



Université catholique de Louvain
Institute of Statistics, Biostatistics and Actuarial Sciences

Stochastic Pathways in Incomplete Markets

—
Volterra Processes, Stopping Times, and Path-Signatures

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A thesis submitted to the
Université catholique de Louvain
in partial fulfillment of the
requirements for the degree of
DOCTOR OF SCIENCES

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September 2026

*Ambition is a state of permanent
dissatisfaction with the present.*

Emanuel DERMAN

To my dad. To Emilie.

Acknowledgements

First of all, I would like to express my deepest gratitude to my supervisor, Prof. Donatien Hainaut, for his guidance, support, and insightful advice throughout my PhD. I am particularly grateful for his trust, for the freedom he gave me in shaping my research directions, and for always supporting my choices, even when they evolved stochastically from week to week. His encouragement, especially during funding applications, and his constant availability have been invaluable.

I am also sincerely grateful to Prof. Eduardo Abi Jaber for accepting to be a member of my jury, for our fruitful collaborations, and for the opportunity to spend six enriching months at Ecole Polytechnique under his supervision. I greatly benefited from his knowledge, generosity, and the many stimulating discussions we shared.

I would like to thank the other members of my jury, Prof. Jennifer Alonso Garcia, Prof. Karim Barigou, and Prof. Frédéric Vrins, for taking the time to evaluate my work, and for their relevant comments and suggestions, which helped improve the quality of this thesis

I would like also to give a special mention to Dr. Jean-Loup Dupret for his kind advice and support throughout my PhD. Thank you for the fruitful discussions, our collaboration, and the enjoyable moments shared during conferences.

I am thankful to my fellow PhD colleagues at ISBA, especially those with whom I shared an office or built close friendships. In particular, I would like to thank the actuarial team including Charlotte, Kenrick, Dr. José Miguel Flores-Contro, Arnaud and Oussama, for the stimulating discussions, but also Mathias, Antoine, Guillaume, Lara, Mathilde, Anas and Kamal. A special thanks to Kenrick and José for this last year: you turned long research days into something far more enjoyable, with a perfect balance of stimulating discussions, good laughs, and just enough randomness to keep things interesting. Extra credit to Kenrick for learning French along the way, still a work in progress, but already très impressionnant !

I also wish to thank the PhD students from Eduardo Abi Jaber's group whom I met during my visiting period in Paris, Shaun, Nathan, Louis-Amand, Elie, Maxime, Dimitri, and Ronan, for their warm welcome and for the many formal and less formal exchanges that made this experience particularly memorable. I hope to see you all again very soon, either in Paris or at future conferences.

My deepest thanks go to my friends and family for their unconditional support throughout these years. I would like to thank Armand, Mathias, Maxime, Guillaume, Alexis, and my group of friends 'La Meute' for all the moments of relaxation, whether around a drink or during football games, and I apologize for being less present at times in recent months. I am especially grateful to my mother for her unwavering support throughout all these years, including her invaluable financial support, which made my studies possible.

I would like to pay a special tribute to my late father. Your values continue to inspire me every day, and I know that you have been with me throughout this journey. I miss you.

Last but certainly not least, I owe my deepest gratitude to Emilie (and Otis), who has been by my side for more than five years, and to whom I dedicate this thesis. Your constant support, love, and belief in me, both in the good times and the more difficult ones, have been a cornerstone of this journey. Thank you for always being there when I need it, supporting me in my projects, and listening to me, even when I speak about my research late in the evening. It is thanks to you that I always found the strength and perseverance to move forward. The future is stochastic, but I'm certain my path will always include you.

Edouard Motte, June 2026

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Introduction

Financial markets are shaped by a multitude of uncertainties that have real consequences for investors and institutions. Episodes such as the 1987 stock market crash, the 2008 global financial crisis, or the sharp market movements during the COVID-19 pandemic illustrate how extreme fluctuations in asset prices, sudden changes in volatility, and correlated losses across markets can threaten portfolios and institutions. Similarly, in insurance markets, unexpected mortality shocks, such as during pandemics, or systematic trends in longevity can dramatically affect insurers liabilities. The growth of complex financial products, ranging from Asian and barrier options to structured derivatives with path-dependent payoffs, introduces additional layers of risk that cannot be captured by simple models or classical hedging techniques. In many situations, markets are incomplete: some risks, like stochastic volatility or interest rates, cannot be fully hedged; life insurance portfolios are exposed to random events such as policyholder deaths; and market frictions, including transaction costs or permanent price impact, prevent perfect replication of payoffs. On the other hand, path-dependence plays a central role in financial and actuarial problems. It can appear in the modeling of risk factors, allowing for more consistent representation of volatility or interest rate dynamics; it can be intrinsic to the products themselves, as in widely traded path-dependent derivatives; or it can arise from the structure of equity-linked life insurance portfolios, where efficient hedging strategies depend on the timing of policyholders' deaths. These path-dependent features pose significant challenges for modeling, pricing and hedging. Addressing them often requires new tools and novel approaches. This thesis is organized around a common theme: the modeling and hedging of path-dependent risks in incomplete financial and actuarial markets. Although the sources of path-dependence differ across the problems considered — memory effects in risk factors, stopping-time information in insurance portfolios, or path-dependent derivative payoffs under market frictions — they lead to non-Markovian dynamics for which classical approaches usually become insufficient. This thesis therefore develops several mathematical tools and novel approaches de-

signed to address these path-dependent problems.

In financial markets, empirical studies revealed memory effects in historical financial time series, especially for asset volatility and interest rates, see Cont (2001); Dai and Singleton (2003); McCarthy et al. (2004). Volatility exhibits roughness and short-range dependence, a feature observed in high-frequency financial data and linked to market microstructure effects such as order splitting and endogenous trading activity, see for instance Jaisson and Rosenbaum (2016); Rosenbaum and Tomas (2021). In contrast, interest rates often display long-range dependence, with persistent autocorrelations reflecting structural and macroeconomic trends over extended horizons, as reported by Dai and Singleton (2003); McCarthy et al. (2004); Benth and Rohde (2019). Beyond historical data, robust and universal patterns emerge in the term structures of option prices across a broad range of maturities. Specifically, for interest rates the term structure of the implied volatilities of cap and floor options is typically hump-shaped, see for instance Dai and Singleton (2003); De Jong et al. (2004); for stock indices such as the S&P 500: the term structure of the At-The-Money (ATM) skew of the implied volatility of call and put options (in log-log scale), exhibits typically a concave shape. In addition, the ATM skew displays an approximately linear decrease at long maturities suggesting a power-law decay, though only for sufficiently large maturities, see Bergomi (2015); Bayer et al. (2016); Delemotte et al. (2023); Guyon and El Amrani (2023); Abi Jaber and Li (2024). In quantitative finance, memory genuinely matters: the past history of risk factors has a quantifiable impact on current dynamics, shaping the behavior of derivative prices and hedging strategies. Volterra processes appear natural to capture these path-dependent features and to model the dynamics of volatility and interest rates, since these processes encode memory effects and inherently non-Markovian behavior. However, this realism comes at a cost. The presence of memory introduces challenges for both pricing and hedging, as classical Markovian techniques are no longer directly applicable and new methodological tools are required. The first part of the thesis addresses path-dependence arising directly from the dynamics of financial risk factors. In this setting, memory effects in volatility and interest rates naturally lead to non-Markovian models, for which Volterra processes provide a flexible modeling framework. In particular, the first chapter concentrates on partial hedging in rough volatility models, addressing a non-Markovian stochastic control problem in an incomplete market. The key idea is to overcome non-Markovianity by lifting the original problem to a high-dimensional but Markovian setting. The second chapter introduces a unified modeling framework for equity volatility and interest rates based on correlated Gaussian Volterra processes. Incorporating both stochastic volatility and stochastic interest rates is essential for the accurate pricing and risk management of financial products such as equity-linked interest rate derivatives or quanto options, as well as insurance products including equity-linked life insurance contracts. The valuation and hedging of these instruments critically depend on the joint dynamics of stock prices,

interest rates, volatility, and their correlations. This chapter demonstrates that the Volterra extension of the Stein–Stein model with stochastic interest rates provides a flexible yet analytically tractable framework for capturing the joint evolution of equity and interest rate markets, thereby enabling efficient pricing and hedging.

While the first part focuses on path-dependence generated by the dynamics of financial risk factors, the second part investigates path-dependence arising from insurance portfolios themselves. In life insurance markets, the evolution of liabilities depends on the random occurrence and timing of policyholder deaths, which progressively reveal information and naturally induce a path-dependence in hedging strategies. Unlike financial markets, where risks are primarily driven by traded assets, life insurance liabilities depend also on random events such as policyholder deaths, which cannot be perfectly hedged, leading by definition to an incomplete market. Insurers typically rely on diversification across large portfolios to mitigate this unhedgeable risk, but diversification alone does not eliminate the need for sound hedging strategies. While a substantial body of literature studies the pricing and hedging of life insurance contracts, far fewer works address the problem at the portfolio level, even though this is the setting that truly matters in practice: insurers manage collections of policies whose aggregate risk determines their solvency and overall exposure. Moreover, the timing of policyholder deaths introduces a crucial information that an insurer can exploit to improve hedging decisions. As the life portfolio evolves through death occurrences, the insurer gains information about its remaining liabilities and can adjust its position accordingly. This information creates opportunities for more effective risk management. Given that insurers' primary objective is to honor their contractual commitments rather than to maximize speculative gains, hedging approaches that focus directly on limiting downside risk become especially important. In this spirit, the loss-based frameworks introduced by Föllmer and Leukert (1999); Föllmer and Leukert (2000), such as quantile hedging, which aims to maximize the probability of meeting future liabilities, and shortfall hedging, which seeks to minimize the expected loss when a shortfall occurs, offer natural tools for insurance applications. These methodologies align closely with the risk-management perspective of insurance companies and with regulatory environments that emphasize solvency, capital adequacy, and the prudent management of extreme losses. The second part of the thesis investigates hedging of insurance liabilities via stopping-time-based strategies. Chapter 3 studies optimal strategies that either maximize the probability of successfully hedging (quantile hedging) or minimize expected shortfall losses for life insurance portfolio, in a model-free financial-actuarial framework. In particular, we show that efficient strategies are adapted to the filtration generated by the insurance portfolio, naturally updating at each stopping time corresponding to a death event. Hence, at a given time, those efficient strategies depend not only on the number of survivals but also on the whole path related to the number of survivals. This approach highlights how path-dependence linked to random events can be incorporated into practical risk management strategies.

The third part of the thesis considers a third source of path-dependence, namely the structure of the financial products themselves. Unlike the previous parts, where path-dependence originated either from the dynamics of risk factors or from the flow of insurance information, here it is intrinsic to the payoff structure of traded derivatives. Numerous widely traded derivatives, such as Asian, look-back, and barrier options, have payoffs determined not only by the terminal price of an underlying asset but by the entire trajectory it follows. Since these products have path-dependent payoffs, pricing and hedging naturally lead to non-Markovian problems. Moreover, in practice, market frictions such as transaction costs or market impact cannot be ignored. Accounting for these frictions is relevant because they prevent perfect replication, influence optimal trading strategies, and reflect realistic trading conditions faced by market participants, especially large traders. These considerations make the resulting control problems associated to hedging highly nonlinear and non-Markovian. Even in simplified Markovian settings, which correspond, for instance, to European payoffs, solving these problems remains analytically challenging as pointed out by Almgren and Li (2016); Guéant and Pu (2017). To address these difficulties, the third part of the thesis investigates a novel approach based on path-signatures, a tool originating from rough-paths theory that has recently attracted interest in mathematical finance (see for e.g. Lyons et al. (2020); Cuchiero and Möller (2025); Abi Jaber and Gérard (2025a)). A key property of path-signatures is that they allow the systematic linearization of nonlinear path-dependent dynamics. This linearization property opens the door to handling path-dependent control problems by lifting them into the space of signatures, where they become more tractable. This approach allows for the application of classical Markovian methods such as the Itô formula or the optimal martingality principle. Chapter 4 provides both theoretical insights and practical tools for pricing and hedging path-dependent options in markets with instantaneous and permanent market impacts via a novel signature approach, enabling the design of effective strategies for hedging complex path-dependent derivatives.

Memory effects play a crucial role in shaping financial phenomena, from asset prices and order flow dynamics to portfolio optimization and derivatives risk-management. Overall, this thesis studies several manifestations of path-dependence arising in incomplete financial and actuarial markets, ranging from memory effects in stochastic risk factors to stopping-times-based hedging in insurance and path-dependent derivative payoffs under market frictions. Across these contexts, the common challenge is that classical Markovian approaches are often insufficient, requiring new mathematical tools to model, price, and hedge effectively. By combining rigorous theoretical analysis with practical implementation considerations, the thesis aims to offer both insight and methodology for tackling complex path-dependent problems arising in financial and actuarial markets.

Outline of the thesis

The three parts of the thesis contain four chapters, which are self-contained research articles that can be read independently from each other. Throughout this thesis, notation may change from one chapter to the next. Unless explicitly stated otherwise, all notation is to be interpreted within the local context of the corresponding chapter. The outline of the dissertation is as follows.

I. Volterra processes in finance: Chapters 1 and 2

Many financial processes exhibit path-dependent behaviors, such as long/short-memory effects, which cannot be adequately captured by traditional Markovian models. Understanding and incorporating these dependencies is essential for accurate risk management. Volterra processes have emerged as fundamental tools to capture persistent memory and short-range dependence observed in financial markets Bayer et al. (2016); Fukasawa (2017); Gatheral et al. (2018). They provide a consistent framework that better reproduces empirical features such as volatility clustering, implied volatility smiles, and term-structures. The first part of this thesis develops both theoretical and practical tools to deal with the challenges posed by Volterra dynamics of the form

$$X_t = X_0 + \int_0^t G(t, s)b(X_s)ds + \int_0^t G(t, s)\sigma(X_s)dW_s, \quad t > 0,$$

with some well-defined functions b and σ , W a standard Brownian motion, and G a well-defined kernel to account for long or short term memory effects. The two chapters in this part illustrate how Markovian approximations, duality techniques, and mathematical objects such as Fredholm (1903) resolvents and determinants can be used to solve non-Markovian pricing and hedging problems.

The first chapter addresses the resolution of a non-Markovian stochastic control problem. More precisely, this chapter studies the partial hedging of derivative claims in affine Volterra volatility models within an incomplete market setting. A non-Markovian stochastic control problem is introduced, where the goal is to minimize the discrepancy between a stochastic target and the terminal value of the hedging portfolio. Due to the non-Markovian nature of the volatility models, classical dynamic programming techniques cannot be directly applied. In the same spirit of Abi Jaber and El Euch (2019a), we introduce a Markovian multifactor approximation of the volatility, and lift the original problem to a high-dimensional but Markovian setting. We prove convergence of the optimal Markovian solution to the original rough solution, and inspired by Ma et al. (2020), we provide approximate solutions using a dual control method. This approach allows to deduce lower and upper bounds on the optimal solution, and enables the derivation of semi-explicit partial hedging strategies in the Volterra Heston model.

The second chapter focuses on the modeling of financial risk factors. In particular, this chapter extends the Stein and Stein (1991) stochastic volatility model to a Volterra setting with stochastic interest rates, extending the framework of Abi Jaber (2022a). Despite the non-Markovian nature of the dynamics, we derive explicit formulas for pricing zero-coupon bonds and interest rate cap/floor, along with a semi-explicit formulas for the characteristic function of the log-forward equity index using Fredholm (1903) resolvents and determinants, enabling efficient pricing and calibration to market data. Beyond analytical tractability, the Volterra Stein-Stein model with stochastic interest rates offers a great flexibility and allows to capture key empirical features such as the humped shape of cap/floor implied volatility, and the concave shape (in log-log scale) of the ATM skew term structure of the S&P500 options. Finally, this chapter also makes the connection with linear-quadratic models as explored by van Haastrecht et al. (2009).

II. Hedging of insurance liabilities via stopping-time-based strategies: Chapter 3

The second part of the thesis focuses on the hedging of equity-linked life insurance portfolios, where the total liability at maturity of the insurers is characterized by its dependence between financial and actuarial risks, and can be represented as

$$H_T^{Total} := N_T \times H_T,$$

where $(N_t)_{t \leq T}$ is a survival process, counting the number of survival policyholders through time, and H_T is a purely financial payoff. As the liability is driven by non-hedgeable actuarial risks and, potentially non-hedgeable financial risks, a perfect hedge of the total liability at maturity is unattainable. The insurer must therefore determine an efficient strategy to manage and hedge her risk exposure.

In this sense, the second part of this thesis show how to construct efficient hedging strategies that are stopping-time-based, in the sense that the strategies naturally update each time a death occurs and depend on the path of the process counting the number of survivals $(N_t)_{t \leq T}$.

More precisely, in the third chapter, we investigate efficient hedging strategies of equity-linked life insurance portfolios, for loss-averse insurers that either wants to maximize the probability of successful hedge (quantile hedging initially introduced by Föllmer and Leukert (1999)) or to minimize shortfall loss (shortfall hedging initially introduced by Föllmer and Leukert (2000)). We consider two classes of admissible controls: financial-only controls, which do not incorporate actuarial information, and combined financial-actuarial controls, which can exploit all available financial and actuarial information. By exploiting the super-replication theory of El Karoui and Quenez (1995) and using a duality approach as in Kramkov and Schachermayer (1999), we theoretically characterize the optimal strategies that are adapted to the portfolio size and depend both on financial and actuarial risks.

Moreover, the efficient strategies are linked to (super) replication hedging strategies of modified payoffs. In complete financial markets, and when restricting to purely financial strategies, we are able to derive fully explicit expressions for the optimal controls. However, limiting the insurer to financial-only strategies is not realistic in practice, as insurers can observe the evolution of the survival process and exploit this information to design more efficient hedging strategies within the class of combined financial–actuarial controls. Building on the explicit form of the financial-only optimal strategies, we construct natural, stopping-time-based explicit hedging strategies for the combined market, which are updated whenever a death occurs. These strategies are therefore practical, interpretable, and dynamically adapted to the insurance portfolio. Numerical illustrations demonstrate that stopping-time-based hedging strategies are efficient, particularly for insurers who are highly loss-averse or hold large portfolios.

III. Path-signatures for risk-management: Chapter 4

Path-signatures, defined as sequences of iterated integrals, constitute a promising and rigorous framework for capturing memory effects (see Chevyrev and Kormilitzin (2016)). The signature of a continuous path (of a semimartingale) $X : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, introduced by Chen (1957), encodes the information contained in the trajectory through its iterated integrals. For each $n \geq 1$, the signature of order n of X over the interval $[0, t]$ is defined by

$$\mathbb{X}_t^n := \int_{0 < u_1 < \dots < u_n < t} \circ dX_{u_1} \otimes \dots \otimes \circ dX_{u_n},$$

where $\circ$ denotes the Stratonovich integral and $\otimes$ the tensor product. The full signature is the collection

$$\mathbb{X}_t = (1, \mathbb{X}_t^1, \mathbb{X}_t^2, \mathbb{X}_t^3, \dots),$$

which lives in the extended tensor algebra over $\mathbb{R}^d$, and provides a universal and algebraic representation of paths (see Cuchiero et al. (2025b)).

Path-signatures possess powerful properties such as universality, injectivity (up to tree-like equivalence), and a rich algebraic structure, making them fundamental tools for the analysis of trajectories and stochastic processes. Although these properties have been extensively studied in algebraic and geometric settings (notably in rough path theory; see Lyons et al. (2007)), their use in applied stochastic analysis remains relatively limited. Yet, path-signatures constitute a particularly well-suited framework for tackling non-Markovian stochastic control problems, as they encode the history of a process in a hierarchical, universal, and tractable manner, thereby overcoming the limitations of classical approaches. By providing a rigorous representation of memory effects, they enable the use of Markovian techniques in an infinite-dimensional state space. Beyond their elegant mathematical structure, path-signatures offer a practical and versatile methodology for

stochastic control, relying on two key principles: first, their ability to faithfully summarize past information in a compact and systematic way; second, their linearization property, which plays a role analogous to that of polynomials in classical analysis, allowing intrinsically nonlinear path-dependent phenomena to be treated in a linear and computationally exploitable form. In mathematical finance, path-signatures have recently attracted increasing attention; see, among others, Lyons et al. (2020); Cuchiero et al. (2023); Abi Jaber and Gérard (2025a). In particular, Abi Jaber and Gérard (2025a) introduced signature volatility models, a universal framework that encompasses a wide range of (possibly non-Markovian) stochastic volatility models while remaining tractable for pricing and hedging in frictionless setting. This framework naturally includes Volterra-type dynamics as particular cases, thereby providing a broader perspective on the Volterra volatility models studied in the first part of this thesis.

The third part of this thesis investigates how path-signatures can be exploited beyond the frictionless setting, by addressing pricing and hedging problems for path-dependent options in the presence of market frictions. Due to path-dependence and market frictions the control problem associated to this financial problem is highly nonlinear and non-Markovian. By lifting the control problem into the space of path-signatures, we recast a nonlinear and non-Markovian problem into a high-dimensional but tractable setting.

