On the pressure gradient error in the σ-coordinate system

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Introduction

Geophysical fluid flow models usually rely on the hydrostatic assumption. Therefore, the pressure is computed from the hydrostatic equilibrium equation, i.e. the formula to which the vertical momentum equation is simplified. Then, the horizontal pressure gradient has to be calculated, for this term is involved in a driving force of the horizontal velocity. Over the last 40 years, the discretisation of this pressure force has been considered a major source of error for reasons that are outlined hereinafter.

If there is not external forcing, if the sea is initially at rest and if the density is a function of the vertical coordinate only, then the horizontal pressure force is zero and the velocity will remain zero. When the z-coordinate system is used, this situation is easily dealt with in a numerical model: the discretised pressure force is zero and the modelled sea is not set in motion. Unfortunately, this is not so when the σ-coordinate system is relied upon, or any other coordinate transformation such that no coordinate surface is horizontal. In other words, the velocity does not remain zero, though there is no doubt that the sea should remain motionless.

The atmospheric and marine science literature is replete with articles suggesting methods to reduce the magnitude of the aforementioned numerical error. The objective of the present working note is not to propose another such method. Instead, one will calculate the truncation error of what is presumably the simplest scheme for estimating the horizontal pressure force in the σ-coordinate system, showing that the so-called *hydrostatic consistency criterion* introduced by Haney (1991) is unrelated to the consistency of the scheme, but impacts, in a somewhat surprising manner, its accuracy.

Discretised horizontal pressure gradient in the σ-coordinate system

Consider the dynamic variables (velocity components, pressure, density) in a vertical x-z plane of a finite difference model based on a staggered C-grid generated in the framework of the σ-coordinate system. For the present study, the sea surface elevation may be neglected. If the seabed is located at the altitude $z = -h$, the transformed coordinates are

$$(\tilde{x},\sigma) = \left(x, \frac{z+h}{h}\right) . \tag{1}$$

Thus, in the transformed space the lower and upper boundaries of the domain correspond to $\sigma = 0$ and $\sigma = 1$, respectively.

Now, consider two water columns where the density and the pressure are discretised. They are located at $x = x_-$ and $x = x_+$, with

$$x_+ - x_- = \Delta x \ , \tag{2}$$

where Δx is the horizontal space increment. The corresponding water depths are denoted

$$h_- = h(x_-) \ , \quad h_+ = h(x_+) \ . \tag{3}$$

The horizontal velocity components are to be calculated on the vertical whose horizontal coordinate is

$$x_0 = \frac{x_+ + x_-}{2} \ . \tag{4}$$

It is generally assumed that the depth varies linearly between the "density water columns". Then, the corresponding space derivative of the depth is

