CREDIT RISK MODELLING WITH FRACTIONAL SELF-EXCITED PROCESSES

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DISCUSSION PAPER | 2020 / 02

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Credit risk modelling with fractional self-excited processes

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January 27, 2020

Abstract

This article proposes a potential approach for credit risk in which the driving process is fractional and self-excited. In this framework, the probability of firm survival is the ratio of the expectation of a super-martingale on its initial value. The super-martingale is ruled by a time-changed Hawkes jump diffusion process. The self-excitation mechanism allows for sudden and repeated increases of default probabilities. As time-change, we use the inverse of a α stable subordinator. For this type of time-change, the Laplace's transform of the super-martingale is solution of a fractional Fokker-Planck equation. This equation is numerically solved by discretization of Caputo and partial derivatives.

Keywords: credit risk, subdiffusion, Fokker-Planck, Caputo Derivative, potential approach
JEL code: C5

1 Introduction

The most common paradigms in credit risk management are structural and reduced models. In the first approach, the balance sheet of a firm is explicitly modelled and the default occurs when assets fall below liabilities, as e.g. in Hainaut and Colwell (2016). In the reduced approach, the default is caused by the first jump of a point process, see e.g. Liang and Wang (2012), Hainaut and Le Courtois (2014) or Eckert et al. (2016). Chen and Panjer (2013) unify discrete structural and reduced models whereas Ballestra and Pacelli (2014) propose an hybrid approach. In this article, we investigate a third alternative called the potential approach. This method has been successfully applied to interest rates by Flesaker and Hughston (1996), Rogers (1997) or Macrina (2014). Instead of modelling the short term rates, discount factors are the ratios of the expectation of a super-martingale on its initial value.

We apply and extend the potential approach to credit risk in two directions. Firstly, we work with a self-excited process that allows sudden and repeated jumps of default probabilities, as those caused by contagion between defaults of several firms. In this framework, the intensity of default arrival depends on recent past shocks. This approach finds its origin in the articles of Hawkes (1971a, b) and Hawkes and Oakes (1974). More recently, Errais and al. (2010) develop a self-excited approach for modelling defaults in a credit portfolio. Ait-Sahalia et al. (2015) use a similar model to measure contagion between international markets. Hainaut (2016 a,b) proposes a model explaining the dynamics of interest rates by self-excited jump-diffusions.

The second extension of the potential approach concerns the auto-correlation between jumps of default probabilities. In a standard version of Hawkes processes, the importance of recent shocks on the intensity decays exponentially over time. This type of memory kernel guarantees that the point process and its intensity form a bivariate Markov process. Properties are in this case found with standard tools from stochastic calculus. Despite its analytical tractability, this process has not a long-term memory of past shocks. A common method to avoid this problem consists to replace the exponential memory kernel by a power decaying function. However, the point processes obtained by this way are not Markov and we cannot rely anymore on Itô's calculus to study their features. This article explores an alternative for introducing a long-term memory. This involves a non-Markov time-change built with the inverse of an alpha stable subordinator. This approach is applied to diffusions in physics for describing the movements

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of heavy particles that can get immobilized (see e.g. Metzler and Klafter 2004 or Eliazar, Klafter 2004). This type of time-changed Brownian motions, called sub-diffusions, are also popular in econophysics (see Scalas 2006) and applied to financial derivatives by Magdziarz (2009, a) for illiquid markets. As shown e.g. in Magdziarz (2009, b), the probability density function of sub-diffusions is solution of a Fractional Fokker-Planck equation. This equation is proposed in Barkai et al. (2000) and Metzler et al. (1999). Articles of Leonenko et al. (2013, a) Leonenko et al. (2013, b) go a step further and explore fractional Pearson diffusions and their correlation.

The structure of the article is as follows. Sections 2 and 3 review the potential approach and the properties of a Hawkes jump-diffusion. In section 4, we find the Fokker-Planck equation for the probability density of the process ruling the term structure of survival probabilities. Section 5 compares forward and backward numerical methods to compute the Laplace's transform of the Hawkes jump-diffusion. In Sections 6, we build a family of exponential super-martingales. Section 7 presents the fractional model obtained by a time-change. The next section introduces the numerical scheme for computing the joint Laplace's transform of the subordinator and of the time-changed Hawkes jump-diffusion. This article is concluded by a numerical analysis.

2 The potential approach for credit risk modelling

All stochastic processes considered in this article are defined on a probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, Q\right)$ where Q is the risk neutral measure and $(\mathcal{F}_t)_{t\geq 0}$ is their natural filtration. The default time of a firm is denoted by τ. τ is a $\mathcal{F}_t$-adapted stopping time. As mentioned in the introduction, the two most common approaches in credit risk management are the structural and reduced models. In the first one, the default is triggered when the market value of assets falls below debts. In the second approach, the default is caused by the first jump of a point process. In this article, we investigate an alternative way, called the potential approach. Instead of modelling an event triggering the default, we propose a dynamics for the probability that the company survives till a certain maturity. This probability is of course in the interval $[0, 1]$ and is a decreasing function of the time horizon, as the exposure to default is proportional to the period during which the firm is in activity. The potential approach is based upon a $\mathcal{F}_t$-adapted stochastic process $(\xi_t)_{t\geq 0}$. The survival probability till time t if the company is still in activity at time s, is denoted by $Q(\tau > t \,|\, \tau > s)$:

$$Q(\tau > t \,|\, \tau > s) \quad = \quad \frac{1}{\xi_s}\mathbb{E}_s\left(\xi_t\right) , \tag{1}$$

where $\mathbb{E}_s(.) := \mathbb{E}\left(\,.\,|\,\mathcal{F}_s\right)$. To ensure that $\frac{1}{\xi_s}\mathbb{E}_s\left(\xi_t\right)$ is a probability with values in $[0, 1]$, decreasing with the time horizon t, $(\xi_t)_{t\geq 0}$ fulfills three conditions:

- ξ_t is a positive process, $\forall t > 0$,
- ξ_t is a super-martingale, $\mathbb{E}_s\left(\xi_t\right) \leq \xi_s$,
- and its expectaction converges to 0, $\lim_{t\to\infty} \mathbb{E}_0\left(\xi_t\right) = 0$.

There exist various ways to build such a process in a Brownian setting. We e.g. refer to Rogers (1997) for examples. In this article, we consider a process ξ_t that is a Hawkes jump diffusion super-martingale. The construction and properties of this process are developed in the three next sections. Next, we will consider a time-changed version with a fractional interpretation.

3 The Hawkes jump diffusion process

We consider a counting process denoted by $(N_t)_{t\geq 0}$ with random independent, identically distributed, exponential marks, noted $(J_k)_{k=1,N_t}$. The probability density function (pdf) of marks is the exponential measure, $\nu(.)$, defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$:

$$\nu(x) = \rho e^{-\rho x} \quad x \in \mathbb{R}^+ ,$$

where $\rho \in \mathbb{R}^{+}$. The expectation and standard deviation of the mark are equal to $\mu := \mathbb{E}(J_k) = \frac{1}{\rho}$. The sum of marks is a pure positive jump process, denoted by $(P_t)_{t\geq 0}$:

$$P_t = \sum_{k=1}^{N_t} J_k \,. \tag{2}$$

We interpret the times of jumps of P_t as times at which major economic events threaten the company's survival. The rate of arrival of these events is a stochastic process $(\lambda_t)_{t\geq 0}$ that depends upon the history of the point process $(P_t)_{t\geq 0}$ and on a Brownian motion noted $(W_t)_{t\geq 0}$. $(\lambda_t)_{t\geq 0}$ takes its value in $\mathbb{R}^{+}$ and is $\mathcal{F}_t$-adapted. We assume that the dynamics of λ_t is driven by the following stochastic differential equation (SDE):

$$d\lambda_t \quad = \quad \kappa\left(\theta - \lambda_t\right)dt + \sigma\sqrt{\lambda_t}dW_t + \eta dP_t \,, \tag{3}$$

where κ, θ, σ and η are in $\mathbb{R}^{+}$. The intensity reverts toward θ and the speed of reversion depends on κ. As the volatility of the Brownian term is proportional to the square root of λ_t, the intensity stays positive. Furthermore, when P_t jumps of J, the intensity is increased by ηJ. Figure 1 shows two simulated sample paths of λ_t. After a schock caused by P_t, the process reverts exponentially to θwhile the Brownian generates some noise.

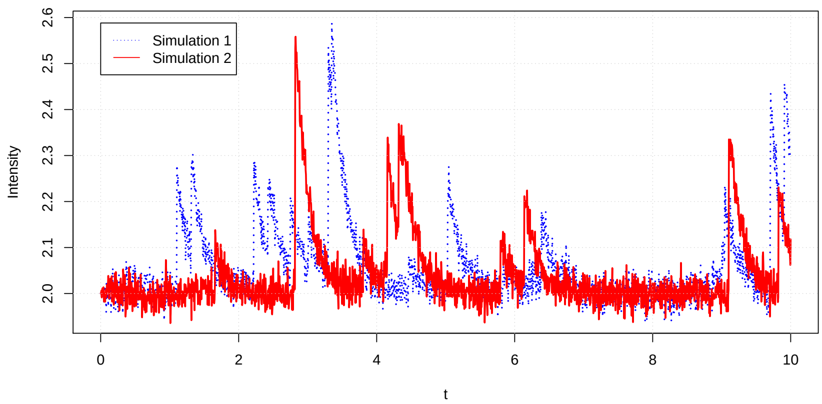

Figure 1: Simulated sample path of the Hawkes-diffusion intensity. $\kappa = 5$, $\theta = 2$, $\sigma = 0.05$, $\eta = 1$ and $\rho = 5$.

According to the Itô's lemma for semi-martingales, any $C^{1,2,1}$ function $f : [0,T] \times \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}$ of time, λ_t and P_t admits the following representation:

$$\begin{aligned} df(t, \lambda_t, P_t) \quad &= \quad \left(\frac{\partial f}{\partial t} + \kappa\left(\theta - \lambda_t\right) \frac{\partial f}{\partial \lambda} dt + \frac{1}{2}\lambda_t \sigma^2 \frac{\partial^2 f}{\partial \lambda^2} \right) dt \\ &\quad + \left(f(t, \lambda_t + \eta J, P_t + J) - f(t, \lambda_t, P_t)\right) dN_t \,. \end{aligned}$$

The infinitesimal generator of this process is denoted by $\mathcal{A}f$ and equal to

$$\begin{aligned} \mathcal{A}f \quad &= \quad \kappa\left(\theta - \lambda_t\right) \frac{\partial f}{\partial \lambda} + \frac{1}{2}\lambda_t \sigma^2 \frac{\partial^2 f}{\partial \lambda^2} \\ &\quad + \lambda_t \mathbb{E}\left(f(t, \lambda_t + \eta J, P_t + J) - f(t, \lambda_t, P_t)\right) \,. \end{aligned} \tag{4}$$

The intensity also admits the following integral form,

$$\lambda_t = \theta + e^{-\kappa t}\left(\lambda_0 - \theta\right) + \sigma \int_0^t e^{-\kappa(t-s)} \sqrt{\lambda_s} dW_s + \eta \int_0^t e^{-\kappa(t-s)} dP_s \,. \tag{5}$$

Deriving this expression with respect to time leads to the SDE (3). On the other hand, if we denote by X_t the pair (λ_t, P_t), for any $C^{2,1}$ function $f : \mathbb{R}^{+} \times \mathbb{R}^{+} \to \mathbb{R}$, the compensated process

$$M_t \quad = \quad f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_u)du \,,$$

is a martingale relative to its natural filtration (see e.g. Proposition 1,6 of Chapter VII in Revuz and Yor (1999)). Thus, for $s > t$, we have that

$$\mathbb{E}_t\left[f(X_s) - \int_0^s \mathcal{A}f(X_u)du\right] = f(X_t) - \int_0^t \mathcal{A}f(X_u)du$$

by the martingale property (where $\mathbb{E}_t(.) = \mathbb{E}(.|\mathcal{F}_t)$). This leads to the Dynkin formula:

$$\mathbb{E}_t\left[f\left(X_s\right)\right] = f(X_t) + \mathbb{E}_t\left[\int_t^s \mathcal{A}f(X_u)du\right]. \tag{6}$$

This last formula allows us to calculate the first two moments of the intensity.

