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Stability and sustainability of urban systems under commuting and transportation costs*

Yuki Takayama, † Kiyohiro Ikeda, ‡ and Jacques-François Thisse §

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Abstract

This paper explores the conditions for the emergence of a system of cities in a general equilibrium setting that accounts for the cost of shipping commodities between cities and the commuting cost borne by consumers within cities. Potential cities are equally distributed over a circular space. We find that the multiplicity of stable spatial equilibria is the rule and not the exception. Using the concept of stability areas to study the transition from one stable equilibrium to the next, we show that decreasing commuting or transportation costs generate equilibrium paths that feature either a megalopolis or hierarchical system of cities.

Keywords: economic geography; cities; racetrack economy; multiplicity of stable equilibria; commuting costs; transportation costs

JEL Classification: F12, R12

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†Address for correspondence: Yuki Takayama (ytakayama@se.kanazawa-u.ac.jp), Institute of Science and Engineering, Kanazawa University, Kakuma, Kanazawa 920-1192, Japan

‡Department of Civil and Environmental Engineering, Tohoku University, Aoba, Sendai 980-8579, Japan

§CORE, Université catholique de Louvain (Belgium), HSE University (Russian Federation), and CEPR.

1 Introduction

This paper explores the conditions for the emergence of a system of cities in a general equilibrium setting that accounts for the transportation cost of goods between cities, the mobility of consumers across space, and the commuting cost borne by consumers within cities. To achieve our goal, we use a bare-bones framework with one sector and a finite number of locations where monocentric cities can be developed; locations are equidistantly distributed along a circle. Consumers are mobile and choose a city where to live and work, as well as a location within this city where they consume a fixed amount of land. We investigate the impact of spatial linkages between cities (trade) and labor markets (commuting and migration) on the structure of stable equilibria and the transition from one equilibrium to another. We pay a special attention to commuting costs as these costs remain without question high. According to the Census Bureau, 139 million American workers have spent a collective 3.4 million years in commuting during the year 2014 (*The Washington Post*, February 25, 2016). This is a sizable number as the opportunity cost of time has risen with income. In addition, Kahneman et al. (2003) have documented the fact that commuting is one of the activities individuals dislike most, which raises the non-pecuniary costs of longer commutes.

The new fundamental ingredient that a multi-location setting brings is the (implicit) existence of a transport network so that accessibility varies across spatially dispersed locations. In particular, the two-city setting makes the (stability) analysis simple as moving away from one region automatically implies that consumers and firms necessarily go to the other. By contrast, when the spatial economy involves several locations, what happens in one location has different impacts on the others because the accessibility to markets varies across cities. In other words, the relative position of cities within the transportation system matters: any change in parameters that directly involves only two cities generates spatial spillover effects that are unlikely to leave the remaining cities unaffected. This in turn further affects the other regions and so on. As a result, firms and consumers may be more or less agglomerated – or dispersed – across a variable number of locations.

Our main findings may be summarized as follows. First, several spatial equilibria may coexist. In this case, it is common place to consider stability as a selection device. However, for various domains of transportation and commuting costs, there exist *multiple stable equilibria*. This makes it hard a priori to predict the evolution of the urban system when the intensity of spatial frictions changes. Furthermore, one may wonder which equilibrium outcome is associated with the data at hand and what are the results to be tested. As a remedy for these problems, we combine two devices. The first one relies on the historic evidence that shows the resilience of cities and the resulting persistence of spatial equilibria to various kinds of shocks (Davis and Weinstein, 2002; Brakman et al., 2004; Bosker and Buringh, 2017). Therefore, among the plethora of stable equilibria, we will pay a special attention to *invariant* equilibria in which the urban system remains the same for non-degenerate intervals of transportation and/or commuting cost values. In what follows, we call this hypothesis the principle of path dependency.

Somewhat unexpectedly, two types of invariant equilibria exist, i.e., the *symmetric* and *quasi-symmetric* patterns. The former suggests itself and involves equidistant cities that have the same size. The latter is symmetric about an oblique axis and involves one or several pairs of cities having the same size. Since they are established within a small space, the two larger cities may be viewed as forming a *megalopolis*. The existence of stable quasi-symmetric equilibria shows that a multi-location setting leads to possible urban systems that are overlooked in the literature. Our analysis uncovers a still different type of equilibrium, that is, *non-invariant* equilibria in which cities are distributed according to a hierarchy. Again, such equilibria can appear only in multi-location settings.

The coexistence of multiple stable equilibria that differ in nature points to the need of a second selection device to assess the impact of shocks on the urban system. In this paper, we use the concept of *stability area* of a spatial equilibrium, which is defined as the domain of parameters over which this equilibrium is stable. When a shock renders the prevailing equilibrium unstable, the stability area of this equilibrium shares a boundary with the stability area of another spatial equilibrium. This one is the natural candidate in the transition to an alternative pattern. By applying this argument to the subsequent stability areas, it is possible to select a path generated by decreasing transportation or commuting costs. We will see that such a path often involves invariant and non-invariant equilibria.

In sum, we square the circle of multiple stable equilibria by combining historic evidence and simple stability analysis. Putting results together shows that the urban system may vastly differ as they are described either by *a finite number of identical, but not necessarily equidistant, cities* or by *a hierarchical system of cities.* It is worth stressing that hierarchical patterns emerge in a seamless space, thus showing that they are the cheer outcome of interactions among agents. Furthermore, since the selected path depends on the initial conditions, all paths do not necessarily contain all possible stable states. For example, the invariant patterns involving m_1 and $m_2 > m_1$ cities may be stable equilibria while the invariant pattern with m cities such that $m_1 < m < m_2$ need not emerge. We also find that the number of locations available for urban development matters for the set of stable equilibria. More specifically, raising the number of locations entails a rapid increase of the number of spatial equilibria. This concurs with the idea that, by restricting the number of potential settlements, different physical environments and the non-replicability of scarce resources needed for establishing cities may lead to different types of urban systems (Henderson et al., 2018).

We then turn our attention to the effects of decreasing commuting and transportation costs and show that these costs have opposite impacts on the location of activities. This extends Murata and Thisse (2005) and Tabuchi and Thisse (2006) who consider two-location settings. The multi-location setting generates a richer set of results that are more likely to emerge than the perennial cases of full agglomeration or full dispersion. When commuting costs are very high, the economy involves a dispersed pattern of small cities because urban costs become too high when the number of cities is smaller. When commuting costs decrease thanks to the adoption of new transport technologies, *cities are fewer and larger* because the home market effect remains a significant agglomeration force when transportation costs are not too low. Note that *cities need not have the same size nor be equidistant* when commuting costs steadily decline. Last, if commuting costs almost vanish (a very unlikely event), all firms and consumers are gathered in a single mega-city because this pattern brings the cost of shipping goods down to zero.

Finally, we turn our attention to the standard thought experiment of economic geography in which transportation costs decrease. According to Krugman (1991) and Fujita et al. (1999), falling transportation costs would foster the geographical concentration of activities. Ikeda et al. (2012) have extended this result to the case of a racetrack economy by showing that, as transportation costs steadily decrease, the number of market centers is reduced by half, doubling the spacing between them. When urban costs are the main dispersion force, this prediction ceases to hold. To be precise, when commuting costs are not too high, *decreasing transportation costs leads to more and smaller cities*. In other words, improving the transportability of goods fosters the dispersion of activities through a growing number of smaller places where urban costs are lower. Since the level of urban costs is unaffected when the population distribution remains the same, it is no surprise that, eventually, dispersion overcomes agglomeration. This is what Brülhart et al. (2019) observe in developed countries, but not in developing countries, where the market potential effect is significantly weaker than what it used to be thanks to the provision of very efficient transportation infrastructures. Furthermore, the paths generated by decreasing transportation costs display a richer set of possible outcomes than what Ikeda et al. (2012) obtain in the core-periphery model.

Before proceeding, the following comment is in order. The multiplicity of stable equilibria could be driven by the fact that a racetrack economy retains a great deal of symmetry. In contrast, many cities have been developed at locations endowed with specific natural advantages or are the outcome of historical accidents, such as the existence of a colonial transport networks that beget a lock-in effect on the location of economic activities. All of this points to the need to work with more general spaces. However, the work of Allen et al. (2020) in trade theory shows how difficult it is to work with a general matrix of spatial frictions. So, one should not expect a silver bullet to solve the dimensionality problem in economic geography. This is why we want to argue in this paper that working with simple geographies remains a useful departure from the canonical two-location setting.

Related literature. Urban economics and economic geography compete for adherents among economists and regional scientists to explain the organization of the space-economy. In urban economics, the dominant model relies on the pioneering work of Henderson (1974) where land developers understand that they may benefit from organizing cities in a way that maximizes the local land rent, while internalizing the external effects generated by the agglomeration of firms at the city center. However, cities trade their outputs at zero transportation costs. As a result, if we understand why cities exist, we remain in the blur about where cities are.[1] In addition, many cities were established at a time where transportation costs were high. This points to the need of a theory of urban systems that accounts for transportation costs between cities. On the other hand, following Krugman (1991) and Fujita et al. (1999), most of the economic geography literature focusses on two locations. Akamatsu et al. (2012) and Ikeda et al. (2012) are noticeable exceptions. However, their setting is very different from ours.

Rossi-Hansberg (2005) is another attempt made to reconcile a large number of locations (formally, a continuum), positive transportation costs, and spatial externalities within an integrated setting. This approach is able to capture a rich set of effects that do not unfold in a two-location economy. However, this comes at the cost of models that are technically difficult to solve. Consequently, analytical results are not many and one has to appeal to numerical analysis to characterize the system of cities (Desmet and Rossi-Hansberg, 2013). Finally, spatial quantita-

[1] Recent contributions, such as Behrens *et al.* (2014) and Davis and Dingel (2019), explain how cities may emerge without appealing to land developers. If these papers make a major step forward by explaining that cities have different sizes because they have different skill compositions, they still ignore between-city transportation costs.

tive models account for both commuting and transportation costs in multi-location settings (Redding and Rossi-Hansberg, 2017). However, these models do not tell us much about the properties of the urban system as their main purpose is to quantify the impact of various shocks on the spatial equilibrium.

The paper is organized as follows. Section 2 presents the model. The set of stable equilibria is characterized in Section 3 when there are four potential cities. In Section 4, we study the properties of invariant patterns in the case of an arbitrary number of locations. Section 5 presents the results of numerical analyses in the case of eight potential cities. Section 6 concludes.

2 The model

2.1 The economy

The economy features $n \geq 2$ potential cities equidistantly distributed along the unit circle $\mathbf{C}$, and one sector producing a differentiated good, one production factor, labor.[2] The advantage of working with a circular space is that it rules out boundary effects that may act as an agglomeration force. City $i \in \mathcal{I} \equiv \{0, 1, \cdots, n-1\}$ is monocentric. It is described by a one-dimensional arc $\mathcal{A}_i$ and has a spaceless central business district (CBD) located at the city center ($x = 0$). The distance between the CBDs of cities i and j is measured by the minimum path length, i.e., $\ell_{ij} = \min\{|i-j|/n, 1-|i-j|/n\}$. The amount of land available at each location of $\mathbf{C}$ is equal to one and the opportunity cost of land is zero.

The differentiated good is produced by using labor under monopolistic competition and increasing returns. Each variety is provided by a single firm and each firm supplies a single variety. To operate, a firm in city i needs a fixed requirement of $\alpha > 0$ units and a marginal requirement of $\beta > 0$ units of labor. Any variety is shipped according to an iceberg transportation technology, i.e., for each unit of the variety shipped from city i to city j, only a fraction $1/\tau_{ij} \leq 1$ arrives at destination. The transportation cost τ_{ij} between locations i and j is given by $\tau_{ij} = \exp(\tau \ell_{ij}) \geq 1$, where $\tau \geq 0$ is the uniform transportation rate.

There is a unit mass of identical consumers. Each consumer consumes one unit of land and supplies one unit of labor. Admittedly, the assumption of a fixed lot size does not allow us to replicate the empirically well-documented fact that the

[2]Since we allow for the spatial extension of cities, we find it reasonable to consider atomic distributions of monocentric cities. We leave for future research the case of cities whose employment centers are endogenous.

population density decreases as the distance to the CBD increases. However, our main results remain qualitatively the same when consumers are free to choose their land consumption. A consumer chooses the city where she wants to live and her residential location in that city. The mass of consumers residing in city i is denoted by $h_i \geq 0$. Hence, at the residential equilibrium, consumers residing in city $i \in \mathcal{I}$ are uniformly distributed over the arc $\mathcal{A}_i \equiv [-h_i/2, h_i/2]$.

Commuting costs have the nature of an iceberg. More specifically, the effective labor supply $l(x)$ by a consumer living at a distance $|x| \leq h_i/2 \leq 1/2$ from the CBD is given by

$$l(x) = 1 - 2\theta|x| \qquad \forall x \in \mathcal{A}_i,$$

where $\theta \geq 0$ denotes the commuting rate expressed in labor unit. We assume that $\theta \leq 1$ for $l(x) \geq 0$ to hold regardless of the spatial distribution of consumers. This modeling strategy captures the fact that individuals who have a long commute are more prone to being absent from work, to arrive late at the workplace and/or to make less work effort (van Ommeren and Gutiérrez-i-Puigarnau, 2011). An iceberg cost is also consistent with the empirical literature that shows that commuting costs increase with income.

The effective labor supply in city i is:

$$L_i(h_i) = \int_{-h_i/2}^{h_i/2} l(x)\mathrm{d}x = h_i\left(1 - \frac{\theta}{2}h_i\right).$$

Since L_i is maximized at $h_i = 1/\theta \geq 1$, $L_i(h_i)$ increases at a decreasing rate. On the other hand, $L_i(h_i)$ decreases when the commuting rate increases.

Let $r_i(x)$ be the land rent at location $x \in \mathcal{A}_i$. At the residential equilibrium, the wage net of both commuting costs and land rent must be equal across $\mathcal{A}_i$. Therefore, we have

$$l(x)w_i - r_i(x) = l(\tilde{x})w_i - r_i(\tilde{x}) \qquad \forall x, \tilde{x} \in \mathcal{A}_i,$$

where w_i denotes the wage rate paid at the CBD of city i. The equilibrium land rent in city i is given by

$$r_i(x) = \theta\left(h_i - 2|x|\right)w_i \qquad \forall x \in \mathcal{A}_i,$$

so that the aggregate land rent in city i is as follows:

$$\int_{-h_i}^{h_i} r_i(x)\mathrm{d}x = \frac{\theta}{2}w_i h_i^2.$$

Land is owned by consumers residing in each city. Therefore, the income net of commuting costs and land rents of a consumer residing at x in city i is equal to

$$y_i = \left(1 - \frac{\theta}{2} h_i\right) w_i.$$

Hence, the level of *urban costs* in city i is given by $w_i - y_i = (\theta/2)h_i w_i$. Assuming that cities are polycentric rather than monocentric does not affect our results. Since a polycentric city has lower urban costs than a monocentric city, allowing cities to be polycentric weakens the intensity of the dispersion force but does not affect the general trends.

Each consumer residing in city i is endowed with CES preferences:

$$U_i = \left[\sum_{j\in\mathcal{I}} \int_0^{M_j} q_{ji}(k)^{(\sigma-1)/\sigma} \mathrm{d}k\right]^{\sigma/(\sigma-1)},$$

where M_j is the mass of varieties produced in city j, $q_{ji}(k)$ the consumption of variety $k \in [0, M_j]$, and σ the constant elasticity of substitution between any two varieties.

The budget constraint is given by

$$y_i = \sum_{j\in\mathcal{I}} \int_0^{M_j} p_{ji}(k) q_{ji}(k) \mathrm{d}k,$$

where $p_{ji}(k)$ denotes the price in city i of variety k produced in city j. This price is independent of the consumer's location in city i.

Utility maximization yields the following demand functions:

$$q_{ji}(k) = p_{ji}(k)^{-\sigma} P_i^{\sigma-1} y_i, \tag{1}$$

where

$$P_i \equiv \left[\sum_{j\in\mathcal{I}} \int_0^{M_j} p_{ji}(k)^{1-\sigma} \mathrm{d}k\right]^{1/(1-\sigma)}$$

denotes the price index in city i.

Since the total income in city i is $h_i y_i = L_i w_i$, city i's total demand $Q_{ji}(k)$ for variety k produced in city j is equal to

$$Q_{ji}(k) = p_{ji}(k)^{-\sigma} P_i^{\sigma-1} L_i w_i. \tag{2}$$

Market clearing implies that the supply $x_i(k)$ of variety k produced in city i is given by

$$x_i(k) = \sum_{j \in \mathcal{I}} \tau_{ij} Q_{ij}(k). \tag{3}$$

Producing $x_i(k)$ units of variety k requires $\alpha + \beta x_i(k)$ units of labor. The total production cost of a firm in city i is thus given by $[\alpha + \beta x_i(k)]\, w_i$. Each firm located in city i maximizes its profits given by

$$\Pi_i(k) = \sum_{j \in \mathcal{I}} p_{ij}(k) Q_{ij}(k) - w_i \left[\alpha + \beta x_i(k)\right].$$

Under monopolistic competition, the first order condition for profit maximization yields the equilibrium price

$$p^*_{ij}(k) = \frac{\sigma}{\sigma - 1} \beta w_i \tau_{ij}, \tag{4}$$

which is the same across varieties $k \in [0, M_i]$. Since all firms set up in city i charge the same price in equilibrium, we drop the variety index k.

