mathbb{R}} \gamma_t^Y(z) (e^z - 1) \nu^Q(dz) \lambda_t^Q}{\left(\sigma^2 + \int_{\mathbb{R}} [(e^z - 1)^2] \nu^Q(dz) \lambda_t^Q \right)}. \quad (18)$$

The process $\tilde{Y}_t$ is the discounted price of a European derivative, like e.g. a call or a put option and is a function of underlying state variable. The option (put or call) price is denoted by $O(t, S_t, \lambda_t^Q)$ and is such that $\tilde{Y}_t = e^{-rt}O(t, S_t, \lambda_t^Q)$. If this function is continuous with respect to time and twice continuously differentiable with respect to S_t then $O(t, S_t, \lambda_t^Q)$ admits the following representation:

$$\begin{aligned} d\tilde{Y}_t &= \frac{\partial}{\partial t}e^{-rt}O + \frac{\partial}{\partial S}e^{-rt}O dS_t + \frac{1}{2}\frac{\partial^2}{\partial S^2}e^{-rt}O d[S_t, S_t]^c + \frac{\partial}{\partial \lambda^Q}e^{-rt}O d\lambda_t^Q \\ &\quad + e^{-rt} \left[O(t, S_t, \lambda_t^Q) - O(t-, S_{t-}, \lambda_{t-}^Q) - \frac{\partial}{\partial S}O \Delta S_t - \frac{\partial}{\partial \lambda^Q}O \Delta \lambda_t^Q \right], \end{aligned}$$

or after developments,

$$\begin{aligned} d\tilde{Y}_t &= -re^{-rt}O + e^{-rt}\frac{\partial}{\partial t}O + \left[\frac{\partial}{\partial S}e^{-rt}O \right] \left(r - \mathbb{E}^Q \left((e^{J^Q} - 1) \mid \mathcal{F}_t \right) \lambda_t^Q \right) S_t dt \\ &\quad + \frac{1}{2} \left[\frac{\partial^2}{\partial S^2}e^{-rt}O \right] \sigma^2 S_t^2 dt + \left[\frac{\partial}{\partial \lambda^Q}e^{-rt}O \right] \alpha \left(\theta^Q - \lambda_t^Q \right) dt + \left[\frac{\partial}{\partial S}e^{-rt}O \right] \sigma S_t dW_t^Q \\ &\quad + e^{-rt} \int_{\mathbb{R}} \left[O \left(t, S_{t-}e^z, \lambda_{t-}^Q + \eta^Q |z| \right) - O(t-, S_{t-}, \lambda_{t-}^Q) \right] \nu^Q(dz) \lambda_{t-}^Q dt \\ &\quad + e^{-rt} \int_{\mathbb{R}} \left[O \left(t, S_{t-}e^z, \lambda_{t-}^Q + \eta^Q |z| \right) - O(t-, S_{t-}, \lambda_{t-}^Q) \right] \left(\chi(dz, dt) - \nu^Q(dz) \lambda_{t-}^Q dt \right). \end{aligned}$$

The last line and the Brownian term of this equation are local martingales. On the other hand, $\tilde{Y}_t$ is also a martingale. Therefore, the sum of all other terms is a finite variation continuous local martingale and we infer the following martingale representation for $\tilde{Y}_t$:

$$\begin{aligned} \tilde{Y}_t &= e^{-rt}O(t, S_t, \lambda_t^Q) = O(0, S_0, \lambda_0^Q) + \int_0^t \frac{\partial O}{\partial S_s} \sigma e^{-rs} S_s dW_s^Q + \\ &\quad \int_0^t \int_{\mathbb{R}} e^{-rs} \left[O \left(s, S_s e^z, \lambda_s^Q + \eta^Q |z| \right) - O(s, S_s, \lambda_s^Q) \right] \left(\chi(dz, ds) - \nu^Q(dz) \lambda_s^Q ds \right). \end{aligned}$$

From this last relation, we can obtain the expressions for σ_t^Y and $\gamma_t^Y(z)$ in the dynamics of $\tilde{Y}_t$ as presented in equation (16). One has:

$$\sigma_t^Y = \frac{\partial O(t, S_t, \lambda_t^Q)}{\partial S_t} \sigma e^{-rt} S_t,$$

and

$$\gamma_t^Y(z) = e^{-rt} \left[O \left(t, S_t e^z, \lambda_t^Q + \eta^Q |z| \right) - O(t, S_t, \lambda_t^Q) \right].$$

This is a heuristic argument but the reader interested by a more rigorous proof of the continuity of option prices may refer to Cont and Tankov (2004). Setting up the minimum variance hedge would require to calculate $\int_{\mathbb{R}} \gamma_s^Y(z) \nu(dz)$. This step being computational intensive, the minimum variance strategy is approached by the following result.

Proposition 6.2. *If we denote the sensitivities of the derivative price to state variables by:*

$$\Delta_S = \frac{\partial O(t, S_t, \lambda_t^Q)}{\partial S_t} \quad \Delta_\lambda = \frac{\partial O(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q},$$

then the optimal number of shares of S_t is approached by the following ratio at a first order:

$$\phi_t^* = \Delta_S + \Delta_\lambda \frac{\eta^Q \lambda_t^Q}{S_t} \frac{\mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right) \right)}{\left(\sigma^2 + \mathbb{E}^Q \left((e^{J^Q} - 1)^2 \right) \lambda_t^Q \right)} + \mathcal{O}(S_t^2), \quad (19)$$

where

$$\mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right) = p^Q \left(\frac{\rho^{+Q}}{\rho^{+Q} - 2} - \frac{2\rho^{+Q}}{\rho^{+Q} - 1} + 1 \right) + (1 - p^Q) \left(\frac{\rho^{-Q}}{\rho^{-Q} - 2} - \frac{2\rho^{-Q}}{\rho^{-Q} - 1} + 1 \right), \quad (20)$$

$$\mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right) \right) = p^Q \left(\frac{\rho^{+Q}}{(1 - \rho^{+Q})^2} - \frac{1}{\rho^{+Q}} \right) - (1 - p^Q) \left(\frac{\rho^{-Q}}{(1 - \rho^{-Q})^2} - \frac{1}{\rho^{-Q}} \right). \quad (21)$$

This result confirms that the optimal hedge ratio is not only the delta: it contains an additional term related to the frequency of jumps. In the next section, we assess the efficiency of this strategy with numerical simulations. A more accurate approximation of the hedge ratio is obtained by developing the function $\gamma_t^Y(z)$ to the second order. But it is time consuming to use.