In the last chapter, we tackle the problem of pricing and hedging path-dependent options in a Bachelier framework with both temporary and permanent market impact, extending the Almgren and Li (2016) setting to path-dependent options. Initially, we restrict the class of path-dependent payoffs to the class of signature payoffs, i.e. payoffs that can be expressed as a linear combination of elements of the time-augmented signature of price process. In this configuration, we study the existence and uniqueness for the control problem expressed in this signature space. Moreover, a verification theorem is derived yielding to hedging strategies in linear feedback form in the time-augmented signature of the control variables, characterized by infinite-dimensional Riccati equations on the extended tensor algebra. For the first time in the context of stochastic optimal control, we establish the connection between non-Markovian control problems and infinite-dimensional Riccati equations on the extended tensor algebra. Then, leveraging the universal approximation properties of signatures (Lyons et al. (2020); Abi Jaber and Gérard (2025a); Cuchiero et al. (2025b)), we apply our signature-based strategies to hedge any path-dependent payoffs that could be beyond the class of signature payoffs. Numerical examples show that, in the context of market frictions, signature-based strategies are efficient to hedge complex path-dependent payoffs such as Asian or lookback call options.

Published material

- Chapter 1 is based on Motte and Hainaut (2024), published in *SIAM Journal on Financial Mathematics*.
- Chapter 2 is based on Abi Jaber, Hainaut, and Motte (2025b), under minor revision for publication in *Quantitative Finance*.
- Chapter 3 is based on Motte and Hainaut (2025), published in *Insurance: Mathematics and Economics*.
- Chapter 4 is based on Abi Jaber, Hainaut, and Motte (2025a), under major revision for publication in *Mathematical Finance*.

Part I

Volterra processes in finance

CHAPTER 1

Partial hedging in rough volatility models

This chapter studies the problem of partial hedging within the framework of rough volatility models in an incomplete market setting. We employ a stochastic control problem formulation to minimize the discrepancy between a stochastic target and the terminal value of a hedging portfolio. As rough volatility models are neither Markovian nor semi-martingales, stochastic control problems associated with rough models are quite complex to solve. Therefore, we propose a multifactor approximation of the rough volatility model and introduce the associated Markov stochastic control problem. We establish the convergence of the optimal solution for the Markov partial hedging problem to the optimal solution of the original problem as the number of factors tends to infinity. Furthermore, the optimal solution of the Markov problem can be derived by solving a Hamilton-Jacobi-Bellman (HJB) equation and more precisely a nonlinear partial differential equation (PDE). Due to the inherent complexity of this nonlinear PDE, an explicit formula for the optimal solution is generally unattainable. By introducing the dual solution of the Markov problem and expressing the primal solution as a function of the dual solution, we derive approximate solutions to the Markov problem using a dual control method. This method enables for sub-optimal choices of dual control to deduce lower and upper bounds on the optimal solution as well as sub-optimal hedging ratios. In particular, semi-explicit formulas for partial hedging strategies in the Volterra Heston model are derived.

1.1 Introduction

Rough volatility models have gained significant popularity in quantitative finance since the pioneering work of Gatheral et al. (2018). These models incorporate short-

range dependence, capturing important empirical stylized facts such as volatility clustering and roughness, which are often neglected in classical volatility models. Moreover, in option pricing, short-range dependence volatility models generate implied volatility surfaces that are consistent with observed volatility surfaces, as shown in subsequent papers Bayer et al. (2016); El Euch and Rosenbaum (2019); Fukasawa (2017); Gatheral et al. (2018); Horvath et al. (2020); Abi Jaber and Illand (2022); Guyon and El Amrani (2023); Rømer (2022). These aspects of market consistency in short-range dependence volatility models are discussed in greater detail in Chapter 2.

In this chapter, we investigate the problem of hedging in rough volatility models. While previous research El Euch and Rosenbaum (2018); Fukasawa et al. (2021) have explored this matter in the context of complete market, where the volatility risk can be hedged either by trading forward variance curve or variance swap, we take a different approach. We relax the complete market assumption and focus on an incomplete market, considering only the underlying asset as hedging instrument. Since the market is incomplete, a perfect hedging strategy does not exist for any given contingent claim, this is why we are interested in partial hedging strategies. Partial hedging strategies introduced by Föllmer and Leukert (1999); Föllmer and Leukert (2000), are powerful techniques for minimizing hedging losses at a fixed cost lower than the super-replication price. Their results have next been applied to various markets and various risk processes, we can mention among others Biagini et al. (2000); Cvitanic (2000); Garnier and Sølna (2020); Jonsson and Sircar (2002); Kamizono (2003); Mania et al. (2008). Notably, Jonsson and Sircar (2002) extends the theory of partial hedging to stochastic volatility environment by formulating the problem as a stochastic control problem. However, the problem of partial hedging in rough volatility models has not yet been investigated, so this chapter aims to fill this gap.

To this end, we introduce a stochastic control problem under rough volatility models. While the literature has studied stochastic control problems in rough volatility models primarily focusing on portfolio optimization, see Abi Jaber et al. (2021a); Aichinger and Desmettre (2023); Bäuerle and Desmettre (2020); Fouque and Hu (2018a); Fouque and Hu (2018b); Han and Wong (2021); Ma et al. (2023), problems involving hedging or stochastic targets have received less attention. The non-Markovian nature of rough volatility processes poses significant challenges in solving these control problems. In portfolio optimization problems, Aichinger and Desmettre (2023); Fouque and Hu (2018a); Fouque and Hu (2018b) propose first order approximate solutions by relying on martingale distortion transformation of the value function, while Bäuerle and Desmettre (2020); Ma et al. (2023) employ a Markov approximation of the rough volatility models. It is the latter technique that is developed in this chapter to solve the control problem. Relying on several papers Abi Jaber (2019); Abi Jaber and El Euch (2019a); Alfonsi and Kebaier (2024); Bayer and Breneis (2023); Hainaut (2022b), we introduce a Markov multi-

factor approximation of rough volatility models based on the representation of the kernel function in terms of a Laplace transform. Then, we consider the Markov control problem associated with the approximate volatility model and show, with the help of convergence results stated in Abi Jaber (2019); Abi Jaber and El Euch (2019a), that instead of solving the initial non-Markovian problem, we can solve the Markovian problem with negligible error.

The introduced Markov stochastic control problem is similar to a stochastic control problem associated to a partial hedging problem in multivariate stochastic volatility. Previous studies Fouque et al. (2017); Jonsson and Sircar (2002); Ma et al. (2020) have shown that the optimal value function for such problems satisfies a nonlinear partial differential equation that cannot be completely linearized, even by switching to the dual formulation of the problem. There are mainly two techniques developed in the literature to overcome this nonlinearity issue. Fouque et al. (2017); Jonsson and Sircar (2002) consider fast-mean reverting volatility models to propose asymptotic solutions, while Ma et al. (2020) considers a dual control method to provide approximate solutions of the optimal solution. In this chapter, we adopt a similar approach to the dual control method introduced by Ma et al. (2020) to propose approximate solutions of the Markov problem for sub-optimal choices of dual control. We prefer this dual method to the asymptotic approach of Fouque et al. (2017); Jonsson and Sircar (2002), as it applies to broad classes of volatility models and is not restricted to fast mean-reverting settings, as assumed in Fouque et al. (2017); Jonsson and Sircar (2002). Moreover, the dual framework enables the quantification of approximation errors through upper and lower bounds and, in certain models such as the Volterra Heston model, yields explicit sub-optimal hedging strategies.

This chapter is outlined as follows. First, in Section 1.2, the mathematical framework is presented. We introduce the class of volatility models considered and we formulate the partial hedging problem. Next, in Sections 1.3 and 1.4.6, we discuss the multifactor approximation of the rough volatility model and introduce the associated Markov stochastic control problem. Moreover, we demonstrate the convergence of the optimal solution of the Markov problem to the optimal solution of the original problem. Then, in Section 1.5.12, we solve the Markov problem by introducing the Hamilton-Jacobi-Bellman (HJB) equation and deduce that the optimal solution satisfies a nonlinear PDE. Consequently, by expressing the primal solution in terms of the dual solution, we derive approximate solutions using a dual control method. Notably, we provide explicit formulas for sub-optimal partial hedging strategies in the Volterra Heston case. Finally, in Section 1.6, we conclude the chapter by presenting a numerical application that focuses on the partial hedging of linear and vanilla options within the Volterra Heston model.

1.2 Statement of the problem

Consider a finite horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ where $\mathbb{P}$ stands for the real measure, the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the canonical filtration of a two-dimensional Brownian motion (W_S, B_ν) and denotes all information known over time. Assume an arbitrage-free financial market in which we have a cash-account and a risky asset denoted respectively by $(S_t^0)_{0 \leq t \leq T}$ and $(S_t)_{0 \leq t \leq T}$. We suppose that those processes have the following dynamics

$$dS_t^0 = r S_t^0 dt, S_0^0 = 1,$$

and

$$dS_t = \mu_t S_t dt + \sqrt{\nu_t} S_t dW_S(t), S_0 = s_0 > 0,$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where $r \in \mathbb{R}$ is the risk-free interest rate, $\mu_t := r + A\nu_t$, $A \in \mathbb{R}_+^1$ and $(\nu_t)_{0 \leq t \leq T}$ a variance process. The variance process satisfies a stochastic Volterra equation of the form

$$\nu_t = \nu_0 + \int_0^t G(t-s)b(\nu_s)ds + \int_0^t G(t-s)\sigma(\nu_s)dW_\nu(s) \quad (1.2.1)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where $\nu_0 \in \mathbb{R}_+$, $b : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz continuous with $b(0) \geq 0$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ η -Hölder continuous with $\sigma(0) = 0$ and $\eta \in [1/2, 1]$, W_ν is a standard Brownian motion such that $W_\nu = \rho W_S + \sqrt{1 - \rho^2} B_\nu$ for $\rho \in (-1, 1)$ and G a kernel assumed to be completely monotone². The Volterra model (1.2.1) is a general model that incorporates well-known models such, for example, the Volterra Heston model introduced in El Euch and Rosenbaum (2018) with variance process given by

$$\nu_t = \nu_0 + \int_0^t G(t-s)\kappa(\theta - \nu_s)ds + \int_0^t G(t-s)\zeta\sqrt{\nu_s}dW_\nu(s),$$

where $\kappa \in \mathbb{R}_+$ is the speed of mean reversion toward the level $\theta \in \mathbb{R}_+$ and $\zeta \in \mathbb{R}_+$ is the vol-of-vol parameter. In the following, we consider the fractional kernel defined by

$$G(t) := \frac{t^{H-1/2}}{\Gamma(H+1/2)} \quad (1.2.2)$$

where H is the Hurst coefficient such that $H \in (0, 1/2)$ in order to consider rough volatility models. Note that using (Abi Jaber and El Euch, 2019a, Corollary B.2.),

¹We assume that the excess of return of the risky asset is proportional to its variance. This specification is not only analytically convenient but also economically motivated by relation between volatility risk and expected return: higher market volatility is typically associated with larger expected excess returns.

²A Kernel $G(\cdot)$ is completely monotone if it is infinitely differentiable on $(0, +\infty)$ such that $(-1)^j G^{(j)}(t) \geq 0$ for $t > 0$ and $j \geq 0$.

since functions $b(\cdot)$ and $\sigma(\cdot)$ are in particular continuous with linear growth, we can prove the existence of an unconstrained weak solution of the stochastic Volterra equation (1.2.1) when the fractional kernel satisfies (1.2.2). Moreover, the existence of a non-negative solution can be obtained by relying on (Abi Jaber and El Euch, 2019a, Theorem B.4.).

Remark 1.1. *The fractional kernel is chosen to consider rough volatility models and to work in a framework where the volatility process is non-Markovian and not a semimartingale. Nevertheless, the theoretical results developed in this chapter extends naturally to a broader class of completely monotone kernels, including, for example, the shifted fractional kernel introduced in Chapter 2.*

Remark 1.2. *If we consider the rough Heston model with $b(x) = \kappa(\theta - x)$ and $\sigma(x) = \zeta\sqrt{x}$, we can prove the uniqueness of a non-negative weak solution of the stochastic Volterra equation (1.2.1) when the kernel satisfies (1.2.2), see for instance Abi Jaber and El Euch (2019b).*

In this financial market with rough volatility, we are concerned with the hedging of any contingent claim given by $\mathcal{F}_T$ -measurable square-integrable random variable of the form

$$H_T := h(S_T),$$

with $h(\cdot)$ a continuous function. El Euch and Rosenbaum (2018) already tackle the question of hedging in rough volatility environment. They prove that perfect hedging is possible in rough Heston model provided that the forward variance curve can be taken as hedging instrument. However, this assumption is quite strong, which is why we are interested in the question of hedging in a financial market with only underlying assets as hedging instruments. As the market is incomplete, we already know that perfect hedging is not possible for any contingent claim but we can still stay on the safe side by super-hedging the contingent claims. However, super-hedging strategies generally lead to super-hedging prices that are too high to be considered in practice. In cases where the initial capital available is smaller than the super-hedging prices, we know that we are not hedged in 100% of the cases. We can nevertheless define hedging strategies that aim at minimizing a loss arising from the hedging operation. This type of hedging strategy is called partial hedging strategy and was introduced in Föllmer and Leukert (1999); Föllmer and Leukert (2000). It is this kind of strategy that we consider in the following. In this perspective, as usual for hedging problems³, we consider a self-financing hedging portfolio denoted by $(V_t)_{0 \leq t \leq T}$ and defined by investment in the assets available in the market (cash-account and the risky asset). The amount invested at time $t \in [0, T]$ in the risky asset is denoted by ξ_t and the portfolio evolves according to

³For more details on how hedging problems can be tackled in complete and incomplete markets, we refer the reader to the book Bouchard and Chassagneux (2016).

the following SDE

$$\begin{aligned} dV_t &= r (V_t - \xi_t S_t) dt + \xi_t dS_t, \\ V_0 &= v_0, \end{aligned}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The hedging ratio process $\xi := (\xi_t)_{0 \leq t \leq T}$ is admissible if ξ is progressively measurable in regards to $\mathcal{F}_t$ such that $E(\int_0^T \xi_t^2 S_t^2 \nu_t dt) < +\infty$. Similarly, we can also define the Profit and Loss (P&L) at maturity T denoted by π_T and defined by

$$\pi_T := V_T - H_T.$$

Let $\mathcal{R}$ be the set of all progressively measurable processes $(\xi_t)_{0 \leq t \leq T}$ valued in $\mathbb{R}$ such that $E(\int_0^T \xi_t^2 S_t^2 \nu_t dt) < +\infty$, since we already mentioned that, in our framework, perfect hedging is not possible for any given contingent claim, there exist payoffs⁴ H_T for which

$$\nexists (\xi_t)_{0 \leq t \leq T} \in \mathcal{R} \text{ s.t. } \pi_T = 0 \text{ a.s.}$$

We thus consider the partial hedging problem and define in this sense an optimal hedging strategy satisfying the following optimization problem

$$l(s_0, \nu_0, v_0) := \inf_{\xi_t \in \mathcal{R}} \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(H_T, V_T) \right) \quad (1.2.3)$$

where $L(\cdot)$ is a continuous proper convex loss function. Note that different choices can be made for the loss function $L(\cdot)$. We can consider symmetric loss functions, like for example, power loss functions of the form

$$L_{power}(x, y) := \frac{1}{p} (x - y)^p, \quad p = 2k, \quad k \in \mathbb{N}_0, \quad (1.2.4)$$

with the particular case $p = 2$ linked to mean-variance minimization problem. For other types of risk-averse traders, we can consider asymmetric functions, such as, for example, exponential or shortfall loss functions. At this stage, the main problem in solving the introduced stochastic control problem is that the volatility model (1.2.1) with the fractional kernel is neither Markovian nor a semi-martingale. Thus, the principle of dynamic programming cannot be applied directly to solve the stochastic control problem. To overcome this problem, and following the idea of Bäuerle and Desmettre (2020), we will consider a Markovian approximation of our initial problem and then solve the Markovian problem using the principle of dynamic programming.

1.3 Markov approximation

The non-Markovian structure of the rough volatility prevents from directly solving the partial hedging problem using classical stochastic control techniques. However,

⁴Perfect hedging strategies exist for linear-form payoffs.

as shown in Bäuerle and Desmettre (2020) in the case of portfolio optimization, the problem can be solved with a small error by considering a Markov approximation of the volatility process. As shown in several papers Abi Jaber (2019); Abi Jaber and El Euch (2019a); Alfonsi and Kebaier (2024); Bayer and Breneis (2023); Hainaut (2022b); Hainaut (2023), the starting point of the Markovian approximation is the representation of the kernel $G(t)$ in terms of a Laplace transform⁵ such that

$$G(t) = \int_0^{+\infty} e^{-tx} \lambda(dx),$$

where λ is a measure on $\mathbb{R}_+$. For the fractional kernel, we have that

$$\begin{aligned} G(t) &= \frac{t^{H-1/2}}{\Gamma(H+1/2)} \\ &= \frac{1}{\underbrace{\Gamma(H+1/2)\Gamma(1/2-H)}_{:=C_H}} \int_0^{+\infty} e^{-tx} x^{-H-1/2} dx \\ &= C_H \int_0^{+\infty} e^{-tx} x^{-H-1/2} dx, \end{aligned}$$

where for $x > 0$,

$$\lambda(x) = C_H x^{-H-1/2} dx.$$

Then, we approximate the integral by a finite sum and consider the approximate kernel $\hat{G}$ defined by

$$\hat{G}(t) := \sum_{i=1}^n w_i e^{-tx_i}, \quad (1.3.1)$$

where $(w_i)_{i=1,\dots,n}$ are the weights and $(x_i)_{i=1,\dots,n}$ the mean reversion terms that should be appropriately defined, we discuss later on the choice of these parameters. In this way, we can approximate the rough variance process by defining a new stochastic process denoted by $(\hat{\nu}_t)_{0 \leq t \leq T}$ as the unique strong continuous solution to the stochastic Volterra equation (1.2.1) with a kernel $\hat{G}(\cdot)$ such that, for $t \in [0, T]$,

$$\hat{\nu}_t = \nu_0 + \int_0^t \hat{G}(t-s) b(\hat{\nu}_s) ds + \int_0^t \hat{G}(t-s) \sigma(\hat{\nu}_s) dW_\nu(s), \quad (1.3.2)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ where $\nu_0 \in \mathbb{R}_+$, $b : \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz continuous with $b(0) \geq 0$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ η -Hölder continuous with $\sigma(0) = 0$ and $\eta \in [1/2, 1]$. As for the rough variance process, the existence of a non-negative weak solution of (1.3.2) can be obtained by relying on (Abi Jaber and El Euch, 2019a, Theorem B.4). Moreover, as, unlike the fractional kernel, the approximate kernel is smooth, the strong existence and uniqueness of $(\hat{\nu}_t)_{0 \leq t \leq T}$ follow from (Abi Jaber

⁵This representation is possible since $G(\cdot)$ is assumed to be completely monotone.

and El Euch, 2019a, Proposition B.3.). The following proposition states that the stochastic Volterra equation (1.3.2) can be reduced to a n -dimensional stochastic differential equation.

Proposition 1.3.1. *The solution of (1.3.2) is given by*

$$\hat{\nu}_t = \nu_0 + \sum_{i=1}^n w_i \nu_t^i \quad (1.3.3)$$

where $(\nu_t)_{0 \leq t \leq T} := \left((\nu_t^1, \nu_t^2, \dots, \nu_t^n) \right)_{0 \leq t \leq T}$ is solution of the n -dimensional SDE defined by

$$\begin{aligned} \nu_t^i &= - \int_0^t x_i \nu_s^i ds + \int_0^t b(\hat{\nu}_s) ds + \int_0^t \sigma(\hat{\nu}_s) dW_\nu(s), \quad i = 1, \dots, n, \\ \nu_0^i &= 0, \end{aligned} \quad (1.3.4)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Proof. We refer the reader to the proof of (Alfonsi and Kebaier, 2024, Proposition 2.1.). $\square$

Based on the approximate variance process (1.3.2) and its Markov representation induced by SDEs (1.3.4), we can define the Markovian approximation of the stochastic control problem introduced in (1.2.3). First, we consider the approximate process $(S_t^n)_{0 \leq t \leq T}$ for which its SDE can either be written in terms of the approximate volatility process $(\hat{\nu}_t)_{0 \leq t \leq T}$ or in terms of its Markov representation. Thus, for $t \in [0, T]$, the dynamic of $(S_t^n)_{0 \leq t \leq T}$ is given by

$$dS_t^n = \hat{\mu}_t S_t^n dt + \sqrt{\hat{\nu}_t} S_t^n dW_S(t), \quad (1.3.5)$$

but also by

$$dS_t^n = \hat{\mu}_t S_t^n dt + \sqrt{\nu_0 + \sum_{i=1}^n w_i \nu_t^i} S_t^n dW_S(t), \quad (1.3.6)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, with $S_0^n = s_0$ and $\hat{\mu}_t = r + A\hat{\nu}_t = r + A \left(\nu_0 + \sum_{i=1}^n w_i \nu_t^i \right)$.