Proposition 3.1. *The expectation of λ_t is given by*

$$\mathbb{E}_0\left[\lambda_t\right] = e^{(\eta\mu-\kappa)t}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)t} - 1\right). \tag{7}$$

If we denote by $\rho_1 = \left(\sigma^2 + 2\eta^2\mu^2\right)$ and $\rho_2 = \frac{\kappa\theta\left(\sigma^2+2\eta^2\mu^2\right)}{\eta\mu-\kappa}$, the variance of λ_t is equal to:

$$\mathbb{V}_0\left[\lambda_t\right] = \frac{\rho_1\lambda_0 + \rho_2}{\eta\mu-\kappa}\left(e^{2(\eta\mu-\kappa)t} - e^{(\eta\mu-\kappa)t}\right) + \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(1 - e^{2(\eta\mu-\kappa)t}\right). \tag{8}$$

Proof. From equation (6), we know that

$$\mathbb{E}_0\left[\lambda_t\right] = \theta + e^{-\kappa t}\left(\lambda_0 - \theta\right) + \eta\mu\int_0^t e^{-\kappa(t-s)}\mathbb{E}_0\left[\lambda_t\right]ds.$$

Deriving this last equation with respect to time allows us to infer that $\mathbb{E}_{,0}\left[\lambda_t\right]$ is the solution of an ordinary differential equation (ODE):

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}_0\left[\lambda_t\right] &= -\kappa e^{-\kappa t}\left(\lambda_0 - \theta\right) + \eta\mu\mathbb{E}_0\left[\lambda_t\right] - \kappa\eta\mu\int_0^t e^{-\kappa(t-s)}\mathbb{E}_0\left[\lambda_t\right]ds \\ &= \eta\mu\mathbb{E}_0\left[\lambda_t\right] + \kappa\left(\theta - \mathbb{E}_0\left[\lambda_t\right]\right).\end{aligned}$$

The solution of this ODE is well given by equation (7). On the other hand, since $\mathbb{E}(J^2) = 2\mu^2$

$$\mathbb{E}_s\left(\left(\lambda_s + \eta J\right)^2 - \lambda_s^2\right) = 2\eta\mu\lambda_s + 2\eta^2\mu^2,$$

and using the Dynkin formula, we find that the second order moment is:

$$\mathbb{E}_0\left[\lambda_t^2\right] = \lambda_0^2 + \mathbb{E}_t\left[\int_0^t 2\kappa\left(\theta - \lambda_s\right)\lambda_s + \lambda_s\sigma^2 + \lambda_s\left(2\eta\mu\lambda_s + 2\eta^2\mu^2\right)ds\right].$$

Deriving this last expression with respect to time, leads to an ODE for the second order moment:

$$\frac{\partial}{\partial t}\mathbb{E}_0\left[\lambda_t^2\right] = 2\left(\kappa\theta + \frac{1}{2}\sigma^2 + \eta^2\mu^2\right)\mathbb{E}_0\left[\lambda_t\right] + 2\left(\eta\mu - \kappa\right)\mathbb{E}_0\left[\lambda_t^2\right].$$

On the other hand, we have that

$$\frac{\partial}{\partial t}\mathbb{E}_0\left[\lambda_t\right]^2 = 2\left(\eta\mu - \kappa\right)\mathbb{E}_{,0}\left[\lambda_t\right]^2 + 2\kappa\theta\mathbb{E}_{,0}\left[\lambda_t\right].$$

If we adopt the notations $\rho_1 = \left(\sigma^2 + 2\eta^2\mu^2\right)$ and $\rho_2 = \frac{\kappa\theta\left(\sigma^2+2\eta^2\mu^2\right)}{\eta\mu-\kappa}$, we therefore infer that the variance is solution of the next ODE:

$$\frac{\partial}{\partial t}\mathbb{V}_0\left[\lambda_t\right] - 2\left(\eta\mu - \kappa\right)\mathbb{V}_0\left[\lambda_t\right] = \left(\rho_1\lambda_0 + \rho_2\right)e^{(\eta\mu-\kappa)t} - \rho_2,$$

that admits (8) as solution. □

From this last proposition we infer that moments are well defined and meaningful when $t \to \infty$ if and only if $\eta\mu - \kappa < 0$. If this inequality is not fulfilled, the process tends to $+\infty$ when time passes. In following developments, we always consider that this condition is satisfied. In this case, the long-term expected value of λ_t is equal to $-\frac{\kappa\theta}{\eta\mu-\kappa}$ whereas the long-term variance is $\frac{\kappa\theta\left(\sigma^2+2\eta^2\mu^2\right)}{2(\eta\mu-\kappa)^2}$. The next proposition presents the link between the variance and autocovariance of the intensity.

Proposition 3.2. *The covariance between λ_s and λ_t when $t \leq s$, is proportional to the variance of*

$$\mathbb{C}_0\left[\lambda_t \lambda_s\right] = e^{(\eta\mu-\kappa)(s-t)} \mathbb{V}_0\left[\lambda_t\right] . \tag{9}$$

Proof. Using nested expectations, we infer that $\mathbb{E}_0\left[\lambda_t \lambda_s\right] = \mathbb{E}_0\left[\lambda_t \mathbb{E}_t\left[\lambda_s\right]\right]$ where

$$\mathbb{E}_t\left[\lambda_s\right] = e^{(\eta\mu-\kappa)(s-t)} \lambda_t + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)(s-t)} - 1\right) .$$

Therefore,

$$\mathbb{E}_0\left[\lambda_t \lambda_s\right] = e^{(\eta\mu-\kappa)(s-t)} \mathbb{E}_0\left[\lambda_t^2\right] + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)(s-t)} - 1\right) \mathbb{E}_0\left[\lambda_t\right] . \tag{10}$$

On the other hand, the expected value of λ_s is linked to $\mathbb{E}_0\left[\lambda_t\right]$ by the relation

$$\mathbb{E}_0\left[\lambda_s\right] = e^{(\eta\mu-\kappa)(s-t)} \mathbb{E}_0\left[\lambda_t\right] + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)(s-t)} - 1\right) .$$

Therefore, the following equality holds:

$$\mathbb{E}_0\left[\lambda_s\right] \mathbb{E}_0\left[\lambda_t\right] = e^{(\eta\mu-\kappa)(s-t)} \mathbb{E}_0\left[\lambda_t\right]^2 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)(s-t)} - 1\right) \mathbb{E}_0\left[\lambda_t\right] . \tag{11}$$

From equations (10) and (11), we infer the covariance (9). □

In order to evaluate probabilities of default, we calculate the Laplace's transform of λ_t. Since this is an affine process, the Laplace's transform may be found by solving forward or backward differential equations. The next section presents an intermediary result that is used later in the construction of the forward equation satisfied by the Laplace transform of λ_t. This forward equation is required for developing the fractional version of the model. From a computational point of view, it is also more efficient to solve the forward equation than the backward one.

4 Fokker-Planck equation for the probability density function (pdf) of λ_t.

In order to study properties of the fractional Hawkes process, we need the forward equation (also called Fokker-Planck equation) satisfied by the pdf of λ_t. At our knowledge, this result is new and compared to a pure jump process, the Fokker-Planck equation of the Hawkes jump-diffusion counts additional terms. Since the expectation $\mathbb{E}_t\left[\lambda_s\right]$ only depends upon the value of the intensity at time t, λ_t is an inhomogeneous Markov process. We denote by $p(t, x|s, y)$ its transition probability density function. This transition pdf is defined as follows:

$$p(t, x|s, y)dx = Q\left(\lambda_t \in [x, x+dx] \mid \lambda_s \in [y, y+dy]\right)$$

for $s \leq t$ and $x, y \in \mathbb{R}^+$.

Proposition 4.1. *$p(t, x|s, y)$ is solution of the following Fokker-Planck equation:*

$$\begin{aligned} \frac{\partial p(t, x|s, y)}{\partial t} = & -\frac{\partial}{\partial x}\left(\kappa(\theta - x) p(t, x|s, y)\right) + \sigma^2 \frac{\partial p(t, x|s, y)}{\partial x} \\ & + \frac{1}{2}\sigma^2 x \frac{\partial^2 p(t, x|s, y)}{\partial x^2} - \eta \mathbb{E}\left[J p(t, x - \eta J|s, y)\right] \\ & + x \mathbb{E}\left[p(t, x - \eta J|s, y) - p(t, x|s, y)\right] , \end{aligned} \tag{12}$$

with the initial condition $p(s, x|s, y) = \delta_{\{x-y\}}$ where δ_z is the Dirac measure located at z.