$$h' = \frac{h_+ - h_-}{\Delta x} \tag{5}$$

and the depth of the "velocity water column" is

$$h_0 = \frac{h_- + h_+}{2} \ . \tag{6}$$

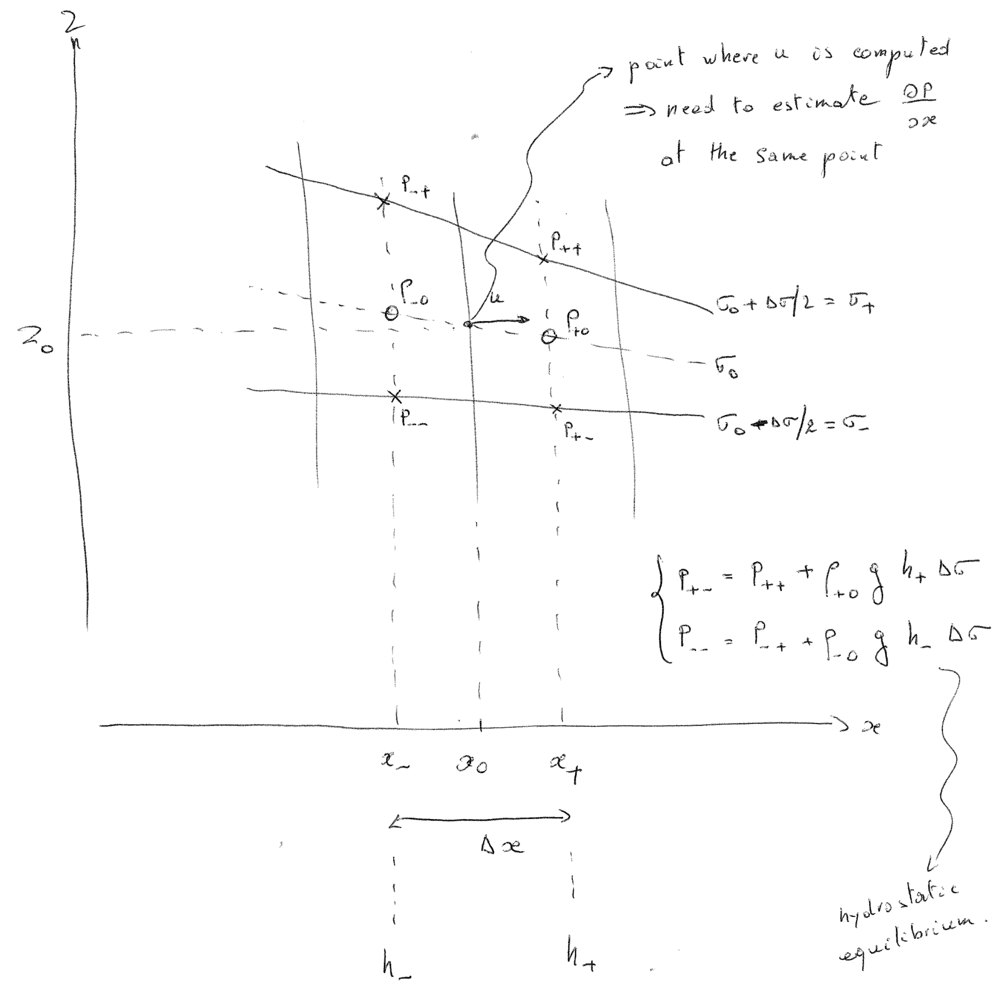

Figure 1. Schematic representation of the location of the discrete variables in the vertical *x-z* plane under consideration.

Now focus on the horizontal velocity point whose altitude is z_0. By virtue of (1), its altitude in the transformed space is

$$\sigma_0 = \frac{z_0 + h_0}{h_0} \ . \tag{7}$$

Let $\Delta\sigma$ represent the vertical grid spacing the transformed space and introduce the following notations

$$\sigma_\pm = \sigma_0 \pm \frac{\Delta\sigma}{2} \tag{8}$$

and

$$z_\pm = -(1-\sigma_\pm)h_\pm \ . \tag{9}$$

Then, at the velocity point located at coordinates (x_0, z_0), the horizontal pressure force may be evaluated by means of the neighbouring discrete pressure values, which are (Figure 1)

$$p_{++} = p(x_+, z_+) \ , \quad p_{+-} = p(x_+, z_-) \ , \quad p_{-+} = p(x_-, z_+) \ , \quad p_{--} = p(x_-, z_-) \ . \tag{10}$$

In the transformed coordinate system (1), the horizontal pressure gradient reads

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \tilde{x}} + \frac{\partial \sigma}{\partial x}\frac{\partial p}{\partial \sigma} \ . \tag{11}$$

It is readily seen that

$$\frac{\partial \sigma}{\partial x} = \frac{1-\sigma}{h}\frac{\partial h}{\partial x} \ . \tag{12}$$

Next, combining (11) and (12), one obtains

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \tilde{x}} + \frac{1-\sigma}{h}\frac{\partial h}{\partial x}\frac{\partial p}{\partial \sigma} \ . \tag{13}$$

This suggests that x-derivative of the pressure at (x_0, z_0) be approximated as follows

$$\frac{\delta_x p_0}{\Delta x} = \frac{(p_{++} + p_{+-}) - (p_{-+} + p_{--})}{2\Delta x} - \frac{(1-\sigma_0)h'}{h_0}\frac{(p_{++} + p_{-+}) - (p_{+-} + p_{--})}{2\Delta\sigma} \ . \tag{14}$$

The horizontal pressure gradient error

If the pressure is a function of the vertical coordinate z only (i.e. if the pressure is independent of x), then expression (14) should be zero. Unfortunately, this is not so. The associated truncation error, which can be obtained easily by resorting to symbolic calculation software, is

$$\frac{\delta_x p_0}{\Delta x} = \underbrace{-\frac{h_0 h'}{12}\left[3\left(\frac{d^2 P}{dz^2}\right)_{x_0, z_0} - h_0(1-\sigma_0)\left(\frac{d^3 P}{dz^3}\right)_{x_0, z_0}\right]}_{\text{independent of } \Delta x \text{ and } \Delta\sigma} \underbrace{\overbrace{(\mu^2 - 1)\,\Delta\sigma^2}^{=\varepsilon}}_{\substack{\text{depends on} \\ \Delta x \text{ and } \Delta\sigma}} + O(\Delta x^4, \Delta\sigma^4) \tag{15}$$

where the dimensionless parameter

$$\mu = \frac{h'\Delta x}{h_0}\frac{1-\sigma_0}{\Delta\sigma} \tag{16}$$

has been introduced by Haney[1] (1991). For his hydrostatic consistency criterion to be satisfied, this dimensionless parameter was to be smaller than unity ($\mu < 1$).

The leading order term of the truncation error (15) contains two contributions. There is one that directly depends on the pressure field and the bathymetry, while the second one, ε, depends on the discretisation parameters. The latter may be rewritten as follows:

$$\varepsilon = (\mu^2 - 1)\Delta\sigma^2 = \left[\frac{h'(1-\sigma_0)}{h_0}\right]^2 \Delta x^2 - \Delta\sigma^2 \; . \tag{17}$$

Clearly, the numerical scheme considered herein is second-order accurate in both space directions and the value of the dimensionless parameter μ has not impact on the order of accuracy. Therefore, the so-called hydrostatic consistency criterion is not related to the consistency of the scheme.