In the same way, denoting the approximate hedging process by $(\xi_t^n)_{0 \leq t \leq T}$, we define the associated hedging portfolio $(V_t^n)_{0 \leq t \leq T}$ satisfying the following SDE

$$\begin{aligned} dV_t^n &= r(V_t^n - \xi_t^n S_t^n) dt + \xi_t^n dS_t^n, \\ V_0^n &= v_0, \end{aligned} \quad (1.3.7)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where $(\xi_t^n)_{0 \leq t \leq T}$ is admissible if $(\xi_t^n)_{0 \leq t \leq T}$ is a progressively measurable process in regards to $\mathcal{F}_t$ such that $E(\int_0^T (\xi_t^n)^2 (S_t^n)^2 \hat{\nu}_t dt) < +\infty$. The approximate P&L is defined by

$$\pi_T^n := V_T^n - H_T^n, \quad (1.3.8)$$

with $H_T^n := h(S_T^n)$, a $\mathcal{F}_T$ -measurable square-integrable random variable. As the market is incomplete, there exist payoffs for which

$$\nexists (\xi_t^n)_{0 \leq t \leq T} \in \mathcal{R}_n \text{ s.t. } \pi_T^n = 0 \text{ a.s. ,}$$

where $\mathcal{R}_n$ the set of all progressively measurable processes $(\xi_t^n)_{0 \leq t \leq T}$ with regards to $\mathcal{F}_t$ valued in $\mathbb{R}$ such that $E(\int_0^T (\xi_t^n)^2 (S_t^n)^2 \hat{\nu}_t dt) < +\infty$. Thus, we introduce the approximate partial hedging problem that can be written either by considering the dependence on the approximate volatility process $(\hat{\nu}_t)_{0 \leq t \leq T}$ or the dependence on its Markovian representation. In fact, by considering the dynamic (1.3.5) of $(S_t^n)_{0 \leq t \leq T}$ written in terms of $(\hat{\nu}_t)_{0 \leq t \leq T}$, we define the stochastic control problem

$$l_{n, \hat{\nu}}(s_0, \nu_0, v_0) := \inf_{\xi_t^n \in \mathcal{R}_n} \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(H_T^n, V_T^n) \right). \quad (1.3.9)$$

Considering now the dynamic (1.3.6) of $(S_t^n)_{0 \leq t \leq T}$ written in terms of $(\boldsymbol{\nu}_t)_{0 \leq t \leq T}$, we have the Markovian stochastic control problem

$$l_{n, \boldsymbol{\nu}}(s_0, \boldsymbol{\nu}_0, v_0) := \inf_{\xi_t^n \in \mathcal{R}_n} \mathbb{E}_{s_0, \boldsymbol{\nu}_0, v_0}^{\mathbb{P}} \left(L(H_T^n, V_T^n) \right) \quad (1.3.10)$$

such that, for $\boldsymbol{\nu}_0 = \mathbf{0}^n$,

$$l_{n, \hat{\nu}}(s_0, \nu_0, v_0) = l_{n, \boldsymbol{\nu}}(s_0, \boldsymbol{\nu}_0, v_0).$$

Thanks to the approximation of the variance process (1.3.2) and its Markov representation, we obtain a Markovian framework in which we solve the stochastic control problem (1.3.10) using the principle of dynamic programming. Nevertheless, before solving the control problem, it is interesting to consider the question of convergence of the approximate solution $l_{n, \boldsymbol{\nu}}(\cdot)$ toward $l(\cdot)$. Indeed, without proof of convergence, solving the approximate problem would be pointless, this is why we dedicate a section to this issue.

1.4 Convergence results

In this section, we prove a convergence between the approximate solution $l_{n, \boldsymbol{\nu}}(\cdot)$ and $l(\cdot)$. The first step is to prove, the weak convergence of the value at time $t \in [0, T]$ of the approximate variance process $\hat{\nu}_t$ to ν_t . To prove it, we rely on Abi Jaber (2019) and Abi Jaber and El Euch (2019a). First, under specific assumptions on the weights $(w_i)_{i=1, \dots, n}$ and mean reversion terms $(x_i)_{i=1, \dots, n}$, we can prove that $\hat{G}$ converges to G in $L^2_{[0, T]}$.

Assumption 1. Fix $r_n > 1$ and suppose that the weights $(w_i)_{i=1,\dots,n}$ and mean reversion terms $(x_i)_{i=1,\dots,n}$ are given by

$$w_i := \frac{(r_n^{1-\alpha} - 1)r_n^{(\alpha-1)(1+n/2)}}{\Gamma(\alpha)\Gamma(2-\alpha)} r_n^{(1-\alpha)i}, \quad x_i := \frac{1 - \alpha r_n^{2-\alpha} - 1}{2 - \alpha r_n^{1-\alpha} - 1} r_n^{i-1-n/2}, \quad i = 1, \dots, n,$$

with $\alpha := H + 1/2$ for $H \in (0, 1/2)$ and $(r_n)_{n \geq 1}$ satisfying

$$r_n \downarrow 1, \quad n \ln(r_n) \rightarrow \infty,$$

as n goes to infinity.

Remark 1.3. As stated in Abi Jaber (2019), Assumption 1 is satisfied if we consider $(r_n)_{n \geq 1}$ of the form

$$r_n = 1 + 10n^{-0.9}, \quad n \geq 1.$$

Therefore, without loss of generality, we will consider this form of auxiliary terms in the numerical results.

Proposition 1.4.1. (Abi Jaber, 2019, Lemma A.3). Suppose that for all $n \geq 1$, $(w_i)_{i=1,\dots,n}$ and $(x_i)_{i=1,\dots,n}$ satisfy Assumption 1 and that $\hat{G}$ is defined by (1.3.1). Then $\hat{G}$ converges in $L^2_{[0,T]}$ to G when n goes to infinity i.e.

$$\|\hat{G} - G\|_{L^2_{[0,T]}} \rightarrow 0,$$

as n goes to infinity.

Proof. We refer to Abi Jaber (2019) for the proof. $\square$

Lemma 1.4.2. Assume that $n \geq 1$, $(w_i)_{i=1,\dots,n}$ and $(x_i)_{i=1,\dots,n}$ satisfy Assumption 1. Let $\hat{G}(\cdot)$ be the approximate fractional kernel defined by (1.3.1), then there exist positive constants δ and C such that

$$\sup_{n \geq 1} \left(\int_0^h |\hat{G}(s)|^2 ds + \int_0^{T-h} |\hat{G}(h+s) - \hat{G}(s)|^2 ds \right) \leq Ch^{2\delta},$$

for any $t, h \geq 0$ with $t+h \leq T$.

Proof. As $(w_i)_{i=1,\dots,n}$ and $(x_i)_{i=1,\dots,n}$ satisfy Assumption 1, we can rewrite those parameters such that for $i = 1, \dots, n$,

$$w_i = \int_{\eta_{i-1}^n}^{\eta_i^n} \lambda(dx), \quad x_i = \frac{1}{w_i} \int_{\eta_{i-1}^n}^{\eta_i^n} x \lambda(dx)$$

with $\eta_i^n = r_n^{i-n/2}$. As $\eta_0^n \neq 0$, the Assumption 3.1. in Abi Jaber and El Euch (2019a) is not satisfied and therefore the result of (Abi Jaber and El Euch, 2019a,

Lemma 5.2.) cannot be directly applied. However, as mentioned in Abi Jaber (2019), the result of this lemma is deduced by adapting the proof of (Abi Jaber and El Euch, 2019a, Lemma 5.2.) using the same small adjustments highlighted in the proof of (Abi Jaber, 2019, Lemma A.3.). $\square$

Based on Proposition 1.4.1 and Lemma 1.4.2, we now establish a weak convergence of the value at time $t \in [0, T]$ of the approximate variance process $\hat{\nu}_t$ toward ν_t , which is a corollary to (Abi Jaber and El Euch, 2019a, Theorem 3.6).

Remark 1.4. In the subsequent results, the notation $\xrightarrow{\mathcal{L}}$ refers to weak convergence.

Theorem 1.4.3. Assume that $n \geq 1$, $(w_i)_{i=1, \dots, n}$ and $(x_i)_{i=1, \dots, n}$ satisfy Assumption 1. Let $\left((\hat{\nu}_t)_{0 \leq t \leq T} \right)_{n \geq 1}$ be a sequence of unique strong solutions to (1.3.2) and suppose that the stochastic Volterra equation (1.2.1) with fractional kernel admits a unique weak solution, then, $\forall t \in [0, T]$,

$$\hat{\nu}_t = \nu_0 + \sum_{i=1}^n w_i \nu_t^i \xrightarrow{\mathcal{L}} \nu_t$$

as n goes to infinity.

Proof. The proof is similar to the proof of (Abi Jaber and El Euch, 2019a, Theorem 3.5) and is an immediate consequence of (Abi Jaber and El Euch, 2019a, Theorem 3.6) for one dimension. Thus, to prove the theorem, we just need to check that the assumptions of (Abi Jaber and El Euch, 2019a, Theorem 3.6) are satisfied. By Lemma 1.4.2, we know that there exist positive constants δ and C such that

$$\sup_{n \geq 1} \left(\int_0^h |\hat{G}(s)|^2 ds + \int_0^{T-h} |\hat{G}(h+s) - \hat{G}(s)|^2 ds \right) \leq Ch^{2\delta},$$

for any $t, h \geq 0$ with $t+h \leq T$. Thus, we just need to check that

$$\int_0^T |G(s) - \hat{G}(s)|^2 ds \rightarrow 0, \quad \hat{\nu}_0 \rightarrow \nu_0$$

as n goes to infinity. The first convergence is obtained by Proposition 1.4.1 since assumptions of this proposition are fulfilled, we know that $\hat{G}$ converges in $L^2_{[0, T]}$ to G , therefore, we have that

$$\int_0^T |G(s) - \hat{G}(s)|^2 ds \rightarrow 0.$$

Similarly, the second convergence is direct since, by definition, we consider for all $n \geq 1$,

$$\hat{\nu}_0 = \nu_0.$$

As the assumptions of (Abi Jaber and El Euch, 2019a, Theorem 3.6) are valid for one dimension, we can conclude that $\hat{\nu}$ is tight for the uniform topology and any point limit ν is a solution of the Volterra equation (1.2.1). Thus, as we assume that the stochastic Volterra equation (1.2.1) with fractional kernel admits a unique weak solution, we deduce that

$$\hat{\nu}_t \xrightarrow{\mathcal{L}} \nu_t,$$

as n goes to infinity. $\square$

Now that we establish a weak convergence for the value at time $t \in [0, T]$ of the variance process, we can go a step further and show that that value of the approximate processes S_t^n and V_t^n converge weakly respectively toward S_t and V_t .

Proposition 1.4.4. *Assume that $n \geq 1$, $(w_i)_{i=1, \dots, n}$ and $(x_i)_{i=1, \dots, n}$ satisfy Assumption 1. Let $\left((\hat{\nu}_t)_{0 \leq t \leq T} \right)_{n \geq 1}$ a sequence of unique strong solutions to (1.3.2) and suppose that the stochastic Volterra equation (1.2.1) with fractional kernel admits a unique weak solution. Consider the approximate processes $(S_t^n)_{0 \leq t \leq T}$ and $(V_t^n)_{0 \leq t \leq T}$ satisfying SDEs (1.3.5) and (1.3.7) such that $\forall t \in [0, T]$, $\xi_t^n = \xi_t$ a.s., then we have the following convergence results*

$$S_t^n \xrightarrow{\mathcal{L}} S_t,$$

$$V_t^n \xrightarrow{\mathcal{L}} V_t,$$

as n goes to infinity.

Proof. The proof is provided in Section 1.8. $\square$

With Proposition 1.4.4, we have shown that the value at time $t \in [0, T]$ of the approximate processes converge weakly to the value at time $t \in [0, T]$ of rough volatility dependent processes. Using these results, we are now able to show the convergence of the approximate solution to the solution of the control problem under rough volatility. To do so, inspired by Bäuerle and Desmettre (2020), we define $l^\xi(s_0, \nu_0, v_0)$ and $l_n^{\xi^n}(s_0, \nu_0, v_0)$ by

$$l^\xi(s_0, \nu_0, v_0) := \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(h(S_T), V_T(\xi)) \right)$$

$$l_n^{\xi^n}(s_0, \nu_0, v_0) := \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(H_T^n, V_T^n(\xi^n)) \right)$$

such that

$$l(s_0, \nu_0, v_0) = \inf_{\xi_t \in \mathcal{R}} l^\xi(s_0, \nu_0, v_0),$$

$$l_{n, \hat{\nu}}(s_0, \nu_0, v_0) = \inf_{\xi_t^n \in \mathcal{R}_n} l_n^{\xi^n}(s_0, \nu_0, v_0).$$

We first consider a lemma before stating the convergence result we want to achieve.

Lemma 1.4.5. *Assume that $n \geq 1$, $(w_i)_{i=1,\dots,n}$ and $(x_i)_{i=1,\dots,n}$ satisfy Assumption*

1. Let $\left((\hat{v}_t)_{0 \leq t \leq T} \right)_{n \geq 1}$ a sequence of unique strong solutions to (1.3.2) and suppose that the stochastic Volterra equation (1.2.1) with fractional kernel admits a unique weak solution. Fix admissible hedging strategies $(\xi_t)_{0 \leq t \leq T}$, $(\xi_t^n)_{0 \leq t \leq T}$ such that $\forall t \in [0, T]$, $\xi_t^n = \xi_t$ a.s., if the sequence $\left(L(H_T^n, V_T^n) \right)_{n \geq 1}$ is uniformly integrable, then

$$l_n^\xi(s_0, \nu_0, v_0) \rightarrow l^\xi(s_0, \nu_0, v_0),$$

as n goes to infinity.

Proof. Using Proposition 1.4.4, since we assume that $\forall t \in [0, T]$, $\xi_t^n = \xi_t$ a.s., we have that

$$S_t^n \xrightarrow{\mathcal{L}} S_t,$$

$$V_t^n \xrightarrow{\mathcal{L}} V_t,$$

as the loss function $L(\cdot)$ and $h(\cdot)$ are continuous, we deduce that

$$L(H_T^n, V_T^n) \xrightarrow{\mathcal{L}} L(h(S_T), V_T),$$

and the uniform integrability of $\left(L(H_T^n, V_T^n) \right)_{n \geq 1}$ implies that

$$\mathbb{E}^\mathbb{P} \left(L(H_T^n, V_T^n) \right) \rightarrow \mathbb{E}^\mathbb{P} \left(L(h(S_T), V_T) \right),$$

as n goes to infinity. Thus we obtain that

$$l_n^\xi(s_0, \nu_0, v_0) \rightarrow l^\xi(s_0, \nu_0, v_0),$$

as n goes to infinity. □

We are now able to consider the statement of the desired convergence result.

Theorem 1.4.6. *Assume that $n \geq 1$, $(w_i)_{i=1,\dots,n}$ and $(x_i)_{i=1,\dots,n}$ satisfy Assumption*

1. Let $\left((\hat{v}_t)_{0 \leq t \leq T} \right)_{n \geq 1}$ a sequence of unique strong solutions to (1.3.2), suppose that the stochastic Volterra equation (1.2.1) with fractional kernel admits a unique weak solution and that the sequence $\left(L(H_T^n, V_T^n) \right)_{n \geq 1}$ is uniformly integrable.

Let $(\xi_t^{n,*})_{0 \leq t \leq T}$ be the optimal hedging ratio associated to the n -approximate stochastic control problem (1.3.10) with $\nu_0 = \mathbf{0}^n$. For every $\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N$,

$$|l(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)| < \varepsilon,$$

i.e.

$$\lim_{n \rightarrow +\infty} l^{\xi^{n,*}}(s_0, \nu_0, v_0) = l(s_0, \nu_0, v_0).$$

Moreover, for $\nu_0 = \mathbf{0}^n$,

$$\lim_{n \rightarrow +\infty} l_{n,\hat{\nu}}(s_0, \nu_0, v_0) = \lim_{n \rightarrow +\infty} l_{n,\nu}(s_0, \nu_0, v_0) = l(s_0, \nu_0, v_0).$$

Proof. Let fix $\varepsilon > 0$. Suppose that $(\xi_t^{n,*})_{0 \leq t \leq T}$ is the optimal hedging ratio associated to the n -approximate stochastic control problem (1.3.10) with $\nu_0 = \mathbf{0}^n$, as $\forall n \geq 1$,

$$l_{n,\hat{\nu}}(s_0, \nu_0, v_0) = l_{n,\nu}(s_0, \nu_0, v_0),$$

for $\nu_0 = \mathbf{0}^n$, then we have that

$$l_{n,\hat{\nu}}(s_0, \nu_0, v_0) = l^{\xi^{n,*}}_n(s_0, \nu_0, v_0).$$

Using Lemma 1.4.5, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} l_{n,\hat{\nu}}(s_0, \nu_0, v_0) &= \lim_{n,m \rightarrow \infty} l^{\xi^{n,*}}_m(s_0, \nu_0, v_0) \\ &= \lim_{n \rightarrow \infty} l^{\xi^{n,*}}(s_0, \nu_0, v_0) \end{aligned}$$

or equivalently

$$\lim_{n \rightarrow \infty} \left(l_{n,\hat{\nu}}(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0) \right) = 0. \quad (1.4.1)$$

Therefore by definition of the limit, $\exists N_1 \in \mathbb{N}$, such that $\forall n \geq N_1$,

$$|l_{n,\hat{\nu}}(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)| < \frac{\varepsilon}{2}. \quad (1.4.2)$$

Moreover, considering $\underline{l}(s_0, \nu_0, v_0)$ defined by

$$\underline{l}(s_0, \nu_0, v_0) := \lim_{n \rightarrow \infty} l_{n,\hat{\nu}}(s_0, \nu_0, v_0) = \lim_{n \rightarrow \infty} \inf_{\xi_t^n \in \mathcal{R}_n} l^{\xi^n}_n(s_0, \nu_0, v_0),$$

thus $\exists N_2 \in \mathbb{N}$, such that $\forall n \geq N_2$,

$$|\underline{l}(s_0, \nu_0, v_0) - l_{n,\hat{\nu}}(s_0, \nu_0, v_0)| \leq \frac{\varepsilon}{2} \quad (1.4.3)$$

Note that since

$$l(s_0, \nu_0, v_0) = \inf_{\xi_t \in \mathcal{R}} \lim_{n \rightarrow \infty} \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(H_T^n, V_T^n) \right),$$

we have that

$$l(s_0, \nu_0, v_0) \geq \underline{l}(s_0, \nu_0, v_0).$$

By choosing $N := \max(N_1, N_2)$, we have that for $n \geq N$ inequalities (1.4.2) and (1.4.3) are satisfied. In this case, we have that

$$\begin{aligned}
 |l(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)| &\leq |l(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)| \\
 &\leq \underbrace{|l(s_0, \nu_0, v_0) - l_{n,\hat{\nu}}(s_0, \nu_0, v_0)|}_{< \frac{\varepsilon}{2}} \\
 &\quad + \underbrace{|l_{n,\hat{\nu}}(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)|}_{< \frac{\varepsilon}{2}} \\
 &< \varepsilon.
 \end{aligned}$$

Therefore, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|l(s_0, \nu_0, v_0) - l^{\xi^{n,*}}(s_0, \nu_0, v_0)| < \varepsilon$$

or equivalently

$$\lim_{n \rightarrow +\infty} l^{\xi^{n,*}}(s_0, \nu_0, v_0) = l(s_0, \nu_0, v_0).$$

Moreover, using (1.4.1), we deduce that

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} l_{n,\nu}(s_0, \nu_0, v_0) &= \lim_{n \rightarrow +\infty} l_{n,\hat{\nu}}(s_0, \nu_0, v_0) = \lim_{n \rightarrow +\infty} l^{\xi^{n,*}}(s_0, \nu_0, v_0) \\
 &= l(s_0, \nu_0, v_0).
 \end{aligned}$$

This completes the proof as we have proved the two stated convergence results. $\square$

These convergence results are crucial for the following. On the one hand, it means that the optimal hedging ratio $(\xi_t^{n,*})_{0 \leq t \leq T}$ associated to the n -approximate stochastic control problem (1.3.10) is ε -optimal for the original problem. On the other hand, we know that the solution of the approximate control problem $l_{n,\nu}(\cdot)$ converges toward the solution of the initial control problem $l(\cdot)$. Therefore, thanks to these results, we know that instead of solving the original non-Markovian problem, we can solve the approximate Markovian problem with an error that can be relatively small if n is large enough.

1.5 Solution of the approximate Markovian problem

We have just shown that we can solve the optimal problem with a small error by solving the Markovian problem. In this section, we thus solve this problem using classical dynamic programming techniques and more precisely the Hamilton-Jacobi-Bellman (HJB) equation. The Markovian problem is equivalent to solving a partial hedging problem in a multidimensional stochastic volatility environment.

The partial hedging problem under stochastic (one dimensional) volatility model has already been investigated in the literature by Jonsson and Sircar (2002). From Jonsson and Sircar (2002), it follows that the problem requires solving a nonlinear partial differential equation and therefore the solution cannot be reduced to an expectation by the Feynman-Kac theorem. In our multidimensional volatility case, we will also observe that the control problem involves solving a nonlinear PDE making it quite complex and not allowing to deduce an explicit form of the optimal solution. Inspired by Ma et al. (2020), we propose a dual control method to obtain approximate solutions of the problem.