Proof. It follows from the definition of the transition probability and from the Kramers-Moyal forward expansion that the probability density $p(t+\Delta, x|s,y)$ at time $t+\Delta$ and $p(t,x|s,y)$ at time t are related by the following relation

$$p(t+\Delta, x|s,y) - p(t,x|s,y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[M_n(t,x,\Delta) p(t,x|s,y) \right] , \tag{13}$$

where $M_n(t,x,\Delta)$ is the moment of order n of $\Delta\lambda_t = \lambda_{t+\Delta} - \lambda_t$:

$$\begin{aligned} M_n(t,x,\Delta) &= \mathbb{E}_t \left[(\lambda_{t+\Delta} - \lambda_t)^n \,|\, \lambda_t = x \right] \\ &= \int_{-\infty}^{+\infty} (z-x)^n \, p(t+\Delta, z \,|\, t, x) dz \,. \end{aligned}$$

On the other hand, since jump sizes are independent from (λ_t, P_t) and from equation (5), we infer that for small enough Δ, the centered moments can be developped as follows.

$$\begin{aligned} \mathbb{E}_t \left[\lambda_{t+\Delta} - \lambda_t | \, \lambda_t = x \right] &= \left(\kappa \left(\theta - x \right) + \eta \mu x \right) \Delta + \mathcal{O}\left(\Delta^2 \right) \\ \mathbb{E}_t \left[(\lambda_{t+\Delta} - \lambda_t)^2 \,|\, \lambda_t = x \right] &= \left(\eta^2 \mathbb{E}\left[J^2 \right] + \sigma^2 \right) x\Delta + \mathcal{O}\left(\Delta^2 \right) \\ &\vdots \\ \mathbb{E}_t \left[(\lambda_{t+\Delta} - \lambda_t)^n \,|\, \lambda_t = x \right] &= \eta^n \mathbb{E}\left[J^n \right] x\Delta + \mathcal{O}\left(\Delta^2 \right) \end{aligned}$$

Injecting these expansions in Equation (13) gives us:

$$\begin{aligned} &\frac{p(t+\Delta, x|s,y) - p(t,x|s,y)}{\Delta} = -\frac{\partial}{\partial x} \left[\kappa \left(\theta - x \right) p(t,x|s,y) \right] \\ &+ \frac{1}{2} \sigma^2 \frac{\partial^2 \left(x p(t,x|s,y) \right)}{\partial x^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \eta^n \mathbb{E}\left[J^n \right] \frac{\partial^n \left(x p(t,x|s,y) \right)}{\partial^n x} + \mathcal{O}\left(\Delta \right) . \end{aligned} \tag{14}$$

Since the n^{th} derivative of $x\,p(t,x|s,y)$ is equal to

$$\frac{\partial^n}{\partial x^n} \left(x\, p(t,x|s,y) \right) = n \frac{\partial^{n-1} p(t,x|s,y)}{\partial x^{n-1}} + x \frac{\partial^n p(t,x|s,y)}{\partial^n x} ,$$

Equation (14) may be rewritten as the following sum:

$$\begin{aligned} &\frac{p(t+\Delta, x|s,y) - p(t,x|s,y)}{\Delta} = -\frac{\partial}{\partial x} \left[\kappa \left(\theta - x \right) p(t,x|s,y) \right] \\ &+ \sigma^2 \frac{\partial p(t,x|s,y)}{\partial x} + \frac{1}{2} \sigma^2 x \frac{\partial^2 p(t,x|s,y)}{\partial x^2} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \eta^n \mathbb{E}\left[J^n \right] \frac{\partial^{n-1} p(t,x|s,y)}{\partial x^{n-1}} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \eta^n \mathbb{E}\left[J^n \right] x \frac{\partial^n p(t,x|s,y)}{\partial^n x} + \mathcal{O}\left(\Delta \right) . \end{aligned} \tag{15}$$

The third term in this equation is equal to the Taylor's expansion of the following expectation:

$$\mathbb{E}\left[-\eta J \sum_{n=0}^{\infty} \frac{(-\eta J)^n}{n!} \frac{\partial^n p(t,x|s,y)}{\partial x^n} \right] = -\eta \mathbb{E}\left[J p(t, x - \eta J | s, y) \right] .$$

Whereas the fourth term in Equation (15) is equal to the Taylor's expansion of :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \eta^n \mathbb{E}\left[J^n \right] x \frac{\partial^n p(t,x|s,y)}{\partial^n x} = x \mathbb{E}\left[p(t, x - \eta J | s, y) - p(t,x|s,y) \right] .$$

□

In the case where the intensity is driven by a Pearson diffusion without self-excited jumps, Leonenko and al. (2013) have shown that the pdf of the intensity is solution of the next differential equation:

$$\begin{aligned} \frac{\partial p(t,x|s,y)}{\partial t} &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|s,y)\right)+\sigma^{2}\frac{\partial p(t,x|s,y)}{\partial x} \\ &\quad +\frac{1}{2}\sigma^{2}x\frac{\partial^{2}p(t,x|s,y)}{\partial x^{2}}\,. \end{aligned} \tag{16}$$

A comparison of equations (12) and (16) reveals that jumps are related to the term $-\eta\mathbb{E}\left[Jp(t,x-\eta J|s,y)\right]$ whereas the self-exciting mechanism adds the term $x\mathbb{E}\left[p(t,x-\eta J|s,y)-p(t,x|s,y)\right]$ in equation (16). In the next section, we use the forward equation ruling the pdf of λ_t in order to evaluate its Laplace's transform.

5 Laplace's transform of λ_t: forward and backward methods

Let us denote by $\phi(.)$, the Laplace's transform of the pdf $p(t,x|s,y)$. This transform is equal to the following expectation:

$$\phi(t,z|s,y)=\mathbb{E}\left(e^{-z\lambda_t}|\lambda_s=y\right)\,,$$

for $z\in\mathbb{R}^+$. However the domain of z may be extended to $\mathbb{R}^-$ given that the expectation and variance of λ_t converges to finite values when $t\to\infty$ if $\eta\mu-\kappa<0$. It is well known in the literature that this Laplace's transform is solution of a backward Kolmogorov equation. Details about this equation are reminded at the end of this section. This transform is also solution of a forward Kolmogorov equation as stated in the following proposition.

Proposition 5.1. *The Laplace transform $\phi(t,z|0,y)$ satisfies the forward equation:*

$$\begin{aligned} \frac{\partial \phi(t,z|0,y)}{\partial t} &= -\kappa\theta z\,\phi(t,z|0,y)-\frac{1}{2}\sigma^{2}z^{2}\frac{\partial\phi(t,z|0,y)}{\partial z} \\ &\quad +\left(1-\kappa z-\mathbb{E}\left(e^{-z\eta J}\right)\right)\frac{\partial\phi(t,z|0,y)}{\partial z} \end{aligned} \tag{17}$$

with the initial conditions $\phi(0,z|0,y)=e^{-zy}$ and $\phi(t,0|0,y)=1$.

Proof Let $\Delta\in\mathbb{R}^+$ be a small step of time. The difference $\phi(t+\Delta,z|0,y)-\phi(t,z|0,y)$ when $\Delta\to 0$ is equal to

$$\lim_{\Delta\to 0}\frac{\phi(t+\Delta,z|0,y)-\phi(t,z|0,y)}{\Delta}=\lim_{\Delta\to 0}\int_0^\infty e^{-zx}\left(\frac{p(t+\Delta,x|0,y)-p(t,x|0,y)}{\Delta}\right)dx\,.$$

Using the forward equation (12) for the pdf of λ_t, we infer that the derivative of $\phi(.)$ with respect to time is the sum of several terms:

$$\begin{aligned} \frac{\partial \phi(t,z|0,y)}{\partial t} &= \int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|0,y)\right)\right)dx + \\ &\int_0^\infty e^{-zx}\left(\sigma^2\frac{\partial p(t,x|s,y)}{\partial x}\right)dx+\int_0^\infty e^{-zx}\left(\frac{1}{2}\sigma^2 x\frac{\partial^2 p(t,x|s,y)}{\partial x^2}\right)dx + \\ &\int_0^\infty e^{-zx}\left(-\eta\mathbb{E}\left[Jp(t,x-\eta J|0,y)\right]\right)dx + \\ &\int_0^\infty e^{-zx}\left(x\mathbb{E}\left[p(t,x-\eta J|0,y)-p(t,x|0,y)\right]\right)dx\,. \end{aligned} \tag{18}$$

Using the relation

$$\int_0^\infty e^{-zx}x\frac{\partial}{\partial x}p(t,x|0,y)dx \;=\; -\phi(t,z|0,y)-z\frac{\partial\phi(t,z|0,y)}{\partial z}\,,$$

and properties of the Laplace's transform, we rewrite the first term of Equation (18) as

$$\begin{aligned}
&\int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|0,y)\right)\right)dx\\
&= -\kappa\theta\left(z\phi(t,z|0,y)-\underbrace{p(t,0|s,y)}_{=0}\right)+\kappa\phi(t,z|0,y)\\
&-\kappa\phi(t,z|0,y)-\kappa z\frac{\partial\phi(t,z|0,y)}{\partial z}\,.
\end{aligned}$$

The second term of Equation (18) is equal to

$$\begin{aligned}
\int_0^\infty e^{-zx}\left(\sigma^2\frac{\partial p(t,x|s,y)}{\partial x}\right)dx &= \underbrace{\left[\sigma^2 e^{-zx}p(t,x|s,y)\right]_{x=0}^{x=\infty}}_{=0}+z\sigma^2\int_0^\infty e^{-zx}p(t,x|s,y)dx\\
&= z\,\phi(t,z|0,y)\,\sigma^2.
\end{aligned}$$

Finally, the third term is proportional to

$$\begin{aligned}
\int_0^\infty e^{-zx}x\frac{\partial^2 p(t,x|s,y)}{\partial x^2}dx &= \underbrace{\left[e^{-zx}x\frac{\partial p(t,x|s,y)}{\partial x}\right]_{x=0}^{x=\infty}}_{=0}+z\int_0^\infty e^{-zx}x\frac{\partial p(t,x|s,y)}{\partial x}dx\\
&-\int_0^\infty e^{-zx}\frac{\partial p(t,x|s,y)}{\partial x}dx\,.
\end{aligned}$$

Therefore, we infer that

$$\frac{1}{2}\sigma^2\int_0^\infty e^{-zx}x\frac{\partial^2 p(t,x|s,y)}{\partial x^2}dx = \frac{1}{2}\sigma^2\left(-2z\phi(t,z|0,y)-z^2\frac{\partial\phi(t,z|0,y)}{\partial z}\right).$$

After a change of variable $x'=x-\eta J$, the fourth term of (18) becomes:

$$\begin{aligned}
&-\eta\int_0^\infty u\left(\int_0^\infty e^{-zx}p(t,x-\eta u|0,y)dx\right)\nu(u)du\\
&= -\eta\mathbb{E}\left(Je^{-z\eta J}\right)\phi(t,z|0,y)\,.
\end{aligned}$$

Given that $p(t,x|0,y)$ is null for $x\leq 0$, the fifth term of (18) is rewritten, after the same change of variable, as:

$$\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-zx}\left(xp(t,x-\eta u|0,y)\right)d\nu(u)du-\int_0^\infty e^{-zx}\left(xp(t,x|0,y)\right)dx\\
&= \int_0^\infty e^{-z\eta u}\,\nu(u)du\int_0^\infty e^{-zx'}x'p(t,x'|0,y)dx'\\
&+\eta\int_0^\infty ue^{-z\eta u}\nu(u)du\int_0^\infty e^{-zx'}p(t,x'|0,y)dx'+\frac{\partial\phi(t,z|0,y)}{\partial z}\\
&= -\mathbb{E}\left(e^{-z\eta J}\right)\frac{\partial\phi(t,z|0,y)}{\partial z}+\eta\mathbb{E}\left(Je^{-z\eta J}\right)\phi(t,z|0,y)+\frac{\partial\phi(t,z|0,y)}{\partial z}\,.
\end{aligned}$$