Proposition 6.3. *If we denote the second order sensitivities of the derivative price to state variables by:*

$$\Gamma_S = \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial S_t^2} \quad \Gamma_\lambda = \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q} \quad \Gamma_{S\lambda} = \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial S_t \partial \lambda_t^Q},$$

then the optimal number of shares of S_t is approached at second order by

$$\begin{aligned} \phi_t^* = \Delta_S &+ \frac{\lambda_t^Q}{S_t} \frac{\Delta_\lambda \eta^Q \mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right) \right)}{\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)} \lambda_t^Q + \frac{\lambda_t^Q}{S_t} \frac{\Gamma_{\lambda \frac{1}{2}} (\eta^Q)^2 \mathbb{E}^Q \left(|J^Q|^2 \left(e^{J^Q} - 1 \right) \right)}{\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)} \lambda_t^Q \\ &+ \lambda_t^Q S_t \frac{\frac{1}{2} \Gamma_S \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^3 \right)}{\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)} \lambda_t^Q + \lambda_t^Q \frac{\eta^Q \Gamma_{S,\lambda} \mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right)^2 \right)}{\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)} \lambda_t^Q + \mathcal{O}(S_t^3), \end{aligned}$$

where $\mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)$ and $\mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right) \right)$ are equal to equations (20) and (21) whereas

$$\begin{aligned} \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^3 \right) &= p^Q \left(\frac{\rho^{+Q}}{\rho^{+Q} - 3} - \frac{3\rho^{+Q}}{\rho^{+Q} - 2} + \frac{3\rho^{+Q}}{\rho^{+Q} - 1} - 1 \right) \\ &+ (1 - p^Q) \left(\frac{\rho^{-Q}}{\rho^{-Q} - 3} - \frac{3\rho^{-Q}}{\rho^{-Q} - 2} + \frac{3\rho^{-Q}}{\rho^{-Q} - 1} - 1 \right), \end{aligned}$$

$$\begin{aligned} \mathbb{E}^Q \left(|J^Q|^2 \left(e^{J^Q} - 1 \right) \right) &= -2p^Q \left(\frac{\rho^{+Q}}{(1 - \rho^{+Q})^3} + \frac{1}{(\rho^{+Q})^2} \right) \\ &- 2(1 - p^Q) \left(\frac{\rho^{-Q}}{(1 - \rho^{-Q})^3} + \frac{1}{(\rho^{-Q})^2} \right), \end{aligned}$$

$$\begin{aligned} \mathbb{E}^Q \left(|J^Q| \left(e^{J^Q} - 1 \right)^2 \right) &= p^Q \left[\frac{\rho^{+Q}}{(2 - \rho^{+Q})^2} - \frac{2\rho^{+Q}}{(1 - \rho^{+Q})^2} + \frac{1}{\rho^{+Q}} \right] \\ &- (1 - p^Q) \left[\frac{\rho^{-Q}}{(2 - \rho^{-Q})^2} - \frac{2\rho^{-Q}}{(1 - \rho^{-Q})^2} + \frac{1}{\rho^{-Q}} \right]. \end{aligned}$$

The first and second orders approximations are compared in section 7.

6.2 Hedging with options

Instead of hedging the position with the underlying asset, we consider here a strategy in which we invest in another European option (call or put). This option denoted by H is only used for hedging purposes. It is written on the same underlying asset but it has different specifications: its maturity T^H is, for instance, longer than T . One denotes by Y^H the payoff of the European option used for hedging. Its expected discounted payoff is then defined by

$$\tilde{Y}_t^H = \mathbb{E} \left(\frac{1}{B_{T^H}} Y^H \mid \mathcal{F}_t \right).$$

The related option price, denoted by $O^H(t, S_t, \lambda_t^Q)$, is linked to $\tilde{Y}_t^H$ by the relation $\tilde{Y}_t^H = e^{-rt} O^H(t, S_t, \lambda_t^Q)$. From previous developments, we know that $\tilde{Y}_t^H$ is a martingale driven by the next SDE:

$$\begin{aligned} d\tilde{Y}_t^H &= d \left(e^{-rt} O^H(t, S_t, \lambda_t^Q) \right) \\ &= \sigma_t^{YH} dW_t^Q + \int_{\mathbb{R}} \gamma_t^{YH}(z) \left(\chi(dz, dt) - \nu^Q(dz) \lambda_t^Q dt \right), \end{aligned}$$

where

$$\begin{aligned} \sigma_t^{YH} &:= \frac{\partial O^H}{\partial S_t} \sigma e^{-rt} S_t, \\ \gamma_t^{YH}(z) &:= e^{-rt} \left[O^H \left(t, S_t e^z, \lambda_t^Q + \eta^Q |z| \right) - O^H(t, S_t, \lambda_t^Q) \right]. \end{aligned}$$

Based on a similar reasoning to the one used in the proof of proposition 6.1, we infer the following optimal hedging policy:

Corollary 6.4. *The minimum variance price of a derivative delivering a payoff Y at time T , hedged with another option is*

$$P_0^* = \tilde{Y}_0 = \mathbb{E}^Q \left(B_T^{-1} Y \mid \mathcal{F}_0 \right), \quad (22)$$

and the minimum variance hedging strategy is given by:

$$\phi_s^{opt} = \frac{\sigma_s^Y \sigma_s^{YH} + \int_{\mathbb{R}} \gamma_s^Y(z) \gamma_s^{YH}(z) \nu^Q(dz) \lambda_s^Q}{(\sigma_s^{YH})^2 + \int_{\mathbb{R}} (\gamma_s^{YH}(z))^2 \nu^Q(dz) \lambda_s^Q}. \quad (23)$$

Without surprise, the optimal price is not affected by the type of underlying asset, chosen to hedge the position. But the optimal hedging ratio depends now upon the product of $\gamma_s^Y(z) \gamma_s^{YH}(z)$. Even if this product can be calculated by DFT, the evaluation of its integral is too computationally intensive to be efficient. For this reason, the hedging policy is approached by a ratio using only the first order derivatives of option prices with respect to state variables:

Corollary 6.5. *If we denote the sensitivities of derivatives to state variables as follows:*

$$\begin{aligned} \Delta_S &= \frac{\partial O(t, S_t, \lambda_t^Q)}{\partial S_t} & \Delta_\lambda &= \frac{\partial O(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q}, \\ \Delta_S^H &= \frac{\partial O^H(t, S_t, \lambda_t^Q)}{\partial S_t} & \Delta_\lambda^H &= \frac{\partial O^H(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q}, \end{aligned}$$

then the optimal hedge is approached at first order by the ratio $\phi_t^{opt} = \frac{\phi_t^{num}}{\phi_t^{den}} + \mathcal{O}(S_t^2)$, where

$$\begin{aligned}\phi_t^{num} &= \Delta_S \Delta_S^H S_t^2 \left(\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right) \lambda_t^Q \right) + \Delta_\lambda \Delta_\lambda^H (\eta^Q)^2 \mathbb{E}^Q (|J^Q|^2) \lambda_t^Q \\ &\quad + (\Delta_S \Delta_\lambda^H + \Delta_S^H \Delta_\lambda) \eta^Q S_t \mathbb{E}^Q (|J^Q| (e^{J^Q} - 1)) \lambda_t^Q,\end{aligned}\tag{24}$$

$$\begin{aligned}\phi_t^{den} &= (\Delta_S^H)^2 S_t^2 \left(\sigma^2 + \mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right) \lambda_t^Q \right) + 2 \Delta_S^H \Delta_\lambda^H \eta^Q S_t \mathbb{E}^Q (|J^Q| (e^{J^Q} - 1)) \lambda_t^Q \\ &\quad + (\Delta_\lambda^H \eta^Q)^2 \mathbb{E}^Q (|J^Q|^2) \lambda_t^Q,\end{aligned}\tag{25}$$

and

$$\mathbb{E}^Q (|J^Q|^2) = p^Q \frac{2}{(\rho^+ Q)^2} + (1 - p^Q) \frac{2}{(\rho^- Q)^2},$$

whereas $\mathbb{E}^Q \left(\left(e^{J^Q} - 1 \right)^2 \right)$, $\mathbb{E}^Q (|J^Q| (e^{J^Q} - 1))$ are provided by equations (20) and (21).

