Eigenvalues of truncated unitary matrices: disk counting statistics

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Abstract

Let T be an $n \times n$ truncation of an $(n+\alpha) \times (n+\alpha)$ Haar distributed unitary matrix. We consider the disk counting statistics of the eigenvalues of T. We prove that as $n \to +\infty$ with α fixed, the associated moment generating function enjoys asymptotics of the form

$$\exp\big(C_1 n + C_2 + o(1)\big),$$

where the constants C_1 and C_2 are given in terms of the incomplete Gamma function. Our proof uses the uniform asymptotics of the incomplete Beta function.

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1 Introduction

Let $n \in \mathbb{N}_{>0}$, $\alpha > 0$, and consider the joint probability measure

$$\frac{1}{n! Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^{n} (1-|z_j|^2)^{\alpha-1} d^2 z_j, \qquad |z_j| \leq 1, \tag{1.1}$$

where Z_n is the normalization constant. Note that the z_j's are constrained to lie in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. A main motivation for studying this point process stems from its connection with random matrices: it is shown in [19] that for $\alpha \in \mathbb{N}_{>0}$, (1.1) is the law of the eigenvalues of an $n \times n$ truncated unitary matrix T, i.e. T is the upper-left $n \times n$ submatrix of a Haar distributed unitary matrix of size $(n+\alpha) \times (n+\alpha)$. By rewriting (1.1) in the form

$$\frac{1}{n! Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)} d^2 z_j, \qquad Q(z) := \begin{cases} -\frac{\alpha-1}{n} \ln(1-|z|^2), & \text{if } |z| < 1, \\ +\infty, & \text{if } |z| \geq 1, \end{cases}$$

we infer that for general $\alpha > 0$ (not necessarily $\alpha \in \mathbb{N}_{>0}$), (1.1) is also the law of a Coulomb gas with n particles at inverse temperature $\beta = 2$ associated with the potential Q [11].

We emphasize that (1.1) is a probability measure only for $\alpha > 0$. If $\alpha = 0$, the above matrix T is an $n \times n$ Haar distributed unitary matrix, for which the n eigenvalues $\mathrm{z}_1, \ldots, \mathrm{z}_n$ lie exactly on the unit circle according to the probability measure proportional to

$$\prod_{1 \leq j < k \leq n} |\mathrm{z}_k - \mathrm{z}_j|^2 \prod_{j=1}^{n} d\theta_j, \qquad \mathrm{z}_j = e^{i\theta_j},\ \theta_j \in [0, 2\pi). \tag{1.2}$$

As is well-known, the equilibrium measure associated with (1.2) is the uniform measure on the unit circle.

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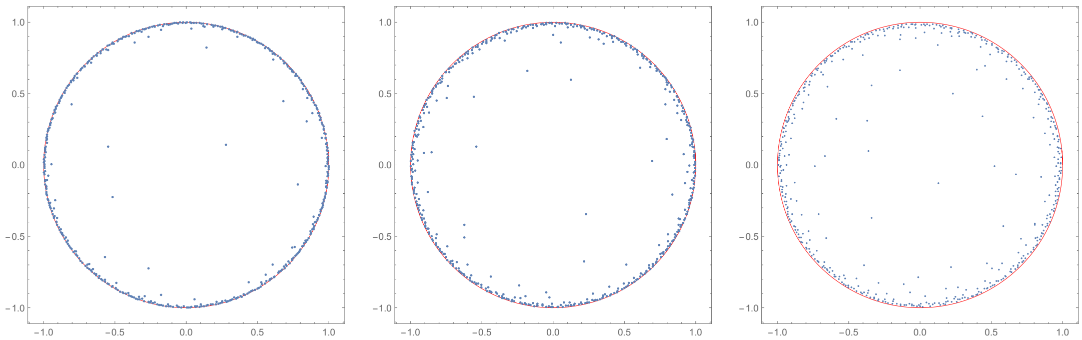

Figure 1: Illustration of the point process (1.1) with $n = 500$ and $\alpha = 2$ (left), $\alpha = 5$ (middle) and $\alpha = 10$ (right). The unit circle is represented in red.

This work focuses on the point process (1.1) as $n \to +\infty$ with $\alpha > 0$ fixed. In this regime, $Q(z) = \mathcal{O}(n^{-1})$ for any fixed $z \in \mathbb{D}$, and the associated equilibrium measure μ is defined as the unique measure minimizing the following energy functional

$$\nu \mapsto I[\nu] = \iint \ln \frac{1}{|z-w|} \nu(d^2z)\nu(d^2w)$$

among all Borel probability measures ν supported on $\mathbb{D}$. This problem is a so-called classical (or unweighted) electrostatics problem, and as such the support of μ must be the boundary of $\mathbb{D}$ [16] (and this, despite the fact that the point process (1.1) is two-dimensional). Because the density of (1.1) is invariant under rotation, we conclude that μ is the uniform measure on the unit circle $\partial\mathbb{D}$.

In the language of random matrix theory, $\partial\mathbb{D}$ is a "hard wall" of (1.1). For two-dimensional Coulomb gases, it is a standard fact that along a hard wall the equilibrium measure is singular with respect to the two-dimensional Lebesgue measure; moreover, a non-zero percentage of the points are expected to accumulate (as $n \to +\infty$) in a very small interface of width $1/n$ around the hard wall, see e.g. [16, 17, 3]. Following [3], we call this small interface "the hard edge regime". A particular feature of (1.1) is that the associated equilibrium measure μ is purely singular (i.e. μ has no absolutely continuous component and $\int_{\partial\mathbb{D}} d\mu = 1$). Hence, for large n and α fixed, most of the points $z_1, \ldots, z_n$ of (1.1) are expected to lie in a $1/n$-neighborhood of $\partial\mathbb{D}$, see also Figure 1.

There already exists a fairly rich literature on truncated unitary matrices. For example, the convergence of the distribution of the maximal modulus $\max_j |z_j|$ to the Weibull distribution has been studied in [13, 14, 17], several characterizations in terms of Painlevé transcendents for expectations of powers of the characteristic polynomial are established in [9], and results on the eigenvectors can be found in [10]. Also, besides (1.1), other two-dimensional point processes whose points are distributed within a narrow interface (or "band") have been considered, see e.g. [12, 2, 7]; however, the point processes considered in these works only feature "soft edges", and are thus very different from (1.1).

Let $\mathrm{N}(r) := \#\{z_j : |z_j| < r\} \in \{0, 1, \ldots, n\}$ be the random variable that counts the number of points of (1.1) in the disk centered at 0 of radius r. The goal of this paper is to understand the large n behavior of the multivariate moment generating function

$$\mathbb{E}\left[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\right] \tag{1.3}$$

where $m \in \mathbb{N}_{>0}$ is arbitrary (but fixed), $u_1, \ldots, u_m \in \mathbb{R}$ and $r_1 < \cdots < r_m$. We consider the hard edge regime, i.e. the radii $r_1, \ldots, r_m$ are merging near 1 at the critical speed $1 - r_j \asymp n^{-1}$ (we also allow $r_m = 1$), see also Figure 2. More precisely, we define

$$r_\ell = \left(1 - \frac{t_\ell}{n}\right)^{1/2}, \qquad t_1 > \ldots > t_m \geq 0. \tag{1.4}$$

Note that if $t_m = 0$, then $r_m = 1$ and trivially $\mathrm{N}(r_m) = n$ with probability one.

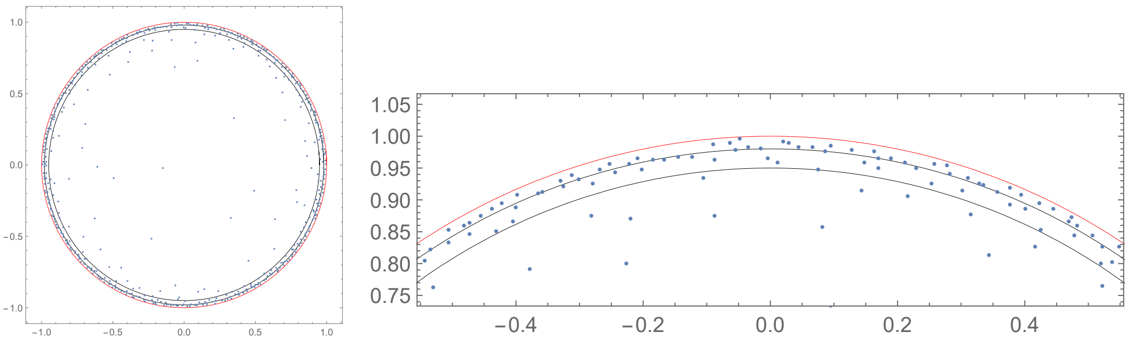

Figure 2: Left: two circles (in black) merging near the unit circle (in red). Right: a zoom is taken around i. For both pictures, $\alpha = 10$ and $n = 500$.