Combining previous elements allows us to obtain Equation (17). □

We detail the numerical scheme that can be used for solving Equation (17). Let us consider the domain $D=[0,t_{max}]\times[-z_{max},z_{max}]$ on which we want to estimate the Laplace's transform of λ_t. Firstly, we choose two steps of discretization, noted Δ_t and Δ_z, in order to define a cube of pairs (t_k,z_j) where

$$t_k = k\,\Delta_t \quad , \quad z_j = -z_{max}+j\,\Delta_z$$

for $k=0,....,n_t$ and $j=0,...,n_z$. The numbers n_t and n_z are integers equal to $n_t=\left\lfloor\frac{t_{max}}{\Delta_t}\right\rfloor$ and $n_z=\left\lfloor\frac{2z_{max}}{\Delta_z}\right\rfloor$. In order to ligthen developments, we denote by $\phi(k,j)$ the approached value of $\phi(t_k,z_j|0,y)$. The partial derivatives of ϕ with respect to time are approached its finite difference estimate:

$$\frac{\partial\phi(k,j)}{\partial t}\approx\frac{\phi(k,j)-\phi(k-1,j)}{\Delta_t}.$$

The partial derivatives of ϕ with respect to z are approached by finite difference estimates depending upon the sign of z:

$$\frac{\partial\phi(k,j)}{\partial z} \approx \frac{\phi(k,j)-\phi(k,j-1)}{\Delta_z} \; for\; z_j>0\,,$$
$$\frac{\partial\phi(k,j)}{\partial z} \approx \frac{\phi(k,j+1)-\phi(k,j)}{\Delta_z} \; for\; z_j<0\,.$$

If we inject these expressions in the Fokker-Planck equation (38), we estimate recursively the $\phi(k,j)$ for $z_j>0$ by:

$$\begin{aligned}\phi(k,j) &= \left(1+\Delta_t\left(\kappa\theta z_j\right)-\frac{\Delta_t}{\Delta_z}\left(1-\kappa z_j-\frac{1}{2}\sigma^2 z_j^2-\mathbb{E}\left(e^{-z_j\eta J}\right)\right)\right)^{-1}\\ &\times\left(\phi(k-1,j)-\frac{\Delta_t}{\Delta_z}\left(1-\kappa z_j-\frac{1}{2}\sigma^2 z_j^2-\mathbb{E}\left(e^{-z_j\eta J}\right)\right)\phi(k,j-1)\right)\,,\end{aligned}$$

whereas for$z_j<0$, we use the recursion:

$$\begin{aligned}\phi(k,j) &= \left(1+\Delta_t\left(\kappa\theta z_j\right)+\frac{\Delta_t}{\Delta_z}\left(1-\kappa z_j-\frac{1}{2}\sigma^2 z_j^2-\mathbb{E}\left(e^{-z_j\eta J}\right)\right)\right)^{-1}\\ &\times\left(\phi(k-1,j)+\frac{\Delta_t}{\Delta_z}\left(1-\kappa z_j-\frac{1}{2}\sigma^2 z_j^2-\mathbb{E}\left(e^{-z_j\eta J}\right)\right)\phi(k,j+1)\right)\,.\end{aligned}$$

Given that $\lambda_0=y$ and $\phi(k,j)=\mathbb{E}\left(e^{-z_j\,\lambda_{t_k}}\right)$, the recursion is initialized by

$$\begin{cases}\phi_\alpha\left(k,\frac{1}{2}n_z+1\right)=1 & k\in\{0,...,n_t\}\\ \phi_\alpha(0,j)=e^{-z_j y} & j\in\{0,...,n_z\}\end{cases}.$$

We compare the forward Equation (17) with its backward equivalent. From the Itô's lemma for semi-martingales, we know that $\phi(.)$ is solution of the backward partial differential equation:

$$0 \;=\; \frac{\partial\phi}{\partial s}+\kappa\left(\theta-y\right)\frac{\partial\phi}{\partial y}+\frac{1}{2}\sigma^2 y\frac{\partial^2\phi}{\partial y^2}+y\mathbb{E}\left(\phi(t,z|s,y+\eta J)-\phi\right)\,. \tag{19}$$

If we do the assumption that $\phi(.)$ is a function of the form $\exp\left(A(s,t)+B(s,t)y\right)$, we can easily show that functions $A(.)$ and $B(.)$ are solutions of a system of ordinary differential equations (ODE) as stated in the next proposition:

Proposition 5.2. *$\phi(t,z|s,y)$ is equal to*

$$\phi(t,z|s,y)=\exp\left(A(s,t)+B(s,t)y\right)\,, \tag{20}$$

where $A(s,t)$ and $B(s,t)$ are functions that satisfy the following ODE's:

$$\begin{aligned}\frac{\partial A(s,t)}{\partial s} &= -\kappa\theta B(s,t)\,,\\ \frac{\partial B(s,t)}{\partial s} &= \kappa B(s,t)-\frac{1}{2}\sigma^2 B(s,t)^2-\mathbb{E}\left(e^{B(s,t)\eta J}-1\right)\,,\end{aligned}$$

with the terminal conditions $A(t,t)=0$ and $B(t,t)=-z$.

From a numerical point of view, it is less computationally intensive to solve the forward equation (17) than the backward one (20) if we need the value of Laplace's transform for multiple maturities and values of z. This operation is a numerical calculation of order $\mathcal{O}\left(2\right)$ instead of $\mathcal{O}(4)$ with the backward equation.

6 Hawkes jump diffusion super-martingales

We have now all materials for constructing the Hawkes-diffusion super-martingale. We first consider a C^2 positive function $\psi:\mathbb{R}\to\mathbb{R}^+$ of λ_t that satisfies the following differential equation:

$$\mathcal{A}\psi(\lambda) \;=\; \beta\psi(\lambda) \tag{21}$$

where β is constant. This $\psi(.)$ is an eigenfunction of the infinitesimal generator, with an eigenvalue, β. By construction $M_t = e^{-\beta t}\psi(\lambda_t)$ is a local martingale. Indeed, according to the Itô's lemma, the expectation of the drift of M_t is equal to

$$\mathbb{E}_t\left(dM_t\right) = \left(-\beta+\mathcal{A}\right)M_t dt = 0,$$

which confirms that M_t is a local martingale. On the other hand, given that $\mathbb{E}_s\left(M_t\right) = M_s$, we also have that

$$\mathbb{E}_s\left(e^{-\beta t}\psi(\lambda_t)\right) = e^{-\beta s}\psi(\lambda_s)\,.$$

We infer from this last relation that $\mathbb{E}_s\left(\psi(\lambda_t)\right) = \psi(\lambda_s)e^{\beta(t-s)}$. For any constant $\beta < \alpha \in \mathbb{R}$, the process $(\xi_t)_{t\geq 0}$ defined as follows:

$$\xi_t := e^{-\alpha t}\psi(\lambda_t) \tag{22}$$

is therefore a positive, super-martingale and its expectation converges to zero when $t \to \infty$. The process ξ_t, that we call the Hawkes jump diffusion super-martingale, is therefore eligible for defining the term structure of default probabilities, such as defined by Equation (1) in Section 2. In practice, we need to find analytical positive eigenfunctions. A natural choice for $\psi(.)$ is the exponential function, as stated in the next proposition.

Proposition 6.1. *If γ and β solves the system of equations*

$$\begin{cases}\beta & = \gamma\,\kappa\,\theta\,, \\ 0 & = -\gamma\,\kappa + \frac{1}{2}\sigma^2\gamma^2 + \left(\frac{\rho}{\rho-\eta\gamma} - 1\right)\,,\end{cases} \tag{23}$$

such that $\eta\gamma < \rho$, the function defined by

$$\psi(\lambda_t) = \exp\left(\gamma\,\lambda_t\right)\,. \tag{24}$$

is a positive eigenfunction of $\mathcal{A}$ with an eigenvalue equal to β.

Proof. If the eigenfunction $\psi(\lambda_t)$ admits the representation (24), then first and second order derivatives of $\psi(.)$ are equal to

$$\begin{aligned}\partial_\lambda\psi(\lambda_t) &= \gamma\psi(\lambda_t)\,, \\ \partial_\lambda^2\psi(\lambda_t) &= \gamma^2\psi(\lambda_t)\,,\end{aligned}$$

and the infinitesimal generator of this function is equal to:

$$\begin{aligned}\mathcal{A}\psi &= \gamma\,\kappa\left(\theta-\lambda_t\right)\psi + \frac{1}{2}\sigma^2\gamma^2\,\lambda_t\,\psi \\ &\quad +\lambda_t\psi\left(\mathbb{E}\left(e^{\gamma\eta J}\right)-1\right)\,.\end{aligned}$$

As the eigenvalue is such that $\mathcal{A}\psi = \beta\psi$, we infer the following equality:

$$\beta = \gamma\,\kappa\left(\theta-\lambda_t\right) + \frac{1}{2}\sigma^2\gamma^2\,\lambda_t + \lambda_t\left(\mathbb{E}\left(e^{\gamma\eta J}\right)-1\right)\,.$$

As $\mathbb{E}\left(e^{\gamma\eta J}\right) = \frac{\rho}{\rho-\eta\gamma}$ for $\gamma\eta < \rho$, Regrouping terms dependent and independent from λ_t leads to the result. $\square$

The system of equations (23) may be solved analytically as shown in the next corollary.

Corollary 6.2. *Let us define the following positive constant*

$$\zeta = \left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right)^2 + 2\eta\sigma^2\left(\eta-\kappa\rho\right)\,.$$

There exists one exponential positive eigenfunctions $\psi_{-}(\lambda_t) = \exp(\gamma_{-}\,\lambda_t)$ *with*

$$\gamma_{-} \;=\; \frac{\left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right) - \sqrt{\zeta}}{\eta\sigma^2} > 0\,.$$

that satisfies the constraint $\gamma\eta < \rho$. *If* $\sigma = 0$ *then*

$$\gamma_{-} = \frac{\kappa\rho - \eta}{\eta\kappa} > 0$$

The eigenvalue of $\psi_{-}(.)$ *is equal to* $\beta_{-} = \gamma_{-}\,\kappa\,\theta > 0$.

Proof The second equation of (23) may be rewritten as a second order equation:

$$0 \;=\; -\frac{1}{2}\eta\sigma^2\gamma^2 + \left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right)\gamma + (\eta - \kappa\rho)\,. \tag{25}$$

The determinant ,ζ , is strictly positive. There exist two solutions to Equation (25) that are denoted by γ_{+} and γ_{-} and given by:

$$\gamma_{\pm} \;=\; \frac{\left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right) \pm \sqrt{\zeta}}{\eta\sigma^2}\,.$$

The Hawkes diffusion process is stable if and only if the condition $\eta - \kappa\rho < 0$ is fulfilled. Therefore, the determinant is bounded by

$$\zeta = \left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right)^2 + 2\eta\sigma^2(\eta - \kappa\rho) \le \left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right)^2.$$

we infer then that $\gamma_{+} > 0$ and $\gamma_{-} > 0$. However the constraint $\gamma\eta < \rho$ is not satisfied by γ_{+}. Indeed given that $\xi > \left(\frac{1}{2}\sigma^2\rho\right)^2$, we have that

$$\begin{aligned}\gamma_{+}\eta &\ge \frac{\left(\frac{1}{2}\sigma^2\rho + \eta\kappa\right) + \frac{1}{2}\sigma^2\rho}{\sigma^2} \\ &> \rho\,.\end{aligned}$$

□

In the next corollary, we extend the class of eligible Hawkes jump diffusion super-martingales that we use for constructing term structures of default probabilities.