The proof is similar to the one of proposition 6.2 and is based on the first order Taylor's developments of $O(t, S_t, \lambda_t^Q)$ and $O^H(t, S_t, \lambda_t^Q)$. Notice that if we ignore the impact of a variation of intensity on options prices, the optimal hedging ratio is approached by $\frac{\Delta_S^H}{\Delta_S^H}$, the classical delta hedge ratio. In the next section, the performance of this method is tested. A second order approximation of the hedge ratio is also available:

Corollary 6.6. *If we denote the second order sensitivities of derivatives to state variables as follows:*

$$\begin{aligned}\Gamma_S &= \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial S_t^2} \quad \Gamma_\lambda = \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial \lambda_t^{Q^2}} \quad \Gamma_{S,\lambda} = \frac{\partial^2 O(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q \partial S_t}, \\ \Gamma_S^H &= \frac{\partial^2 O^H(t, S_t, \lambda_t^Q)}{\partial S_t^2} \quad \Gamma_\lambda^H = \frac{\partial^2 O^H(t, S_t, \lambda_t^Q)}{\partial \lambda_t^{Q^2}} \quad \Gamma_{S,\lambda}^H = \frac{\partial^2 O^H(t, S_t, \lambda_t^Q)}{\partial \lambda_t^Q \partial S_t},\end{aligned}$$

then the optimal hedge is approached at second order by

$$\phi_s^{opt} = \frac{\Delta_S \Delta_S^H \sigma^2 S_t^2 + \mathbb{E}^Q (e^{2rt} \gamma_s^Y(J^Q) \gamma_s^{YH}(J^Q)) \lambda_s^Q}{(\Delta_S^H)^2 \sigma^2 S_t^2 + \mathbb{E}^Q ((e^{rt} \gamma_s^{YH}(J^Q))^2) \lambda_s^Q} + \mathcal{O}(S_t^3), \tag{26}$$

where

$$\begin{aligned}e^{rt} \gamma_t^Y(z) &= \eta^Q \Delta_\lambda |z| + \frac{1}{2} (\eta^Q)^2 \Gamma_\lambda |z|^2 + \eta^Q S_t \Gamma_{S,\lambda} (e^z - 1) |z| \\ &\quad + S_t \Delta_S (e^z - 1) + \frac{1}{2} S_t^2 \Gamma_S (e^z - 1)^2,\end{aligned}$$

$$\begin{aligned}e^{rt} \gamma_t^{YH}(z) &= \eta^Q \Delta_\lambda^H |z| + \frac{1}{2} (\eta^Q)^2 \Gamma_\lambda^H |z|^2 + \eta^Q S_t \Gamma_{S,\lambda}^H (e^z - 1) |z| \\ &\quad + S_t \Delta_S^H (e^z - 1) + \frac{1}{2} S_t^2 \Gamma_S^H (e^z - 1)^2.\end{aligned}$$

The expectations $\mathbb{E}^Q (e^{2rt} \gamma_s^Y(J^Q) \gamma_s^{YH}(J^Q))$ and $\mathbb{E}^Q ((e^{rt} \gamma_s^{YH}(J^Q))^2)$ admit closed form expressions but their length and complexity make them difficult to implement. For these reasons, the expectations are computed numerically.

7 Numerical illustration

Figure 4 compares minimum variance ratios of first and second orders with the delta hedge ratio for 6 months call and put options. These options are evaluated with parameters of table 2 and the risk free rate is equal to 1%. S_0 is set to 100 and strikes range from 80 to 120. The hedging instrument is the underlying asset. In the upper graphs, the intensity is assumed equal to $\lambda_0 = \lambda_\infty = 5.62$, which corresponds to periods of low jumps activity as revealed by the sample path of λ_t , presented in figure 1. During these periods, the hedge ratio of first order is very close to Δ_S , the delta hedge ratio. We may then fear that there is nearly no benefit to take into account the first order sensitivity of option prices to the jump frequency into the hedging strategy. The difference between the hedge ratio of second order and Δ_S is more pronounced for options deep in or out of the money. The two last graphs of figure 4 exhibit the same ratios in period of high activity of jumps. When λ_t is equal $\lambda_\infty + 4\eta|J| = 47.05$ which a level of market stress attained during economic slowdowns, the curves of hedge ratios vs strikes tend to be flatter than in a normal economic conjuncture.

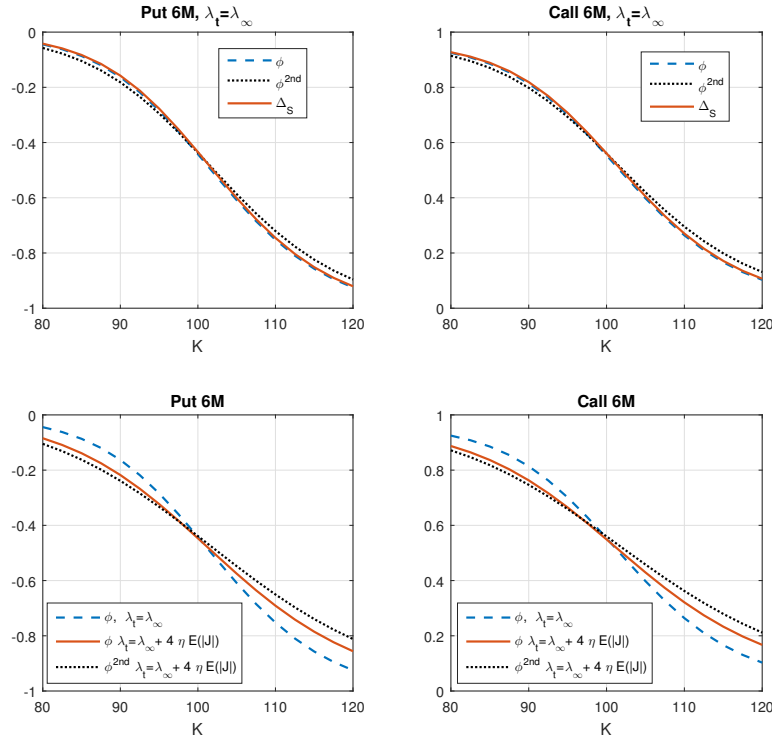


Figure 4: The upper graphs show the optimal hedge ratios of first and second orders for put and call option with different levels of moneyness. The intensity is equal to its asymptotic level, $\lambda_\infty = 5.62$. The lower graphs present the same ratios but when $\lambda_\infty + 4\eta E(|J|) = 47.05$.

The figure 5 compares the minimum variance hedge ratios of first and second orders with the delta hedge ratio when the short position is covered by another option. The shorted option for this test is a 6 months put, priced with parameters of table 2. The hedging instrument is here an “at the money” put option, with a maturity of one year. In this case, the difference between variance hedge ratios and the delta hedge ratio is significant, particularly if the hedged and hedging options have different strikes.

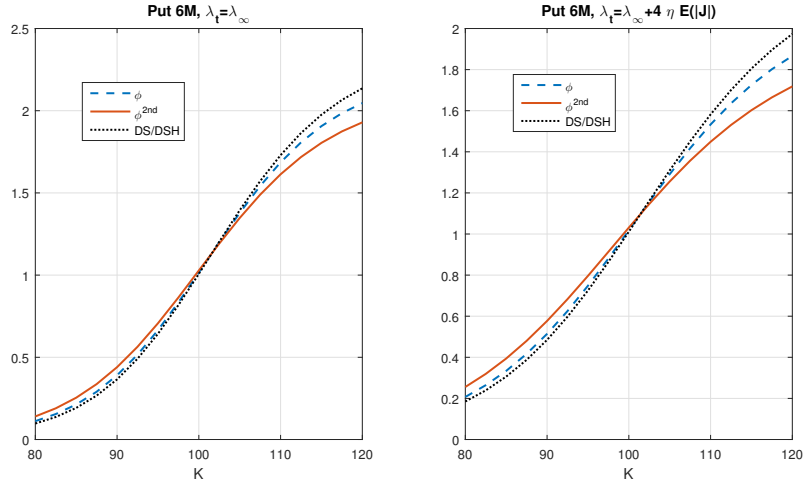


Figure 5: These graphs show the optimal hedge ratios of first and second orders for put and call options with different levels of moneyness. The intensity is equal to its asymptotic level, $\lambda_\infty = 5.62$ or to $\lambda_\infty + 4\eta\mathbb{E}(|J|) = 47.05$.

