THE SOLITON RESOLUTION CONJECTURE FOR THE BOUSSINESQ EQUATION

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Abstract. We analyze the Boussinesq equation on the line with Schwartz initial data belonging to the physically relevant class of global solutions. In a recent paper, we determined ten main asymptotic sectors describing the large (x,t)-behavior of the solution, and for each of these sectors we provided the leading order asymptotics in the case when no solitons are present. In this paper, we give a formula valid in the asymptotic sector $x/t \in (1,M]$, where M is a large positive constant, in the case when solitons are present. Combined with earlier results, this validates the soliton resolution conjecture for the Boussinesq equation everywhere in the (x,t)-plane except in a number of small transition zones.

Résumé. Nous analysons l'équation de Boussinesq sur la ligne avec des données initiales de Schwartz appartenant à la classe physiquement pertinente de solutions globales. Dans un article récent, nous avons déterminé dix principaux secteurs asymptotiques décrivant le comportement de la solution pour (x,t) grand, et pour chacun de ces secteurs, nous avons fourni les termes asymptotiques dominants dans le cas où aucun soliton n'est présent. Dans cet article, nous donnons une formule valable dans le secteur asymptotique $x/t \in (1,M]$, où M est une grande constante positive, dans le cas où des solitons sont présents. Combiné avec des résultats antérieurs, cela valide la conjecture de résolution des solitons pour l'équation de Boussinesq partout dans le plan (x,t), à l'exception d'un certain nombre de petites zones de transition.

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1. Introduction

This paper is concerned with the asymptotic behavior as $t \to \infty$ of solutions of the Boussinesq [6] equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx}, \tag{1.1}$$

which models two-way propagation of dispersive waves of small amplitude in shallow water [28]. A Lax pair for (1.1) was constructed in [44], soliton solutions were studied in [2, 3, 23], and the existence of global solutions of certain initial-boundary value problems was investigated in [29, 31, 43]. Determining the long-time asymptotics for the solution of (1.1) was listed as an important open problem by Deift in 2006 [18]; we recently addressed this problem in [8].

Equation (1.1) is linearly unstable due to exponentially growing Fourier components of large frequency. As a consequence, it is ill-posed and is also referred to as the "bad" Boussinesq equation. The "good" Boussinesq equation is the equation obtained by changing

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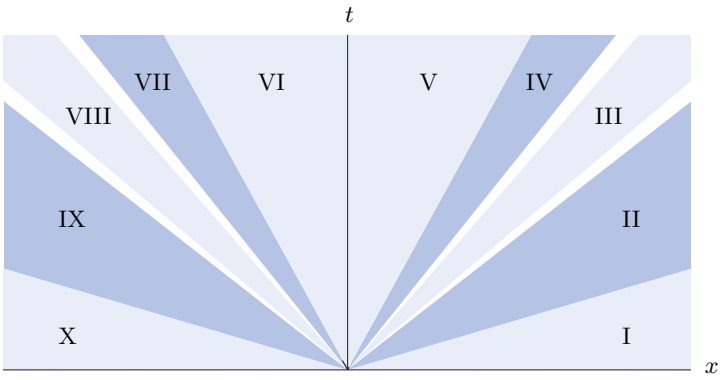

Figure 1. *The ten asymptotic sectors for (1.1) in the xt-plane.*

the sign of the u_{xxxx}-term in (1.1); this change of signs makes the equation locally well-posed and papers dedicated to the study of the "good" Boussinesq equation include [1, 4, 7, 13, 20–22, 30, 34–36, 39–41].

In [8], we developed an inverse scattering approach to the initial value problem for (1.1). Along with other results, we established in [8] uniqueness and existence of global solutions to (1.1) for a broad class of initial data relevant for water waves. For these global solutions, we identified ten main asymptotic sectors describing the large (x,t)-behavior of the solution (see Figure 1), and for each of these sectors we gave the leading order asymptotics in [8, Theorem 2.14] in the case when no solitons are present. For conciseness, we omitted the proofs of some of these asymptotic formulas in [8].

In this paper, we establish an asymptotic formula for $u(x,t)$ valid in the sector $x/t \in (1, M]$, where M is a large positive constant, in the case when solitons are present. Following [8], we refer to this sector as Sector II, see Figure 1. If no solitons are present, our formula reduces to the formula announced in [8], thereby also providing a proof of this formula.

1.1. **The soliton resolution conjecture.** The soliton resolution conjecture for an integrable equation asserts, roughly speaking, that a solution of the equation with generic initial data eventually behaves as a finite number of solitons of order 1 superimposed on a decaying radiative background. Earlier works on soliton resolution for equations with 2×2 Lax pairs include [5, 14–17, 24, 26, 27, 32, 33, 38, 42]. As mentioned above, the Boussinesq equation (1.1) has the peculiar feature of being linearly unstable because large frequency Fourier modes grow (or decay) exponentially in time. The inverse scattering transform can be viewed as a nonlinear Fourier transform, and as such it allows us to identify the unstable high-frequency modes in a nonlinearly precise way [8]. The upshot is that if the initial data are such that unstable high-frequency modes are present, the solution will contain components that grow exponentially in time and the soliton resolution conjecture will automatically fail. This observation is consistent with the interpretation of the Boussinesq equation as a model for water waves, which relies on the absence of high-frequency modes. Consequently, the natural formulation of the soliton resolution conjecture for (1.1) restricts attention to initial data which contain no unstable high-frequency modes (in a nonlinear sense made precise in Assumption (*iii*) below). Combined with the results of [9, 11, 12], our results here validate this soliton resolution conjecture for the Boussinesq equation everywhere in the (x,t)-plane except in a number of transition zones. In what follows, we explain this claim in further detail.

The one-soliton solutions of (1.1) are given by

$$u(x,t) = A \operatorname{sech}^2(\sqrt{A/6}(x - x_0 - ct)), \tag{1.2}$$

where $A \geq 0$ is the amplitude, $c = \pm\sqrt{1+2A/3}$ is the velocity, and $x_0 \in \mathbb{R}$ is a free parameter related to translation invariance [23]. We observe that c can be both positive and negative, corresponding to the fact that (1.1) describes waves traveling in both directions. We also observe that $|c| > 1$, meaning that all one-solitons travel with speeds greater than 1. One can show that all breather solitons of (1.1) also propagate with speeds greater than 1. Thus, by choosing M large enough, we may assume that any finite number of right-moving solitons are contained in the sector $x/t \in (1, M]$, i.e., in Sector II. Similarly, we may assume that any finite number of left-moving solitons are contained in Sector IX, which is characterized by $x/t \in [-M, -1)$, see Figure 1. The soliton resolution conjecture for the Boussinesq equation then asserts that for generic initial data with no unstable high-frequency modes, the corresponding solution $u(x,t)$ of (1.1) behaves as follows: In Sectors II and IX, the large t behavior of u is described by a soliton component of order 1 plus a decaying radiation component, whereas everywhere else, $u(x,t)$ asymptotes to a decaying radiative wave. Since (1.1) is invariant under the reflection symmetry $x \mapsto -x$, it is enough to consider positive values of x when verifying this conjecture.

In [11], we showed that the long-time behavior of $u(x,t)$ in Sector IV, which is characterized by $x/t \in (1/\sqrt{3}, 1)$, is a radiative decaying wave described by a sum of two modulated sine-waves of order $O(t^{-1/2})$; we also observed that the only effect of solitons on the asymptotics in Sector IV is to introduce phase shifts in these modulated sine-waves. In [9], we obtained a similar asymptotic formula in Sector V. For brevity, we did not include solitons in [9], but the same arguments as in [11] show that the only effect of the solitons in Sector V is again to introduce phase shifts in the radiative terms. In [12], we obtained the asymptotics of $u(x,t)$ in Sector I. The formula in [12] was stated under the assumption of no solitons, but the arguments in the present paper show that they remain valid unchanged, provided that M is large enough (so that all right-moving solitons are contained in Sector II). In the present paper, we obtain explicitly the leading order soliton component of order 1 in Sector II, and show that the subleading term consists of a radiation component that decays like $t^{-1/2}$. Taken together, these results validate the soliton resolution conjecture for (1.1) everywhere in the (x,t)-plane with the exception of a few narrow transition zones; more precisely, for any $\epsilon > 0$, they validate it for all $\zeta := x/t \in \mathbb{R}$ such that ζ stays a distance ϵ away from the points $0, \pm 1/\sqrt{3}$, and ± 1. We strongly expect that the soliton resolution conjecture holds also for ζ in the transition zones around $0, \pm 1/\sqrt{3}$, ± 1, and that this can be proved using the framework of [8]; however, a treatment of these transition zones is beyond the scope of the present paper.

1.2. **Organization of the paper.** The main results are stated in Section 2. The row-vector Riemann–Hilbert (RH) problem that underlies our inverse scattering approach to (1.1) is recalled in Section 3. Sections 4–7 implement four transformations of this RH problem which bring it to a form suitable for the evaluation of its large t behavior. In Section 8, we construct a local parametrix and in Section 9 we transform the RH problem one last time to arrive at a small-norm problem. The asymptotic behavior of $u(x,t)$ in Sector II is obtained in Section 10. In Section 11, we solve the RH problem corresponding to pure multi-soliton solutions of (1.1). Another RH problem relevant for the asymptotics is analyzed in Section 12. Appendix A considers a model RH problem that is used in the construction of the local parametrix.

2. Main results

In this paper, the initial data

$$u_0(x) := u(x,0) \quad \text{and} \quad u_1(x) := u_t(x,0) \tag{2.1}$$

are real-valued functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} u_1(x)dx = 0$. The last assumption guarantees that the total mass $\int_{\mathbb{R}} u(x,t)dx$ is independent of t. It is shown in [8] that the solution to the initial value problem for (1.1) can be expressed in terms of the solution n of a row-vector RH problem whose definition involves two spectral functions $r_1(k)$ and $r_2(k)$. The functions $r_1(k)$ and $r_2(k)$ are defined via a nonlinear Fourier transform of the initial data, which we briefly recall here for convenience.

2.1. **Definition of r_1 and r_2.** Let $\{l_j(k), z_j(k)\}_{j=1}^3$ be given by

$$l_j(k) = i\frac{\omega^j k + (\omega^j k)^{-1}}{2\sqrt{3}}, \qquad z_j(k) = i\frac{(\omega^j k)^2 + (\omega^j k)^{-2}}{4\sqrt{3}}, \qquad k \in \mathbb{C}\setminus\{0\}, \tag{2.2}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Let $v_0(x) := \int_{-\infty}^x u_1(x')dx'$, and define $P(k)$ and $\mathsf{U}(x,k)$ by

$$P(k) = \begin{pmatrix} 1 & 1 & 1 \\ l_1(k) & l_2(k) & l_3(k) \\ l_1(k)^2 & l_2(k)^2 & l_3(k)^2 \end{pmatrix}, \quad \mathsf{U}(x,k) = P(k)^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{u_{0x}}{4} - \frac{iv_0}{4\sqrt{3}} & -\frac{u_0}{2} & 0 \end{pmatrix} P(k).$$

Let $\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3)$, and let $X(x,k), X^A(x,k), Y(x,k), Y^A(x,k)$ be the unique solutions to the Volterra integral equations

$$\begin{aligned} X(x,k) &= I - \int_x^\infty e^{(x-x')\mathcal{L}(k)}(\mathsf{U}X)(x',k)e^{-(x-x')\mathcal{L}(k)}dx', \\ X^A(x,k) &= I + \int_x^\infty e^{-(x-x')\mathcal{L}(k)}(\mathsf{U}^T X^A)(x',k)e^{(x-x')\mathcal{L}(k)}dx', \\ Y(x,k) &= I + \int_{-\infty}^x e^{(x-x')\mathcal{L}(k)}(\mathsf{U}Y)(x',k)e^{-(x-x')\mathcal{L}(k)}dx', \\ Y^A(x,k) &= I - \int_{-\infty}^x e^{-(x-x')\mathcal{L}(k)}(\mathsf{U}^T Y^A)(x',k)e^{(x-x')\mathcal{L}(k)}dx', \end{aligned}$$

where U^T denotes the transpose of U. Let $s(k)$ and $s^A(k)$ be given by

$$s(k) = I - \int_{\mathbb{R}} e^{-x\mathcal{L}(k)}(\mathsf{U}X)(x,k)e^{x\mathcal{L}(k)}dx, \quad s^A(k) = I + \int_{\mathbb{R}} e^{x\mathcal{L}(k)}(\mathsf{U}^T X^A)(x,k)e^{-x\mathcal{L}(k)}dx.$$

Let $\mathbb{D} = \{k \in \mathbb{C} \,|\, |k| < 1\}$ be the open unit disk and let $\Gamma = \cup_{j=1}^9 \Gamma_j$ be the contour displayed in Figure 2. Let $\hat{\Gamma}_j = \Gamma_j \cup \partial\mathbb{D}$ denote the union of Γ_j and the unit circle $\partial\mathbb{D}$. The functions $\{r_j(k)\}_1^2$ are defined by

$$\begin{cases} r_1(k) = \frac{(s(k))_{12}}{(s(k))_{11}}, & k \in \hat{\Gamma}_1 \setminus (\mathcal{Q}\cup\{0\}), \\ r_2(k) = \frac{(s^A(k))_{12}}{(s^A(k))_{11}}, & k \in \hat{\Gamma}_4 \setminus (\mathcal{Q}\cup\{0\}), \end{cases} \tag{2.3}$$

where the set $\mathcal{Q} := \{\kappa_j\}_{j=1}^6$ consists of the six points $\kappa_j := e^{\frac{\pi i(j-1)}{3}}$, $j = 1, \dots, 6$.

2.2. **Solitons.** In [10], we generalized the approach of [8] to allow for solutions with solitons. The solitons manifest themselves in the formulation of the RH problem as a set of poles Z and a set of corresponding residue constants $\{c_{k_0}\}_{k_0\in\mathsf{Z}} \subset \mathbb{C}$ which are defined as follows.

The functions s_{11} and s_{11}^A appearing in (2.3) are analytic in the interiors of the sets $\bar{D}_1 \cup \bar{D}_2$ and $\bar{D}_4 \cup \bar{D}_5$, respectively, where the open sets $\{D_j\}_{j=1}^6$ are as indicated in Figure 2, see [8]. The solitons are generated by the possible zeros of $s_{11}(k)$ and $s_{11}^A(k)$. We will restrict ourselves to the generic case when there are no zeros on the contour Γ. In view of the symmetries $s_{11}(k) = s_{11}(\omega/k)$ and $s_{11}^A(k) = \overline{s_{11}(\bar{k}^{-1})}$ which were established in [8], it is then sufficient to consider the zeros of s_{11} in D_2. The set Z is defined as the zero-set

of $s_{11}(k)$. Since $s_{11}(k) \to 1$ as $k \to \infty$, Z is necessarily a finite set. We will consider the generic case when the zeros of s_{11} are simple and $s_{22}^A \neq 0$ at every point of $\mathsf{Z} \setminus \mathbb{R}$.

We divide the open set D_2 into three parts: $D_2 = D_{\text{reg}} \sqcup D_{\text{sing}} \sqcup (D_2 \cap \mathbb{R})$, where $D_2 \cap \mathbb{R} = (-1, 0) \cup (1, \infty)$ and

$$D_{\text{reg}} := D_2 \cap \big(\{k \,|\, |k| > 1, \operatorname{Im} k > 0\} \cup \{k \,|\, |k| < 1, \operatorname{Im} k < 0\}\big),$$
$$D_{\text{sing}} := D_2 \cap \big(\{k \,|\, |k| > 1, \operatorname{Im} k < 0\} \cup \{k \,|\, |k| < 1, \operatorname{Im} k > 0\}\big).$$

If u_0 and u_1 are compactly supported, the residue constants $\{c_{k_0}\}_{k_0 \in \mathsf{Z}}$ are defined by

$$c_{k_0} := \begin{cases} -\frac{s_{13}(k_0)}{\dot{s}_{11}(k_0)}, & k_0 \in \mathsf{Z} \setminus \mathbb{R}, \\ -\frac{s_{12}(k_0)}{\dot{s}_{11}(k_0)}, & k_0 \in \mathsf{Z} \cap \mathbb{R}. \end{cases} \tag{2.4}$$

If u_0 and v_0 are not compactly supported, then the expressions in (2.4) are in general not well-defined and we instead use the following more complicated definition of the residue constants: Define the vector-valued function $w(x, k)$ by

$$w = \begin{pmatrix} Y_{21}^A X_{32}^A - Y_{31}^A X_{22}^A \\ Y_{31}^A X_{12}^A - Y_{11}^A X_{32}^A \\ Y_{11}^A X_{22}^A - Y_{21}^A X_{12}^A \end{pmatrix}. \tag{2.5}$$

If $k_0 \in \mathsf{Z} \setminus \mathbb{R}$, then $c_{k_0} \in \mathbb{C}$ is defined as the unique constant such that

$$\frac{w(x, k_0)}{\dot{s}_{11}(k_0)} = c_{k_0} e^{(l_1(k_0) - l_3(k_0))x} [X(x, k_0)]_1 \qquad \text{for all } x \in \mathbb{R}, \tag{2.6a}$$

where we write $[A]_j$ for the jth column of an $n \times m$ matrix A. If $k_0 \in \mathsf{Z} \cap \mathbb{R}$, then $c_{k_0} \in \mathbb{C}$ is defined as the unique constant such that

$$\frac{[Y(x, k_0)]_2}{\dot{s}_{22}^A(k_0)} = c_{k_0} e^{(l_1(k_0) - l_2(k_0))x} [X(x, k_0)]_1 \qquad \text{for all } x \in \mathbb{R}. \tag{2.6b}$$

It is proved in [10] that the c_{k_0} are well-defined by the relations in (2.6).

Simple zeros of $s_{11}(k)$ in $\mathsf{Z} \setminus \mathbb{R}$ with positive and negative real parts give rise to right- and left-moving breather solitons, respectively [10]. Similarly, positive and negative simple zeros of $s_{11}(k)$ in $\mathsf{Z} \cap \mathbb{R}$ give rise to right- and left-moving bell-shaped solitons, respectively. It is shown in [10] that zeros in D_{sing} generate solitons with singularities. It is also shown in [10] that a soliton corresponding to $k_0 \in \mathsf{Z} \cap \mathbb{R}$ is non-singular and real-valued if and only if the associated residue constant c_{k_0} satisfies $i(\omega^2 k_0^2 - \omega) c_{k_0} \geq 0$. Since we are only interested in singularity-free real-valued solutions, we will assume that $\mathsf{Z} \subset D_{\text{reg}} \cup (-1, 0) \cup (1, \infty)$ and that $i(\omega^2 k_0^2 - \omega) c_{k_0} \geq 0$ for each $k_0 \in \mathsf{Z} \cap \mathbb{R}$.

2.3. **Assumptions.** Our results will hold under the following assumptions:

(i) Finite number of non-singular solitons: assume that $s_{11}(k)$ has a simple zero at each point in Z, where $\mathsf{Z} \subset D_{\text{reg}} \cup (-1, 0) \cup (1, \infty)$ is a finite set, and that $s_{11}(k)$ has no other zeros in $\bar{D}_2 \cup \partial\mathbb{D} \setminus (\mathcal{Q} \cup \{0\})$. If $k_0 \in \mathsf{Z} \setminus \mathbb{R}$, then suppose that $s_{22}^A(k_0) \neq 0$. If $k_0 \in \mathsf{Z} \cap \mathbb{R}$, then suppose that $i(\omega^2 k_0^2 - \omega) c_{k_0} \geq 0$.

(ii) Generic behavior of s and s^A near $k = \pm 1$: we assume for $k_\star = 1$ and $k_\star = -1$ that

$$\lim_{k \to k_\star} (k - k_\star) s(k)_{11} \neq 0, \quad \lim_{k \to k_\star} (k - k_\star) s(k)_{13} \neq 0, \quad \lim_{k \to k_\star} s(k)_{31} \neq 0, \quad \lim_{k \to k_\star} s(k)_{33} \neq 0,$$
$$\lim_{k \to k_\star} (k - k_\star) s^A(k)_{11} \neq 0, \quad \lim_{k \to k_\star} (k - k_\star) s^A(k)_{31} \neq 0, \quad \lim_{k \to k_\star} s^A(k)_{13} \neq 0, \quad \lim_{k \to k_\star} s^A(k)_{33} \neq 0.$$

(iii) No unstable high-frequency modes: we assume that $r_1(k) = 0$ for all $k \in [0, i]$, where $[0, i]$ is the vertical segment from 0 to i.

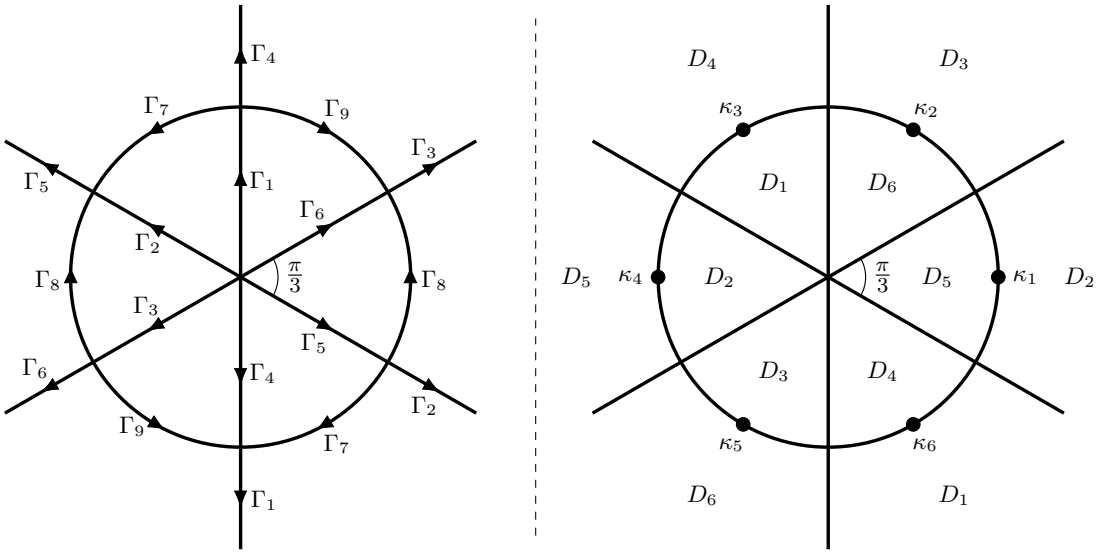

Figure 2. *The contour* $\Gamma = \cup_{j=1}^{9}\Gamma_j$ *in the complex* k*-plane (left) and the open sets* D_n*,* $n = 1, \dots, 6$*, together with the sixth roots of unity* κ_j*,* $j = 1, \dots, 6$ *(right).*

Regarding Assumption (*iii*), we note that equation (1.1) possesses solutions that blow up at any given positive time [8]. Assumption (*iii*) guarantees that the solution of the initial value problem (1.1)–(2.1) does not carry unstable nonlinear high-frequency modes and therefore exists globally. As discussed in the introduction, this assumption is physically motivated because the derivation of (1.1) requires the wavelength to be long compared to the depth of the water; the equation therefore only remains valid as a model for water waves as long as the high-frequency modes are absent (or at least heavily suppressed). If Assumptions (*i*)–(*iii*) hold true, then the solution $u(x,t)$ of the initial value problem (1.1)–(2.1) exists globally [10]. By [10, Theorems 2.6 and 2.11], there is a wide class of initial data satisfying (*i*)–(*iii*).

2.4. **Statement of the main result.** Let $\zeta := x/t$ and suppose that $\zeta > 1$. For $1 \le j < i \le 3$, let $\Phi_{ij}(\zeta, k)$ be given by

$$\Phi_{ij}(\zeta, k) = (l_i(k) - l_j(k))\zeta + (z_i(k) - z_j(k)), \qquad k \in \mathbb{C} \setminus \{0\}. \tag{2.7}$$

The saddle points $\{k_j = k_j(\zeta)\}_{j=1}^{4}$ of $k \mapsto \Phi_{21}(\zeta, k)$ are given by

$$k_1 = \frac{1}{4}\left(\zeta - \sqrt{8+\zeta^2} + i\sqrt{2}\sqrt{4 - \zeta^2 + \zeta\sqrt{8+\zeta^2}}\right), \qquad k_2 = \bar{k}_1, \tag{2.8a}$$

$$k_3 = \frac{1}{4}\left(\zeta + \sqrt{8+\zeta^2} + \sqrt{2}\sqrt{-4 + \zeta^2 + \zeta\sqrt{8+\zeta^2}}\right), \qquad k_4 = k_3^{-1}, \tag{2.8b}$$

and satisfy $|k_1| = 1$, $k_3 \in (1, +\infty)$, and $\arg k_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$. Define $z_\star$ by

$$z_\star = z_\star(\zeta) := \sqrt{2}e^{\frac{\pi i}{4}}\sqrt{\frac{4 - 3k_1\zeta - k_1^3\zeta}{4k_1^4}},$$

where the branch of the square root is such that $-ik_1 z_\star > 0$. Let $\Gamma(k)$ be the Gamma function, and define

$$\nu = \nu(k_1) := -\frac{1}{2\pi}\ln(1 + r_1(k_1)r_2(k_1)) \le 0. \tag{2.9}$$

Define also

$$\arg d_0 = \arg d_0(\zeta,t) := \nu \ln\left|\frac{(\frac{1}{\omega^2 k_1}-k_1)(\frac{1}{\omega k_1}-k_1)}{3(\frac{1}{k_1}-k_1)^2 z_\star^2}\right| - \nu \ln t + \frac{1}{2\pi}\int_i^{k_1} \ln\left|\frac{(k_1-s)^2(\frac{1}{\omega^2 k_1}-s)(\frac{1}{\omega k_1}-s)}{(\frac{1}{k_1}-s)^2(\omega k_1 - s)(\omega^2 k_1 - s)}\right| d\ln(1+r_1(s)r_2(s)), \tag{2.10}$$

where the path of the integral $\int_i^{k_1}$ starts at i, follows the unit circle in the counterclockwise direction, and ends at k_1.

Our main result establishes the behavior of $u(x,t)$ as $(x,t)\to\infty$ in Sector II. Its formulation involves the the unique solution $M_{\rm sol}(x,t,k)$ of RH problem 2.7, which is stated and discussed in Section 2.8 below.