Corollary 6.3. *For any* $\delta_{-}, \alpha_{-} \in \mathbb{R}$ *such that* $0 < \delta_{-} < \gamma_{-}$ *and* $\alpha_{-} > \beta_{-} > 0$, *the process* $(\xi_t)_{t\ge 0}$ *defined by*

$$\xi_t = e^{-\alpha_{-}t + \delta_{-}\lambda_t}\,, \tag{26}$$

is a positive super-martingale with an expectation converging to zero.

Proof. Firstly, let us consider a process of the form $\xi_t = e^{-\beta_{-}t + \delta_{-}\lambda_t}$. Using the Itô's lemma, the infinitesimal variation of ξ_t is given by

$$\begin{aligned}d\xi_t =& \xi_t\left(-\beta_{-} + \kappa(\theta - \lambda_t)\delta_{-} + \frac{1}{2}\sigma^2\lambda_t\delta_{-}^2\right)dt \\ &+ \sigma\delta_{-}\sqrt{\lambda_t}\xi_t dW_t + \xi_t(e^{\delta_{-} - \eta J} - 1)dN_t\,,\end{aligned}$$

and $\mathbb{E}_t(d\xi_t) < 0$ if and only if

$$-\beta_{-} + \kappa(\theta - \lambda_t)\delta_{-} + \frac{1}{2}\sigma^2\lambda_t\delta_{-}^2 + \lambda_t\left(\mathbb{E}(e^{\delta_{-}\eta J}) - 1\right) < 0\,. \tag{27}$$

Since $\beta_{-} > 0$ and $\mathbb{E}(e^{\delta_{-}\eta J}) - 1 = \frac{\rho}{\rho - \delta_{-}\eta} - 1$,the inequality (27) is fulfilled only if the following terms are negative:

$$\begin{aligned}&-\beta_{-} + \kappa\theta\delta_{-} < 0\,, \\ &-\kappa\delta_{-} + \frac{1}{2}\sigma^2\delta_{-}^2 + \left(\frac{\rho}{\rho - \delta_{-}\eta} - 1\right) < 0\,.\end{aligned} \tag{28}$$

Given that $\beta_- = \kappa\theta\gamma_-$, the first relation is satisfied if $\delta_- < \gamma_-$. if δ_- such that $\rho - \delta_-\eta > 0$ and if $\delta_- > 0$, the second inequality becomes:

$$-\frac{1}{2}\eta\rho\sigma^2\delta_-^2 + \left(\frac{1}{2}\rho\sigma^2 + \eta\kappa\right)\delta_- + (\eta - \kappa\rho) < 0\,.$$

This condition is satisfied if $\delta_- < \gamma_-$ or $\delta_- > \gamma_+$. Since $\delta_- < \gamma_-$, we reject values of δ_- bigger than γ_+. We have also that $\rho - \delta_-\eta > 0$ given that $\rho > \gamma_-\eta$. If $\delta_- < 0$ then the second inequality of Equation (28) becomes:

$$-\frac{1}{2}\eta\rho\sigma^2\delta_-^2 + \left(\frac{1}{2}\rho\sigma^2 + \eta\kappa\right)\delta_- + (\eta - \kappa\rho) > 0\,.$$

This condition is fulfilled only if $\gamma_- < \delta_- < \gamma_+$, but in this case the condition $\delta_- < \gamma_-$ is systematically breached. We infer that when δ_- is in the interval $(0, \gamma_-)$, the process ξ_t is a super-martingale. It is therefore evident that for $\alpha_- > \beta_-$, $\xi_t = e^{-\alpha_- t + \delta_- \lambda_t}$ is also a super-martingale.
□

If we consider that ξ_t is defined by equation (26), the term structure of default probabilities is defined by the following equation:

$$Q(\tau > t \,|\, \tau > s) \quad = \quad \frac{e^{-\alpha_- t}\mathbb{E}_s\left(e^{\delta_-\lambda_t}\right)}{e^{-\alpha_- s + \delta_- \lambda_s}}\,. \tag{29}$$

The expectation $\mathbb{E}_s\left(e^{\delta_-\lambda_t}\right)$ is obtained by solving numerically the forward equation (17).

Notice that if we set $\delta_- = \gamma_-$, the conditional survival probability admits a closed form expression. In this particular case, we have that

$$\mathbb{E}_s\left(e^{\gamma_-\lambda_t}\right) \quad = \quad e^{\beta_-(t-s) + \gamma_-\lambda_s}\,,$$

the survival probability decays exponentially at a rate $(\alpha_- - \beta_-)$ and is in this particular case independent upon the level of λ_s:

$$Q(\tau > t \,|\, \tau > s) \quad = \quad \frac{e^{-\alpha_- t + \beta_-(t-s) + \gamma_-\lambda_s}}{e^{-\alpha_- s + \gamma_-\lambda_s}} = e^{-(\alpha_- - \beta_-)(t-s)}\,. \tag{30}$$

7 The fractional potential credit risk model

We have seen in the previous section that the autocovariance of the intensity, λ_t, decays exponentially with time. This exponential structure is sometimes inappropriate for modelling intensity processes with a longer memory of past events. The solution that we explore to circumvent this drawback consists to introduce periods during which the intensity is motionless. A way for modelling these motionless periods consists to observe the process on a new scale of time. This section reviews the features of this process.

We use as time-change, a subordinator that is the inverse of an α stable process $(U_t)_{t\geq 0}$, defined on $(\Omega, \mathcal{G}, Q)$. This particular type of Lévy processes has a simple moment generating function given by:

$$\mathbb{E}_0\left(e^{-uU_t}\right) \quad = \quad e^{-t\,u^{\alpha}}\,.$$

The process U_t is a $\frac{1}{\alpha}$ self-similar process, meaning that:

$$U_{at} \quad \overset{d}{=} \quad (at)^{\frac{1}{\alpha}}\,U_1\,.$$

The α-stable processes are strictly increasing and may therefore be used as subordinator but they cannot duplicate motionless periods since they have an infinite activity. Therefore, we use U_t for defining a subordinator, noted $(S_t)_{t\geq 0}$ that is its inverse hitting time:

$$S_t = \inf\{\tau > 0 \,:\, U_\tau \geq t\}\,.$$

Under the assumption that $U_0 = 0$, we have that $S_0 = 0$. By definition and due to the self-similarity property, the cumulative distribution function (cdf) of S_t admits the following representation:

$$\begin{aligned} Q(S_t \leq \tau) &= P(U_\tau \geq t) \\ &= P(\tau^{\frac{1}{\alpha}} U_1 \geq t) \\ &= Q\left(\left(\frac{t}{U_1}\right)^\alpha \leq \tau\right). \end{aligned}$$

The distribution of S_t is then the same as the random variable $\left(\frac{t}{U_1}\right)^\alpha$. In the rest of the article, the pdf of S_t is denoted by $g(t,\tau) = \frac{d}{d\tau} q(\tau \leq S_t \leq \tau + d\tau)$ and $p_U(t,u)$ is the pdf of U_t. On the other-hand, the self-similarity leads to the relation:

$$\begin{aligned} Q(U_\tau \leq t) &= Q(\tau^{\frac{1}{\alpha}} U_1 \leq t) \\ &= Q(U_1 \leq t\tau^{-\frac{1}{\alpha}}). \end{aligned}$$

If we derive this last expression with respect to t, we obtain that

$$p_U(\tau, t) = \tau^{-\frac{1}{\alpha}} p_U(1, t\tau^{-\frac{1}{\alpha}}),$$

and infer an important relation linking the pdf of S_t to the pdf of U_t:

$$\begin{aligned} \tau\, g(t,\tau) &= \frac{t}{\alpha}\left(\tau^{-\frac{1}{\alpha}} p_U(1, t\tau^{-\frac{1}{\alpha}})\right) \\ &= \frac{t}{\alpha} p_U(\tau, t). \end{aligned} \tag{31}$$

On the other hand, the pdf of S_t is related to the pdf of U_t through the relation

$$\begin{aligned} g(t,\tau) &= \frac{\partial}{\partial \tau} Q(S_t \leq \tau) = -\frac{\partial}{\partial \tau} Q(U_\tau \leq t) \\ &= -\frac{\partial}{\partial \tau} \int_0^t p_U(\tau, u) du. \end{aligned}$$

Recalling that the Laplace transform of a function $\int_0^t f(u)du$ is equal to $\omega^{-1}\tilde{f}(\omega)$, the Laplace transform $\tilde{g}(\omega,\tau)$ of $g(t,\tau)$ with respect to time t is therefore equal to:

$$\begin{aligned} \tilde{g}(\omega,\tau) &= -\frac{\partial}{\partial \tau} \widetilde{\int_0^t p_U(\tau, u) du} \\ &= -\frac{\partial}{\partial \tau}\left(\omega^{-1} e^{-\tau\,\omega^\alpha}\right) \\ &= \omega^{\alpha-1} e^{-\tau\,\omega^\alpha}. \end{aligned} \tag{32}$$

The Laplace's transform of S_t conditionally to the information available at time zero is given by :

$$\begin{aligned} \mathbb{E}_0\left[e^{-\omega S_t}\right] &= \int_0^\infty e^{-\omega\tau} g(t,\tau) d\tau \\ &= E_\alpha(-\omega t^\alpha) \end{aligned} \tag{33}$$

where E_α is the Mittag-Leffler function (for a proof see e.g. Piryatinska et al. 2005). The Mittag-Leffler function is an infinite sum:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},$$

where $\Gamma(.)$ is the gamma function. This result clearly reveals that S_t is not a Lévy process since its Laplace's transform does not have an exponential form. However, we can easily compute the moments of S_t by deriving and cancelling its Laplace's transform:

$$\mathbb{E}_0\left(S_t^n\right) = \frac{n! t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}.$$

The Mittag-Leffler function is closely related to the concept of fractional or Caputo's derivative that is also involved in the construction of the fractional Hawkes process. The Caputo's derivative of order $\alpha \in]0,1[$ for a function $h(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^1$ with respect to t is defined by

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} h(s,x) ds - \frac{h(0,x)}{t^\alpha} . \tag{34}$$

An alternative writing is the following:

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} h(s,x) ds \tag{35}$$

When $\alpha = 1$, this derivative corresponds to the derivative with respect to time. Let $\tilde{h}(\omega, x)$ be the usual Laplace's transform of a function $h(t,x)$ with respect to time t. A direct calculation shows that the Laplace's transform of the Caputo's derivative $\frac{\partial^\alpha}{\partial t^\alpha} h(t,x)$ is equal to:

$$\widetilde{\frac{\partial^\alpha h}{\partial t^\alpha}}(\omega, x) = \omega^\alpha \tilde{h}(\omega, x) - \omega^{\alpha-1} h(0,x) ,$$

which reduces to the familiar form when $\alpha = 1$. Notice that the Caputo's fractional derivative of a power function is given by

$$\frac{\partial^\alpha}{\partial t^\alpha} t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & p \geq 1 \, , \, p \in \mathbb{R} \\ 0 & p \leq 0 \, , \, p \in \mathbb{N} \end{cases} .$$

On the other hand, the solution of the fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} y(t) = \lambda y(t) \qquad 0 < \alpha < 1$$

with the initial condition $y(0) = b_0$ is precisely the Mittag-Leffler function $y(t) = E_\alpha(\lambda t^\alpha)$.