In Theorem 1.2 below, we prove that

$$\mathbb{E}\Bigg[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\Bigg] = \exp\Big(C_1 n + C_2 + o(1)\Big), \qquad \text{as } n \to +\infty,$$

and we give explicit expressions for the constants C_1 and C_2 in terms of the following functions

$$\mathcal{H}_\alpha(x;\vec{t},\vec{u}) := 1 + \sum_{\ell=1}^{m} (e^{u_\ell} - 1) \exp\Bigg[\sum_{j=\ell+1}^{m} u_j\Bigg] \mathrm{Q}(\alpha, t_\ell x), \tag{1.5}$$

$$\mathcal{G}_\alpha(x;\vec{t},\vec{u}) := \frac{1}{\mathcal{H}_\alpha(x;\vec{t},\vec{u})} \frac{x^{\alpha-1}}{\Gamma(\alpha)} \sum_{\ell=1}^{m} (e^{u_\ell} - 1) \exp\Bigg[\sum_{j=\ell+1}^{m} u_j\Bigg] t_\ell^\alpha e^{-t_\ell x} \frac{1 - \alpha - t_\ell x}{2}, \tag{1.6}$$

where $x \in (0,1]$, $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{C}^m$, $\vec{t} = (t_1, \ldots, t_m)$ is such that $t_1 > \ldots > t_m \geq 0$, $\Gamma(a) := \int_0^{+\infty} s^{a-1} e^{-s} ds$ is the Gamma function, and Q is the normalized incomplete Gamma function:

$$\mathrm{Q}(a,x) := \frac{\Gamma(a,x)}{\Gamma(a)}, \qquad \Gamma(a,x) := \int_x^{+\infty} s^{a-1} e^{-s} ds. \tag{1.7}$$

Note that $\mathcal{H}_\alpha$ is also well-defined at $x = 0$, while $\mathcal{G}_\alpha$ is well-defined at $x = 0$ only for $\alpha \geq 1$.

The statement of our main theorem involves $\ln \mathcal{H}_\alpha$, and the function $\mathcal{H}_\alpha$ also appears in the denominator of (1.6). The following lemma implies that $\ln \mathcal{H}_\alpha$ and $\mathcal{G}_\alpha$ are well-defined and real-valued for $x \in (0,1]$, $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$, and $t_1 > \ldots > t_m \geq 0$. In this paper, ln always denotes the principal branch of the logarithm.

Lemma 1.1. *$\mathcal{H}_\alpha(x;\vec{t},\vec{u}) > 0$ for all $x \in (0,1]$, $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $t_1 > \ldots > t_m \geq 0$.*

Proof. Since

$$\partial_{u_1} \mathcal{H}_\alpha(x;\vec{t},\vec{u}) = e^{u_1 + \ldots + u_m} \mathrm{Q}(\alpha, t_1 x) > 0,$$

it only remains to verify that $\mathcal{H}_\alpha|_{u_1 = -\infty} \geq 0$. Setting $u_1 = -\infty$ in (1.5) and rearranging the terms, we find

$$\mathcal{H}_\alpha(x;\vec{t},\vec{u})|_{u_1=-\infty} = \big[1 - \mathrm{Q}(\alpha, t_m x)\big] + \sum_{\ell=2}^{m} e^{u_\ell + \ldots + u_m} [\mathrm{Q}(\alpha, t_\ell x) - \mathrm{Q}(\alpha, t_{\ell-1} x)].$$

Recall that $t_1 > \ldots > t_m \geq 0$ and $x \in (0,1]$. Hence, since $\mathbb{R} \ni s \mapsto \mathrm{Q}(\alpha, s)$ decreases from 1 to 0, the m terms in the above right-hand side are all ≥ 0, which proves $\mathcal{H}_\alpha|_{u_1=-\infty} \geq 0$. ☐

Theorem 1.2. *Let $m \in \mathbb{N}_{>0}$, $\alpha > 0$ and $t_1 > \cdots > t_m \geq 0$ be fixed parameters. For $n \in \mathbb{N}_{>0}$, define*

$$r_\ell = \Big(1 - \frac{t_\ell}{n}\Big)^{1/2}, \qquad \ell = 1, \ldots, m. \tag{1.8}$$

Let $\mathrm{N}(r) := \#\{z_j : |z_j| < r\} \in \{0, 1, \ldots, n\}$ *be the random variable that counts the number of points of* (1.1) *in the disk centered at* 0 *of radius* r*. For any fixed* $x_1, \ldots, x_m \in \mathbb{R}$*, there exists* $\delta > 0$ *such that*

$$\mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\Big] = \exp\Big(C_1 n + C_2 + \mathcal{O}(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n})\Big), \qquad \text{as } n \to +\infty \tag{1.9}$$

uniformly for $u_1 \in \{z \in \mathbb{C} : |z - x_1| \le \delta\}, \ldots, u_m \in \{z \in \mathbb{C} : |z - x_m| \le \delta\}$*, where* $\hat{\alpha} := \min\{\alpha, 1\}$*,* $\mathbf{1}_{\alpha=1} = 1$ *if* $\alpha = 1$ *and* $\mathbf{1}_{\alpha=1} = 0$ *otherwise, and*

$$\begin{aligned} C_1 &= \int_0^1 \ln \mathcal{H}_\alpha(x; \vec{t}, \vec{u})\, dx, \\ C_2 &= \int_0^1 \mathcal{G}_\alpha(x; \vec{t}, \vec{u})\, dx + \frac{\ln \mathcal{H}_\alpha(1; \vec{t}, \vec{u}) - \sum_{j=1}^m u_j}{2}. \end{aligned} \tag{1.10}$$

In particular, since $\mathbb{E}\big[\prod_{j=1}^m e^{u_j \mathrm{N}(r_j)}\big]$ *is analytic in* $u_1, \ldots, u_m \in \mathbb{C}$ *and is positive for* $u_1, \ldots, u_m \in \mathbb{R}$*, the asymptotic formula* (1.9) *combined with Cauchy's formula implies that*

$$\partial_{u_1}^{k_1} \ldots \partial_{u_m}^{k_m} \Big\{ \ln \mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\Big] - \Big(C_1 n + C_2\Big)\Big\} = \mathcal{O}(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n}), \qquad \text{as } n \to +\infty, \tag{1.11}$$

for any $k_1, \ldots, k_m \in \mathbb{N}$*, and* $u_1, \ldots, u_m \in \mathbb{R}$*.*

Let $(\mathbb{N}^m)_{>0} := \{\vec{j} = (j_1, \ldots, j_m) \in \mathbb{N}^m : j_1 + \cdots + j_m \ge 1\}$. For $\vec{j} \in (\mathbb{N}^m)_{>0}$, the joint cumulant $\kappa_{\vec{j}} = \kappa_{\vec{j}}(r_1, \ldots, r_m; n, \alpha)$ of $\mathrm{N}(r_1), \ldots, \mathrm{N}(r_m)$ is defined by

$$\kappa_{\vec{j}} = \kappa_{j_1, \ldots, j_m} := \partial_{\vec{u}}^{\vec{j}} \ln \mathbb{E}[e^{u_1 \mathrm{N}(r_1) + \cdots + u_m \mathrm{N}(r_m)}]\Big|_{\vec{u}=\vec{0}}, \tag{1.12}$$

where $\partial_{\vec{u}}^{\vec{j}} := \partial_{u_1}^{j_1} \ldots \partial_{u_m}^{j_m}$ and $\vec{0} := (0, \ldots, 0)$. For instance, we have

$$\mathbb{E}[\mathrm{N}(r)] = \kappa_1(r), \qquad \mathrm{Var}[\mathrm{N}(r)] = \kappa_2(r) = \kappa_{(1,1)}(r, r), \qquad \mathrm{Cov}[\mathrm{N}(r_1), \mathrm{N}(r_2)] = \kappa_{(1,1)}(r_1, r_2).$$

Corollary 1.3. *Let* $m \in \mathbb{N}_{>0}$*,* $\vec{j} \in (\mathbb{N}^m)_{>0}$*,* $\alpha > 0$*, and* $t_1 > \cdots > t_m \ge 0$ *be fixed. For* $n \in \mathbb{N}_{>0}$*, define* $\{r_\ell\}_{\ell=1}^m$ *by* (1.8)*.*