As the previous analysis does not provide any information about the efficiency of hedging strategies, we proceed by simulations to measure it. The hedged position is a short put option, with a 6 months maturity and a strike of 105. The hedging instruments are the underlying asset or an “at the money” put option, with a 1 year maturity. We simulate 1000 sample paths for S_t and the hedging portfolio is rebalanced every 5 days (a business week). λ_0 is set to $\lambda_\infty + 2\eta|J| = 26.33$, which corresponds to an intensity observed for a slightly stressed market. The first left and right graphs of figure 6 show the empirical distributions of the 6 months log return and of gains in % of the fee whether the position is not hedged. Related statistics are provided in table 5. The standard deviation of 6 months log-returns is around 16.98% for 6 months (or 24.01% on an annual basis). The value at risk (VaR) of log returns, for a confidence level of 5% reaches 26.60%. This seems quite high but is a direct consequence of the jump clustering and of the high initial intensity. If λ_0 is set to λ_∞ , the VaR falls to 22.12%. In absence of hedging, the gain is positive for 64.80% of scenarios. However, the 5% VaR is twice bigger (231%) than the option fee. This confirms the riskiness of a naked short position in presence of jump clustering effects.

	6M log-return $\lambda_0 = \lambda_\infty + 2\eta J $	6M log-return $\lambda_0 = \lambda_\infty$	Gain/Loss (% of the fee), no hedge
Average	0.90%	2.18%	9.35%
St. dev.	16.98%	14.46%	121.01%
Skew	-0.43	-0.39	-1.74
Kurtosis	6.14	4.74	6.46
VaR @ 5%	-26.60%	-22.12%	-231.28%

Table 5: The two first columns present statistics about simulated log-returns over a period of 6 months. The initial frequency is set to λ_∞ or to $\lambda_\infty + 2\eta|J|$. The last column reports statistics about the distribution of gains or losses, whether the naked position is not hedged.

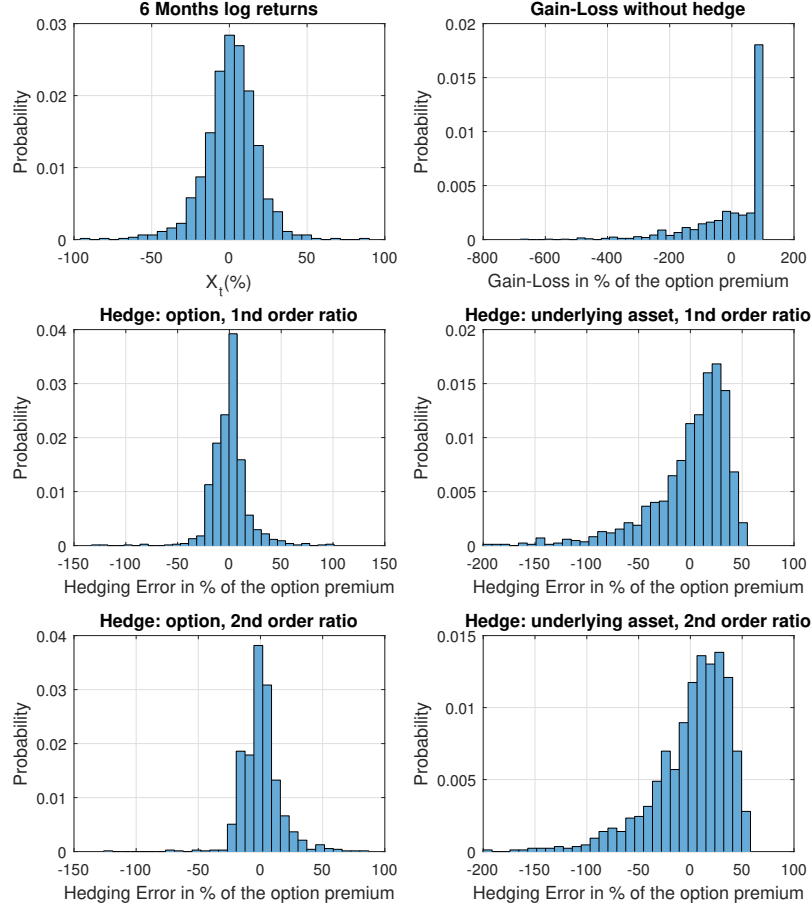


Figure 6: First line: empirical distributions of 6M log-returns and gains-losses of a naked short position in a 6M put with a strike of 105. Second line: empirical distributions of hedging errors, expressed in % of the option premium. The hedging portfolio is rebalanced every 5 days. The hedge ratio used is the minimum variance ratio of first order. And the hedging instruments in the left and right graphs are respectively the underlying asset and a 1 year “at the money” put option. Last line: empirical distributions of hedging errors obtained with the hedge ratio of second order.

In figure 6, the graphs of the second line exhibit empirical distributions of hedging errors after 6 months, expressed as a percentage of the initial option fee. The portfolios used in these simulations are built with the hedging ratio of first order. Table 6 provides statistics about these errors. When the underlying asset is used for hedging, the distribution of errors is asymmetric and displays a very negative fat tail. Even if the standard deviation is limited to 37%, the 5% Value at Risk (VaR) reaches nearly 77% of the option fee. This analysis clearly reveals two things. Firstly, it confirms our intuition that the difference of performances between delta and minimum variance hedges is not significant. Secondly, hedging portfolios based composed of cash and of the underlying asset are risky and unable to limit the exposure of an options seller to market jump clustering. The most efficient strategy consists in hedging the position with another option. The standard deviation is reduced by 50% and the 5% VaR falls by nearly 75%. The minimum variance portfolio also performs better than the pure delta hedge: the average error is closer to zero and the standard deviation falls from 19.42% to 17.74%.

The last graphs show empirical distributions of hedging errors, for hedges based on the minimum variance

ratio of second order. If the short position is covered with the underlying asset, the standard deviation (37.24%) remains very close to the one obtained with pure delta hedging strategies (37.80%). But the VaR is reduced by nearly 5% which is a non-negligible improvement. However, the best performance is obtained by hedging the short position with another option. With such a strategy, the standard deviation and the VaR of errors fall respectively to 15.56% and to 19.35%.

Hedging ratio	Hedge : underlying asset			Hedge : option		
	Δ_S	ϕ	ϕ^{2nd}	$\frac{\Delta_S}{\Delta_H}$	ϕ^{opt}	$\phi^{opt\ 2nd}$
Option fee	8.56					
Average	0.78%	0.59%	1.17%	-0.19%	0.05%	0.34%
St. dev.	37.80%	37.90%	37.24%	19.42%	17.74%	15.56%
Skew	-1.69	-1.72	-1.41	0.39	-0.07	0.2115
Kurtosis	6.64	6.82	5.75	12.32	12.82	10.54
VaR @ 5%	-77.56%	-76.75%	-72.87%	-22.31%	-20.73%	-19.35%

Table 6: This table reports the statistics about hedging errors displayed in figure 6, obtained with different hedging ratios and instruments. The hedging portfolio is rebalanced every 5 days.