We use the process $(S_t)_{t \geq 0}$ as a new time scale for observing the super-martingale ξ_t, that is used for defining the term structure of default probabilities in Equation (29). Figure 2 allows to understand the impact of this time-change on the sample path of the intensity. We clearly observe that the subordinated process is randomly immobilized. The autocovariance structure of λ_{S_t} is studied in Hainaut (2019) for a Hawkes process without diffusion. This time-changed process inherits from the main features of ξ_t: it is a positive super-martingale and its expectation converges to 0. In following developments, we assume that the term structure of survival probabilities is defined by a time-changed version of Equation (1):

$$Q(\tau > T \,|\, \tau > t) = \frac{1}{\xi_{S_t}} \mathbb{E}_t \left(\xi_{S_T} \right) ,$$

where ξ_t is the Hawkes-diffusion super-martingale, previously studied:

$$\xi_{S_t} = e^{-\alpha_- S_t + \delta_- \lambda_{S_t}} .$$

Notice that neither the subordinator S_t nor the time-changed process ξ_{S_t} are Markov and semi-martingales. Therefore, we cannot rely anymore on stochastic calculus for studying the properties of our model.

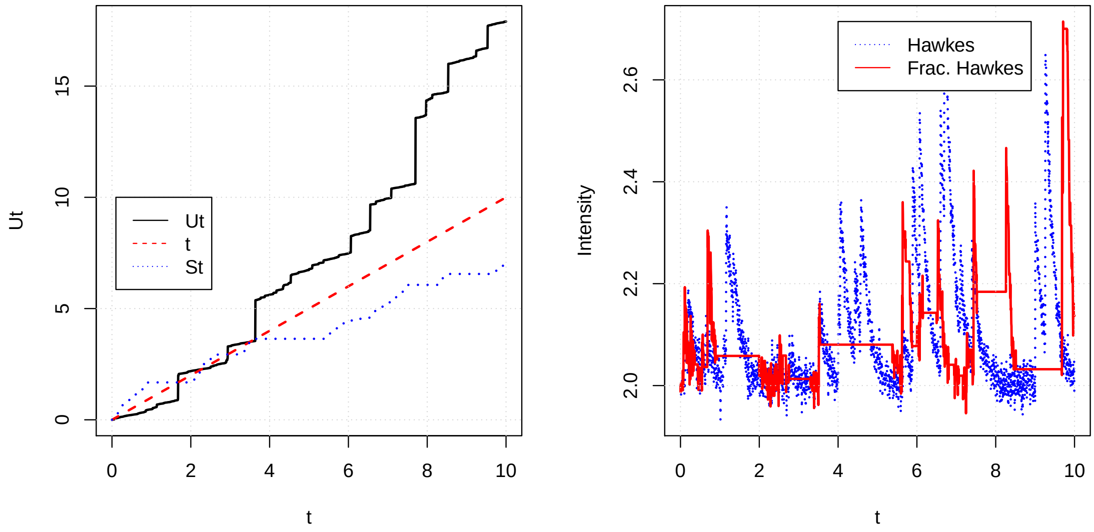

Figure 2: Simulated sample path of the time-changed fractional Hawkes-diffusion. $\alpha = 0.8$, $\kappa = 5$, $\theta = 2$, $\sigma = 0.05$, $\eta = 1$ and $\rho = 5$.

Let us recall that the filtration of S_t, denoted $(\mathcal{G}_t)_{t\geq 0}$ is independent from the filtration $(\mathcal{F}_t)_{t\geq 0}$ de $(\lambda_t)_{t\geq 0}$. The survival probability from 0 to t can therefore be calculated with nested expectations:

$$Q(0,t) \quad = \quad \frac{\mathbb{E}_0\left(e^{-\alpha_- S_t + \delta_- \lambda_{S_t}}\right)}{e^{\delta_- \lambda_0}} \,. \tag{36}$$

The expectations in the numerator do not have any closed form expression, excepted if we set $\delta_- = \gamma_-$. In this particular case, we know that $\mathbb{E}_0\left(e^{\gamma_- \lambda_t}\right) = e^{\beta_- t + \gamma_- \lambda_0}$. Furthermore, we have that$\mathbb{E}_0\left(e^{-(\alpha_- - \beta_-)S_t}\right) = E_\alpha\left(-\left(\alpha_- - \beta_-\right)t^\alpha\right)$. Using nested expectations and since λ_t is independent from S_t, we directly infer that

$$\begin{aligned}
\mathbb{E}_0\left(e^{-\alpha_- S_t + \gamma_- \lambda_{S_t}}\right) &= \mathbb{E}\left(e^{-\alpha_- S_t}\mathbb{E}\left(e^{\gamma_- \lambda_{S_t}} | \mathcal{F}_0 \vee \mathcal{G}_t\right) | \mathcal{F}_0\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(e^{-(\alpha_- - \beta_-)\,S_t}\, e^{\gamma_- \lambda_0} | \mathcal{F}_0 \vee \mathcal{G}_t\right) | \mathcal{F}_0\right) \\
&= E_\alpha\left(-\left(\alpha_- - \beta_-\right)t^\alpha\right)\, e^{\gamma_- \lambda_0} \,.
\end{aligned}$$

Combining this result with Equation (36) leads to the following closed form expression for the survival probability up to time t:

$$Q(0,t) = P(\tau \geq t \,|\, \tau > 0) \quad = \quad E_\alpha\left(-\left(\alpha_- - \beta_-\right)t^\alpha\right) \,.$$

When $\delta_- < \gamma_-$, there exists several numerical alternatives for computing the expectations $\mathbb{E}_0\left(e^{-\alpha_- S_t + \delta_- \lambda_{S_t}}\right)$. One of them consists to simulate sample path of S_t and to evaluate in each scenario the Laplace's transform of λ_{S_t} using the result of propositions (5.1) or (5.2). Given that the pdf $g(t,u)$ of S_t admits a closed-form expression (see e.g. Meerschaert and Straka, 2013), we can evaluate numerically the integral

$$\mathbb{E}_0\left(e^{-\alpha_- S_t + \delta_- \lambda_{S_t}}\right) = \int_0^\infty \mathbb{E}\left(e^{-\alpha_- u + \delta_- \lambda_u} \,|\, \lambda_0 = y\right) g(t,u) du \,,$$

These two approaches are however computing intensive as they both require to numerically solve many times forward or backward equations. A more efficient approach consists to solve a fractional Fokker-Planck equation that is introduced in the next section.

8 Fractional Fokker-Planck equation for the joint Laplace transform of (S_t, λ_t)

The expectation $\mathbb{E}_0\left(e^{-\alpha_- S_t+\delta_- \lambda_{S_t}}\right)$ involved in the calculation of the survival probability, is the joint Laplace transform of (S_t, λ_t), denoted as

$$\phi_\alpha(t, z_1, z_2) = \mathbb{E}_0\left(e^{-z_1 S_t - z_2 \lambda_{S_t}}\right) ,$$

evaluated at points $(z_1, z_2) = (\alpha_-, -\delta_-)$. The joint density of S_t and λ_{S_t} is a function denoted by $p_\alpha(t, x_1, x_2|s, y_1, y_2)$ and defined as follows:

$$\begin{aligned}
p_\alpha(t, x_1, x_2|0, y_2) &= P\left(S_t \in [x_1, x_1 + dx_1], \lambda_{S_t} \in [x_2, x_2 + dx_2] \,|S_0 = 0, \lambda_0 = y_2\right) , \\
&= g(t, x_1) p(x_1, x_2|0, y_2) .
\end{aligned}$$

where $g(t, .)$ is the pdf of S_t. The Laplace's transform of $p_\alpha(t, x_1, x_2|0, y_2)$ with respect to time t is thus given by an integral:

$$\begin{aligned}
\tilde{p}_\alpha(\omega, x_1, x_2) &= \int_0^\infty e^{-\omega t} g(t, x_1) p(x_1, x_2|0, y_2) dt \\
&= p(x_1, x_2|0, y_2) \tilde{g}(\omega, x_1) ,
\end{aligned}$$

where $\tilde{g}(\omega, x_1) = \omega^{\alpha-1} e^{-x_1 \omega^\alpha}$, is the Laplace transform of the pdf of S_t with respect to time. Therefore, $\tilde{p}_\alpha(.)$ is equal to the following product

$$\tilde{p}_\alpha(\omega, x_1, x_2) = p(x_1, x_2|0, y_2) \omega^{\alpha-1} e^{-x_1 \omega^\alpha} . \tag{37}$$

This result will be helpful in the construction of the fractional forward (or Fokker-Planck) equation satisfied by the bivariate Laplace's transform of the pair (S_t, λ_{S_t}).

Proposition 8.1. *The joint Laplace's transform $\phi_\alpha(t, z_1, z_2)$, is solution of the following fractional differential equation:*

$$\begin{aligned}
\frac{\partial^\alpha \phi_\alpha(t, z_1, z_2)}{\partial t^\alpha} &= -\left(z_1 + \kappa\theta z_2\right) \phi_\alpha(t, z_1, z_2) \\
&\quad + \left(1 - \kappa z_2 - \frac{1}{2}\sigma^2 z_2^2 - \mathbb{E}\left(e^{-z_2 \eta J}\right)\right) \frac{\partial \phi_\alpha(t, z_1, z_2)}{\partial z_2} ,
\end{aligned} \tag{38}$$

with the initial conditions $\phi_\alpha(0, z_1, z_2) = e^{-z_2 y}$ and $\phi_\alpha(t, z_1, 0) = \mathbb{E}\left(e^{-z_1 S_t}|\lambda_0 = y\right) = E_\alpha\left(-z_1 t^\alpha\right)$.