Assumption 2. *For the rest of the chapter, we still assume that $b(\cdot)$ is Lipschitz continuous with $b(0) \geq 0$ and $\sigma(\cdot)$ is η -Hölder continuous with $\sigma(0) = 0$, $\eta \in [1/2, 1]$, but we additionally assume that these functions satisfy sufficient conditions for the approximate volatility process $(\hat{\nu}_t)_{0 \leq t \leq T} = (\nu_0 + \sum_{i=1}^n w_i \nu_t^i)_{0 \leq t \leq T}$ to remain strictly positive i.e. for $t \in [0, T]$,*

$$\hat{\nu}_t > 0,$$

almost surely. As shown in Appendix 1.9.1, this assumption is satisfied in the Heston case i.e. $b(x) = \kappa(\theta - x)$ and $\sigma(x) = \zeta\sqrt{x}$ if

$$2\kappa\theta > \zeta^2 \sum_{i=1}^n w_i, \quad \nu_0 > 0, \quad (1.5.1)$$

such that for all $\varepsilon > 0$, $i \in \{1, \dots, n\}$ and $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{P}} \left(1_{\{t \leq \tau_\varepsilon\}} w_i \nu_t^i \hat{\nu}_t^{-(m+1)} \right) \leq \mathbb{E}^{\mathbb{P}} \left(1_{\{t \leq \tau_\varepsilon\}} \hat{\nu}_t^{-m} \right), \quad (1.5.2)$$

with $\tau_\varepsilon := \min\{t \geq 0 : \hat{\nu}_t \leq \varepsilon\}$ and $m := \frac{2\kappa\theta - \zeta^2 \sum_{i=1}^n w_i}{\zeta^2 \sum_{i=1}^n w_i}$.

Remark 1.5. *The Feller condition (1.5.1) depends on n and therefore may fail to hold for sufficiently large values of n , especially in regimes where the Hurst index H is small (see Figure 1.1 in Section 1.6). Nevertheless, it should be kept in mind that this condition is only sufficient. Hence, even when it is not satisfied, the process may still remain strictly positive.*

For $n \geq 1$, the approximate partial hedging problem (1.3.10) involves solving a Markovian stochastic control problem with value function of the form⁶:

$$l_n(t, s, \boldsymbol{\nu}, v) := \inf_{\xi_t^n \in \mathcal{R}_n} \mathbb{E}^{\mathbb{P}} \left(L(H_T^n, V_T^n) | S_t^n = s, \boldsymbol{\nu}_t = \boldsymbol{\nu}, V_t^n = v \right), \quad (1.5.3)$$

for $(t, s, \boldsymbol{\nu}, v) \in [0, T] \times \mathbb{R}_+ \times \underbrace{\{\boldsymbol{\nu} \in \mathbb{R}^n : \nu_0 + \sum_{i=1}^n w_i \nu_i > 0\}}_{:=S} \times \mathbb{R}$ and $\mathcal{R}_n$ the set of

all progressively measurable processes $(\xi_t^n)_{0 \leq t \leq T}$ with regards to $\mathcal{F}_t$ valued in $\mathbb{R}$

⁶For $t = 0$, we have $l_n(t = 0, s_0, \boldsymbol{\nu}_0, v_0) = l_{n, \boldsymbol{\nu}}(s_0, \boldsymbol{\nu}_0, v_0)$.

such that $\mathbb{E}(\int_0^T (\xi_t^n)^2 (S_t^n)^2 \hat{\nu}_t dt) < +\infty$. As the stochastic control problem (1.5.3) is Markovian, we can solve it using the HJB equation. Assuming that $l_n(t, s, \boldsymbol{\nu}, v)$ is locally bounded on $[0, T) \times \mathbb{S}$ and the hamiltonian associated to the problem (1.5.3) is finite and continuous on $[0, T) \times \mathbb{S} \times \mathbb{R}^{2+n} \times \mathcal{S}_{2+n}$, classic results from dynamic programming, see in (Touzi, 2012, Theorems 7.4. and 7.6.), imply that $l_n(t, s, \boldsymbol{\nu}, v)$ is the viscosity solution of the following HJB⁷:

$$\begin{aligned} & -\partial_t l_n - \inf_{\xi^n \in \mathbb{R}} \left\{ \partial_s l_n \hat{\mu} s + \sum_{i=1}^n \partial_{\nu_i} l_n \left(-x_i \nu_i + b(\hat{\nu}) \right) + \partial_v l_n \left(r v + (\hat{\mu} - r) \xi^n s \right) \right. \\ & \quad + \frac{1}{2} \partial_{ss} l_n \hat{\nu} s^2 + \frac{1}{2} \partial_{vv} l_n (\xi^n)^2 \hat{\nu} s^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} l_n \sigma^2(\hat{\nu}) + \partial_{sv} l_n \xi^n \hat{\nu} s^2 \\ & \quad \left. + \rho \sum_{i=1}^n \partial_{\nu_i s} l_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu}) + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \xi^n \sqrt{\hat{\nu}} s \sigma(\hat{\nu}) \right\} = 0, \\ & l_n(T, s, \boldsymbol{\nu}, v) = L(h(s), v). \end{aligned} \quad (1.5.4)$$

Proposition 1.5.1. *The primal optimal control $(\xi_t^{n,*})_{0 \leq t \leq T}$ is given by*

$$\xi_t^{n,*} = - \frac{\partial_v l_n (\hat{\mu}_t - r) S_t^n + \partial_{sv} l_n \hat{\nu}_t (S_t^n)^2 + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \sqrt{\hat{\nu}_t} S_t^n \sigma(\hat{\nu}_t)}{\partial_{vv} l_n \hat{\nu}_t (S_t^n)^2}$$

with $\hat{\nu}_t = \nu_0 + \sum_{i=1}^n w_i \nu_t^i$ and the associated solution $l_n(\cdot)$ solves a nonlinear PDE of the form

$$\begin{aligned} 0 &= \partial_t l_n + \mathcal{L}_{s, \boldsymbol{\nu}} l_n + \partial_v l_n r v \\ & \quad - \frac{\left(\partial_v l_n (\hat{\mu} - r) s + \partial_{sv} l_n \hat{\nu} s^2 + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu}) \right)^2}{2 \partial_{vv} l_n \hat{\nu} s^2}, \\ & l_n(T, s, \boldsymbol{\nu}, v) = L(h(s), v). \end{aligned} \quad (1.5.5)$$

with $\mathcal{L}_{s, \boldsymbol{\nu}}$ the generator associated to S^n and $\boldsymbol{\nu}$.

Proof. To deduce the optimal control associated with the Markovian control problem, we solve the HJB equation (1.5.4). The HJB equation has a solution if the infimum is different from $-\infty$, it is the case if $\partial_{vv} l_n \geq 0$. In this case, assuming

⁷For the sake of clarity, we write l_n instead of $l_n(t, s, \boldsymbol{\nu}, v)$ and $\hat{\nu}$ instead of $\nu_0 + \sum_{i=1}^n w_i \nu_i$

that $\partial_{vv}l_n \geq 0$, the infimum is obtained by the first order condition i.e. by cancelling the derivative of the function with respect to ξ^n . Therefore, the optimal ξ^n denoted by $\xi^{n,*}$ is such that

$$\partial_{vv}l_n \xi^n \hat{\nu} s^2 + \partial_v l_n (\hat{\mu} - r)s + \partial_{sv}l_n \hat{\nu} s^2 + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu}) = 0,$$

as $\hat{\nu} > 0$, we deduce that

$$\xi^{n,*} = -\frac{\partial_v l_n (\hat{\mu} - r)s + \partial_{sv}l_n \hat{\nu} s^2 + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu})}{\partial_{vv}l_n \hat{\nu} s^2}.$$

Plugging the optimal control into the HJB equation and consider the generator $\mathcal{L}_{s,\nu}$ defined by

$$\begin{aligned} \mathcal{L}_{s,\nu} := & \partial_s \hat{\mu} s + \sum_{i=1}^n \partial_{\nu_i} \left(-x_i \nu_i + b(\hat{\nu}) \right) + \frac{1}{2} \partial_{ss} \hat{\nu} s^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} \sigma^2(\hat{\nu}) \\ & + \rho \sum_{i=1}^n \partial_{\nu_i s} \sqrt{\hat{\nu}} s \sigma(\hat{\nu}), \end{aligned}$$

the optimal solution satisfies the nonlinear PDE given by

$$\partial_t l_n + \mathcal{L}_{s,\nu} l_n + \partial_v l_n r v - \frac{\left(\partial_v l_n (\hat{\mu} - r)s + \partial_{sv}l_n \hat{\nu} s^2 + \rho \sum_{i=1}^n \partial_{\nu_i v} l_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu}) \right)^2}{2 \partial_{vv}l_n \hat{\nu} s^2} = 0.$$

□

The PDE satisfied by the optimal solution is nonlinear, therefore we cannot reduce l_n as an expectation using the Feynman-Kac theorem. One could in principle attempt to solve the PDE numerically using finite-difference methods. However, this quickly becomes computationally intractable in the present high-dimensional setting, where n may be large.

At this stage, machine learning approaches could be considered for the numerical resolution of the problem. One may first mention the work of Bertoli and Bishop (2015), which combines dynamic programming with a Least Squares Monte Carlo (LSMC) method to solve the control problem. Physics-Informed Neural Networks (PINNs) could also be considered as an alternative approach for approximating the solution of the nonlinear PDE, as proposed for instance by Dupret and Hainaut (2026) in the context of control problems. Nevertheless, such methods can also be computationally unstable in very high dimensions. Along the same lines,

one may also mention the recent work of Hainaut and Dupret (2025), which proposes a novel approach combining policy iteration with Gaussian process regression to solve stochastic control problems. Another possible approach would consist in exploiting the connection between nonlinear PDEs and backward stochastic differential equations (BSDEs), and then solving the associated BSDE numerically using regression-based methods or Deep BSDE techniques, see for instance Pham et al. (2021). However, these approaches remain computationally demanding and do not provide explicit expressions for the optimal strategy.

As already motivated previously, our objective here is instead to adopt a dual control approach in order to derive an explicit suboptimal hedging strategy together with quantitative error controls. The first step therefore consists in switching to the dual formulation, which partially linearizes the original problem and leads to a more tractable framework for the subsequent approximation analysis.

To this end, we apply the Legendre-Fenchel transform to the problem (1.5.3) and consider the concave dual $\hat{l}_n(\cdot)$ of $l_n(\cdot)$ with respect to the variable v as the additive inverse of the Legendre-Fenchel transform, such that

$$\begin{aligned}\hat{l}_n(t, s, \boldsymbol{\nu}, z) &:= - \sup_v \{zv - l_n(t, s, \boldsymbol{\nu}, v)\}, \\ &= \inf_v \{l_n(t, s, \boldsymbol{\nu}, v) - zv\},\end{aligned}$$

for $(t, s, \boldsymbol{\nu}, z) \in [0, T] \times \mathbb{S}$. We observe that as $l_n(t, s, \boldsymbol{\nu}, v)$ is convex in v then $\hat{l}_n(t, s, \boldsymbol{\nu}, z)$ is concave in z . We also associate the terminal value to the dual solution $\hat{l}_n$ given by

$$\hat{l}_n(T, s, \boldsymbol{\nu}, z) = \hat{L}(h(s), z) = \inf_v \{L(h(s), v) - zv\}.$$

Based on the PDE satisfied by the primal solution $l_n(t, s, \boldsymbol{\nu}, v)$, we deduce the PDE satisfied by $\hat{l}_n(t, s, \boldsymbol{\nu}, z)$.

Proposition 1.5.2. *The dual solution $\hat{l}_n(t, s, \boldsymbol{\nu}, z)$ satisfies the nonlinear PDE*

$$\begin{aligned}0 = & \partial_t \hat{l}_n + \mathcal{L}_{s, \boldsymbol{\nu}} \hat{l}_n - zr \partial_z \hat{l}_n + \frac{1}{2\hat{\nu} s^2} z^2 (\hat{\mu} - r)^2 s^2 \partial_{zz} \hat{l}_n - z (\hat{\mu} - r) s \partial_{sz} \hat{l}_n, \\ & - \frac{1}{\sqrt{\hat{\nu}}} \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n z (\hat{\mu} - r) \sigma(\hat{\nu}) - \frac{1}{2\partial_{zz} \hat{l}_n} \sigma(\hat{\nu})^2 (1 - \rho^2) \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i z} \hat{l}_n \partial_{\nu_j z} \hat{l}_n,\end{aligned}\tag{1.5.6}$$

with the associated terminal value $\hat{l}_n(T, s, \boldsymbol{\nu}, z) = \hat{L}(h(s), z)$.

Proof. The proof is provided in Section 1.8. □

This dual solution is also the solution of a stochastic control problem. To prove it, we introduce the dual process $(Z_t)_{0 \leq t \leq T}$ controlled by the dual control process $(\gamma_t)_{0 \leq t \leq T}$ and defined, for $t \in [0, T]$, by the following SDE

$$dZ_t = -rZ_t dt - Z_t \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} dW_s(t) + \gamma_t dB_\nu(t),$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The dual control process $(\gamma_t)_{0 \leq t \leq T}$ is admissible if $(\gamma_t)_{0 \leq t \leq T}$ is a progressively measurable and square integrable process in regards to $\mathcal{F}_t$. We now define the dual stochastic control problem. In addition, we show that there is no duality gap as the primal solution can be written in terms of the dual solution.

Proposition 1.5.3. *The dual solution $\hat{l}_n(t, s, \nu, z)$ is the solution of a stochastic control problem such that*

$$\hat{l}_n(t, s, \nu, z) = \sup_{\gamma_t \in \mathcal{D}} \mathbb{E}_{t,s,\nu,z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T) \right),$$

with $\mathcal{D}$ the set of all progressively measurable and square integrable processes in regards to $\mathcal{F}_t$ valued in $\mathbb{R}$. Moreover the optimal dual control $(\gamma_t^*)_{0 \leq t \leq T}$ is given by

$$\gamma_t^* = -\sigma(\hat{\nu}_t) \sqrt{1 - \rho^2} \sum_{i=1}^n \frac{\partial_{\nu_i z} \hat{l}_n}{\partial_{zz} \hat{l}_n}. \quad (1.5.7)$$

Proof. Assume that $\hat{l}_n^{bis}(t, s, \nu, z)$ is defined by

$$\hat{l}_n^{bis}(t, s, \nu, z) := \sup_{\gamma_t \in \mathcal{D}} \mathbb{E}_{t,s,\nu,z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T) \right). \quad (1.5.8)$$

We just have to prove that the HJB equation associated to $\hat{l}_n^{bis}$ matches the PDE (1.5.6). The HJB equation associated to the control problem (1.5.8) is given by

$$\begin{aligned} 0 &= \partial_t \hat{l}_n^{bis} + \mathcal{L}_{s,\hat{\nu}} \hat{l}_n^{bis} - z r \partial_z \hat{l}_n^{bis} \\ &+ \sup_{\gamma \in \mathbb{R}} \left\{ \frac{1}{2} \partial_{zz} \hat{l}_n^{bis} \left(z^2 \frac{(\hat{\mu} - r)^2}{\hat{\nu}^2} + \gamma^2 \right) - \partial_{zs} \hat{l}_n^{bis} z \frac{(\hat{\mu} - r)}{\sqrt{\hat{\nu}}} \sqrt{\hat{\nu}} s \right. \\ &\left. + \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n^{bis} \left(-\rho z \frac{(\hat{\mu} - r)}{\sqrt{\hat{\nu}_t}} \sigma(\hat{\nu}) + \sqrt{1 - \rho^2} \gamma \sigma(\hat{\nu}) \right) \right\}, \\ \hat{l}_n^{bis}(T, s, \nu, z) &= \hat{L}(h(s), z). \end{aligned}$$

The supremum is different of $+\infty$ if $\partial_{zz} \hat{l}_n^{bis} \leq 0$. In this case, using the first order condition, the optimal dual control $(\gamma_t^*)_{0 \leq t \leq T}$ is given by

$$\gamma^* = -\frac{\sqrt{1 - \rho^2} \sigma(\hat{\nu}) \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n^{bis}}{\partial_{zz} \hat{l}_n^{bis}}.$$

Thus, $\hat{l}_n^{bis}$ is the solution of the following PDE

$$\begin{aligned} 0 = & \partial_t \hat{l}_n^{bis} + \mathcal{L}_{s,\nu} \hat{l}_n^{bis} - z r \partial_z \hat{l}_n^{bis} + \frac{1}{2} \partial_{zz} \hat{l}_n^{bis} z^2 \frac{(\hat{\mu} - r)^2}{\hat{\nu}^2} - \partial_{zs} \hat{l}_n^{bis} z (\hat{\mu} - r) s \\ & - \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n^{bis} \rho z \frac{(\hat{\mu} - r)}{\sqrt{\hat{\nu}}} \sigma(\hat{\nu}) - \frac{1}{2 \partial_{zz} \hat{l}_n^{bis}} \sigma^2(\hat{\nu}) (1 - \rho^2) \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i z} \hat{l}_n^{bis} \partial_{\nu_j z} \hat{l}_n^{bis}, \end{aligned} \quad (1.5.9)$$

We can observe that (1.5.9) is exactly the same PDE as (1.5.6) and as the two PDE's have the same terminal value, we can conclude by unicity that

$$\hat{l}_n(t, s, \nu, z) = \hat{l}_n^{bis}(t, s, \nu, z) = \sup_{\gamma_t \in \mathcal{D}} \mathbb{E}_{t,s,\nu,z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T) \right).$$

In this case, the optimal dual control is given by

$$\gamma_t^* = - \frac{\sqrt{1 - \rho^2} \sigma(\hat{\nu}_t) \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n}{\partial_{zz} \hat{l}_n}.$$

□

Proposition 1.5.4. *By choosing $z(t, s, \nu, v)$ solution of*

$$\partial_z \hat{l}_n(t, s, \nu, z) + v = 0, \quad (1.5.10)$$

then

$$l_n(t, s, \nu, v) = \hat{l}_n(t, s, \nu, z) + zv, \quad (1.5.11)$$

with $z = z(t, s, \nu, v)$ the value at time t of the dual process $(Z_t)_{0 \leq t \leq T}$. Moreover, the optimal primal control $(\xi_t^{n,*})_{0 \leq t \leq T}$ can be expressed in term of dual solution such that, for $t \in [0, T]$,

$$\xi_t^{n,*} = \frac{z(t, S_t^n, \nu_t, V_t^n) \partial_{zz} \hat{l}_n (\hat{\mu}_t - r) S_t^n - \partial_{sz} \hat{l}_n \hat{\nu}_t (S_t^n)^2 - \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n \sqrt{\hat{\nu}_t} S_t^n \sigma(\hat{\nu}_t)}{\hat{\nu}_t (S_t^n)^2}, \quad (1.5.12)$$

with $z(t, s, \nu, v)$ solution of (1.5.10).

Proof. Consider $l_n^{bis}(t, s, \nu, v)$ to be the dual of the dual solution and defined by

$$l_n^{bis}(t, s, \nu, v) := \sup_z \left\{ \hat{l}_n(t, s, \nu, z) + zv \right\}.$$

Our goal is to prove that the dual of the dual is the primal. Using the first order condition since $\hat{l}_n(t, s, \nu, z)$ is concave in z , we have that $z(t, s, \nu, v)$ is solution of

$$\partial_z \hat{l}_n(t, s, \nu, z) + v = 0, \quad (1.5.13)$$

in this case l_n^{bis} reduces to

$$l_n^{bis}(t, s, \boldsymbol{\nu}, v) = \hat{l}_n(t, s, \boldsymbol{\nu}, z) + zv$$

with z satisfying (1.5.13). Now, we just have to prove that $l_n^{bis}(t, s, \boldsymbol{\nu}, v) = l_n(t, s, \boldsymbol{\nu}, v)$. The proof of this equality is similar to the proof of Proposition 1.5.2 this is why we have decided not to go into detail but it is easy to show that $l_n^{bis}(t, s, \boldsymbol{\nu}, v)$ satisfies the same PDE then the primal solution $l_n(t, s, \boldsymbol{\nu}, v)$ given by (1.5.5). Moreover, the two PDEs have the same terminal value. In fact, we can rewrite the terminal value of l_n^{bis} as

$$\begin{aligned} L^{bis}(h(s), v) &= \sup_z \left\{ \inf_v \left(L(h(s), v) - zv \right) + zv \right\} \\ &= \sup_z \left\{ zv - \sup_v \left(zv - L(h(s), v) \right) \right\} \end{aligned}$$

As $\sup_v \left(zv - L(h(s), v) \right)$ is the Legendre transform of $L(h(s), v)$, $L^{bis}(h(s), v)$ is the Legendre transform of the Legendre transform of $L(h(s), v)$. By the Fenchel-Moreau Theorem, as $L(\cdot)$ is a proper continuous convex function, we obtain that

$$L^{bis}(h(s), v) = L(h(s), v).$$

Therefore, as $l_n(t, s, \boldsymbol{\nu}, v)$ and $l_n^{bis}(t, s, \boldsymbol{\nu}, v)$ satisfy the same PDE with the same terminal value, we conclude by unicity that

$$l_n^{bis}(t, s, \boldsymbol{\nu}, v) = l_n(t, s, \boldsymbol{\nu}, v).$$

Thus we obtain that given $z(t, s, \boldsymbol{\nu}, v)$ is solution of

$$\partial_z \hat{l}_n(t, s, \boldsymbol{\nu}, z) + v = 0,$$

then

$$l_n(t, s, \boldsymbol{\nu}, v) = \hat{l}_n(t, s, \boldsymbol{\nu}, z) + zv.$$

Finally, for $t \in [0, T]$, using the form of the optimal primal control $\xi_t^{n,*}$ and the expression of the primal value $l_n(\cdot)$ as function of the dual value $\hat{l}_n(\cdot)$, we deduce that

$$\xi_t^{n,*} = \frac{z(t, S_t^n, \boldsymbol{\nu}_t, V_t^n) \partial_{zz} \hat{l}_n(\hat{\mu}_t - r) S_t^n - \partial_{sz} \hat{l}_n \hat{\nu}_t (S_t^n)^2 - \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n \sqrt{\hat{\nu}_t} S_t^n \sigma(\hat{\nu}_t)}{\hat{\nu}_t (S_t^n)^2}$$

□

As the primal solution and the optimal primal control are functions of the dual solution, we can derive those expressions in the case where the dual solution admits

a closed formula. However, in our case, we observe that by switching to the dual problem, although the nonlinear term of the PDE is less important, the PDE (1.5.6) satisfied by the dual solution $\hat{l}_n$ remains nonlinear. Thus, as the nonlinearity problem persists, we are not able, in general, to reduce the dual solution as an expectation. The partial hedging problem is still complicated to solve. There is nevertheless a specific case for which a closed formula of the dual solution can be obtained. Indeed, if we consider a linear payoff defined as

$$H_T^{linear} := \alpha + \beta S_T, \quad (1.5.14)$$

a power loss function and a Volterra Heston model then the solution of the dual problem is obtained by closed formula. This particular case is a toy case since most of the payoffs are generally not linear functions of the underlyings. However, it will allow to quantify the errors made when considering sub-optimal hedging strategies deduced by the dual control method and therefore to benchmark this dual control approach.