2.2 The market equilibrium

We first assume that consumers are immobile across cities, but are free to choose their location within each city. In other words, the spatial distribution $\mathbf{h} \equiv (h_i)_{i \in \mathcal{I}}$ is given. The market equilibrium conditions involve the differentiated product and labor market clearing conditions and the zero-profit condition associated with free entry. The first condition is given by (3), while the second is such that

$$(\alpha + \beta x_i) M_i = L_i.$$

The third condition implies that a firm's operating profits are entirely absorbed by the sum of wages paid its workers:

$$(\alpha + \beta x_i) w_i = \sum_{i \in \mathcal{I}} p_{ij} Q_{ij}.$$

These three conditions and (4) yield the equilibrium output, $x^*_i = (\sigma - 1)\alpha/\beta$, and the equilibrium mass of firms in city i, $M^*_i(\mathbf{h}) = L_i/(\alpha\sigma)$. Hence, the mass of firms established in city i increases when the commuting rate θ decreases because more labor becomes available for production.

Substituting x_i^*, $M_i^*(\mathbf{h})$ and (4) into (1) yields the price index in city i:

$$P_i^*(\mathbf{h}) = \frac{\beta\sigma}{\sigma - 1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{I}} L_j w_j^{1-\sigma}\phi_{ji}\right)^{\frac{1}{1-\sigma}},$$

where $\phi_{ji} \equiv \tau_{ji}^{1-\sigma} \in [0,1]$ measures the freeness of trade between cities i and j. Since $\tau_{ij} = \exp(\tau\ell_{ij})$, we have $\phi_{ij} \equiv \phi^{t_{ij}}$ where $t_{ij} \equiv \min\{|i-j|, n-|i-j|\}$ and $\phi \equiv \exp(-(\sigma-1)\tau/n) \in [0,1]$ decreases when the transportation rate τ increases, while ϕ_{ij} decreases when the distance ℓ_{ij} rises (transportation costs are prohibitively high when $\phi = 0$ and zero when $\phi = 1$). In what follows, we will use ϕ as the index of transportability of the differentiated good.

Furthermore, it follows from (2) and (3) that the wage equation corresponding to city i is given by

$$L_i w_i = \sum_{j\in\mathcal{I}} \frac{L_i w_i^{1-\sigma}\phi_{ij}}{\sum_{k\in\mathcal{I}} L_k w_k^{1-\sigma}\phi_{kj}} L_j w_j, \qquad \forall i \in \mathcal{I}. \tag{5}$$

Set $\mathbf{w} \equiv (w_i)_{i\in\mathcal{I}}$. Given the solution $\mathbf{w}^*(\mathbf{h}) \equiv (w_i^*(\mathbf{h}))_{i\in\mathcal{I}}$ to (5), the indirect utility of a consumer residing in city i is given by

$$v_i(\mathbf{h}) = \zeta\Delta_i(\mathbf{h})^{\frac{1}{\sigma-1}} y_i \geq 0 \quad \forall i \in \mathcal{I} \tag{6}$$

where $y_i(\mathbf{h}) = (1-\theta h_i/2)w_i^*(\mathbf{h})$, $\Delta_i(\mathbf{h}) \equiv \sum_{j\in\mathcal{I}} L_j w_j^{1-\sigma}\phi_{ji}$, and $\zeta \equiv \frac{\sigma-1}{\beta\sigma}\left(\frac{1}{\alpha\sigma}\right)^{1/(\sigma-1)}$.

To illustrate, consider the case in which there is an even number m of identical and equidistant cities of size $1/m$. In this case, (6) implies that the equilibrium indirect utility in a populated city is given by

$$v^* = \zeta\left[\left(\frac{2m-\theta}{2m}\right)^{\sigma} \cdot \frac{1}{m} \cdot **\Phi_0**\right]^{\frac{1}{\sigma-1}}, \tag{7}$$

where $\mathcal{I}_m \equiv \{i \mid h_i > 0\}$ is the set of populated cities and

$$\Phi_0 \equiv \sum_{i\in\mathcal{I}_m} \phi^{t_{0i}} = \begin{cases} \frac{\left(1-\phi^{n/2}\right)\left(1+\phi^{n/m}\right)}{1-\phi^{n/m}} \geq 1 & \text{for } 0 \leq \phi < 1, \\ m & \text{for } \phi = 1 \end{cases}$$

Since the total mass of varieties supplied in the economy is equal to $(2m-\theta)/2m$, *a higher number of cities leads to a wider range of varieties* because a larger amount of labor becomes available for the production of the differentiated product through lower commuting costs. However, the mass of varieties produced in each city, i.e.,

$(2m-\theta)/2m^2$, decreases with the actual number of cities. The factor $(\frac{2m-\theta}{2m})^\sigma \frac{1}{m}$ on the right-hand side of (7) stands for these two effects; it may increase or decrease with m. Regarding the factor Φ_0, it accounts for the global accessibility of city i to the other populated cities, which always increases with m.

Differentiating (7) with respect to m yields

$$\frac{\mathrm{d}v^*}{\mathrm{d}m} = \begin{cases} f(m) \equiv \dfrac{v^*}{\sigma-1}\left[\dfrac{(\sigma+1)\theta - 2m}{m(2m-\theta)} - \dfrac{2n}{m^2}\dfrac{2\phi^{n/m}}{1-\phi^{2n/m}}\ln\phi\right] & \text{for } 0 \leq \phi < 1, \\ \dfrac{\sigma\zeta}{\sigma-1}\left(\dfrac{2m-\theta}{2m}\right)^{\frac{1}{\sigma-1}}\dfrac{\theta}{2m^2} > 0 & \text{for } \phi = 1. \end{cases}$$

Since

$$\lim_{\phi\to 1} f(m) = \frac{(\sigma-1)\theta + 2m}{(\sigma-1)m(2m-\theta)} v^* > 0,$$

while the first term of $f(m)$ is independent of ϕ and the second one decreases in ϕ, $f(m)$ is positive for all $0 \leq \phi < 1$. Therefore, v^* increases with m. Put differently, the dispersion of production and consumption in a growing number of cities makes consumers better-off. This has the following somewhat unexpected implication: in the absence of agglomeration economies, the concentration of activities in a smaller number of large cities is detrimental to consumers for all values of ϕ. This runs against the prediction that agglomeration is welfare-enhancing in the core-periphery model when transportation costs are low because the winners are able to compensate the losers.

To reduce the proliferation of parameters, we choose the unit of the good for $\alpha = 1$. Since the number of firms is continuous, we also choose the unit of the real line along which this number is measured for $\beta = 1$ to hold.

2.3 Spatial equilibria

Assume now that consumers are mobile across cities and attracted by cities where their indirect utility is higher. In what follows, we describe the migration of consumers by the replicator dynamics, which has been routinely used in economic geography and evolutionary game theory (Fujita et al., 1999; Sandholm, 2010):

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = F_i(\mathbf{h}) \equiv F_i(\mathbf{h}) = [v_i(\mathbf{h}) - \bar{v}(\mathbf{h})]\, h_i \qquad \text{for all } i \in \mathcal{I}, \tag{8}$$

where

$$\bar{v}(\mathbf{h}) = \sum_{i\in\mathcal{I}} h_j v_j(\mathbf{h}),$$

denotes the average utility level. The pattern $\bar{\mathbf{h}}$ is a *steady-state* of (8) if $\left[v_i(\bar{\mathbf{h}}) - \bar{v}(\bar{\mathbf{h}})\right] \bar{h}_i = 0$ for all $i \in \mathcal{I}$.

Given $\mathbf{F}(\mathbf{h}) \equiv (F_i(\mathbf{h}))_{i\in\mathcal{I}}$, the stability of a steady-state $\bar{\mathbf{h}}$ of (8) is studied by linearizing the system (8) in a neighborhood of $\bar{\mathbf{h}}$:

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \nabla\mathbf{F}(\bar{\mathbf{h}}) \cdot (\mathbf{h} - \bar{\mathbf{h}}),$$

where $\nabla\mathbf{F}(\mathbf{h})$ is the Jacobian matrix of $\mathbf{F}(\mathbf{h})$.

A steady-state $\bar{\mathbf{h}}$ is said to be (locally) *stable* if any small perturbations away from $\bar{\mathbf{h}}$ dies out over time; otherwise it is said to be unstable. Let $\delta\mathbf{h} \equiv \mathbf{h} - \bar{\mathbf{h}}$ be a small perturbation of the steady-state $\bar{\mathbf{h}}$. Since $\mathrm{d}\bar{\mathbf{h}}/\mathrm{d}t = 0$, we obtain the following differential equations for $\delta\mathbf{h}$:

$$\frac{\mathrm{d}\delta\mathbf{h}}{\mathrm{d}t} = \nabla\mathbf{F}(\bar{\mathbf{h}})\delta\mathbf{h}$$

whose solution is given by

$$\delta\mathbf{h} = \sum_{i\in\mathcal{I}} c_i \exp(\lambda_i t)\boldsymbol{\eta}_i,$$

where c_i is a constant, λ_i is the i-th eigenvalue of $\nabla\mathbf{F}(\bar{\mathbf{h}})$, and $\boldsymbol{\eta}_i$ is the associated eigenvector. It follows immediately from this expression that $\bar{\mathbf{h}}$ is *stable* (i.e., $\lim_{t\to\infty} \delta\mathbf{h} = \mathbf{0}$) if all the eigenvalues λ_i have negative real parts.

For any given ϕ and θ, $\mathbf{h}^*(\phi, \theta)$ is a *spatial equilibrium* when no consumer may get a higher utility level by moving to another city. Formally, $\mathbf{h}^*(\phi, \theta)$ is an equilibrium if v^* exists such that the following two conditions hold:

$$\begin{cases} v^* - v_i(\mathbf{h}^*(\phi,\theta)) = 0 & \text{if} \quad h_i^*(\phi,\theta) > 0 \\ v^* - v_i(\mathbf{h}^*(\phi,\theta)) \geq 0 & \text{if} \quad h_i^*(\phi,\theta) = 0 \end{cases} \qquad \forall i \in \mathcal{I}, \tag{9}$$

and

$$\sum_{i\in\mathcal{I}} h_i^*(\phi,\theta) = 1, \tag{10}$$

where $v^* > 0$ denotes the equilibrium utility level. The condition (9) means that no consumer may get a higher utility level by moving to another city, while (10) is the population constraint.

Clearly, (9) implies

$$\left[v^* - v_i(\mathbf{h}^*(\phi,\theta))\right] h_i^* = 0 \quad \text{for all } i \in \mathcal{I}.$$

Therefore, a spatial equilibrium is always a steady-state of (8), but a steady-state need not be a spatial equilibrium.

Note that the uniform distribution $h_i = 1/n$ is always a spatial equilibrium because the uniform distribution of potential cities along the circle $\mathbf{C}$ implies $v_i(\mathbf{h}_n) = \bar{v}(\mathbf{h}_n)$ for all $i \in \mathcal{I}$. Therefore, we have:[3]

Proposition 1. *For any given $0 \leq \phi \leq 1$ and $0 \leq \theta \leq 1$, there exists at least one spatial equilibrium.*

2.4 Invariant equilibria

In general, a spatial equilibrium $\mathbf{h}^*(\phi, \theta)$ varies with (ϕ, θ). However, there may exist a non-zero measure set S of the positive orthant over which $\mathbf{h}^*(\phi, \theta)$ is an invariant steady-state for any $(\phi, \theta) \in S$. Such a spatial equilibrium is said to be *invariant*. On the other hand, a spatial equilibrium $\mathbf{h}^*(\phi, \theta)$ that varies with (ϕ, θ) is called a *non-invariant* pattern. At such an equilibrium, the size of populated cities changes continuously with ϕ and θ.

Let $\mathbf{h}_m$ be a pattern with $1 \leq m \leq n$ populated cities. Since the equilibrium always involves at least one city, we may assume without loss of generality that city $i = 0$ is always populated ($h_0 > 0$).

Set

$$\Phi_i \equiv \sum_{j \in \mathcal{I}_m} \phi^{t_{ji}} \in [0, m]$$

for any potential city $i \in \mathcal{I}$. This index measures the accessibility of city i to the populated cities and varies with the locations of those cities. When $\Phi_i = 0$, city i has no access to the others while it has the highest accessibility when Φ_i is equal to m.

The following proposition summarizes the main properties of invariant patterns.

Proposition 2.

(a) *A pattern $\mathbf{h}_m$ is an invariant steady-state if and only if the following two conditions hold:*

$$h_i = \frac{1}{m} \quad \text{for all } i \in \mathcal{I}_m, \qquad h_i = 0 \quad \text{for all } i \in \mathcal{I}_0 \equiv \mathcal{I} \backslash \mathcal{I}_m, \tag{11}$$

$$\Phi_0 = \Phi_i \quad \text{for all } i \in \mathcal{I}_m. \tag{12}$$

[3]More generally, if the individual payoffs are continuous in the population distribution, it can be shown that there alawys exists a spatial equilibrium for any finite set of locations and any matrix of transportation costs.

(b) *An invariant steady-state* $\mathbf{h}_m$ *is a spatial equilibrium if and only if the following inequality is satisfied:*

$$\Phi_0 \geq \left(\frac{2m}{2m-\theta}\right)^{\frac{\sigma(\sigma-1)}{2\sigma-1}} \Phi_i \quad \text{for all } i \in \mathcal{I}_0. \tag{13}$$

(c) *An invariant steady-state* $\mathbf{h}_m$ *is a stable spatial equilibrium if the following two conditions hold:*

$$\Phi_0 > \left(\frac{2m}{2m-\theta}\right)^{\frac{\sigma(\sigma-1)}{2\sigma-1}} \Phi_i \quad \text{for all } i \in \mathcal{I}_0, \tag{14}$$

all the eigenvalues of the Jacobian $\left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m}$ *have negative real parts.* (15)

Proof: See Appendix A.

Proposition 2(a) shows that *all populated cities have the same size and the same accessibility index.* In this case, populated cities have the same total volume of trade because they have the same global accessibility to their trading partners. Note that under (11) and (12), $v_i(\mathbf{h}_m) = v_j(\mathbf{h}_m)$ when i and j are populated cities. However, $v_i(\mathbf{h}_m)$ and $v_j(\mathbf{h}_m)$ need not be equal when i and/or j are unpopulated because the accessibility to cities i and j (Φ_i and Φ_j) to the populated cities need not be the same. Proposition 2(b) implies that every unpopulated city has a lower accessibility than a populated city when $\theta > 0$. Proposition 2(c) says that $\mathbf{h}_m$ must be a strict equilibrium for $\mathbf{h}_m$ to be stable. As a result, the unpopulated cities can be ignored in the stability analyses developed below.

To illustrate, let us describe what the above concepts are in the core-periphery model developed by Krugman (1991). Denoting by λ the share of people in region 1, $\lambda = 1/2$ is always an invariant spatial equilibrium, which is stable when transportation costs are sufficiently high. On the other hand, $\lambda = 1$ ($\lambda = 0$) is always an invariant steady-state, but it becomes a stable spatial equilibrium only if transportation costs are small enough. In Murata and Thisse (2005), $\lambda = 1/2$ is always an invariant spatial equilibrium, which is stable when transportation costs are sufficiently low and/or commuting costs are high enough. By contrast, $\lambda = 1$ is always an invariant steady-state, but it is a stable spatial equilibrium only if transportation costs are high and/or commuting costs are low.

In the foregoing, we did not say anything about the relative position of occupied cities. Using (12), it can be shown that there are two classes of patterns that satisfy the conditions (11) and (12) (Ikeda et al., 2019). The former is defined by the *symmetric* patterns in which the populated cities are equally distributed over the circle **C**. In the latter, patterns are *quasi-symmetric*, which means that the distribution of cities is symmetric about one oblique axis that goes through the center of the circle. The critical difference with the symmetric pattern is that *cities are no longer equidistant along the circle.* In an quasi-symmetric pattern, two cities are sufficiently close to each other to form an urban cluster, which is isolated from the other clusters. An illustration is provided by Figure 1(a).

3 Sustainable and stable patterns with 4 potential cities

Our setting may exhibit several stable equilibria that differ in nature. In order to gain insights about these equilibria, we consider a racetrack economy with 4 potential cities ($n = 4$), prior to the general analysis undertaken in Section 4. We first consider the *invariant* patterns in which populated cities have the same size for any ϕ and θ. Then, we turn our attention to *non-invariant* patterns in which populated cities have different sizes.

3.1 The candidate spatial equilibria

Figure 1(a) depicts the four invariant patterns for $n = 4$ cities, while Figure 1(b) depicts three non-invariant patterns, denoted $\mathbf{h}_3^{\text{non}}$, $\mathbf{h}_4^{\text{non}}$, and $\mathbf{h}_4^{q\text{non}}$.