To conclude this section, we evaluate the impact of the rebalancing frequency on the efficiency of the hedge. Table 7 reports the average and standard deviation of (relative) hedging errors, for hedging frequencies from 3 to 8 days. The hedging instrument is here an option and we use the minimum variance ratio of second order. As we could expect, increasing the frequency of reallocations reduces the risk and the average hedging error. However, due to the numerical noise that is it at the origin of inaccuracies in the calculation of options prices, we do not observe a clear improvement of the hedging efficiency for frequencies lower than 4 days.

Hedging ratio	$\phi^{opt, 2nd}$	
Frequency of rebalancing	Average	St. dev.
3 days	0.31%	15.29%
4 days	0.29%	14.62%
5 days	0.34%	15.56%
6 days	0.37%	16.47%
7 days	0.43%	17.92%
8 days	0.52%	19.31%

Table 7: This table reports the statistics about hedging errors (1000 simulations) for different frequencies of rebalancing. The short option is a 6 months put priced with parameters of table 2. The hedging instrument is an “at the money” put option, with a maturity of one year. The hedge ratio used is the minimum variance ratio of second order.

8 Conclusion

This article studies the influence of jump clustering on the efficiency of quadratic hedging strategies for options. Firstly, we propose a parsimonious self-excited jump-diffusion model that can replicate the jump clustering phenomenon observed in the stock markets. Contagion between shocks is obtained by assuming that a jump of the stock price increases momentarily the probability of observing a new shock. We establish next the conditions that ensure the stability of the stock price process and the moment generating function of log-returns. We also find a family of affine changes of measure that preserves the dynamics of prices under the risk neutral measure. A “Peak-Over-Threshold” procedure is applied to a time series of S&P500 stock index returns to estimate parameters and to motivate empirically the proposed dynamics.

The second part of this work focuses on the pricing and hedging of European options. Prices and Greeks are evaluated by a Discrete Fourier transform. Next, we present four minimum variance-hedging strategies. Two hedging instruments are considered: the underlying asset and an option. In both cases, we propose two hedging ratios based on a first or a second order Taylor’s development of option prices.

The efficiency of hedging strategies is appraised by Monte Carlo simulations from which we draw several conclusions. Firstly, jump clustering can cause huge losses in absence of any hedge. For example, the VaR of a naked short position in a 6 months put on the S&P 500 exceeds 200% of the option fee. Secondly, we observe a similarity between the delta hedge ratio and the minimum variance ratio of first order, when the position is hedged with the underlying asset. In this case, there is no benefit from taking into account the sensitivity of option prices to variations of the jump intensity. Using the hedge ratio of second order slightly reduces the 5% VaR from 77% to 72%. However, the exposure to potential losses remains too high to be considered as a serious hedging policy. Finally, the most efficient solution that reduces significantly the standard deviation of hedging errors, consists of using another derivative as hedging instrument. In this case, minimum variance ratios of first and second orders clearly outperform a pure delta hedging strategy. With the ratio of second order, the exposure to potential losses (measured by the 5% VaR) nearly falls by 75%, compared to a hedge with the underlying asset.

Appendix

Proof of proposition 2.1. Define the martingale $M = (M_t)_t$ with $M_t := \lambda_t - \mathbb{E}(\lambda_t | \mathcal{F}_0)$. Then $[\lambda, \lambda]_t = [M, M]_t = M_t^2 - 2 \int_0^t M_s - dM_s$ and $\mathbb{V}(\lambda_t | \mathcal{F}_0) = \mathbb{E}([M, M]_t | \mathcal{F}_0) = \mathbb{E}([\lambda, \lambda]_t | \mathcal{F}_0)$. The quadratic variation of λ is given by

$$\begin{aligned} [\lambda, \lambda]_t &= \left[\int_0^t \eta e^{-\alpha(t-u)} dL_u, \int_0^t \eta e^{-\alpha(t-u)} dL_u \right]_t \\ &= \int_0^t \left(\eta e^{-\alpha(t-u)} \right)^2 |J_u|^2 dN_u. \end{aligned} \quad (27)$$

Finally, from this last equation and as $\mathbb{E}(|J|^2) = p \frac{2}{(\rho^+)^2} + (1-p) \frac{2}{(\rho^-)^2}$, the result follows. ■

Proof of proposition 2.2. If we note $f = \mathbb{E}(e^{\omega_1 X_s + \omega_2 \lambda_s} | \mathcal{F}_t)$, using the Itô’s lemma allows us to infer that f satisfies the following integro-differential equation:

$$\begin{aligned} 0 &= f_t + f_X \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}(e^J - 1) \right) + f_{XX} \frac{\sigma^2}{2} + \alpha(\theta - \lambda_t) f_\lambda + \\ &\quad + \lambda_t \int_0^{+\infty} [f(t, X_t + z, \lambda_t + \eta |z|) - f(\cdot)] d\nu(z). \end{aligned} \quad (28)$$

Conjecture that f is an exponential affine function of λ_t and X_t :

$$f = \exp(A(t, s) + B(t, s)\lambda_t + C(t, s)X_t),$$

where $A(t, s)$, $B(t, s)$, $C(t, s)$ are time dependent functions, then the partial derivatives of f are given by:

$$f_t = \left(\frac{\partial}{\partial t} A(t, s) + \frac{\partial}{\partial t} B(t, s)\lambda_t + \frac{\partial}{\partial t} C(t, s)X_t \right) f,$$

$$f_X = C(t, s)f, \quad f_{XX} = C(t, s)^2 f, \quad f_\lambda = B(t, s)f$$

Now, rewriting the integrand of equation (28) as follows:

$$\int_0^{+\infty} [f(t, X_t + z, \lambda_t + \eta |z|) - f(\cdot)] d\nu(z) = f[\psi(C(t, s), B(t, s)\eta) - 1]$$

and injecting all these expressions into equation (28) lead to:

$$\begin{aligned} 0 = & \left(\frac{\partial}{\partial t} A + \frac{\partial}{\partial t} B \lambda_t + \frac{\partial}{\partial t} C X_t \right) + C \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}(e^J - 1) \right) \\ & + C^2 \frac{\sigma^2}{2} + \alpha(\theta - \lambda_t)B + \lambda_t [\psi(C, B\eta) - 1], \end{aligned}$$

from which we guess that $C(t, s) = \omega_1$. Regrouping terms allows to infer that

$$\begin{aligned} 0 = & \frac{\partial}{\partial t} A + \omega_1 \left(\mu - \frac{\sigma^2}{2} \right) + \omega_1^2 \frac{\sigma^2}{2} + \alpha\theta B \\ & + \lambda_t \left(\frac{\partial}{\partial t} B - \alpha B - \omega_1 \mathbb{E}(e^J - 1) + [\psi(\omega_1, B\eta) - 1] \right). \end{aligned}$$

■

Proof of proposition 4.1. If we note m_t , the logarithm of M_t then

$$\begin{aligned} m_t = & \kappa_1(\xi) \lambda_t + \xi L_t + \kappa_2(\xi) t \\ & - \frac{1}{2} \int_0^t \varphi(s)^2 ds - \int_0^t \varphi(s) dW_s, \end{aligned} \tag{29}$$

and its infinitesimal dynamics is given by

$$\begin{aligned} dm_t = & \kappa_1(\xi) \alpha(\theta - \lambda_t) dt + \kappa_2(\xi) dt + (\kappa_1(\xi) \eta + \xi) |J| dN_t \\ & - \frac{1}{2} \varphi(t)^2 dt - \varphi(t) dW_t. \end{aligned}$$