Lemma 1.5.5. *Suppose a power loss of the form $L(h, v) = \frac{1}{p}(h - v)^p$ with $p = 2k$, $k \in \mathbb{N}_0$, then*

$$\hat{L}(h(s), z) = -\frac{1}{q}z^q - h(s)z,$$

with $q = \frac{p}{p-1}$.

Proof. The proof is provided in Section 1.8. □

Proposition 1.5.6. *Consider a linear payoff H_T^{linear} given by (1.5.14), a power loss of the form $L(h(s), v) = \frac{1}{p}(h(s) - v)^p$ with $p = 2k$, $k \in \mathbb{N}_0$, suppose that the volatility is modeled by a Volterra Heston model i.e. $b(x) = \kappa(\theta - x)$ and $\sigma(x) = \zeta\sqrt{x}$ with $2\kappa\theta > \zeta^2 \sum_{i=1}^n w_i$ and $\nu_0 > 0$ such that (1.5.2) is satisfied. Therefore,*

$$\hat{l}_n(t, s, \boldsymbol{\nu}, z) = -\frac{z^q}{q} \exp\left(C_t + \sum_{i=1}^n D_t^i \nu^i\right) - \left(e^{-r(T-t)}\alpha + \beta s\right)z,$$

where C_t and $(D_t)_{i=1, \dots, n}$ are time-dependent functions solution of Riccati ODEs given respectively by

$$\begin{aligned} \partial_t C_t &= rq - \frac{1}{2}q(q-1)A^2\nu_0 - \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) - q\rho A\zeta\nu_0\right) \\ &\quad - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \zeta^2 \nu_0 \left(1 - (1 - \rho^2) \frac{q}{q-1}\right), \\ C_T &= 0, \end{aligned}$$

and for $i = 1, \dots, n$,

$$\begin{aligned} \partial_t D_t^i &= x_i D_t^i + w_i \sum_{j=1}^n D_t^j (\kappa + q \rho A \zeta) - \frac{1}{2} \zeta^2 w_i \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \left(1 - (1 - \rho^2) \frac{q}{q-1} \right) \\ &\quad - \frac{1}{2} w_i q(q-1) A^2, \\ D_T^i &= 0. \end{aligned}$$

Moreover, the primal solution is given by

$$l_n(t, s, \boldsymbol{\nu}, v) = -\frac{z(t, s, \boldsymbol{\nu}, v)^q}{q} \exp \left(C_t + \sum_{i=1}^n D_t^i \nu^i \right) - z(t, s, \boldsymbol{\nu}, v) \left(e^{-r(T-t)} \alpha + \beta s - v \right),$$

with

$$z(t, s, \boldsymbol{\nu}, v) = \left(v - \left(e^{-r(T-t)} \alpha + \beta s \right) \right)^{\frac{1}{q-1}} \exp \left(-\frac{1}{q-1} \left(C_t + \sum_{i=1}^n D_t^i \nu^i \right) \right)$$

and the associated optimal hedging ratio is such that

$$\xi_t^{n,*} = \frac{1}{\hat{\nu}_t S_t^n} \left(e^{-r(T-t)} \alpha + \beta S_t^n - V_t^n \right) \left(\frac{1}{p-1} (\hat{\mu}_t - r) - \rho \zeta \hat{\nu}_t \sum_{i=1}^n D_t^i \right) + \beta.$$

Proof. The full proof is in Section 1.8. Here is a summary of the steps in the proof. Firstly, to obtain the form of $\hat{l}_n(\cdot)$, we just need to consider the following ansatz

$$\hat{l}_n(t, s, \boldsymbol{\nu}, z) = -\frac{z^q}{q} \exp \left(C_t + \sum_{i=1}^n D_t^i \nu^i \right) - \left(e^{-r(T-t)} \alpha + \beta s \right) z,$$

and plug it into PDE (1.5.6). Then using the form of $\hat{l}_n(\cdot)$ and the relation (1.5.11) between the primal and the dual, we deduce the form of $l_n(\cdot)$. Finally, the form of the optimal hedging ratio is deduced using the form (1.5.12) of $\xi_t^{n,*}$ and the closed form of the dual solution. $\square$

Remark 1.6. Using the same assumptions as Proposition 1.5.6 but instead of the Volterra Heston model, consider the classical Heston model, i.e. $n = 1$, then C_t and D_t admit closed formulas, see for instance Appendix A in Ma et al. (2020).

For linear payoffs, the nonlinearity of the PDE satisfied by the dual solution⁸ disappears because the Vega⁹ is zero which is not the case for more general payoffs. As it is not possible to obtain explicit forms of the primal value function and the

⁸In the Volterra Heston case with a power loss function.

⁹Measure of option's price sensitivity to changes in the volatility of the risky asset.

optimal hedging ratios for general payoffs, we are interested in deducing approximate solutions close enough to the optimal one. To approximate the solution of our problem in the general case, we rely on the expression of the primal as a function of the dual. For that, define a set $U \subseteq \mathbb{R}$ such that U is a convex compact subset of $\mathbb{R}$ with non-empty interior and denote $\mathcal{U}$ a set of progressively measurable and square integrable processes valued in U such that $\mathcal{U} \subseteq \mathcal{D}$. Using the relation between the primal and the dual solution, we deduce that

$$\begin{aligned} l_n(t, s, \boldsymbol{\nu}, v) &= \sup_z \left\{ \sup_{\gamma_t \in \mathcal{D}} \mathbb{E}^{\mathbb{P}}_{t, s, \boldsymbol{\nu}, z} \left(\hat{L}(H_T^n, Z_T) \right) + zv \right\} \\ &\geq \sup_z \left\{ \sup_{\gamma_t \in \mathcal{U}} \mathbb{E}^{\mathbb{P}}_{t, s, \boldsymbol{\nu}, z} \left(\hat{L}(H_T^n, Z_T) \right) + zv \right\}. \end{aligned} \quad (1.5.15)$$

Based on inequality (1.5.15) and inspired by the dual control method stated in Ma et al. (2020), we will define lower and upper bounds for the primal solution of the Markov partial hedging problem. For that, for every fixed admissible dual control $(\gamma_t)_{0 \leq t \leq T}$, we define

$$Y(t, s, \boldsymbol{\nu}, z; \gamma) := \mathbb{E}^{\mathbb{P}}_{t, s, \boldsymbol{\nu}, z} \left(\hat{L}(H_T^n, Z_T(\gamma)) \right), \quad (1.5.16)$$

We derive now a theorem that states how to deduce upper and lower bounds for the primal solution.

Theorem 1.5.7. *Let $\mathcal{U} \subseteq \mathcal{D}$ be a set of admissible dual controls and define $\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z)$ by*

$$\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) := \sup_{\gamma_t \in \mathcal{U}} Y(t, s, \boldsymbol{\nu}, z; \gamma) \leq \hat{l}_n(t, s, \boldsymbol{\nu}, z).$$

Therefore, defining $\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v)$ by

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) := \sup_z \left\{ \hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) + zv \right\},$$

we have that

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) \leq l_n(t, s, \boldsymbol{\nu}, v).$$

Moreover, suppose that $\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v)$ is twice continuously differentiable, strictly concave and $z(t, s, \boldsymbol{\nu}, v)$ is the solution of

$$\partial_z \hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) + v = 0. \quad (1.5.17)$$

We define the primal control function by $\bar{\xi}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z)$ such that

$$\bar{\xi}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = \frac{z(t, s, \boldsymbol{\nu}, v) \partial_{zz} \hat{l}_n^{\mathcal{U}}(\hat{\mu} - r)s - \partial_{sz} \hat{l}_n^{\mathcal{U}} \hat{\nu} s^2 - \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n^{\mathcal{U}} \sqrt{\hat{\nu}} s \sigma(\hat{\nu})}{\hat{\nu} s^2}, \quad (1.5.18)$$

with $z(t, s, \boldsymbol{\nu}, v)$ solution of (1.5.17). We consider the approximate hedging process

$$\xi_t^{n, \mathcal{U}} := \bar{\xi}_t^{\mathcal{U}}(t, S_t^n, \boldsymbol{\nu}_t, V_t^{n, \mathcal{U}}) \quad (1.5.19)$$

and the associated self-financing portfolio denoted by $(V_t^{n, \mathcal{U}})_{0 \leq t \leq T}$ satisfying SDE (1.3.7) with $\xi_t^n = \xi_t^{n, \mathcal{U}}$. If we define

$$\bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) := \mathbb{E}_{t, s, \boldsymbol{\nu}, v}^{\mathbb{P}} \left(L(H_T^n, V_T^{n, \mathcal{U}}) \right),$$

then the primal solution satisfies

$$l_n(t, s, \boldsymbol{\nu}, v) \leq \bar{l}_n(t, s, \boldsymbol{\nu}, v).$$

Therefore, we obtain lower and upper bounds for the primal solution such that

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) \leq l_n(t, s, \boldsymbol{\nu}, v) \leq \bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v).$$

Proof. The proof is almost direct. Fix a set of admissible dual controls $\mathcal{U} \subseteq \mathcal{D}$, as

$$\begin{aligned} \hat{l}_n(t, s, \boldsymbol{\nu}, z) &= \sup_{\gamma_t \in \mathcal{D}} \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T(\gamma)) \right), \\ &= \sup_{\gamma_t \in \mathcal{D}} Y(t, s, \boldsymbol{\nu}, z; \gamma), \end{aligned}$$

we immediately obtain that

$$\underline{\hat{l}}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) \leq \hat{l}_n(t, s, \boldsymbol{\nu}, z).$$

Moreover, we observe that

$$\begin{aligned} l_n(t, s, \boldsymbol{\nu}, v) &= \sup_z \left\{ \hat{l}_n(t, s, \boldsymbol{\nu}, v) + zv \right\} \\ &\geq \sup_z \left\{ \underline{\hat{l}}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) + zv \right\} \\ &:= \underline{\hat{l}}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v). \end{aligned}$$

The inequality of the upper bound is obvious since using the definition of the optimal solution,

$$\begin{aligned} l_n(t, s, \boldsymbol{\nu}, v) &= \inf_{\xi_t^n \in \mathcal{R}_n} \mathbb{E}_{t, s, \boldsymbol{\nu}, v}^{\mathbb{P}} \left(L(H_T^n, V_T^n) \right) \\ &= \mathbb{E}_{t, s, \boldsymbol{\nu}, v}^{\mathbb{P}} \left(L(H_T^n, V_T^{n, *}) \right) \\ &\leq \mathbb{E}_{t, s, \boldsymbol{\nu}, v}^{\mathbb{P}} \left(L(H_T^n, V_T^{n, \mathcal{U}}) \right) \\ &= \bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v). \end{aligned}$$

Therefore, we prove

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) \leq l_n(t, s, \boldsymbol{\nu}, v) \leq \bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v).$$

□

Remark 1.7. *The way we define the upper bound is different than in Ma et al. (2020). We have made this choice in order to prove a convergence result of the bounds toward the primal solution when considering large dual control subsets $\mathcal{U}$. Note also that the approximate hedging ratio $(\xi_t^{n, \mathcal{U}})_{0 \leq t \leq T}$ defined by (1.5.19) has the same form as the optimal hedging ratio $(\xi_t^{n, *})_{0 \leq t \leq T}$ defined in (1.5.12) with the difference that we consider the subset of admissible dual control $\mathcal{U}$ instead of $\mathcal{D}$.*

Theorem 1.5.7 is important since it allows to approximate the primal by different sub-optimal choices of dual controls and thus, enables to easily deduce sub-optimal hedging strategies that can be computed relying on Monte Carlo simulations. Note that there is a wide range of possible subset choices, but depending on the choice of the subset $\mathcal{U}$, the computation time of the bounds can be quite substantial. We refer to Ma et al. (2020) for the algorithm allowing to compute the bounds via a Monte Carlo approach. In this chapter, we decide to only focus on a particular dual control subset that allows to obtain explicit formulas for the lower bound and the approximate hedging ratio. We discuss later the choice of the dual control subset.

Theorem 1.5.7 introduces a sub-optimal hedging strategy associated with the approximate hedging ratio $(\xi_t^{n, \mathcal{U}})_{0 \leq t \leq T}$ and the upper bound $\bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v)$ that can be implemented in practice and for which we can obtain a bound on the error made by considering this strategy instead of the optimal since:

$$\begin{aligned} |\bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) - l_n(t, s, \boldsymbol{\nu}, v)| &\leq |\bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) - \underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v)| \\ &:= C_{UL}^{\mathcal{U}}. \end{aligned}$$

In practice, by computing upper and lower bounds, we can deduce an upper bound on the error between the dual control approximate solution and the optimal solution of the Markov problem. Therefore, it enables to verify that the error is acceptable and that the proposed dual control method is relevant. Moreover, we show that if we consider a set sequence of admissible dual control $(\mathcal{U}_i)_{i \in \mathbb{N}}$ such that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ and $\lim_{i \rightarrow +\infty} \mathcal{U}_i = \mathcal{D}$ then the approximate solution of the Markov problem also converges to the primal solution.

Proposition 1.5.8. *Consider a compact set sequence of admissible dual controls $(\mathcal{U}_i)_{i \in \mathbb{N}}$ such that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ with $\lim_{i \rightarrow +\infty} \mathcal{U}_i = \mathcal{D}$. For $n \geq 1$, suppose that the sequence of functions $\left(\bar{l}_n^{\mathcal{U}_i}(\cdot) \right)_{i \in \mathbb{N}}$ is twice continuously differentiable with second derivatives that converge uniformly in $\mathbb{R}$ and the sequence $\left(L(H_T^n, V_T^n, \mathcal{U}_i) \right)_{i \in \mathbb{N}}$ is*

uniformly integrable, then $\forall t \in [0, T]$,

$$\lim_{i \rightarrow +\infty} \underbrace{|\bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) - \underline{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v)|}_{=C_{UL}^{\mathcal{U}_i}} = 0,$$

i.e.

$$\lim_{i \rightarrow +\infty} \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) = l_n(t, s, \boldsymbol{\nu}, v).$$

Proof. Consider a compact set sequence of admissible dual controls $(\mathcal{U}_i)_{i \in \mathbb{N}}$ such that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ and $\lim_{i \rightarrow +\infty} \mathcal{U}_i = \mathcal{D}$ and fix $n \geq 1$. First, we can show the convergence of $\bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v)$ toward $\hat{l}_n(t, s, \boldsymbol{\nu}, v)$. In fact, as for $i \in \mathbb{N}$, $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$, we have that $\forall t \in [0, T]$,

$$\bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, z) \leq \bar{l}_n^{\mathcal{U}_{i+1}}(t, s, \boldsymbol{\nu}, z) \leq \hat{l}_n^{\mathcal{D}}(t, s, \boldsymbol{\nu}, z),$$

and as $(\mathcal{U}_i)_{i=1, \dots, n}$ is a sequence of compact set, the infimum function over $\mathcal{U}_i$ is continuous for $i \in \mathbb{N}$. Thus taking the limit of $i \rightarrow +\infty$, we have that $\forall t \in [0, T]$,

$$\lim_{i \rightarrow +\infty} \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, z) = \hat{l}_n(t, s, \boldsymbol{\nu}, z).$$

In this case, we deduce the convergence of the lower bound of the primal solution toward the primal solution since $\forall t \in [0, T]$,

$$\begin{aligned} \lim_{i \rightarrow +\infty} \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) &= \lim_{i \rightarrow +\infty} \left(\sup_z \left\{ \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, z) + zv \right\} \right) \\ &= \sup_z \left\{ \lim_{i \rightarrow +\infty} \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, z) + zv \right\} \\ &= \sup_z \left\{ \hat{l}_n(t, s, \boldsymbol{\nu}, z) + zv \right\} \\ &= l_n(t, s, \boldsymbol{\nu}, v). \end{aligned}$$

It remains to show the convergence of the upper bound to the primal solution. For this purpose, we need to show that the approximate hedge ratio converges to the optimal hedge ratio. As the sequence of functions $\left(\bar{l}_n^{\mathcal{U}_i}(\cdot) \right)_{i \in \mathbb{N}}$ is twice continuously differentiable with second derivatives that converge uniformly in $\mathbb{R}$, standard result in Analysis states that $\forall t \in [0, T]$,

$$\begin{aligned} \lim_{i \rightarrow +\infty} \bar{\xi}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) &= \lim_{i \rightarrow +\infty} \frac{z \partial_{zz} \bar{l}_n^{\mathcal{U}_i}(\hat{\mu} - r)s - \partial_{sz} \bar{l}_n^{\mathcal{U}_i} \hat{\nu} s^2 - \rho \sum_{i=1}^n \partial_{\nu_i z} \bar{l}_n^{\mathcal{U}_i} \sqrt{\hat{\nu}} s \sigma(\hat{\nu})}{\hat{\nu} s^2} \\ &= \frac{z \partial_{zz} \hat{l}_n(\hat{\mu} - r)s - \partial_{sz} \hat{l}_n \hat{\nu} s^2 - \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n \sqrt{\hat{\nu}} s \sigma(\hat{\nu})}{\hat{\nu} s^2} \\ &= \xi_n^*(t, s, \boldsymbol{\nu}, v), \end{aligned}$$

with $z = z(t, s, \boldsymbol{\nu}, v)$ solution of

$$\lim_{i \rightarrow \infty} \underbrace{\partial_z \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, z)}_{=\partial_z \hat{l}_n(t, s, \boldsymbol{\nu}, z)} + v = 0.$$

Thus, since we have that

$$\xi_t^{n, \mathcal{U}_i} \xrightarrow{\mathcal{L}} \xi_t^{n, *},$$

as i goes to infinity, using similar arguments as for the proof of Proposition 1.4.4, we deduce that

$$V_T^{n, \mathcal{U}_i} \xrightarrow{\mathcal{L}} V_T^{n, *},$$

as i goes to infinity. Moreover, as $\left(L(H_T^n, V_T^{n, \mathcal{U}_i}) \right)_{i \in \mathbb{N}}$ is uniformly integrable, using similar arguments as for the proof of Lemma 1.4.5, we obtain that, $\forall t \in [0, T]$,

$$\lim_{i \rightarrow +\infty} \bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) = l_n(t, s, \boldsymbol{\nu}, v).$$

We therefore conclude that $\forall t \in [0, T]$,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\bar{l}_n^{\mathcal{U}_i}(t, s, \boldsymbol{\nu}, v) - \bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) \right) &= l_n(t, s, \boldsymbol{\nu}, v) - l_n(t, s, \boldsymbol{\nu}, v) \\ &= 0. \end{aligned}$$

That concludes the proof since we have proved the stated proposition. $\square$

The previous proposition shows that if we consider a large enough set of admissible dual controls, the approximate solution converges to the primal solution of the Markov problem. In practice, we observe that even if the set of admissible dual controls $\mathcal{U}$ is small, the error is small, which seems to show that the choice of the dual control does not significantly impact the value of the primal solution.

Let us go back to the original hedging problem under rough volatility, the initial control problem posed was

$$l(s_0, \nu_0, v_0) = \inf_{\xi_t \in \mathcal{R}} \mathbb{E}_{s_0, \nu_0, v_0}^{\mathbb{P}} \left(L(h(S_T), V_T) \right).$$

The proposed sub-optimal hedging strategy is $(\xi_t^{n, \mathcal{U}})_{0 \leq t \leq T}$ associated with the approximate initial value function $\bar{l}_n^{\mathcal{U}}(s_0, \boldsymbol{\nu}_0, v_0)$ defined by

$$\bar{l}_n^{\mathcal{U}}(s_0, \boldsymbol{\nu}_0, v_0) := \bar{l}_n^{\mathcal{U}}(t = 0, s_0, \boldsymbol{\nu}_0, v_0),$$

with $\boldsymbol{\nu}_0 = \mathbf{0}^n$. It is a two-fold approximate solution, on the one hand by the Markov approximation of the volatility process and on the other hand by the sub-optimal choice of the dual control. However, we can show that the error with respect to the optimal solution of the initial problem can be small if n and $\mathcal{U}$ are large enough.