[Figure 1 about here.]

As expected, the class of *symmetric* invariant patterns comprises the uniform distribution $\mathbf{h}_4 = (1/4, 1/4, 1/4, 1/4)$, the 2-city symmetric pattern $\mathbf{h}_2 = (1/2, 0, 1/2, 0)$, and full agglomeration $\mathbf{h}_1 = (1, 0, 0, 0)$. Less expected (at least to us), another type of invariant pattern, which we call *quasi-symmetric*, may also emerge as an equilibrium outcome. It is given by $\mathbf{h}_2^q = (1/2, 1/2, 0, 0)$ where the two populated cities are now located at $x = 0$ and $x = 1/4$ instead of $x = 0$ and $x = 1/2$ in the symmetric pattern. The city locations are no longer symmetric about the horizontal axis like in $\mathbf{h}_2$, but about the oblique axis passing through the center of the circle and the point $x = 1/8$ (see the bottom of Figure 1(a)). Clearly, a quasi-symmetric pattern displays a proclivity toward agglomeration that is absent in the corresponding symmetric pattern. Furthermore, the equilibrium utility level at $\mathbf{h}_2^q$ is higher than that at $\mathbf{h}_2$ because transportation costs are lower in the former than in the latter, while commuting costs and the mass of supplied varieties are the same because cities host the same population.

This is not yet the end of the story as non-invariant patterns may also arise. First, the *hierarchy* $\mathbf{h}_4^{\text{non}}$ such that $h_0 > h_1 = h_3 > h_2 > 0$ in which the largest city is at $x = 0$ is a spatial equilibrium. The primate city is flanked by two medium size cities at $x = 1/4$ and $x = 3/4$, while the smallest city is established at $x = 1/2$. The pattern $\mathbf{h}_4^{\text{non}}$ may be viewed as the hierarchical counterpart of $\mathbf{h}_4$ since the city size decreases as the distance to the largest city increases. Note also that $\mathbf{h}_4^{\text{non}}$ is also more concentrated than $\mathbf{h}_4$ as $h_0 > 1/4$ and $h_2 < 1/4$. The medium size cities are situated at the intermediate locations $x = 1/4$ and $x = 3/4$ because these locations allow them to sustain a substantial volume of trade with the largest city while bearing intermediate urban costs. The smallest city at $x = 1/2$ trades less with the others because it has a fairly bad access to them, especially the largest city. However, this disadvantage is compensated by very low urban costs.

A related, but distinct, hierarchal pattern appears in $\mathbf{h}_3^{\text{non}}$ with $h_0 > h_1 = h_3 > h_2 = 0$. The city system is now formed by a big urban center flanked by two adjacent sub-centers. Note $\mathbf{h}_3^{\text{non}}$ is more concentrated than $\mathbf{h}_4^{\text{non}}$ as the city at $x = 1/2$ vanishes. What makes these non-invariant configurations $\mathbf{h}_3^{\text{non}}$ and $\mathbf{h}_4^{\text{non}}$ especially appealing is that they both involve an urban hierarchy in an otherwise symmetric setting with no boundary. A still different hierarchical pattern $\mathbf{h}_4^{q\text{non}}$ is displayed in Figure 1(b). It is given by the pseudo-hierarchy formed by two large cities established at $x = 0$ and $x = 1/4$, while two small cities are located at $x = 1/2$ and $x = 3/4$, with $h_0 = h_1 > h_2 = h_3$. The two large cities form the core of the economy while the two small cities constitute its periphery. This configuration, denoted $\mathbf{h}_4^{q\text{non}}$, corresponds to a hierarchical version of the quasi-symmetric pattern $\mathbf{h}_2^q$. Observe the difference with Henderson-like models where cities producing the same good must have the same size.

The following proposition shows that the above patterns are spatial equilibria over some subsets of $[0, 1] \times [0, 1]$:

Proposition 3. *There exist values of ϕ and θ such that the invariant patterns* ($\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_1$, and $\mathbf{h}_2^q$) *and the non-invariant patterns* ($\mathbf{h}_4^{\text{non}}$, $\mathbf{h}_4^{q\text{non}}$, and $\mathbf{h}_3^{\text{non}}$) *are spatial equilibria.*

Proof: See Appendix B.

Relying on the analysis conducted in Section 4 for the derivation of the expressions used here, we investigate the stability of those spatial equilibria. Since our numerical analysis reveals that the non-invariant patterns $\mathbf{h}_4^{\text{non}}$ and $\mathbf{h}_4^{q\text{non}}$ never emerge as stable equilibria, we will focus on the three symmetric patterns $\mathbf{h}_4$, $\mathbf{h}_2$, and $\mathbf{h}_1$, the quasi-symmetric patterns $\mathbf{h}_2^q$, and the hierarchical pattern $\mathbf{h}_3^{\text{non}}$.

The uniform distribution $(1/4, 1/4, 1/4, 1/4)$ is always a spatial equilibrium because $v_i = v_j$ holds for any $i, j \in \mathcal{I}$. By contrast, the other four patterns need not be spatial equilibria. More specifically, the patterns $\mathbf{h}_2, \mathbf{h}_2^q, \mathbf{h}_1$ are sustainable when the conditions

$$\begin{cases} 0 \le \theta \le \theta^s(\phi, \mathbf{h}_m) & \text{for } \theta^s(\phi, \mathbf{h}_m) \le 1 \\ 0 \le \theta \le 1 & \text{for } \theta^s(\phi, \mathbf{h}_m) \ge 1 \end{cases} \qquad \text{for } \mathbf{h}_m = \mathbf{h}_2, \mathbf{h}_2^q, \mathbf{h}_1$$

hold. In other words, the invariant pattern $\mathbf{h}_m$ is a spatial equilibrium if and only if

$$\theta \le \min\left\{\theta^s(\phi, \mathbf{h}_m), 1\right\}.$$

- We show in Appendix C.1 that the sustain points corresponding to $\mathbf{h}_2, \mathbf{h}_2^q, \mathbf{h}_1$ are given by the following expressions:

$$\begin{cases} \theta^s(\phi, \mathbf{h}_2) = 4\left[1 - \left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\ \theta^s(\phi, \mathbf{h}_2^q) = 4\left(1 - \phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right), \\ \theta^s(\phi, \mathbf{h}_1) = 2\left(1 - \phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right). \end{cases} \tag{16}$$

By inspection, $\theta^s(\phi, \cdot)$ exceeds 1 when transportation costs are high. In this case, $\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_2^q$, and $\mathbf{h}_1$ are all spatial equilibria. Furthermore, since $\theta^s(\phi, \cdot)$ is a monotonically decreasing positive function of ϕ, which is equal to 0 when $\phi = 1$, when transportation costs steadily decrease, $\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_2^q$, and $\mathbf{h}_1$ are spatial equilibria over shrinking intervals of commuting cost values. It is readily verified that $\mathbf{h}_2^q$ has the largest sustain point:

$$\theta^s(\phi, \mathbf{h}_2) < \theta^s(\phi, \mathbf{h}_2^q), \qquad \theta^s(\phi, \mathbf{h}_1) < \theta^s(\phi, \mathbf{h}_2^q) \qquad \text{for } 0 < \phi < 1. \tag{17}$$

There is no break point for the agglomerated pattern $\mathbf{h}_1$ because there always exists at least one city. Furthermore, the patterns $\mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^q$ are stable steady-states when the following condition holds:

$$\max\{\theta^b(\phi, \mathbf{h}_m), 0\} < \theta \le 1 \qquad \text{for } \mathbf{h}_m = \mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^q.$$

- We show in Appendix C.2 that the break point is given

$$\theta^b(\phi, \mathbf{h}_m) = m \frac{\frac{2\sigma-1}{\sigma-1} K}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) K + \frac{\sigma}{2}} \geq 0, \tag{18}$$

  where

$$K \equiv \begin{cases} \frac{1-\phi}{1+\phi} & \text{for } \mathbf{h}_4 \text{ and } \mathbf{h}_2^q, \\ \frac{1-\phi^2}{1+\phi^2} & \text{for } \mathbf{h}_2. \end{cases},$$

Since $\phi = 1$ implies $K = 0$, we have $\theta^b(1, \mathbf{h}_m) = 0$, which means that $\mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^q$ are stable stead-states when transportation costs are negligible. By implication of (18), the break point θ^b is a decreasing positive function of ϕ. As a consequence, $\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_2^q$ are stable over intervals whose left bound decrease with the level of transportation costs.

Since the break points (18) can be ranked as follows:

$$\theta^b(\phi, \mathbf{h}_2^q) < \theta^b(\phi, \mathbf{h}_2) < \theta^b(\phi, \mathbf{h}_4) \qquad \text{for } 0 < \phi < 1, \tag{19}$$

it follows from (16) and (18) that $\theta^b(\phi, \mathbf{h}_2^q) < \theta^s(\phi, \mathbf{h}_1) < \theta^s(\phi, \mathbf{h}_2^q)$ holds for any $0 < \phi < 1$. As a result, when ϕ varies, the quasi-symmetric pattern ($\mathbf{h}_2^q$) emerges before agglomeration ($\mathbf{h}_1$), dispersion ($\mathbf{h}_4$) ceases to be stable before the symmetric 2-city pattern ($\mathbf{h}_2$), which becomes unstable before the quasi-symmetric pattern ($\mathbf{h}_2^q$).

3.2 Sustainability and stability

In this section, we discuss the sustainability and stability of the above-mentioned five patterns. We call the *stability area* of $\mathbf{h}_m$ the (ϕ, θ)-domain over which $\mathbf{h}_m$ is a stable spatial equilibrium (i.e., $\theta^b(\phi, \mathbf{h}_m) < \theta \leq \theta^s(\phi, \mathbf{h}_m)$). We have determined the stability area of each pattern for $\sigma = 6$, i.e., a value that is in accordance with various estimations of the elasticity of substitution (Bergstrand et al., 2013). In Figure 2, the stability areas corresponding to the five stable equilibria are described by the shaded areas of the parameter space $[0, 1] \times [0, 1]$. The stability areas for the four invariant patterns have been obtained analytically, while that associated with the non-invariant pattern is obtained by carrying out a series of computational analyses with respect to ϕ by changing the value of θ through fine intervals.

[Figure 2 about here.]

Consider first the two polar cases $\mathbf{h}_1$ and $\mathbf{h}_4$. Full agglomeration ($\mathbf{h}_1$) is a stable equilibrium in the area situated at the lower-left corner of Figure 2(a), which is colored in red. The uniform distribution ($\mathbf{h}_4$) is a stable equilibrium in the area situated at the upper-right corner colored in grey. These two areas do not cover the whole parameter set. To be precise, $\mathbf{h}_1$ and $\mathbf{h}_4$ are not stable equilibria when the parameters ϕ and θ belong to the white area of Figure 2(a). This area can be covered by the stability areas of the other three patterns, that is, the symmetric invariant pattern $\mathbf{h}_2$ in Figure 2(b), the quasi-symmetric invariant pattern $\mathbf{h}_2^q$ in Figure 2(c), and the hierarchical non-invariant pattern $\mathbf{h}_3^{\text{non}}$ in Figure 2(d). Since these stability areas overlap, there are *multiple stable equilibria* for some parameter configurations. Figure 3 describes (i) the set of (ϕ, θ)-values associated with a single stable equilibrium (panel (a)) and (ii) the set of (ϕ, θ)-values that generate multiple stable equilibria (panel (b)). Observe that the multiplicity of equilibria tends to increase when transportation costs fall and commuting costs decrease.

[Figure 3 about here.]

Assume that ϕ and θ are such that $\mathbf{h}_1$ and $\mathbf{h}_4$ are not stable. Using (17) and (19), the interval $(\theta^b(\phi, \mathbf{h}_2), \theta^s(\phi, \mathbf{h}_2))$ over which $\mathbf{h}_2$ is a stable spatial equilibrium is included in the interval $(\theta^b(\phi, \mathbf{h}_2^q), \theta^s(\phi, \mathbf{h}_2^q))$ for which $\mathbf{h}_2^q$ is a stable spatial equilibrium. In other words, if $\theta^b(\phi, \mathbf{h}_2^q) < \theta^s(\phi, \mathbf{h}_2^q)$, the quasi-symmetric pattern $\mathbf{h}_2^q$ is a stable equilibrium for a wider range of commuting cost values than the symmetric pattern $\mathbf{h}_2$. Similarly, the stability area of the quasi-symmetric pattern $\mathbf{h}_2^q$ covers the stability areas of the non-invariant pattern $\mathbf{h}_3^{\text{non}}$. Putting these two results together suggests that,*when the agglomerated and dispersed patterns do not prevail, the quasi-symmetric pattern $\mathbf{h}_2^q$ is more likely to emerge as the equilibrium outcome.*

3.3 The impact of commuting costs

To show how an equilibrium path is determined in response to a change in spatial frictions, we first consider the case of decreasing commuting costs. Assume that the economy is characterized by low transportation costs ($\phi = 0.8$) and high commuting costs ($\theta = 0.7$). At these values, Figure 2 shows that there exists a unique stable equilibrium given by $\mathbf{h}_4$. When commuting costs start decreasing, the principle of path dependency implies that the economy remains at $\mathbf{h}_4$. When point **A** of Figure 2(a), where θ is about 0.4, is reached, the pattern $\mathbf{h}_4$ ceases to be stable. Since the neighboring stability area is that of the quasi-symmetric equilibrium $\mathbf{h}_2^q$, the economy shifts to this new pattern. In other words, the population located in cities 2 and 3 migrate to cities 0 and 1 whose size doubles. When θ decreases further, the economy remains quasi-symmetric until point **B** of Figure 2(c) is reached, where θ is about 0.2. When commuting costs take on a value slightly smaller than 0.2, $\mathbf{h}_2^q$ is no longer stable. As a result, the economy enters the stability area of $\mathbf{h}_1$. The whole population is now concentrated in city 0 located at $x = 0$ (see Figure 2(a)). There is full agglomeration because commuting is very inexpensive. In sum, *the equilibrium path*, which is obtained by combining the principle of path dependency and the contiguity of stability areas, *is formed by the concatenation of three invariant stable equilibria.* Note that the economy follows a similar path when transportation costs are high ($\phi = 0.2$), but the θ-domain for which $\mathbf{h}_4$ ($\mathbf{h}_1$) prevails is much narrower (wider).

We may thus conclude that, starting from dispersion, firms and consumers get more and more agglomerated as commuting costs steadily fall. Furthermore, when agents start pulling up together, firms and consumers are gathered in two identical cities situated at $x = 0$ and $x = 1/4$ that trade along the shorter route linking them. Accordingly, as commuting costs decrease, the most likely equilibrium path is as follows:

$$\text{Dispersion } (\mathbf{h}_4) \longrightarrow \text{Quasi-symmetric pattern } (\mathbf{h}_2^q) \longrightarrow \text{Agglomeration } (\mathbf{h}_1).$$

3.4 The impact of transportation costs

We now consider the effect of decreasing transportation costs for a given value of commuting costs. Inspecting the stability areas of Figure 2 shows that agglomeration, then the quasi-symmetric pattern and, finally, dispersion are stable equilibria as transportation costs fall from prohibitive to negligible values (ϕ increases from 0 to 1). Hence, an increasing number of smaller cities becomes the stable outcome as shipping goods is inexpensive. In addition, as θ decreases, the range of ϕ-values for the uniform distribution to be a stable equilibrium shrinks, whereas the range of values associated with agglomeration widens.

Hence, as transportation costs decrease, the most likely sequence of stable equilibria is as follows:

$$\text{Agglomeration } (\mathbf{h}_1) \longrightarrow \text{Quasi-symmetric pattern } (\mathbf{h}_2^q) \longrightarrow \text{Dispersion } (\mathbf{h}_4).$$

Thus, starting from agglomeration, firms and consumers get more and more dispersed when transportation costs steadily decrease. Indeed, lowering transportation costs leads to the convergence of prices, thus weakening the agglomeration force stressed in economic geography. By contrast, as long as the population distribution remains the same, the intensity of the dispersion force is unaffected. Therefore, it is no surprise that, eventually, the latter overcomes the former.

To conclude, when urban costs, rather than the existence of immobile farmers, are the dispersion force, decreasing transportation costs leads to a conclusion that runs against the main prediction of economic geography: *decreasing commuting costs within cities, rather than transportation costs between cities, foster the agglomeration of activities*. Furthermore, there is a whole domain of parameters in which the economy involves two neighboring big cities, which form together a *megalopolis*, and a large empty region. That $\mathbf{h}_2^q$ may be a stable spatial equilibrium may come as a surprise because the general belief holds that a spatial equilibrium does not involve two large cities situated in close proximity. More specifically, a large city would generate a shadow effect that would prevent the development of another large and close city. However, the empirical evidence is not conclusive. For example, Cuberes et al. (2019) use data on U.S. counties and metro areas to show that proximity to large urban centers need not prevent the growth of neighboring places.