Theorem 2.1 (Asymptotics in Sector II with solitons)**.** *Let $u_0, u_1 \in \mathcal{S}(\mathbb{R})$ be real-valued initial data such that $\int_{\mathbb{R}} u_1 dx = 0$ and such that Assumptions (i)–(iii) are fulfilled. Let $\mathcal{I}$ be a fixed compact subset of $(1,\infty)$. The global solution $u(x,t)$ of the initial value problem for (1.1) with initial data u_0, u_1 enjoys the following asymptotics as $t\to\infty$:*

$$u(x,t) = u_{\rm sol}(x,t) + \frac{u_{\rm rad}(x,t)}{\sqrt{t}} + O\left(\frac{\ln t}{t}\right), \tag{2.11}$$

uniformly for $\zeta := \frac{x}{t} \in \mathcal{I}$, where the soliton term $u_{\rm sol}$ is given by

$$u_{\rm sol}(x,t) := -i\sqrt{3}\frac{\partial}{\partial x}\lim_{k\to\infty} k\Big(\big((1,1,1)M_{\rm sol}(x,t,k)\big)_3 - 1\Big), \tag{2.12}$$

and the coefficient $u_{\rm rad}$ of the radiation term is given by

$$u_{\rm rad}(x,t) := \frac{\sqrt{6\pi}e^{-\frac{\pi\nu}{2}}\,{\rm Im}\,(k_1)}{-ik_1 z_\star}(1,1,1)M_{\rm sol}(x,t,k_1)\begin{pmatrix} 0 & \frac{e^{\frac{\pi i}{4}}e^{-t\Phi_{21}(\zeta,k_1)}}{e^{i\arg d_0} r_2(k_1)\Gamma(i\nu)} & 0 \\ \frac{e^{-\frac{\pi i}{4}}e^{t\Phi_{21}(\zeta,k_1)}}{e^{-i\arg d_0} r_1(k_1)\Gamma(-i\nu)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times M_{\rm sol}(x,t,k_1)^{-1}\begin{pmatrix} \omega^2 i k_1^{-1} - \omega i k_1 \\ \omega i k_1^{-1} - \omega^2 i k_1 \\ i k_1^{-1} - i k_1 \end{pmatrix}. \tag{2.13}$$

The proof of Theorem 2.1 is given in Sections 3–10, with the final steps presented in Section 10.1.

The asymptotic formula of Theorem 2.1 is very general. However, the presence of the solution $M_{\rm sol}(x,t,k)$ of RH problem 2.7 makes it somewhat challenging to interpret. In what follows, we therefore present more explicit versions of the asymptotic formula (2.11) which make the resolution into individual solitons more evident. We also give an explicit expression for the leading term $u_{\rm sol}(x,t)$.

2.5. **Formula valid away from the asymptotic solitons.** Away from the directions of the asymptotic solitons, the asymptotic formula (2.11) can be simplified as follows. Let $S_0 \subset (1,+\infty)$ be the (finite) set of all $\zeta \in (1,+\infty)$ for which there exists a k_0 such that either $k_0 \in (\mathsf{Z}\setminus\mathbb{R}) \cap \{k \,|\, {\rm Re}\,\Phi_{31}(\zeta,k) = 0\}$ or $k_0 \in \mathsf{Z}\cap\mathbb{R}\cap\{k\,|\,{\rm Re}\,\Phi_{21}(\zeta,k)=0\}$. Then S_0 is the set of velocities of all asymptotic solitons propagating to the right, see (2.24). For each $\epsilon > 0$, let S_ϵ be the ϵ-neighborhood of S_0 given by

$$S_\epsilon = \bigcup_{\zeta_0\in S_0} \{\zeta \in (1,+\infty)\,|\,|\zeta-\zeta_0| \le \epsilon\}. \tag{2.14}$$

If $\zeta \in (1,+\infty) \setminus S_\epsilon$, then there is no asymptotic soliton traveling with velocity ζ, so we expect the soliton term $u_{\rm sol}(x,t)$ in (2.11) to be exponentially small and the radiation term $u_{\rm rad}(x,t)/\sqrt{t}$ to simplify. Our next theorem, whose statement involves the function

$$\mathcal{P}(\zeta,k) = \prod_{\substack{k_0 \in \mathsf{Z}\setminus\mathbb{R} \\ \operatorname{Re}\Phi_{31}(\zeta,k_0)<0}} \frac{(k-k_0)(k-k_0^{-1})(k-\omega^2\bar{k}_0)(k-\omega\bar{k}_0^{-1})}{(k-\bar{k}_0)(k-\bar{k}_0^{-1})(k-\omega k_0)(k-\omega^2 k_0^{-1})} \times \prod_{\substack{k_0 \in \mathsf{Z}\cap\mathbb{R} \\ \operatorname{Re}\Phi_{21}(\zeta,k_0)<0}} \frac{(k-\omega^2 k_0)(k-\omega k_0^{-1})}{(k-\omega k_0)(k-\omega^2 k_0^{-1})}, \tag{2.15}$$

shows that this is indeed the case.

Theorem 2.2 (Asymptotics in Sector II away from solitons)**.** *Under the assumptions of Theorem 2.1, we have, for each $\epsilon > 0$,*

$$u(x,t) = \frac{A(\zeta)}{\sqrt{t}} \cos\alpha(\zeta,t) + O\left(\frac{\ln t}{t}\right) \qquad \text{as } t \to \infty, \tag{2.16}$$

uniformly for $\zeta = \frac{x}{t} \in \mathcal{I} \setminus S_\epsilon$, where

$$A(\zeta) := 2\sqrt{3}\frac{\sqrt{-\nu}\sqrt{-1-2\cos(2\arg k_1)}}{-ik_1 z_\star}\operatorname{Im} k_1,$$

$$\alpha(\zeta,t) := \frac{3\pi}{4} + \arg r_2(k_1) + \arg\Gamma(i\nu) + \arg d_0 + \arg\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)} + t\operatorname{Im}\Phi_{21}(\zeta,k_1).$$

The proof of Theorem 2.2 is given in Section 10.2.

2.6. **Explicit expression for the leading term.** RH problem 2.7 for $M_{\rm sol}(x,t,k)$ can be solved explicitly and this leads to an explicit expression for the leading term $u_{\rm sol}(x,t)$ in Theorem 2.1. In general, $u_{\rm sol}(x,t)$ is not a pure multi-soliton solution of (1.1). However, it can be described as a modulated multi-soliton, meaning that it is a multi-soliton with parameters depending on the slowly varying variable $\zeta = x/t$.

To present the expression for $u_{\rm sol}(x,t)$, we first state the formula for the pure multi-soliton of (1.1). Consider the following data:

(1) two integers $n_b \geq 0$ and $n_s \geq 0$,
(2) a finite set $\{\lambda_j\}_1^{n_b} \subset D_{\rm reg}$ and associated constants $\{c_{\lambda_j}\}_{j=1}^{n_b} \in \mathbb{C}$,
(3) a finite set $\{k_j\}_1^{n_s} \subset (-1,0)\cup(1,\infty)$ and associated constants $\{c_{k_j}\}_{j=1}^{n_s} \in \mathbb{C}$ satisfying $i(\omega^2 k_j^2 - \omega)c_{k_j} \geq 0$ for $j = 1,\dots,n_s$.

We show in Lemma 11.1 that associated to each such collection of data, there is a real-valued Schwartz class solution u_s of (1.1) for $(x,t) \in \mathbb{R}\times[0,+\infty)$ given by

$$u_s\Big(x,t;(\lambda_j,c_{\lambda_j})_{j=1}^{n_b},(k_j,c_{k_j})_{j=1}^{n_s}\Big) := 6\frac{\partial^2}{\partial x^2}\ln\det(I-B), \tag{2.17}$$

where the matrix $B = B\big(x,t;(\lambda_j,c_{\lambda_j})_{j=1}^{n_b},(k_j,c_{k_j})_{j=1}^{n_s}\big)$ is the $(2n_b+n_s)\times(2n_b+n_s)$-matrix given in block form as

$$B := \begin{pmatrix} B_{\lambda\lambda} & B_{\lambda\bar{\lambda}} & B_{\lambda k} \\ B_{\bar{\lambda}\lambda} & B_{\bar{\lambda}\bar{\lambda}} & B_{\bar{\lambda}k} \\ B_{k\lambda} & B_{k\bar{\lambda}} & B_{kk} \end{pmatrix} \tag{2.18}$$

with the submatrices

$$(B_{\lambda\lambda})_{jl} = \frac{\tilde{c}_{\lambda_l} e^{x(l_1(\lambda_j)-l_3(\lambda_l))+t(z_1(\lambda_j)-z_3(\lambda_l))}}{l_1(\lambda_j)-l_3(\lambda_l)} \qquad \text{for } j,l \in \{1,\dots,n_b\},$$

$$(B_{\lambda\bar\lambda})_{jl}=\frac{\bar{\tilde c}_{\lambda_l}e^{x(l_1(\lambda_j)-l_2(\bar\lambda_l))+t(z_1(\lambda_j)-z_2(\bar\lambda_l))}}{l_1(\lambda_j)-l_2(\bar\lambda_l)}\qquad \text{for } j,l\in\{1,\dots,n_b\},$$

$$(B_{\lambda k})_{jl}=\frac{\tilde c_{k_l}e^{x(l_1(\lambda_j)-l_2(k_l))+t(z_1(\lambda_j)-z_2(k_l))}}{l_1(\lambda_j)-l_2(k_l)}\qquad \text{for } j\in\{1,\dots,n_b\} \text{ and } l\in\{1,\dots,n_s\},$$

$$(B_{\bar\lambda\lambda})_{jl}=\frac{\tilde c_{\lambda_l}e^{x(l_3(\bar\lambda_j)-l_3(\lambda_l))+t(z_3(\bar\lambda_j)-z_3(\lambda_l))}}{l_3(\bar\lambda_j)-l_3(\lambda_l)}\qquad \text{for } j,l\in\{1,\dots,n_b\},$$

$$(B_{\bar\lambda\bar\lambda})_{jl}=\frac{\bar{\tilde c}_{\lambda_l}e^{x(l_3(\bar\lambda_j)-l_2(\bar\lambda_l))+t(z_3(\bar\lambda_j)-z_2(\bar\lambda_l))}}{l_3(\bar\lambda_j)-l_2(\bar\lambda_l)}\qquad \text{for } j,l\in\{1,\dots,n_b\},$$

$$(B_{\bar\lambda k})_{jl}=\frac{\tilde c_{k_l}e^{x(l_3(\bar\lambda_j)-l_2(k_l))+t(z_3(\bar\lambda_j)-z_2(k_l))}}{l_3(\bar\lambda_j)-l_2(k_l)}\qquad \text{for } j\in\{1,\dots,n_b\} \text{ and } l\in\{1,\dots,n_s\},$$

$$(B_{k\lambda})_{jl}=\frac{\tilde c_{\lambda_l}e^{x(l_1(k_j)-l_3(\lambda_l))+t(z_1(k_j)-z_3(\lambda_l))}}{l_1(k_j)-l_3(\lambda_l)}\qquad \text{for } j\in\{1,\dots,n_s\} \text{ and } l\in\{1,\dots,n_b\},$$

$$(B_{k\bar\lambda})_{jl}=\frac{\bar{\tilde c}_{\lambda_l}e^{x(l_1(k_j)-l_2(\bar\lambda_l))+t(z_1(k_j)-z_2(\bar\lambda_l))}}{l_1(k_j)-l_2(\bar\lambda_l)}\qquad \text{for } j\in\{1,\dots,n_s\} \text{ and } l\in\{1,\dots,n_b\},$$

$$(B_{kk})_{jl}=\frac{\tilde c_{k_l}e^{x(l_1(k_j)-l_2(k_l))+t(z_1(k_j)-z_2(k_l))}}{l_1(k_j)-l_2(k_l)}\qquad \text{for } j,l\in\{1,\dots,n_s\},$$

and

$$\tilde c_{\lambda_l}:=\frac{i(\lambda_l^2-1)}{2\sqrt{3}\lambda_l^2}c_{\lambda_l}\ \text{ for } l\in\{1,\dots,n_b\},\qquad \tilde c_{k_l}:=\frac{i(k_l^2-\omega^2)}{2\sqrt{3}k_l^2}\omega^2 c_{k_l}\ \text{ for } l\in\{1,\dots,n_s\}.$$

The function $u_s\big(x,t;(\lambda_j,c_{\lambda_j})_{j=1}^{n_b},(k_j,c_{k_j})_{j=1}^{n_s}\big)$ in (2.17) is a multi-soliton solution of (1.1) supporting n_b breather solitons and n_s bell-shaped solitons, see Remark 2.4.

The leading term $u_{\rm sol}(x,t)$ in Theorem 2.1 is given by replacing the constants c_{λ_j} and c_{k_j} in (2.17) by functions depending on $\zeta=x/t$. More precisely, we have the following result whose proof is presented in Section 11.1.

Theorem 2.3 (Explicit expression for $u_{\rm sol}$)**.** *Under the assumptions of Theorem 2.1, the leading term in (2.11) is given by*

$$u_{\rm sol}(x,t)=u_s\bigg(x,t;\bigg(\lambda_j,\frac{c_{\lambda_j}}{\hat\Delta_{11}(\zeta,\lambda_j)\hat\Delta_{33}^{-1}(\zeta,\lambda_j)}\bigg)_{j=1}^{n_b},\bigg(k_j,\frac{c_{k_j}}{\hat\Delta_{11}(\zeta,k_j)\hat\Delta_{22}^{-1}(\zeta,k_j)}\bigg)_{j=1}^{n_s}\bigg),\tag{2.19}$$

where u_s is the function in (2.17), $\zeta=x/t$, $\{\lambda_j\}_1^{n_b}:=\mathsf{Z}\cap D_{\rm reg}$, $\{k_j\}_1^{n_s}:=\mathsf{Z}\cap\mathbb{R}$, the constants $\{c_{\lambda_j}\}_1^{n_b}$ and $\{c_{k_j}\}_1^{n_s}$ are defined by (2.6), and the functions $\hat\Delta_{jj}(\zeta,k)$, $j=1,2,3$, are defined for $\zeta\in\mathcal{I}$ and $k\in\mathbb{C}\setminus\partial\mathbb{D}$ by

$$\hat\Delta_{11}(\zeta,k):=\hat\Delta_{33}(\zeta,\omega k),\quad \hat\Delta_{22}(\zeta,k):=\hat\Delta_{33}(\zeta,\omega^2k),\quad \hat\Delta_{33}(\zeta,k):=\frac{\delta(\zeta,\omega k)}{\delta(\zeta,\omega^2k)}\frac{\delta(\zeta,\frac{1}{\omega^2k})}{\delta(\zeta,\frac{1}{\omega k})},$$

$$\delta(\zeta,k):=\exp\bigg\{\frac{-1}{2\pi i}\int_i^{k_1}\frac{\ln(1+r_1(s)r_2(s))}{s-k}ds\bigg\}.\tag{2.20}$$

In (2.20), the principal branch is adopted for the logarithm and the path of integration follows the unit circle in the counterclockwise direction.

We see from (2.19) and (2.20) that if $r_1\equiv r_2\equiv 0$, then $u_{\rm sol}(x,t)$ is a pure multi-soliton solution of (1.1). In general, $u_{\rm sol}(x,t)$ is a modulated version of this multi-soliton, where the

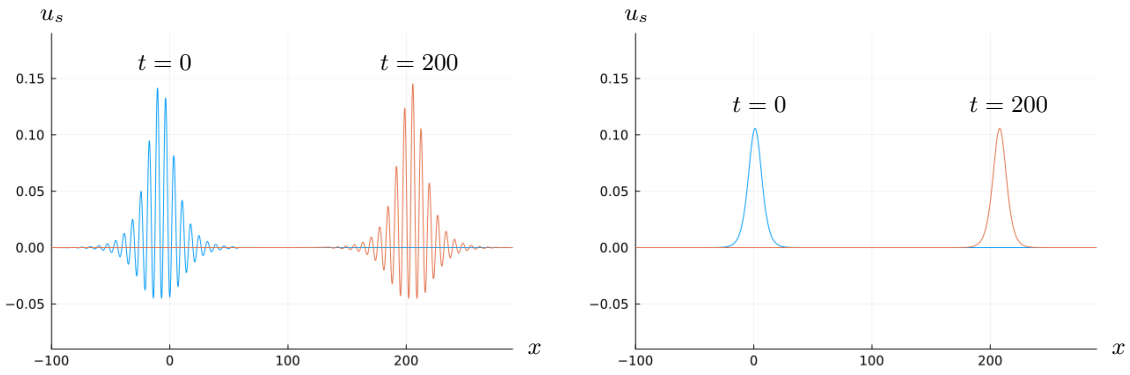

Figure 3. *Left: The single breather soliton* $u_s(x,t;(\lambda_1,c_{\lambda_1}),\emptyset)$ *for* $\lambda_1 = 1.2+0.01i$ *and* $c_{\lambda_1} = 0.1$ *at time* $t=0$ *(blue) and* $t=200$ *(red).*
Right: The one-soliton $u_s(x,t;\emptyset,(k_1,c_{k_1}))$ *for* $k_1 = 1.3$ *and* $c_{k_1} = 0.1e^{-i\arg(i(\omega^2k_1^2-\omega))}$ *at time* $t=0$ *(blue) and* $t=200$ *(red).*

constants c_{λ_j} and c_{k_j} have been replaced by the slowly varying functions $\frac{c_{\lambda_j}}{\hat{\Delta}_{11}(\zeta,\lambda_j)\hat{\Delta}_{33}^{-1}(\zeta,\lambda_j)}$ and $\frac{c_{k_j}}{\hat{\Delta}_{11}(\zeta,k_j)\hat{\Delta}_{22}^{-1}(\zeta,k_j)}$, respectively.[1]

Remark 2.4. *To understand the multi-soliton formula (2.17) it is useful to keep the following in mind (see [10, Appendix A]): If* $n_b = 1$ *and* $n_s = 0$*, then the function* $u_s(x,t;(\lambda_1,c_{\lambda_1}),\emptyset)$ *in (2.17) reduces to a single breather soliton propagating in the direction* $x/t = \zeta_{\lambda_1}$*, where* ζ_{λ_1} *is such that* $\operatorname{Re}\Phi_{31}(\zeta_{\lambda_1},\lambda_1) = 0$ *(recall here that* $\lambda_1 \in D_{\mathrm{reg}}$*), i.e.*

$$\zeta_{\lambda_1} := \frac{(1+|\lambda_1|^2)(\lambda_1^2+\omega\bar{\lambda}_1^2)}{2|\lambda_1|^2(\omega\lambda_1+\bar{\lambda}_1)} \in (-\infty,-1)\cup(1,+\infty), \tag{2.21}$$

while if $n_b = 0$ *and* $n_s = 1$*, then the function* $u_s(x,t;\emptyset,(k_1,c_{k_1}))$ *in (2.17) reduces to the pure one-soliton (assuming that* $c_{k_1} \neq 0$*)*

$$u_s(x,t;\emptyset,(k_1,c_{k_1})) = A\operatorname{sech}^2\left(\sqrt{\frac{A}{6}}(x-x_0-\zeta_{k_1}t)\right), \tag{2.22}$$
$$A := \frac{3}{8}(k_1-k_1^{-1})^2 \geq 0, \qquad x_0 := \frac{2k_1}{k_1^2-1}\ln\left(\frac{i(\omega^2k_1^2-\omega)c_{k_1}}{\sqrt{3}k_1(k_1^2-1)}\right) \in \mathbb{R},$$

propagating in the direction $x/t = \zeta_{k_1}$*, where the principal branch is used for the logarithm and* ζ_{k_1} *is such that* $\operatorname{Re}\Phi_{21}(\zeta_{k_1},k_1) = 0$ *(recall here that* $k_1 \in (-1,0)\cup(1,+\infty)$*), i.e.*

$$\zeta_{k_1} := \frac{k_1+k_1^{-1}}{2} \in (-\infty,-1)\cup(1,+\infty), \tag{2.23}$$

see Figure 3. For general values of $n_b \geq 0$ *and* $n_s \geq 0$*, the multi-soliton* u_s *can be viewed as a nonlinear superposition of* n_b *breather solitons and* n_s *one-solitons.*

2.7. **Formula valid near an asymptotic soliton.** Our last theorem provides a simplification of the asymptotic formula (2.11) in the direction of an asymptotic soliton. Under the assumptions of Theorem 2.1, let $\{\lambda_j\}_1^{n_b} := \mathsf{Z}\cap D_{\mathrm{reg}}$ and $\{k_j\}_1^{n_s} := \mathsf{Z}\cap\mathbb{R}$ denote the

[1]We emphasize that the x-derivatives in (2.17) should be evaluated before this replacement is made, i.e., the x-derivatives in (2.17) do not act on the functions $\hat{\Delta}_{11}$, $\hat{\Delta}_{22}$, and $\hat{\Delta}_{33}$.

zeros of $s_{11}(k)$ in $D_{\rm reg}$ and $(-1,0)\cup(1,+\infty)$, respectively. For $j=1,\dots,n_b$, we define

$$\zeta_{\lambda_j} := \frac{(1+|\lambda_j|^2)(\lambda_j^2+\omega\bar\lambda_j^2)}{2|\lambda_j|^2(\omega\lambda_j+\bar\lambda_j)} \in (-\infty,-1)\cup(1,+\infty),$$

and for $j=1,\dots,n_s$, we define

$$\zeta_{k_j} := \frac{k_j+k_j^{-1}}{2} \in (-\infty,-1)\cup(1,+\infty).$$

Note that $\operatorname{Re}\Phi_{31}(\zeta,\lambda_j)=0$ if and only if $\zeta=\zeta_{\lambda_j}$ and $\operatorname{Re}\Phi_{21}(\zeta,k_j)=0$ if and only if $\zeta=\zeta_{k_j}$, so the set S_0 defined above equation (2.14) is given by

$$S_0 = \{\zeta_{\lambda_1},\dots,\zeta_{\lambda_{n_b}},\zeta_{k_1},\dots,\zeta_{k_{n_s}}\}\cap(1,+\infty). \tag{2.24}$$

In light of (2.21) and (2.23), we expect each zero λ_j to generate an asymptotic breather soliton propagating at velocity $x/t=\zeta_{\lambda_j}$, and each zero k_j to generate an asymptotic one-soliton propagating at velocity $x/t=\zeta_{k_j}$. If some of the velocities ζ_{λ_j} and/or ζ_{k_j} coincide and are equal to say ζ_0, then we expect the asymptotics in the direction $x/t=\zeta_0$ to be described by a multi-soliton that combines all the breathers and one-solitons traveling at velocity ζ_0. The following theorem, which complements Theorem 2.2, confirms that this is indeed the case.

Theorem 2.5 (Asymptotics in the direction of an asymptotic soliton)**.** *Under the assumptions of Theorem 2.1, if $\zeta_0\in S_0$ and $\epsilon>0$ is so small that $\zeta_0-\epsilon>1$ and $\{\zeta\in\mathbb{R}\,|\,|\zeta-\zeta_0|\le\epsilon\}$ contains no other point in S_0, then, as $t\to\infty$,*

$$u(x,t) = u_s\left(x,t;\left(\lambda_j,\frac{c_{\lambda_j}}{\hat\Delta_{11}(\zeta,\lambda_j)\hat\Delta_{33}^{-1}(\zeta,\lambda_j)}\right)_{j\in I_0},\left(k_j,\frac{c_{k_j}}{\hat\Delta_{11}(\zeta,k_j)\hat\Delta_{22}^{-1}(\zeta,k_j)}\right)_{j\in J_0}\right) + \frac{u_{\rm rad}(x,t)}{\sqrt{t}} + O\left(\frac{\ln t}{t}\right) \tag{2.25}$$

uniformly for all $\zeta:=\frac{x}{t}$ such that $|\zeta-\zeta_0|\le\epsilon$, where I_0 consists of all $j\in\{1,\dots,n_b\}$ such that $\zeta_{\lambda_j}=\zeta_0$, J_0 consists of all $j\in\{1,\dots,n_s\}$ such that $\zeta_{k_j}=\zeta_0$, and $u_{\rm rad}(x,t)$ is given by (2.13). In particular, if the set I_0 is empty and $J_0=\{j\}$ contains one element, then, for any $C_0>0$,

$$u(x,t) = A_j\operatorname{sech}^2\left(\sqrt{\frac{A_j}{6}}(x-x_{0j}-\zeta_{k_j}t)\right)+\frac{u_{\rm rad}(x,t)}{\sqrt{t}}+O\left(\frac{\ln t}{t}\right) \tag{2.26}$$

as $t\to\infty$ uniformly for $x\in[\zeta_{k_j}t-C_0,\zeta_{k_j}t+C_0]$, where

$$A_j := \frac{3}{8}(k_j-k_j^{-1})^2\ge 0$$

and

$$x_{0j} := \frac{2k_j}{k_j^2-1}\ln\left(\frac{i(\omega^2k_j^2-\omega)}{\sqrt{3}k_j(k_j^2-1)}\frac{c_{k_j}}{\hat\Delta_{11}(\zeta_{k_j},k_j)\hat\Delta_{22}^{-1}(\zeta_{k_j},k_j)}\right)\in\mathbb{R}$$

with the principal branch used for the logarithm.

Remark 2.6. *Since the function $k\to\zeta_k:=\frac{k+k^{-1}}{2}$ is a bijection from $(1,+\infty)$ to $(1,+\infty)$, the set J_0 in (2.25) contains at most one element.*

The proof of Theorem 2.5 is presented in Section 12.1.

2.8. **The RH problem for M_{sol}.** The statement of Theorem 2.1 involves the 3×3-matrix valued function $M_{\mathrm{sol}}(x,t,k)$. This function is defined as the unique solution of a RH problem which we now state. For each $k_0 \in \mathsf{Z}\setminus\mathbb{R}$, we define $d_{k_0}\in\mathbb{C}$ by

$$d_{k_0} = \frac{\bar{k}_0^2-1}{\omega^2(\omega^2-\bar{k}_0^2)}\bar{c}_{k_0}, \qquad k_0\in\mathsf{Z}\setminus\mathbb{R}. \tag{2.27}$$

For $\zeta\in\mathcal{I}$ and $k\in\mathbb{C}\setminus\partial\mathbb{D}$, we define $\delta(\zeta,k)$, $\hat{\Delta}_{11}(\zeta,k)$, $\hat{\Delta}_{22}(\zeta,k)$, and $\hat{\Delta}_{33}(\zeta,k)$ as in (2.20) and we set

$$\hat{\Delta}(\zeta,k) := \begin{pmatrix} \hat{\Delta}_{11}(\zeta,k) & 0 & 0\\ 0 & \hat{\Delta}_{22}(\zeta,k) & 0\\ 0 & 0 & \hat{\Delta}_{33}(\zeta,k)\end{pmatrix}. \tag{2.28}$$

We also define the set $\hat{\mathsf{Z}}$ by

$$\hat{\mathsf{Z}} = \mathsf{Z}\cup\mathsf{Z}^*\cup\omega\mathsf{Z}\cup\omega\mathsf{Z}^*\cup\omega^2\mathsf{Z}\cup\omega^2\mathsf{Z}^*\cup\mathsf{Z}^{-1}\cup\mathsf{Z}^{-*}\cup\omega\mathsf{Z}^{-1}\cup\omega\mathsf{Z}^{-*}\cup\omega^2\mathsf{Z}^{-1}\cup\omega^2\mathsf{Z}^{-*}, \tag{2.29}$$

where $\mathsf{Z}^{-1}=\{k_0^{-1}\,|\,k_0\in\mathsf{Z}\}$, $\mathsf{Z}^*=\{\bar{k}_0\,|\,k_0\in\mathsf{Z}\}$, and $\mathsf{Z}^{-*}=\{\bar{k}_0^{-1}\,|\,k_0\in\mathsf{Z}\}$.