(a) The joint cumulant $\kappa_{\vec{j}}$ *satisfies*

$$\kappa_{\vec{j}} = \partial_{\vec{u}}^{\vec{j}} C_1\big|_{\vec{u}=\vec{0}}\, n + \partial_{\vec{u}}^{\vec{j}} C_2\big|_{\vec{u}=\vec{0}} + \mathcal{O}\big(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n}\big), \qquad n \to +\infty, \tag{1.13}$$

where C_1, C_2 *are as in Theorem* 1.2 *and* $\hat{\alpha} := \min\{\alpha, 1\}$*. In particular, for any* $1 \le \ell < k \le m$*,*

$$\begin{aligned} &\mathbb{E}[\mathrm{N}(r_\ell)] = b_1(t_\ell) n + c_1(t_\ell) + \mathcal{O}\big(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n}\big), \\ &\mathrm{Var}[\mathrm{N}(r_\ell)] = b_{(1,1)}(t_\ell, t_\ell) n + c_{(1,1)}(t_\ell, t_\ell) + \mathcal{O}\big(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n}\big), \\ &\mathrm{Cov}(\mathrm{N}(r_\ell), \mathrm{N}(r_k)) = b_{(1,1)}(t_\ell, t_k) n + c_{(1,1)}(t_\ell, t_k) + \mathcal{O}\big(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \tfrac{\mathbf{1}_{\alpha=1}\ln n}{n}\big) \end{aligned}$$

as $n \to +\infty$*, where*

$$\begin{aligned} b_1(t_\ell) &= \int_0^1 \mathrm{Q}(\alpha, t_\ell x) dx = \mathrm{Q}(\alpha, t_\ell) + \alpha \frac{1 - \mathrm{Q}(\alpha+1, t_\ell)}{t_\ell}, \\ c_1(t_\ell) &= \frac{t_\ell^\alpha}{\Gamma(\alpha)} \int_0^1 x^{\alpha-1} e^{-t_\ell x} \frac{1 - \alpha - t_\ell x}{2} dx + \frac{\mathrm{Q}(\alpha, t_\ell) - 1}{2}, \end{aligned} \tag{1.14}$$

and, for $\ell \le k$*,*

$$\begin{aligned} b_{(1,1)}(t_\ell, t_k) &= \int_0^1 \mathrm{Q}(\alpha, t_\ell x)\big(1 - \mathrm{Q}(\alpha, t_k x)\big) dx, \\ c_{(1,1)}(t_\ell, t_k) &= \int_0^1 \Big\{(\alpha - 1 + t_k x) t_k^\alpha e^{-t_k x} \mathrm{Q}(\alpha, t_\ell x) - (\alpha - 1 + t_\ell x) t_\ell^\alpha e^{-t_\ell x}\big(1 - \mathrm{Q}(\alpha, t_k x)\big)\Big\} \frac{x^{\alpha-1} dx}{2\,\Gamma(\alpha)} \end{aligned} \tag{1.15}$$

$$+\frac{1}{2}\mathrm{Q}(\alpha,t_\ell)\big(1-\mathrm{Q}(\alpha,t_k)\big). \tag{1.16}$$

(b) Assume furthermore that $t_m>0$. As $n\to+\infty$, the random variable $(\mathcal{N}_1,\dots,\mathcal{N}_m)$, where

$$\mathcal{N}_\ell := \frac{\mathrm{N}(r_\ell)-b_1(t_\ell)n}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)n}}, \qquad \ell=1,\dots,m, \tag{1.17}$$

convergences in distribution to a multivariate normal random variable of mean $(0,\dots,0)$ whose covariance matrix Σ is given by

$$\Sigma_{\ell,k}=\Sigma_{k,\ell}=\frac{b_{(1,1)}(t_\ell,t_k)}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)b_{(1,1)}(t_k,t_k)}}, \qquad 1\le\ell\le k\le m.$$

Remark 1.4. *If $t_m=0$, then $b_1(t_m)=1$ and $c_1(t_m)=b_{(1,1)}(t_m,t_m)=c_{(1,1)}(t_m,t_m)=0$, which is consistent with the fact that $\mathrm{N}(r_m)=n$ with probability 1. This is the reason why we required $t_m>0$ in Corollary 1.3 (b). The graphs of $t\mapsto b_1(t)$ and $t\mapsto b_{(1,1)}(t,t)$ are shown in Figure 3 for several values of α.*

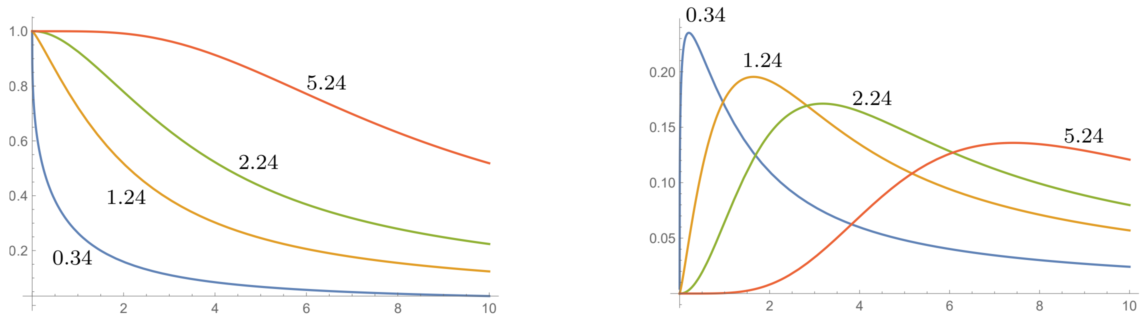

Figure 3: The coefficients $t\mapsto b_1(t)$ (left) and $t\mapsto b_{(1,1)}(t,t)$ (right) for the indicated values of α. (Recall from Corollary 1.3 that $b_1(t)n$ and $b_{(1,1)}(t,t)n$ are the leading terms in the large n asymptotics of $\mathbb{E}[\mathrm{N}((1-\frac{t}{n})^{1/2})]$ and $\mathrm{Var}[\mathrm{N}((1-\frac{t}{n})^{1/2})]$, respectively.)

Proof of Corollary 1.3. Assertion (a) (except for the second equality in (1.14)) is a direct consequence of (1.11) and (1.12). The second equality in (1.14) is obtained using (1.7), Fubini's theorem, and $\Gamma(\alpha+1)=\alpha\,\Gamma(\alpha)$:

$$\int_0^1 \mathrm{Q}(\alpha,t_\ell x)dx=\int_0^1 dx\int_{t_\ell x}^{+\infty}dy\frac{e^{-y}y^{\alpha-1}}{\Gamma(\alpha)}=\bigg(\int_0^{t_\ell}dy\int_0^{\frac{y}{t_\ell}}dx+\int_{t_\ell}^{+\infty}dy\int_0^1 dx\bigg)\frac{e^{-y}y^{\alpha-1}}{\Gamma(\alpha)}$$
$$=\frac{\alpha}{t_\ell}\int_0^{t_\ell}\frac{e^{-y}y^\alpha}{\Gamma(\alpha+1)}dy+\int_{t_\ell}^{+\infty}\frac{e^{-y}y^{\alpha-1}}{\Gamma(\alpha)}dy=\alpha\frac{1-\mathrm{Q}(\alpha+1,t_\ell)}{t_\ell}+\mathrm{Q}(\alpha,t_\ell).$$

We now turn to the proof of (b). Using (1.9) with $u_\ell=\frac{iv_\ell}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)n}}$ and $v_\ell\in\mathbb{R}$ fixed, we get

$$\mathbb{E}[e^{i\sum_{\ell=1}^m v_\ell\mathcal{N}_\ell}]=\mathbb{E}[e^{\sum_{\ell=1}^m u_\ell\mathrm{N}(r_\ell)}]e^{-\sum_{\ell=1}^m u_\ell b_1(t_\ell)n}$$
$$=e^{C_1(\vec{u})n+C_2(\vec{u})+\mathcal{O}(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}})}e^{-\sum_{\ell=1}^m u_\ell\partial_{u_\ell}C_1|_{\vec{u}=\vec{0}}n}$$

as $n\to+\infty$, where the dependence of C_1 and C_2 in $\vec{u}$ has been made explicit. Since $C_j|_{\vec{u}=\vec{0}}=0$ for $j=1,2$ and $u_\ell=\mathcal{O}(n^{-1/2})$, we thus have

$$\mathbb{E}[e^{i\sum_{\ell=1}^m v_\ell\mathcal{N}_\ell}]=e^{\frac{1}{2}\sum_{\ell,k=1}^m u_\ell u_k\partial_{u_\ell}\partial_{u_k}C_1|_{\vec{u}=\vec{0}}n+\mathcal{O}(|\vec{u}|^3n+|\vec{u}|+n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}})}$$
$$=e^{\frac{1}{2}\sum_{\ell,k=1}^m\frac{iv_\ell}{\sqrt{b_{(1,1)}(t_\ell,t_\ell)}}\frac{iv_k}{\sqrt{b_{(1,1)}(t_k,t_k)}}b_{(1,1)}(t_{\min(\ell,k)},t_{\max(\ell,k)})+o(1)}\to e^{-\frac{1}{2}\sum_{\ell,k=1}^m v_\ell\Sigma_{\ell,k}v_k}$$

as $n\to+\infty$. In other words, $\mathbb{E}[e^{i\sum_{\ell=1}^m v_\ell\mathcal{N}_\ell}]$ converges pointwise to $e^{-\frac{1}{2}\sum_{\ell,k=1}^m v_\ell\Sigma_{\ell,k}v_k}$ as $n\to+\infty$, which implies Assertion (b) by Lévy's continuity theorem. $\square$

Comparison with other works on counting statistics. There has been a lot of interest recently on counting statistics of two-dimensional point processes. We will not attempt to survey this literature here, but refer the interested reader to [5] and the introduction of [1]. Our main goal in this subsection is to compare Theorem 1.2 with the works [8, 3].