4 Sustainability and stability of symmetric invariant patterns

In what follows, we discuss the sustainability and stability of *symmetric* invariant patterns $\mathbf{h}_m \equiv (h_i)_{i\in\mathcal{I}_m}$ when the number m of populated cities is an even number. In this case, $h_i = 1/m$ for any $i \in \mathcal{I}_m \equiv \{0, \frac{n}{m}, 2\frac{n}{m}, ..., (m-1)\frac{n}{m}\}$ and $h_i = 0$ for any $i \in \mathcal{I}_0$. Recall that city $i = 0$ is always populated ($h_0 > 0$), so that the neighboring cities 1 and $n-1$ are unpopulated when $m < n$. The analysis is developed in terms of the commuting rate θ. To be precise, we derive a *sustain point* $\theta^s(\phi, \mathbf{h}_m)$, that is, a threshold at which $\mathbf{h}_m$ becomes or ceases to be a spatial equilibrium (the condition (14) of Proposition 2 is satisfied or not) and a *break point* $\theta^b(\phi, \mathbf{h}_m)$, that is, a threshold at which $\mathbf{h}_m$ becomes or ceases to be stable (the condition (15) is satisfied or not). We will show that these two thresholds are such that the invariant pattern $\mathbf{h}_m$ is a stable spatial equilibrium over the interval $(\theta^b(\phi, \mathbf{h}_m), \theta^s(\phi, \mathbf{h}_m)]$ whenever this interval is non-empty.

We show in Appendix C that there exist a unique sustain point $\theta^s(\phi, \mathbf{h}_m)$ and a unique break point $\theta^b(\phi, \mathbf{h}_m)$ such that the invariant $\mathbf{h}_m$ is a stable equilibrium if and only if $\theta^b(\phi, \mathbf{h}_m) < \theta \leq \theta^s(\phi, \mathbf{h}_m)$. Since $\theta^b(\phi, \mathbf{h}_m)$ is not necessarily smaller than $\theta^s(\phi, \mathbf{h}_m)$, an invariant pattern with m cities need not be stable nor a spatial equilibrium.

By appealing to the symmetry of $\mathbf{h}_m$, we have the following lemmas, which provide a necessary and sufficient condition for $\mathbf{h}_m$ to satisfy (14) and (15), respectively.

Lemma 1. *Consider a symmetric invariant pattern $\mathbf{h}_m$ with $m < n$. Then, $\mathbf{h}_m$ satisfies* (14) *if and only if $v_0(\mathbf{h}_m) > v_1(\mathbf{h}_m)$.*

Proof: Since $\Phi_1 = \max_{i \in \mathcal{I}_0} \Phi_i$, we have $v_1(\mathbf{h}_m) = \max_{i \in \mathcal{I}_0} v_i(\mathbf{h}_m)$. Therefore, $\mathbf{h}_m$ satisfies the condition (14) if and only if $v_0(\mathbf{h}_m) > v_1(\mathbf{h}_m)$.

In other words, comparing the utility levels at locations 0 and 1 is sufficient to determine whether $\mathbf{h}_m$ is a spatial equilibrium.

Lemma 2. *Suppose m is even. Then, a symmetric invariant pattern $\mathbf{h}_m$ satisfies* (15) *if and only if the commuting rate θ is such that*

$$K(\phi, \mathbf{h}_m) \cdot \left[\frac{2\sigma - 1}{\sigma - 1}(1 - \theta h_0) - (\sigma - 1)\frac{\theta h_0}{2} \right] - \sigma \frac{\theta h_0}{2} < 0, \tag{20}$$

holds where $K(\phi, \mathbf{h}_m)$ is defined by

$$K(\phi, \mathbf{h}_m) \equiv \max_{i \in \mathcal{I}_m \setminus \{0\}} f_i = \frac{(1 - \phi^{\frac{n}{m}})^2}{1 - 2\cos\left(\frac{2\pi}{m}\right)\phi^{\frac{n}{m}} + \phi^{\frac{2n}{m}}} \frac{1 + \phi^{\frac{n}{2}}}{1 - \phi^{\frac{n}{2}}} \geq 0. \tag{21}$$

Proof: See Appendix D.

Plugging (21) into (20) shows that the necessary and sufficient condition for the symmetric pattern with m cities to be an equilibrium involves both commuting and transportation parameters. In addition, (21) gives us an explicit form for K as a function of ϕ. In particular, it can be shown that the right-hand side of (21) is decreasing in ϕ. Therefore, the break point $\theta^b(\phi, \mathbf{h}_m)$ decreases when transportation costs fall. In other words, *lowering transportation costs allows the symmetric pattern with m cities to remain a stable equilibrium for lower commuting costs.* Last, (20) also depends on n, which means that the number of natural sites that can host cities may affect the structure of the urban system.

The expressions provided in Appendix C for the sustain and break points may be rewritten as follows (see (C.1) and (C.2)):

$$\theta^s(\phi, \mathbf{h}_m) = 2m \left[1 - \left(\frac{\Phi_1}{\Phi_0} \right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} \right], \qquad \theta^b(\phi, \mathbf{h}_m) = m \frac{\frac{2\sigma-1}{\sigma-1} K(\phi, \mathbf{h}_m)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) K(\phi, \mathbf{h}_m) + \frac{\sigma}{2}}.$$

Note that $\theta^b(\phi, \mathbf{h}_m) > 0$ for all $0 \leq \phi < 1$. The expression $\theta^s(\phi, \mathbf{h}_m)$ implies that a pattern $\mathbf{h}_m$ in which an unpopulated city j has an accessibility index Φ_j higher than Φ_0 cannot be a spatial equilibrium. Indeed, since it has a good access to the populated cities, city j should host firms and consumers. In other words, a pattern involving m cities is sustainable only if the corresponding populated cities share the highest accessibility in $\mathbf{h}_m$.

Since $\mathrm{d}\theta^b(\phi, \mathbf{h}_m)/\mathrm{d}m > 0$ for any given $0 < \phi < 1$, the break point $\theta^b(\phi, \mathbf{h}_m)$ satisfies the following property.

Proposition 4. *Assume that m_1 and m_2 are even with $m_1 < m_2$. Then, $\theta^b(\phi, \mathbf{h}_{m_2}) > \theta^b(\phi, \mathbf{h}_{m_1})$ for any $0 < \phi < 1$.*

Thus, we may rank the break points by decreasing numbers of cities. As θ decreases and crosses the break point $\theta^b(\phi, \mathbf{h}_n)$ from above, Proposition 4 implies that the next stable equilibrium necessarily involves a smaller number $m < n$ of equidistant cities. Furthermore, since we focus on stable equilibria, this proposition is sufficient for the following result to hold: the smaller the number of populated cities at a stable equilibrium, the lower the value of the commuting rate θ at which the corresponding pattern ceases to be stable.

In sum, for a given value of ϕ, *when commuting costs steadily fall, the market outcome may involve the step-wise agglomeration of activities in a decreasing number of larger cities.* In particular, since $\theta^s(\phi, \mathbf{h}_1) > 0$ always holds, the economy has a single mega-city when commuting costs are very low, while all potential cities are populated when commuting costs are very high.

In the next proposition, we determine the admissible values of $1 \leq m \leq n$ and the necessary and sufficient conditions on θ for the symmetric pattern $\mathbf{h}_m$ to be a stable equilibrium.

Proposition 5.

(i) *The dispersed pattern $\mathbf{h}_n$ is a stable equilibrium if and only if θ is larger than $\theta^b(\phi, \mathbf{h}_n)$.*

(ii) *For any even number $m < n$, the symmetric pattern $\mathbf{h}_m$ is a stable equilibrium for $\phi \geq 0$ if and only if $\theta^b(\phi, \mathbf{h}_m) < \theta \leq \theta^s(\phi, \mathbf{h}_m)$.*

(iii) *The agglomerated pattern $\mathbf{h}_1$ is a stable equilibrium for $\phi \geq 0$ if and only if θ is smaller than $\theta^s(\phi, \mathbf{h}_1)$.*

Unlike the break points, the sustain points cannot be ranked according to the value of m because the ranking may change with ϕ. Nevertheless, the above proposition has some interesting consequences. The symmetric pattern with m cities is stable and sustainable over the interval $(\theta^b(\phi, \mathbf{h}_m), \theta^s(\phi, \mathbf{h}_m)]$ if and only if $\theta^b(\phi, \mathbf{h}_m) < \theta^s(\phi, \mathbf{h}_m)$. When $\theta^b(\phi, \mathbf{h}_m)$ is not smaller than $\theta^s(\phi, \mathbf{h}_m)$, the symmetric pattern with m cities is a spatial equilibrium for $\theta \leq \theta^s(\phi, \mathbf{h}_m)$, but this equilibrium is unstable. Since the symmetric pattern with m cities is never an equilibrium when $\theta > \theta^s(\phi, \mathbf{h}_m)$, the equilibrium path of the economy bypasses the symmetric pattern with m cities when the inequality $\theta^b(\phi, \mathbf{h}_m) \geq \theta^s(\phi, \mathbf{h}_m)$ holds. By implication, $\mathbf{h}_{m_1}$ and $\mathbf{h}_{m_2}$ may be stable equilibria while $\mathbf{h}_m$ such that $m_1 < m < m_2$ may not be a stable equilibrium. To put differently, *when θ decreases, the transition from m_2 to m_1 cities need not go through all the values of m belonging to* (m_1, m_2).

In sum, three cases may arise when θ crosses $\theta^b(\phi, \mathbf{h}_m)$ from above: (i) there is multiplicity of stable invariant equilibria that have less than m cities; (ii) there is a unique stable symmetric or quasi-symmetric invariant equilibrium with a number of cities smaller than m; and (iii) the economy may display a path of non-invariant equilibria in which the size of cities changes with the level of commuting and transportation costs. Note also that invariant and non-invariant equilibria may coexist.

Last, when consumers are free to choose their lot size, the values of the sustain and break points $\theta^s(\phi, \mathbf{h}_m)$ and $\theta^b(\phi, \mathbf{h}_m)$ change because the common utility level associated with $\mathbf{h}_m$ depends on the substitutability between land and the consumption good. Cities still have the same population size but their "physical" size may differ from $1/m$.

5 Stable and sustainable patterns under 8 potential cities

To gain further insights about the emergence of quasi-symmetric and non-invariant patterns, we consider a racetrack economy with 8 potential cities. Both theoretical and numerical analyses are conducted to determine the stable equilibria. We have checked that results are similar when $n = 16$.

5.1 The candidate spatial equilibria

Figure 4(a) depicts the invariant patterns for $n = 8$. There are 4 symmetric configurations given by the uniform distribution $\mathbf{h}_8$, the 4-city pattern $\mathbf{h}_4 = (1/4, 0, 1/4, 0, 1/4, 0, 1/4, 0)$, the 2-city pattern $\mathbf{h}_2 = (1/2, 0, 0, 0, 1/2, 0, 0, 0)$, and agglomeration $\mathbf{h}_1$. There are 4 quasi-symmetric ones for $m = 4$ and 2. The quasi-symmetric pattern for $m = 4$ is such that $\mathbf{h}_4^q = (1/4, 1/4, 0, 0, 1/4, 1/4, 0, 0)$. In this configuration, one city is located at $x = 0$ and another one at $x = 1/2$ like in the symmetric pattern. However, the two remaining cities are located at $x = 1/8$ and $x = 5/8$ instead of $x = 1/4$ and $x = 3/4$, respectively. When $m = 2$, there are three quasi-symmetric 2-city patterns where the second city is located at $x = 1/8$, $x = 1/4$, and $x = 3/8$, respectively (see Figure 4(a)). In these configurations, denoted $\mathbf{h}_2^{1/8}$, $\mathbf{h}_2^{1/4}$, and $\mathbf{h}_2^{3/8}$, the population is more concentrated than in the symmetric one $\mathbf{h}_2$.

[Figure 4 about here.]

Patterns other than those shown in Figure 4(a) are all non-invariant (Ikeda et al., 2019). Among them, we have the hierarchical patterns $\mathbf{h}_m^{\text{non}}$ ($m = 6, 5, 4, 3$) displayed in Figure 4(b) where a bigger circle means a city hosting a larger population. Note that cities having the same rank have the same size.

5.2 Sustainability and stability of invariant patterns

When $n = 8$, the sustain and break points associated with symmetric invariant patterns are given by the following expressions (see (C.1) and (C.2) in Appendix C):

$$\theta^b(\phi, \mathbf{h}_8) = 8\frac{\frac{2\sigma-1}{\sigma-1}K_8}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right)K_8 + \frac{\sigma}{2}},$$

$$\theta^b(\phi, \mathbf{h}_4) = 4\frac{\frac{2\sigma-1}{\sigma-1}K_4}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right)K_4 + \frac{\sigma}{2}}, \qquad \theta^s(\phi, \mathbf{h}_4) = 8\left[1 - \left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right],$$

$$\theta^b(\phi, \mathbf{h}_2) = 2\frac{\frac{2\sigma-1}{\sigma-1}K_2}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right)K_2 + \frac{\sigma}{2}}, \qquad \theta^s(\phi, \mathbf{h}_2) = 4\left[1 - \left(\frac{\phi(1+\phi^2)}{1+\phi^4}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right],$$

$$\theta^s(\phi, \mathbf{h}_1) = 2\left(1 - \phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right),$$

where (21) implies

$$\begin{aligned} K_8 &\equiv K(\phi, \mathbf{h}_8) = \Psi(\phi^2)\frac{1+\sqrt{2}\phi+\phi^2}{(1+\phi)^2}, \\ K_4 &\equiv K(\phi, \mathbf{h}_4) = \Psi(\phi^2), \\ K_2 &\equiv K(\phi, \mathbf{h}_2) = \Psi(\phi^4), \end{aligned}$$

with

$$\Psi(z) \equiv \frac{1-z}{1+z}.$$

By Proposition 4, we have:

$$\theta^b(\phi, \mathbf{h}_8) > \theta^b(\phi, \mathbf{h}_4) > \theta^b(\phi, \mathbf{h}_2) \qquad \text{for } 0 < \phi < 1.$$

As for the quasi-symmetric patterns, the expressions (C.1) and (C.2) determined in Appendix C yield the following break and sustain points:

$$\begin{aligned}
\theta^b(\phi, \mathbf{h}_2^{1/8}) &= 2\frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi)+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_2^{1/8}) &= 4\left(1-\phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right), \\
\theta^b(\phi, \mathbf{h}_2^{1/4}) &= 2\frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^2)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^2)+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_2^{1/4}) &= 4\left[1-\left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\
\theta^b(\phi, \mathbf{h}_2^{3/8}) &= 2\frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^3)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^3)+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_2^{3/8}) &= 4\left\{1-\left[\frac{\phi(1+\phi)}{1+\phi^3}\right]^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right\}, \\
\theta^b(\phi, \mathbf{h}_4^{q}) &= 4\frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^3)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^3)+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_4^{q}) &= 8\left\{1-\left[\frac{\phi(1+\phi)}{1+\phi^3}\right]^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right\}.
\end{aligned}$$

Therefore, we have:

$$\theta^b(\phi, \mathbf{h}_2^{1/8}) < \theta^b(\phi, \mathbf{h}_2^{1/4}) < \theta^b(\phi, \mathbf{h}_2^{3/8}) \qquad \text{for } 0 < \phi < 1, \tag{22}$$

$$\theta^s(\phi, \mathbf{h}_2^{1/8}) > \theta^s(\phi, \mathbf{h}_2^{1/4}) \qquad \theta^s(\phi, \mathbf{h}_2^{1/8}) > \theta^s(\phi, \mathbf{h}_2^{3/8}) \qquad \text{for } 0 < \phi < 1. \tag{23}$$

5.3 Stability areas

We now investigate the stability of the spatial equilibria discussed above, that is, the eight invariant patterns and the two types of non-invariant patterns depicted in Figure 4. Like in Section 3, we use the principle of path dependency and the contiguity of stability areas to select a stable spatial equilibrium when there is multiplicity. The stability areas for the invariant patterns were obtained analytically by using the same approach as in Section 3 and those for the non-invariant patterns by carrying out a series of numerical analyses. The stability areas are drawn in Figure 5 for $\sigma = 6$. Simulations show that the stability areas are similar for $\sigma = 4$ and $\sigma = 8$.

[Figure 5 about here.]

The shaded areas in Figure 5(a) describe the stability areas in the parameter space (ϕ, θ) of the symmetric invariant patterns for $m = 8$, $m = 4$, $m = 2$, and $m = 1$ cities. The uniform distribution $\mathbf{h}_8$ has a relatively small stability area at the upper-right corner of the parameter space, while agglomeration $\mathbf{h}_1$ has a large stability area at the lower-left corner, which encompasses more than half of the square. However, the stability areas of $\mathbf{h}_1$ and $\mathbf{h}_8$ do not cover the white domain. The upper-left of this domain is covered by the stability areas of $\mathbf{h}_2$ and $\mathbf{h}_4$ in Figure 5(a).[4] The stability areas of the quasi-symmetric invariant patterns $\mathbf{h}_2^{1/8}$, $\mathbf{h}_2^{1/4}$, and $\mathbf{h}_2^{3/8}$ are depicted in Figure 5(b). Note that $\mathbf{h}_4^q$ is never a stable equilibrium.