RH problem 2.7 (RH problem for M_{sol})**.** *Find a 3×3-matrix valued function $M_{\mathrm{sol}}(x,t,k)$ with the following properties:*

(a) *$M_{\mathrm{sol}}(x,t,\cdot):\mathbb{C}\setminus\hat{\mathsf{Z}}\to\mathbb{C}^{3\times 3}$ is analytic, where $\hat{\mathsf{Z}}$ is given by (2.29).*

(b) *For $k\in\mathbb{C}\setminus\hat{\mathsf{Z}}$, M_{sol} obeys the symmetries*

$$M_{\mathrm{sol}}(x,t,k) = \mathcal{A}M_{\mathrm{sol}}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}M_{\mathrm{sol}}(x,t,k^{-1})\mathcal{B}, \tag{2.30}$$

where

$$\mathcal{A} := \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \qquad \textit{and} \qquad \mathcal{B} := \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}. \tag{2.31}$$

(c) *$M_{\mathrm{sol}}(x,t,k) = I + O(k^{-1})$ as $k\to\infty$.*

(d) *At each point $k_0\in\mathsf{Z}\setminus\mathbb{R}$, the first two columns of M_{sol} are analytic, while the third column has (at most) a simple pole. Furthermore,*

$$\operatorname*{Res}_{k=k_0}[M_{\mathrm{sol}}(x,t,k)]_3 = \frac{c_{k_0}e^{-\theta_{31}(x,t,k_0)}}{\hat{\Delta}_{11}(\zeta,k_0)\hat{\Delta}_{33}^{-1}(\zeta,k_0)}[M_{\mathrm{sol}}(x,t,k_0)]_1. \tag{2.32}$$

At each point $\bar{k}_0$ with $k_0\in\mathsf{Z}\setminus\mathbb{R}$, the first and third columns of M_{sol} are analytic, while the second column has (at most) a simple pole. Furthermore,

$$\operatorname*{Res}_{k=\bar{k}_0}[M_{\mathrm{sol}}(x,t,k)]_2 = \frac{d_{k_0}e^{\theta_{32}(x,t,\bar{k}_0)}}{\hat{\Delta}_{33}(\zeta,\bar{k}_0)\hat{\Delta}_{22}^{-1}(\zeta,\bar{k}_0)}[M_{\mathrm{sol}}(x,t,\bar{k}_0)]_3, \tag{2.33}$$

where d_{k_0} is given by (2.27). At each point $k_0\in\mathsf{Z}\cap\mathbb{R}$, the first and third columns of M_{sol} are analytic, while the second column has (at most) a simple pole. Furthermore,

$$\operatorname*{Res}_{k=k_0}[M_{\mathrm{sol}}(x,t,k)]_2 = \frac{c_{k_0}e^{-\theta_{21}(x,t,k_0)}}{\hat{\Delta}_{11}(\zeta,k_0)\hat{\Delta}_{22}^{-1}(\zeta,k_0)}[M_{\mathrm{sol}}(x,t,k_0)]_1. \tag{2.34}$$

Suppose $\{r_1,r_2,\mathsf{Z},\{c_{k_0}\}_{k_0\in\mathsf{Z}}\}$ are scattering data associated to some initial data u_0,u_1 fulfilling the assumptions of Theorem 2.1. Then the solution of RH problem 2.7 exists, is unique, and satisfies $\det M_{\mathrm{sol}}(x,t,k)=1$; moreover, the soliton term $u_{\mathrm{sol}}(x,t)$ in the asymptotic formula (2.11) defined in terms of $M_{\mathrm{sol}}(x,t,k)$ via (2.12) is a smooth function of x and t, see Lemma 12.1.

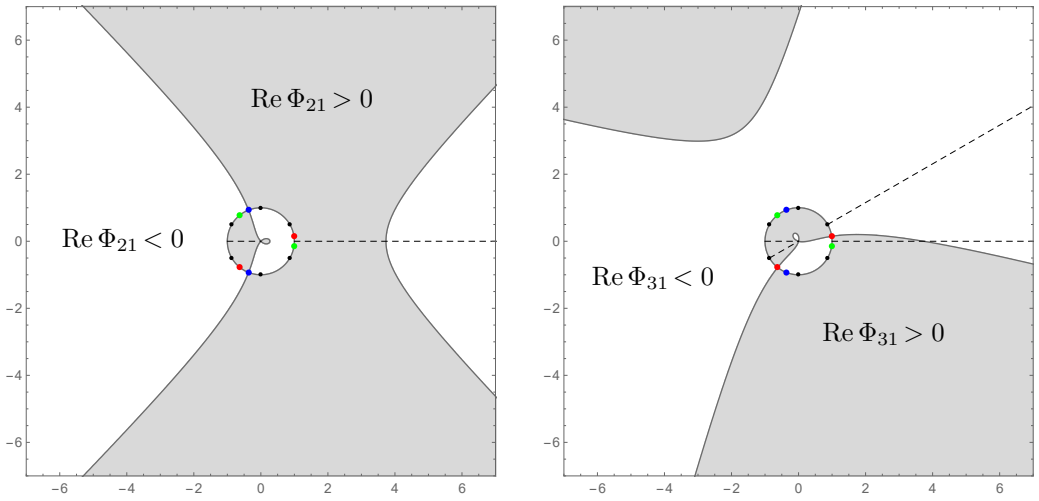

Figure 4. *From left to right: The signature tables for* Φ_{21} *and* Φ_{31} *for* $\zeta = 2$*. The grey regions correspond to* $\{k \,|\, \operatorname{Re}\Phi_{ij} > 0\}$ *and the white regions to* $\{k \,|\, \operatorname{Re}\Phi_{ij} < 0\}$*. In the left picture, the dashed line segments represent* $(-1,0) \cup (1,\infty)$*, while in the right picture the dashed line segments are parts of* $\partial D_{\mathrm{reg}}$*. The saddle points* k_1, k_2 *of* Φ_{21} *are blue, the saddle points* $\omega k_1, \omega k_2$ *of* Φ_{31} *are red, and the saddle points* $\omega^2 k_1, \omega^2 k_2$ *of* Φ_{32} *are green. The black dots are the points* $i\kappa_j$*,* $j = 1, \dots, 6$*.*

2.9. **Discussion of the asymptotic formulas.** Let us first discuss the soliton term $u_{\mathrm{sol}}(x,t)$ in (2.11). The solution M_{sol} depends on the reflection coefficients $r_1(k)$ and $r_2(k)$ only via the function $\hat{\Delta}$ that appears in the residue conditions (2.32)–(2.34). Pure soliton solutions of the Boussinesq equation (1.1) arise when the reflection coefficients $r_1(k)$ and $r_2(k)$ vanish identically. If we temporarily set r_1 and r_2 to zero in RH problem 2.7, then $\hat{\Delta}$ is identically equal to the identity matrix I and the function u_{sol} defined in (2.12) is a non-decaying solution of (1.1); more precisely, it is the (multi-)soliton solution of (1.1) generated by the scattering data $\{r_1 = 0, r_2 = 0, \mathsf{Z}, \{c_{k_0}\}_{k_0 \in \mathsf{Z}}\}$. In the general case, the functions $r_1(k)$ and $r_2(k)$ do not vanish identically. In this case, the function u_{sol} defined in (2.12) is a perturbation of the pure multi-soliton generated by $\{0, 0, \mathsf{Z}, \{c_{k_0}\}_{k_0 \in \mathsf{Z}}\}$. The perturbation, which is introduced via the function $\hat{\Delta}$ (see (2.25)), encodes the effect of the radiative background on the solitons.

We next discuss the leading term in (2.16). This term is a modulated sine-wave of order $O(t^{-1/2})$, where the effect of the solitons is encoded in the phase shift $\arg \frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}$. The first product in the definition (2.15) of $\mathcal{P}(\zeta,k)$ runs over all points $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ such that $\operatorname{Re}\Phi_{31}(\zeta,k_0) < 0$; these points correspond to breather solitons, and the condition $\operatorname{Re}\Phi_{31}(\zeta,k_0) < 0$ ensures that only those breather solitons traveling at velocities greater than ζ are taken into account, see Figure 4. Similarly, the second product in (2.15) runs over all points $k_0 \in \mathsf{Z} \cap \mathbb{R}$ such that $\operatorname{Re}\Phi_{21}(\zeta,k_0) < 0$; these points correspond to one-solitons, and the condition $\operatorname{Re}\Phi_{21}(\zeta,k_0) < 0$ ensures that only those one-solitons traveling at velocities larger than ζ are taken into account. In other words, the asymptotic behavior of $u(x,t)$ in the direction $x/t = \zeta > 1$ is given by a decaying modulated sine-wave and the phase of this sine-wave receives a shift from each right-moving soliton traveling at a speed greater than ζ.

The function $\mathcal{P}(\zeta,k)$ is in general discontinuous as a function of ζ, because as ζ decreases below the velocity of an asymptotic soliton, a new factor suddenly appears in one of the

products in (2.15). This is consistent with the fact that the asymptotic formula (2.16) only is valid away from the directions of the asymptotic solitons, i.e., away from the points of discontinuity of $\mathcal{P}(\zeta,k)$. The additional complexity of the expression for the term of $O(t^{-1/2})$ in Theorems 2.1 and 2.5 compared to the analogous term in Theorem 2.2 can be viewed as a price we have to pay to smoothen out these discontinuities. Indeed, the asymptotic formulas (2.11) and (2.25) are uniformly valid also near the directions of the asymptotic solitons and the radiation term $u_{\rm rad}(x,t)/\sqrt{t}$ in (2.11) and (2.25) depends smoothly on ζ.

2.10. **Notation.** We use $C>0$ and $c>0$ to denote generic constants that may change within a computation. If A is an $n\times m$ matrix, we define $|A|\geq 0$ by $|A|^2=\Sigma_{i,j}|A_{ij}|^2$. For a piecewise smooth contour $\gamma\subset\mathbb{C}$ and $1\leq p\leq\infty$, if $|A|$ belongs to $L^p(\gamma)$, we write $A\in L^p(\gamma)$ and define $\|A\|_{L^p(\gamma)}:=\||A|\|_{L^p(\gamma)}$. We will use $D_\epsilon(k)$ to denote the open disk of radius ϵ centered at $k\in\mathbb{C}$. The Schwartz space $\mathcal{S}(\mathbb{R})$ contains all smooth functions f on $\mathbb{R}$ such that f and all its derivatives have rapid decay as $x\to\pm\infty$.

3. The RH problem for n

Our proofs of Theorems 2.1 and 2.2 are based on a Deift-Zhou steepest descent analysis of a 1×3 RH problem. This RH problem, whose solution is denoted by n, was first derived in [8], and extended to the case when solitons are present in [10]. Its jump contour $\Gamma=\cup_{j=1}^9\Gamma_j$ is oriented as in Figure 2. For each $j\in\{1,\dots,6\}$, we write $\Gamma_j=\Gamma_{j'}\cup\Gamma_{j''}$, where $\Gamma_{j'}=\Gamma_j\setminus\mathbb{D}$ and $\Gamma_{j''}:=\Gamma_j\setminus\Gamma_{j'}$ with Γ_j as in Figure 2. We let $\theta_{ij}(x,t,k)=t\,\Phi_{ij}(\zeta,k)$. By Assumption (iii), we have $r_1(k)=0$ for $k\in[0,i]$, and thus the jump matrix $v(x,t,k)$ is given for $k\in\Gamma$ by (see [8])

$$
v_{1'}=\begin{pmatrix}1&-r_1(k)e^{-\theta_{21}}&0\\0&1&0\\0&0&1\end{pmatrix},\ v_{1''}=\begin{pmatrix}1&0&0\\r_1(\frac{1}{k})e^{\theta_{21}}&1&0\\0&0&1\end{pmatrix},\ v_{2'}=\begin{pmatrix}1&0&0\\0&1&-r_2(\frac{1}{\omega k})e^{-\theta_{32}}\\0&0&1\end{pmatrix},
$$

$$
v_{2''}=\begin{pmatrix}1&0&0\\0&1&0\\0&r_2(\omega k)e^{\theta_{32}}&1\end{pmatrix},\ v_{3'}=\begin{pmatrix}1&0&0\\0&1&0\\-r_1(\omega^2k)e^{\theta_{31}}&0&1\end{pmatrix},\ v_{3''}=\begin{pmatrix}1&0&r_1(\frac{1}{\omega^2k})e^{-\theta_{31}}\\0&1&0\\0&0&1\end{pmatrix},
$$

$$
v_{4'}=\begin{pmatrix}1&-r_2(\frac{1}{k})e^{-\theta_{21}}&0\\0&1&0\\0&0&1\end{pmatrix},\ v_{4''}=\begin{pmatrix}1&0&0\\r_2(k)e^{\theta_{21}}&1&0\\0&0&1\end{pmatrix},\ v_{5'}=\begin{pmatrix}1&0&0\\0&1&-r_1(\omega k)e^{-\theta_{32}}\\0&0&1\end{pmatrix},
$$

$$
v_{5''}=\begin{pmatrix}1&0&0\\0&1&0\\0&r_1(\frac{1}{\omega k})e^{\theta_{32}}&1\end{pmatrix},\ v_{6'}=\begin{pmatrix}1&0&0\\0&1&0\\-r_2(\frac{1}{\omega^2k})e^{\theta_{31}}&0&1\end{pmatrix},\ v_{6''}=\begin{pmatrix}1&0&r_2(\omega^2k)e^{-\theta_{31}}\\0&1&0\\0&0&1\end{pmatrix},
$$

$$
v_7=\begin{pmatrix}1&-r_1(k)e^{-\theta_{21}}&r_2(\omega^2k)e^{-\theta_{31}}\\-r_2(k)e^{\theta_{21}}&1+r_1(k)r_2(k)&\big(r_2(\frac{1}{\omega k})-r_2(k)r_2(\omega^2k)\big)e^{-\theta_{32}}\\r_1(\omega^2k)e^{\theta_{31}}&\big(r_1(\frac{1}{\omega k})-r_1(k)r_1(\omega^2k)\big)e^{\theta_{32}}&f(\omega^2k)\end{pmatrix},
$$

$$
v_8=\begin{pmatrix}f(k)&r_1(k)e^{-\theta_{21}}&\big(r_1(\frac{1}{\omega^2k})-r_1(k)r_1(\omega k)\big)e^{-\theta_{31}}\\r_2(k)e^{\theta_{21}}&1&-r_1(\omega k)e^{-\theta_{32}}\\\big(r_2(\frac{1}{\omega^2k})-r_2(\omega k)r_2(k)\big)e^{\theta_{31}}&-r_2(\omega k)e^{\theta_{32}}&1+r_1(\omega k)r_2(\omega k)\end{pmatrix},
$$

$$
v_9=\begin{pmatrix}1+r_1(\omega^2k)r_2(\omega^2k)&\big(r_2(\frac{1}{k})-r_2(\omega k)r_2(\omega^2k)\big)e^{-\theta_{21}}&-r_2(\omega^2k)e^{-\theta_{31}}\\\big(r_1(\frac{1}{k})-r_1(\omega k)r_1(\omega^2k)\big)e^{\theta_{21}}&f(\omega k)&r_1(\omega k)e^{-\theta_{32}}\\-r_1(\omega^2k)e^{\theta_{31}}&r_2(\omega k)e^{\theta_{32}}&1\end{pmatrix},
\tag{3.1}
$$

where $v_j, v_{j'}, v_{j''}$ are the restrictions of v to Γ_j, $\Gamma_{j'}$, and $\Gamma_{j''}$, respectively, and

$$
f(k):=1+r_1(k)r_2(k)+r_1(\tfrac{1}{\omega^2k})r_2(\tfrac{1}{\omega^2k}),\qquad k\in\partial\mathbb{D}.
\tag{3.2}
$$

The function v obeys the symmetries

$$v(x,t,k) = \mathcal{A}v(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}v(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma, \tag{3.3}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by (2.31). Note that v depends on x and t only via the phase functions Φ_{21}, Φ_{31}, and Φ_{32}.

Let Z be as in Assumption (i) and define $\hat{\mathsf{Z}}$ by (2.29). We denote by $\Gamma_\star = \{i\kappa_j\}_{j=1}^6 \cup \{0\}$ the set of intersection points of Γ.

RH problem 3.1 (RH problem for n)**.** *Find a 1×3-row-vector valued function $n(x,t,k)$ with the following properties:*

(a) $n(x,t,\cdot) : \mathbb{C}\setminus(\Gamma\cup\hat{\mathsf{Z}}) \to \mathbb{C}^{1\times 3}$ *is analytic.*

(b) *The limits of $n(x,t,k)$ as k approaches $\Gamma\setminus\Gamma_\star$ from the left and right exist, are continuous on $\Gamma\setminus\Gamma_\star$, and are denoted by n_+ and n_-, respectively. Moreover,*

$$n_+(x,t,k) = n_-(x,t,k)v(x,t,k) \qquad \text{for} \quad k\in\Gamma\setminus\Gamma_\star. \tag{3.4}$$

(c) $n(x,t,k) = O(1)$ *as $k\to k_\star\in\Gamma_\star$.*

(d) *For $k\in\mathbb{C}\setminus(\Gamma\cup\hat{\mathsf{Z}})$, n obeys the symmetries*

$$n(x,t,k) = n(x,t,\omega k)\mathcal{A}^{-1} = n(x,t,k^{-1})\mathcal{B}. \tag{3.5}$$

(e) $n(x,t,k) = (1,1,1) + O(k^{-1})$ *as $k\to\infty$.*

(f) *At each point of $\hat{\mathsf{Z}}$, one entry of n has (at most) a simple pole while two entries are analytic. Moreover, the following residue conditions hold: for each $k_0\in\mathsf{Z}\setminus\mathbb{R}$,*

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_3(k) = c_{k_0}e^{-\theta_{31}(k_0)}n_1(k_0), && \operatorname*{Res}_{k=\omega k_0} n_2(k) = \omega c_{k_0}e^{-\theta_{31}(k_0)}n_3(\omega k_0),\\
&\operatorname*{Res}_{k=\omega^2 k_0} n_1(k) = \omega^2 c_{k_0}e^{-\theta_{31}(k_0)}n_2(\omega^2 k_0), && \operatorname*{Res}_{k=k_0^{-1}} n_3(k) = -k_0^{-2}c_{k_0}e^{-\theta_{31}(k_0)}n_2(k_0^{-1}),\\
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_1(k) = -\tfrac{\omega^2}{k_0^2}c_{k_0}e^{-\theta_{31}(k_0)}n_3(\omega^2 k_0^{-1}), && \operatorname*{Res}_{k=\omega k_0^{-1}} n_2(k) = -\tfrac{\omega}{k_0^2}c_{k_0}e^{-\theta_{31}(k_0)}n_1(\omega k_0^{-1}),\\
&\operatorname*{Res}_{k=\bar{k}_0} n_2(k) = d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_3(\bar{k}_0), && \operatorname*{Res}_{k=\omega\bar{k}_0} n_1(k) = \omega d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_2(\omega\bar{k}_0),\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0} n_3(k) = \omega^2 d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_1(\omega^2\bar{k}_0), && \operatorname*{Res}_{k=\bar{k}_0^{-1}} n_1(k) = -\bar{k}_0^{-2}d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_3(\bar{k}_0^{-1}),\\
&\operatorname*{Res}_{k=\omega^2\bar{k}_0^{-1}} n_2(k) = -\tfrac{\omega^2}{\bar{k}_0^2}d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_1(\omega^2\bar{k}_0^{-1}), && \operatorname*{Res}_{k=\omega\bar{k}_0^{-1}} n_3(k) = -\tfrac{\omega}{\bar{k}_0^2}d_{k_0}e^{\theta_{32}(\bar{k}_0)}n_2(\omega\bar{k}_0^{-1}),
\end{aligned} \tag{3.6}$$

and, for each $k_0\in\mathsf{Z}\cap\mathbb{R}$,

$$\begin{aligned}
&\operatorname*{Res}_{k=k_0} n_2(k) = c_{k_0}e^{-\theta_{21}(k_0)}n_1(k_0), && \operatorname*{Res}_{k=\omega k_0} n_1(k) = \omega c_{k_0}e^{-\theta_{21}(k_0)}n_3(\omega k_0),\\
&\operatorname*{Res}_{k=\omega^2 k_0} n_3(k) = \omega^2 c_{k_0}e^{-\theta_{21}(k_0)}n_2(\omega^2 k_0), && \operatorname*{Res}_{k=k_0^{-1}} n_1(k) = -k_0^{-2}c_{k_0}e^{-\theta_{21}(k_0)}n_2(k_0^{-1}),\\
&\operatorname*{Res}_{k=\omega^2 k_0^{-1}} n_2(k) = -\tfrac{\omega^2}{k_0^2}c_{k_0}e^{-\theta_{21}(k_0)}n_3(\omega^2 k_0^{-1}), && \operatorname*{Res}_{k=\omega k_0^{-1}} n_3(k) = -\tfrac{\omega}{k_0^2}c_{k_0}e^{-\theta_{21}(k_0)}n_1(\omega k_0^{-1}),
\end{aligned} \tag{3.7}$$

where the (x,t)-dependence of n and of θ_{ij} has been suppressed for conciseness.

Since u_0 and u_1 satisfy Assumptions (i)–(iii), it follows from [10, Theorems 2.6 and 2.11] that RH problem 3.1 has a unique solution $n(x,t,k)$ for each $(x,t)\in\mathbb{R}\times[0,\infty)$, that

$$n_3^{(1)}(x,t) := \lim_{k\to\infty} k(n_3(x,t,k)-1)$$

is well-defined and smooth for $(x,t) \in \mathbb{R} \times [0,\infty)$, and that

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x} n_3^{(1)}(x,t) \tag{3.8}$$

is a real-valued Schwartz class solution of (1.1) on $\mathbb{R}\times[0,\infty)$ with initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$.

Let $\mathcal{I}$ be a fixed compact subset of $(1,\infty)$. Recall from Section 2 that $\{k_j\}_{j=1}^4$ are the saddle points of Φ_{21}. From the identities $\Phi_{31}(\zeta,k) = -\Phi_{21}(\zeta,\omega^2 k)$ and $\Phi_{32}(\zeta,k) = \Phi_{21}(\zeta,\omega k)$, we infer that $\{\omega k_j\}_{j=1}^4$ are the saddle points of Φ_{31} and that $\{\omega^2 k_j\}_{j=1}^4$ are the saddle points of Φ_{32}. For the sector considered in this paper, namely $\zeta \in \mathcal{I}$, it turns out that only the saddle points $\{\omega^j k_1, \omega^j k_2\}_{j=0}^2$ play a role in the asymptotic analysis of n. The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} are shown in Figure 5 (see also Figure 4 for the signature tables of Φ_{21} and Φ_{31} in a larger subset of the complex plane).

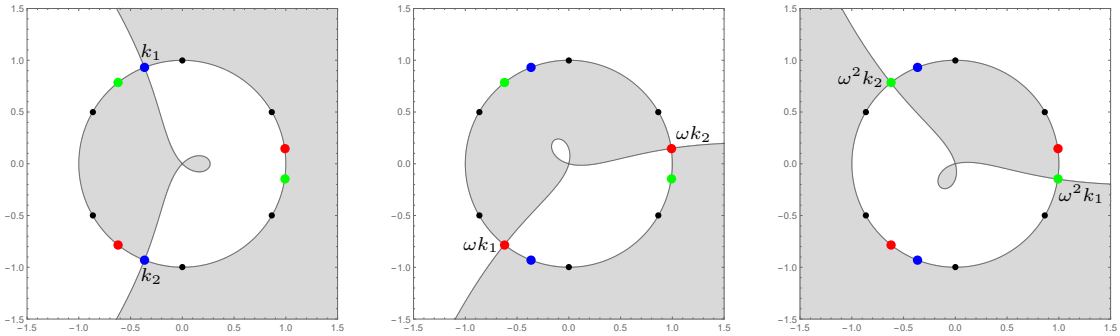

Figure 5. *From left to right: The signature tables for Φ_{21}, Φ_{31}, and Φ_{32} for $\zeta = 2$. The grey regions correspond to $\{k \,|\, \operatorname{Re}\Phi_{ij} > 0\}$ and the white regions to $\{k \,|\, \operatorname{Re}\Phi_{ij} < 0\}$. The saddle points k_1, k_2 of Φ_{21} are blue, the saddle points $\omega k_1, \omega k_2$ of Φ_{31} are red, and the saddle points $\omega^2 k_1, \omega^2 k_2$ of Φ_{32} are green. The black dots are the points $i\kappa_j$, $j = 1,\dots,6$.*

We will use the Deift–Zhou [19] steepest descent method applied to RH problem 3.1 to obtain asymptotics for u. We will proceed via several transformations $n \to n^{(1)} \to n^{(2)} \to n^{(3)} \to n^{(4)} \to \hat{n}$, such that the RH problems satisfied by $n^{(1)}, n^{(2)}, n^{(3)}, n^{(4)}, \hat{n}$ are equivalent to the original RH problem 3.1. We will write $\Gamma^{(j)}, \hat{\Gamma}$ and $v^{(j)}, \hat{v}$ for the contours and jump matrices of the RH problems for $n^{(j)}, \hat{n}$, respectively. At each step, the symmetries (3.5) and (3.3) will be preserved, so that, for $j = 1,2,3,4$,

$$v^{(j)}(x,t,k) = \mathcal{A} v^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} v^{(j)}(x,t,k^{-1})^{-1}\mathcal{B}, \qquad k \in \Gamma^{(j)}, \tag{3.9}$$

$$n^{(j)}(x,t,k) = n^{(j)}(x,t,\omega k)\mathcal{A}^{-1} = n^{(j)}(x,t,k^{-1})\mathcal{B}, \qquad k \in \mathbb{C}\setminus\Gamma^{(j)}, \tag{3.10}$$

and similarly for $\hat{n}$ and $\hat{v}$. Thanks to the symmetries (3.9)–(3.10), the transformations $n \to n^{(1)}$, $n^{(j)} \to n^{(j+1)}$, and $n^{(4)} \to \hat{n}$ will only be defined explicitly in the sector

$$\mathsf{S} := \{k \in \mathbb{C} \,|\, \arg k \in [\tfrac{\pi}{3}, \tfrac{2\pi}{3}]\}. \tag{3.11}$$

In the rest of the complex plane, they will be defined using the $\mathcal{A}$- and $\mathcal{B}$-symmetries.

Any point $k_0 \in \mathsf{Z}$ with $c_{k_0} = 0$ can be removed from Z without affecting RH problem 3.1. We will therefore without loss of generality henceforth assume that $c_{k_0} \neq 0$ for all $k_0 \in \mathsf{Z}$.