In [8], the following Mittag-Leffler ensemble is considered:

$$\frac{1}{n!\hat{Z}_n} \prod_{1\leq j<k\leq n} |z_k - z_j|^2 \prod_{j=1}^{n} |z_j|^{2a} e^{-n|z_j|^{2b}} d^2 z_j, \qquad z_1,\ldots,z_n \in \mathbb{C}, \tag{1.18}$$

where $b > 0$ and $a > -1$ are parameters of the model. The associated equilibrium measure μ^{ML} is supported on the disk $\{|z| \leq b^{-\frac{1}{2b}}\}$ and given by $\mu^{\mathrm{ML}}(d^2z) = \frac{b^2}{\pi}|z|^{2b-2}d^2z$. Note that μ^{ML} is absolutely continuous with respect to the Lebesgue measure d^2z (this contrasts with the equilibrium measure μ of (1.1), which is purely singular). Let $m \in \mathbb{N}_{>0}$, $r \in (0, b^{-\frac{1}{2b}})$, $\mathfrak{s}_1, \ldots, \mathfrak{s}_m \in \mathbb{R}$, $a > -1$ and $b > 0$ be fixed parameters such that $\mathfrak{s}_1 < \ldots < \mathfrak{s}_m$. The main result of [8] is the large n asymptotics of the m-point moment generating function of the disk counting statistics of (1.18) when the radii are merging either in the bulk, i.e. $r_\ell = r\big(1 + \frac{\sqrt{2}\,\mathfrak{s}_\ell}{r^b\sqrt{n}}\big)^{\frac{1}{2b}}$ for all $\ell \in \{1,\ldots,m\}$, or at the soft edge, i.e. $r_\ell = b^{-\frac{1}{2b}}\big(1 + \sqrt{2b}\frac{\mathfrak{s}_\ell}{\sqrt{n}}\big)^{\frac{1}{2b}}$ for all $\ell \in \{1,\ldots,m\}$. In both cases, it is shown in [8] that

$$\mathbb{E}^{\mathrm{ML}}\bigg[\prod_{j=1}^{m} e^{u_j \mathrm{N}(r_j)}\bigg] = \exp\bigg(D_1 n + D_2\sqrt{n} + D_3 + \frac{D_4}{\sqrt{n}} + \mathcal{O}\Big(\frac{(\ln n)^2}{n}\Big)\bigg), \qquad \text{as } n \to +\infty,$$

and the constants $D_1, \ldots, D_4$ are determined explicitly. The constant D_1 is particularly simple; for example, in the bulk regime it is given by $D_1 = \int_{|z|\leq r} \mu^{\mathrm{ML}}(d^2z) \sum_{j=1}^m u_j = br^{2b}\sum_{j=1}^m u_j$. The constant D_2 is more complicated and given by

$$D_2 = \begin{cases} \sqrt{2}\, br^b \displaystyle\int_{-\infty}^{+\infty} \bigg(\ln \mathcal{H}^{\mathrm{ML}}(x;\vec{\mathfrak{s}},\vec{u}) - \chi_{(-\infty,0)}(x)\sum_{j=1}^m u_j\bigg)dx, & \text{for the bulk regime,} \\ \sqrt{2b} \displaystyle\int_{-\infty}^{0} \bigg(\ln \mathcal{H}^{\mathrm{ML}}(x;\vec{\mathfrak{s}},\vec{u}) - \sum_{j=1}^m u_j\bigg)dx, & \text{for the soft edge regime,} \end{cases}$$

where $\chi_{(-\infty,0)}(x) = 1$ if $x < 0$ and $\chi_{(-\infty,0)}(x) = 0$ otherwise, $\vec{\mathfrak{s}} := (\mathfrak{s}_1, \ldots, \mathfrak{s}_m)$, and

$$\mathcal{H}^{\mathrm{ML}}(x;\vec{\mathfrak{s}},\vec{u}) := 1 + \sum_{\ell=1}^{m}(e^{u_\ell} - 1)\exp\bigg[\sum_{j=\ell+1}^{m} u_j\bigg]\frac{\mathrm{erfc}(x - \mathfrak{s}_\ell)}{2}.$$

We find it curious that the above function has the same structure as the function $\mathcal{H}_\alpha$ in (1.5); namely, they are both of the form

$$1 + \sum_{\ell=1}^{m}(e^{u_\ell} - 1)\exp\bigg[\sum_{j=\ell+1}^{m} u_j\bigg]X_\ell(x) \tag{1.19}$$

where $X_\ell(x) := \mathrm{Q}(\alpha, t_\ell x)$ in the present paper and $X_\ell(x) := \mathrm{erfc}(x - \mathfrak{s}_\ell)/2$ in [8]. Note also that

- $\mathcal{H}_\alpha$ already appears in the leading constant C_1, while $\mathcal{H}^{\mathrm{ML}}$ appears in D_2,
- $t_1, \ldots, t_m$ are dilation parameters of X_ℓ, in the sense that they appear in the multiplicative form "$t_\ell x$" in $\mathrm{Q}(\alpha, t_\ell x)$, while $\mathfrak{s}_1, \ldots, \mathfrak{s}_m$ are translation parameters of X_ℓ, in the sense that they appear in the additive form "$x - \mathfrak{s}_\ell$" in $\mathrm{erfc}(x - \mathfrak{s}_\ell)/2$.

Let $0 < \rho < b^{-\frac{1}{2b}}$ be fixed. The following point process was considered in [3]:

$$\frac{1}{n!\check{Z}_n} \prod_{1\leq j<k\leq n} |z_k - z_j|^2 \prod_{j=1}^{n} |z_j|^{2\alpha} e^{-n|z_j|^{2b}} d^2 z_j, \qquad |z_j| \leq \rho. \tag{1.20}$$

The only (but important) difference between the point processes (1.18) and (1.20) is that in (1.20) the points are constrained to lie in the disk $\{|z| \leq \rho\}$. Because $\rho < b^{-\frac{1}{2b}}$, the circle $\{|z| = \rho\}$ is a hard wall of (1.20) and it is shown in [3] that the associated equilibrium measure μ_h^{ML} is given by

$$\mu_h^{\mathrm{ML}}(d^2z) = \mu_{\mathrm{reg}}^{\mathrm{ML}}(d^2z) + \mu_{\mathrm{sing}}^{\mathrm{ML}}(d^2z),$$

$$\mu^{\mathrm{ML}}_{\mathrm{reg}}(d^2z) := 2b^2 r^{2b-1} dr \frac{d\theta}{2\pi}, \quad \mu^{\mathrm{ML}}_{\mathrm{sing}}(d^2z) := c_\rho \delta_\rho(r) dr \frac{d\theta}{2\pi}, \tag{1.21}$$

where $z = re^{i\theta}$, $r > 0$, $\theta \in (-\pi, \pi]$ and $c_\rho := \int_{|z|>\rho} \mu^{\mathrm{ML}}(d^2z) = \int_\rho^{b^{-\frac{1}{2b}}} 2b^2 r^{2b-1} dr = 1 - b\rho^{2b}$. For the hard edge regime $r_\ell = \rho(1 - \frac{t_\ell}{n})^{\frac{1}{2b}}$ with $t_1 > \cdots > t_m \geq 0$, it is proved in [3] that

$$\mathbb{E}^{\mathrm{ML}}_h\left[\prod_{j=1}^m e^{u_j \mathrm{N}(r_j)}\right] = \exp\left(E_1 n + E_2 \ln n + E_3 + \frac{E_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\right), \qquad \text{as } n \to +\infty, \tag{1.22}$$

where $E_1 = \int_{|z|\leq\rho} \mu^{\mathrm{ML}}_{\mathrm{reg}}(d^2z) \times \sum_{j=1}^m u_j + \int_{b\rho^{2b}}^1 \ln \mathcal{H}^{\mathrm{ML}}_h(x; \vec{t}, \vec{u}) dx$ with

$$\mathcal{H}^{\mathrm{ML}}_h(x; \vec{t}, \vec{u}) = 1 + \sum_{\ell=1}^m (e^{u_\ell} - 1) \exp\left[\sum_{j=\ell+1}^m u_j\right] e^{-\frac{t_\ell}{b}(x - b\rho^{2b})}.$$

This function is also in the form (1.19), with $X_\ell(x) = e^{-\frac{t_\ell}{b}(x-b\rho^{2b})} = \mathrm{Q}(1, \frac{t_\ell}{b}(x - b\rho^{2b}))$. It is also interesting to note the presence of the term $E_2 \ln n$ in (1.22), while in (1.9) there is no term proportional to $\ln n$. We believe the reason for this is that μ^{ML}_h has a non-trivial component $\mu^{\mathrm{ML}}_{\mathrm{reg}}$ which is absolutely continuous with respect to d^2z, while the equilibrium measure μ of (1.1) is purely singular. This belief is supported by the following fact: when $\rho \to 0$, the measure μ^{ML}_h becomes purely singular (because $c_\rho \to 1$), and $E_2 \to 0$ (as can be easily checked from [3, Theorem 1.3]).