The inequalities (22) and (23) imply that the stability area of $\mathbf{h}_2^{1/8}$ includes those of the other two quasi-symmetric 2-city patterns. Furthermore, Figures 5(a) and 5(b) show that the stability area of $\mathbf{h}_2$ is included in that of $\mathbf{h}_2^{1/8}$. Last, the equilibrium utility level reached at $\mathbf{h}_2^{1/8}$ is higher than that achieved at $\mathbf{h}_2$, $\mathbf{h}_2^{1/4}$ and $\mathbf{h}_2^{3/8}$ because varieties are shipped over a shorter distance while urban costs take on the same value. In other words, $\mathbf{h}_2^{1/8}$ Pareto-dominates $\mathbf{h}_2$, $\mathbf{h}_2^{1/4}$, and $\mathbf{h}_2^{3/8}$. Combing these results strongly suggests that, among the 2-city patterns, $\mathbf{h}_2^{1/8}$ is the most natural candidate.

What about the white area in Figure 5(c) which includes no stable invariant equilibrium? This domain is covered by the stability areas of the two types of non-invariant hierarchical patterns defined in Section 5.1 for $m = 6, 5, 4, 3$ cities and depicted in Figures 5(d), 5(e), and 5(f). Thus, there exists at least one stable equilibrium for any admissible (ϕ, θ). More importantly, *non-invariant patterns may emerge as the only stable equilibria over a non-zero measure set of parameters.* This is to be contrasted with what we saw in Section 3.3 where such equilibria always coexist with invariant equilibria. Indeed, for parameters that belong to the white area of Figure 5(c), the urban system involves a hierarchy of cities which involves different numbers of city-types. Consequently, *the number of potential cities matters for the nature of the equilibrium urban system.*

5.4 The impact of transportation and commuting costs

Figure 6(a) shows that, for low values of ϕ, there is a unique spatial equilibrium involving $m = 1$ city, while there is a unique stable equilibrium with $m = 8$ cities for high values of ϕ. Other stable equilibria, such as the quasi-symmetric and hierarchical patterns, coexist for intermediate values of ϕ. More specifically, Figure 6(a) depicts the equilibrium path for $0 \leq \phi \leq 1$ obtained by numerical analysis for $\sigma = 6$ and $\theta = 0.2$. As shown by this figure, the uniform distribution encounters a bifurcation point $\mathbf{A}$, when ϕ is decreased from 1. From this bifurcation point, two bifurcating curves branch. The red curve corresponds to the stable 2-center $\mathbf{h}_2^{1/8}$ at $\mathbf{B}$, while the blue curve corresponds to the stable agglomerated pattern $\mathbf{h}_1$ at $\mathbf{C}$.

[4] The stability area for $\mathbf{h}_4$ is extremely small.

Along the bifurcating paths, there are also stable *non-invariant* patterns such as $\mathbf{h}_m^{\text{non}}$ whose number of cities decreases from $m = 6$ to 3. These patterns also involve a hierarchy of cities having different sizes. In sum, *lowering transportation costs increases the number of populated cities*, which rises from 1 to 6, one by one, and, then, jumps up from 6 to 8.

[Figure 6 about here.]

The intuition behind this finding is as follows. When transportation costs are prohibitive, individuals consume mainly the locally produced varieties. However, they are willing to bear high commuting costs generated by a large population because they have a preference for variety. Therefore, as transportation costs steadily decrease, importing varieties from other cities become cheaper, so that the market solves the congestion problem by spreading the production over a growing number of smaller cities in which the individual labor supply rises. Eventually, when transport costs are sufficiently low, production and consumption are evenly dispersed across all potential cities because such a configuration allows consumers to produce the highest mass of varieties. Figure 6(b) shows that, as ϕ rises, *the dispersion of activities obeys a hierarchical principle* in that the urban system involves a growing number of cities whose size shrinks. During this process, consumers are able to produce a wider range of varieties because they supply more labor thanks to the drop in the time spent in commuting.

We now consider the same thought experiment where the commuting rate decreases from 1 to 0. Recall that we consider transportation across cities that belong to the same country. As a result, we may expect ϕ to be significantly larger than in the case of international trade costs. Therefore, we assume that $\phi = 0.8$ while keeping $\sigma = 6$. Figure 7, which depicts the sequence of stable equilibria for $0 \leq \theta \leq 1$, shows that the number of cities jumps down from 8 to 6 and, then, decreases from 6 to 1, one by one. To be precise, Figure 7 shows that *a steady decrease in commuting costs leads to the gradual concentration of firms and consumers*. Like in Section 3, in between the extreme cases of dispersion and agglomeration, the economy obeys the hierarchical principle with one (or two) primate city whose size grows when θ falls, while the size of the other cities decreases as the distance to the primate city rises. Furthermore, the migration of consumers toward the primate city implies that the small cities gradually disappear from the urban system. Unexpectedly perhaps, some cities, such as those at $x = 1/8$ and $x = 7/8$, may grow during the first phases of the agglomeration process before declining at the benefit of the biggest cities. Note also that, during this process, an expanding arc of the racetrack, which used to host small cities, ends up being empty. In other words, *the size of the empty region grows.*

[Figure 7 about here.]

In sum, as θ decreases, the most likely sequence of stable equilibria is as follows:

$$\begin{aligned}&\text{Dispersion } (\mathbf{h}_8) \longrightarrow \text{Hierarchical patterns } (\mathbf{h}_m^{\text{non}};\ m = 6, 5, 4, 3)\\ &\longrightarrow \text{Quasi-symmetric 2-city agglomeration } (\mathbf{h}_2^{1/8}) \longrightarrow \text{Agglomeration } (\mathbf{h}_1).\end{aligned}$$

The comparison of the cases $n = 4$ and $n = 8$ shows that *a larger number of potential sites leads to the emergence of non-invariant patterns that do not appear when n is smaller.* Indeed, the spatial equilibria that arise between the extreme cases of agglomeration and dispersion are hierarchical. These equilibria vastly differ from the symmetric patterns $\mathbf{h}_4$ and $\mathbf{h}_2$ that are a priori the natural candidate equilibria. What is more, the sequence of bifurcations also differs from what Akamatsu et al. (2012) and Ikeda et al. (2012) obtained with n regions and a population of immobile and evenly distributed farmers. This shows once more that accounting for urban costs leads to very different and richer conclusions than those obtained in Krugman's setting where land and commuting are disregarded.

6 Conclusion

In this paper, we have proposed a new approach to determine the path of stable spatial equilibria in a multi-location setting that involves migration, transportation and commuting costs. In other words, we recognize that firms and people use land, which anchor cities to specific locations, while shipping goods between cities remains costly. Furthermore, even though working with several rather than two locations renders the analysis more complex, modeling space as a racetrack has led to new results. First, there exist new and empirically relevant equilibria that cannot emerge in a two-location setting. Second, the multiplicity of stable equilibria is not an exotica. Third, by combining the concepts of stability areas and path dependency, we have been able to select plausible equilibrium paths that display urban patterns that are either quasi-symmetric or hierarchical. By contrast, symmetric patterns, which seem a priori the most natural candidate equilibria in our setting, seldom emerge when the economy is subject to various shocks. Our analysis also confirms the result that transportation and commuting costs have opposite effects on the number and size of cities.

Our analysis remains incomplete in several respects. First, our model disregards several general equilibrium effects that shape the actual space-economy. For example, we assume that exogenous markups and homogeneous firms. Yet, it has been shown that shocks to transportation and commuting costs foster tougher competition and firm selection when preferences are no longer modeled by the CES (Behrens et al., 2017). Second, the stability analysis of the non-invariant patterns, especially those that display an urban hierarchy, remains so far out of reach from the analytical point of view. One way out is to introduce a second and immobile population like in Krugman (1991). However, we are left with the following question: why are some agents mobile and the others immobile? Similarly, introducing location-specific factors, such as city-specific commuting rates, renders the stability analysis especially hard. On the other hand, the model can be extended to deal with the production of non-tradable services supplied to the population of each city. Third, in equilibrium cities have the same size or form a hierarchy in which cities get smaller as the distance to the biggest city rises. Instead, we would like to obtain patterns in which cities having different sizes alternate. The most natural way to get such urban patterns is to add at least a second sector, like in Tabuchi and Thisse (2011). This is something we hope to accomplish in the future.

A final comment is in order. Our paper relies on *internal* increasing returns. Yet, empirical evidence shows the existence of significant agglomeration economies that take the form of *external* increasing returns (Duranton and Puga, 2004). Accounting for such effects makes the analysis much more difficult from the analytical viewpoint. However, in the case of symmetric invariant patterns we can show that agglomeration economies slow down the process of dispersion associated with the decrease of transportation costs. We find it reasonable to expect the same to hold for other stable equilibria.

References

Akamatsu, T., Mori, T., Osawa, M., and Takayama, Y. 2019. Endogenous agglomeration in a many-region world, MPRA paper, No.97496.

Akamatsu, T., Takayama, Y. and Ikeda, K. 2012. Spatial discounting, Fourier, and racetrack economy: A recipe for the analysis of spatial agglomeration models, J. Econ. Dynamics and Control 36(11), 1729-1759.

Allen T., Arkolakis, C. and Takahashi Y. 2020. Universal gravity, J. Pol. Econ., forthcoming.

Behrens, K., Duranton, G. and Robert-Nicoud, F. 2014. Productive cities: Sorting, selection, and agglomeration, J. Pol. Econ. 122(3), 507-553.

Behrens, K., Mion, G., Murata, Y., and Suedekum, J. 2017. Spatial frictions, J. Urban Econ. 97, 40-70.

Bergstrand, J.H., Egger, P. and Larch, M. 2013. Gravity Redux: Estimation of gravity-equation coefficients, elasticities of substitution, and general equilibrium comparative statics under asymmetric bilateral trade costs, J. Int. Econ. 89(1), 110-121.

Bosker, M. and Buringh, E. 2017. City seeds: Geography and the origins of the European city system, J. Urban Econ. 98, 139-157.

Brakman, S., Garretsen, H. and Schramm, M. 2004. The strategic bombing of German cities during World War II and its impact on city growth, J. Econ. Geog. 4(2), 201-218.

Brülhart, M., Desmet, K. and Klink, G-P. 2019. The shrinking advantage of market potential, CEPR DP14157.

Cuberes, G., Desmet, K. and Rappaport, J. 2019. Urban growth shadows, Southern Methodist University, memo.

Davis, D.R. and Dingel, J. 2019. A spatial knowledge economy, Am. Econ. Rev. 109(1), 153-170.

Davis, D.R. and Weinstein, D.E. 2002. Bones, bombs and break points: The geography of economic activity, Am. Econ. Rev. 92(5), 1269-1289.

Davis, P.J. (1979) Circulant Matrices, John Wiley & Sons.

Desmet, K. and Rossi-Hansberg, E. 2013. Urban accounting and welfare, Am. Econ. Rev. 103(6), 2296-2327.

Duranton, G. and Puga, D. 2004. Micro-foundations of urban increasing returns: Theory. In: Handbook of Regional and Urban Economics, Volume 4, edited by J.V. Henderson and J.-F. Thisse, 2063-2117. Elsevier.

Fujita, M., Krugman, P. and Venables, A.J. 1999. The Spatial Economy: Cities, Regions and International Trade, MIT Press.

Henderson, J.V. 1974. The sizes and types of cities, Am. Econ. Rev. 64(4), 640-656.

Henderson, J.V., Squires, T., Storeygard, A., and Weil, D. 2018. The global distribution of economic activity: Nature, history, and the role of trade, Q. J. Econ. 133(1), 357-406.

Horn, R.A. and Johnson, C.R. 2013. Matrix Analysis, Cambridge University Press, 2nd edition.

Ikeda, K., Akamatsu, T. and Kono, T. 2012. Spatial period-doubling agglomeration of a core-periphery model with a system of cities, J. Econ. Dynamics and Control 36(5), 754-778.

Ikeda, K., Onda, M. and Takayama, Y. 2019. Bifurcation theory of a racetrack economy in a spatial economy model, Networks and Spatial Econ. 19, 57-82.

Kahneman, D., Krueger, A.B., Schkade, D.A., Schwarz, N. and Stone, A.A. 2004. A survey method for characterizing daily life experience: The day reconstruction method, Science 306, 1776-80.

Krugman, P. 1991. Increasing returns and economic geography, J. Pol. Econ. 99(3), 483-499.

Murata, Y. and Thisse, J.-F. 2005. A simple model of economic geography à la Helpman-Tabuchi, J. Urban Econ. 58(1), 137-155.

Rossi-Hansberg, E. 2005. A spatial theory of trade, Am. Econ. Rev. 95(5), 1464-1491.

Sandholm, W.H. 2010. Population Games and Evolutionary Dynamics, MIT Press.

Tabuchi, T., and Thisse, J.-F. 2006. Regional specialization, urban hierarchy, and commuting costs, Int. Econ. Rev. 47(4), 1295-1317.

Tabuchi, T., and Thisse, J.-F. 2011. A new economic geography model of central places, J. Urban Econ. 69(2), 240-252.

van Ommeren, J.N. and Gutiérrez-i-Puigarnau, E. 2011. Are workers with a long commute less productive? An empirical analysis of absenteeism. Reg. Sci. Urban Econ. 41(1), 1-8.

Appendix

A. Proof of Proposition 2

Proof of Proposition 2(a) We first consider the case where $\phi = 0$ to obtain a necessary condition for a pattern $\mathbf{h}_m$ to be an invariant steady-state. When $\phi = 0$, the indirect utility $v_i(\mathbf{h}_m)$ is given by

$$v_i(\mathbf{h}_m) = \zeta \left(1 - \frac{\theta}{2} h_i\right)^{\frac{\sigma}{\sigma-1}} h_i^{\frac{1}{\sigma-1}}.$$

Therefore, for $\mathbf{h}_m$ to be a steady-state for any θ, that is, $v_i(\mathbf{h}_m) = v_0(\mathbf{h}_m)$ for all $i \in \mathcal{I}_m - \{0\}$, it must be that

$$h_i = \frac{1}{m} \quad \text{for all } i \in \mathcal{I}_m. \tag{A.1}$$

In other words, $\mathbf{h}_m$ is an invariant steady-state only if all populated cities have the same size.

Next, we consider the general case of Proposition 2(a). Using (A.1), it is readily verified that the wage equation (5) implies that the wage bill is the same across all occupied cities:

$$w_i L_i = \begin{cases} w_0 L_0 & \text{if} \quad i \in \mathcal{I}_m, \\ 0 & \text{if} \quad i \in \mathcal{I}_0. \end{cases}$$

Substituting this expression into (5), we obtain the equilibrium wage at the potential city i:

$$w_i^\sigma = \sum_{j \in \mathcal{I}_m} \frac{\phi_{ij}}{w_0^{-\sigma} \sum_{k \in \mathcal{I}_m} \phi_{kj}} = \frac{\Phi_i}{\Phi_0} w_0^\sigma.$$

Using the above expressions, we can rewrite the indirect utility (6) as follows:

$$v_i(\mathbf{h}_m) = \zeta \left(1 - \frac{\theta}{2} h_i\right) L_0^{\frac{1}{\sigma-1}} \left(\frac{\Phi_i}{\Phi_0}\right)^{\frac{1}{\sigma}} \Phi_i^{\frac{1}{\sigma-1}} \quad \text{for all } i \in \mathcal{I}, \tag{A.2}$$

where $L_0 \equiv h_0(1 - \frac{1}{2}\theta h_0)$ is the labor supply in a populated city where $h_i = 1/m$ for $i \in \mathcal{I}_m$, while $h_i = 0$ otherwise.