The random measure of J is noted $\Xi(\cdot)$ and is such that $J = \int_0^\infty \Xi(dz)$. Applying the Ito's lemma for semi-martingales to M_t give us the next relation:

$$\begin{aligned} dM_t = & M_t dm_t + \frac{1}{2} M_t d[m_t, m_t]^c \\ & + M_t \int_0^\infty \left(e^{(\kappa_1(\xi)\eta + \xi)|z|} - 1 - (\kappa_1(\xi)\eta + \xi)|z| \right) \Xi(dz) dN_t, \end{aligned}$$

that is developed as follows:

$$\begin{aligned} dM_t = & M_t \left(\kappa_1(\xi) \alpha(\theta - \lambda_t) dt + \kappa_2(\xi) dt + (\kappa_1(\xi) \eta + \xi) |J| dN_t \right. \\ & \left. - \frac{1}{2} \varphi(t)^2 dt - \varphi(t) dW_t \right) \\ & + \frac{1}{2} M_t d[m_t, m_t]^c + M_t \int_0^\infty \left(e^{(\kappa_1(\xi)\eta + \xi)|z|} - 1 - (\kappa_1(\xi)\eta + \xi)|z| \right) \Xi(dz) dN_t, \end{aligned}$$

and after the introduction of the compensator of the jump process,

$$\begin{aligned} dM_t &= \kappa_1(\xi) \alpha \theta M_t dt + \kappa_2(\xi) M_t dt - \varphi(t) M_t dW_t \\ &\quad - M_t \lambda_t \left(\kappa_1(\xi) \alpha - \int_0^\infty \left(e^{(\kappa_1(\xi) \eta + \xi) |z|} - 1 \right) \nu(dz) \right) dt \\ &\quad + M_t \int_0^\infty \left(e^{(\kappa_1(\xi) \eta + \xi) |z|} - 1 \right) (\Xi(dz) dN_t - \lambda_t \nu(dz) dt) \end{aligned}$$

because $\frac{1}{2} M_t d[m_t, m_t]^c = \frac{1}{2} \varphi(t)^2 M_t dt$. Since the integral with respect to $\Xi(dz) dN_t - \lambda_t \nu(dz) dt$ is a local martingale, M is also a local martingale if and only if the relations (6) hold. ■

Proof of proposition 4.2. If m_t is the logarithm of M_t , as defined by equation (29), the mgf of λ_T under $Q^{\xi, \varphi}$ is equal to

$$\begin{aligned} \mathbb{E}^Q \left(e^{\omega \lambda_T^Q} | \mathcal{F}_t \right) &= \mathbb{E} \left(e^{m_T - m_t + \omega \psi(0, \kappa_1(\xi) \eta + \xi) \lambda_T} | \mathcal{F}_t \right) \\ &= e^{-m_t} \mathbb{E} \left(e^{m_T + \omega \psi(0, \kappa_1(\xi) \eta + \xi) \lambda_T} | \mathcal{F}_t \right). \end{aligned}$$

If $f(\cdot)$ denotes $\mathbb{E} \left(e^{m_T + \omega \psi(0, \kappa_1(\xi) \eta + \xi) \lambda_T} | \mathcal{F}_t \right)$, according to the Itô's lemma, it solves the next stochastic differential equation:

$$\begin{aligned} 0 &= f_t + (\kappa_1(\xi) \alpha (\theta - \lambda_t) + \kappa_2(\xi)) f_Y + \alpha (\theta - \lambda_t) f_\lambda \\ &\quad - \frac{1}{2} \varphi(t)^2 f_m dt + \frac{1}{2} \varphi(t)^2 f_{mm} \\ &\quad + \lambda_t \int_{-\infty}^{+\infty} [f(t, \lambda_t + \eta |z|, Y_t + (\kappa_1 \eta + \xi) |z|) - f(\cdot)] d\nu(z). \end{aligned} \tag{30}$$

Conjecture that $f(\cdot)$ is an exponential affine function of state variables:

$$f = \exp(A(t, T) + \psi(0, \kappa_1(\xi) \eta + \xi) B(t, T) \lambda_t + C(t, T) m_t),$$

with the terminal conditions $A(T, T) = 0$, $B(T, T) = \omega$ and $C(T, T) = 1$. If we note $\psi^b = \psi(0, \kappa_1(\xi) \eta + \xi)$ for a little while, the partial derivatives of $f(\cdot)$ are

$$\begin{aligned} f_t &= \left(\frac{\partial}{\partial t} A + \psi^b \lambda_t \frac{\partial}{\partial t} B + \frac{\partial}{\partial t} C Y_t \right) f, \\ f_m &= C f \quad f_{mm} = C^2 f \quad f_\lambda = B \psi^b f. \end{aligned}$$

Inserting these expressions into the equation (30), leads to the following relation (after grouping terms)

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \left(\alpha \theta \psi^b B + \alpha \theta \kappa_1(\xi) C + \kappa_2(\xi) C \right) - \frac{1}{2} \varphi(t)^2 C + \frac{1}{2} \varphi(t)^2 C^2 \\ &\quad + \lambda_t \left(\psi^b \frac{\partial}{\partial t} B - \kappa_1(\xi) \alpha C - \alpha \psi^b B + \int_0^{+\infty} \left[e^{B \psi^b \eta |z| + C(\kappa_1(\xi) \eta + \xi) |z|} - 1 \right] d\nu(z) \right) \\ &\quad + Y_t \left(\frac{\partial}{\partial t} C \right), \end{aligned}$$

we infer that $C(t, s) = 1$ as $\frac{\partial}{\partial t} C(t, s) = 0$. And we get that

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \alpha \theta \psi^b B + \alpha \theta \kappa_1(\xi) + \kappa_2(\xi), \\ 0 &= \psi^b \frac{\partial}{\partial t} B - \kappa_1(\xi) \alpha - \alpha \psi^b B + \int_0^{+\infty} \left[e^{B \psi^b \eta |z| + (\kappa_1(\xi) \eta + \xi) |z|} - 1 \right] d\nu(z). \end{aligned}$$

Using conditions (6), this system is simplified as follows:

$$\begin{aligned}\frac{\partial}{\partial t}A &= -\alpha [\theta\psi^b] B, \\ \frac{\partial}{\partial t}B &= \alpha B - \left[\frac{\psi(0, B[\eta\psi^b] + (\kappa_1(\xi)\eta + \xi))}{\psi^b} - 1 \right].\end{aligned}$$

and we can conclude by comparison with the results of proposition 2.2. ■

Proof of proposition 4.3. By construction, the moment-generating function for jumps under the risk-neutral measure is the ratio

$$\psi^Q(z, 0) = \frac{\psi(z, (\kappa_1(\xi)\eta + \xi))}{\psi(0, \kappa_1(\xi)\eta + \xi)}.$$