A transition regime between the hard edge and the bulk was also considered in [3]. This regime is called “the semi-hard edge regime” and corresponds to the case when the radii are at a distance of order $1/\sqrt{n}$ from the hard edge. More precisely, for $r_\ell = \rho(1 + \frac{\sqrt{2}\,\mathfrak{s}_\ell}{\rho^b \sqrt{n}})^{\frac{1}{2b}}$ with $\mathfrak{s}_1 < \cdots < \mathfrak{s}_m < 0$, we have

$$\mathbb{E}^{\mathrm{ML}}_h\left[\prod_{j=1}^m e^{u_j \mathrm{N}(r_j)}\right] = \exp\left(F_1 n + F_2 \sqrt{n} + F_3 + \frac{F_4}{\sqrt{n}} + \mathcal{O}\left(\frac{(\ln n)^4}{n}\right)\right),$$

where $F_1 = \int_{|z|\leq\rho} \mu^{\mathrm{ML}}_{\mathrm{reg}}(d^2z) \sum_{j=1}^m u_j = b\rho^{2b} \sum_{j=1}^m u_j$ and

$$F_2 = \sqrt{2}\, b\rho^b \int_{-\infty}^{+\infty} \left(\ln \mathcal{H}^{\mathrm{ML}}_{\mathrm{sh}}(x; \vec{\mathfrak{s}}, \vec{u}) - \chi_{(-\infty,0)}(x) \sum_{j=1}^m u_j\right) dx,$$

with

$$\mathcal{H}^{\mathrm{ML}}_{\mathrm{sh}}(x; \vec{\mathfrak{s}}, \vec{u}) = 1 + \sum_{\ell=1}^m (e^{u_\ell} - 1) \exp\left[\sum_{j=\ell+1}^m u_j\right] \frac{\mathrm{erfc}(x - \mathfrak{s}_\ell)}{\mathrm{erfc}(x)}.$$

The function $\mathcal{H}^{\mathrm{ML}}_{\mathrm{sh}}$ is also in the form (1.19), with $X_\ell(x) = \frac{\mathrm{erfc}(x-\mathfrak{s}_\ell)}{\mathrm{erfc}(x)}$. The above discussion is summarized in Table 4.

Point process	Regime	$\mathbb{E}\left[\prod_{j=1}^m e^{u_j \mathrm{N}(r_j)}\right]$ as $n \to +\infty$	$X_\ell(x)$	Ref
(1.1)	Hard edge	$\exp\left(C_1 n + C_2 + \mathcal{O}(n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}} + \frac{\mathbf{1}_{\alpha=1}\ln n}{n})\right)$	$\mathrm{Q}(\alpha, t_\ell x)$	Theorem 1.2
(1.18)	Bulk & Soft edge	$\exp\left(D_1 n + D_2\sqrt{n} + D_3 + \frac{D_4}{\sqrt{n}} + \mathcal{O}(\frac{(\ln n)^2}{n})\right)$	$\frac{\mathrm{erfc}(x - \mathfrak{s}_\ell)}{2}$	[8]
(1.20)	Hard edge	$\exp\left(E_1 n + E_2 \ln n + E_3 + \frac{E_4}{\sqrt{n}} + \mathcal{O}(n^{-\frac{3}{5}})\right)$	$e^{-\frac{t_\ell}{b}(x-b\rho^{2b})}$	[3]
(1.20)	Semi-hard edge	$\exp\left(F_1 n + F_2\sqrt{n} + F_3 + \frac{F_4}{\sqrt{n}} + \mathcal{O}(\frac{(\ln n)^4}{n})\right)$	$\frac{\mathrm{erfc}(x - \mathfrak{s}_\ell)}{\mathrm{erfc}(x)}$	[3]

Table 4: Comparison of our main result with results from [8, 3].

2 Preliminaries

Let $\mathcal{E}_n := \mathbb{E}\big[\prod_{\ell=1}^m e^{u_\ell \mathrm{N}(r_\ell)}\big]$, and define

$$w(z) = (1-|z|^2)^{\alpha-1}\omega(|z|), \qquad \omega(x) := \prod_{\ell=1}^m \begin{cases} e^{u_\ell}, & \text{if } x < r_\ell, \\ 1, & \text{if } x \geq r_\ell. \end{cases} \tag{2.1}$$

By rewriting $\prod_{1\leq j<k\leq n}|z_k - z_j|^2$ as the product of two Vandermonde determinants, and then using standard algebraic manipulations, we get

$$\begin{aligned} \mathcal{E}_n &= \frac{1}{n! Z_n}\int_{\mathbb{D}}\cdots\int_{\mathbb{D}} \prod_{1\leq j<k\leq n}|z_k - z_j|^2 \prod_{j=1}^n w(z_j) d^2 z_j \\ &= \frac{1}{Z_n}\det\left(\int_{\mathbb{D}} z^j \bar{z}^k w(z) d^2 z\right)_{j,k=0}^{n-1}. \end{aligned} \tag{2.2}$$

Since w is rotation-invariant, only the diagonal elements in (2.2) are non-zero. Indeed, after writing $z = xe^{i\theta}$, $x \geq 0, \theta \in [0, 2\pi)$ and integrating over θ, we find

$$\mathcal{E}_n = \frac{1}{Z_n}(2\pi)^n \prod_{j=1}^n \int_0^1 x^{2j-1} w(x) dx. \tag{2.3}$$

In the same way, we have $Z_n = (2\pi)^n \prod_{j=1}^n \int_0^1 x^{2j-1}(1-x^2)^{\alpha-1}dx$ (see also (2.14) below). Substituting this identity and (2.1) in (2.3), we then find

$$\mathcal{E}_n = \prod_{j=1}^n \frac{\int_0^1 x^{2j-1}(1-x^2)^{\alpha-1}\omega(x)dx}{\int_0^1 x^{2j-1}(1-x^2)^{\alpha-1}dx}. \tag{2.4}$$

It will be convenient for us to rewrite ω (defined in (2.1)) as follows:

$$\omega(x) = 1 + \sum_{\ell=1}^m \omega_\ell \mathbf{1}_{[0,r_\ell)}(x), \qquad \omega_\ell = \begin{cases} e^{u_\ell + \cdots + u_m} - e^{u_{\ell+1}+\cdots+u_m}, & \text{if } \ell < m, \\ e^{u_m} - 1, & \text{if } \ell = m. \end{cases} \tag{2.5}$$

Using (2.5) in (2.4) yields the following expression for $\ln \mathcal{E}_n$:

$$\ln \mathcal{E}_n = \sum_{j=1}^n \ln\left(1 + \sum_{\ell=1}^m \omega_\ell F_{n,j,\ell}\right), \tag{2.6}$$

$$F_{n,j,\ell} := \frac{\int_0^{r_\ell} x^{2j-1}(1-x^2)^{\alpha-1}dx}{\int_0^1 x^{2j-1}(1-x^2)^{\alpha-1}dx} = \frac{\mathrm{B}(r_\ell^2, j, \alpha)}{\mathrm{B}(j,\alpha)}, \qquad j = 1,\ldots,n, \;\; \ell = 1,\ldots,m, \tag{2.7}$$

where $\mathrm{B}(j,\alpha)$ is the Beta function

$$\mathrm{B}(j,\alpha) := \int_0^1 y^{j-1}(1-y)^{\alpha-1}dy = \frac{\Gamma(j)\Gamma(\alpha)}{\Gamma(j+\alpha)}, \tag{2.8}$$

and $\mathrm{B}(v,j,\alpha)$ is the incomplete Beta function

$$\mathrm{B}(v,j,\alpha) := \int_0^v y^{j-1}(1-y)^{\alpha-1}dy. \tag{2.9}$$

Many properties of these functions are stated e.g. in [15, Sections 5.12 and 8.17]. It is also convenient for us to consider the normalized incomplete Beta function, which is given by

$$I(v,j,\alpha) := \frac{\mathrm{B}(v,j,\alpha)}{\mathrm{B}(j,\alpha)}, \tag{2.10}$$

so that $F_{n,j,\ell} = I(r_\ell^2, j, \alpha)$.