It follows from (A.2) that $\mathbf{h}_m$ is a steady-state if and only if

$$\frac{v_0(\mathbf{h}_m)}{v_i(\mathbf{h}_m)} = \left(\frac{\Phi_0}{\Phi_i}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} = 1 \quad \text{for all } i \in \mathcal{I}_m.$$

As a result, $\mathbf{h}_m$ is a steady-state for any ϕ and θ if and only if (A.1) and the following condition hold:

$$\Phi_0 = \Phi_i \quad \text{for all } i \in \mathcal{I}_m.$$

Proof of Proposition 2(b) It follows from (A.2) that a necessary and sufficient condition for an invariant steady-state $\mathbf{h}_m$ $(m < n)$ to be sustainable is given by the following inequality:

$$\frac{v_0(\mathbf{h}_m)}{\max_{i\in\mathcal{I}_0} v_i(\mathbf{h}_m)} = \frac{\left(\frac{2m-\theta}{2m}\right)\Phi_0^{\frac{2\sigma-1}{\sigma(\sigma-1)}}}{(\max_{i\in\mathcal{I}_0}\Phi_i)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}} \geq 1.$$

Proof of Proposition 2(c) By permuting appropriately the components of $\mathbf{h}_m$, we obtain:

$$\hat{\mathbf{h}}_m = (\mathbf{h}_{m+}, \mathbf{h}_{m0}),$$

where $\mathbf{h}_{m+} = (h_i)_{i\in\mathcal{I}_m}$ and $\mathbf{h}_{m0} = (h_i)_{i\in\mathcal{I}_0}$. In line with Ikeda et al. (2012), we may rearrange the Jacobian matrix (E.1) of the adjustment process as follows:

$$\begin{aligned}
\hat{\mathbf{J}} &= \begin{bmatrix} \mathbf{J}_+ & \mathbf{J}_{+0} \\ \mathbf{0} & \mathbf{J}_0 \end{bmatrix} \qquad\qquad \text{(A.3)}\\
\mathbf{J}_+ &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = h_0\,(\mathbf{I} - h_0\mathbf{E})\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - h_0 v_0(\mathbf{h}_m)\mathbf{E},\\
\mathbf{J}_{+0} &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i\in\mathcal{I}_m,\ j\in\mathcal{I}_0} = h_0\,(\mathbf{I} - h_0\mathbf{E})\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i\in\mathcal{I}_m,\ j\in\mathcal{I}_0}\\
\mathbf{J}_0 &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_0} = \mathrm{diag}\left[(v_i(\mathbf{h}_m) - v_0(\mathbf{h}_m))_{i\in\mathcal{I}_0}\right].
\end{aligned}$$

This implies that the eigenvalues of $\nabla\mathbf{F}(\mathbf{h}_m)$ are given by the eigenvalues of $\mathbf{J}_+$ and $\mathbf{J}_0$. Since the eigenvalues of $\mathbf{J}_0$ are $(v_i(\mathbf{h}_m) - v_0(\mathbf{h}_m))_{i\in\mathcal{I}_0}$, all the eigenvalues of $\hat{\mathbf{J}}$ have the negative real part if and only if

$$\begin{aligned}
&\text{all the eigenvalues of } \mathbf{J}_+ \text{ have negative real part,}\\
&v_i(\mathbf{h}_m) < v_0(\mathbf{h}_m) \quad \forall i \in \mathcal{I}_0.
\end{aligned}$$

B. Proof of Proposition 3

We show that the invariant and non-invariant patterns are spatial equilibria over some subsets of $[0, 1] \times [0, 1]$.

(i) Proposition 2(b) implies that there exist values of ϕ and θ such that the invariant patterns of Figure 1(a) are spatial equilibria.

(ii) For the non-invariant pattern $\mathbf{h}_4^{qnon} = (h_0, h_0, h_2, h_2)$, the indirect utilities satisfy the conditions:

$$
\begin{aligned}
v_0(\mathbf{h}_4^{qnon}) &= v_1(\mathbf{h}_4^{qnon}), \qquad v_2(\mathbf{h}_4^{qnon}) = v_3(\mathbf{h}_4^{qnon}) \\
\frac{v_0(\mathbf{h}_4^{qnon})}{v_2(\mathbf{h}_4^{qnon})} &= \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_2}\frac{w_0}{w_2}\left(\frac{L_0 w_0^{1-\sigma}+\phi L_2 w_2^{1-\sigma}}{\phi L_0 w_0^{1-\sigma}+L_2 w_2^{1-\sigma}}\right)^{1/(\sigma-1)}.
\end{aligned}
\tag{B.1}
$$

Using (5) implies

$$
\left(\frac{w_0}{w_2}\right)^{\sigma} = \frac{L_0 w_0^{1-\sigma}+\phi L_2 w_2^{1-\sigma}}{\phi L_0 w_0^{1-\sigma}+L_2 w_2^{1-\sigma}},
\tag{B.2}
$$

so that (B.1) may be rewritten as follows:

$$
\frac{v_0(\mathbf{h}_4^{qnon})}{v_2(\mathbf{h}_4^{qnon})} = \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_2}\left(\frac{w_0}{w_2}\right)^{(2\sigma-1)/(\sigma-1)}.
$$

It follows from this expression and (B.2) that $\mathbf{h}_4^{qnon}$ is a spatial equilibrium (i.e., $v_0(\mathbf{h}_4^{qnon}) = v_2(\mathbf{h}_4^{qnon})$) if and only if the following condition holds:

$$
\left(\frac{l_2}{l_0}\right)^{\sigma(\sigma-1)/(2\sigma-1)} = \frac{\frac{h_1}{h_3}+\phi\left(\frac{l_3}{l_1}\right)^{\sigma^2/(2\sigma-1)}}{\phi\frac{h_1}{h_3}+\left(\frac{l_3}{l_1}\right)^{\sigma^2/(2\sigma-1)}},
$$

where $l_i \equiv 1-\frac{\theta}{2}h_i$ and $h_2 = 1/2 - h_0$. Since there are three parameters, this equation is satisfied over a non-negligible domain of ϕ, θ and $h_0 > 1/2$.

(iii) For the non-invariant pattern $\mathbf{h}_3^{\text{non}} = (h_0, h_1, 0, h_1)$, the indirect utilities are given by the following expressions:

$$
\begin{aligned}
v_1(\mathbf{h}_3^{\text{non}}) &= v_3(\mathbf{h}_3^{\text{non}}) \\
\frac{v_0(\mathbf{h}_3^{\text{non}})}{v_1(\mathbf{h}_3^{\text{non}})} &= \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_1}\frac{w_0}{w_1}\left(\frac{L_0 w_0^{1-\sigma}+2\phi L_1 w_1^{1-\sigma}}{\phi L_0 w_0^{1-\sigma}+(1+\phi^2)L_1 w_1^{1-\sigma}}\right)^{1/(\sigma-1)}, \qquad \text{(B.3)} \\
\frac{v_0(\mathbf{h}_3^{\text{non}})}{v_2(\mathbf{h}_3^{\text{non}})} &= \left(1-\frac{\theta}{2}h_0\right)\frac{w_0}{w_2}\left(\frac{L_0 w_0^{1-\sigma}+2\phi L_1 w_1^{1-\sigma}}{\phi^2 L_0 w_0^{1-\sigma}+2\phi L_1 w_1^{1-\sigma}}\right)^{1/(\sigma-1)}, \qquad \text{(B.4)}
\end{aligned}
$$

where v_2 (w_2) is the indirect utility (wage) that prevails at 2 when one worker is located there.

Since (5) implies

$$\begin{aligned}\left(\frac{w_0}{w_1}\right)^{\sigma} &= \frac{L_0 w_0^{1-\sigma} + 2\phi L_1 w_1^{1-\sigma}}{\phi L_0 w_0^{1-\sigma} + (1+\phi^2) L_1 w_1^{1-\sigma}},\\ \left(\frac{w_0}{w_2}\right)^{\sigma} &= \frac{L_0 w_0^{1-\sigma} + 2\phi L_1 w_1^{1-\sigma}}{\phi^2 L_0 w_0^{1-\sigma} + 2\phi L_1 w_1^{1-\sigma}},\end{aligned}$$

(B.3) and (B.4) are equivalent to

$$\begin{aligned}\frac{v_0(\mathbf{h}_3^{\text{non}})}{v_1(\mathbf{h}_3^{\text{non}})} &= \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_1}\left(\frac{w_0}{w_1}\right)^{(2\sigma-1)/(\sigma-1)},\\ \frac{v_0(\mathbf{h}_3^{\text{non}})}{v_2(\mathbf{h}_3^{\text{non}})} &= \left(1-\frac{\theta}{2}h_1\right)\left(\frac{w_0}{w_2}\right)^{(2\sigma-1)/(\sigma-1)},\end{aligned}$$

where we have used

$$\begin{aligned}v_0(\mathbf{h}_3^{\text{non}}) &= \zeta\Delta_0(\mathbf{h}_3^{\text{non}})^{\frac{1}{\sigma-1}} y_0\\ &= \zeta\left(L_0 w_0^{1-\sigma} + 2L_1 w_1^{1-\sigma}\phi\right)^{\frac{1}{\sigma-1}}\left(1-\frac{\theta}{2}h_0\right) w_0\end{aligned}$$

and

$$\begin{aligned}v_2(\mathbf{h}_3^{\text{non}}) &= \zeta\Delta_2(\mathbf{h}_3^{\text{non}})^{\frac{1}{\sigma-1}} y_2\\ &= \zeta\left(L_0 w_0^{1-\sigma}\phi^2 + 2L_1 w_1^{1-\sigma}\phi\right)^{\frac{1}{\sigma-1}} w_2\end{aligned}$$

because $L_2 = h_2(1-\theta h_2/2) = 0$.

Hence, $\mathbf{h}_3^{\text{non}}$ is a spatial equilibrium (i.e., $v_0(\mathbf{h}_3^{\text{non}}) = v_1(\mathbf{h}_3^{\text{non}})$ and $v_0(\mathbf{h}_3^{\text{non}}) \geq v_2(\mathbf{h}_3^{\text{non}})$) if and only if the following conditions hold:

$$\begin{aligned}\left(\frac{l_1}{l_0}\right)^{\sigma(\sigma-1)/(2\sigma-1)} &= \frac{\frac{h_0}{h_1} + 2\phi\left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)}}{\phi\frac{h_0}{h_1} + (1+\phi^2)\left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)}},\\ \left(\frac{l_2}{l_0}\right)^{\sigma(\sigma-1)/(2\sigma-1)} &\leq 1 + \frac{(1-\phi^2)\frac{h_0}{h_1}}{\phi^2\frac{h_0}{h_1} + 2\phi\left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)}},\end{aligned}$$

where $h_1 = (1-h_0)/2$. Since there exists a non-negligible domain of ϕ, θ and $h_0 > 1/3$ satisfying these two equations, $\mathbf{h}_3^{\text{non}}$ is a spatial equilibrium over this domain.

(iv) For the non-invariant pattern $\mathbf{h}_4^{\text{non}} = (h_0, h_1, h_2, h_1)$, the indirect utilities are given by the following expressions:

$$\begin{aligned}
v_1(\mathbf{h}_3^{\text{non}}) &= v_3(\mathbf{h}_3^{\text{non}}) \\
\frac{v_0(\mathbf{h}_3^{\text{non}})}{v_1(\mathbf{h}_3^{\text{non}})} &= \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_1}\frac{w_0}{w_1}\left[\frac{\Delta_0(\mathbf{h}_4^{\text{non}})}{\Delta_1(\mathbf{h}_4^{\text{non}})}\right]^{1/(\sigma-1)}, \\
\frac{v_0(\mathbf{h}_3^{\text{non}})}{v_2(\mathbf{h}_3^{\text{non}})} &= \frac{1-\frac{\theta}{2}h_0}{1-\frac{\theta}{2}h_2}\frac{w_0}{w_2}\left[\frac{\Delta_0(\mathbf{h}_4^{\text{non}})}{\Delta_2(\mathbf{h}_4^{\text{non}})}\right]^{1/(\sigma-1)},
\end{aligned}$$

where $\Delta_i(\mathbf{h}) \equiv \sum_{j\in\mathcal{I}} L_j w_j^{1-\sigma}\phi_{ji}$.

It follows from (5) that

$$L_0 w_0 \left(\frac{w_2}{w_0}\right)^{\sigma} + L_2 w_2 \left(\frac{w_0}{w_2}\right)^{\sigma} = \frac{\Delta_2(\mathbf{h}_3^{\text{non}})}{\Delta_0(\mathbf{h}_3^{\text{non}})} L_0 w_0 + \frac{\Delta_0(\mathbf{h}_3^{\text{non}})}{\Delta_2(\mathbf{h}_3^{\text{non}})} L_2 w_2, \tag{B.5}$$

$$L_0 w_0 \left(\frac{w_1}{w_0}\right)^{\sigma} + L_2 w_2 \left(\frac{w_1}{w_2}\right)^{\sigma} = \frac{\Delta_2(\mathbf{h}_4^{\text{non}})}{\Delta_0(\mathbf{h}_4^{\text{non}})} L_0 w_0 + \frac{\Delta_1(\mathbf{h}_4^{\text{non}})}{\Delta_2(\mathbf{h}_4^{\text{non}})} L_2 w_2. \tag{B.6}$$

Therefore, $\mathbf{h}_4^{\text{non}}$ is a spatial equilibrium (i.e., $v_0(\mathbf{h}_4^{\text{non}}) = v_3(\mathbf{h}_3^{\text{non}})$) if and only if (B.5) and (B.6) and the following conditions hold:

$$\frac{\Delta_0(\mathbf{h}_4^{\text{non}})}{\Delta_1(\mathbf{h}_4^{\text{non}})} = \left(\frac{l_1}{l_0}\frac{w_1}{w_0}\right)^{\sigma-1}, \qquad \frac{\Delta_0(\mathbf{h}_4^{\text{non}})}{\Delta_2(\mathbf{h}_4^{\text{non}})} = \left(\frac{l_2}{l_0}\frac{w_2}{w_0}\right)^{\sigma-1}. \tag{B.7}$$

The conditions (B.5)–(B.7) may be rewritten as follows:

$$\left(\frac{w_2}{w_0}\right)^{2\sigma-1} = \left(\frac{l_0}{l_2}\right)^{\sigma-1}, \qquad \left(\frac{w_1}{w_0}\right)^{2\sigma-1} = \left(\frac{l_0}{l_1}\right)^{\sigma-1}. \tag{B.8}$$

Substituting (B.8) into (B.7), we have:

$$\begin{aligned}
\left(\frac{l_1}{l_0}\right)^{\sigma(\sigma-1)/(2\sigma-1)} &= \frac{h_0 + 2\phi h_1 \left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)} + \phi^2 h_2 \left(\frac{l_2}{l_0}\right)^{\sigma^2/(2\sigma-1)}}{\phi h_0 + (1+\phi^2) h_1 \left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)} + \phi h_2 \left(\frac{l_2}{l_0}\right)^{\sigma^2/(2\sigma-1)}}, \\
\left(\frac{l_2}{l_0}\right)^{\sigma(\sigma-1)/(2\sigma-1)} &= \frac{h_0 + 2\phi h_1 \left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)} + \phi^2 h_2 \left(\frac{l_2}{l_0}\right)^{\sigma^2/(2\sigma-1)}}{\phi^2 h_0 + 2\phi h_1 \left(\frac{l_1}{l_0}\right)^{\sigma^2/(2\sigma-1)} + h_2 \left(\frac{l_2}{l_0}\right)^{\sigma^2/(2\sigma-1)}}.
\end{aligned}$$

Once again, these two equations are satisfied over a non-negligible domain of ϕ, θ, h_0, and $h_1 (< h_0)$.

C. Sustain and break points for invariant patterns

1. We first derive the sustain point $\theta^s(\phi, \mathbf{h}_m)$ of the invariant pattern $\mathbf{h}_m$ for $1 \leq m \leq n$. We know that the uniform distribution $\mathbf{h}_n$ is always a spatial equilibrium. However, the invariant patterns $\mathbf{h}_m$ with $m < n$ are not necessarily sustainable.

It follows from (13) that the sustain point $\theta^s(\mathbf{h}_m, \phi)$ with $m < n$ cities is uniquely determined by

$$\theta^s(\phi, \mathbf{h}_m) \equiv 2m \left[1 - \left(\frac{\max_{i \in \mathcal{I}_0} \Phi_i}{\Phi_0} \right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} \right]. \tag{C.1}$$

2. We next determine the break point $\theta^b(\phi, \mathbf{h}_m)$ for $m > 1$. As shown in Appendix F, the break point $\theta^b(\phi, \mathbf{h}_m)$ is given by

$$\theta^b(\phi, \mathbf{h}_m) = \min_{i \in \mathcal{I}_m} \theta_i(\phi, \mathbf{h}_m),$$

$$\theta_i(\phi, \mathbf{h}_m) \equiv m \frac{\frac{2\sigma-1}{\sigma-1} f_i(\mathbf{h}_m, \phi)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) f_i(\mathbf{h}_m, \phi) + \frac{\sigma}{2}},$$

where the function $-1 \leq f_i(\mathbf{h}_m, \phi) \leq 1$ is defined in Appendix F.

Furthermore, the break point $\theta^b(\phi, \mathbf{h}_m)$ decreases with ϕ when $\mathbf{h}_m$ is a symmetric invariant pattern. Indeed, we have

$$\frac{\partial \theta_i(\theta, \phi, \mathbf{h}_m)}{\partial \phi} = m \frac{\frac{2\sigma-1}{\sigma-1} \frac{\sigma}{2}}{\left[\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) f_i(\mathbf{h}_m, \phi) + \frac{\sigma}{2}\right]^2} \frac{\partial f_i(\mathbf{h}_m, \phi)}{\partial \phi} < 0,$$

because $\frac{\partial f_i(\mathbf{h}_m, \phi)}{\partial \phi} < 0$ for all $i \in \mathcal{I}$ if m is even (Akamatsu et al., 2019, Lemma A.1).

D. Proof of Lemma 2

The eigenvalues of the Jacobian matrix $(\partial F_i(\mathbf{h}_m)/\partial h_j)_{i,j \in \mathcal{I}_m}$ are given by (F.2). Hence, $\mathbf{h}_m$ satisfies (15) if and only if the following condition holds:

$$K(\phi, \mathbf{h}_m) \cdot \left[\frac{2\sigma - 1}{\sigma - 1}(1 - \theta h_0) - (\sigma - 1)\frac{\theta h_0}{2} \right] - \sigma \frac{\theta h_0}{2} < 0,$$

where $K(\phi, \mathbf{h}_m) \equiv \max_{i \in \mathcal{I}_m \setminus \{0\}} f_i$.