Proposition 1.5.9. *Consider a compact set sequence of admissible dual controls $(\mathcal{U}_i)_{i \in \mathbb{N}}$ such that $\mathcal{U}_i \subseteq \mathcal{U}_{i+1}$ with $\lim_{i \rightarrow +\infty} \mathcal{U}_i = \mathcal{D}$. Suppose that the assumptions of Theorem 1.4.6 and Proposition 1.5.8 are satisfied. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall n \geq N$, $\exists M \in \mathbb{N}$ such that $\forall i \geq M$,*

$$|l(s_0, \nu_0, v_0) - \bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)| < \varepsilon$$

and

$$|l(s_0, \nu_0, v_0) - l^{\xi^{n, \mathcal{U}_i}}(s_0, \nu_0, v_0)| < \varepsilon,$$

with $\nu_0 = \mathbf{0}^n$. It means that the approximate hedging ratio $(\xi_t^{n, \mathcal{U}_i})_{0 \leq t \leq T}$ associated to $\bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)$ is ε -optimal for the original problem.

Proof. The proof is almost direct. Fix $\varepsilon > 0$, from Theorem 1.4.6, we know that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|l(s_0, \nu_0, v_0) - l_{n, \nu}(s_0, \nu_0, v_0)| < \frac{\varepsilon}{2},$$

with $\nu_0 = \mathbf{0}^n$, and

$$|l(s_0, \nu_0, v_0) - l^{\xi^{n, *}}(s, \nu_0, v)| < \frac{\varepsilon}{2}.$$

Moreover, from Proposition 1.5.8, we have that $\forall n \geq 1$, $\exists M_1 \in \mathbb{N}$ such that $\forall i \geq M_1$,

$$|l_{n, \nu}(s_0, \nu_0, v_0) - \bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)| < \frac{\varepsilon}{2},$$

and since $\xi_t^{n, \mathcal{U}_i} \xrightarrow{\mathcal{L}} \xi_t^{n, *}$ as $i \rightarrow +\infty$, we deduce, using similar arguments as for the proof of Proposition 1.4.4 and Lemma 1.4.5, that $\forall n \geq 1$, $\exists M_2 \in \mathbb{N}$ such that $\forall i \geq M_2$,

$$|l^{\xi^{n, *}}(s_0, \nu_0, v_0) - l^{\xi^{n, \mathcal{U}_i}}(s_0, \nu_0, v_0)| < \frac{\varepsilon}{2}.$$

Therefore choosing $M := \max(M_1, M_2)$, for $\nu_0 = \mathbf{0}^n$, we know that $\forall n \geq N$, $\exists M \in \mathbb{N}$ such that $\forall i \geq M$,

$$\begin{aligned} |l(s_0, \nu_0, v_0) - \bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)| &= |l(s_0, \nu_0, v_0) - l_{n, \nu}(s_0, \nu_0, v_0) \\ &\quad + l_{n, \nu}(s_0, \nu_0, v_0) - \bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)| \\ &\leq |l(s_0, \nu_0, v_0) - l_{n, \nu}(s_0, \nu_0, v_0)| \\ &\quad + |l_{n, \nu}(s_0, \nu_0, v_0) - \bar{l}_n^{\mathcal{U}_i}(s_0, \nu_0, v_0)| \\ &< \varepsilon, \end{aligned}$$

and

$$\begin{aligned}
|l(s_0, \nu_0, v_0) - l^{\xi^{n, \mathcal{U}_i}}(s_0, \nu_0, v_0)| &= |l(s_0, \nu_0, v_0) - l^{\xi^{n, *}}(s_0, \nu_0, v_0) \\
&\quad + l^{\xi^{n, *}}(s_0, \nu_0, v_0) - l^{\xi^{n, \mathcal{U}_i}}(s, \nu_0, v_0)| \\
&\leq |l(s_0, \nu_0, v_0) - l^{\xi^{n, *}}(s_0, \nu_0, v_0)| \\
&\quad + |l^{\xi^{n, *}}(s_0, \nu_0, v_0) - l^{\xi^{n, \mathcal{U}_i}}(s_0, \nu_0, v_0)| \\
&< \varepsilon.
\end{aligned}$$

□

Proposition 1.5.9 states that, if n and the dual control set $\mathcal{U}$ are large enough, then the error with respect to the original problem should be quite small. In practice, for a fixed set of dual control $\mathcal{U}$, the error is controlled by the number of factors n and the gap $C_{UL}^{\mathcal{U}}$ between lower and upper bounds.

Remark 1.8. *The convergence rate with respect to the number of factors n is difficult to quantify explicitly in general. In practice, the required value of n depends on the regularity of the underlying rough volatility process. Smoother dynamics (i.e. H closer to 0.5) are expected to yield faster convergence, whereas rougher regimes require larger values of n . In particular, the standard Brownian setting ($H = 0.5$) corresponds to $n = 1$, while for rougher cases such as ($H = 0.1$), numerical approximations suggest that $n \approx 20$ is already sufficient in practice, see for instance Abi Jaber (2019).*

Appropriate choice of the dual control subset $\mathcal{U} \subseteq \mathcal{D}$

Now, we consider a particular subset of dual controls for which explicit formulas can be obtained. Thus for the following, the dual control subset considered is defined as

$$\mathcal{U} = \left\{ (\gamma_t)_{0 \leq t \leq T} = \left(c \times Z_t \times \sigma(\hat{\nu}_t) \right)_{0 \leq t \leq T}, c \in U \subseteq \mathbb{R} \right\} \subseteq \mathcal{D}. \quad (1.5.20)$$

We notice that the chosen form of the dual controls belonging to $\mathcal{U}$ is similar to the form of the optimal dual control (1.5.7). This particular subset (1.5.20) enables to interpret the sub-optimal hedging strategy as well as obtain closed forms for the lower bound and the approximate hedging ratio. First, assuming this subset of admissible dual control, we observe that

$$\begin{aligned}
\bar{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) &= \sup_z \left\{ \sup_{\gamma_t \in \mathcal{U}} Y(t, s, \boldsymbol{\nu}, z; \gamma) + zv \right\} \\
&= \sup_z \left\{ \max_{c \in U} Y(t, s, \boldsymbol{\nu}, z; c) + zv \right\} \quad (1.5.21)
\end{aligned}$$

$$= \max_{c \in U} \sup_z \left\{ Y(t, s, \boldsymbol{\nu}, z; c) + zv \right\}. \quad (1.5.22)$$

Defining, for $c \in U$, $\underline{l}_n(t, s, \boldsymbol{\nu}, v; c)$ by

$$\underline{l}_n(t, s, \boldsymbol{\nu}, v; c) := \sup_z \left\{ Y(t, s, \boldsymbol{\nu}, z; c) + zv \right\},$$

we have that

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = \underline{l}_n(t, s, \boldsymbol{\nu}, v; c^*)$$

with

$$c^* := \arg \max_{c \in U} \underline{l}_n(t, s, \boldsymbol{\nu}, v; c).$$

In this case, the value at time $t \in [0, T]$ of the sub-optimal hedging portfolio $V_t^{n, \mathcal{U}}$ is easily interpreted as the price at time t of a modified payoff. This is in line with Föllmer and Leukert (1999); Föllmer and Leukert (2000) who present partial hedging strategies as perfect hedging of knock-out options.

Proposition 1.5.10. *If we consider a subset of admissible dual control $\mathcal{U}$ of the form (1.5.20), for $t \in [0, T]$, if $z = z(t, s, \boldsymbol{\nu}, v)$ solution of (1.5.17) with $v = V_t^{n, \mathcal{U}}$, then*

$$V_t^{n, \mathcal{U}} = \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{Q}(c^*)} \left(e^{-r(T-t)} \left(-\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \right) \right),$$

where $(\mathbb{Q}(c))_{c \in U}$ called risk-neutral measures are $\mathbb{P}$ -equivalent measures such that, for $c \in U$, the processes have the following dynamics

$$dS_t^n = rS_t^n + \sqrt{\hat{\nu}_t} S_t^n dW_S^{\mathbb{Q}(c)}(t),$$

for $i = 1, \dots, n$,

$$d\nu_t^i = \left(-x_i \nu_t^i + b(\hat{\nu}_t) + \sigma(\hat{\nu}_t) \left(-\rho \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} + \sqrt{1 - \rho^2} c \sigma(\hat{\nu}_t) \right) \right) dt + \sigma(\hat{\nu}_t) dW_\nu^{\mathbb{Q}(c)}(t),$$

and

$$dZ_t = Z_t \left(\left(-r + \frac{(\hat{\mu}_t - r)^2}{\hat{\nu}_t} + c^2 \sigma^2(\hat{\nu}_t) \right) dt - \frac{(\mu - r)}{\sqrt{\hat{\nu}_t}} dW_S^{\mathbb{Q}(c)}(t) + c \sigma(\hat{\nu}_t) dB_\nu^{\mathbb{Q}(c)}(t) \right)$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q}(c))$, with two-dimensional Brownian motion $(W_S^{\mathbb{Q}(c)}, B_\nu^{\mathbb{Q}(c)})$ such that $W_\nu^{\mathbb{Q}(c)} = \rho W_S^{\mathbb{Q}(c)} + \sqrt{1 - \rho^2} B_\nu^{\mathbb{Q}(c)}$. In particular, for a power loss function of the form $L(h(s), v) = \frac{1}{p}(h(s) - v)^p$ with $p = 2k$, $k \in \mathbb{N}_0$, the value sub-optimal hedging portfolio at time $t \in [0, T]$ is given by

$$V_t^{n, \mathcal{U}} = \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{Q}(c^*)} \left(e^{-r(T-t)} \left(H_T^n + Z_T^{q-1} \right) \right),$$

with $q = \frac{p}{p-1}$.

Proof. From the Theorem 1.5.7, we know, using (1.5.17), that, at time $t \in [0, T]$, $z = z^* = z(t, s, \boldsymbol{\nu}, v)$ with $z(t, s, \boldsymbol{\nu}, v)$ solution of

$$v = -\partial_z \hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z),$$

Thus, by (1.5.21), we have that

$$\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = Y(t, s, \boldsymbol{\nu}, z^*; c^*(z^*)) + z^* v,$$

with

$$c^*(z) := \arg \max_{c \in U} Y(t, s, \boldsymbol{\nu}, z; c).$$

But using (1.5.22), we also have that

$$\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = Y(t, s, \boldsymbol{\nu}, z^*(c^*); c^*) + z^*(c^*) v$$

with $z^*(c)$ solution of

$$v = -\partial_z Y(t, s, \boldsymbol{\nu}, z; c),$$

we conclude by unicity that, at time $t \in [0, T]$, $z = z^* = z(t, s, \boldsymbol{\nu}, v) = z^*(c^*)$ and $c^*(z^*) = c^*$. Therefore, we obtain, at time $t \in [0, T]$ and for $z = z^*(c^*)$ with $v = V_t^{n, \mathcal{U}}$, the following relation

$$V_t^{n, \mathcal{U}} = -\partial_z \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T(c^*)) \right).$$

Using the theorem of exchanging expectation and derivative, we have that

$$\begin{aligned} V_t^{n, \mathcal{U}} &= \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{P}} \left(-\partial_{Z_t} \hat{L}(H_T^n, Z_T(c^*)) \right) \\ &= \mathbb{E}_{t, s, \boldsymbol{\nu}, z}^{\mathbb{P}} \left(-\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \times \partial_{Z_t} Z_T(c^*) \right). \end{aligned}$$

But as, for $t \in [0, T]$, the dual control is given by $\gamma_t = c \times Z_t \times \sigma(\hat{\nu}_t)$, $c \in U$, we have that

$$dZ_t(c) = Z_t(c) \left(-r dt - \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} dW_S(t) + c \sigma(\hat{\nu}_t) dB_v(t) \right)$$

and therefore, for $t \in [0, T]$,

$$\begin{aligned} \frac{Z_T(c)}{Z_t(c)} &= e^{-r(T-t)} \\ &\times \underbrace{\exp \left(-\frac{1}{2} \int_t^T \left(\frac{(\hat{\mu}_s - r)^2}{\hat{\nu}_s} + c^2 \sigma^2(\hat{\nu}_s) \right) ds - \int_t^T \frac{(\hat{\mu}_s - r)}{\sqrt{\hat{\nu}_s}} dW_S(s) + \int_t^T c \sigma(\hat{\nu}_s) dB_v(s) \right)}_{:= \frac{dQ(c)}{d\mathbb{P}} |_{\mathcal{F}_t}}. \end{aligned}$$

Since $\left(\frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}}, c\sigma(\hat{\nu}_t)\right)_{0 \leq t \leq T}$ is a 2-dimensional vector of adapted and square integrable processes, using Girsanov's Theorem, we can define $\mathbb{P}$ -equivalent probability measures $(\mathbb{Q}(c))_{c \in U}$ with change of measure defined by $\frac{d\mathbb{Q}(c)}{d\mathbb{P}}|_{\mathcal{F}_t}$ such that

$$\begin{aligned} dW_S^{\mathbb{Q}(c)}(t) &= dW_S(t) + \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} dt, \\ dB_\nu^{\mathbb{Q}(c)}(t) &= dB_\nu(t) - c\sigma(\hat{\nu}_t) dt. \end{aligned}$$

Therefore, dynamics of processes under the $\mathbb{Q}(c)$ -measure are given by

$$dS_t^n = rS_t^n + \sqrt{\hat{\nu}_t} S_t^n dW_s^{\mathbb{Q}(c)}(t),$$

for $i = 1, \dots, n$,

$$d\nu_t^i = \left(-x_i \nu_t^i + b(\hat{\nu}_t) + \sigma(\hat{\nu}_t) \left(-\rho \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} + \sqrt{1 - \rho^2} c\sigma(\hat{\nu}_t) \right) \right) dt + \sigma(\hat{\nu}_t) dW_\nu^{\mathbb{Q}(c)}(t),$$

and

$$dZ_t = Z_t \left(\left(-r + \frac{(\hat{\mu}_t - r)^2}{\hat{\nu}_t} + c^2 \sigma^2(\hat{\nu}_t) \right) dt - \frac{(\hat{\mu}_t - r)}{\sqrt{\hat{\nu}_t}} dW_S^{\mathbb{Q}(c)}(t) + c\sigma(\hat{\nu}_t) dB_\nu^{\mathbb{Q}(c)}(t) \right)$$

where $W_S^{\mathbb{Q}(c)}$, $W_\nu^{\mathbb{Q}(c)}$ are standard Brownian motions under $\mathbb{Q}(c)$ -measure with $d\langle W_S^{\mathbb{Q}(c)}, W_\nu^{\mathbb{Q}(c)} \rangle_t = \rho dt$ and $B_\nu^{\mathbb{Q}(c)}$ is a standard Brownian motion, independent from $W_S^{\mathbb{Q}(c)}$ such that $W_\nu^{\mathbb{Q}(c)} = \rho W_S^{\mathbb{Q}(c)} + \sqrt{1 - \rho^2} B_\nu^{\mathbb{Q}(c)}$. In this case we have that

$$\begin{aligned} V_t^{n, \mathcal{U}} &= \mathbb{E}_{t, s, \nu, z}^{\mathbb{P}} \left(-\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \times \partial_{Z_t} Z_T(c^*) \right) \\ &= \mathbb{E}_{t, s, \nu, z}^{\mathbb{P}} \left(-\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \times \frac{Z_T(c^*)}{Z_t(c^*)} \right) \\ &= \mathbb{E}_{t, s, \nu, z}^{\mathbb{P}} \left(e^{-r(T-t)} - \partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \times \frac{d\mathbb{Q}(c^*)}{d\mathbb{P}}|_{\mathcal{F}_t} \right) \\ &= \mathbb{E}_{t, s, \nu, z}^{\mathbb{Q}(c^*)} \left(e^{-r(T-t)} \left(-\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) \right) \right). \end{aligned}$$

Moreover, if we consider a power loss, we know that

$$\hat{L}(H_T^n, Z_T(c^*)) = -\frac{Z_T^q(c^*)}{q} - H_T^n Z_T(c^*),$$

we deduce that in this case,

$$\partial_{Z_T} \hat{L}(H_T^n, Z_T(c^*)) = -Z_T^{q-1}(c^*) - H_T^n,$$

Therefore, we obtain that

$$V_t^{n,\mathcal{U}} = \mathbb{E}_{t,s,\boldsymbol{\nu},z}^{\mathbb{Q}(c^*)} \left(e^{-r(T-t)} \left(H_T^n + Z_T^{q-1}(c^*) \right) \right).$$

□

Still assuming that the subset of admissible dual controls $\mathcal{U}$ has the form (1.5.20), we next show that, for the Volterra Heston model, the lower bound as well as the approximate hedging ratio associated with a power loss function have explicit forms.

Proposition 1.5.11. *Consider a power loss of the form $L(h(s), v) = \frac{1}{p}(h(s) - v)^p$ with $p = 2k, k \in \mathbb{N}_0$, suppose that the volatility is modeled by Volterra Heston model i.e. $b(x) = \kappa(\theta - x)$ and $\sigma(x) = \zeta\sqrt{x}$ with $2\kappa\theta > \zeta^2 \sum_{i=1}^n w_i$ and $\nu_0 > 0$ such that (1.5.2) is satisfied. Moreover, assume that the subset of admissible dual control is given by (1.5.20). Therefore $Y(t, s, \boldsymbol{\nu}, z; \gamma)$ defined by (1.5.16) is such that*

$$Y(t, s, \boldsymbol{\nu}, z; c) = -\frac{1}{q} z^q \exp(C_t(c) + \sum_{i=1}^n D_t^i(c) \nu^i) - \mathbb{E}_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c)} (e^{-r(T-t)} H_T^n) z,$$

where $C_t(c)$ and $(D_t^i(c))_{i=1,\dots,n}$ are time-dependent functions, solutions of Riccati equations given respectively by

$$\begin{aligned} \partial_t C_t(c) &= r q - \frac{1}{2} q(q-1) (A^2 + \zeta^2 c^2) \nu_0 \\ &\quad - \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) + q \zeta \nu_0 \left(-\rho A + \sqrt{1 - \rho^2 \zeta^2 c} \right) \right) \\ &\quad - \frac{1}{2} \nu_0 \zeta^2 \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j, \\ C_T(c) &= 0, \end{aligned}$$

and for $i = 1, \dots, n$,

$$\begin{aligned} \partial_t D_t^i(c) &= x_i D_t^i + w_i \sum_{j=1}^n D_t^j \left(\kappa - q \left(-\rho A \zeta + \sqrt{1 - \rho^2 \zeta^2 c} \right) \right) - \frac{1}{2} w_i \zeta^2 \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \\ &\quad - \frac{1}{2} w_i q(q-1) (A^2 + \zeta^2 c^2), \\ D_T^i(c) &= 0. \end{aligned}$$

In this case, the lower bound satisfies

$$\begin{aligned} \underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) &= -\frac{1}{q} z(t, s, \boldsymbol{\nu}, v)^q \exp(C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu_t^i) \\ &\quad + z(t, s, \boldsymbol{\nu}, v) \left(v - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c^*)} (e^{-r(T-t)} H_T^n) \right), \end{aligned} \tag{1.5.23}$$

with

$$z(t, s, \boldsymbol{\nu}, v) = \exp \left(-\frac{1}{q-1} (C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu^i) \right) \times \left(v - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c^*)} (e^{-r(T-t)} H_T^n) \right)^{\frac{1}{q-1}}. \quad (1.5.24)$$

Moreover, if $z = z(t, s, \boldsymbol{\nu}, v)$,

$$\tilde{L}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) = Y(t, s, \boldsymbol{\nu}, z; c^*).$$

Proof. The proof is similar to the proof of Proposition 1.5.6 and is essentially obtained by using the Feynman-Kac formula. First, using Lemma 1.5.5, we have that

$$\begin{aligned} Y(t, s, \boldsymbol{\nu}, z; c) &= \mathbb{E}_{t,s,\boldsymbol{\nu},z}^{\mathbb{P}} \left(\hat{L}(H_T^n, Z_T) \right) \\ &= \mathbb{E}_{t,s,\boldsymbol{\nu},z}^{\mathbb{P}} \left(-\frac{1}{q} Z_T^q - H_T^n Z_T \right) \\ &= \underbrace{\mathbb{E}_{t,\boldsymbol{\nu},z}^{\mathbb{P}} \left(-\frac{1}{q} Z_T^q \right)}_{:=Y_1(t,\boldsymbol{\nu},z)} - \underbrace{\mathbb{E}_{t,s,\boldsymbol{\nu},z}^{\mathbb{P}} \left(H_T^n Z_T \right)}_{:=Y_2(t,s,\boldsymbol{\nu},z)}. \end{aligned}$$

Let focus on Y_1 , using Feynman-Kac formula, we have that

$$\begin{aligned} 0 &= \partial_t Y_1 - \partial_z Y_1 r z + \frac{1}{2} \partial_{zz} Y_1 z^2 \left(\frac{(\hat{\mu} - r)^2}{\hat{\nu}} + c^2 \hat{\nu} \zeta^2 \right) + \sum_{i=1}^n \partial_{\nu_i} Y_1 (-x_i \nu_i + \kappa(\theta - \hat{\nu})) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} Y_1 \zeta^2 \hat{\nu} + \sum_{i=1}^n \partial_{\nu_i z} Y_1 z \left(-\rho \frac{(\hat{\mu} - r)}{\sqrt{\hat{\nu}}} \zeta \sqrt{\hat{\nu}} + \sqrt{1 - \rho^2} \zeta^2 \hat{\nu} c \right), \\ Y_1(T, z) &= -\frac{1}{q} z^q. \end{aligned}$$