Hence, to analyze the right-hand side of (2.6), we need the asymptotics of $I(v,j,\alpha)$ when $(v-1) \asymp n^{-1}$ and simultaneously $j \in \{1,\ldots,n\}$ and α fixed. The large n behavior of $I(r_\ell^2, j, \alpha)$ depends crucially on whether j remains bounded or not as $n \to +\infty$. We will therefore split the sum (2.6) in two parts as follows

$$\ln \mathcal{E}_n = S_0 + S_1, \qquad \text{where} \quad S_0 = \sum_{j=1}^{\frac{n}{M}-1} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell F_{n,j,\ell}\Big), \qquad S_1 = \sum_{j=\frac{n}{M}}^{n} \ln\Big(1+\sum_{\ell=1}^{m} \omega_\ell F_{n,j,\ell}\Big),$$

and $M := n(\lceil \frac{n}{n^{\frac{2}{2+\alpha}}} \rceil)^{-1} \asymp n^{\frac{2}{2+\alpha}}$ is a new parameter such that $\mathbb{N} \ni n/M \to +\infty$ as $n \to +\infty$. (A more naive choice for M would be $M = n/M'$ where M' is large but fixed, but this choice does not yield a good control over certain error terms in the proof. The precise reason as to why we choose $M \asymp n^{\frac{2}{2+\alpha}}$ is technical and will become apparent at the end of Section 3.)

The following lemma establishes an exact identity that will be useful to handle the sum S_0, i.e. to obtain the large n asymptotics of $F_{n,j,\ell}$ when j is "not very large".

Lemma 2.1. *Let $j \in \mathbb{N}_{>0}$, $\alpha > 0$ and $v \in [0,1]$. Then we have the exact identity*

$$I(v,j,\alpha) = 1 - \frac{(1-v)^\alpha}{\mathrm{B}(j,\alpha)} \sum_{p=0}^{j-1} (-1)^p \binom{j-1}{p} \frac{(1-v)^p}{\alpha+p}.$$

Proof. The statement follows from [15, eqs 8.17.4 and 8.17.7]. We also provide a short proof here for convenience. Substituting $x^{j-1} = \sum_{p=0}^{j-1}(-1)^p\binom{j-1}{p}(1-x)^p$ in (2.10) yields

$$\begin{aligned} I(v,j,\alpha) &= \frac{1}{\mathrm{B}(j,\alpha)} \sum_{p=0}^{j-1} (-1)^p \binom{j-1}{p} \int_0^v (1-x)^{\alpha+p-1} dx \\ &= \frac{1}{\mathrm{B}(j,\alpha)} \bigg(\sum_{p=0}^{j-1} \binom{j-1}{p} \frac{(-1)^p}{\alpha+p} - (1-v)^\alpha \sum_{p=0}^{j-1} (-1)^p \binom{j-1}{p} \frac{(1-v)^p}{\alpha+p} \bigg). \end{aligned}$$

Replacing v by 1 above yields $1 = \frac{1}{\mathrm{B}(j,\alpha)} \sum_{p=0}^{j-1} \binom{j-1}{p} \frac{(-1)^p}{\alpha+p}$, and the claim follows. ☐

To analyze S_1, we will use the uniform asymptotics of the incomplete Beta function (this is the main novelty of the proof, as earlier works such as [8, 3] on the Mittag-Leffler ensemble rely instead on the uniform asymptotics of the incomplete gamma function). The following lemma is due to Temme [18, Section 11.3.3.1] (this result can also be found in e.g. [15, Section 8.18(ii)]).

Lemma 2.2 (Temme [18])**.** *Let $N \in \mathbb{N}_{>0}$. As $j \to +\infty$ with $\alpha > 0$ fixed,*

$$I(v,j,\alpha) = \frac{\Gamma(j+\alpha)}{\Gamma(j)} d_0 F_0 \bigg(1 + \sum_{k=1}^{N-1} \frac{d_k F_k}{d_0 F_0} + \mathcal{O}(j^{-N})\bigg)$$

uniformly for v in compact subsets of $(0,1]$. The coefficients $F_k = F_k(v,j,\alpha)$ are defined by

$$F_k = \frac{k-1+\alpha - j\ln(v^{-1})}{j} F_{k-1} + \frac{(k-1)\ln(v^{-1})}{j} F_{k-2}, \qquad k \geq 2, \tag{2.11}$$

with the initial assignments

$$F_0 = j^{-\alpha} \mathrm{Q}(\alpha, j\ln(v^{-1})), \qquad F_1 = \frac{\alpha - j\ln(v^{-1})}{j} F_0 + \frac{(\ln(v^{-1}))^\alpha v^j}{j\Gamma(\alpha)}, \tag{2.12}$$

where Q is defined in (1.7) and the coefficients $d_k = d_k(v,\alpha)$ are defined through the generating function

$$\left(\frac{1-e^{-t}}{t}\right)^{\alpha-1} = \sum_{k=0}^{\infty} d_k (t - \ln(v^{-1}))^k. \tag{2.13}$$

In particular,

$$d_0 = \left(\frac{1-v}{\ln(v^{-1})}\right)^{\alpha-1}, \qquad d_1 = \frac{(\alpha-1)(v-1+v\ln(v^{-1}))}{(v-1)^2} \left(\frac{1-v}{\ln(v^{-1})}\right)^{\alpha}.$$

Remark 2.3. *(Determinants with circular root-type singularities.) Note from* (2.2) *that $\mathcal{E}_n$ can be seen as a ratio of two determinants. The determinant on the numerator involves w, and this weight has a root-type singularity along the unit circle (i.e. along the hard edge). Other determinants with circular root-type singularities have been considered in [4]; however, the singularities in [4] lie in the bulk, and the asymptotics of the corresponding determinants involve the so-called associated Hermite polynomials (this contrasts drastically with the asymptotics of $\mathcal{E}_n$, which are given in Theorem 1.2).*

Remark 2.4. *(Partition function.) Asymptotic expansions of partition functions of two-dimensional point processes are a classical topic of interest, see e.g. [5, Section 5.3]. For rotation-invariant (and determinantal) ensembles with soft edges, precise formulas up to and including the term of order 1 have been obtained in the recent work [6]. The class of ensembles considered in [6] includes* (1.1) *when α is proportional to n, see [6, Section 4.2]. As mentioned earlier, for α fixed, the ensemble* (1.1) *has a hard edge and is therefore not considered in [6]. As a minor aside, we compute here the partition function of* (1.1) *with α fixed using a similar formula as* (2.3)*. As in* (2.3) *(but with $w(u)$ replaced by $(1-|u|^2)^{\alpha-1}$), we get*

$$Z_n = (2\pi)^n \prod_{j=0}^{n-1} \int_0^1 u^{2j+1}(1-u^2)^{\alpha-1}du = \pi^n \prod_{j=1}^{n} \mathrm{B}(j,\alpha) = \pi^n \Gamma(\alpha)^n \frac{G(n+1)G(1+\alpha)}{G(n+1+\alpha)}, \tag{2.14}$$

where for the last identity we have used the functional equation for the Barnes G-function to write

$$\prod_{j=1}^{n} \Gamma(j+\alpha) = \frac{G(n+\alpha+1)}{G(1+\alpha)}. \tag{2.15}$$

Using the expansions (see [15, Eqs. 5.17.5 and 5.11.8])

$$\begin{aligned} &\ln G(z+1) = \frac{z^2}{4} + z\ln\Gamma(z+1) - \left(\frac{z(z+1)}{2} + \frac{1}{12}\right)\ln z + \zeta'(-1) - \frac{1}{12} + \mathcal{O}(z^{-2}), \\ &\ln\Gamma(z+h) = (z+h-\tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln(2\pi) + \frac{1-6h+6h^2}{12z} + \mathcal{O}(z^{-2}), \end{aligned} \tag{2.16}$$

as $z\to+\infty$ for any fixed $h\in\mathbb{C}$, we then get

$$Z_n = \exp\left(-\,\alpha\, n\ln n + \big(\alpha + \ln(\pi\Gamma(\alpha))\big)n - \frac{\alpha^2}{2}\ln n + \ln G(1+\alpha) - \frac{\alpha}{2}\ln(2\pi) + \mathcal{O}(n^{-1})\right), \quad \text{as } n\to+\infty.$$

3 Proof of Theorem 1.2

As mentioned in Section 2, it is convenient to split the sum (2.6) into two parts:

$$\ln \mathcal{E}_n = S_0 + S_1, \tag{3.1}$$

where

$$S_0 = \sum_{j=1}^{\frac{n}{M}-1} \ln\left(1+\sum_{\ell=1}^{m}\omega_\ell F_{n,j,\ell}\right), \qquad S_1 = \sum_{j=\frac{n}{M}}^{n} \ln\left(1+\sum_{\ell=1}^{m}\omega_\ell F_{n,j,\ell}\right), \tag{3.2}$$

and $M := n(\lceil \frac{n}{n^{\frac{2}{2+\alpha}}}\rceil)^{-1} \asymp n^{\frac{2}{2+\alpha}}$. Define also $\Omega := e^{u_1+\cdots+u_m}$. We first obtain the large n asymptotics of S_0 using Lemma 2.1. Let $\mathsf{T}_0 := \sum_{\ell=1}^m \omega_\ell t_\ell^\alpha$.