Since f_i is an eigenvalue of $\mathbf{D}_m \equiv (\phi_{ij}/\Phi_0)_{i,j \in \mathcal{I}_m}$, it follows from Akamatsu et al. (2019, Lemma C.1) that f_i is given by

$$f_i = \frac{1}{\Phi_0} \frac{(1 - \phi^{\frac{2n}{m}})[1 - (-1)^i \phi^{\frac{n}{2}}]}{1 - 2\phi^{\frac{n}{m}} \cos\left(\frac{2\pi}{m}\right) + \phi^{\frac{2n}{m}}},$$

where

$$\Phi_0 = 1 + \phi^{\frac{n}{2}} + 2\sum_{k=1}^{m/2-1} \phi^{\frac{n}{m}k} = \frac{(1+\phi^{\frac{n}{m}})(1-\phi^{\frac{n}{2}})}{1-\phi^{\frac{n}{m}}} \geq 0.$$

Furthermore, it also follows from Akamatsu et al. (2019, Lemma C.1) that $f_1 = \max_{i\in\mathcal{I}_m\setminus\{0\}} f_i$. Therefore, we have Lemma 2.

E. Jacobian matrix of the adjustment process

For any (invariant or non-invariant) pattern $\mathbf{h}$, the Jacobian $\nabla\mathbf{F}(\mathbf{h})$ of the adjustment process is given by

$$\nabla\mathbf{F}(\mathbf{h}) = \mathbf{R}(\mathbf{h})\cdot\nabla\mathbf{v}(\mathbf{h}) + \mathbf{J}(\mathbf{h}), \tag{E.1}$$

where the matrices $\mathbf{R}(\mathbf{h})$ and $\mathbf{J}(\mathbf{h})$ are defined as follows:

$$\begin{aligned}\mathbf{R}(\mathbf{h}) &\equiv \text{diag}[\mathbf{h}]\cdot(\mathbf{I}-\mathbf{E}\cdot\text{diag}[\mathbf{h}]),\\ \mathbf{J}(\mathbf{h}) &\equiv \text{diag}[\mathbf{v}(\mathbf{h})-\bar{v}(\mathbf{h})\mathbf{1}] - \mathbf{h}\mathbf{v}(\mathbf{h})^\top.\end{aligned}$$

In these matrices, we have:
(i) the Jacobian $\nabla\mathbf{v}(\mathbf{h})$ of the indirect utility vector, which is given by

$$\begin{aligned}\nabla\mathbf{v}(\mathbf{h}) = \text{diag}[\mathbf{v}(\mathbf{h})]\cdot\Bigg\{&\frac{1}{\sigma-1}\mathbf{M}^\top\cdot\text{diag}[\mathbf{L}]^{-1}\cdot\text{diag}[\mathbf{1}-\theta\mathbf{h}]\\ &+(\mathbf{I}-\mathbf{M})\cdot\text{diag}[\mathbf{w}]^{-1}\cdot\nabla\mathbf{w}(\mathbf{h}) - \frac{\theta}{2}\text{diag}\left[\mathbf{1}-\frac{\theta}{2}\mathbf{h}\right]^{-1}\Bigg\},\end{aligned}$$

where $\mathbf{1}$ is the vector whose elements equal 1, $\mathbf{E}$ is the $n\times n$ matrix whose elements equal to 1, $\mathbf{I}$ is the identity matrix, $\mathbf{D}\equiv(\phi_{ij})_{i,j\in\mathcal{I}}$, while

$$\begin{aligned}\mathbf{\Delta} &\equiv \mathbf{D}\cdot\text{diag}[\mathbf{w}]^{1-\sigma}\cdot\mathbf{L},\\ \mathbf{M} &\equiv \text{diag}[\mathbf{L}]\cdot\text{diag}[\mathbf{w}]^{1-\sigma}\cdot\mathbf{D}\cdot\text{diag}[\mathbf{\Delta}]^{-1};\end{aligned}$$

(ii) the Jacobian $\nabla\mathbf{w}(\mathbf{h})$ of the wage vector, which is given by

$$\nabla\mathbf{w}(\mathbf{h}) = -\left(\frac{\partial W_i(\mathbf{h})}{\partial w_j}\right)^{-1}_{i,j\in\mathcal{I}}\left(\frac{\partial W_i(\mathbf{h})}{\partial h_j}\right)_{i,j\in\mathcal{I}},$$

with

$$\left(\frac{\partial W_i(\mathbf{h})}{\partial w_j}\right)_{i,j\in\mathcal{I}} = (\mathbf{I}-\mathbf{M})\cdot\text{diag}[\mathbf{L}] + (\sigma-1)\left(\text{diag}[\mathbf{MY}] - \mathbf{M}\cdot\text{diag}[\mathbf{Y}]\cdot\mathbf{M}^\top\right)\cdot\text{diag}[\mathbf{w}]^{-1})$$

$$\left(\frac{\partial W_i(\mathbf{h})}{\partial h_j}\right)_{i,j\in\mathcal{I}} = \text{diag}[\mathbf{1}-\theta\mathbf{h}]\cdot(\text{diag}[\mathbf{w}] - \text{diag}[\mathbf{L}]^{-1}\cdot\text{diag}[\mathbf{MY}]),$$

$$+ (\mathbf{M}\cdot\text{diag}[\mathbf{Y}]\cdot\mathbf{M}^\top\cdot\text{diag}[\mathbf{L}]^{-1} - \mathbf{M}\cdot\text{diag}[\mathbf{w}])\cdot\text{diag}[\mathbf{1}-\theta\mathbf{h}],$$

where $\mathbf{Y} \equiv (L_i w_i)_{i\in\mathcal{I}}$ and $W_i(\mathbf{h})$ is defined as follows:

$$W_i(\mathbf{h}) = L_i w_i - \sum_{j\in\mathcal{I}} \frac{L_i w_i^{1-\sigma}\phi_{ij}}{\sum_{k\in\mathcal{I}} L_k w_k^{1-\sigma}\phi_{kj}} L_j w_j.$$

Note that the wage equation (5) is equivalent to $W_i(\mathbf{h}) = 0$ for all $i \in \mathcal{I}$.

In the case where $\mathbf{h} = \mathbf{h}_m$, the Jacobian matrix (E.1) of the adjustment process can be rearranged as (A.3) with

$$\left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = h_0(\mathbf{I}-h_0\mathbf{E})\cdot\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - h_0 v_0(\mathbf{h}_m)\mathbf{E},$$

$$\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = \text{diag}\left[(v_i(\mathbf{h}_m))_{i\in\mathcal{I}_m}\right]\cdot\left\{\frac{1}{\sigma-1}\frac{1}{L_0}\mathbf{D}_m\right.$$

$$\left. + \frac{1}{w_0}(\mathbf{I}-\mathbf{D}_m)\cdot\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - \frac{\theta h_0}{2L_0}\mathbf{I}\right\},$$

where $\mathbf{D}_m = (\phi_{ij}/\Phi_{m0})_{i,j\in\mathcal{I}_m}$. Since $\partial W_i(\mathbf{h}_m)/\partial h_j = 0$ for all $j \in \mathcal{I}_0$, $(\partial w_i(\mathbf{h}_m)/\partial h_j)_{i,j\in\mathcal{I}_m}$ can be expressed as follows:

$$\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = -\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial w_j}\right)^{-1}_{i,j\in\mathcal{I}_m}\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m},$$

$$\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial w_j}\right)_{i,j\in\mathcal{I}_m} = L_0(\mathbf{I}-\mathbf{D}_m)\left\{\sigma\mathbf{I} + (\sigma-1)\mathbf{D}_m\right\},$$

$$\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = -w_0(1-\theta h_0)(\mathbf{I}-\mathbf{D}_m)\mathbf{D}_m.$$

F. The break point

Let $\mathbf{D}_m \equiv (\phi_{ij}/\Phi_{m0})_{i,j\in\mathcal{I}_m}$ where ϕ_{ij}/Φ_{m0} measures the relative accessibility of city $i \in \mathcal{I}_m$. As shown in Appendix E, the Jacobian matrix $(\partial F_i(\mathbf{h}_m)/\partial h_j)_{i,j\in\mathcal{I}_m}$ is such that

$$\left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = h_0\left(\mathbf{I}-h_0\mathbf{E}\right)\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - h_0 v_0(\mathbf{h}_m)\mathbf{E},$$

where

$$\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = \text{diag}\left[(v_i(\mathbf{h}_m))_{i\in\mathcal{I}_m}\right]\cdot\left\{\frac{1}{\sigma-1}\frac{1-\theta h_0}{L_0}\mathbf{D}_m + \frac{1}{w_0}(\mathbf{I}-\mathbf{D}_m)\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - \frac{\theta h_0}{2L_0}\mathbf{I}\right\} \quad \text{(F.1)}$$

and

$$\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = (1-\theta h_0)\frac{w_0}{L_0}\left[\sigma\mathbf{I}+(\sigma-1)\mathbf{D}_m\right]^{-1}\mathbf{D}_m.$$

The symmetry of invariant patterns in a racetrack economy implies that $\mathbf{D}_m$ is a block-circulant matrix with circulant blocks (BCCB). Furthermore, since the matrices $\mathbf{I}$ and $\mathbf{E}$ are also BCCB, the three matrices have the same eigenvectors (Davis, 1979). Therefore, the eigenvalues $\mathbf{g}_m = (g_i(\mathbf{h}_m))_{i\in\mathcal{I}_m}$ of the Jacobian $(\partial F_i(\mathbf{h}_m)/\partial h_j)_{i,j\in\mathcal{I}_m}$ are given by

$$g_i(\mathbf{h}_m) = \begin{cases} -v_0(\mathbf{h}_m) < 0 & \text{if} \quad i = 0, \\ h_0 e_i(\mathbf{h}_m) & \text{if} \quad i \in \mathcal{I}_m\backslash\{0\}, \end{cases} \quad \text{(F.2)}$$

where the eigenvector for $g_0(\mathbf{h}_m)$ is $\mathbf{1}$. Furthermore,

$$e_i(\mathbf{h}_m) = \frac{v_0(\mathbf{h}_m)}{L_0\left\{\sigma+(\sigma-1)f_i\right\}} \cdot\left\{\left[\frac{2\sigma-1}{\sigma-1}(1-\theta h_0)-(\sigma-1)\frac{\theta h_0}{2}\right]f_i - \sigma\frac{\theta h_0}{2}\right\}$$

are the eigenvalues of the Jacobian (F.1) of the indirect utilities, while $f_i(\mathbf{h}_m)$ is the i-th eigenvalue of the matrix $\mathbf{D}_m$. Applying Gershgorin circle theorem yields $-1 \leq f_i \leq 1$ for any $\phi \in [0, 1]$ and $i \in \mathcal{I}_m$ (Horn and Johnson, 2013). As a result, if $\mathbf{h}_1$ is a spatial equilibrium, it is always stable. Indeed, there is no break point because g_0 is negative.

Furthermore, $\mathbf{h}_m$ is stable if and only if $e_i(\mathbf{h}_m) < 0$ for all $i \in \mathcal{I}_m\backslash\{0\}$. This condition is equivalent to

$$K(\phi,\mathbf{h}_m)\cdot\left[\frac{2\sigma-1}{\sigma-1}(1-\theta h_0)-(\sigma-1)\frac{\theta h_0}{2}\right] - \sigma\frac{\theta h_0}{2} < 0,$$

where $K(\phi,\mathbf{h}_m) \equiv \max_{i\in\mathcal{I}_m\backslash\{0\}} f_i$.

Last, consider an invariant pattern $\mathbf{h}_m$ with $m > 1$. Proposition 2 (c) implies that this pattern is stable if and only if $e_i(\mathbf{h}_m) < 0$ for all $i \in \mathcal{I}_m\backslash\{0\}$. Let $\theta_i(\phi,\mathbf{h}_m)$

be the solution of $e_i(\mathbf{h}_m) = 0$. Therefore, $e_i(\mathbf{h}_m) < 0$ if and only if

$$\theta > \theta_i(\phi, \mathbf{h}_m) = m \frac{\frac{2\sigma-1}{\sigma-1} f_i(\mathbf{h}_m, \phi)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) f_i(\mathbf{h}_m, \phi) + \frac{\sigma}{2}}.$$

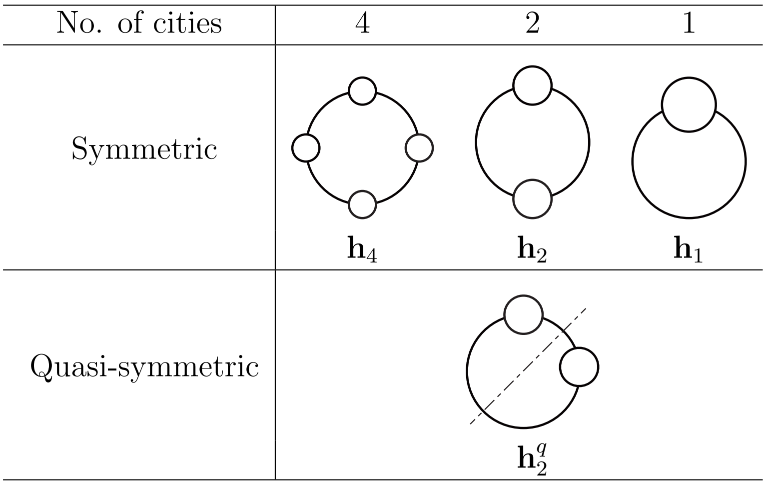
Since θ_i is an increasing for f_i, the break point $\theta^b(\phi, \mathbf{h}_m)$ is uniquely determined by

$$\theta^b(\phi, \mathbf{h}_m) = m \frac{\frac{2\sigma-1}{\sigma-1} K(\phi, \mathbf{h}_m)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) K(\phi, \mathbf{h}_m) + \frac{\sigma}{2}},$$

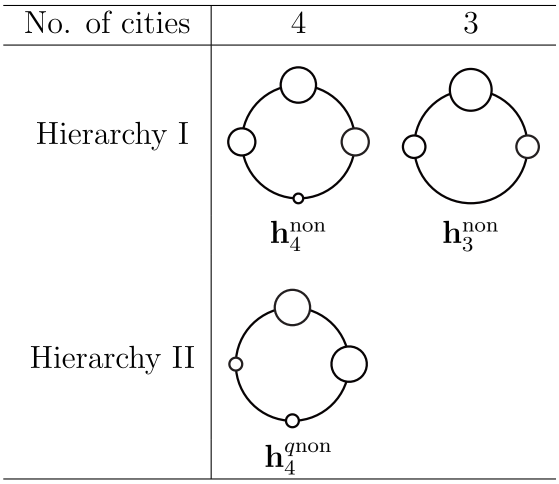
where $K(\phi, \mathbf{h}_m) \equiv \max_{i \in \mathcal{I}_m \setminus \{0\}} f_i(\phi, \mathbf{h}_m)$.

Consequently, a necessary and sufficient condition for an invariant pattern $\mathbf{h}_m$ to be stable is given by

$$\theta > \theta^b(\phi, \mathbf{h}_m).$$

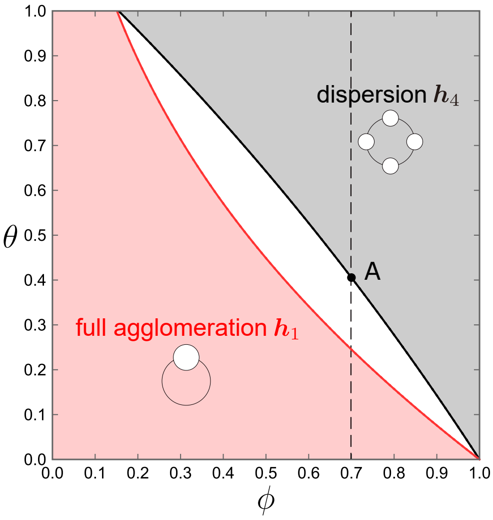


(a) Invariant patterns

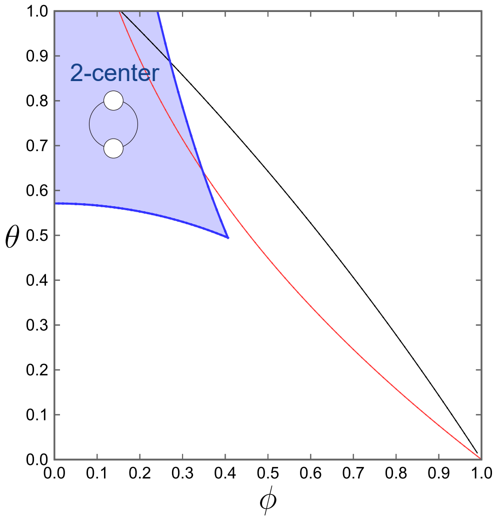


(b) Non-invariant patterns

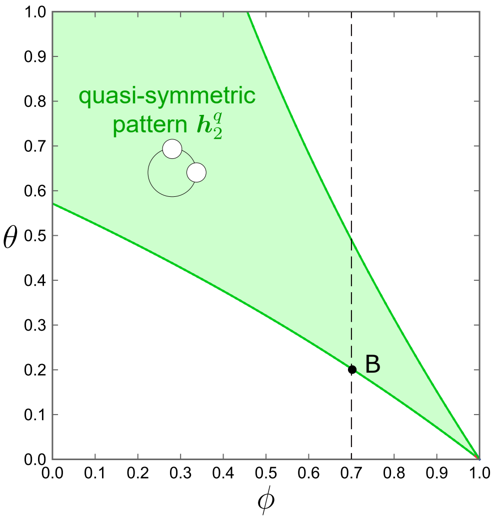
Figure 1: Invariant patterns for the racetrack economy with 4 cities.
(A larger circle expresses a city with larger population.)