4. The $n \to n^{(1)}$ transformation

We now begin the proof of Theorem 2.1. We recall the following facts from [10, Theorem 2.3]: $r_1 \in C^\infty(\hat{\Gamma}_1)$, $r_2 \in C^\infty(\hat{\Gamma}_4 \setminus \{\pm\omega^2\})$, $r_1(\kappa_j) \neq 0$ for $j = 1,\dots,6$, $r_2(k)$ has simple

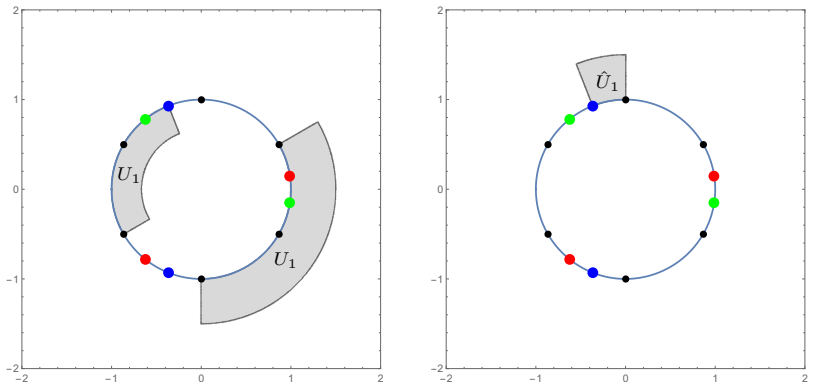

Figure 6. *The open subsets U_1 and $\hat{U}_1$ of the complex k-plane.*

zeros at $k = \pm\omega$ and simple poles at $k = \pm\omega^2$, and r_1, r_2 have rapid decay as $|k| \to \infty$. Moreover,

$$r_1(\tfrac{1}{\omega k}) + r_2(\omega k) + r_1(\omega^2 k) r_2(\tfrac{1}{k}) = 0, \qquad k \in \partial\mathbb{D} \setminus \{\pm\omega\}, \tag{4.1}$$

$$r_2(k) = \tilde{r}(k)\overline{r_1(\bar{k}^{-1})}, \qquad \tilde{r}(k) := \frac{\omega^2 - k^2}{1 - \omega^2 k^2}, \qquad k \in \hat{\Gamma}_4 \setminus \{0, \pm\omega^2\}, \tag{4.2}$$

$$r_1(1) = r_1(-1) = 1, \qquad r_2(1) = r_2(-1) = -1. \tag{4.3}$$

Define $\hat{r}_1(k)$ and $\hat{r}_2(k)$ for $k \in \partial\mathbb{D}$ with $\arg k \in [\frac{\pi}{2}, \arg k_1]$ by

$$\hat{r}_j(k) := \frac{r_j(k)}{1 + r_1(k) r_2(k)} = \frac{r_j(k)}{1 + \tilde{r}(k)|r_1(k)|^2}, \qquad j = 1, 2, \tag{4.4}$$

where we have used (4.2) for the second equality. Since $\arg k_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$ for $\zeta > 1$, and since $\tilde{r}(e^{i\theta}) > 0$ for $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3})$, we have $1 + r_1(k) r_2(k) = 1 + \tilde{r}(k)|r_1(k)|^2 > 1$ for $|k| = 1, \arg k \in [\frac{\pi}{2}, \arg k_1]$. It follows that $\hat{r}_1$, $\hat{r}_2$ are well-defined. To implement the transformation $n \to n^{(1)}$, we need analytic approximations of the functions $r_1, r_2, \hat{r}_1$, and $\hat{r}_2$.

For $K > 1$, we define open sets $U_1 = U_1(\zeta, K), \hat{U}_1 = \hat{U}_1(\zeta, K) \subset \mathbb{C}$ as follows:

$$\begin{aligned} U_1 = \big(&\{k \,|\, \arg k \in [-\pi, -\tfrac{5\pi}{6}) \cup (\arg k_1, \pi],\ K^{-1} < |k| < 1\} \\ &\cup \{k \,|\, \arg k \in (-\tfrac{\pi}{2}, \tfrac{\pi}{6}),\ 1 < |k| < K\}\big) \cap \{k \,|\, \operatorname{Re}\Phi_{21} > 0\}, \\ \hat{U}_1 = &\{k \,|\, \arg k \in (\tfrac{\pi}{2}, \arg k_1),\ 1 < |k| < K\} \cap \{k \,|\, \operatorname{Re}\Phi_{21} > 0\}, \end{aligned}$$

see also Figure 6. Let $N \geq 1$ be an integer.

Lemma 4.1 (Decomposition lemma)**.** *There exist $K > 1$ and decompositions*

$$r_1(k) = r_{1,a}(x,t,k) + r_{1,r}(x,t,k), \qquad k \in \partial U_1 \cap \partial\mathbb{D},$$

$$\hat{r}_1(k) = \hat{r}_{1,a}(x,t,k) + \hat{r}_{1,r}(x,t,k), \qquad k \in \partial\hat{U}_1 \cap \partial\mathbb{D},$$

such that the functions $r_{1,a}, r_{1,r}, \hat{r}_{1,a}, \hat{r}_{1,r}$ have the following properties:

(a) *For each $\zeta \in \mathcal{I}$ and $t \geq 1$, $r_{1,a}(x,t,k)$ is defined and continuous for $k \in \bar{U}_1$ and analytic for $k \in U_1$, while $\hat{r}_{1,a}(x,t,k)$ is defined and continuous for $k \in \bar{\hat{U}}_1$ and analytic for $k \in \hat{U}_1$.*

(b) *For $\zeta \in \mathcal{I}$ and $t \geq 1$, the functions $r_{1,a}$ and $\hat{r}_{1,a}$ satisfy*

$$\Big| r_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{r_1^{(j)}(k_\star)}{j!}(k - k_\star)^j \Big| \leq C|k - k_\star|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \qquad k \in \bar{U}_1,\ k_\star \in \mathcal{R},$$

$$\left|\hat{r}_{1,a}(x,t,k) - \sum_{j=0}^{N} \frac{\hat{r}_1^{(j)}(k_\star)}{j!}(k-k_\star)^j\right| \le C|k-k_\star|^{N+1} e^{\frac{t}{4}|\operatorname{Re}\Phi_{21}(\zeta,k)|}, \qquad k \in \bar{\hat{U}}_1,\ k_\star \in \hat{\mathcal{R}},$$

where $\mathcal{R} = \{k_1, \pm\omega, \omega^2 k_2, \pm e^{\frac{\pi i}{6}}, \pm e^{-\frac{\pi i}{6}}, \pm 1, -i, \omega^2 k_1, \omega k_2\}$, $\hat{\mathcal{R}} = \{i, k_1\}$, *and the constant* C *is independent of* ζ, t, k.

(c) *For each* $1 \le p \le \infty$, *the* L^p*-norm of* $r_{1,r}(x,t,\cdot)$ *on* $\partial U_1 \cap \partial\mathbb{D}$ *is* $O(t^{-N})$ *and the* L^p*-norm of* $\hat{r}_{1,r}(x,t,\cdot)$ *on* $\partial\hat{U}_1 \cap \partial\mathbb{D}$ *is* $O(t^{-N})$ *uniformly for* $\zeta \in \mathcal{I}$ *as* $t \to \infty$.

Proof. On each connected component of $\partial U_1 \cap \partial\mathbb{D}$ and $\partial\hat{U}_1 \cap \partial\mathbb{D}$, the function $\theta \mapsto \operatorname{Im}\Phi_{21}(\zeta, e^{i\theta}) = (\zeta - \cos\theta)\sin\theta$ is monotone and $\operatorname{Re}\Phi_{21}$ is identically zero. We can thus prove the statement using the method from [19]. This method is standard by now, so we omit details. □

The above lemma establishes decompositions of r_1 and $\hat{r}_1$; we now obtain decompositions of r_2 and $\hat{r}_2$ using the symmetry (4.2). Let $U_2 := \{k|\bar{k}^{-1} \in U_1\}$ and $\hat{U}_2 := \{k|\bar{k}^{-1} \in \hat{U}_1\}$. We define decompositions $r_2 = r_{2,a} + r_{2,r}$ and $\hat{r}_2 = \hat{r}_{2,a} + \hat{r}_{2,r}$ by

$$r_{2,a}(k) := \tilde{r}(k)\overline{r_{1,a}(\bar{k}^{-1})}, \quad k \in U_2, \qquad r_{2,r}(k) := \tilde{r}(k)\overline{r_{1,r}(\bar{k}^{-1})}, \quad k \in \partial U_2 \cap \partial\mathbb{D},$$
$$\hat{r}_{2,a}(k) := \tilde{r}(k)\overline{\hat{r}_{1,a}(\bar{k}^{-1})}, \quad k \in \hat{U}_2, \qquad \hat{r}_{2,r}(k) := \tilde{r}(k)\overline{\hat{r}_{1,r}(\bar{k}^{-1})}, \quad k \in \partial\hat{U}_2 \cap \partial\mathbb{D}.$$

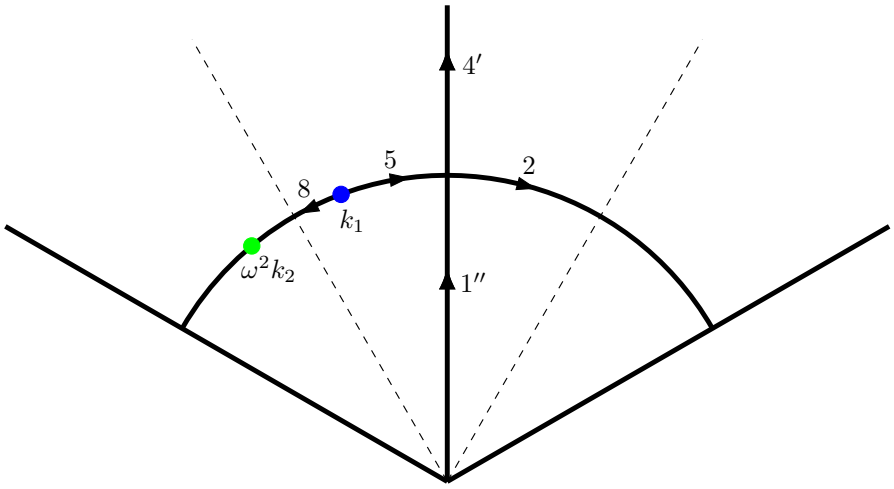

Figure 7. *The contour* $\Gamma^{(0)}$ *(solid), the boundary of* S *(dashed) and the saddle points* k_1 *(blue) and* $\omega^2 k_2$ *(green).*

We are now in a position to define the first transformation $n \to n^{(1)}$. As explained in Section 3, we will focus our attention on the sector S defined in (3.11). Let $\Gamma^{(0)}$ be as in Figure 7. Note that

$$\Gamma_8^{(0)} := \{e^{i\theta} \,|\, \theta \in (\arg k_1, \tfrac{2\pi}{3})\}, \qquad \Gamma_2^{(0)} := \{e^{i\theta} \,|\, \theta \in (\tfrac{\pi}{3}, \tfrac{\pi}{2})\}.$$

The contour $\Gamma^{(0)}$ coincides as a set with $\Gamma \cap \mathsf{S}$, but is oriented and labeled differently.

On $\Gamma_2^{(0)}$, we will use the factorization

$$v_9 = v_3^{(1)} v_2^{(1)} v_1^{(1)},$$

where

$$v_3^{(1)} = \begin{pmatrix} 1 & 0 & -r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ r_{1,a}(\frac{1}{k})e^{\theta_{21}} & 1 & r_{1,a}(\omega k)e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_1^{(1)} = \begin{pmatrix} 1 & r_{2,a}(\frac{1}{k})e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ -r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{2,a}(\omega k)e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_2^{(1)} = I + v_{2,r}^{(1)}, \qquad v_{2,r}^{(1)} = \begin{pmatrix} r_{1,r}(\omega^2 k)r_{2,r}(\omega^2 k) & g_2(\omega k)e^{-\theta_{21}} & -r_{2,r}(\omega^2 k)e^{-\theta_{31}} \\ g_1(\omega k)e^{\theta_{21}} & g(\omega k) & h_1(\omega k)e^{-\theta_{32}} \\ -r_{1,r}(\omega^2 k)e^{\theta_{31}} & h_2(\omega k)e^{\theta_{32}} & 0 \end{pmatrix}, \tag{4.5}$$

and

$$\begin{aligned} h_1(k) &= r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k), \qquad h_2(k) = r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k})r_{1,r}(\omega k), \\ g_1(k) &= r_{1,r}(\tfrac{1}{\omega^2 k}) - r_{1,r}(\omega k)\big(r_{1,r}(k) + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k)\big), \\ g_2(k) &= r_{2,r}(\tfrac{1}{\omega^2 k}) - r_{2,r}(\omega k)\big(r_{2,r}(k) + r_{2,a}(\tfrac{1}{\omega^2 k})r_{1,r}(\omega k)\big), \\ g(k) &= r_{1,r}(k)\big(r_{1,r}(\omega k)r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) \\ &\quad + r_{1,a}(\tfrac{1}{\omega^2 k})r_{2,r}(\omega k)\big(r_{1,r}(\omega k)r_{2,a}(\tfrac{1}{\omega^2 k}) + r_{2,r}(k)\big) + r_{1,r}(\tfrac{1}{\omega^2 k})r_{2,r}(\tfrac{1}{\omega^2 k}). \end{aligned}$$

On $\Gamma_5^{(0)}$, we will use the factorization

$$v_7^{-1} = v_6^{(1)}v_5^{(1)}v_4^{(1)}, \qquad v_5^{(1)} = \begin{pmatrix} 1 + r_1(k)r_2(k) & 0 & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} & 0 \\ 0 & 0 & 1 \end{pmatrix} + v_{5,r}^{(1)},$$

$$v_6^{(1)} = \begin{pmatrix} 1 & 0 & r_{1,a}(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ \hat{r}_{2,a}(k)e^{\theta_{21}} & 1 & -r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \; v_4^{(1)} = \begin{pmatrix} 1 & \hat{r}_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ r_{2,a}(\frac{1}{\omega^2 k})e^{\theta_{31}} & -r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \tag{4.6}$$

where we have used (4.1). Finally, on $\Gamma_8^{(0)}$, we will use the factorization

$$v_7 = v_7^{(1)}v_8^{(1)}v_9^{(1)}, \quad v_8^{(1)} = I + v_{8,r}^{(1)}, \quad v_{8,r}^{(1)} = \begin{pmatrix} 0 & -r_{1,r}(k)e^{-\theta_{21}} & h_2(\omega^2 k)e^{-\theta_{31}} \\ -r_{2,r}(k)e^{\theta_{21}} & r_{1,r}(k)r_{2,r}(k) & g_2(\omega^2 k)e^{-\theta_{32}} \\ h_1(\omega^2 k)e^{\theta_{31}} & g_1(\omega^2 k)e^{\theta_{32}} & g(\omega^2 k) \end{pmatrix},$$

$$v_7^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -r_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ r_{1,a}(\omega^2 k)e^{\theta_{31}} & r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix}, \quad v_9^{(1)} = \begin{pmatrix} 1 & -r_{1,a}(k)e^{-\theta_{21}} & r_{2,a}(\omega^2 k)e^{-\theta_{31}} \\ 0 & 1 & r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.7}$$

We omit the long expression for $v_{5,r}^{(1)}$, but note that Lemma 4.1 implies that

$$\|v_{j,r}^{(1)}\|_{(L^1\cap L^\infty)(\Gamma_j^{(0)})} = O(t^{-1}) \qquad \text{as } t \to \infty, \; j = 2, 5, 8.$$

Let $\Gamma^{(1)}$ be the contour shown in Figure 8. The function $n^{(1)}$ is defined by

$$n^{(1)}(x, t, k) = n(x, t, k)G^{(1)}(x, t, k), \qquad k \in \mathbb{C} \setminus \Gamma^{(1)}, \tag{4.8}$$

where $G^{(1)}$ is analytic in $\mathbb{C} \setminus \Gamma^{(1)}$. It is given for $k \in \mathsf{S}$ by

$$G^{(1)} = \begin{cases} v_3^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_2^{(1)}, \\ (v_1^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_2^{(1)}, \\ v_6^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_5^{(1)}, \\ (v_4^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_5^{(1)}, \end{cases} \; G^{(1)} = \begin{cases} v_7^{(1)}, & k \text{ on the } - \text{ side of } \Gamma_8^{(1)}, \\ (v_9^{(1)})^{-1}, & k \text{ on the } + \text{ side of } \Gamma_8^{(1)}, \\ I, & \text{otherwise}, \end{cases} \tag{4.9}$$

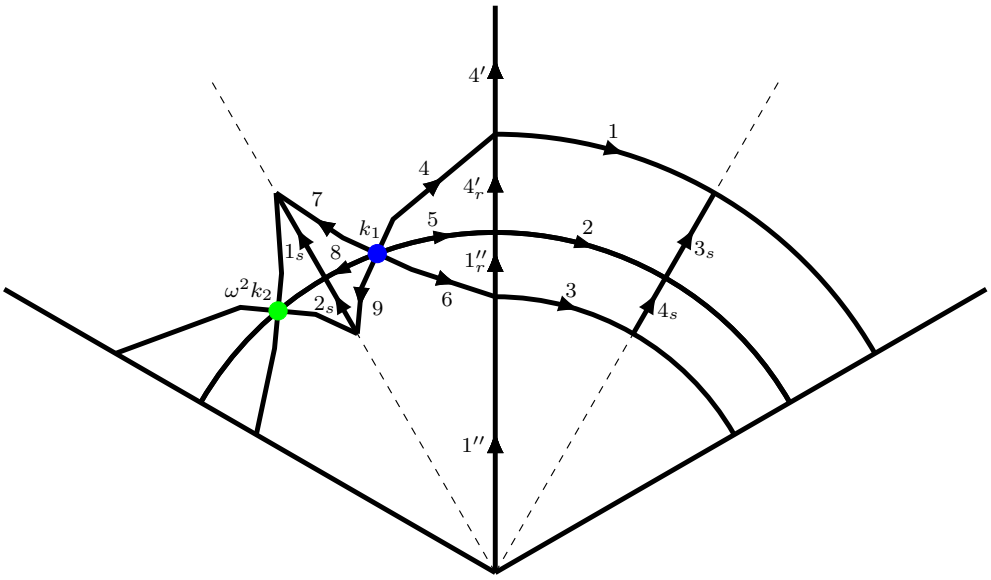

Figure 8. *The contour $\Gamma^{(1)}$ (solid), the boundary of S (dashed), and the saddle points k_1 (blue) and $\omega^2 k_2$ (green).*

and extended to $\mathbb{C}\setminus\Gamma^{(1)}$ by

$$G^{(1)}(x,t,k) = \mathcal{A}G^{(1)}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}G^{(1)}(x,t,k^{-1})\mathcal{B}. \tag{4.10}$$

The next lemma follows from the signature tables of Figure 5.

Lemma 4.2. *$G^{(1)}(x,t,k)$ and $G^{(1)}(x,t,k)^{-1}$ are uniformly bounded for $k\in\mathbb{C}\setminus\Gamma^{(1)}$, $t\geq 1$, and $\zeta\in\mathcal{I}$. Furthermore, $G^{(1)}(x,t,k)=I$ for all large enough $|k|$.*

Using (4.8), we infer that $n_+^{(1)} = n_-^{(1)}v_j^{(1)}$ on $\Gamma_j^{(1)}$, where $v_j^{(1)}$ is given by (4.5) for $j=1,2,3$, by (4.6) for $j=4,5,6$, by (4.7) for $j=7,8,9$, and

$$v_{1''}^{(1)} = v_{1''}, \quad v_{4'}^{(1)} = v_{4'}, \quad v_{1''_r}^{(1)} = (v_3^{(1)})^{-1}v_{1''}v_6^{(1)}, \quad v_{4'_r}^{(1)} = v_1^{(1)}v_{4'}(v_4^{(1)})^{-1},$$

$$v_{1_s}^{(1)} = G_-^{(1)}(k)^{-1}G_+^{(1)}(k) = v_7^{(1)}(k)^{-1}\mathcal{A}\mathcal{B}v_9^{(1)}(\tfrac{1}{\omega k})^{-1}\mathcal{B}\mathcal{A}^{-1} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ (v_{1_s}^{(1)})_{31} & 0 & 1\end{pmatrix}, \tag{4.11}$$

$$(v_{1_s}^{(1)})_{31} := -\big(r_{1,a}(\omega^2 k) + r_{2,a}(k)r_{1,a}(\tfrac{1}{\omega k}) + r_{2,a}(\tfrac{1}{\omega^2 k})\big)e^{\theta_{31}}.$$

In general, $(v_{1_s}^{(1)})_{31}\neq 0$ for $k\in\Gamma_{1_s}^{(1)}$, even though $r_1(\omega^2 k)+r_2(k)r_1(\frac{1}{\omega k})+r_2(\frac{1}{\omega^2 k})=0$ for $k\in\partial\mathbb{D}$ by (4.1). The matrices $v_{2_s}, v_{3_s}, v_{4_s}$ are similar to v_{1_s} so their explicit expressions are omitted.

5. The $n^{(1)}\to n^{(2)}$ transformation

For the second transformation, we will use the following factorizations (the matrices with subscripts u and d will be used to deform contours up and down, respectively, see (5.3) below):

$$v_4^{(1)} = v_4^{(2)}v_{4u}^{(2)}, \quad v_4^{(2)} = \begin{pmatrix} 1 & \hat{r}_{1,a}(k)e^{-\theta_{21}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad v_{4u}^{(2)} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ r_{2,a}(\frac{1}{\omega^2 k})e^{\theta_{31}} & -r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1\end{pmatrix},$$

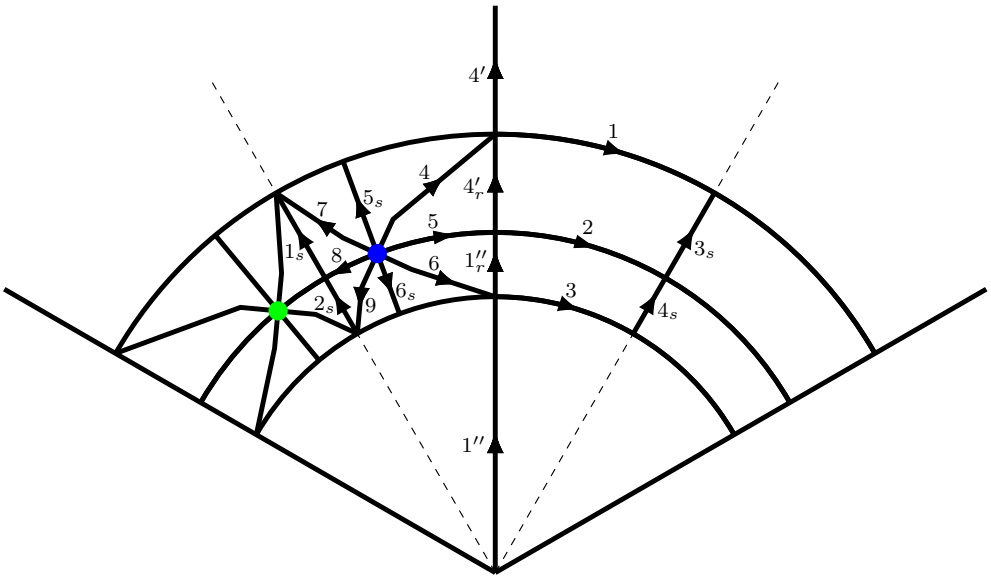

Figure 9. *The contour $\Gamma^{(2)}$ (solid), the boundary of S (dashed), and the saddle points k_1 (blue) and $\omega^2 k_2$ (green).*

$$v_6^{(1)} = v_{6d}^{(2)} v_6^{(2)}, \quad v_{6d}^{(2)} = \begin{pmatrix} 1 & 0 & r_{1,a}(\frac{1}{\omega^2 k})e^{-\theta_{31}} \\ 0 & 1 & -r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}, \quad v_6^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \hat{r}_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_7^{(1)} = v_{7u}^{(2)} v_7^{(2)}, \quad v_7^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ -r_{2,a}(k)e^{\theta_{21}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_9^{(1)} = v_9^{(2)} v_{9d}^{(2)}, \quad v_9^{(2)} = \begin{pmatrix} 1 & -r_{1,a}(k)e^{-\theta_{21}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v_{7u}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (r_{1,a}(\omega^2 k) + r_{1,a}(\frac{1}{\omega k})r_{2,a}(k))e^{\theta_{31}} & r_{1,a}(\frac{1}{\omega k})e^{\theta_{32}} & 1 \end{pmatrix},$$

$$v_{9d}^{(2)} = \begin{pmatrix} 1 & 0 & (r_{2,a}(\omega^2 k) + r_{1,a}(k)r_{2,a}(\frac{1}{\omega k}))e^{-\theta_{31}} \\ 0 & 1 & r_{2,a}(\frac{1}{\omega k})e^{-\theta_{32}} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

Let $\Gamma^{(2)}$ be the contour shown in Figure 9. The function $n^{(2)}$ is defined by

$$n^{(2)}(x,t,k) = n^{(1)}(x,t,k) G^{(2)}(x,t,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(2)}, \tag{5.2}$$

where $G^{(2)}$ is given for $k \in \mathsf{S}$ by

$$G^{(2)}(x,t,k) = \begin{cases} (v_{4u}^{(2)})^{-1}, & k \text{ above } \Gamma_4^{(2)}, \\ v_{6d}^{(2)}, & k \text{ below } \Gamma_6^{(2)}, \\ v_{7u}^{(2)}, & k \text{ above } \Gamma_7^{(2)}, \\ (v_{9d}^{(2)})^{-1}, & k \text{ below } \Gamma_9^{(2)}, \\ I, & \text{otherwise}, \end{cases} \tag{5.3}$$

and $G^{(2)}$ is defined to $\mathbb{C} \setminus \Gamma^{(2)}$ using the $\mathcal{A}$- and $\mathcal{B}$-symmetries (as in (4.10)). The following lemma follows from Lemma 4.1 and Figure 5.

Lemma 5.1. *$G^{(2)}(x,t,k)$ and $G^{(2)}(x,t,k)^{-1}$ are uniformly bounded for $k \in \mathbb{C} \setminus \Gamma^{(2)}$, $t \geq 1$, and $\zeta \in \mathcal{I}$. Furthermore, $G^{(2)}(x,t,k) = I$ for all large enough $|k|$.*

The jumps $v_j^{(2)}$ of $n^{(2)}$ are given for $j = 4, 6, 7, 9$ by (5.1) and by

$$v_j^{(2)} := v_j^{(1)},\ j = 1, 2, 3, 5, 8, 1_s, 2_s, 3_s, 4_s, \qquad v_{5_s}^{(2)} := v_{4u}^{(2)} v_{7u}^{(2)}, \qquad v_{6_s}^{(2)} := v_{9d}^{(2)} v_{6d}^{(2)}. \tag{5.4}$$

On the subcontours of $\Gamma^{(2)} \cap \mathsf{S}$ that are unlabeled in Figure 9, the matrix $v^{(2)}$ is close to I as $t \to \infty$, so we do not write it down. On $\Gamma^{(2)} \setminus \mathsf{S}$, $v^{(2)}$ can be computed using (3.9). The proof of the next lemma is similar to [8, Proof of Lemma 8.5], so we omit it.

Lemma 5.2. *The L^∞-norm of $v^{(2)} - I$ on $\Gamma^{(2)}_{4'_r} \cup \Gamma^{(2)}_{1''_r}$ is $O(t^{-N})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$.*

As in [11, Lemma 6.3], we can (and do) choose the analytic approximations of Lemma 4.1 so that the L^∞-norm of $v_j^{(2)} - I$ on $\Gamma_j^{(2)}$, $j = 1_s, \dots, 6_s$, is $O(t^{-N})$ as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$. Thus $v^{(2)} - I = O(t^{-N})$ as $t \to \infty$, uniformly for $k \in \Gamma^{(2)} \setminus \big(\cup_{j=0}^2 \cup_{\ell=1}^2 D_\epsilon(\omega^j k_\ell) \cup \partial\mathbb{D}\big)$.