Lemma 3.1. *Let $x_1,\ldots,x_m\in\mathbb{R}$ be fixed. There exists $\delta>0$ such that*

$$S_0 = \left(\frac{n}{M}-1\right)\ln\Omega - \frac{\mathsf{T}_0}{\Omega\,\Gamma(2+\alpha)}\frac{n}{M^{1+\alpha}} - \frac{\mathsf{T}_0}{\Omega\,\Gamma(1+\alpha)}\frac{\alpha-2}{2}\frac{1}{M^\alpha} + \mathcal{O}\left(\frac{n}{M^{2+\alpha}} + \frac{n}{M^{1+2\alpha}} + \frac{M}{n\,M^\alpha}\right),$$

as $n\to+\infty$, uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\le\delta\},\ldots,u_m\in\{z\in\mathbb{C}:|z-x_m|\le\delta\}$.

Proof. Let $K := n/M - 1$. Recalling that $F_{n,j,\ell} = I(r_\ell^2, j, \alpha)$ and $r_\ell = (1-\frac{t_\ell}{n})^{1/2}$, and using Lemma 2.1, we infer that

$$F_{n,j,\ell} = 1 - \frac{t_\ell^\alpha}{n^\alpha \mathrm{B}(j,\alpha)} \sum_{p=0}^{j-1} (-1)^p \binom{j-1}{p} \frac{t_\ell^p}{(\alpha+p)n^p} = 1 - \frac{t_\ell^\alpha}{\alpha\, n^\alpha \mathrm{B}(j,\alpha)} + \mathcal{O}\Big(\frac{n^{-\alpha-1}j}{\mathrm{B}(j,\alpha)}\Big), \tag{3.3}$$

as $n \to +\infty$ uniformly for $j \in \{1,\dots,K\}$ and $\ell \in \{1,\dots,m\}$. Using (2.16), we infer that $\mathrm{B}(j,\alpha) = \mathcal{O}(j^{-\alpha})$ as $j \to +\infty$ with α fixed, so that the error term in (3.3) can be replaced by $\mathcal{O}((j/n)^{\alpha+1})$. Hence, since $1 + \sum_{\ell=1}^m \omega_\ell = e^{u_1+\dots+u_m} = \Omega$,

$$S_0 = \sum_{j=1}^K \ln\Big(1 + \sum_{\ell=1}^m \omega_\ell \Big[1 - \frac{t_\ell^\alpha}{\alpha\, n^\alpha \mathrm{B}(j,\alpha)} + \mathcal{O}\Big(\Big(\frac{j}{n}\Big)^{1+\alpha}\Big)\Big]\Big) = \sum_{j=1}^K \ln\Big(\Omega - \frac{\mathsf{T}_0}{\alpha\, n^\alpha \mathrm{B}(j,\alpha)} + \mathcal{O}\Big(\Big(\frac{j}{n}\Big)^{1+\alpha}\Big)\Big).$$

The $\mathcal{O}$-term after the first equality is clearly independent of $u_1,\dots,u_m$, and therefore the $\mathcal{O}$-term after the second equality is uniform for $u_1 \in \{z \in \mathbb{C} : |z-x_1| \le \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z-x_m| \le \delta\}$, for any fixed $\delta > 0$. We can (and do) choose $\delta > 0$ sufficiently small such that Ω remains bounded away from $(-\infty, 0]$ for $u_1 \in \{z \in \mathbb{C} : |z-x_1| \le \delta\}, \dots, u_m \in \{z \in \mathbb{C} : |z-x_m| \le \delta\}$, so that

$$S_0 = \sum_{j=1}^K \Big(\ln\Omega - \frac{\mathsf{T}_0}{\alpha\,\Omega\, \mathrm{B}(j,\alpha)\, n^\alpha} + \mathcal{O}\Big(\Big(\frac{j}{n}\Big)^{1+\alpha} + \Big(\frac{j}{n}\Big)^{2\alpha}\Big)\Big).$$

Furthermore,

$$\sum_{j=1}^K \Big(\Big(\frac{j}{n}\Big)^{1+\alpha} + \Big(\frac{j}{n}\Big)^{2\alpha}\Big) = \mathcal{O}\Big(K\Big(\frac{K}{n}\Big)^{1+\alpha} + K\Big(\frac{K}{n}\Big)^{2\alpha}\Big), \qquad \text{as } n \to +\infty.$$

The above $\mathcal{O}$-term can also be written as $\mathcal{O}(\frac{n}{M^{2+\alpha}} + \frac{n}{M^{1+2\alpha}})$, and thus

$$S_0 = \Big(\frac{n}{M} - 1\Big)\ln\Omega - \frac{\mathsf{T}_0}{\alpha\,\Omega\, n^\alpha} \sum_{j=1}^K \frac{1}{\mathrm{B}(j,\alpha)} + \mathcal{O}\Big(\frac{n}{M^{2+\alpha}} + \frac{n}{M^{1+2\alpha}}\Big), \qquad \text{as } n \to +\infty. \tag{3.4}$$

By (2.8) and the functional relation $\Gamma(z+1) = z\Gamma(z)$,

$$\sum_{j=1}^K \frac{1}{\mathrm{B}(j,\alpha)} = \sum_{j=1}^K \frac{\Gamma(j+\alpha)}{\Gamma(j)\Gamma(\alpha)} = \alpha \sum_{j=1}^K \binom{j+\alpha-1}{j-1}. \tag{3.5}$$

We now use the so-called "parallel summation formula" $\sum_{j=0}^{K-1}\binom{j+\alpha}{j} = \binom{K+\alpha}{K-1}$ to find

$$\sum_{j=1}^K \frac{1}{\mathrm{B}(j,\alpha)} = \frac{K\,\Gamma(1+K+\alpha)}{\Gamma(1+K)\Gamma(\alpha)} \frac{1}{1+\alpha}.$$

This formula can easily be expanded as $K = \frac{n}{M} - 1 \to +\infty$ using (2.16):

$$\frac{1}{n^\alpha} \sum_{j=1}^K \frac{1}{\mathrm{B}(j,\alpha)} = \frac{1}{M^\alpha} \frac{1}{(1+\alpha)\Gamma(\alpha)} \Big\{\frac{n}{M} + \frac{(1+\alpha)(\alpha-2)}{2} + \mathcal{O}\Big(\frac{M}{n}\Big)\Big\}. \tag{3.6}$$

Substituting (3.6) in (3.4) yields the claim. $\square$

We now turn to the analysis of S_1. We will rely on Lemma 2.2, as well as on the following Riemann sum approximation lemma (whose proof is omitted).

Lemma 3.2. *Let $A = A(n)$, $B = B(n)$ be bounded functions of $n \in \{1,2,\dots\}$, such that*

$$a_n := An \qquad \text{and} \qquad b_n := Bn$$

are integers. Assume also that $B - A$ is positive and remains bounded away from 0 as $n \to +\infty$. Let f be a function independent of n, which is $C^2([A,B])$ for all $n \in \{1,2,\dots\}$. Then as $n \to +\infty$, we have

$$\sum_{j=a_n}^{b_n} f(\tfrac{j}{n}) = n\int_A^B f(x)dx + \frac{f(A)+f(B)}{2} + \mathcal{O}\Big(\frac{f'(A)+f'(B)}{n} + \sum_{j=a_n}^{b_n-1} \frac{\mathfrak{m}_{j,n}(f'')}{n^2}\Big), \tag{3.7}$$

where, for a given function g continuous on $[A,B]$ and $j \in \{a_n,\dots,b_n-1\}$, $\mathfrak{m}_{j,n}(g) := \max_{x\in[\frac{j}{n},\frac{j+1}{n}]}|g(x)|$.