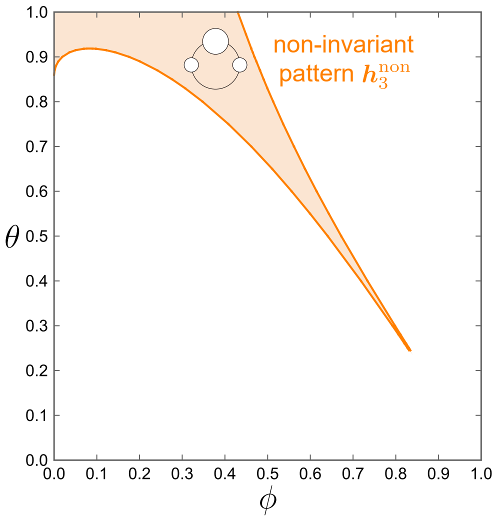
(a) Symmetric invariant patterns I



(b) Symmetric invariant patterns II

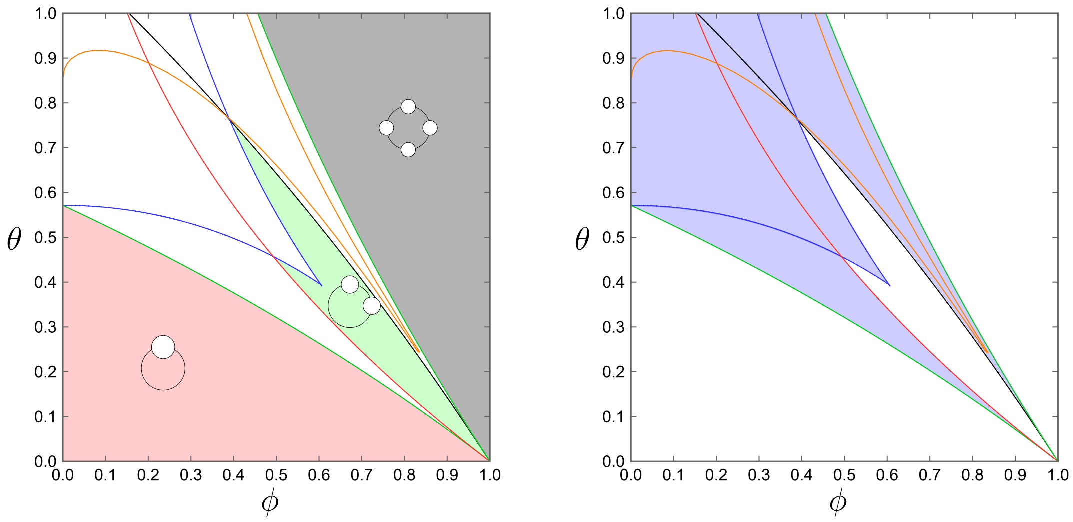


(c) Quasi-symmetric invariant pattern



(d) Non-invariant pattern

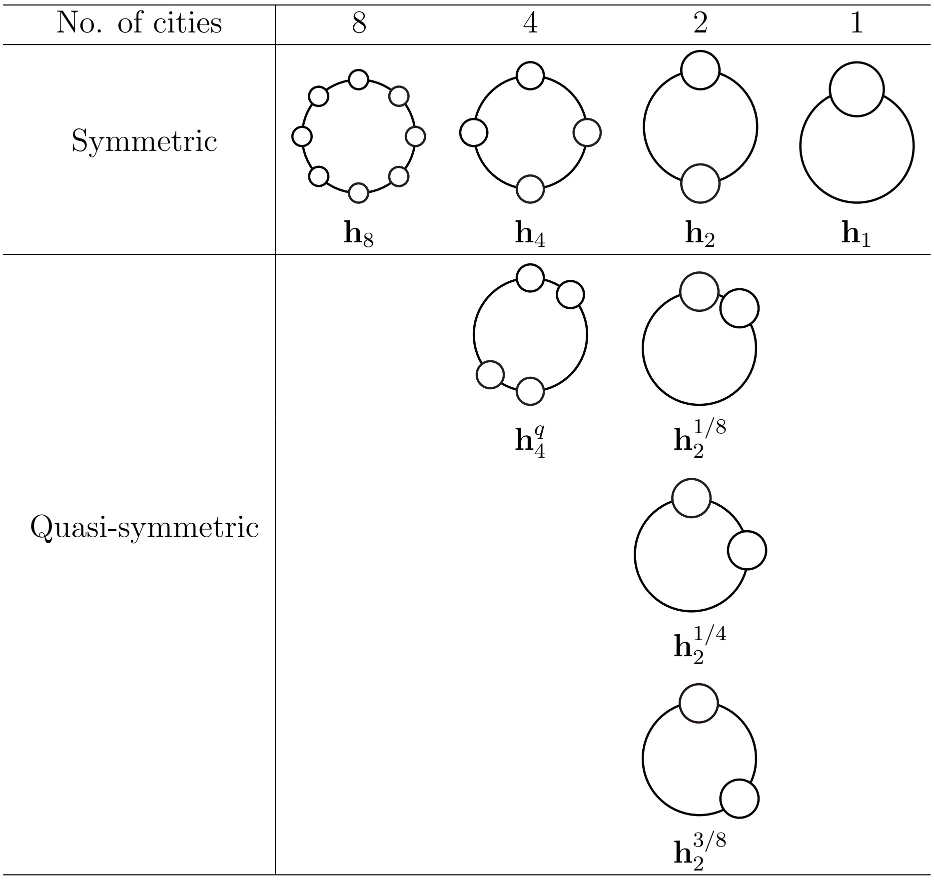
Figure 2: Stability areas of (ϕ, θ) for the five patterns for 4 cities ($\sigma = 6.0$)



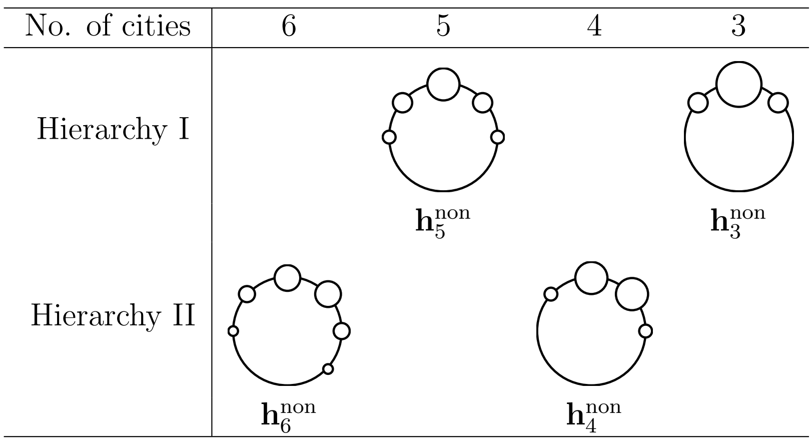
(a) Unique stable equilibrium

(b) Multiple stable equilibria

Figure 3: Zoning of (ϕ, θ) based on the multiplicity of stable equilibria for 4 cities ($\sigma = 6.0$)

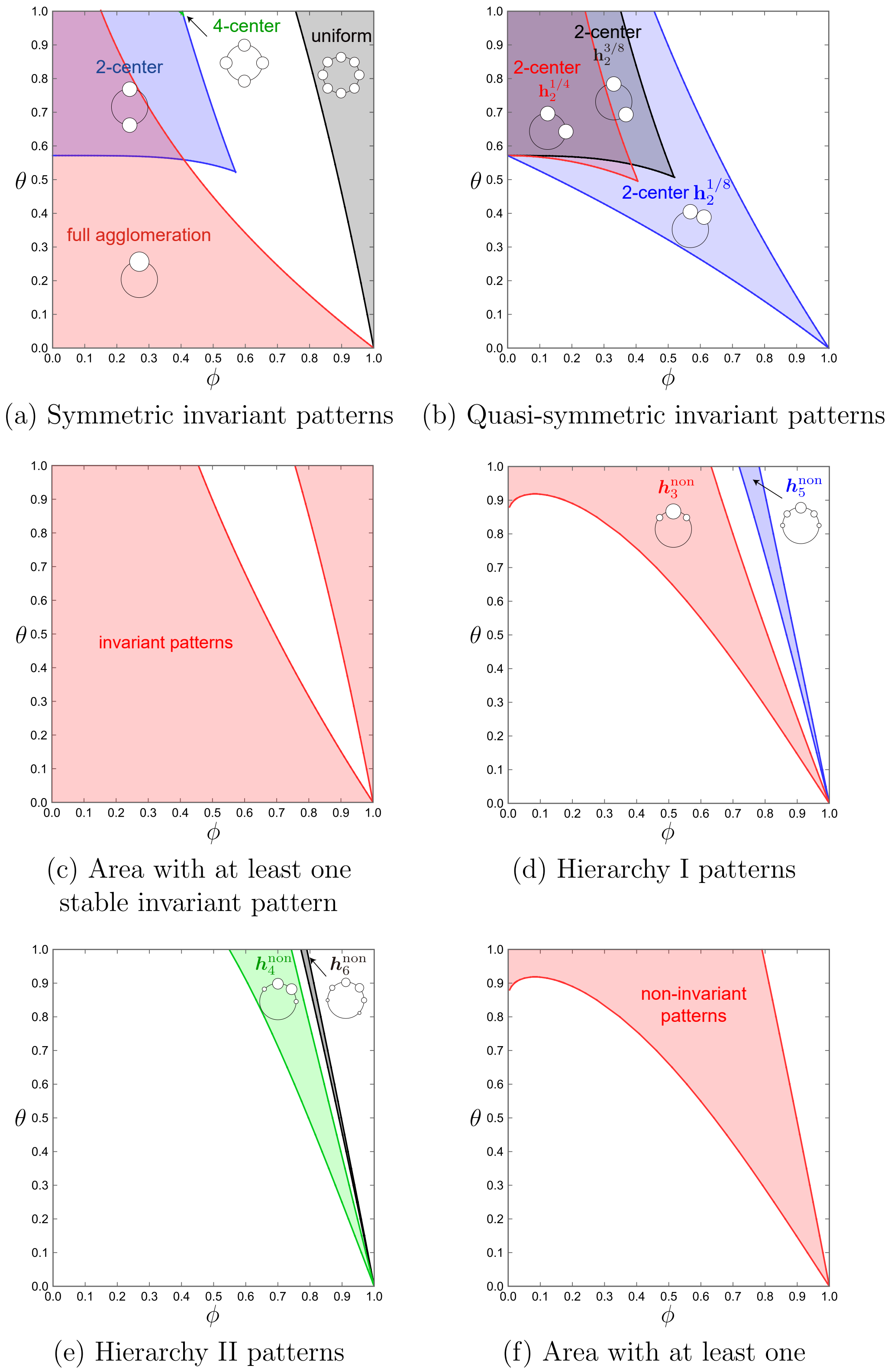


(a) Invariant patterns



(b) Non-invariant patterns (hierarchical patterns)

Figure 4: Invariant and non-invariant patterns for the racetrack economy with 8 cities. (A larger circle expresses a city with larger population.)



(a) Symmetric invariant patterns

(b) Quasi-symmetric invariant patterns

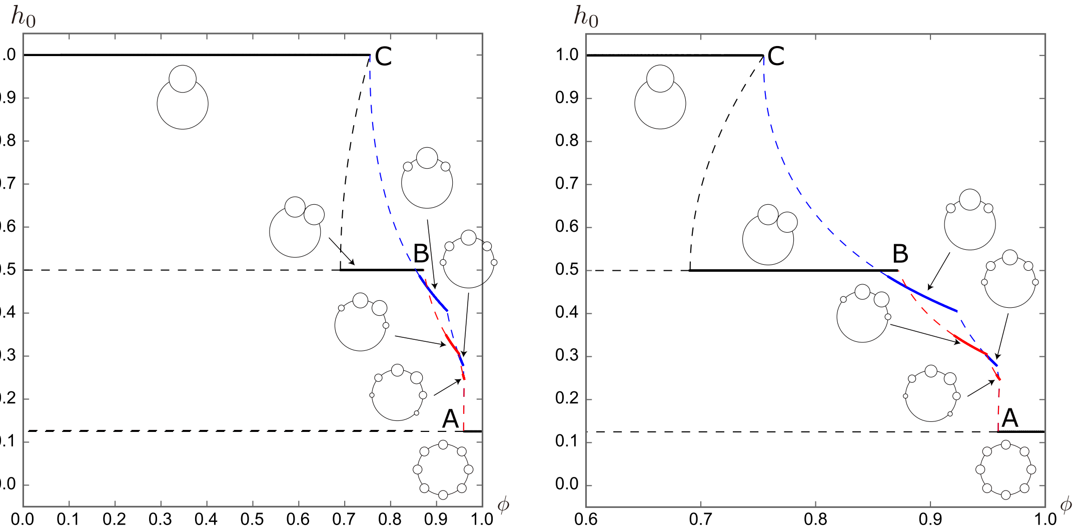
(c) Area with at least one stable invariant pattern

(d) Hierarchy I patterns

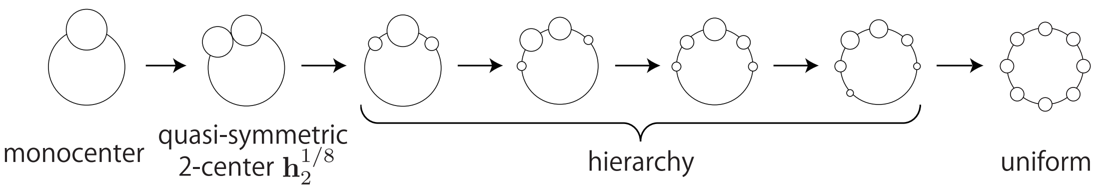
(e) Hierarchy II patterns

(f) Area with at least one stable non-invariant pattern

Figure 5: Stability areas of patterns of interest in the parameter space (ϕ, θ) for 8 cities. $(\sigma = 6.0)$

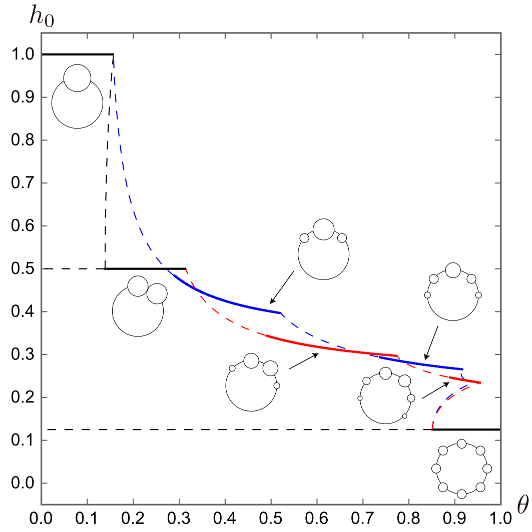


(a) Equilibrium curves (left: $0 < \phi < 1$, right: $0.6 < \phi < 1$)

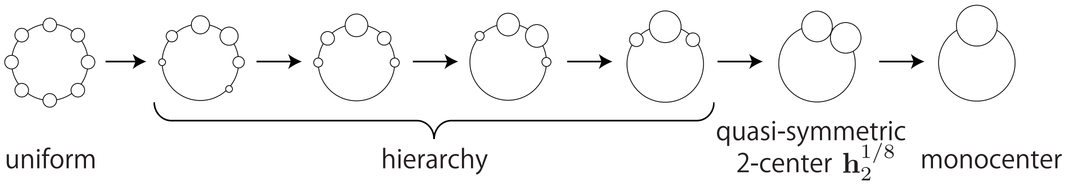


(b) Transition of stable agglomeration patterns when ϕ increases

Figure 6: Equilibrium path for $0 < \phi < 1$. ($\theta = 0.2$, $\sigma = 6.0$)
(solid curve: stable equilibria, dashed curve: unstable equilibria)



(a) Equilibrium curves



(b) Transition of stable agglomeration patterns when θ decreases

Figure 7: Equilibrium path for $0 < \theta < 1$. ($\phi = 0.8, \sigma = 6.0$)
(solid curve: stable equilibria, dashed curve: unstable equilibria)