6. The $n^{(2)} \to n^{(3)}$ transformation

The goal of the transformation $n^{(2)} \to n^{(3)}$ is to make $v^{(3)} - I$ uniformly small as $t \to \infty$ for $k \in \partial\mathbb{D}$.

For each $\zeta \in \mathcal{I}$, we seek an analytic function $\delta(\zeta, \cdot) : \mathbb{C} \setminus \Gamma_5^{(2)} \to \mathbb{C}$ obeying the jump condition

$$\delta_+(\zeta, k) = \delta_-(\zeta, k)(1 + r_1(k) r_2(k)), \qquad k \in \Gamma_5^{(2)},$$

and the normalization condition $\delta(\zeta, k) = 1 + O(k^{-1})$ as $k \to \infty$. As mentioned previously (see the text below (4.4)), $1 + r_1(k) r_2(k) > 1$ for all $k \in \Gamma_5^{(2)}$. Hence, by taking the logarithm and using Plemelj's formula, we find that

$$\delta(\zeta, k) = \exp\left\{ \frac{-1}{2\pi i} \int_i^{k_1} \frac{\ln(1 + r_1(s) r_2(s))}{s - k} ds \right\}, \qquad k \in \mathbb{C} \setminus \Gamma_5^{(2)}, \tag{6.1}$$

where the principal branch is taken for the logarithm, and where the path of integration follows the unit circle in the counterclockwise direction.

Lemma 6.1. *The function $\delta(\zeta, k)$ has the following properties:*

(a) *$\delta(\zeta, k)$ can be written as*

$$\delta(\zeta, k) = e^{-i\nu \ln_{k_1}(k - k_1)} e^{-\chi(\zeta, k)}, \tag{6.2}$$

$$\chi(\zeta, k) = \frac{-1}{2\pi i} \int_i^{k_1} \ln_s(k - s) d\ln(1 + r_1(s) r_2(s)), \tag{6.3}$$

where $\nu \le 0$ is defined in (2.9) and the path of integration follows the unit circle in the counterclockwise direction. For $s \in \Gamma_5^{(2)}$, $k \mapsto \ln_s(k - s) = \ln|k - s| + i \arg_s(k - s)$ has a cut along $\{e^{i\theta} \,|\, \theta \in [\frac{\pi}{2}, \arg s],\ \arg s \in [\frac{\pi}{2}, \frac{2\pi}{3}]\} \cup (i, i\infty)$ and $\arg_s(1) = 2\pi$.

(b) *For each $\zeta \in \mathcal{I}$, $\delta(\zeta, k)$ and $\delta(\zeta, k)^{-1}$ are analytic functions of $k \in \mathbb{C} \setminus \Gamma_5^{(2)}$. Moreover,*

$$\sup_{\zeta \in \mathcal{I}} \sup_{k \in \mathbb{C} \setminus \Gamma_5^{(2)}} |\delta(\zeta, k)^{\pm 1}| < \infty.$$

(c) *For $k \in \partial\mathbb{D} \setminus \Gamma_5^{(2)}$,*

$$\operatorname{Re} \chi(\zeta, k) = \frac{\pi + \arg_i k}{2} \nu + \frac{1}{2\pi} \int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(1 + r_1(s) r_2(s)), \tag{6.4}$$

where $\arg_i k \in (\frac{\pi}{2}, \frac{5\pi}{2})$ *and* $\arg s \in (-\pi, \pi)$. *In particular, for* $k \in \partial\mathbb{D} \setminus \Gamma_5^{(2)}$, $|\delta(\zeta, k)|$ *is constant and given by*

$$|\delta(\zeta,k)| = \exp\left(\nu\frac{\arg k_1}{2} - \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\arg s}{2} d\ln(1 + r_1(s)r_2(s))\right), \tag{6.5}$$

where $\arg k_1 \in [\frac{\pi}{2}, \frac{2\pi}{3}]$.

(*d*) *As* $k \to k_1$ *along a path which is nontangential to* $\Gamma_5^{(2)}$, *we have*

$$|\chi(\zeta,k) - \chi(\zeta,k_1)| \le C|k-k_1|(1 + |\ln|k-k_1||), \tag{6.6}$$

where C *is independent of* $\zeta \in \mathcal{I}$.

Proof. Part (*a*) follows from (6.1) and an integration by parts. Part (*b*) follows from (6.2) and the fact that $\operatorname{Im}\nu = 0$. For part (*c*), we note from (6.3) that, for $k \in \partial\mathbb{D} \setminus \Gamma_5^{(2)}$,

$$\begin{aligned}\operatorname{Re}\chi(\zeta,k) &= \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \arg_s(k-s) d\ln(1 + r_1(s)r_2(s)) \\ &= \frac{1}{2\pi}\int_{\Gamma_5^{(2)}} \frac{\pi + \arg_i k + \arg s}{2} d\ln(1 + r_1(s)r_2(s)),\end{aligned}$$

and (6.4) now follows from the definition (2.9) of ν and the assumption that $r_1 = 0$ on $[0, i]$. By (6.2),

$$|\delta(\zeta,k)| = e^{\nu \arg_{k_1}(k-k_1)} e^{-\operatorname{Re}\chi(\zeta,k)}.$$

After substituting (6.4) and $\arg_{k_1}(k-k_1) = \frac{\pi + \arg_i k + \arg k_1}{2}$ into the above equation, we find (6.5). Part (*d*) follows from (6.3) and standard estimates. □

For $k \in \mathbb{C} \setminus \partial\mathbb{D}$, we define

$$\Delta_{33}(\zeta,k) = \frac{\delta(\zeta,\omega k)}{\delta(\zeta,\omega^2 k)} \frac{\delta(\zeta,\frac{1}{\omega^2 k})}{\delta(\zeta,\frac{1}{\omega k})} \mathcal{P}(\zeta,k), \quad \Delta_{11}(\zeta,k) = \Delta_{33}(\zeta,\omega k), \quad \Delta_{22}(\zeta,k) = \Delta_{33}(\zeta,\omega^2 k),$$

where we recall that $\mathcal{P}$ is defined in (2.15). Note that $\mathcal{P}$ and Δ_{33} admit the symmetries

$$\mathcal{P}(\zeta,k) = \mathcal{P}(\zeta,\tfrac{1}{k}) = \overline{\mathcal{P}(\zeta,\bar{k})}^{-1}, \qquad \Delta_{33}(\zeta,k) = \Delta_{33}(\zeta,\tfrac{1}{k}) = \overline{\Delta_{33}(\zeta,\bar{k})}^{-1}. \tag{6.7}$$

The function Δ defined by

$$\Delta(\zeta,k) = \begin{pmatrix} \Delta_{11}(\zeta,k) & 0 & 0 \\ 0 & \Delta_{22}(\zeta,k) & 0 \\ 0 & 0 & \Delta_{33}(\zeta,k) \end{pmatrix}, \quad \zeta \in \mathcal{I},\ k \in \mathbb{C} \setminus \partial\mathbb{D}, \tag{6.8}$$

obeys the symmetries

$$\Delta(\zeta,k) = \mathcal{A}\Delta(\zeta,\omega k)\mathcal{A}^{-1} = \mathcal{B}\Delta(\zeta,\tfrac{1}{k})\mathcal{B} = \mathcal{B}\overline{\Delta(\zeta,\bar{k})}^{-1}\mathcal{B}, \tag{6.9}$$

and satisfies

$$\Delta_+^{-1}(\zeta,k) = \Delta_-^{-1}(\zeta,k) \begin{pmatrix} 1 + r_1(k)r_2(k) & 0 & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad k \in \Gamma_5^{(2)}, \tag{6.10a}$$

$$\Delta_+^{-1}(\zeta,k) = \Delta_-^{-1}(\zeta,k), \qquad k \in \Gamma_2^{(2)} \cup \Gamma_8^{(2)}. \tag{6.10b}$$

The next lemma follows from (6.8) and Lemma 6.1.

Lemma 6.2. *For any* $\epsilon > 0$, $\Delta(\zeta,k)$ *and* $\Delta(\zeta,k)^{-1}$ *are uniformly bounded for* $k \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \cup_{k_0 \in \hat{\mathsf{Z}}} D_\epsilon(k_0))$ *and* $\zeta \in \mathcal{I}$. *Furthermore,* $\Delta(\zeta,k) = I + O(k^{-1})$ *as* $k \to \infty$.

We define $n^{(3)}$ by

$$n^{(3)}(x,t,k) = n^{(2)}(x,t,k)\Delta(\zeta,k), \qquad k \in \mathbb{C} \setminus \Gamma^{(3)}, \tag{6.11}$$

where $\Gamma^{(3)} = \Gamma^{(2)}$. The boundary values of $n^{(3)}$ obey $n^{(3)}_+ = n^{(3)}_- v^{(3)}$ on $\Gamma^{(3)}$, where $v^{(3)} = \Delta_-^{-1} v^{(2)} \Delta_+$. Let $\mathcal{X}^\epsilon := \cup_{j=1}^4 \mathcal{X}_j^\epsilon$ be a small cross centered at k_1, where

$$\mathcal{X}_1^\epsilon = \Gamma_4^{(3)} \cap D_\epsilon(k_1), \quad \mathcal{X}_2^\epsilon = \Gamma_7^{(3)} \cap D_\epsilon(k_1), \quad \mathcal{X}_3^\epsilon = \Gamma_9^{(3)} \cap D_\epsilon(k_1), \quad \mathcal{X}_4^\epsilon = \Gamma_6^{(3)} \cap D_\epsilon(k_1),$$

are oriented outwards from k_1. Let $\hat{\mathcal{X}}^\epsilon = \mathcal{X}^\epsilon \cup \omega\mathcal{X}^\epsilon \cup \omega^2\mathcal{X}^\epsilon \cup (\mathcal{X}^\epsilon)^{-1} \cup (\omega\mathcal{X}^\epsilon)^{-1} \cup (\omega^2\mathcal{X}^\epsilon)^{-1}$ be the union of six small crosses centered at $\{\omega^j k_1, \omega^j k_2\}_{j=0}^2$.

Lemma 6.3. *$v^{(3)}$ converges to the identity matrix I as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}$ and $k \in \Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon$. More precisely, for $\zeta \in \mathcal{I}$,*

$$\|v^{(3)} - I\|_{(L^1 \cap L^\infty)(\Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon)} \le Ct^{-1}. \tag{6.12}$$

Proof. Using (4.5), (4.6), (4.7), and (5.4) together with the properties (6.10) of Δ, we infer that

$$v_j^{(3)} = \Delta_-^{-1} v_j^{(1)} \Delta_+ = I + \Delta_-^{-1} v_{j,r}^{(1)} \Delta_+, \quad j = 2,5,8. \tag{6.13}$$

By Lemmas 4.1 and 6.2, the matrices $\Delta_-^{-1} v_{j,r}^{(1)} \Delta_+$, $j = 2,5,8$, are small as $t \to \infty$. Using also Lemma 5.2, we infer that the L^1 and L^∞ norms of $v^{(3)} - I$ are $O(t^{-N})$ as $t \to \infty$, uniformly for $\zeta \in \mathcal{I}$ and $k \in \mathsf{S} \cap \Gamma^{(3)} \setminus \hat{\mathcal{X}}^\epsilon$. The claim (6.12) now follows from (3.9). □

The next lemma, which follows from (3.6)–(3.7) and the definition (6.11) of $n^{(3)}$, establishes the residue conditions of $n^{(3)}$ at the points in Z.

Lemma 6.4 (Residue conditions for $n^{(3)}$)**.** *At each point of $\hat{\mathsf{Z}}$, one entry of $n^{(3)}$ has a simple pole while two entries are analytic. Moreover, the following residue conditions hold at the points in Z (suppressing the (x,t)-dependence for conciseness):*

(a) For each $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ with $\operatorname{Re} \Phi_{31}(\zeta, k_0) < 0$,

$$\operatorname*{Res}_{k=k_0} n_1^{(3)}(k) = \frac{c_{k_0}^{-1} e^{\theta_{31}(k_0)} n_3^{(3)}(k_0)}{\Delta_{33}'(k_0)(\Delta_{11}^{-1})'(k_0)}, \qquad \operatorname*{Res}_{k=\bar{k}_0} n_3^{(3)}(k) = \frac{d_{k_0}^{-1} e^{-\theta_{32}(\bar{k}_0)} n_2^{(3)}(\bar{k}_0)}{\Delta_{22}'(\bar{k}_0)(\Delta_{33}^{-1})'(\bar{k}_0)}.$$

(b) For each $k_0 \in \mathsf{Z} \cap \mathbb{R}$ with $\operatorname{Re} \Phi_{21}(\zeta, k_0) < 0$,

$$\operatorname*{Res}_{k=k_0} n_1^{(3)}(k) = \frac{c_{k_0}^{-1} e^{\theta_{21}(k_0)} n_2^{(3)}(k_0)}{\Delta_{22}'(k_0)(\Delta_{11}^{-1})'(k_0)}.$$

(c) For each $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ with $\operatorname{Re} \Phi_{31}(\zeta, k_0) \ge 0$,

$$\operatorname*{Res}_{k=k_0} n_3^{(3)}(k) = \frac{c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(3)}(k_0)}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)}, \qquad \operatorname*{Res}_{k=\bar{k}_0} n_2^{(3)}(k) = \frac{d_{k_0} e^{\theta_{32}(\bar{k}_0)} n_3^{(3)}(\bar{k}_0)}{\Delta_{33}(\bar{k}_0)\Delta_{22}^{-1}(\bar{k}_0)}.$$

(d) For each $k_0 \in \mathsf{Z} \cap \mathbb{R}$ with $\operatorname{Re} \Phi_{21}(\zeta, k_0) \ge 0$,

$$\operatorname*{Res}_{k=k_0} n_2^{(3)}(k) = \frac{c_{k_0} e^{-\theta_{21}(k_0)} n_1^{(3)}(k_0)}{\Delta_{11}(k_0)\Delta_{22}^{-1}(k_0)}.$$

The residue conditions of $n^{(3)}$ at the points of $\hat{\mathsf{Z}} \setminus \mathsf{Z}$ can be obtained from those in Lemma 6.4 and the $\mathcal{A}$- and $\mathcal{B}$-symmetries (3.5).

7. The $n^{(3)} \to n^{(4)}$ transformation

The purpose of the transformation $n^{(3)} \to n^{(4)}$ is to remove the poles of $n^{(3)}$ at the points of $\hat{\mathsf{Z}}$. Let

$$\hat{M}_{\rm sol}(x,t,k) := M_{\rm sol}(x,t,k)\mathsf{P}(\zeta,k), \tag{7.1}$$

where $M_{\rm sol}(x,t,k)$ is the unique solution of RH problem 2.7 and

$$\mathsf{P}(\zeta,k) := \operatorname{diag}\left(\mathcal{P}(\zeta,\omega k), \mathcal{P}(\zeta,\omega^2 k), \mathcal{P}(\zeta,k)\right). \tag{7.2}$$

We define $n^{(4)}$ by

$$n^{(4)}(x,t,k) = n^{(3)}(x,t,k)\hat{M}_{\rm sol}(x,t,k)^{-1}, \qquad k \in \mathbb{C} \setminus \Gamma^{(4)}, \tag{7.3}$$

where $\Gamma^{(4)} = \Gamma^{(3)}$. We deduce from (6.7) and (2.30) that

$$\hat{M}_{\rm sol}(x,t,k) = \mathcal{A}\hat{M}_{\rm sol}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\hat{M}_{\rm sol}(x,t,k^{-1})\mathcal{B}, \tag{7.4}$$

and hence $n^{(4)}$ obeys the $\mathcal{A}$- and $\mathcal{B}$-symmetries in (3.10). The jump matrix for $n^{(4)}$ is given by $v^{(4)} = \hat{M}_{\rm sol} n^{(4)} \hat{M}_{\rm sol}^{-1}$.

Lemma 7.1. *At each point of* Z*, one column of* $\hat{M}_{\rm sol}$ *has a simple pole while two columns are analytic. Moreover,* $\hat{M}_{\rm sol}(x,t,k)$ *satisfies the residue conditions in Lemma 6.4 with* $n_j^{(3)}$ *replaced by* $[\hat{M}_{\rm sol}(x,t,k)]_j$ *for* $j = 1,2,3$.

Proof. We see from (2.28) and (6.8) that $\hat{\Delta}(\zeta,k) = \Delta(\zeta,k)\mathsf{P}(\zeta,k)^{-1}$ and clearly $\hat{\Delta}$ has no zeros or poles in $\hat{\mathsf{Z}}$. Let $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ be such that $\operatorname{Re}\Phi_{31}(\zeta,k_0) < 0$. Then $\mathsf{P}_{11}(\zeta,k)$ has a simple pole at k_0, $\mathsf{P}_{33}(\zeta,k)$ has a simple zero at k_0, and $\mathsf{P}_{22}(\zeta,k)$ has neither a zero nor a pole at k_0. Consequently, in view of part (*d*) of RH problem 2.7, $[\hat{M}_{\rm sol}]_j = \mathsf{P}_{jj}[M_{\rm sol}]_j$ has a simple pole at k_0 for $j = 1$, while it is analytic for $j = 2$ and $j = 3$. Moreover, using (2.32) and omitting the (x,t)-dependence for brevity, we find

$$\begin{aligned}\operatorname*{Res}_{k=k_0}[\hat{M}_{\rm sol}(k)]_1 &= \operatorname*{Res}_{k=k_0}\mathsf{P}_{11}(k)[M_{\rm sol}(k_0)]_1 = \frac{1}{(\mathsf{P}_{11}^{-1})'(k_0)}\frac{c_{k_0}^{-1}e^{\theta_{31}(k_0)}}{\hat{\Delta}_{11}^{-1}(k_0)\hat{\Delta}_{33}(k_0)}\operatorname*{Res}_{k=k_0}[M_{\rm sol}(k)]_3 \\ &= \frac{c_{k_0}^{-1}e^{\theta_{31}(k_0)}}{(\mathsf{P}_{11}^{-1})'(k_0)\hat{\Delta}_{11}^{-1}(k_0)\hat{\Delta}_{33}(k_0)}\frac{[\hat{M}_{\rm sol}(k_0)]_3}{\mathsf{P}_{33}'(k_0)} = \frac{c_{k_0}^{-1}e^{\theta_{31}(k_0)}[\hat{M}_{\rm sol}(k_0)]_3}{(\Delta_{11}^{-1})'(k_0)\Delta_{33}'(k_0)},\end{aligned}$$

which proves the claim for $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ with $\operatorname{Re}\Phi_{31}(\zeta,k_0) < 0$. The other points in Z can be treated in a similar way. □

It follows from Lemma 7.1 and the symmetries (7.4) that $(1,1,1)\hat{M}_{\rm sol}$ has the same pole structure and obeys the same residue conditions as $n^{(3)}$. The following lemma can be viewed as a consequence of this fact.

Lemma 7.2. *The function* $n^{(4)}$ *is analytic at every point of* $\hat{\mathsf{Z}}$.

Proof. We give the details in the case of points $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ such that $\operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0$; the other points in Z can be handled similarly and then the $\mathcal{A}$- and $\mathcal{B}$-symmetries yields analyticity also at the points in $\hat{\mathsf{Z}} \setminus \mathsf{Z}$.

Let $k_0 \in \mathsf{Z} \setminus \mathbb{R}$ be such that $\operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0$. Since $\hat{M}_{\rm sol}$ has unit determinant, it follows from Lemma 6.4 with $n_j^{(3)}$ replaced by $[\hat{M}_{\rm sol}(x,t,k)]_j$ that

$$\hat{M}_{\rm sol}^{-1}(k) = \begin{pmatrix} -\frac{c_{k_0}e^{-\theta_{31}(k_0)}}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)(k-k_0)}V + O(1) \\ O(1) \\ V + O(k-k_0) \end{pmatrix} \qquad \text{as } k \to k_0,$$

where the row-vector $V = (V_1, V_2, V_3) \in \mathbb{C}^3$ is the (transpose of the) cross product of the two vectors $[\hat{M}_{\rm sol}(k_0)]_1$ and $[\hat{M}_{\rm sol}(k_0)]_2$, i.e.,

$$V_1 = (M_{\rm sol}(k_0))_{21}(M_{\rm sol}(k_0))_{32} - (M_{\rm sol}(k_0))_{22}(M_{\rm sol}(k_0))_{31},$$

etc. On the other hand, by Lemma 6.4,

$$n^{(3)}(k) = \begin{pmatrix} n_1^{(3)}(k_0) + O(k-k_0) & O(1) & \frac{c_{k_0} e^{-\theta_{31}(k_0)} n_1^{(3)}(k_0)}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)(k-k_0)} + O(1) \end{pmatrix} \qquad \text{as } k \to k_0.$$

Hence, as $k \to k_0$,

$$n^{(4)} = n^{(3)}\hat{M}_{\rm sol}^{-1} = \begin{pmatrix} n_1^{(3)}(k_0) + O(k-k_0) & O(1) & \frac{c_{k_0} e^{-\theta_{31}(k_0)}}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)(k-k_0)} n_1^{(3)}(k_0) + O(1) \end{pmatrix}$$
$$\times \begin{pmatrix} -\frac{c_{k_0} e^{-\theta_{31}(k_0)}}{\Delta_{11}(k_0)\Delta_{33}^{-1}(k_0)(k-k_0)} V + O(1) \\ O(1) \\ V + O(k-k_0) \end{pmatrix} = O(1),$$

showing that $n^{(4)}$ is analytic at k_0. □

8. Local parametrix near k_1

The matrix $v^{(4)} - I$ is not uniformly small on $\hat{\mathcal{X}}^\epsilon$. In this section, we construct a local parametrix around k_1 which approximates $n^{(4)}$ near k_1. The local parametrices near the five other saddle points $\{k_2, \omega k_1, \omega k_2, \omega^2 k_1, \omega^2 k_2\}$ will then be constructed using the $\mathcal{A}$- and $\mathcal{B}$-symmetries. In Section 9, we will prove that these parametrices approximate $n^{(4)}$ in $\cup_{j=0}^2 \big(D_\epsilon(\omega^j k_1) \cup D_\epsilon(\omega^j k_2)\big)$.

As $k \to k_1$, we have

$$\Phi_{21}(\zeta, k) - \Phi_{21}(\zeta, k_1) = \Phi_{21,k_1}(k-k_1)^2 + O((k-k_1)^3), \qquad \Phi_{21,k_1} = \frac{4 - 3k_1\zeta - k_1^3\zeta}{4k_1^4}.$$

We define $z = z(\zeta, t, k)$ by

$$z = z_\star \sqrt{t}(k-k_1)\hat{z}, \qquad \hat{z} = \sqrt{\frac{2i(\Phi_{21}(\zeta,k) - \Phi_{21}(\zeta,k_1))}{z_\star^2 (k-k_1)^2}},$$

where the principal branch is chosen for $\hat{z} = \hat{z}(\zeta, k)$, and

$$z_\star = \sqrt{2} e^{\frac{\pi i}{4}} \sqrt{\Phi_{21,k_1}}, \qquad -ik_1 z_\star > 0.$$

Note that $\hat{z}(\zeta, k_1) = 1$, and $t(\Phi_{21}(\zeta,k) - \Phi_{21}(\zeta,k_1)) = -\frac{iz^2}{2}$. Let $\epsilon > 0$ be fixed and so small that the map z is conformal from $D_\epsilon(k_1)$ to a neighborhood of 0. As $k \to k_1$,

$$z = z_\star \sqrt{t}(k-k_1)(1 + O(k-k_1)),$$

and for all $k \in D_\epsilon(k_1)$, we have

$$\ln_{k_1}(k-k_1) = \ln_0[z_\star(k-k_1)\hat{z}] - \ln \hat{z} - \ln z_\star$$

where $\ln_0(k) := \ln|k| + i \arg_0 k$, $\arg_0(k) \in (0, 2\pi)$, and ln is the principal logarithm. Decreasing $\epsilon > 0$ if needed, $k \mapsto \ln \hat{z}$ is analytic in $D_\epsilon(k_1)$, $\ln \hat{z} = O(k-k_1)$ as $k \to k_1$, and

$$\ln z_\star = \ln|z_\star| + i \arg z_\star = \ln|z_\star| + i\big(\tfrac{\pi}{2} - \arg k_1\big),$$

where $\arg k_1 \in (\frac{\pi}{2}, \frac{2\pi}{3})$. By (6.2), we have

$$\delta(\zeta, k) = e^{-i\nu \ln_{k_1}(k-k_1)} e^{-\chi(\zeta,k)} = e^{-i\nu \ln_0 z} t^{\frac{i\nu}{2}} e^{i\nu \ln \hat{z}} e^{i\nu \ln z_\star} e^{-\chi(\zeta,k)}.$$

From now on, for conciseness we will use the notation $z_{(0)}^{-i\nu} := e^{-i\nu \ln_0 z}$, $\hat{z}^{i\nu} := e^{i\nu \ln \hat{z}}$ and $z_\star^{i\nu} := e^{i\nu \ln z_\star}$, so that

$$\delta(\zeta,k) = z_{(0)}^{-i\nu} t^{\frac{i\nu}{2}} \hat{z}^{i\nu} z_\star^{i\nu} e^{-\chi(\zeta,k)}.$$

Using this, we find

$$\frac{\Delta_{11}(\zeta,k)}{\Delta_{22}(\zeta,k)} = \frac{\delta(\frac{1}{k})^2\delta(\omega k)\delta(\omega^2 k)}{\delta(k)^2\delta(\frac{1}{\omega^2 k})\delta(\frac{1}{\omega k})}\frac{\mathcal{P}(\zeta,\omega k)}{\mathcal{P}(\zeta,\omega^2 k)} = z_{(0)}^{2i\nu} d_0(\zeta,t) d_1(\zeta,k), \tag{8.1}$$

$$d_0(\zeta,t) = e^{2\chi(\zeta,k_1)} t^{-i\nu} z_\star^{-2i\nu} \frac{\delta(\frac{1}{k_1})^2\delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})}, \tag{8.2}$$

$$d_1(\zeta,k) = e^{2(\chi(\zeta,k)-\chi(\zeta,k_1))}\hat{z}^{-2i\nu}\frac{\delta(\frac{1}{k})^2\delta(\omega k)\delta(\omega^2 k)}{\delta(\frac{1}{\omega^2 k})\delta(\frac{1}{\omega k})}\left(\frac{\delta(\frac{1}{k_1})^2\delta(\omega k_1)\delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1})\delta(\frac{1}{\omega k_1})}\right)^{-1}\frac{\mathcal{P}(\zeta,\omega k)}{\mathcal{P}(\zeta,\omega^2 k)}.$$

Define

$$Y(\zeta,t) = d_0(\zeta,t)^{-\frac{\tilde{\sigma}_3}{2}}\tilde{r}(k_1)^{-\frac{1}{4}\tilde{\sigma}_3} e^{-\frac{t}{2}\Phi_{21}(\zeta,k_1)\tilde{\sigma}_3}\left(\tfrac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}\right)^{-\frac{1}{2}\tilde{\sigma}_3},$$

where $\tilde{\sigma}_3 = \operatorname{diag}(1,-1,0)$, and note that $\tilde{r}(k_1)^{\pm\frac{1}{4}} > 0$ (the choices of the branches of $d_0(\zeta,t)^{\frac{1}{2}}$ and $\left(\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}\right)^{\frac{1}{2}}$ are unimportant for the analysis below). For $k \in D_\epsilon(k_1)$, we define

$$\tilde{n}(x,t,k) = n^{(4)}(x,t,k)\hat{M}_{\mathrm{sol}}(x,t,k)Y(\zeta,t).$$

Let $\tilde{v}$ be the jump matrix of $\tilde{n}$, and let $\tilde{v}_j$ be the restriction of $\tilde{v}$ to $\Gamma_j^{(4)} \cap D_\epsilon(k_1)$. By (5.1), (6.11), and (8.1), we have

$$\tilde{v}_4 = \begin{pmatrix} 1 & \tilde{r}(k_1)^{\frac{1}{2}}\hat{r}_{1,a}(k)\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)} d_1^{-1} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_7 = \begin{pmatrix} 1 & 0 & 0 \\ -\tilde{r}(k_1)^{-\frac{1}{2}} r_{2,a}(k)\frac{\mathcal{P}(\zeta,\omega^2 k_1)}{\mathcal{P}(\zeta,\omega k_1)} d_1 z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_9 = \begin{pmatrix} 1 & -\tilde{r}(k_1)^{\frac{1}{2}} r_{1,a}(k)\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)} d_1^{-1} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}_6 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{r}(k_1)^{-\frac{1}{2}}\hat{r}_{2,a}(k)\frac{\mathcal{P}(\zeta,\omega^2 k_1)}{\mathcal{P}(\zeta,\omega k_1)} d_1 z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $q := -\tilde{r}(k_1)^{-\frac{1}{2}} r_2(k_1)$. We have $-\tilde{r}(k_1)^{\frac{1}{2}} r_1(k_1) = \bar{q}$, $1 + r_1(k_1)r_2(k_1) = 1 + |q|^2$ and $\nu = -\frac{1}{2\pi}\ln(1+|q|^2)$. Note also that $d_1(\zeta,k_1) = \frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}$.