Proof. The Laplace's transform is equal to the following double integrals:

$$\begin{aligned}
\phi_\alpha(t, z_1, z_2) &= \int_0^\infty \mathbb{E}\left(e^{-z_1 u - z_2 \lambda_u} \mid \lambda_0 = y\right) g(t, u) du \\
&= \int_0^\infty \int_0^\infty e^{-z_1 u} e^{-z_2 x} \underbrace{g(t, u)\, p(u, x|0, y)}_{=p_\alpha(t, u, x\,|0, y_2)} \, dx\, du .
\end{aligned}$$

From Equation (37), we infer that the Laplace's transform of $\phi_\alpha(t, z_1, z_2)$ with respect to t si equal to

$$\begin{aligned}
\tilde{\phi}_\alpha(\omega, z_1, z_2) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-z_1 u} e^{-z_2 x} e^{-\omega t}\, p_\alpha(t, u, x\,|0, y_2)\, dx\, du\, dt \\
&= \int_0^\infty \int_0^\infty e^{-z_1 u} e^{-z_2 x}\, \tilde{p}_\alpha(\omega, u, x\,|0, y_2)\, dx\, du \\
&= \omega^{\alpha-1} \int_0^\infty \int_0^\infty e^{-z_1 u} e^{-z_2 x}\, p(u, x|0, y_2) e^{-u\,\omega^\alpha}\, dx\, du \\
&= \omega^{\alpha-1} \int_0^\infty e^{-u(z_1+\omega^\alpha)} \int_0^\infty e^{-z_2 x}\, p(u, x|0, y_2)\, dx\, du \\
&= \omega^{\alpha-1} \int_0^\infty e^{-u(z_1+\omega^\alpha)}\, \phi(u, z_2)\, du .
\end{aligned}$$

Integrating by parts this last integral gives us:

$$\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\phi(u,z_2)\,du = \qquad (39)$$
$$\frac{1}{(z_1+\omega^\alpha)}\left(\phi(0,z_2)+\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial u}du\right).$$

From proposition 5.1, we know that $\phi(.)$ is solution of the Fokker-Planck equation (17). Inserting the forward expression of $\frac{\partial\phi(u,z_2)}{\partial u}$ in Equation (39) leads to:

$$\begin{aligned}(z_1+\omega^\alpha)\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\phi(u,z_2)\,du &= \phi(0,z_2)-\kappa\theta z_2\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\phi(u,z_2)du\\ &\quad -\frac{1}{2}\sigma^2 z_2^2\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial z_2}du\\ &\quad +\left(1-\kappa z_2-\mathbb{E}\left(e^{-z_2\eta J}\right)\right)\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial z_2}du\,.\end{aligned}$$

This last equality can be rewritten as:

$$\begin{aligned}(z_1+\omega^\alpha)\,\tilde{\phi}_\alpha(\omega,z_1,z_2) &= \omega^{\alpha-1}\,\phi(0,z_2)-\kappa\theta z_2\,\underbrace{\omega^{\alpha-1}\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\phi(u,z_2)du}_{\tilde{\phi}_\alpha(\omega,z_1,z_2)}\\ &\quad -\frac{1}{2}\sigma^2 z_2^2\omega^{\alpha-1}\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial z_2}du\\ &\quad +\left(1-\kappa z_2-\mathbb{E}\left(e^{-z_2\eta J}\right)\right)\omega^{\alpha-1}\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial z_2}du\,.\end{aligned} \qquad (40)$$

Since the derivative of $\tilde{\phi}_\alpha(\omega,z_1,z_2)$ with respect to z_2 is

$$\frac{\partial\tilde{\phi}_\alpha(\omega,z_1,z_2)}{\partial z_2}=\omega^{\alpha-1}\int_0^\infty e^{-u(z_1+\omega^\alpha)}\,\frac{\partial\phi(u,z_2)}{\partial z_2}\,du\,,$$

an given that $\phi(0,z_2)=\phi_\alpha(0,z_1,z_2)$, Equation (40) becomes

$$\begin{aligned}\omega^\alpha\tilde{\phi}_\alpha(\omega,z_1,z_2)-\omega^{\alpha-1}\,\phi(0,z_2) &= -z_1\tilde{\phi}_\alpha(\omega,z_1,z_2)-\kappa\theta z_2\tilde{\phi}_\alpha(\omega,z_1,z_2)\\ &\quad -\frac{1}{2}\sigma^2 z_2^2\frac{\partial\tilde{\phi}_\alpha(\omega,z_1,z_2)}{\partial z_2}\\ &\quad +\left(1-\kappa z_2-\mathbb{E}\left(e^{-z_2\eta J}\right)\right)\frac{\partial\tilde{\phi}_\alpha(\omega,z_1,z_2)}{\partial z_2}\,.\end{aligned}$$

The left-hand term is the Laplace's transform of the Caputo's derivative of $p_\alpha(t,x)$. Therefore we infer Equation (38). □

Before concluding this section, we detail the numerical scheme for solving Equation (38) by a finite difference approach. Let us consider the domain $D=[0,t_{max}]\times[-z_{max},z_{max}]^2$ on which we want to estimate the joint Laplace's transform of S_t and λ_{S_t}. Firstly, we choose two steps of discretization, noted Δ_t and Δ_z, in order to define a cube of triplets (t_k,z_j,z_l) where

$$t_k=k\,\Delta_t \quad,\quad z_j=-z_{max}+j\,\Delta_z \quad,\quad z_l=-z_{max}+l\Delta_z\,,$$

for $k=0,....,n_t$ and $j,l=0,...,n_z$. The numbers n_t and n_z are integers equal to $n_t=\left\lfloor\frac{t_{max}}{\Delta_t}\right\rfloor$ and $n_z=\left\lfloor\frac{2z_{max}}{\Delta_z}\right\rfloor$. In order to ligthen developments, we denote by $\phi_\alpha(k,j,l)$ the approached value of $\phi_\alpha(t_k,z_j,z_l)$.

The Caputo's derivative is numerically approached by the following finite difference sum:

$$\begin{aligned}\frac{\partial^\alpha \phi_\alpha(k,j,l)}{\partial t^\alpha} &\approx \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{m=0}^{k-1}(k-m)^{-\alpha}\left(\phi_\alpha(m+1,j,l)-\phi_\alpha(m,j,l)\right) \\ &= \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\left(\phi_\alpha(k,j,l)-\phi_\alpha(k-1,j,l)\right)+ \\ &\quad \frac{1}{\Gamma(1-\alpha)}\sum_{m=0}^{k-2}(k-m)^{-\alpha}(\Delta_t)^{-\alpha}\left(\phi_\alpha(m+1,j,l)-\phi_\alpha(m,j,l)\right).\end{aligned} \tag{41}$$

Whereas the partial derivative of ϕ_α with respect to z_2 is approached by finite difference approximations:

$$\begin{aligned}\frac{\partial \phi_\alpha(k,j,l)}{\partial z_2} &\approx \frac{\phi_\alpha(k,j,l)-\phi_\alpha(k,j,l-1)}{\Delta_z} \quad for\ z_l>0, \\ &\approx \frac{\phi_\alpha(k,j,l+1)-\phi_\alpha(k,j,l)}{\Delta_z} \quad for\ z_l<0.\end{aligned} \tag{42}$$

If we inject expressions (41) and (42) of derivatives in the Fokker-Planck equation (38), we estimate recursively the $\phi_\alpha(k,.,.)$ for $z_l>0$ by:

$$\begin{aligned}&\phi_\alpha(k,j,l)= \\ &\left(1+(\Delta_t)^\alpha\,\Gamma(1-\alpha)\left((z_j+\kappa\theta z_l)-\frac{1}{\Delta_z}\left(1-\kappa z_l-\mathbb{E}\left(e^{-z_l\eta J}\right)-\frac{1}{2}\sigma^2 z_l^2\right)\right)\right)^{-1} \\ &\times\Bigg[\phi_\alpha(k-1,j,l)-\sum_{m=0}^{k-2}(k-m)^{-\alpha}\left(\phi_\alpha(m+1,j,l)-\phi_\alpha(m,j,l)\right) \\ &-\frac{(\Delta_t)^\alpha\,\Gamma(1-\alpha)}{\Delta_z}\left(1-\kappa z_l-\mathbb{E}\left(e^{-z_l\eta J}\right)-\frac{1}{2}\sigma^2 z_l^2\right)\phi_\alpha(k,j,l-1)\Bigg],\end{aligned}$$

whereas for$z_j<0$, we use the recursion:

$$\begin{aligned}&\phi_\alpha(k,j,l)= \\ &\left(1+(\Delta_t)^\alpha\,\Gamma(1-\alpha)\left((z_j+\kappa\theta z_l)+\frac{1}{\Delta_z}\left(1-\kappa z_l-\mathbb{E}\left(e^{-z_l\eta J}\right)-\frac{1}{2}\sigma^2 z_l^2\right)\right)\right)^{-1} \\ &\times\Bigg[\phi_\alpha(k-1,j,l)-\sum_{m=0}^{k-2}(k-m)^{-\alpha}\left(\phi_\alpha(m+1,j,l)-\phi_\alpha(m,j,l)\right) \\ &+\frac{(\Delta_t)^\alpha\,\Gamma(1-\alpha)}{\Delta_z}\left(1-\kappa z_l-\mathbb{E}\left(e^{-z_l\eta J}\right)-\frac{1}{2}\sigma^2 z_l^2\right)\phi_\alpha(k,j,l+1)\Bigg].\end{aligned}$$

Given that $S_0=0$, $\lambda_{S_0}=y$ and $\phi_\alpha(k,j,0)=\mathbb{E}\left(e^{-z_j\,S_{t_k}}\right)$, the recursion is initialized as follows:

$$\begin{cases}\phi_\alpha(0,j,l)=e^{-l\,\Delta_z\,y} & j,l\in\{0,...,n_z\}, \\ \phi_\alpha(k,j,0)=E_\alpha\left(-(j\Delta_z)(k\Delta_t)^\alpha\right) & k\in\{0,...,n_t\}.\end{cases}$$

In the next section, we compare the Laplace's transform of ξ_{S_t} to ξ_t, using this numerical method.

9 Numerical illustrations

Table 1 presents the parameters of the Hawkes jump diffusion process, used in the numerical illustrations. These parameters are chosen in order to obtain realistic term structure of default probabilities. Firstly, we have computed the expectation of ξ_t over 5 years by solving the forward equation (17). We have run the numerical method of Section 5 with n_t =100 time steps per year and n_z =100 steps of discretization between -0.90 γ_- to 0.90 γ_-. The initial value of the intensity is set to its long term mean: $\lambda_0=\lim_{t\to\infty}\mathbb{E}(\lambda_t)$.

θ	0.05	η	10
ρ	2.00	$\lim_{t\to\infty} \mathbb{E}(\lambda_t)$	0.30
σ	0.10	γ_-	0.0333
κ	6.00	β_-	0.0099

Table 1: Parameters of the Hawkes diffusion process and of the exponential eigenfunction.

The left graph of Figure 3 shows the sensitivity of $\mathbb{E}_0(\xi_t)$ to δ_- at different times. This expectation is a decreasing and convex function of δ_-. The right graph of Figure 3 shows term structures of $\mathbb{E}_0(\xi_t)$ for time horizons up to 5 years, and for values of $0 < \delta_- < \gamma_-$. As ξ_t is a super-martingale, the curves are decreasing with time. Curves are convex for small values of δ_- and slightly concave when $\delta_- \to \gamma_-$. Figure 4 shows the same graphs for the time-changed process, ξ_{S_t}, with a fractional order $\alpha = 0.90$. The left graph of Figure 4 reveals that the time-change increases the convexity of the curve of $\mathbb{E}_0(\xi_t)$ in function of δ_-, around the neighborhood of zero. The remainder of the numerical analysis focuses on survival probabilities computed with the fractional model.