Lemma 3.3. *For any fixed $x_1,\ldots,x_m\in\mathbb{R}$, there exists $\delta>0$ such that*

$$S_1 = n\bigg\{\int_0^1 f_0(x)dx - \frac{1}{M}\ln\Omega + \frac{\mathsf{T}_0}{\Omega\,\Gamma(\alpha+2)}\frac{1}{M^{\alpha+1}}\bigg\} + \int_0^1 f_1(x)dx + \frac{f_0(1)+\ln\Omega}{2} - \frac{(2-\alpha)\mathsf{T}_0}{2\Omega\,\Gamma(\alpha+1)}\frac{1}{M^\alpha} + \mathcal{O}\Big(\frac{n}{M^{1+2\alpha}}+\frac{n}{M^{2+\alpha}}+\frac{1+M^{1-\alpha}+\mathbf{1}_{\alpha=1}\ln M}{n}\Big),$$

as $n\to+\infty$ uniformly for $u_1\in\{z\in\mathbb{C}:|z-x_1|\le\delta\},\ldots,u_m\in\{z\in\mathbb{C}:|z-x_m|\le\delta\}$, where $\mathbf{1}_{\alpha=1}=1$ if $\alpha=1$ and $\mathbf{1}_{\alpha=1}=0$ otherwise, and

$$f_0(x) := \ln(1+\Sigma_0(x)), \qquad f_1(x) := \frac{\Sigma_1(x)}{1+\Sigma_0(x)}, \tag{3.8}$$

$$\Sigma_0(x) := \sum_{\ell=1}^m \omega_\ell \mathrm{Q}(\alpha,t_\ell x), \qquad \Sigma_1(x) := \frac{x^\alpha}{x}\sum_{\ell=1}^m \omega_\ell \frac{e^{-t_\ell x}t_\ell^\alpha}{\Gamma(\alpha)}\frac{1-\alpha-t_\ell x}{2}. \tag{3.9}$$

Proof. By Lemma 2.2, since $n/M\to+\infty$, we have

$$F_{n,j,\ell} = \frac{\Gamma(j+\alpha)}{\Gamma(j)}d_0(r_\ell^2,\alpha)F_0(r_\ell^2,j,\alpha)\bigg(1+\sum_{k=1}^3\frac{d_k(r_\ell^2,\alpha)F_k(r_\ell^2,j,\alpha)}{d_0(r_\ell^2,\alpha)F_0(r_\ell^2,j,\alpha)}+\mathcal{O}(j^{-4})\bigg)$$

as $n\to+\infty$ uniformly for $\ell\in\{1,\ldots,m\}$ and $j\in\{\frac{n}{M},\ldots,n\}$. Moreover, using (2.11), (2.12), (2.13) and (2.16), we get

$$F_{n,j,\ell} = \mathrm{Q}(\alpha,t_\ell j/n) + \frac{(j/n)^\alpha}{j}\frac{e^{-t_\ell j/n}t_\ell^\alpha}{\Gamma(\alpha)}\frac{1-\alpha-t_\ell j/n}{2} + \mathcal{O}\Big(\frac{1}{n^2}\frac{(j/n)^\alpha}{(j/n)^2}\Big), \qquad \text{as } n\to+\infty,$$

uniformly for $\ell\in\{1,\ldots,m\}$ and $j\in\{\frac{n}{M},\ldots,n\}$. Hence, for small enough $\delta>0$,

$$\ln\bigg(1+\sum_{\ell=1}^m\omega_\ell F_{n,j,\ell}\bigg) = f_0(j/n) + \frac{f_1(j/n)}{n} + \mathcal{O}\Big(\frac{1}{n^2}\frac{(j/n)^\alpha}{(j/n)^2}\Big), \qquad \text{as } n\to+\infty, \tag{3.10}$$

uniformly for $j\in\{\frac{n}{M},\ldots,n\}$ and $u_1\in\{z\in\mathbb{C}:|z-x_1|\le\delta\},\ldots,u_m\in\{z\in\mathbb{C}:|z-x_m|\le\delta\}$. Note that

$$\frac{1}{n^2}\sum_{j=\frac{n}{M}}^n\frac{(j/n)^\alpha}{(j/n)^2} = \mathcal{O}\Big(\frac{1+M^{1-\alpha}+\mathbf{1}_{\alpha=1}\ln M}{n}\Big).$$

Furthermore, by Lemma 3.2 with $A=\frac{1}{M}$, $B=1$,

$$\sum_{j=\frac{n}{M}}^n f_0(j/n) = n\int_{M^{-1}}^1 f_0(x)dx + \frac{f_0(M^{-1})+f_0(1)}{2} + \mathcal{O}\Big(\frac{M^{1-\alpha}+1}{n}\Big), \tag{3.11}$$

$$\frac{1}{n}\sum_{j=\frac{n}{M}}^n f_1(j/n) = \int_{M^{-1}}^1 f_1(x)dx + \mathcal{O}\Big(\frac{M^{1-\alpha}+1}{n}\Big), \tag{3.12}$$

where, to estimate the $\mathcal{O}$-terms, we have used $n^{-1}f_0'(A)\lesssim n^{-1}A^{\alpha-1}=\mathcal{O}(\frac{M^{1-\alpha}}{n})$, $n^{-1}f_0'(B)=\mathcal{O}(n^{-1})$, $n^{-1}f_1(A)\lesssim n^{-1}A^{\alpha-1}=\mathcal{O}(\frac{M^{1-\alpha}}{n})$, and $n^{-1}f_1(B)=\mathcal{O}(n^{-1})$. Also, using (3.8)–(3.9), as $n\to\infty$ we get

$$n\int_{M^{-1}}^1 f_0(x)dx = n\int_0^1 f_0(x)dx - \frac{n}{M}\ln\Omega + \frac{\mathsf{T}_0}{\Omega\,\Gamma(\alpha+2)}\frac{n}{M^{1+\alpha}} + \mathcal{O}\Big(\frac{n}{M^{1+2\alpha}}+\frac{n}{M^{2+\alpha}}\Big),$$

$$f_0(M^{-1}) = \ln\Omega - \frac{\mathsf{T}_0}{\Omega\,\Gamma(\alpha+1)}\frac{1}{M^\alpha} + \mathcal{O}\Big(\frac{1}{M^{1+\alpha}}+\frac{1}{M^{2\alpha}}\Big),$$

$$\int_{M^{-1}}^1 f_1(x)dx = \int_0^1 f_1(x)dx - \frac{\mathsf{T}_0(1-\alpha)}{2\Omega\,\Gamma(\alpha+1)}\frac{1}{M^\alpha} + \mathcal{O}\Big(\frac{1}{M^{1+\alpha}}+\frac{1}{M^{2\alpha}}\Big).$$

Substituting the above in (3.11)–(3.12) and then in (3.10) yields

$$S_1 = \sum_{j=\frac{n}{M}}^n f_0(j/n) + \sum_{j=\frac{n}{M}}^n\frac{f_1(j/n)}{n} + \mathcal{O}\Big(\frac{1+M^{1-\alpha}+\mathbf{1}_{\alpha=1}\ln M}{n}\Big)$$

$$= n\left\{ \int_0^1 f_0(x)dx - \frac{1}{M}\ln\Omega + \frac{\mathsf{T}_0}{\Omega\,\Gamma(\alpha+2)}\frac{1}{M^{\alpha+1}} \right\} + \int_0^1 f_1(x)dx + \frac{f_0(1)+\ln\Omega}{2}$$
$$- \frac{(2-\alpha)\mathsf{T}_0}{2\Omega\,\Gamma(\alpha+1)}\frac{1}{M^\alpha} + \mathcal{O}\left(\frac{n}{M^{1+2\alpha}} + \frac{n}{M^{2+\alpha}} + \frac{1+M^{1-\alpha}+\mathbf{1}_{\alpha=1}\ln M}{n} \right)$$

□

Proof of Theorem 1.2. Combining Lemmas 3.1 and 3.3 yields

$$\ln\mathcal{E}_n = n\int_0^1 f_0(x)dx + \int_0^1 f_1(x)dx + \frac{f_0(1)-\ln\Omega}{2} + \mathcal{O}\left(\frac{n}{M^{1+2\alpha}} + \frac{n}{M^{2+\alpha}} + \frac{1+M^{1-\alpha}+\mathbf{1}_{\alpha=1}\ln M}{n} \right).$$

The above $\mathcal{O}$-term can be rewritten as

$$\mathcal{O}\left(\frac{n}{M^{1+\alpha+\hat{\alpha}}} + \frac{M^{1-\hat{\alpha}}+\mathbf{1}_{\alpha=1}\ln M}{n} \right),$$

where $\hat{\alpha} = \min\{\alpha, 1\}$. Since $M \asymp n^{\frac{2}{2+\alpha}}$, we have

$$\frac{n}{M^{1+\alpha+\hat{\alpha}}} \asymp \frac{M^{1-\hat{\alpha}}}{n} \asymp n^{-\frac{2\hat{\alpha}+\alpha}{2+\alpha}},$$

which finishes the proof of Theorem 1.2. □

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