The above discussion suggests that $\tilde{n} = n^{(4)}\hat{M}_{\mathrm{sol}}Y$ is well approximated for large t by $(1,1,1)\hat{M}_{\mathrm{sol}}Y\tilde{m}^{k_1}$, where $\tilde{m}^{k_1}(x,t,k)$ is analytic for $k \in D_\epsilon(k_1)\backslash\mathcal{X}^\epsilon$, obeys the jump relation $\tilde{m}_+^{k_1} = \tilde{m}_-^{k_1}\tilde{v}_{\mathcal{X}^\epsilon}^X$ on $\mathcal{X}^\epsilon$ where

$$\tilde{v}_{\mathcal{X}_1^\epsilon}^X = \begin{pmatrix} 1 & \frac{\tilde{r}(k_1)^{\frac{1}{2}} r_1(k_1)}{1+r_1(k_1)r_2(k_1)} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-\bar{q}}{1+|q|^2} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}^X_{\mathcal{X}_2^\epsilon} = \begin{pmatrix} 1 & 0 & 0 \\ -\tilde{r}(k_1)^{-\frac{1}{2}} r_2(k_1) z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ q z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}^X_{\mathcal{X}_3^\epsilon} = \begin{pmatrix} 1 & -\tilde{r}(k_1)^{\frac{1}{2}} r_1(k_1) z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \bar{q} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{v}^X_{\mathcal{X}_4^\epsilon} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tilde{r}(k_1)^{-\frac{1}{2}} r_2(k_1)}{1+r_1(k_1)r_2(k_1)} z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-q}{1+|q|^2} z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and satisfies $\tilde{m}^{k_1}(x,t,k) \to I$ as $t \to \infty$ uniformly for $k \in \partial D_\epsilon(k_1)$. This motivates us to define m^{k_1} for $k \in D_\epsilon(k_1)$ by

$$m^{k_1}(x,t,k) = \hat{M}_{\mathrm{sol}}(x,t,k) Y(\zeta,t) m^X(q, z(\zeta,t,k)) Y(\zeta,t)^{-1} \hat{M}_{\mathrm{sol}}(x,t,k)^{-1}, \tag{8.3}$$

where m^X is the solution to the model RH problem of Lemma A.1. We expect $n^{(4)}$ to be well approximated by $(1,1,1)m^{k_1}$ for large t; in Section 9 we will show that this is indeed the case.

Lemma 8.1. *The function $Y(\zeta,t)$ is uniformly bounded:*

$$\sup_{\zeta \in \mathcal{I}} \sup_{t \geq 2} |Y(\zeta,t)^{\pm 1}| \leq C. \tag{8.4}$$

Moreover, the functions $d_0(\zeta,t)$ and $d_1(\zeta,k)$ satisfy

$$|d_0(\zeta,t)| = e^{2\pi\nu}, \qquad \zeta \in \mathcal{I},\ t \geq 2, \tag{8.5}$$

$$|d_1(\zeta,k) - \tfrac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}| \leq C|k-k_1|(1+|\ln|k-k_1||), \qquad \zeta \in \mathcal{I},\ k \in \mathcal{X}^\epsilon. \tag{8.6}$$

Proof. The estimates (8.4) and (8.6) follow from Lemma 6.1. Let us prove (8.5). Since $\nu \in \mathbb{R}$, by (8.2) we have

$$|d_0(\zeta,t)| = e^{2\operatorname{Re}\chi(\zeta,k_1)} e^{2\nu \arg z_\star} \left| \frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1) \delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1}) \delta(\frac{1}{\omega k_1})} \right|, \tag{8.7}$$

and we recall that $\arg z_\star = \frac{\pi}{2} - \arg k_1$. Using (6.5), we find

$$\left| \frac{\delta(\frac{1}{k_1})^2 \delta(\omega k_1) \delta(\omega^2 k_1)}{\delta(\frac{1}{\omega^2 k_1}) \delta(\frac{1}{\omega k_1})} \right| = \exp\left(\nu \arg k_1 - \frac{1}{2\pi} \int_{\Gamma_5^{(2)}} \arg s \, d\ln(1 + r_1(s) r_2(s)) \right). \tag{8.8}$$

We obtain (8.5) after substituting (6.4) evaluated at k_1 and (8.8) into (8.7). □

We establish in Lemma 12.2 that for any small enough $\epsilon > 0$, $\hat{M}_{\mathrm{sol}}(x,t,k)$ is uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}$, and $k \in \mathbb{C} \setminus \mathcal{D}_{\mathrm{sol}}$, where $\mathcal{D}_{\mathrm{sol}}$ is defined in (12.1). Using this fact, the proof of the following lemma is similar to [8, Proof of Lemma 9.11], so we omit it.

Lemma 8.2. *For each $t \geq 2$ and $\zeta \in \mathcal{I}$, $m^{k_1}(x,t,k)$ defined in (8.3) is analytic for $k \in D_\epsilon(k_1) \setminus \mathcal{X}^\epsilon$. Furthermore, $m^{k_1}(x,t,k)$ is uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}$ and $k \in D_\epsilon(k_1) \setminus \mathcal{X}^\epsilon$. On $\mathcal{X}^\epsilon$, m^{k_1} obeys the jump condition $m^{k_1}_+ = m^{k_1}_- v^{k_1}$, where v^{k_1} satisfies*

$$\begin{cases} \|v^{(4)} - v^{k_1}\|_{L^1(\mathcal{X}^\epsilon)} \leq C t^{-1} \ln t, \\ \|v^{(4)} - v^{k_1}\|_{L^\infty(\mathcal{X}^\epsilon)} \leq C t^{-1/2} \ln t, \end{cases} \qquad \zeta \in \mathcal{I},\ t \geq 2. \tag{8.9}$$

Furthermore, as $t \to \infty$,

$$\|m^{k_1}(x,t,\cdot) - I\|_{L^\infty(\partial D_\epsilon(k_1))} = O(t^{-1/2}), \tag{8.10}$$

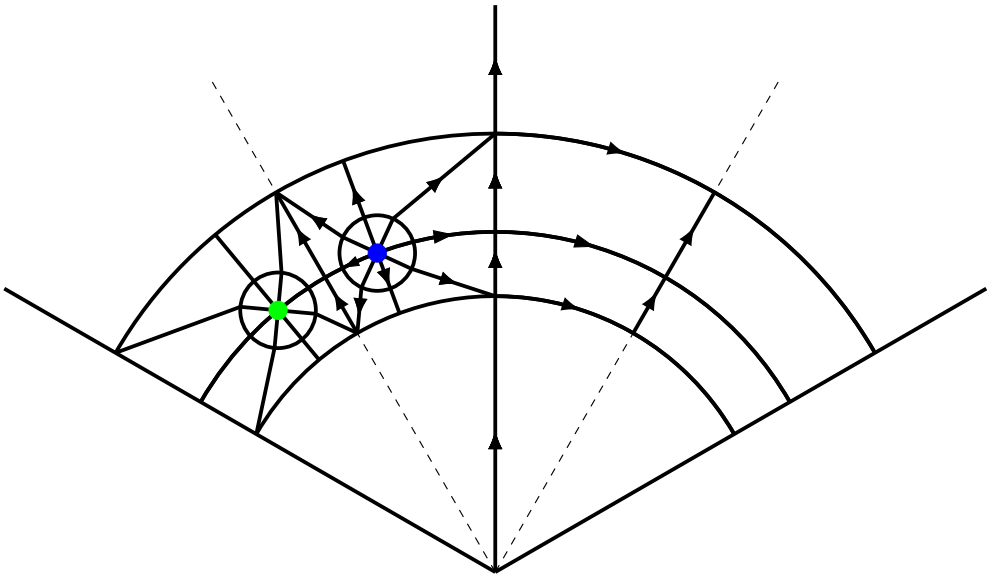

Figure 10. *The contour* $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ *for* $\arg k \in [\frac{\pi}{6}, \frac{5\pi}{6}]$ *(solid), the boundary of* S *(dashed) and, from right to left, the saddle points* k_1 *(blue) and* $\omega^2 k_2$ *(green).*

$$m^{k_1} - I = \frac{\hat{M}_{\mathrm{sol}} Y m_1^X Y^{-1} \hat{M}_{\mathrm{sol}}^{-1}}{z_\star \sqrt{t}(k-k_1)\hat{z}(\zeta,k)} + O(t^{-1}), \tag{8.11}$$

uniformly for $\zeta \in \mathcal{I}$*, and* $m_1^X = m_1^X(q)$ *is given by* (A.3).

9. The $n^{(4)} \to \hat{n}$ transformation

We use the symmetries

$$m^{k_1}(x,t,k) = \mathcal{A} m^{k_1}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B} m^{k_1}(x,t,k^{-1})\mathcal{B}$$

to extend the domain of definition of m^{k_1} from $D_\epsilon(k_1)$ to $\mathcal{D}$, where $\mathcal{D} := D_\epsilon(k_1) \cup \omega D_\epsilon(k_1) \cup \omega^2 D_\epsilon(k_1) \cup D_\epsilon(k_1^{-1}) \cup \omega D_\epsilon(k_1^{-1}) \cup \omega^2 D_\epsilon(k_1^{-1})$. We will show that

$$\hat{n} := \begin{cases} n^{(4)}(m^{k_1})^{-1}, & k \in \mathcal{D}, \\ n^{(4)}, & \text{elsewhere}, \end{cases} \tag{9.1}$$

is close to $(1,1,1)$ for large t and $\zeta \in \mathcal{I}$. Let $\hat{\Gamma} = \Gamma^{(4)} \cup \partial\mathcal{D}$ be the contour displayed in Figure 10, where each circle that is part of $\partial\mathcal{D}$ is oriented clockwise, and define

$$\hat{v} = \begin{cases} v^{(4)}, & k \in \hat{\Gamma} \setminus \bar{\mathcal{D}}, \\ m^{k_1}, & k \in \partial\mathcal{D}, \\ m_-^{k_1} v^{(4)} (m_+^{k_1})^{-1}, & k \in \hat{\Gamma} \cap \mathcal{D}. \end{cases} \tag{9.2}$$

Let $\hat{\Gamma}_\star$ be the set of self-intersection points of $\hat{\Gamma}$. The matrix $\hat{n}$ satisfies the following RH problem: (a) $\hat{n} : \mathbb{C} \setminus \hat{\Gamma} \to \mathbb{C}^{1\times 3}$ is analytic, (b) $\hat{n}_+ = \hat{n}_- \hat{v}$ for $k \in \hat{\Gamma} \setminus \hat{\Gamma}_\star$, (c) $\hat{n}(x,t,k) = (1,1,1) + O(k^{-1})$ as $k \to \infty$, and (d) $\hat{n}(x,t,k) = O(1)$ as $k \to k_\star \in \hat{\Gamma}_\star$.

Lemma 9.1. *Let* $\hat{w} = \hat{v} - I$*. The following estimates hold uniformly for* $t \geq 2$ *and* $\zeta \in \mathcal{I}$*:*

$$\|\hat{w}\|_{(L^1\cap L^\infty)(\hat{\Gamma}\setminus(\partial\mathcal{D}\cup\hat{\mathcal{X}}^\epsilon))} \leq Ct^{-1}, \tag{9.3a}$$

$$\|\hat{w}\|_{(L^1\cap L^\infty)(\partial\mathcal{D})} \leq Ct^{-1/2}, \tag{9.3b}$$

$$\|\hat{w}\|_{L^1(\hat{\mathcal{X}}^\epsilon)} \le Ct^{-1}\ln t, \tag{9.3c}$$

$$\|\hat{w}\|_{L^\infty(\hat{\mathcal{X}}^\epsilon)} \le Ct^{-1/2}\ln t. \tag{9.3d}$$

Proof. Note that $\hat{\Gamma}\setminus(\partial\mathcal{D}\cup\hat{\mathcal{X}}^\epsilon) = \Gamma^{(4)}\setminus\hat{\mathcal{X}}^\epsilon$. Hence (9.3a) follows from (9.2), Lemma 6.3, and the uniform boundedness of m^{k_1}. The inequality (9.3b) is a consequence of (9.2) and (8.10). The estimates (9.3c) and (9.3d) follow from (9.2), (8.9), and the uniform boundedness of m^{k_1}. □

We define the Cauchy transform $\hat{\mathcal{C}}h$ of a function h defined on $\hat{\Gamma}$ by

$$(\hat{\mathcal{C}}h)(z) = \frac{1}{2\pi i}\int_{\hat{\Gamma}}\frac{h(z')dz'}{z'-z}, \qquad z\in\mathbb{C}\setminus\hat{\Gamma}. \tag{9.4}$$

The estimates in Lemma 9.1 show that

$$\begin{cases}\|\hat{w}\|_{L^1(\hat{\Gamma})} \le Ct^{-1/2},\\ \|\hat{w}\|_{L^\infty(\hat{\Gamma})} \le Ct^{-1/2}\ln t,\end{cases} \qquad t\ge 2,\ \zeta\in\mathcal{I}, \tag{9.5}$$

and hence, employing the inequality $\|f\|_{L^p} \le \|f\|_{L^1}^{1/p}\|f\|_{L^\infty}^{(p-1)/p}$, we get

$$\|\hat{w}\|_{L^p(\hat{\Gamma})} \le Ct^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}}, \qquad t\ge 2,\ \zeta\in\mathcal{I}, \tag{9.6}$$

for each $1\le p\le\infty$. From (9.6), we infer that $\hat{w}\in L^2(\hat{\Gamma})\cap L^\infty(\hat{\Gamma})$, and thus the operator $\hat{\mathcal{C}}_{\hat{w}} = \hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)} : L^2(\hat{\Gamma}) + L^\infty(\hat{\Gamma}) \to L^2(\hat{\Gamma})$, $h\mapsto \hat{\mathcal{C}}_{\hat{w}}h := \hat{\mathcal{C}}_-(h\hat{w})$ is well-defined. Let $\mathcal{B}(L^2(\hat{\Gamma}))$ be the space of bounded linear operators on $L^2(\hat{\Gamma})$. We deduce from (9.6) that there exists $T>0$ such that $I-\hat{\mathcal{C}}_{\hat{w}(x,t,\cdot)}\in\mathcal{B}(L^2(\hat{\Gamma}))$ is invertible for all $t\ge T$ and $\zeta\in\mathcal{I}$. Hence $\hat{n}$ is the solution of a small-norm RH problem, and we have

$$\hat{n}(x,t,k) = (1,1,1) + \hat{\mathcal{C}}(\hat{\mu}\hat{w}) = (1,1,1) + \frac{1}{2\pi i}\int_{\hat{\Gamma}}\hat{\mu}(x,t,s)\hat{w}(x,t,s)\frac{ds}{s-k} \tag{9.7}$$

for $t\ge T$ and $\zeta\in\mathcal{I}$, where

$$\hat{\mu} = (1,1,1) + (I-\hat{\mathcal{C}}_{\hat{w}})^{-1}\hat{\mathcal{C}}_{\hat{w}}(1,1,1) \in (1,1,1) + L^2(\hat{\Gamma}). \tag{9.8}$$

Furthermore, using (9.6), (9.8), and the fact that $\|\hat{\mathcal{C}}_-\|_{\mathcal{B}(L^p(\hat{\Gamma}))}<\infty$ for $p\in(1,\infty)$, we find that for any $p\in(1,\infty)$ there exists $C_p>0$ such that

$$\|\hat{\mu}-(1,1,1)\|_{L^p(\hat{\Gamma})} \le C_p t^{-\frac{1}{2}}(\ln t)^{\frac{p-1}{p}} \tag{9.9}$$

holds for all $\zeta\in\mathcal{I}$ and large enough t.

Let us now obtain the long-time asymptotics of $\hat{n}$. Define $\hat{n}^{(1)}$ as the nontangential limit

$$\hat{n}^{(1)}(x,t) := \lim_{k\to\infty}{}^{\angle} k(\hat{n}(x,t,k)-(1,1,1)) = -\frac{1}{2\pi i}\int_{\hat{\Gamma}}\hat{\mu}(x,t,k)\hat{w}(x,t,k)dk.$$

Lemma 9.2. *As $t\to\infty$,*

$$\hat{n}^{(1)}(x,t) = -\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)dk + O(t^{-1}\ln t). \tag{9.10}$$

Proof. Since

$$\begin{aligned}\hat{n}^{(1)}(x,t) = &-\frac{(1,1,1)}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)dk - \frac{(1,1,1)}{2\pi i}\int_{\hat{\Gamma}\setminus\partial\mathcal{D}}\hat{w}(x,t,k)dk\\ &-\frac{1}{2\pi i}\int_{\hat{\Gamma}}(\hat{\mu}(x,t,k)-(1,1,1))\hat{w}(x,t,k)dk,\end{aligned}$$

the statement follows from (9.9) and Lemma 9.1. □

We define $\{F^{(l)}\}_{l\in\mathbb{Z}}$ by

$$F^{(l)}(\zeta,t) = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_1)} k^{l-1}\hat{w}(x,t,k)dk = -\frac{1}{2\pi i}\int_{\partial D_\epsilon(k_1)} k^{l-1}(m^{k_1}-I)dk.$$

Using (8.11) and $\hat{z}(\zeta,k_1)=1$, and the fact that $\partial D_\epsilon(k_1)$ is oriented clockwise, we obtain

$$\begin{aligned} F^{(l)}(\zeta,t) &= k_1^{l-1}\frac{\hat{M}_{\mathrm{sol}}(x,t,k_1)Y(\zeta,t)m_1^X Y(\zeta,t)^{-1}\hat{M}_{\mathrm{sol}}(x,t,k_1)^{-1}}{z_\star\sqrt{t}} + O(t^{-1}) \\ &= -ik_1^l Z(\zeta,t) + O(t^{-1}) \end{aligned} \tag{9.11}$$

as $t\to\infty$ uniformly for $\zeta\in\mathcal{I}$, where

$$\begin{aligned} Z(\zeta,t) &= \frac{\hat{M}_{\mathrm{sol}}(x,t,k_1)Y(\zeta,t)m_1^X Y(\zeta,t)^{-1}\hat{M}_{\mathrm{sol}}(x,t,k_1)^{-1}}{-ik_1 z_\star\sqrt{t}} \\ &= \frac{\hat{M}_{\mathrm{sol}}(x,t,k_1)}{-ik_1 z_\star\sqrt{t}}\begin{pmatrix} 0 & \frac{\beta_{12}\tilde{r}(k_1)^{-\frac{1}{2}}}{d_0 e^{t\Phi_{21}(\zeta,k_1)}}\frac{\mathcal{P}(\zeta,\omega^2k_1)}{\mathcal{P}(\zeta,\omega k_1)} & 0 \\ \frac{d_0\tilde{r}(k_1)^{\frac{1}{2}}\beta_{21}}{e^{-t\Phi_{21}(\zeta,k_1)}}\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\hat{M}_{\mathrm{sol}}(x,t,k_1)^{-1}. \end{aligned}$$

Since $\hat{M}_{\mathrm{sol}} = M_{\mathrm{sol}}\mathsf{P}$ with P given by (7.2), the expression for Z simplifies to

$$Z(\zeta,t) = \frac{M_{\mathrm{sol}}(x,t,k_1)}{-ik_1 z_\star\sqrt{t}}\begin{pmatrix} 0 & \frac{\beta_{12}\tilde{r}(k_1)^{-\frac{1}{2}}}{d_0 e^{t\Phi_{21}(\zeta,k_1)}} & 0 \\ \frac{d_0\tilde{r}(k_1)^{\frac{1}{2}}\beta_{21}}{e^{-t\Phi_{21}(\zeta,k_1)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_{\mathrm{sol}}(x,t,k_1)^{-1}. \tag{9.12}$$

Lemma 9.3. *For $l\in\mathbb{Z}$ and $j=0,1,2$, we have*

$$-\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1)} k^l\hat{w}(x,t,k)dk = \omega^{j(l+1)}\mathcal{A}^{-j}F^{(l+1)}(\zeta,t)\mathcal{A}^j, \tag{9.13}$$

$$-\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1^{-1})} k^l\hat{w}(x,t,k)dk = -\omega^{j(l+1)}\mathcal{A}^{-j}\mathcal{B}F^{(-l-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j. \tag{9.14}$$

Proof. The assertions are a consequence of the symmetries $\hat{w}(x,t,k) = \mathcal{A}\hat{w}(x,t,\omega k)\mathcal{A}^{-1} = \mathcal{B}\hat{w}(x,t,k^{-1})\mathcal{B}$ which hold for $k\in\partial\mathcal{D}$. For further details we refer to [11, Proof of Lemma 12.4]. □

Using Lemma 9.3, we find

$$\begin{aligned} \frac{-1}{2\pi i}\int_{\partial\mathcal{D}}\hat{w}(x,t,k)dk &= \sum_{j=0}^2\frac{-1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1)}\hat{w}(x,t,k)dk - \sum_{j=0}^2\frac{1}{2\pi i}\int_{\omega^j\partial D_\epsilon(k_1^{-1})}\hat{w}(x,t,k)dk \\ &= \sum_{j=0}^2\omega^j\mathcal{A}^{-j}F^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^2\omega^j\mathcal{A}^{-j}\mathcal{B}F^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j. \end{aligned}$$

Therefore, (9.10) and (9.11) imply that $\hat{n}^{(1)} = (1,1,1)\hat{m}^{(1)}$, where

$$\begin{aligned} \hat{m}^{(1)}(x,t) &= \sum_{j=0}^2\omega^j\mathcal{A}^{-j}F^{(1)}(\zeta,t)\mathcal{A}^j - \sum_{j=0}^2\omega^j\mathcal{A}^{-j}\mathcal{B}F^{(-1)}(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t) \\ &= -ik_1\sum_{j=0}^2\omega^j\mathcal{A}^{-j}Z(\zeta,t)\mathcal{A}^j + ik_1^{-1}\sum_{j=0}^2\omega^j\mathcal{A}^{-j}\mathcal{B}Z(\zeta,t)\mathcal{B}\mathcal{A}^j + O(t^{-1}\ln t) \end{aligned} \tag{9.15}$$

as $t\to\infty$, uniformly for $\zeta\in\mathcal{I}$.