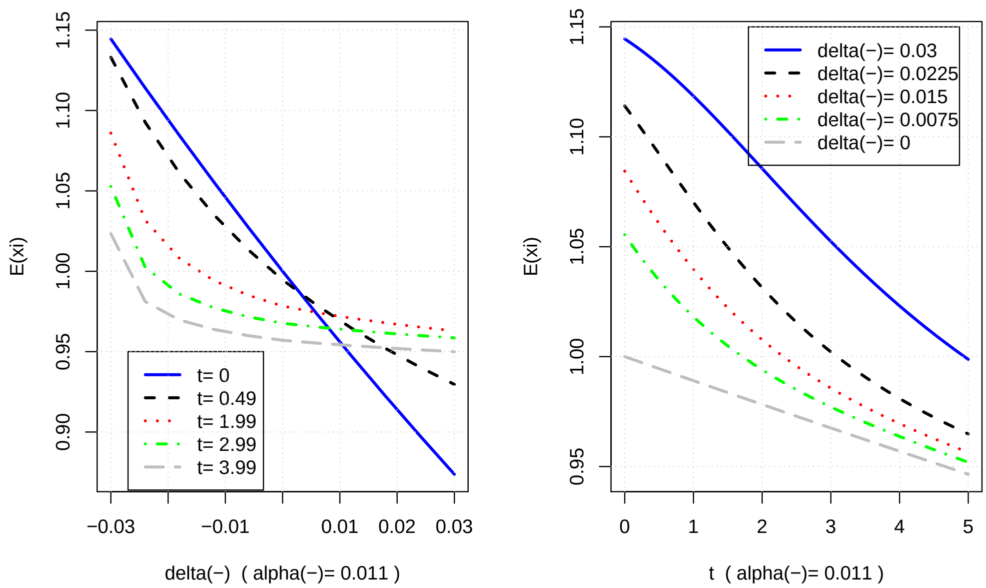

Figure 3: Curves of $\mathbb{E}_0(\xi_t)$ for $\alpha_- = 1.10\,\beta_-$ and δ_- from 0 to $0.90\,\gamma_-$. Time horizon up to $t = 5$ years.

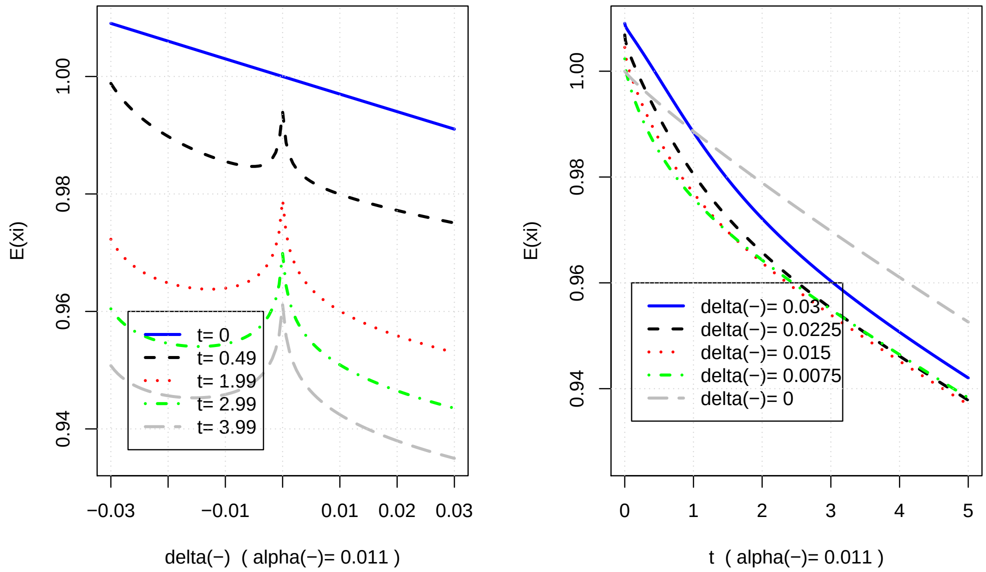

Figure 4: Curves of $\mathbb{E}(\xi_t)$ for $\alpha_- = 1.10\,\beta_-$ and δ_- from 0 to $0.90\,\gamma_-$. Time horizon up to $t = 5$ years. Fractional order: $\alpha = 0.90$.

The left graph of Figure 5 emphasizes the influence of the fractional order α on the term structure of survival probabilities. We observe that these probabilities are inversely proportional to α. The lower is the fractional order; the higher is the probability that the company does not enter bankruptcy over the five first years. The right graph of Figure 5 shows that increasing the parameter α_- globally raises the probabilities of default.

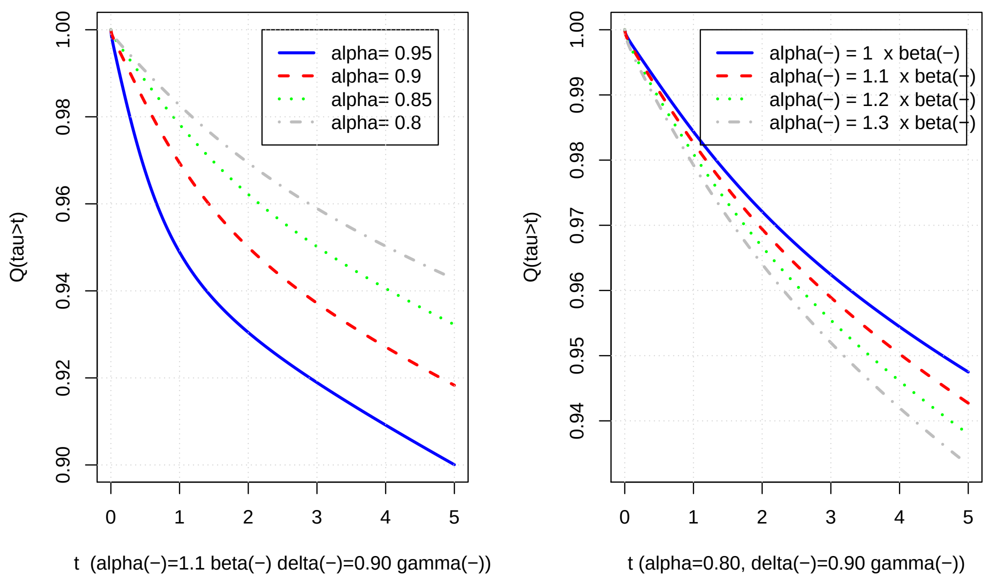

Figure 5: Curves of $Q(t)$ for various fractional order α and α_-. Time horizon up to $t = 5$.

The left graph of Figure 6 reveals that increasing δ_-, reduces the probabilities of survival at short term while at long term, survival curves for different δ_- are nearly parallel. The right graph of Figure 5 shows that the fractional model may generate term structures with a high convexity over the first years when the fractional order is low. This type of curves is sometimes observed for companies facing momentary financial problems. The market considers that the probability of bankruptcy is high at short-term. But if companies overcome this crisis, their long term viability is high. This type of term structure is difficult to replicate with structural or reduced models.

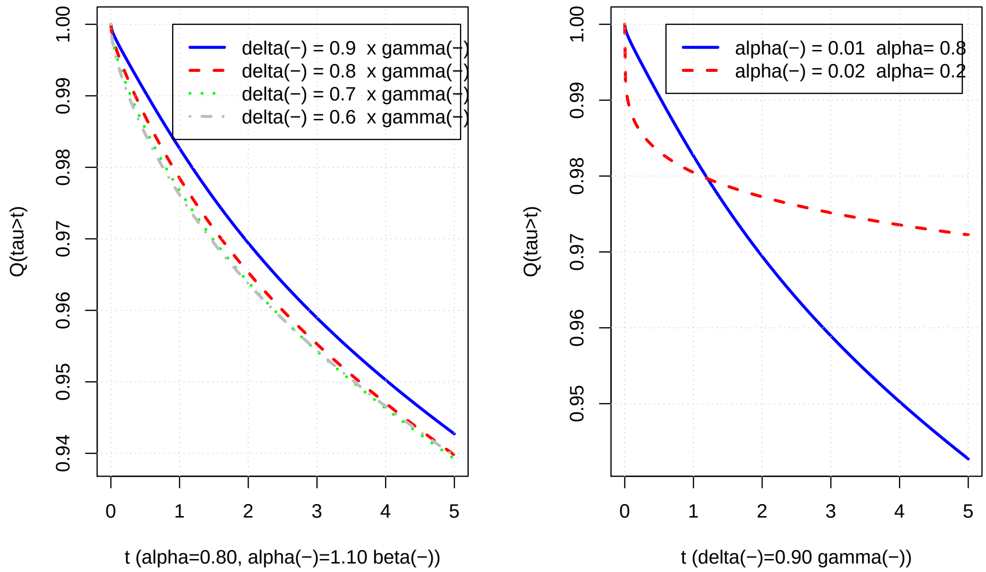

Figure 6: Curves of $Q(t)$ for $\alpha_- = 1.10\,\beta_-$ and various δ_- and fractional orders, α.

Figure 7 analyzes the role played by the initial intensity on the term structure. As we could expect, the higher is this intensity, the lower is the probability of survival. As λ_t and λ_{S_t} are self-exciting processes, our model is well able to explain a worsening of a firm credit quality due to a deterioration of the global conjoncture, here represented by λ_t or λ_{S_t}.

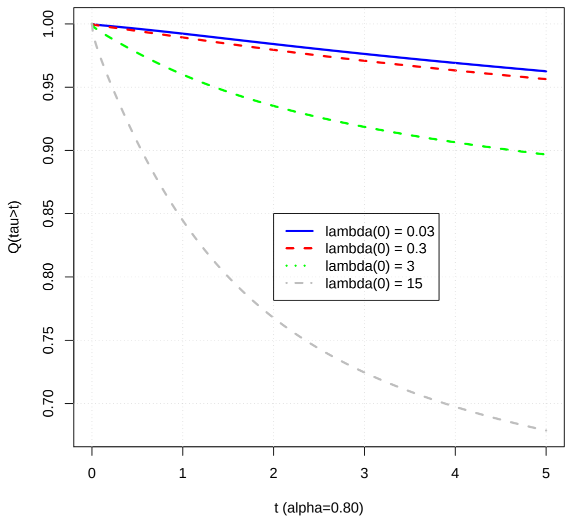

Figure 7: Curves of $Q(t)$ for $\alpha_- = 1.10\,\beta_-$, $\delta_- = 0.90\,\gamma_-$ and various initial intensity levels. Time horizon up to $t = 5$.

10 Conclusions

This article is the first one to propose an application of fractional Hawkes jump-diffusion to credit risk modelling. These processes present several interesting features. Firstly, the intensity may be seen as a systematic risk factor driving the probability of failure of firms. The mechanism of self-excitation in this intensity ensures that the model explains periods during which we observe a sudden and repeated increases of default probabilities. Secondly, the fractional model lengthens the memory of the intensity process by randomly freezing it during relatively long periods. Thirdly, the model is numerically tractable.

We have seen that survival probabilities in the potential approach to credit risk are solutions a forward or Fokker-Planck Equation (FPE). Compared to Pearson Diffusions, the FPE has a new term related to the self-excited dynamics. Whereas, the derivative with respect to time is replaced by the Caputo's derivative. The numerical illustrations reveal that the fractional model can explain a wide variety of term structures of survival probabilities.

This work opens the way to further research. Firstly, the potential approach can be extended to a multivariate framework with self-excitation and contagion between several Hawkes jump-diffusions. Secondly, it is maybe possible to develop structural or intensity approaches for credit risk, with fractional Hawkes-jump diffusions.

Acknowledgment

This work was supported by the "Fonds de la Recherche Scientifique" - FNRS under Grant n° 33658713.

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