If we denote $\varkappa = (\kappa_1(\xi)\eta + \xi)$, then $\rho^{+Q} = \rho^+ - \varkappa$, $\rho^{-Q} = \rho^- + \varkappa$ and the numerator and denominator in this equation are

$$\begin{aligned}\psi(z, \varkappa) &= \frac{p\rho^+(\rho^- + \varkappa - z) + (1-p)\rho^-(\rho^+ - \varkappa - z)}{(\rho^+ - \varkappa - z)(\rho^- + \varkappa - z)} \\ &= \frac{p\rho^+(\rho^{-Q} - z) + (1-p)\rho^-(\rho^{+Q} - z)}{(\rho^{+Q} - z)(\rho^{-Q} - z)}, \\ \psi(0, \varkappa) &= \frac{p\rho^+(\rho^- + \varkappa) + (1-p)\rho^-(\rho^+ - \varkappa)}{(\rho^+ - \varkappa)(\rho^- + \varkappa)} \\ &= \frac{p\rho^+\rho^{-Q} + (1-p)\rho^-\rho^{+Q}}{\rho^{+Q}\rho^{-Q}}.\end{aligned}$$

And, then since

$$\psi^Q(z, 0) = \frac{\frac{p\rho^+\rho^{-Q}}{(p\rho^+\rho^{-Q} + (1-p)\rho^-\rho^{+Q})}(\rho^{-Q} - z)\rho^{+Q} + \frac{(1-p)\rho^-\rho^{+Q}}{(p\rho^+\rho^{-Q} + (1-p)\rho^-\rho^{+Q})}\rho^{-Q}(\rho^{+Q} - z)}{(\rho^{+Q} - z)(\rho^{-Q} - z)},$$

one can appropriately rearrange this equation to complete the proof. ■

Proof of proposition 4.4. If m_t is the logarithm of M_t , as defined by equation (29), the mgf of X_T under $Q^{\xi, \varphi}$ is equal to

$$\begin{aligned}\mathbb{E}^Q(e^{\omega X_T} | \mathcal{F}_t) &= \mathbb{E}(e^{m_T - m_t + \omega X_T} | \mathcal{F}_t) \\ &= e^{-m_t} \mathbb{E}(e^{m_T + \omega X_T} | \mathcal{F}_t).\end{aligned}$$

If $f(\cdot)$ denotes $\mathbb{E}(e^{m_T + \omega X_T} | \mathcal{F}_t)$, according to the Itô's lemma, it solves the next stochastic differential equation:

$$\begin{aligned}0 &= f_t + (\kappa_1(\xi)\alpha(\theta - \lambda_t) + \kappa_2(\xi))f_m + \alpha(\theta - \lambda_t)f_\lambda - \frac{1}{2}\varphi(t)^2 f_{mm} \\ &\quad + \frac{1}{2}\varphi(t)^2 f_{mm} + f_X \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}(e^J - 1) \right) + f_{XX} \frac{\sigma^2}{2} - f_{Xm} \varphi \sigma \\ &\quad \lambda_t \int_{-\infty}^{+\infty} f(t, X_t + z, \lambda_t + \eta|z|, Y_t + (\kappa_1(\xi)\eta + \xi)|z|) - f d\nu(z).\end{aligned}\tag{31}$$

We assume that $f(\cdot)$ is an exponential affine function of state variables:

$$f = \exp(A(t, T) + B(t, T)\psi(0, \kappa_1\eta + \xi)\lambda_t + C(t, T)X_t + D(t, T)m_t),$$

with the terminal conditions $A(T, T) = 0$, $B(T, T) = 0$, $C(T, T) = \omega$ and $D(T, T) = 1$. If we note $\psi^b = \psi(0, \kappa_1 \eta + \xi)$, the partial derivatives of $f(\cdot)$ are given by

$$f_t = \left(\frac{\partial}{\partial t} A + \psi^b \lambda_t \frac{\partial}{\partial t} B + \frac{\partial}{\partial t} C X_t + \frac{\partial}{\partial t} D m_t \right) f,$$

$$\begin{aligned} f_X &= C f, & f_{XX} &= C^2 f, \\ f_m &= D f, & f_{mm} &= D^2 f, \\ f_{Xm} &= C D f, & f_\lambda &= B \psi^b f. \end{aligned}$$

Inserting these expressions in equations (31), leads to the following relation (after grouping terms),

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \left(\alpha \theta \psi^b B + \left(\mu - \frac{\sigma^2}{2} \right) C + \frac{\sigma^2}{2} C^2 + \alpha \theta \kappa_1(\xi) D + \kappa_2 D \right) \\ &\quad - C D \varphi \sigma + \lambda_t \left(\psi^b \frac{\partial}{\partial t} B - \kappa_1(\xi) \alpha D - \alpha \psi^b B - C \mathbb{E}(e^J - 1) \right) \\ &\quad + \lambda_t \int_0^{+\infty} \left[e^{B \psi^b \eta |z| + C z + D(\kappa_1(\xi) \eta + \xi) |z|} - 1 \right] d\nu(z) + X_t \left(\frac{\partial}{\partial t} C \right) + Y_t \left(\frac{\partial}{\partial t} D \right), \end{aligned}$$

we infer that $D(t, s) = 1$, $C(t, s) = \omega$ as $\frac{\partial}{\partial t} D(t, s) = 0$ and $\frac{\partial}{\partial t} C(t, s) = 0$. On the other hand as $\kappa_2(\xi) = -\kappa_1(\xi) \alpha \theta$, this last equation becomes:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \left(\alpha \theta \psi^b B + \left(\mu - \frac{\sigma^2}{2} \right) \omega + \frac{\sigma^2}{2} \omega^2 \right) \\ &\quad - \omega \varphi \sigma + \lambda_t \left(\psi^b \frac{\partial}{\partial t} B - \kappa_1(\xi) \alpha - \alpha \psi^b B - \omega \mathbb{E}(e^J - 1) \right) \\ &\quad + \lambda_t \int_0^{+\infty} \left[e^{B \psi^b \eta |z| + \omega z + (\kappa_1(\xi) \eta + \xi) |z|} - 1 \right] d\nu(z). \end{aligned}$$

If we remember that $\varphi = \varphi_1 + \varphi_2 \lambda_t$ where $\varphi_1 = \frac{\mu - r}{\sigma}$ and $\varphi_2 = \frac{1}{\sigma} [\psi^b (\psi^Q(1, 0) - 1) - (\psi(1, 0) - 1)]$, we obtain the following expression:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \left(\alpha \theta \psi^b B + \left(\mu - \frac{\sigma^2}{2} - \varphi_1 \sigma \right) \omega + \frac{\sigma^2}{2} \omega^2 \right) \\ &\quad + \lambda_t \left(\psi^b \frac{\partial}{\partial t} B - \kappa_1(\xi) \alpha - \alpha \psi^b B - \omega (\mathbb{E}(e^J - 1) + \varphi_2 \sigma) \right) \\ &\quad + \lambda_t \int_0^{+\infty} \left[e^{B \psi^b \eta |z| + \omega z + (\kappa_1(\xi) \eta + \xi) |z|} - 1 \right] d\nu(z). \end{aligned}$$

And finally as $\kappa_1 \alpha = \psi^b - 1$, we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} A + \alpha \theta \psi^b B + \left(\mu - \frac{\sigma^2}{2} - \varphi_1 \sigma \right) \omega + \frac{\sigma^2}{2} \omega^2 \\ 0 &= \frac{\partial}{\partial t} B - \alpha B - \omega \frac{(\psi(1, 0) - 1 + \varphi_2 \sigma)}{\psi^b} + \left[\frac{\psi(\omega, B \psi^b \eta + (\kappa_1(\xi) \eta + \xi))}{\psi^b} - 1 \right]. \end{aligned}$$