10. Asymptotics of u

10.1. **Proof of Theorem 2.1.** By (3.8), we have $u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x} n_3^{(1)}(x,t)$, where $n_3(x,t,k) = 1 + n_3^{(1)}(x,t)k^{-1} + O(k^{-2})$ as $k \to \infty$. By inverting the transformations $n \to n^{(1)} \to n^{(2)} \to n^{(3)} \to n^{(4)} \to \hat{n}$ using (4.8), (5.2), (6.11), (7.3), and (9.1), we get

$$n = \hat{n} M_{\mathrm{sol}} \mathsf{P} \Delta^{-1} (G^{(2)})^{-1} (G^{(1)})^{-1}, \qquad k \in \mathbb{C} \setminus (\hat{\Gamma} \cup \bar{\mathcal{D}}),$$

where $G^{(1)}$, $G^{(2)}$, Δ are defined in (4.9), (5.3), and (6.8), respectively. The functions $\Delta(\zeta,k)$ and $\mathsf{P}(\zeta,k)$ are in general not smooth as functions of ζ, because the number of factors appearing in the products in the definition (2.15) of $\mathcal{P}$ may vary discontinuously with ζ; note however that these products disappear when considering $\hat{\Delta}(\zeta,k) = \Delta(\zeta,k)\mathsf{P}(\zeta,k)^{-1}$, so that $\hat{\Delta}(\zeta,k)$ is smooth as a function of ζ. Let $n_{\mathrm{sol}}(x,t,k) := (1,1,1)M_{\mathrm{sol}}(x,t,k)$. Since $\partial_x = t^{-1}\partial_\zeta$, we find

$$\begin{aligned} u(x,t) &= -i\sqrt{3}\frac{\partial}{\partial x}\left(\hat{n}_3^{(1)}(x,t) + \big(n_{\mathrm{sol}}^{(1)}(x,t)\big)_3 + \angle\lim_{k\to\infty} k(\hat{\Delta}_{33}(\zeta,k)^{-1} - 1)\right) \\ &= u_{\mathrm{sol}}(x,t) - i\sqrt{3}\frac{\partial}{\partial x}\hat{n}_3^{(1)}(x,t) + O(t^{-1}), \qquad t \to \infty, \end{aligned} \tag{10.1}$$

where $\hat{n}_3^{(1)}$ and $n_{\mathrm{sol}}^{(1)}$ are defined through the expansions

$$\begin{aligned} \hat{n}_3(x,t,k) &= 1 + \hat{n}_3^{(1)}(x,t)k^{-1} + O(k^{-2}), \\ n_{\mathrm{sol}}(x,t,k) &= (1,1,1) + n_{\mathrm{sol}}^{(1)}(x,t)k^{-1} + O(k^{-2}) \end{aligned}$$

as $k \to \infty$, and $u_{\mathrm{sol}}(x,t) := -i\sqrt{3}\frac{\partial}{\partial x}(n_{\mathrm{sol}}^{(1)}(x,t))_3$. By (9.15), we have

$$\begin{aligned} \hat{n}_3^{(1)} &= -ik_1 \sum_{j=0}^{2} \omega^j (1,1,1) Z(\zeta,t) \mathcal{A}^j[I]_3 + ik_1^{-1} \sum_{j=0}^{2} \omega^j (1,1,1) Z(\zeta,t) \mathcal{B}\mathcal{A}^j[I]_3 + O(t^{-1}\ln t) \\ &= -ik_1 (1,1,1) Z(\zeta,t) \begin{pmatrix} \omega \\ \omega^2 \\ 1 \end{pmatrix} + ik_1^{-1} (1,1,1) Z(\zeta,t) \begin{pmatrix} \omega^2 \\ \omega \\ 1 \end{pmatrix} + O(t^{-1}\ln t) \\ &= (1,1,1) Z(\zeta,t) \begin{pmatrix} \omega^2 ik_1^{-1} - \omega ik_1 \\ \omega ik_1^{-1} - \omega^2 ik_1 \\ ik_1^{-1} - ik_1 \end{pmatrix} + O(t^{-1}\ln t). \end{aligned}$$

Combining the above formula with (10.1), we get

$$u(x,t) = u_{\mathrm{sol}}(x,t) - i\sqrt{3}\frac{\partial}{\partial x}(1,1,1) Z(\zeta,t) \begin{pmatrix} \omega^2 ik_1^{-1} - \omega ik_1 \\ \omega ik_1^{-1} - \omega^2 ik_1 \\ ik_1^{-1} - ik_1 \end{pmatrix} + O(t^{-1}\ln t). \tag{10.2}$$

Substituting in the expression (9.12) for Z, we arrive at

$$\begin{aligned} u(x,t) = u_{\mathrm{sol}}(x,t) - i\sqrt{3}\frac{n_{\mathrm{sol}}(x,t,k_1)}{-ik_1 z_\star \sqrt{t}} &\begin{pmatrix} 0 & \frac{\beta_{12}\tilde{r}(k_1)^{-\frac{1}{2}} \frac{d}{d\zeta}[\Phi_{21}(\zeta,k_1(\zeta))]}{-d_0 e^{t\Phi_{21}(\zeta,k_1)}} & 0 \\ \frac{d_0 \tilde{r}(k_1)^{\frac{1}{2}} \beta_{21} \frac{d}{d\zeta}[\Phi_{21}(\zeta,k_1(\zeta))]}{e^{-t\Phi_{21}(\zeta,k_1)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\times M_{\mathrm{sol}}(x,t,k_1)^{-1} \begin{pmatrix} \omega^2 ik_1^{-1} - \omega ik_1 \\ \omega ik_1^{-1} - \omega^2 ik_1 \\ ik_1^{-1} - ik_1 \end{pmatrix} + O(t^{-1}\ln t). \end{aligned}$$

A long but straightforward computation using (8.2), (6.2), and (6.3) shows that $\arg d_0(\zeta,t)$ can be rewritten as in (2.10). Using (A.4) as well as the relations $|d_0(\zeta,t)| = e^{2\pi\nu}$, $q = -\tilde r(k_1)^{-\frac12} r_2(k_1)$, $\bar q = -\tilde r(k_1)^{\frac12} r_1(k_1)$, and

$$\frac{d}{d\zeta}[\Phi_{21}(\zeta,k_1(\zeta))] = i\operatorname{Im} k_1, \tag{10.3}$$

the asymptotic formula (2.11) follows. This completes the proof of Theorem 2.1.

10.2. **Proof of Theorem 2.2.** Let us next derive the asymptotic formula of Theorem 2.2. Fix $\epsilon > 0$ and let $S_\epsilon \subset \mathcal{I}$ be the set defined in (2.14). By Lemma 7.1, $\hat M_{\rm sol}(x,t,k)$ satisfies the residue conditions in Lemma 6.4 with $n_j^{(3)}$ replaced by $[\hat M_{\rm sol}(x,t,k)]_j$. For $\zeta \in \mathcal{I}\setminus S_\epsilon$, each residue condition in Lemma 6.4 has a coefficient that is uniformly exponentially small. It follows that $\hat M_{\rm sol}(x,t,k_1) = I + O(t^{-N})$ and $u_{\rm sol}(x,t) = O(t^{-N})$ for every $N \geq 1$ as $t\to\infty$, uniformly for $\zeta \in \mathcal{I}\setminus S_\epsilon$. Thus, for $\zeta \in \mathcal{I}\setminus S_\epsilon$, (10.2) simplifies to

$$u(x,t) = -i\sqrt{3}\frac{\partial}{\partial x}\hat n_3^{(1)} + O(t^{-1}\ln t),$$

where

$$\begin{aligned}\hat n_3^{(1)} &= Z_{12}\big(\omega i k_1^{-1} - \omega^2 i k_1\big) + Z_{21}\big(\omega^2 i k_1^{-1} - \omega i k_1\big) + O(t^{-1}\ln t)\\ &= \frac{\beta_{12}(\omega i k_1^{-1} - \omega^2 i k_1)}{-ik_1 z_\star \sqrt{t} d_0 e^{t\Phi_{21}(\zeta,k_1)}\tilde r(k_1)^{\frac12}}\frac{\mathcal{P}(\zeta,\omega^2 k_1)}{\mathcal{P}(\zeta,\omega k_1)}\\ &\quad + \frac{d_0 e^{t\Phi_{21}(\zeta,k_1)}\tilde r(k_1)^{\frac12}\beta_{21}(\omega^2 i k_1^{-1} - \omega i k_1)}{-ik_1 z_\star\sqrt{t}}\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)} + O(t^{-1}\ln t)\end{aligned}$$

as $t\to\infty$. Since $|d_0(\zeta,t)| = e^{2\pi\nu}$, we have $\frac{\beta_{12}}{d_0}\frac{\mathcal{P}(\zeta,\omega^2k_1)}{\mathcal{P}(\zeta,\omega k_1)} = -\bar d_0\bar\beta_{21}\frac{\overline{\mathcal{P}(\zeta,\omega k_1)}}{\overline{\mathcal{P}(\zeta,\omega^2 k_1)}}$. Using also that $-ik_1z_\star > 0$ and $(\omega i k_1^{-1} - \omega^2 i k_1)/\tilde r(k_1)^{\frac12} = \tilde r(k_1)^{\frac12}(\omega^2 i k_1^{-1} - \omega i k_1) \in \mathbb{R}$, we infer that

$$\hat n_3^{(1)} = 2i\operatorname{Im}\frac{\beta_{12}(\omega i k_1^{-1} - \omega^2 i k_1)\mathcal{P}(\zeta,\omega^2 k_1)}{-ik_1 z_\star\sqrt{t} d_0 e^{t\Phi_{21}(\zeta,k_1)}\tilde r(k_1)^{\frac12}\mathcal{P}(\zeta,\omega k_1)} + O(t^{-1}\ln t) \qquad \text{as } t\to\infty.$$

Employing the identities

$$\beta_{12} = \frac{\sqrt{2\pi}e^{\frac{\pi i}{4}}e^{\frac{3\pi\nu}{2}}}{q\Gamma(i\nu)},\quad d_0(\zeta,t) = e^{2\pi\nu}e^{i\arg d_0(\zeta,t)},\quad |q| = \sqrt{e^{-2\pi\nu}-1},\quad \left|\frac{\mathcal{P}(\zeta,\omega^2k_1)}{\mathcal{P}(\zeta,\omega k_1)}\right| = 1,$$

$$|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{-\nu}\sqrt{e^{-\pi\nu}-e^{\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{-\nu}e^{\frac{\pi\nu}{2}}|q|},\quad \frac{\omega i k_1^{-1} - \omega^2 i k_1}{\tilde r(k_1)^{\frac12}} = -\sqrt{-1-2\cos(2\arg k_1)},$$

we obtain

$$\begin{aligned}\hat n_3^{(1)} = &-2i\frac{\sqrt{-\nu}\sqrt{-1-2\cos(2\arg k_1)}}{-ik_1z_\star\sqrt{t}}\\ &\times\sin\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - \arg\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2k_1)} - t\operatorname{Im}\Phi_{21}(\zeta,k_1)\Big)\\ &+ O(t^{-1}\ln t)\qquad \text{as } t\to\infty,\end{aligned}$$

uniformly for $\zeta\in\mathcal{I}\setminus S_\epsilon$. Combining the above formula with (10.1) and using (10.3), we get

$$\begin{aligned}u(x,t) = &-i\sqrt{3}\frac{\partial}{\partial x}\hat n_3^{(1)}(x,t) + O(t^{-1})\\ = &-2\sqrt{3}\frac1t\frac{\partial}{\partial\zeta}\Big(\sin\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - \arg\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2k_1)}\end{aligned}$$

$$
\begin{aligned}
&\quad - t\mathrm{Im}\,\Phi_{21}(\zeta,k_1)\Big) \times \frac{\sqrt{-\nu}\sqrt{-1-2\cos(2\arg k_1)}}{-ik_1 z_\star\sqrt{t}}\Big) + O(t^{-1}\ln t),\\
&= \frac{A(\zeta)}{\sqrt{t}}\cos\Big(\frac{\pi}{4} - \arg q - \arg\Gamma(i\nu) - \arg d_0 - \arg\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}\\
&\qquad\qquad - t\mathrm{Im}\,\Phi_{21}(\zeta,k_1)\Big) + O(t^{-1}\ln t),\\
&= \frac{A(\zeta)}{\sqrt{t}}\cos\Big(\frac{3\pi}{4} + \arg r_2(k_1) + \arg\Gamma(i\nu) + \arg d_0 + \arg\frac{\mathcal{P}(\zeta,\omega k_1)}{\mathcal{P}(\zeta,\omega^2 k_1)}\\
&\qquad\qquad + t\mathrm{Im}\,\Phi_{21}(\zeta,k_1)\Big) + O(t^{-1}\ln t),
\end{aligned}
$$

as $t\to\infty$, uniformly for $\zeta\in\mathcal{I}\backslash S_\epsilon$, where $A(\zeta)$ is given in the statement of Theorem 2.1, and where in the last step we have used $\arg q = \pi + \arg r_2(k_1) \mod 2\pi$ and $\cos(-x)=\cos(x)$. This finishes the proof of Theorem 2.2.

11. The pure multi-soliton RH problem

Pure multi-soliton solutions of (1.1) are obtained by considering RH problem 3.1 for $n(x,t,k)$ in the special case when $r_1 = r_2 = 0$, i.e., when the jump matrix v is identically equal to I. The next lemma establishes the explicit formula (2.17) for the pure multi-solitons of (1.1).

Lemma 11.1. *Let Z be a finite subset of $D_{\mathrm{reg}}\cup(-1,0)\cup(1,\infty)$ and let $\{c_{k_0}\}_{k_0\in\mathsf{Z}}\subset\mathbb{C}$ be such that*

$$
i(\omega^2 k_0^2 - \omega)c_{k_0} \geq 0 \quad \text{for every } k_0\in\mathsf{Z}\cap\mathbb{R}. \tag{11.1}
$$

Define $\{d_{k_0}\}_{k_0\in\mathsf{Z}\backslash\mathbb{R}}$ as in (2.27) and write $\mathsf{Z} = \{\lambda_j\}_1^{n_b}\sqcup\{k_j\}_1^{n_s}$ where $\{\lambda_j\}_1^{n_b} := \mathsf{Z}\cap D_{\mathrm{reg}}$ and $\{k_j\}_1^{n_s} := \mathsf{Z}\cap\mathbb{R}$. Then the following hold:

(a) *RH problem 3.1 with $v\equiv I$ has a unique solution $n(x,t,k)$ for every $(x,t)\in\mathbb{R}\times[0,\infty)$.*

(b) *The function u_s defined by*

$$
u_s\Big(x,t;(\lambda_j,c_{\lambda_j})_{j=1}^{n_b},(k_j,c_{k_j})_{j=1}^{n_s}\Big) := -i\sqrt{3}\frac{\partial}{\partial x}\lim_{k\to\infty} k\big(n_3(x,t,k)-1\big)
$$

is a real-valued Schwartz class solution of (1.1) for $(x,t)\in\mathbb{R}\times[0,\infty)$.

(c) *$u_s\Big(x,t;(\lambda_j,c_{\lambda_j})_{j=1}^{n_b},(k_j,c_{k_j})_{j=1}^{n_s}\Big)$ is given explicitly by (2.17).*

Proof. Assertions (a) and (b) are a consequence of [10, Theorem 2.6 and Remark 2.7]; we emphasize that the proof of [10, Theorem 2.6] uses condition (11.1).

Let us prove (c). In the proof, we omit the (x,t)-dependence of $n(x,t,k)$ and $\theta_{ij}(x,t,k)$ for brevity. Utilizing the residue conditions (3.6) and (3.7) as well as the normalization condition $n(x,t,k) = (1,1,1) + O(k^{-1})$ as $k\to\infty$, we obtain

$$
\begin{aligned}
n_3(k) = 1 + \sum_{l=1}^{n_b}\Big\{&\frac{c_{\lambda_l}e^{-\theta_{31}(\lambda_l)}n_1(\lambda_l)}{k-\lambda_l} + \frac{-\lambda_l^{-2}c_{\lambda_l}e^{-\theta_{31}(\lambda_l)}n_2(\lambda_l^{-1})}{k-\lambda_l^{-1}}\\
&+ \frac{\omega^2 d_{\lambda_l}e^{\theta_{32}(\bar\lambda_l)}n_1(\omega^2\bar\lambda_l)}{k-\omega^2\bar\lambda_l} + \frac{-\omega\bar\lambda_l^{-2}d_{\lambda_l}e^{\theta_{32}(\bar\lambda_l)}n_2(\omega\bar\lambda_l^{-1})}{k-\omega\bar\lambda_l^{-1}}\Big\}\\
+\sum_{l=1}^{n_s}\Big\{&\frac{\omega^2 c_{k_l}e^{-\theta_{21}(k_l)}n_2(\omega^2 k_l)}{k-\omega^2 k_l} + \frac{-\omega k_l^{-2}c_{k_l}e^{-\theta_{21}(k_l)}n_1(\omega k_l^{-1})}{k-\omega k_l^{-1}}\Big\}.
\end{aligned}
\tag{11.2}
$$

The symmetries (3.5) imply that

$$
n_3(k) = n_2(\omega k) = n_1(\omega^2 k) = n_3(k^{-1}),
$$

so we may write (11.2) as

$$\begin{aligned} n_3(k) = 1 + \sum_{l=1}^{n_b} \Bigg\{ \left(\frac{1}{k-\lambda_l} + \frac{-\lambda_l^{-2}}{k-\lambda_l^{-1}} \right) c_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l) \\ + \left(\frac{\omega^2}{k-\omega^2\bar{\lambda}_l} + \frac{-\omega\bar{\lambda}_l^{-2}}{k-\omega\bar{\lambda}_l^{-1}} \right) d_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l) \Bigg\} \\ + \sum_{l=1}^{n_s} \left\{ \frac{\omega^2}{k-\omega^2 k_l} + \frac{-\omega k_l^{-2}}{k-\omega k_l^{-1}} \right\} c_{k_l} e^{-\theta_{21}(k_l)} n_3(\omega k_l). \end{aligned} \tag{11.3}$$

Since

$$\frac{\frac{i(\lambda_l^2-1)}{2\sqrt{3}\lambda_l^2}}{l_3(k)-l_3(\lambda_l)} = \frac{1}{k-\lambda_l} + \frac{-\lambda_l^{-2}}{k-\lambda_l^{-1}},$$

we deduce that

$$n_3(k) = 1 + \sum_{l=1}^{n_b} \left\{ \frac{\tilde{c}_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l)}{l_3(k)-l_3(\lambda_l)} + \frac{\bar{\tilde{c}}_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l)}{l_3(k)-l_3(\omega^2\bar{\lambda}_l)} \right\} + \sum_{l=1}^{n_s} \frac{\tilde{c}_{k_l} e^{-\theta_{21}(k_l)} n_3(\omega k_l)}{l_3(k)-l_3(\omega^2 k_l)}, \tag{11.4}$$

where $\tilde{c}_{\lambda_l} := \frac{i(\lambda_l^2-1)}{2\sqrt{3}\lambda_l^2} c_{\lambda_l}$ and $\tilde{c}_{k_l} := \frac{i(k_l^2-\omega^2)}{2\sqrt{3}k_l^2}\omega^2 c_{k_l}$, and we have used that $\frac{i(\bar{\lambda}_l^2-\omega^2)}{2\sqrt{3}\bar{\lambda}_l^2}\omega^2 d_{\lambda_l} = \bar{\tilde{c}}_{\lambda_l}$. Evaluating (11.4) at $k = \omega\lambda_j$, $k = \bar{\lambda}_j$, and $k = \omega k_j$, we find the system

$$\begin{aligned} n_3(\omega\lambda_j) &= 1 + \sum_{l=1}^{n_b} \left\{ \frac{\tilde{c}_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l)}{l_1(\lambda_j)-l_3(\lambda_l)} + \frac{\bar{\tilde{c}}_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l)}{l_1(\lambda_j)-l_2(\bar{\lambda}_l)} \right\} + \sum_{l=1}^{n_s} \frac{\tilde{c}_{k_l} e^{-\theta_{21}(k_l)} n_3(\omega k_l)}{l_1(\lambda_j)-l_2(k_l)}, \\ n_3(\bar{\lambda}_j) &= 1 + \sum_{l=1}^{n_b} \left\{ \frac{\tilde{c}_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l)}{l_3(\bar{\lambda}_j)-l_3(\lambda_l)} + \frac{\bar{\tilde{c}}_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l)}{l_3(\bar{\lambda}_j)-l_2(\bar{\lambda}_l)} \right\} + \sum_{l=1}^{n_s} \left\{ \frac{\tilde{c}_{k_l} e^{-\theta_{21}(k_l)} n_3(\omega k_l)}{l_3(\bar{\lambda}_j)-l_2(k_l)} \right\}, \\ n_3(\omega k_j) &= 1 + \sum_{l=1}^{n_b} \left\{ \frac{\tilde{c}_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l)}{l_1(k_j)-l_3(\lambda_l)} + \frac{\bar{\tilde{c}}_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l)}{l_1(k_j)-l_2(\bar{\lambda}_l)} \right\} + \sum_{l=1}^{n_s} \left\{ \frac{\tilde{c}_{k_l} e^{-\theta_{21}(k_l)} n_3(\omega k_l)}{l_1(k_j)-l_2(k_l)} \right\}. \end{aligned}$$

Multiplying the first equation by $e^{xl_1(\lambda_j)+tz_1(\lambda_j)}$, the second by $e^{xl_3(\bar{\lambda}_j)+tz_3(\bar{\lambda}_j)}$, and the third by $e^{xl_1(k_j)+tz_1(k_j)}$, we obtain

$$(I-B) \begin{pmatrix} e^{xl_1(\lambda_1)+tz_1(\lambda_1)} n_3(\omega\lambda_1) \\ \vdots \\ e^{xl_1(\lambda_{n_b})+tz_1(\lambda_{n_b})} n_3(\omega\lambda_{n_b}) \\ e^{xl_3(\bar{\lambda}_1)+tz_3(\bar{\lambda}_1)} n_3(\bar{\lambda}_1) \\ \vdots \\ e^{xl_3(\bar{\lambda}_{n_b})+tz_3(\bar{\lambda}_{n_b})} n_3(\bar{\lambda}_{n_b}) \\ e^{xl_1(k_1)+tz_1(k_1)} n_3(\omega k_1) \\ \vdots \\ e^{xl_1(k_{n_s})+tz_1(k_{n_s})} n_3(\omega k_{n_s}) \end{pmatrix} = W, \quad \text{where } W = W(x,t) := \begin{pmatrix} e^{xl_1(\lambda_1)+tz_1(\lambda_1)} \\ \vdots \\ e^{xl_1(\lambda_{n_b})+tz_1(\lambda_{n_b})} \\ e^{xl_3(\bar{\lambda}_1)+tz_3(\bar{\lambda}_1)} \\ \vdots \\ e^{xl_3(\bar{\lambda}_{n_b})+tz_3(\bar{\lambda}_{n_b})} \\ e^{xl_1(k_1)+tz_1(k_1)} \\ \vdots \\ e^{xl_1(k_{n_s})+tz_1(k_{n_s})} \end{pmatrix}$$

and $B = B(x,t)$ is the matrix defined in (2.18). The existence and uniqueness of n imply that $I-B$ is invertible. Since, by (11.4),

$$\lim_{k\to\infty} k\big(n_3(k)-1\big) = -2i\sqrt{3}\Bigg\{ \sum_{l=1}^{n_b} \Big(\tilde{c}_{\lambda_l} e^{-\theta_{31}(\lambda_l)} n_3(\omega\lambda_l) + \bar{\tilde{c}}_{\lambda_l} e^{\theta_{32}(\bar{\lambda}_l)} n_3(\bar{\lambda}_l) \Big)$$

$$+\sum_{l=1}^{n_s}\tilde{c}_{k_l}e^{-\theta_{21}(k_l)}n_3(\omega k_l)\Big\}, \tag{11.5}$$

we find

$$\lim_{k\to\infty}k\big(n_3(k)-1\big)=-2i\sqrt{3}V^T(I-B)^{-1}W=-2i\sqrt{3}\mathrm{tr}\left\{(I-B)^{-1}WV^T\right\},$$

where

$$V=V(x,t):=\begin{pmatrix}\tilde{c}_{\lambda_1}e^{-xl_3(\lambda_1)-tz_3(\lambda_1)}\\ \vdots\\ \tilde{c}_{\lambda_{n_b}}e^{-xl_3(\lambda_{n_b})-tz_3(\lambda_{n_b})}\\ \bar{\tilde{c}}_{\lambda_1}e^{-xl_2(\bar{\lambda}_1)-tz_2(\bar{\lambda}_1)}\\ \vdots\\ \bar{\tilde{c}}_{\lambda_{n_b}}e^{-xl_2(\bar{\lambda}_{n_b})-tz_2(\bar{\lambda}_{n_b})}\\ \tilde{c}_{k_1}e^{-xl_2(k_1)-tz_2(k_1)}\\ \vdots\\ \tilde{c}_{k_{n_s}}e^{-xl_2(k_{n_s})-tz_2(k_{n_s})}\end{pmatrix}.$$

Since $WV^T=B_x$, this yields

$$\lim_{k\to\infty}k\big(n_3(k)-1\big)=-2i\sqrt{3}\mathrm{tr}\left\{(I-B)^{-1}B_x\right\}=2i\sqrt{3}\frac{\partial}{\partial x}\ln\det(I-B(x,t)),$$

where we have used Jacobi's formula in the second step. This implies assertion (c). □

11.1. **Proof of Theorem 2.3.** The definition (2.12) of the leading term $u_{\mathrm{sol}}(x,t)$ involves the solution M_{sol} of RH problem 2.7. Comparing RH problem 2.7 for M_{sol} with RH problem 3.1, we see that, for any fixed $(x,t)\in\mathbb{R}\times[0,\infty)$, $n_{\mathrm{sol}}:=(1,1,1)M_{\mathrm{sol}}$ satisfies RH problem 3.1 with $v\equiv I$ and with the replacements:

$$c_{k_0}\text{ is replaced by }\frac{c_{k_0}}{\hat{\Delta}_{11}(\zeta,k_0)\hat{\Delta}_{33}^{-1}(\zeta,k_0)}\text{ for }k_0\in\mathsf{Z}\setminus\mathbb{R},\text{ and} \tag{11.6a}$$

$$c_{k_0}\text{ is replaced by }\frac{c_{k_0}}{\hat{\Delta}_{11}(\zeta,k_0)\hat{\Delta}_{22}^{-1}(\zeta,k_0)}\text{ for }k_0\in\mathsf{Z}\cap\mathbb{R}. \tag{11.6b}$$

Indeed, n_{sol} clearly obeys the residue conditions of RH problem 3.1 at every $k_0\in\mathsf{Z}$. Moreover, the symmetries (6.7) imply that $\hat{\Delta}(\zeta,k)=\mathcal{B}\overline{\hat{\Delta}(\zeta,\bar{k})}^{-1}\mathcal{B}$, and hence

$$\overline{\hat{\Delta}_{11}(\zeta,k)\hat{\Delta}_{33}^{-1}(\zeta,k)}=\hat{\Delta}_{33}(\zeta,\bar{k})\hat{\Delta}_{22}^{-1}(\zeta,\bar{k}), \tag{11.7a}$$

$$\overline{\hat{\Delta}_{11}(\zeta,k)\hat{\Delta}_{22}^{-1}(\zeta,k)}=\hat{\Delta}_{11}(\zeta,\bar{k})\hat{\Delta}_{22}^{-1}(\zeta,\bar{k}). \tag{11.7b}$$

The relation (11.7a) together with the definition (2.27) of d_{k_0} implies that n_{sol} obeys also the residue condition at $\bar{k}_0$ for every $k_0\in\mathsf{Z}\setminus\mathbb{R}$. The remaining residue conditions then follow from the symmetries in (2.30). Furthermore, the relation (11.7b) implies that the positivity condition (11.1) is preserved by the above replacements. We may therefore apply Lemma 11.1 with the c_{k_0} replaced as in (11.6) to immediately obtain the explicit formula (2.19) for u_{sol}. This completes the proof of Theorem 2.3.

12. The RH problem for M_{sol}

In this section, we let $u_0,u_1\in\mathcal{S}(\mathbb{R})$ be real-valued initial data such that $\int_{\mathbb{R}}u_1dx=0$ and such that Assumptions (i)–(iii) hold. Let $\{r_1,r_2,\mathsf{Z},\{c_{k_0}\}_{k_0\in\mathsf{Z}}\}$ be the associated scattering data defined by (2.3) and (2.6).