Suppose that Y_1 has the following form

$$Y_1 = -\frac{1}{q} z^q \exp(C_t + \sum_{i=1}^n D_t^i \nu^i),$$

In this case, as we consider that $\hat{\mu} = r + A\hat{\nu}$, we have that

$$\begin{aligned} 0 &= Y_1 \left(\partial_t C_t + \sum_{i=1}^n \partial_t D_t^i \nu_i + \sum_{i=1}^n D_t^i (-x_i \nu_i + \kappa(\theta - \hat{\nu})) + \zeta^2 \hat{\nu} \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \right. \\ &\quad \left. - r q + \frac{1}{2} q(q-1) (A^2 + c^2 \zeta^2) \hat{\nu} + q \left(-\rho A \hat{\nu} \zeta + \sqrt{1 - \rho^2} \zeta^2 \hat{\nu} c \right) \sum_{i=1}^n D_t^i \right), \end{aligned}$$

as $\hat{\nu} = \nu_0 + \sum_{i=1}^n w_i \nu_i$, we have

$$\begin{aligned}
0 = & \left(\partial_t C_t + \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) + q \left(-\rho A \nu_0 \zeta + \sqrt{1 - \rho^2 \zeta^2} \nu_0 c \right) \right) - r q \right. \\
& + \frac{1}{2} q(q-1) (A^2 + c^2 \zeta^2) \nu_0 + \frac{1}{2} \zeta^2 \nu_0 \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \Big) \\
& + \sum_{i=1}^n \nu_i \left(\partial_t D_t^i - x_i D_t^i - w_i \kappa \sum_{j=1}^n D_t^j + \zeta^2 w_i \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \right. \\
& \left. + \frac{1}{2} w_i q(q-1) (A^2 + c^2 \zeta^2) + q w_i \left(-\rho A \zeta + \sqrt{1 - \rho^2 \zeta^2} c \right) \sum_{j=1}^n D_t^j \right) \Big).
\end{aligned}$$

We obtain that C_t and $(D_t^i)_{i=1, \dots, n}$ solve Riccati type ODEs of the form

$$\begin{aligned}
\partial_t C_t = & r q - \frac{1}{2} q(q-1) (A^2 + c^2 \zeta^2) \nu_0 - \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) \right. \\
& \left. + q \left(-\rho A \nu_0 \zeta + \sqrt{1 - \rho^2 \zeta^2} \nu_0 c \right) \right) - \frac{1}{2} \zeta^2 \nu_0 \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j, \quad (1.5.25)
\end{aligned}$$

$$C_T = 0$$

and for $i = 1, \dots, n$,

$$\begin{aligned}
\partial_t D_t^i = & x_i D_t^i + w_i \sum_{j=1}^n D_t^j \left(\kappa - q \left(-\rho A \zeta + \sqrt{1 - \rho^2 \zeta^2} c \right) \right) - \zeta^2 w_i \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \\
& - \frac{1}{2} w_i q(q-1) (A^2 + c^2 \zeta^2), \quad (1.5.26) \\
D_T^i = & 0.
\end{aligned}$$

Let now consider the second function Y_2 . Using the Feynman-Kac formula, we

obtain that Y_2 satisfies

$$\begin{aligned}
0 = & \partial_t Y_2 - \partial_z Y_2 r z + \frac{1}{2} \partial_{zz} Y_2 z^2 \left(\frac{(\hat{\mu} - r)^2}{\hat{\nu}} + c^2 \zeta^2 \hat{\nu} \right) \\
& + \sum_{i=1}^n \partial_{\nu_i} Y_2 \left(-x_i \nu_i + \kappa(\theta - \hat{\nu}) \right) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} Y_2 \zeta^2 \hat{\nu} \\
& + \sum_{i=1}^n \partial_{\nu_i z} Y_2 z \left(-\rho \frac{(\hat{\mu} - r)}{\sqrt{\hat{\nu}}} \zeta \sqrt{\hat{\nu}} + \sqrt{1 - \rho^2} \zeta^2 \hat{\nu} c \right), \\
& + \partial_s Y_2 \hat{\mu} s + \frac{1}{2} \partial_{ss} Y_2 \hat{\nu} s^2 + \sum_{i=1}^n \partial_{\nu_i s} Y_2 s \hat{\nu} \zeta \rho - \partial_{sz} Y_2 z \frac{\hat{\mu} - r}{\sqrt{\hat{\nu}}} s \sqrt{\hat{\nu}},
\end{aligned}$$

$$Y_2(T, s, z) = h(s)z.$$

Suppose that Y_2 has the form

$$Y_2 = g(t, s, \boldsymbol{\nu}) z.$$

Therefore, we have that

$$\begin{aligned}
0 = & z \left(\partial_t g - g r + \sum_{i=1}^n \partial_{\nu_i} g \left(-x_i \nu_i + \kappa(\theta - \hat{\nu}) \right) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} g \zeta^2 \hat{\nu} \right. \\
& + \sum_{i=1}^n \partial_{\nu_i} g \left(-\rho(\hat{\mu} - r) \zeta + \sqrt{1 - \rho^2} \zeta^2 \hat{\nu} c \right) + \partial_s g \hat{\mu} s + \frac{1}{2} \partial_{ss} g \hat{\nu} s^2 \\
& \left. + \sum_{i=1}^n \partial_{\nu_i s} g s \hat{\nu} \zeta \rho - \partial_s g (\hat{\mu} - r) s \right).
\end{aligned}$$

The function $g(t, s, \boldsymbol{\nu})$ satisfies the following PDE

$$\begin{aligned}
0 = & \partial_t g - g r + \partial_s g r s + \sum_{i=1}^n \partial_{\nu_i} g \left(-x_i \nu_i + \kappa(\theta - \hat{\nu}) + \hat{\nu} \zeta (-\rho A + \sqrt{1 - \rho^2} \zeta c) \right) \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i \nu_j} g \zeta^2 \hat{\nu} + \frac{1}{2} \partial_{ss} g \hat{\nu} s^2 + \sum_{i=1}^n \partial_{\nu_i s} g s \hat{\nu} \zeta \rho,
\end{aligned}$$

$$g(T, s, \boldsymbol{\nu}) = h(s).$$

We deduce, using once again the Feynman-Kac theorem (this time in the other sense), that

$$g(t, s, \boldsymbol{\nu}) = \mathbb{E}_{t, s, \boldsymbol{\nu}}^{\mathbb{Q}(c)}(e^{-r(T-t)} H_T^n),$$

with under $\mathbb{Q}(c)$ -measure,

$$dS_t^n = r S_t^n + \sqrt{\hat{\nu}_t} S_t^n dW_S^{\mathbb{Q}(c)}(t),$$

$$d\nu_t^i = \left(-x_i(\nu_t^i - \nu_0^i) + \kappa(\theta - \hat{\nu}) + \hat{\nu}\zeta(-\rho A + \sqrt{1 - \rho^2}\zeta c) \right) dt + \zeta\sqrt{\hat{\nu}_t} dW_\nu^{\mathbb{Q}}(t), \quad i = 1, \dots, n,$$

where $W_S^{\mathbb{Q}(c)}$ and $W_\nu^{\mathbb{Q}(c)}$ are standard brownian motions under $\mathbb{Q}(c)$ -measure with $d\langle W_S^{\mathbb{Q}(c)}, W_\nu^{\mathbb{Q}(c)} \rangle_t = \rho dt$.

Finally, combining the different results, we obtain the announced result

$$Y(t, s, \boldsymbol{\nu}, z; c) = -\frac{1}{q} z^q \exp(C_t(c) + \sum_{i=1}^n D_t^i(c) \nu^i) - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c)}(e^{-r(T-t)} H_T^n) z.$$

Furthermore, as by definition,

$$\underline{l}_n(t, s, \boldsymbol{\nu}, v; c) = \sup_z \left\{ Y(t, s, \boldsymbol{\nu}, z; c) + zv \right\},$$

using the first order condition, we obtain that, given $c \in U$, the value at time t of $z(t, s, \boldsymbol{\nu}, v; c)$ satisfies

$$z(t, s, \boldsymbol{\nu}, v; c) = \exp \left(-\frac{1}{q-1} (C_t(c) + \sum_{i=1}^n D_t^i(c) \nu^i) \right) \times \left(v - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c)}(e^{-r(T-t)} H_T^n) \right)^{\frac{1}{q-1}}$$

and then

$$\begin{aligned} \underline{l}_n(t, s, \boldsymbol{\nu}, v; c) &= -\frac{1}{q} z(t, s, \boldsymbol{\nu}, v; c)^q \exp(C_t(c) + \sum_{i=1}^n D_t^i(c) \nu^i) \\ &\quad + z(t, s, \boldsymbol{\nu}, v; c) \left(v - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c)}(e^{-r(T-t)} H_T^n) \right). \end{aligned}$$

Thus, as

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = \underline{l}_n(t, s, \boldsymbol{\nu}, v; c^*),$$

we deduce the announced result. Finally, as in the proof of Proposition 1.5.10, we show that $c^* = c^*(z^*)$ and $z(t, s, \boldsymbol{\nu}, v; c^*) = z(t, s, \boldsymbol{\nu}, v)$, we conclude that if $z = z(t, s, \boldsymbol{\nu}, v)$ then

$$\underline{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) = Y(t, s, \boldsymbol{\nu}, z; c^*).$$

□

Remark 1.9. Compared to the optimal solution in the case of linear payoff (see Proposition 1.5.6), we observe that the form of the solution is the same, the only difference lies in the expression of the coefficients C_t and $(D_t^i)_{i=1,\dots,n}$. In the classical Heston model, i.e. $n = 1$, then C_t and D_t admit closed formulas, see for instance Appendix A in Ma et al. (2020).

Proposition 1.5.12. *Using the same assumptions as the Proposition 1.5.11 and denoting $g(t, s, \boldsymbol{\nu}) = \mathbb{E}_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c^*)}(e^{-r(T-t)} H_T^n)$, the approximate hedging ratio $\xi_t^{n,\mathcal{U}}$ defined by (1.5.19) is such that*

$$\begin{aligned} \xi_t^{n,\mathcal{U}} = & \frac{1}{\hat{\nu}_t S_t^n} \left(g(t, S_t^n, \boldsymbol{\nu}_t) - V_t^{n,\mathcal{U}} \right) \left(\frac{1}{p-1} (\hat{\mu}_t - r) - \rho \zeta \hat{\nu}_t \sum_{i=1}^n D_t^i(c^*) \right) \\ & + \partial_s g(t, S_t^n, \boldsymbol{\nu}_t) + \frac{\rho \zeta}{S_t^n} \sum_{i=1}^n \left(\partial_{\nu_i} g(t, S_t^n, \boldsymbol{\nu}_t) \right), \end{aligned} \quad (1.5.27)$$

where $V_t^{n,\mathcal{U}}$ is the value at time t of the self-financing portfolio associated to the sub-optimal hedging strategy.

Proof. The proof relies on the definition of the approximate hedging ratio and on Proposition 1.5.11. By definition, we have that

$$\bar{\xi}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, v) = \frac{z \partial_{zz} \hat{l}_n^{\mathcal{U}} (\hat{\mu} - r) s - \partial_{sz} \hat{l}_n^{\mathcal{U}} \hat{\nu} s^2 - \rho \sum_{i=1}^n \partial_{\nu_i} \hat{l}_n^{\mathcal{U}} \sqrt{\hat{\nu}} s \sigma(\hat{\nu})}{\hat{\nu} s^2},$$

with $z = z(t, s, \boldsymbol{\nu}, v)$. With our assumptions, we know, by Proposition 1.5.11, that, for $z = z(t, s, \boldsymbol{\nu}, v)$,

$$\hat{l}_n^{\mathcal{U}}(t, s, \boldsymbol{\nu}, z) = -\frac{1}{q} z^q \exp(C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu^i) - E_{t,s,\boldsymbol{\nu}}^{\mathbb{Q}(c^*)}(e^{-r(T-t)} H_T^n) z$$

thus, we can easily deduce the partial derivatives of the process $\hat{l}_n^{\mathcal{U}}(\cdot)$ such that

$$\partial_z \hat{l}_n^{\mathcal{U}} = -z^{q-1} \exp(C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu^i) - g(t, s, \boldsymbol{\nu}),$$

and

$$\begin{aligned} \partial_{zz} \hat{l}_n^{\mathcal{U}} &= -(q-1) z^{q-2} \exp(C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu^i) \\ \partial_{zs} \hat{l}_n^{\mathcal{U}} &= -\partial_s g(t, s, \boldsymbol{\nu}) \\ \partial_{z\nu_i} \hat{l}_n^{\mathcal{U}} &= -z^{q-1} D_t^i(c^*) \exp(C_t(c^*) + \sum_{i=1}^n D_t^i(c^*) \nu^i) - \partial_{\nu_i} g(t, s, \boldsymbol{\nu}), \end{aligned}$$

therefore, the approximate hedging ratio process $\xi_t^{n,\mathcal{U}} = \bar{\xi}_n^{\mathcal{U}}(t, S_t^n, \boldsymbol{\nu}_t, V_t^{n,\mathcal{U}})$ is given

by

$$\begin{aligned}
\xi_t^{n,\mathcal{U}} = & \frac{1}{\hat{\nu}_t \left(S_t^n \right)^2} \left(- (q-1) z(t, S_t^n, \boldsymbol{\nu}_t, V_t^{n,\mathcal{U}})^{q-1} \exp(C_t(c^*)) \right. \\
& + \sum_{i=1}^n D_t^i(c^*) \nu_t^i (\hat{\mu}_t - r) S_t^n + \partial_s g(t, S_t^n, \boldsymbol{\nu}_t) \hat{\nu}_t \left(S_t^n \right)^2 \\
& + \rho \zeta S_t^n \hat{\nu}_t \sum_{i=1}^n \left(z(t, S_t^n, \boldsymbol{\nu}_t, V_t^{n,\mathcal{U}})^{q-1} D_t^i(c^*) \exp(C_t(c^*)) \right. \\
& \left. \left. + \sum_{i=1}^n D_t^i(c^*) \nu_t^i + \partial_{\nu_i} g(t, S_t^n, \boldsymbol{\nu}_t) \right) \right). \tag{1.5.28}
\end{aligned}$$

Using the form (1.5.24) of the function $z(t, s, \boldsymbol{\nu}, v)$, the approximate hedging ratio becomes

$$\begin{aligned}
\xi_t^{n,\mathcal{U}} = & \frac{1}{\hat{\nu}_t S_t^n} \left(g(t, S_t^n, \boldsymbol{\nu}_t) - V_t^{n,\mathcal{U}} \right) \left(\frac{1}{p-1} (\hat{\mu}_t - r) - \rho \zeta \hat{\nu}_t \sum_{i=1}^n D_t^i(c^*) \right) \\
& + \partial_s g(t, S_t^n, \boldsymbol{\nu}_t) + \frac{\rho \zeta}{S_t^n} \sum_{i=1}^n \left(\partial_{\nu_i} g(t, S_t^n, \boldsymbol{\nu}_t) \right).
\end{aligned}$$

That concludes the proof since we proved the announced result. $\square$

For the partial hedging problem in the Volterra Heston model with power loss function, the lower bound $\underline{l}_n^{\mathcal{U}}(\cdot)$ and the approximate hedging ratio $\xi_t^{n,\mathcal{U}}$ admit semi-explicit formulas. In particular, we observe that $\xi_t^{n,\mathcal{U}}$ is split into three parts: the first part is linked to a sharpe ratio, the second part corresponds to the Delta and the third part is linked to a Vega. Moreover, as the multifactor Heston model allows a closed form for the characteristic function of the log-price, it enables, using a Fourier pricing method, to efficiently compute the price but also the Greeks of vanilla options. Thus, for vanilla options, the lower bound as well as the approximate hedging ratio can be quickly computed. Note that the characteristic function of the log-price is presented in the Appendix 1.9.2.

1.6 Numerical results

In this section, we illustrate the partial hedging method discussed in this chapter for the Volterra Heston model. Without loss of generality, we consider the following parameters (under the real measure $\mathbb{P}$) for the Volterra Heston model:

$$r = 0.02, A = 1, S_0 = 100, \nu_0 = 0.09, \theta = 0.09, \kappa = 2.95, \zeta = 0.3 \text{ and } \rho = -0.7. \tag{1.6.1}$$

Remark 1.10. With those parameters and $T = 0.25$, the Feller's condition for the multifactor Heston model (see Appendix 1.9.1) is satisfied for different Hurst indexes H and number of factors n . For example, if $H = 0.1$, the sufficient condition of strict positivity of the volatility is satisfied for all $n \leq 20$. Figure 1.1 shows the region for which the Feller's condition is satisfied as a function of the Hurst coefficient H and the number of factors n .

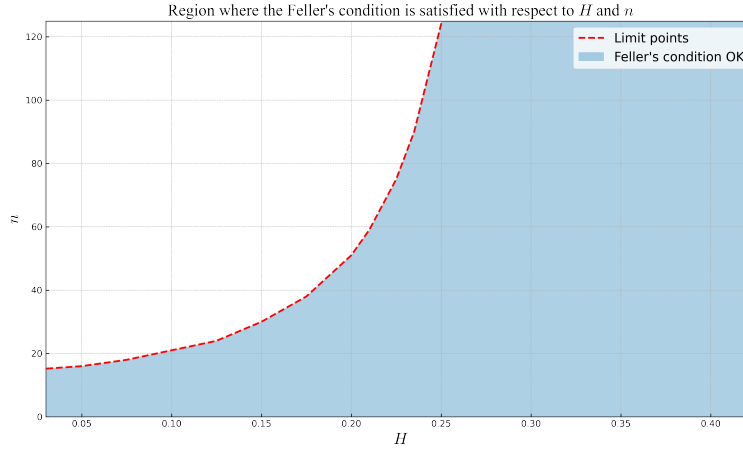


Figure 1.1: Region where the Feller's condition is satisfied with respect to H and n for the multifactor Heston with parameters (1.6.1) and $T = 0.25$.

As already mentioned, path-dependent, rough volatility models and their Markovian approximation, including the Volterra Heston model, allow to capture stylized facts observed on the financial markets. In particular, in option pricing, they allow to represent the implied volatility smile as well as the ATM (at-the-money) skew. Using the characteristic function of the log-price in the multifactor Heston, we can price vanilla options using a Fourier method. From these prices, we can deduce the implied volatility for several strikes and time-to-maturity but also compute the ATM skew i.e. the derivative of the ATM implied volatility with respect to the log strikes. As discussed in Abi Jaber (2019), the difference between the characteristics (implied volatility smile and ATM skew) of the rough Heston and its Markov approximation can be negligible even with a reduced number of factors (for example, $n = 20$). Thus, as $n = 20$ is sufficient for a relevant approximation of the rough Heston model, without loss of generality, we take $n = 20$ for most of the numerical results regarding partial hedging strategies. For more analysis and discussions about the rough Heston and its Markov approximation, we refer among others to Abi Jaber (2019); Abi Jaber and El Euch (2019a); Abi Jaber and De Carvalho (2024); El Euch and Rosenbaum (2019); Hainaut (2022a).

1.6.1 Partial hedging of linear payoff

In this section, we consider the hedging of a linear payoff with a quadratic loss function. We take this toy case because we have shown that with a quadratic loss, the approximate partial hedging problem is solved with a closed formula. Thus, the objective of this section is to check the convergence of the approximate solution when $n \rightarrow +\infty$ but also to verify that the lower and upper bounds of the optimal solution deduced using the dual control method are close enough to this optimal solution. The linear payoff considered for the following is given by

$$H_T^{linear} = \alpha + \beta S_T,$$

and its replication price at time $t \in [0, T]$ is

$$H_t^{linear} = e^{-r(T-t)}\alpha + \beta S_t.$$

We will thus consider several initial values of the hedging portfolio such as

$$V_0 \leq H_0^{linear}.$$

We first look at the convergence of the approximate optimal solution l_n when $n \rightarrow +\infty$. Figures 1.2 and 1.3 present the evolution of l_n and $\xi_0^{n,*}$ with respect to n for different values of H^{10} and with $V_0 = 0.8 \times H_0^{linear}$. We observe that, as expected from the theoretical results, the approximate solution and initial hedging ratio converge for different Hurst coefficients.

We next compare the optimal solution with the upper and lower bounds deduced by the dual control method. As in the theoretical part, we consider dual controls of the form:

$$\gamma_t = c \times Z_t \times \zeta \sqrt{\hat{\nu}_t}, \quad c \in U = \mathbb{R}, \quad (1.6.2)$$

because it allows to keep closed formulas under the approximate rough Heston model. In this case, the associated lower bound is given by (1.5.23) and the upper bound can be computed with the hedging ratio given by (1.5.27).

Table 1.1 compares the results for different values of V_0 . Firstly, we observe that the optimal solution, the lower and upper bounds are very close, as we notice an absolute error that varies between $O(10^{-4})$ and $O(10^{-2})$, this allows to validate the relevance of the bounds deduced by using the dual control method. We also notice that the closer V_0 is to the replication price, the more the quadratic loss decreases and finally, we observe that the quadratic loss is less when considering $H = 0.1$ compared to $H = 0.5$. This seems to indicate that for a linear payoff, the rougher is the volatility, the lower is the quadratic loss. The orders of the absolute errors between the bounds and the primal solutions are consistent with the order of the

¹⁰Results are generated to check the convergence but we have to keep in mind that the sufficient condition of strict positivity of the volatility is not satisfied for all n and H .