Using conditions (6), this system is simplified as follows:

$$\begin{aligned} \frac{\partial}{\partial t} A &= -\omega \left(r - \frac{\sigma^2}{2} \right) - \omega^2 \frac{\sigma^2}{2} - \alpha (\theta \psi^b) B \\ \frac{\partial}{\partial t} B &= \alpha B + \omega (\psi^Q(1, 0) - 1) - [\psi^Q(\omega, B \eta^Q) - 1]. \end{aligned}$$

We infer the dynamics of X_t under the measure $Q^{\xi, \varphi}$ by comparison with the results of proposition 2.2. As $dS_t = d(e^{X_t})$, the equation (10) is proven by applying the Itô's lemma. ■

Proof of proposition 5.1. As $\Delta_\omega = \frac{2\pi}{M\Delta_k}$ and $\omega_j = (j-1)\Delta_\omega$, the product $k_j\omega_m$ is equal to

$$\begin{aligned} k_j\omega_m &= -\frac{M}{2} (m-1) \Delta_k \Delta_\omega + (j-1) (m-1) \Delta_k \Delta_\omega \\ &= -(m-1) \pi + (j-1) (m-1) \frac{2\pi}{M}, \end{aligned}$$

and then

$$e^{-i\omega_m k_j} = (-1)^{(m-1)} e^{-i(j-1)(m-1)\frac{2\pi}{M}}.$$

then the expression (13) may be rewritten as equations (14). ■

Proof of proposition 6.1. Let us denote by

$$\epsilon_T(P_0, \phi) = P_0 + \int_0^T \phi_s d\tilde{S}_s - \tilde{Y}_T,$$

the residual error for a given strategy ϕ_s and for a initial value P_0 . For every admissible strategy, the expectation of the square of this error is the sum of

$$\mathbb{E}^Q (\epsilon_T(P_0, \phi)^2 | \mathcal{F}_0) = (P_0 - \tilde{Y}_0)^2 + \mathbb{E}^Q \left(\left(\int_0^T \phi_s d\tilde{S}_s + \tilde{Y}_0 - \tilde{Y}_T \right)^2 | \mathcal{F}_0 \right).$$

that is minimized for $P_0^* = \tilde{Y}_0 = \mathbb{E}^Q (B_T^{-1}Y | \mathcal{F}_0)$. If $P_0 = \tilde{Y}_0$, the expected residual error simplifies as follows:

$$\mathbb{E}^Q (\epsilon_T(P_0, \phi)^2 | \mathcal{F}_0) = \mathbb{E}^Q \left(\left(\int_0^T \phi_s d\tilde{S}_s + \tilde{Y}_0 - \tilde{Y}_T \right)^2 | \mathcal{F}_0 \right),$$

and $H_T = \int_0^T \phi_s d\tilde{S}_s + \tilde{Y}_0 - \tilde{Y}_T$ is a martingale with

$$\mathbb{E}(H_T | \mathcal{F}_0) = H_0 = \phi_0 \tilde{S}_0 = 0,$$

as $\phi_0 = 0$. Its variance is then equal to $\mathbb{E}(H_T^2)$ that is also the expectation of its quadratic variation (corollary 3 p73 Protter 2005). On the other hand, by construction H_T is also equal to the following sum

$$\begin{aligned} H_T &= \int_0^T (\phi_s \sigma \tilde{S}_s - \sigma_s^Y) dW_s^Q + \int_0^T \int_{\mathbb{R}} [\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z)] \chi(dz, ds) \\ &\quad - \int_0^T \int_{\mathbb{R}} [\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z)] \nu^Q(dz) \lambda_s^Q ds. \end{aligned}$$

and its quadratic variation is equal to

$$[H_T, H_T] = \int_0^T (\phi_s \sigma \tilde{S}_s - \sigma_s^Y)^2 ds + \int_0^T \int_{\mathbb{R}} [\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z)]^2 \chi(dz, ds).$$

The variance of H_T is the expectation of this quadratic bracket, $\mathbb{V}(H_T | \mathcal{F}_0) = \mathbb{E}^Q ([H_T, H_T] | \mathcal{F}_0)$:

$$\mathbb{V}(H_T | \mathcal{F}_0) = \int_0^T \mathbb{E}^Q \left((\phi_s \sigma \tilde{S}_s - \sigma_s^Y)^2 + \int_{\mathbb{R}} [\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z)]^2 \nu^Q(dz) \lambda_s^Q \right) ds,$$

$$\mathbb{V}(H) = \int_0^T \mathbb{E}^Q \left(\left(\phi_s \sigma \tilde{S}_s - \sigma_s^Y \right)^2 + \int_{\mathbb{R}} \left[\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z) \right]^2 \nu^Q(dz) \lambda_s^Q | \mathcal{F}_0 \right) ds,$$

and the optimal hedging strategy is obtained by solving the next optimization problem:

$$\min_{\phi_s} \left(\phi_s \sigma \tilde{S}_s - \sigma_s^Y \right)^2 + \int_{\mathbb{R}} \left[\phi_s \tilde{S}_s (e^z - 1) - \gamma_s^Y(z) \right]^2 \nu^Q(dz) \lambda_s^Q.$$

■

Proof of proposition 6.2. The first order Taylor's development of $O(t, S_t, \lambda_t^Q)$ yields the following approached expression for $\gamma_t^Y(z)$:

$$\gamma_t^Y(z) = e^{-rt} (\Delta_S (S_t e^z - S_t) + \Delta_\lambda \eta^Q |z|) + \mathcal{O}(z^2),$$

and then

$$\int_{\mathbb{R}} \gamma_s^Y(z) (e^z - 1) \nu^Q(dz) \approx \Delta_S e^{-rt} S_t \mathbb{E}^Q \left((e^{J^Q} - 1)^2 \right) + e^{-rt} \Delta_\lambda \eta^Q \mathbb{E}^Q \left(|J^Q| (e^{J^Q} - 1) \right).$$

If we insert these results in the expression for the mean variance strategy (18), we can conclude. ■

Proof of proposition 6.3. The second order Taylor's development of $O(t, S_t, \lambda_t^Q)$ yields the following approached expression for $\gamma_t^Y(z)$:

$$\begin{aligned} \gamma_t^Y(z) = e^{-rt} & \left[\eta^Q \Delta_\lambda |z| + \frac{1}{2} (\eta^Q)^2 \Gamma_\lambda |z|^2 \right] + e^{-rt} \eta^Q S_t \Gamma_{S,\lambda} (e^z - 1) |z| \\ & e^{-rt} \left[S_t \Delta_S (e^z - 1) + \frac{1}{2} S_t^2 \Gamma_S (e^z - 1)^2 \right], \end{aligned}$$

then

$$\begin{aligned} \int_{\mathbb{R}} \gamma_s^Y(z) (e^z - 1) \nu^Q(dz) \approx & \Delta_S e^{-rt} S_t \mathbb{E}^Q \left((e^{J^Q} - 1)^2 \right) + e^{-rt} \Delta_\lambda \eta^Q \mathbb{E}^Q \left(|J^Q| (e^{J^Q} - 1) \right) \\ & + \frac{1}{2} e^{-rt} (\eta^Q)^2 \Gamma_\lambda \mathbb{E}^Q \left(|J^Q|^2 (e^{J^Q} - 1) \right) + \frac{1}{2} e^{-rt} S_t^2 \Gamma_S \mathbb{E}^Q \left((e^{J^Q} - 1)^3 \right) \\ & + e^{-rt} \eta^Q S_t \Gamma_{S,\lambda} \mathbb{E}^Q \left(|J^Q| (e^{J^Q} - 1)^2 \right). \end{aligned}$$

and we can conclude. ■

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