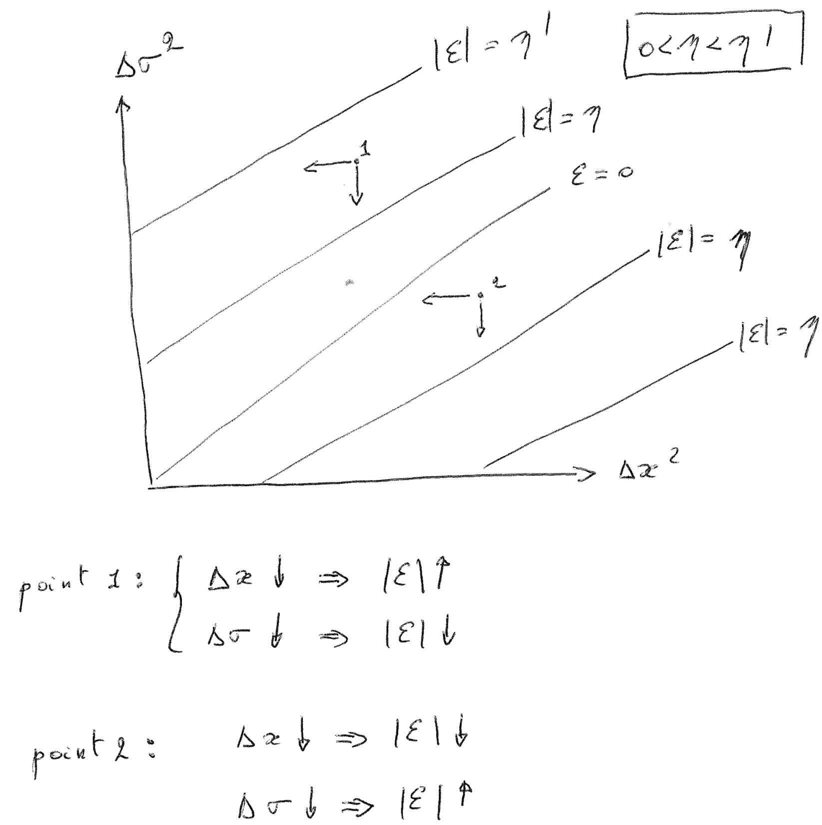

Figure 2. Isolines of the dimensionless parameter $\varepsilon = (\mu^2 - 1)\Delta\sigma^2$ as a function of the square of the space increments Δx^2 and $\Delta\sigma^2$. As is exemplified at points 1 and 2, increasing the resolution in one space direction does not necessarily improve the discrete estimate of the horizontal pressure gradient.

As is illustrated in Figure 2, increasing the vertical resolution (i.e. reducing the value of $\Delta\sigma$) does not necessarily cause the absolute value of ε to decrease. Similar considerations

[1] It must be realised that Haney's notations where different from those employed herein: he set σ=0 and σ=1 at the sea surface and the seabed, respectively.

hold valid for the horizontal resolution. In other words, the absolute value of the dominant term of the truncation depends in a complex manner on the horizontal and vertical resolution (Table 1), implying that enhancing the resolution doe not always increase the accuracy of the discrete approximation of the horizontal pressure force.

Table 1. Sign of the derivative of $|\varepsilon|$ with respect to either of the space increment for various value of the dimensionless parameter μ, which is defined by relation (16). As $|\varepsilon|$ is the factor depending on the space increment in the leading term of the truncation error (15), the results below indicate that enhancing the resolution in one space direction does not necessarily improve the accuracy of the discrete estimate of the horizontal pressure force. For instance, if $\mu < 1$ (i.e. if Haney's hydrostatic consistency criterion is satisfied), increasing the horizontal resolution is unlikely to reduce the pressure gradient error, whereas increasing the vertical resolution might impact favourably the discrete horizontal pressure gradient estimate.

$\mu < 1$	$\mu = 1$	$\mu > 1$
$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta x} < 0$	$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta x} > 0$	$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta x} > 0$
$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta\sigma} > 0$	$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta\sigma} > 0$	$\dfrac{\partial\lvert\varepsilon\rvert}{\partial\Delta\sigma} < 0$

This working note focuses on the dominant term of the truncation error of the horizontal pressure gradient in the case where the pressure is horizontally homogeneous. Therefore, the conclusions arrived at above may not remain valid if the horizontal and vertical grid spacings are no longer sufficiently small for this approach to be relevant.

Reference

Haney R.L., 1991, On the pressure gradient force over steep topography in sigma coordinate ocean models, *Journal of Physical Oceanography*, 21, 610-619