Lemma 12.1. *For any $(x,t) \in \mathbb{R}\times[0,\infty)$, the solution of RH problem 2.7 exists, is unique, and satisfies $\det M_{\mathrm{sol}}(x,t,k) = 1$. Moreover, the function $u_{\mathrm{sol}}(x,t)$ defined in (2.12) is a smooth function of x and t.*

Proof. In view of (11.7), the coefficients in (2.32)–(2.34) obey the appropriate complex conjugation properties for [10, Lemma A.2] to apply, which implies that the solution of RH problem 2.7 exists and is unique. The identity $\det M_{\mathrm{sol}}(x,t,k) = 1$ can be proved by transforming the RH problem into an equivalent RH problem in which the poles are replaced by jumps on small circles and noting that the jumps have unit determinant. The fact that $u_{\mathrm{sol}}(x,t)$ defined in (2.12) is a smooth function of x and t follows from [10, Theorem 2.6 and Remark 2.7]. □

For each $k_0 \in \mathsf{Z}$, we let $D_\epsilon(k_0)$ be a small open disk centered at k_0 of radius $\epsilon > 0$. We let $D_\epsilon(k_0)^{-1}$, $D_\epsilon(k_0)^*$, and $D_\epsilon(k_0)^{-*}$ be the images of $D_\epsilon(k_0)$ under the maps $k \mapsto k^{-1}$, $k \mapsto \bar{k}$, and $k \mapsto \bar{k}^{-1}$, respectively. If $k_0 \in \mathbb{R}$, then $D_\epsilon(k_0)^* = D_\epsilon(k_0)$. Let $\partial D_\epsilon(k_0)$ and $\partial D_\epsilon(k_0)^*$ be oriented counterclockwise, and let $\partial D_\epsilon(k_0)^{-1}$ and $\partial D_\epsilon(k_0)^{-*}$ be oriented clockwise. Let

$$\mathcal{D}_{\mathrm{sol}} = \bigcup_{k_0\in\mathsf{Z}} \bigcup_{j=0,1,2} \Big(\omega^j D_\epsilon(k_0) \cup \omega^j D_\epsilon(k_0)^{-1} \cup \omega^j D_\epsilon(k_0)^* \cup \omega^j D_\epsilon(k_0)^{-*}\Big), \tag{12.1}$$

$$\partial\mathcal{D}_{\mathrm{sol}} = \bigcup_{k_0\in\mathsf{Z}} \bigcup_{j=0,1,2} \Big(\omega^j \partial D_\epsilon(k_0) \cup \omega^j \partial D_\epsilon(k_0)^{-1} \cup \omega^j \partial D_\epsilon(k_0)^* \cup \omega^j \partial D_\epsilon(k_0)^{-*}\Big),$$

so that $\partial\mathcal{D}_{\mathrm{sol}}$ is the union of $6|\mathsf{Z}\cap\mathbb{R}|+12|\mathsf{Z}\setminus\mathbb{R}|$ small circles. By decreasing $\epsilon > 0$ if necessary, we may assume that these circles do not intersect each other. Let $\mathcal{I}$ be a compact subset of $(1,\infty)$ and let $\mathsf{P}(\zeta,k)$ be the matrix-valued function defined in (7.2). The following uniform bound on $\hat{M}_{\mathrm{sol}}$ is used in the proof of Lemma 8.2 and in the proof of Theorem 2.5 (which is given in Section 12.1 below).

Lemma 12.2. *For any fixed and small enough $\epsilon > 0$, $\hat{M}_{\mathrm{sol}}(x,t,k) := M_{\mathrm{sol}}(x,t,k)\mathsf{P}(\zeta,k)$ and its inverse are uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}$, and $k \in \mathbb{C} \setminus \mathcal{D}_{\mathrm{sol}}$.*

Proof. First observe that $\det M_{\mathrm{sol}} \equiv 1$ by Lemma 12.1 and that $\det \mathsf{P} \equiv 1$ by (7.2) and (2.15). Thus it is enough to bound $\hat{M}_{\mathrm{sol}}$.

We know from Lemma 7.1 that $\hat{M}_{\mathrm{sol}}(x,t,k)$ satisfies the residue conditions in Lemma 6.4 with $n_j^{(3)}$ replaced by $[\hat{M}_{\mathrm{sol}}(x,t,k)]_j$. We apply a transformation $\hat{M}_{\mathrm{sol}} \to \tilde{M}_{\mathrm{sol}}$ which replaces each of these residue conditions by a jump on the corresponding circle in $\partial\mathcal{D}_{\mathrm{sol}}$. The function $\tilde{M}_{\mathrm{sol}}$ is defined to equal $\hat{M}_{\mathrm{sol}}$ except in $\mathcal{D}_{\mathrm{sol}}$ where we define $\tilde{M}_{\mathrm{sol}}(x,t,k)$ as follows: If $k_0 \in \mathsf{Z}\setminus\mathbb{R}$, we define $\tilde{M}_{\mathrm{sol}}$ for $k \in D_\epsilon(k_0) \cup D_\epsilon(k_0)^*$ by

$$\tilde{M}_{\mathrm{sol}}(x,t,k) = \begin{cases} \hat{M}_{\mathrm{sol}}(x,t,k)Q_1^<(x,t,k) & \text{if } k \in D_\epsilon(k_0) \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0, \\ \hat{M}_{\mathrm{sol}}(x,t,k)Q_7^<(x,t,k) & \text{if } k \in D_\epsilon(k_0)^* \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0, \\ \hat{M}_{\mathrm{sol}}(x,t,k)Q_1^>(x,t,k) & \text{if } k \in D_\epsilon(k_0) \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0, \\ \hat{M}_{\mathrm{sol}}(x,t,k)Q_7^>(x,t,k) & \text{if } k \in D_\epsilon(k_0)^* \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0, \end{cases} \tag{12.2}$$

where

$$Q_1^< = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-c_{k_0}^{-1}e^{\theta_{31}(x,t,k_0)}}{\Delta_{33}'(\zeta,k_0)(\Delta_{11}^{-1})'(\zeta,k_0)(k-k_0)} & 0 & 1 \end{pmatrix}, \quad Q_7^< = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-d_{k_0}^{-1}e^{-\theta_{32}(x,t,\bar{k}_0)}}{\Delta_{22}'(\zeta,\bar{k}_0)(\Delta_{33}^{-1})'(\zeta,\bar{k}_0)(k-\bar{k}_0)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$Q_1^> = \begin{pmatrix} 1 & 0 & \frac{-c_{k_0}e^{-\theta_{31}(x,t,k_0)}}{\Delta_{11}(\zeta,k_0)\Delta_{33}^{-1}(\zeta,k_0)(k-k_0)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_7^> = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-d_{k_0}e^{\theta_{32}(x,t,\bar{k}_0)}}{\Delta_{33}(\zeta,\bar{k}_0)\Delta_{22}^{-1}(\zeta,\bar{k}_0)(k-\bar{k}_0)} & 1 \end{pmatrix},$$

and if $k_0 \in \mathsf{Z} \cap \mathbb{R}$, we define $\tilde{M}_{\mathrm{sol}}$ for $k \in D_\epsilon(k_0)$ by

$$\tilde{M}_{\mathrm{sol}}(x,t,k) = \begin{cases} \hat{M}_{\mathrm{sol}}(x,t,k) P_1^<(x,t,k) & \text{if } k \in D_\epsilon(k_0) \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) < 0, \\ \hat{M}_{\mathrm{sol}}(x,t,k) P_1^\geq(x,t,k) & \text{if } k \in D_\epsilon(k_0) \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) \geq 0, \end{cases} \tag{12.3}$$

where

$$P_1^< = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-c_{k_0}^{-1} e^{\theta_{21}(x,t,k_0)}}{\Delta_{22}'(\zeta,k_0)(\Delta_{11}^{-1})'(\zeta,k_0)(k-k_0)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1^\geq = \begin{pmatrix} 1 & \frac{-c_{k_0} e^{-\theta_{21}(x,t,k_0)}}{\Delta_{11}(\zeta,k_0)\Delta_{22}^{-1}(\zeta,k_0)(k-k_0)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here prime denotes differentiation with respect to the k-variable. We then extend $\tilde{M}_{\mathrm{sol}}$ to all of $\mathcal{D}_{\mathrm{sol}}$ by means of the $\mathcal{A}$- and $\mathcal{B}$-symmetries (2.30).

It follows from the above definition of $\tilde{M}_{\mathrm{sol}}$ and Lemma 6.4 (with $n_j^{(3)}$ replaced by $[\hat{M}_{\mathrm{sol}}(x,t,k)]_j$) that $\tilde{M}_{\mathrm{sol}}$ has no poles at the points in $\hat{\mathsf{Z}}$. Moreover, on $\partial\mathcal{D}_{\mathrm{sol}}$, the boundary values of $\tilde{M}_{\mathrm{sol}}$ exist, are continuous, and satisfy $(\tilde{M}_{\mathrm{sol}})_+ = (\tilde{M}_{\mathrm{sol}})_- v_{\mathrm{sol}}$, where

$$v_{\mathrm{sol}}(x,t,k) := \begin{cases} Q_1^<(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \setminus \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0, \\ Q_7^<(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \setminus \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0, \\ Q_1^\geq(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \setminus \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0, \\ Q_7^\geq(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z} \setminus \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0, \\ P_1^<(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) < 0, \\ P_1^\geq(x,t,k) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z} \cap \mathbb{R}, \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) \geq 0, \end{cases} \tag{12.4}$$

and v_{sol} is extended to all of $\partial\mathcal{D}_{\mathrm{sol}}$ by means of the $\mathcal{A}$- and $\mathcal{B}$-symmetries (3.9). As in Section 2, let $S_0 \subset \mathcal{I}$ be the (finite) set of all $\zeta \in \mathcal{I}$ for which there exists a k_0 such that either $k_0 \in (\mathsf{Z} \setminus \mathbb{R}) \cap \{k \,|\, \operatorname{Re}\Phi_{31}(\zeta,k) = 0\}$ or $k_0 \in \mathsf{Z} \cap \mathbb{R} \cap \{k \,|\, \operatorname{Re}\Phi_{21}(\zeta,k) = 0\}$. In other words, S_0 is the set of velocities of the asymptotic solitons encoded in u_{sol} that lie in $\mathcal{I}$. The jump matrix v_{sol} is exponentially small as $t \to \infty$ with $\zeta \in \mathcal{I} \setminus S_0$ fixed. Thus, standard small-norm estimates show that $\tilde{M}_{\mathrm{sol}}(x,t,k)$, and hence also $\hat{M}_{\mathrm{sol}}(x,t,k)$, is uniformly bounded for $t \geq 2$ and $k \in \mathbb{C} \setminus \mathcal{D}_{\mathrm{sol}}$ as long as $\zeta \in \mathcal{I}$ stays a bounded distance away from S_0. This argument fails if $\zeta \in S_0$, because then v_{sol} is of order 1. To establish the desired estimate uniformly for all $\zeta \in \mathcal{I}$, we therefore proceed as follows.

Let $w_{\mathrm{sol}} = v_{\mathrm{sol}} - I$. Let $\mathcal{C}$ be the Cauchy operator on $\partial\mathcal{D}_{\mathrm{sol}}$ (defined as in (9.4) with $\hat{\Gamma}$ replaced by $\partial\mathcal{D}_{\mathrm{sol}}$) and let $\mathcal{C}_{w_{\mathrm{sol}}} h = \mathcal{C}_-(h w_{\mathrm{sol}})$. Then $\tilde{M}_{\mathrm{sol}} = I + \mathcal{C}(\mu_{\mathrm{sol}} w_{\mathrm{sol}})$ where $\mu_{\mathrm{sol}} = I + (I - \mathcal{C}_{w_{\mathrm{sol}}})^{-1} \mathcal{C}_{w_{\mathrm{sol}}} I \in I + L^2(\partial\mathcal{D}_{\mathrm{sol}})$. Thus it is enough to show that, for every sufficienty small $\epsilon > 0$, $\|(I - \mathcal{C}_{w_{\mathrm{sol}}})^{-1}\|_{\mathcal{B}(L^2(\partial\mathcal{D}_{\mathrm{sol}}))}$ is uniformly bounded for $t \geq 2$ and $\zeta \in \mathcal{I}$.

If $\zeta_1, \zeta_2 \in S_0$ are such that no other point in S_0 lies between ζ_1 and ζ_2, then the integers

$$|\{k_0 \in \mathsf{Z} \setminus \mathbb{R} \,|\, \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0\}| \quad \text{and} \quad |\{k_0 \in \mathsf{Z} \cap \mathbb{R} \,|\, \operatorname{Re}\Phi_{21}(\zeta,k_0) < 0\}| \tag{12.5}$$

are independent of $\zeta \in [\zeta_1, \zeta_2)$.[2] Since S_0 is a finite set, it is therefore sufficient to prove that $\|(I - \mathcal{C}_{w_{\mathrm{sol}}})^{-1}\|_{\mathcal{B}(L^2(\partial\mathcal{D}_{\mathrm{sol}}))}$ is uniformly bounded for $t \geq 2$ and $\zeta \in \mathcal{J}$ whenever $\mathcal{J}$ is a subinterval of $\mathcal{I}$ such that the integers in (12.5) are independent of $\zeta \in \mathcal{J}$.

[2]The interval includes ζ_1 but not ζ_2 because $\operatorname{Re}\partial_\zeta\Phi_{31}(\zeta,k_0) > 0$ for each $k_0 \in \mathsf{Z}\setminus\mathbb{R}$ and $\operatorname{Re}\partial_\zeta\Phi_{21}(\zeta,k_0) > 0$ for each $k_0 \in \mathsf{Z} \cap \mathbb{R}$.

To each set $\mathsf{A} = \{A_{k_0}\}_{k_0 \in \mathsf{Z}} \subset \mathbb{C}^{|\mathsf{Z}|}$, we associate a jump matrix $\mathsf{v}(k;\mathsf{A})$ on $\partial\mathcal{D}_{\rm sol}$ by setting

$$\mathsf{v}(k;\mathsf{A}) := \begin{cases} \mathsf{Q}_1^<(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z}\setminus\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0 \text{ for } \zeta \in \mathcal{J}, \\ \mathsf{Q}_7^<(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z}\setminus\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) < 0 \text{ for } \zeta \in \mathcal{J}, \\ \mathsf{Q}_1^\geq(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z}\setminus\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0 \text{ for } \zeta \in \mathcal{J}, \\ \mathsf{Q}_7^\geq(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0)^*,\ k_0 \in \mathsf{Z}\setminus\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{31}(\zeta,k_0) \geq 0 \text{ for } \zeta \in \mathcal{J}, \\ \mathsf{P}_1^<(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z}\cap\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) < 0 \text{ for } \zeta \in \mathcal{J}, \\ \mathsf{P}_1^\geq(k;\mathsf{A}) & \text{if } k \in \partial D_\epsilon(k_0),\ k_0 \in \mathsf{Z}\cap\mathbb{R}, \text{ and } \operatorname{Re}\Phi_{21}(\zeta,k_0) \geq 0 \text{ for } \zeta \in \mathcal{J}, \end{cases}$$

where

$$\mathsf{Q}_1^< = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{A_{k_0}}{k-k_0} & 0 & 1 \end{pmatrix}, \quad \mathsf{Q}_7^< = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\bar A_{k_0}}{k-k_0}\frac{\omega^2(\omega^2-\bar k_0^2)}{\bar k_0^2-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{Q}_1^\geq = \begin{pmatrix} 1 & 0 & \frac{A_{k_0}}{k-k_0} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathsf{Q}_7^\geq = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\bar A_{k_0}}{k-k_0}\frac{\bar k_0^2-1}{\omega^2(\omega^2-\bar k_0^2)} & 1 \end{pmatrix}, \quad \mathsf{P}_1^< = \begin{pmatrix} 1 & 0 & 0 \\ \frac{A_{k_0}}{k-k_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{P}_1^\geq = \begin{pmatrix} 1 & \frac{A_{k_0}}{k-k_0} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and v is extended to all of $\partial\mathcal{D}_{\rm sol}$ by means of the $\mathcal{A}$- and $\mathcal{B}$-symmetries (3.9).

Note that $\overline{\theta_{31}(x,t,k_0)} = -\theta_{32}(x,t,\bar k_0)$ and that, by (6.9), $\Delta_{11}(\zeta,k) = \overline{\Delta_{22}(\zeta,\bar k)}^{-1}$ and $\Delta_{33}(\zeta,k) = \overline{\Delta_{33}(\zeta,\bar k)}^{-1}$. Hence there is a compact subset K of $\mathbb{C}^{|\mathsf{Z}|}$ such that

$$\{v_{\rm sol}(x,t,\cdot)\,|\,t \geq 2, \zeta \in \mathcal{J}\} \subset \{\mathsf{v}(\cdot;\mathsf{A})\,|\,\mathsf{A} \in K\}. \tag{12.6}$$

Let U be the open subset of $\mathcal{B}(L^2(\partial\mathcal{D}_{\rm sol}))$ consisting of all invertible operators. Let $\mathsf{w} = \mathsf{v} - I$. By the existence results for RH problem 2.7, we may assume[3] that the operator $I - \mathcal{C}_{\mathsf{w}(\cdot;\mathsf{A})}$ belongs to U for each $\mathsf{A} \in K$. The map $\mathsf{A} \mapsto \mathsf{w}(\cdot;\mathsf{A}) : \mathbb{C}^{|\mathsf{Z}|} \to L^\infty(\partial\mathcal{D}_{\rm sol})$ is continuous, and the map

$$\mathsf{w} \mapsto I - \mathcal{C}_{\mathsf{w}} : L^\infty(\partial\mathcal{D}_{\rm sol}) \to U \subset \mathcal{B}(L^2(\partial\mathcal{D}_{\rm sol}))$$

is continuous by the estimate $\|\mathcal{C}_{\mathsf{w}}\|_{\mathcal{B}(L^2(\partial\mathcal{D}_{\rm sol}))} \leq C\|\mathsf{w}\|_{L^\infty(\partial\mathcal{D}_{\rm sol})}$. Moreover, the map $T \mapsto T^{-1}$ is continuous on U. By composing these maps, we see that the map that sends A to $\|(I-\mathcal{C}_{\mathsf{w}(\cdot;\mathsf{A})})^{-1}\|_{\mathcal{B}(L^2(\partial\mathcal{D}_{\rm sol}))}$ is continuous $K \to \mathbb{R}$ and hence bounded on K. It now follows from (12.6) that $\|(I-\mathcal{C}_{w_{\rm sol}})^{-1}\|_{\mathcal{B}(L^2(\partial\mathcal{D}_{\rm sol}))}$ is uniformly bounded for $t \geq 2$ and $\zeta \in \mathcal{J}$. □

12.1. **Proof of Theorem 2.5.** Let $\zeta_0 \in S_0$ and let $\epsilon > 0$ be so small that $\zeta_0 - \epsilon > 1$ and $\mathcal{I}_0 := \{\zeta \in \mathbb{R}\,|\,|\zeta-\zeta_0| \leq \epsilon\}$ contains no other point in S_0. Let $M^0_{\rm sol}$ be the unique solution of RH problem 2.7 with Z replaced by $\mathsf{Z}^0 := \{k_0 \in \mathsf{Z}\,|\,\zeta_{k_0} = \zeta_0\}$, i.e., $M^0_{\rm sol}$ solves the same RH problem as $M_{\rm sol}$ except that only those residue conditions that give rise to asymptotic solitons propagating at velocity ζ_0 are included. Let $\hat M_{\rm sol} = M_{\rm sol}\mathsf{P}$ and $\hat M^0_{\rm sol} = M^0_{\rm sol}\mathsf{P}$, where P is given by (7.2). Define $\tilde M_{\rm sol}$ and $v_{\rm sol}$ in terms of $\hat M_{\rm sol}$ as in the proof of Lemma 12.2. Define $\tilde M^0_{\rm sol}$ and $v^0_{\rm sol}$ analogously in terms of $\hat M^0_{\rm sol}$. Then $v^0_{\rm sol}$ is the restriction of $v_{\rm sol}$ to the subcontour $\partial\mathcal{D}^0_{\rm sol}$ of $\partial\mathcal{D}_{\rm sol}$, where $\mathcal{D}^0_{\rm sol}$ is given by (12.1) with Z replaced by Z^0.

Shrinking the disks that make up $\mathcal{D}_{\rm sol}$ if necessary, the jump matrix $v_{\rm sol}$ is $O(e^{-ct})$ on $\partial\mathcal{D}_{\rm sol}\setminus\partial\mathcal{D}^0_{\rm sol}$ uniformly for $\zeta \in \mathcal{I}_0$ as $t \to \infty$. The matrix $\tilde M_{\rm sol}(\tilde M^0_{\rm sol})^{-1}$ has no jump on $\partial\mathcal{D}^0_{\rm sol}$ and the jump on $\partial\mathcal{D}_{\rm sol} \setminus \partial\mathcal{D}^0_{\rm sol}$ is given by $\tilde M^0_{\rm sol} v_{\rm sol}(\tilde M^0_{\rm sol})^{-1}$. Since $\tilde M^0_{\rm sol}$ and $(\tilde M^0_{\rm sol})^{-1}$ are uniformly bounded for $t \geq 2$, $\zeta \in \mathcal{I}_0$, and $k \in \mathbb{C}\setminus\mathcal{D}^0_{\rm sol}$ by Lemma 12.2, we conclude that the

[3]We write "we may assume", because K cannot be chosen completely arbitrarily. Indeed, for Lemma 12.1 to apply, the following conditions must be satisfied: For $k_0 \in \mathsf{Z}\cap\mathbb{R}$ such that $A_{k_0} \neq 0$, we must have $i(\omega^2 k_0^2 - \omega)\mathrm{c}_{k_0} \geq 0$, where $\mathrm{c}_{k_0} := -1/(\Delta'_{22}(\zeta,k_0)(\Delta_{11}^{-1})'(\zeta,k_0)A_{k_0})$ if $\operatorname{Re}\Phi_{21}(\zeta,k_0) < 0$ for $\zeta \in \mathcal{J}$, and $\mathrm{c}_{k_0} := -\Delta_{11}(\zeta,k_0)\Delta_{22}^{-1}(\zeta,k_0)A_{k_0}$ if $\operatorname{Re}\Phi_{21}(\zeta,k_0) \geq 0$ for $\zeta \in \mathcal{J}$.

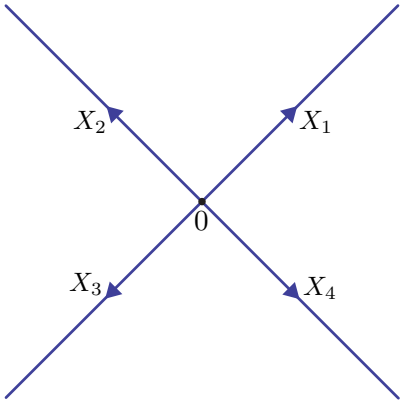

Figure 11. *The contour* $X = X_1 \cup X_2 \cup X_3 \cup X_4$.

jump of $\tilde{M}_{\rm sol}(\tilde{M}^0_{\rm sol})^{-1}$ is exponentially small as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}_0$. Consequently, by standard estimates for small-norm RH problems,

$$\lim_{k\to\infty} k\Big(\big((1,1,1)M_{\rm sol}(x,t,k)\big)_3 - 1\Big) = \lim_{k\to\infty} k\Big(\big((1,1,1)M^0_{\rm sol}(x,t,k)\big)_3 - 1\Big) + O(e^{-ct})$$

as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}_0$, and this formula can be differentiated with respect to x without changing the error term. By (2.12) and Theorem 2.3 (applied with Z replaced by Z^0), this implies that

$$u_{\rm sol}(x,t) = u_s\bigg(x,t; \bigg(\lambda_j, \frac{c_{\lambda_j}}{\hat{\Delta}_{11}(\zeta,\lambda_j)\hat{\Delta}_{33}^{-1}(\zeta,\lambda_j)}\bigg)_{j\in I_0}, \bigg(k_j, \frac{c_{k_j}}{\hat{\Delta}_{11}(\zeta,k_j)\hat{\Delta}_{22}^{-1}(\zeta,k_j)}\bigg)_{j\in J_0}\bigg) + O(e^{-ct})$$

as $t \to \infty$ uniformly for $\zeta \in \mathcal{I}_0$. Substituting this formula into (2.11), we arrive at the asymptotic formula (2.25).

Finally, note that if $C_0 > 0$, then

$$\frac{c_{k_j}}{\hat{\Delta}_{11}(\zeta,k_j)\hat{\Delta}_{22}^{-1}(\zeta,k_j)} = \frac{c_{k_j}}{\hat{\Delta}_{11}(\zeta_{k_j},k_j)\hat{\Delta}_{22}^{-1}(\zeta_{k_j},k_j)} + O(t^{-1}) \qquad \text{as } t \to \infty$$

uniformly for $x \in [\zeta_{k_j}t - C_0, \zeta_{k_j}t + C_0]$. Hence the asymptotic formula (2.26) follows from (2.22) and (2.25). The proof of Theorem 2.5 is complete.

Appendix A. A model RH problem

The model RH problem considered in this appendix is needed for the local parametrix near k_1. The solution of this RH problem is constructed in terms of parabolic cylinder functions as in [25]. We omit the proof here, but refer to [9, Appendix A] for the construction of the solution to a similar (but more complicated) RH problem.

Let $X = X_1 \cup \cdots \cup X_4 \subset \mathbb{C}$ be the cross defined by

$$\begin{aligned} X_1 &= \big\{se^{\frac{i\pi}{4}} \,\big|\, 0 \le s < \infty\big\}, & X_2 &= \big\{se^{\frac{3i\pi}{4}} \,\big|\, 0 \le s < \infty\big\}, \\ X_3 &= \big\{se^{-\frac{3i\pi}{4}} \,\big|\, 0 \le s < \infty\big\}, & X_4 &= \big\{se^{-\frac{i\pi}{4}} \,\big|\, 0 \le s < \infty\big\}, \end{aligned} \tag{A.1}$$

and oriented away from the origin, see Figure 11.

Lemma A.1 (Model RH problem needed near $k = k_1$)**.** *Let* $q \in \mathbb{C}$*, and define*

$$\nu = -\tfrac{1}{2\pi}\ln(1 + |q|^2) \le 0.$$

Define the jump matrix $v^X(z)$ *for* $z \in X$ *by*

$$\begin{pmatrix} 1 & \frac{-\bar{q}}{1+|q|^2} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_1, \qquad \begin{pmatrix} 1 & 0 & 0 \\ q z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_2,$$

$$\begin{pmatrix} 1 & \bar{q} z_{(0)}^{-2i\nu} e^{\frac{iz^2}{2}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_3, \qquad \begin{pmatrix} 1 & 0 & 0 \\ \frac{-q}{1+|q|^2} z_{(0)}^{2i\nu} e^{-\frac{iz^2}{2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } z \in X_4, \tag{A.2}$$

where $z_{(0)}^{i\nu}$ *has a branch cut along* $[0, +\infty)$*, such that* $z_{(0)}^{i\nu} = |z|^{i\nu} e^{-\nu \arg_0(z)}$*,* $\arg_0(z) \in (0, 2\pi)$*. Then the RH problem*

(a) $m^X(\cdot) = m^X(q, \cdot) : \mathbb{C} \setminus X \to \mathbb{C}^{3\times 3}$ *is analytic;*
(b) *on* $X \setminus \{0\}$*, the boundary values of* m^X *exist, are continuous, and satisfy* $m_+^X = m_-^X v^X$*;*
(c) $m^X(z) = I + O(z^{-1})$ *as* $z \to \infty$*, and* $m^X(z) = O(1)$ *as* $z \to 0$*;*

has a unique solution $m^X(q, z)$*. This solution satisfies*

$$m^X(z) = I + \frac{m_1^X}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad m_1^X := \begin{pmatrix} 0 & \beta_{12} & 0 \\ \beta_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.3}$$

where the error term is uniform for $\arg z \in [-\pi, \pi]$ *and* q *in compact subsets of* $\mathbb{C}$*, and*

$$\beta_{12} := \frac{\sqrt{2\pi} e^{\frac{\pi i}{4}} e^{\frac{3\pi\nu}{2}}}{q\Gamma(i\nu)}, \qquad \beta_{21} := \frac{\sqrt{2\pi} e^{-\frac{\pi i}{4}} e^{-\frac{5\pi\nu}{2}}}{-\bar{q}\Gamma(-i\nu)}. \tag{A.4}$$

(Note that $\beta_{12}\beta_{21} = \nu$ *because* $|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{-\nu}\sqrt{e^{-\pi\nu} - e^{\pi\nu}}} = \frac{\sqrt{2\pi}}{\sqrt{-\nu} e^{\frac{\pi\nu}{2}} |q|}$*.)*

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