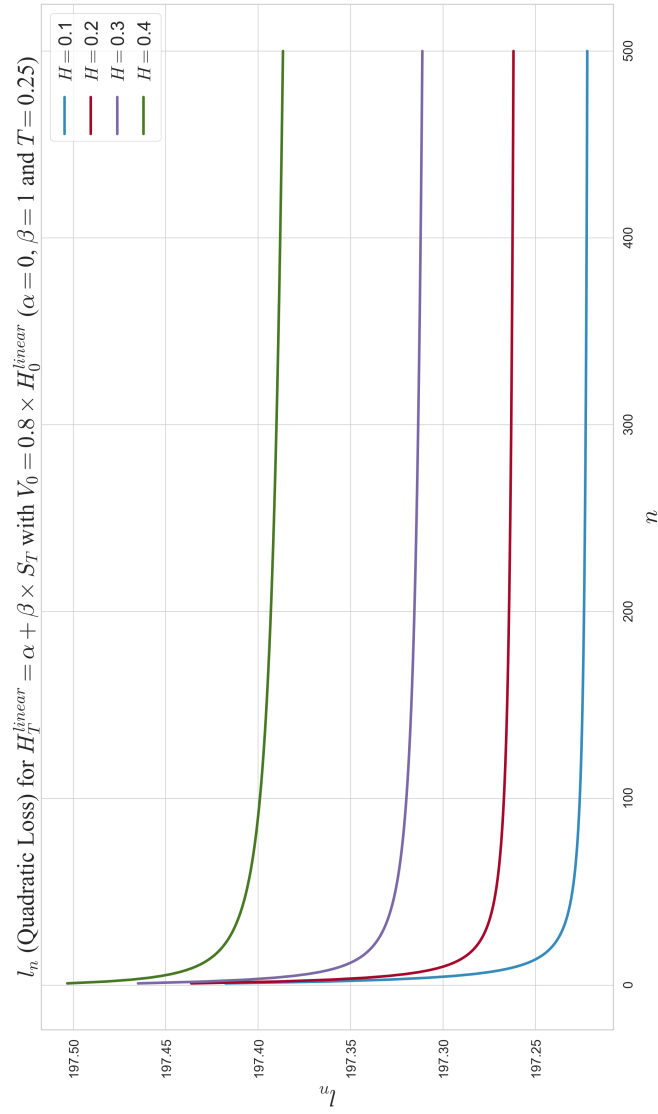


Figure 1.2: Convergence analysis of l_n for quadratic loss and linear payoff with $\alpha = 0, \beta = 1$ and $T = 0.25$

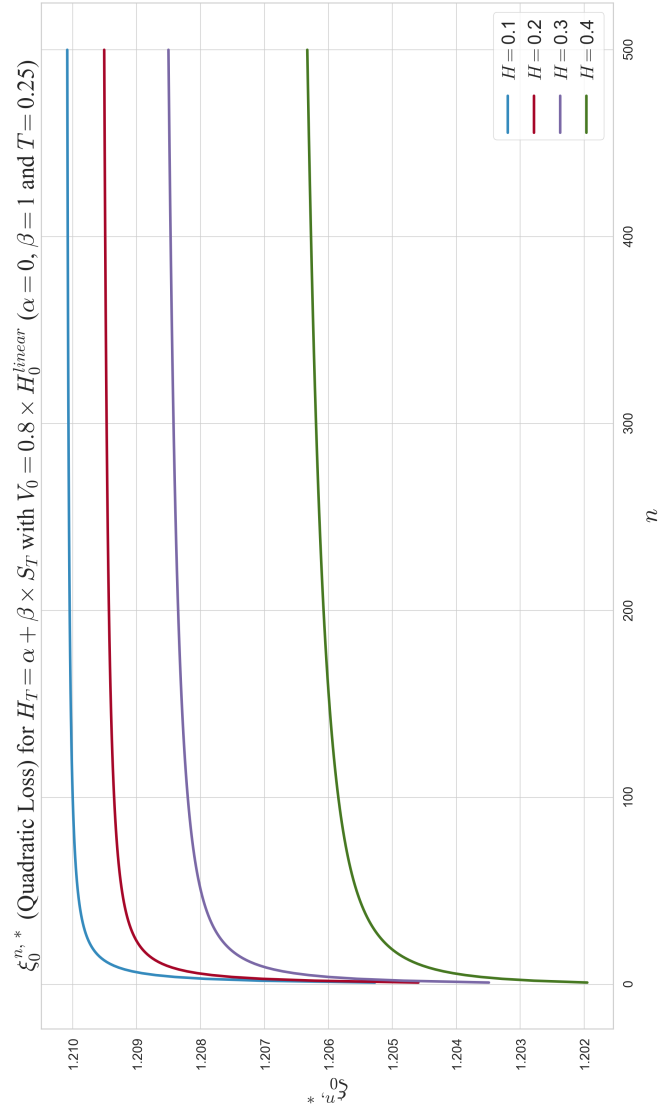


Figure 1.3: Convergence analysis of $\xi_0^{n,*}$ for quadratic loss and linear payoff with $\alpha = 0, \beta = 1$ and $T = 0.25$

$Coeff$	H	l_n	$\bar{l}_n^{\mathcal{U}}$	$\bar{l}_n^{\mathcal{U}}$	$ l_n - \bar{l}_n^{\mathcal{U}} $	$ \bar{l}_n^{\mathcal{U}} - l_n $	$ \bar{l}_n^{\mathcal{U}} - \bar{l}_n^{\mathcal{U}} $
0.75	0.1	308.1899	308.1888	308.1925	1.11×10^{-3}	2.64×10^{-3}	3.71×10^{-3}
0.75	0.5	308.3811	308.3759	308.3853	5.26×10^{-3}	4.28×10^{-3}	9.42×10^{-3}
0.5	0.1	1232.7597	1232.7553	1232.7693	4.41×10^{-3}	9.62×10^{-3}	1.43×10^{-2}
0.5	0.5	1233.5242	1233.5234	1233.5304	8.23×10^{-4}	6.21×10^{-3}	7.45×10^{-3}
0.25	0.1	2773.7095	2773.6996	2773.7263	9.91×10^{-3}	1.68×10^{-2}	2.67×10^{-2}
0.25	0.5	2775.4295	2775.4226	2775.4338	6.96×10^{-3}	4.32×10^{-3}	1.12×10^{-2}

Table 1.1: Comparison of the optimal solution, the lower bound and the upper bound for quadratic loss and linear payoff with $V_0 = Coef \times H_0^{linear}$, $\alpha = 0$, $\beta = 1$ and $T = 0.25$ (for $H = 0.1$; $n = 20$; for $H = 0.5$; $n = 1$, $w_1 = 1$ and $x_1 = 0$).

absolute errors made in applying the dual control method to portfolio optimization problems as in Ma et al. (2020); Ma et al. (2023). This can be explained by the fact that the problem reduces to a portfolio optimization problem when considering partial hedging of linear payoff. Indeed, assuming for simplicity that $r = 0$, we can easily observe that our problem has the form of a portfolio optimization problem such that

$$l_n(t, s, \boldsymbol{\nu}, v) = \inf_{\xi_t^n \in \mathcal{R}_n} \mathbb{E}^{\mathbb{P}}_{t,s,\boldsymbol{\nu},v} \left(\frac{1}{2} \left(\alpha + \beta S_t^n - V_t^n + \int_t^T (\beta - \xi_s^n) dS_s^n \right)^2 \right).$$

Therefore, we observe that in the case of a linear payoff, there is no stochastic target to reach, which is not the case if we consider nonlinear payoffs like, for example, vanilla options.

1.6.2 Partial hedging of vanilla options

We now focus on more relevant payoffs, namely vanilla options. As for the linear case, we take a quadratic loss function. Notice that we only consider Call options, but similar results can be deduced for Put options. For this type of payoff, we cannot derive an optimal solution to the partial hedging problem, but by using the dual control method, we can derive upper and lower bounds. To do this, for the same reasons as in the previous section, we consider dual controls of the form (1.6.2). Moreover, as the characteristic function of the log-price in the multifactor Heston model is available, the lower bound as well as the approximate hedge ratio are computed using a Fourier method. For our numerical results, we decide to consider an initial portfolio value V_0 proportional to the Black-Scholes (BS) price of a Call option with as constant volatility, the mean-reverting level of the rough model. Thus, we consider that

$$V_0 = Cof \times BS_{call}(\sigma = \sqrt{\theta}).$$

Furthermore, as in Jonsson and Sircar (2002), we benchmark the proposed partial hedging strategy with the optimal strategy assuming a constant volatility model of the BS type. Notice that for the BS model, the optimal hedge ratio denoted by ξ_t^{BS} associated with a quadratic loss has an explicit formula that is presented in Appendix 1.9.3, and has a similar form to $\xi_t^{n,\mathcal{U}}$ except that it does not depend on the volatility process. Figure 1.4 presents the lower bound $\ell_n^{\mathcal{U}}$ as a function of the number of factors n , for different values of H . As in the linear payoff case, we notice a convergence as the number of factors increases.

Figure 1.5 compares the histogram of the simulated P&L generated by the sub-optimal strategy ($\xi_t^{n,\mathcal{U}}$) and the benchmark strategy (ξ_t^{BS} , $\sigma = \sqrt{\theta}$) with a daily hedging frequency. We observe that the sub-optimal strategy generates a more centered distribution with a thinner left tail and lower variance than the benchmark strategy. This is also confirmed by Table 1.3 which shows, for different strikes

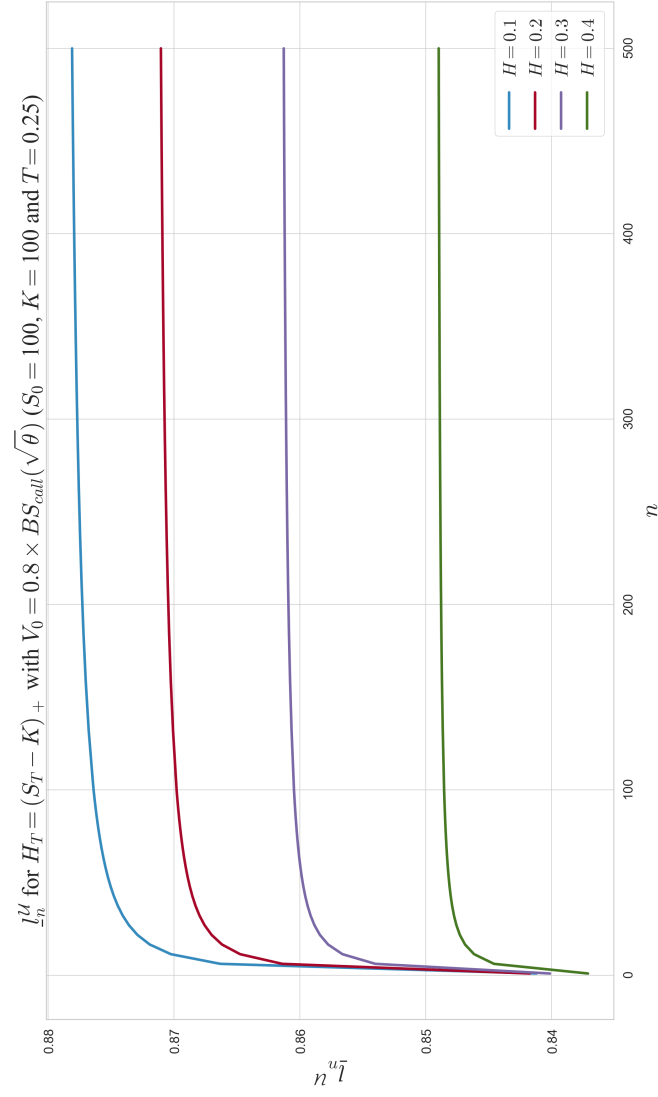


Figure 1.4: Convergence analysis for l_n^L for quadratic loss and ATM call with $V_0 = 0.8 \times BS_{call}(\sigma = \sqrt{\theta})$ and $T = 0.25$.

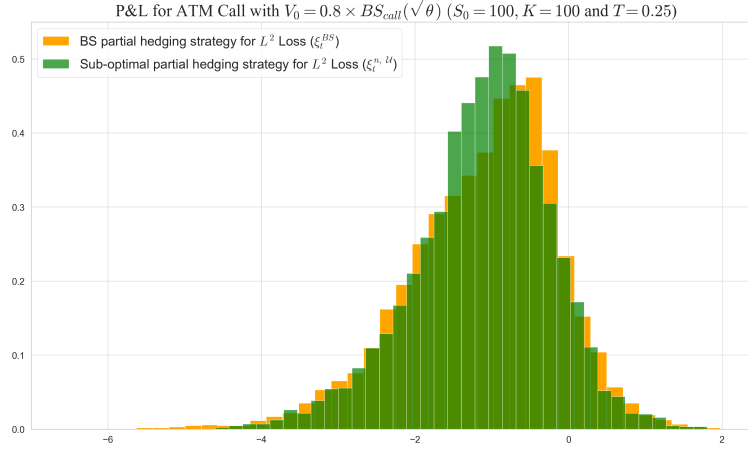


Figure 1.5: Histogram of the P&Ls associated to the sub-optimal strategy $(\xi_t^{n,\mathcal{U}})$ and the benchmark strategy $(\xi_t^{BS}, \sigma = \sqrt{\theta})$, ATM Call with $V_0 = 0.8 \times BS_{call}(\sigma = \sqrt{\theta})$, $H = 0.1$, $n = 20$, $T = 0.25$ and $\Delta_t = 1/365$.

and hedging frequencies, that the sub-optimal strategy outperforms the benchmark strategy in terms of the variance of the P&L. Note that in terms of P&L expectation, the benchmark slightly outperforms the sub-optimal strategy, but the gap between the two strategies is not very significant, as revealed by Table 1.2.

Strike	$\mathbb{E}^{\mathbb{P}}(P\&L_T)$ with $\xi_t^{n,\mathcal{U}}$	$\mathbb{E}^{\mathbb{P}}(P\&L_T)$ with ξ_t^{BS}
S_0	-1.1376	-1.1284
$0.9 \times S_0$	-2.6539	-2.6413
$1.1 \times S_0$	-0.0801	-0.0757

Table 1.2: Expectation of the P&Ls associated to the sub-optimal strategy $(\xi_t^{n,\mathcal{U}})$ and the benchmark strategy $(\xi_t^{BS}, \sigma = \sqrt{\theta})$ with $V_0 = 0.8 \times BS_{call}(\sigma = \sqrt{\theta})$, $H = 0.1$, $n = 20$, $T = 0.25$ and $\Delta_t = 1/365$.

Table 1.4 compares the results for different values of V_0 . In contrast to the linear payoff, the gap between the upper and lower bounds is more pronounced. There are several reasons explaining this. As the vanilla options are nonlinear payoffs, the partial hedging problem cannot be reduced to a portfolio optimization problem. Therefore, we have a stochastic target leading to a more noisy problem. In this case, we notice that the approximate hedging ratio depends more on the choice of the sub-optimal dual control. In fact, in contrast with the linear case, we remark that the approximate hedging ratio (1.5.27) is affected by the choice of dual control,

Strike	Δ_t	$\mathbb{V}(P\&L_T)$ with $\xi_t^{n,\mathcal{U}}$	$\mathbb{V}(P\&L_T)$ with ξ_t^{BS}	Rel. diff.
S_0	1/365	1.0157	1.1520	13.42%
S_0	1/730	0.8456	0.9901	17.08%
$0.9 \times S_0$	1/365	0.7931	1.0756	35.61%
$0.9 \times S_0$	1/730	0.6421	0.9553	48.78%
$1.1 \times S_0$	1/365	0.9086	1.0588	16.53%
$1.1 \times S_0$	1/730	0.7479	0.9031	20.76%

Table 1.3: Variance of the P&Ls associated to the sub-optimal strategy $(\xi_t^{n,\mathcal{U}})$ and the benchmark strategy $(\xi_t^{BS}, \sigma = \sqrt{\theta})$ with $V_0 = 0.8 \times BS_{call}(\sigma = \sqrt{\theta})$, $H = 0.1$, $n = 20$, and $T = 0.25$.

$Coeff$	H	$\underline{l}_n^{\mathcal{U}}$	$\bar{l}_n^{\mathcal{U}}$	$l_{BS(\sqrt{\theta})}$	$ \bar{l}_n^{\mathcal{U}} - \underline{l}_n^{\mathcal{U}} $	$ l_{BS(\sqrt{\theta})} - \bar{l}_n^{\mathcal{U}} $
1	0.1	0.0927	0.4659	0.5678	0.3732	0.1019
1	0.5	0.0147	0.2601	0.2943	0.2454	0.0306
0.75	0.1	1.2929	1.5342	1.6358	0.2413	0.1016
0.75	0.5	1.2621	1.4210	1.4454	0.1589	0.0244
0.5	0.1	4.7731	4.9361	5.0357	0.1631	0.0996
0.5	0.5	4.7994	4.9231	4.9436	0.1237	0.0205
0.25	0.1	10.5916	10.6751	10.7555	0.0835	0.0804
0.25	0.5	10.6839	10.7528	10.7870	0.0689	0.0342

Table 1.4: Comparison of the lower bound, the upper bound and the benchmark ($l_{BS(\sqrt{\theta})}$ with $\sigma = \sqrt{\theta}$) for quadratic loss, ATM Call with $V_0 = Coef \times BS_{Call}(\sigma = \sqrt{\theta})$, $T = 0.25$, $\Delta_t = 1/365$ (for $H = 0.1$: $n = 20$; for $H = 0.5$: $n = 1$, $w_1 = 1$ and $x_1 = 0$).

notably through the risk-neutral measure $\mathbb{Q}(c^*)$ used to compute the Greeks. The difference between the bounds can also be explained by the tracking error i.e. the error made by not continuously hedging the portfolio. Indeed, we observe in Table 1.5 that by increasing the frequency of portfolio rebalancing, the gap between the bounds decreases since the variance of the simulated P&L decreases as the hedging frequency increases (see also Table 1.3).

Benchmarking the sub-optimal strategy against the Black-Scholes strategy in terms of quadratic loss, we observe at Tables 1.4 and 1.5 that the hedging strategy associated with the upper bound outperforms the benchmark strategy, and the gap between the two strategies is much more pronounced when $H = 0.1$. Thus, the rougher is the volatility trajectory, the worse the benchmark strategy performs compared to the sub-optimal strategy.

<i>Coe f</i>	Δ_t	$\underline{l}_n^{\mathcal{U}}$	$\bar{l}_n^{\mathcal{U}}$	$l_{BS(\sqrt{\theta})}$	$ \bar{l}_n^{\mathcal{U}} - \underline{l}_n^{\mathcal{U}} $	$ l_{BS(\sqrt{\theta})} - \bar{l}_n^{\mathcal{U}} $
1	1/365	0.0927	0.4659	0.5678	0.3732	0.1019
1	1/730	0.0927	0.3773	0.4831	0.2846	0.1058
0.75	1/365	1.2929	1.5342	1.6358	0.2413	0.1016
0.75	1/730	1.2929	1.4562	1.5549	0.1633	0.0987
0.5	1/365	4.7731	4.9361	5.0357	0.1631	0.0996
0.5	1/730	4.7731	4.8513	4.9598	0.0712	0.1085
0.25	1/365	10.5916	10.6751	10.7555	0.0835	0.0804
0.25	1/730	10.5916	10.6333	10.7143	0.0418	0.0813

Table 1.5: Comparison of the lower bound, the upper bound and the benchmark ($l_{BS(\sqrt{\theta})}$ with $\sigma = \sqrt{\theta}$) for quadratic loss, ATM Call for different hedging frequencies with $H = 0.1$, $n = 20$, $V_0 = Coef \times BS_{Call}(\sigma = \sqrt{\theta})$ and $T = 0.25$.

Finally, it is also interesting to study the impact of the correlation ρ on the hedging strategy. Figures 1.6 and 1.7 present the impact of the correlation ρ on the bounds. We observe that the loss is reduced as ρ decreases because the closer the absolute value of the correlation is to 1, the more the volatility risk can be hedged by taking a hedging strategy on the underlying. We also observe that the relative difference between the upper bound and the benchmark loss widens as the correlation decreases. This can be explained by the fact that the closer the absolute value correlation is to 1, the larger is the difference between the approximate hedging ratio $\xi_t^{n,\mathcal{U}}$ and the Black-Scholes hedging ratio ξ_t^{BS} .

1.7 Concluding remarks

This chapter discusses partial hedging strategies in rough volatility models. We formulate the problem as a stochastic control problem but, due to the non-Markovian nature of the rough volatility models, this problem is considerably difficult to solve.

Thanks to a Markov multifactor approximation of the volatility process, we introduce a Markov stochastic control problem. We show, using convergence results, that, instead of solving the original problem, we can solve the Markov problem with a small error. The optimal solution to this problem is characterized by a Hamilton-Jacobi-Bellman (HJB) equation. However, even by switching to the dual formulation of the problem, we need to solve a nonlinear PDE to obtain the optimal solution. Therefore, in general, we cannot derive an explicit form of the optimal solution.

In order to obtain explicit hedging strategies, we introduce a dual control method. We derive lower and upper bounds as well as sub-optimal hedging ratios for sub-optimal choices of dual control. Moreover, if the subset of admissible dual controls

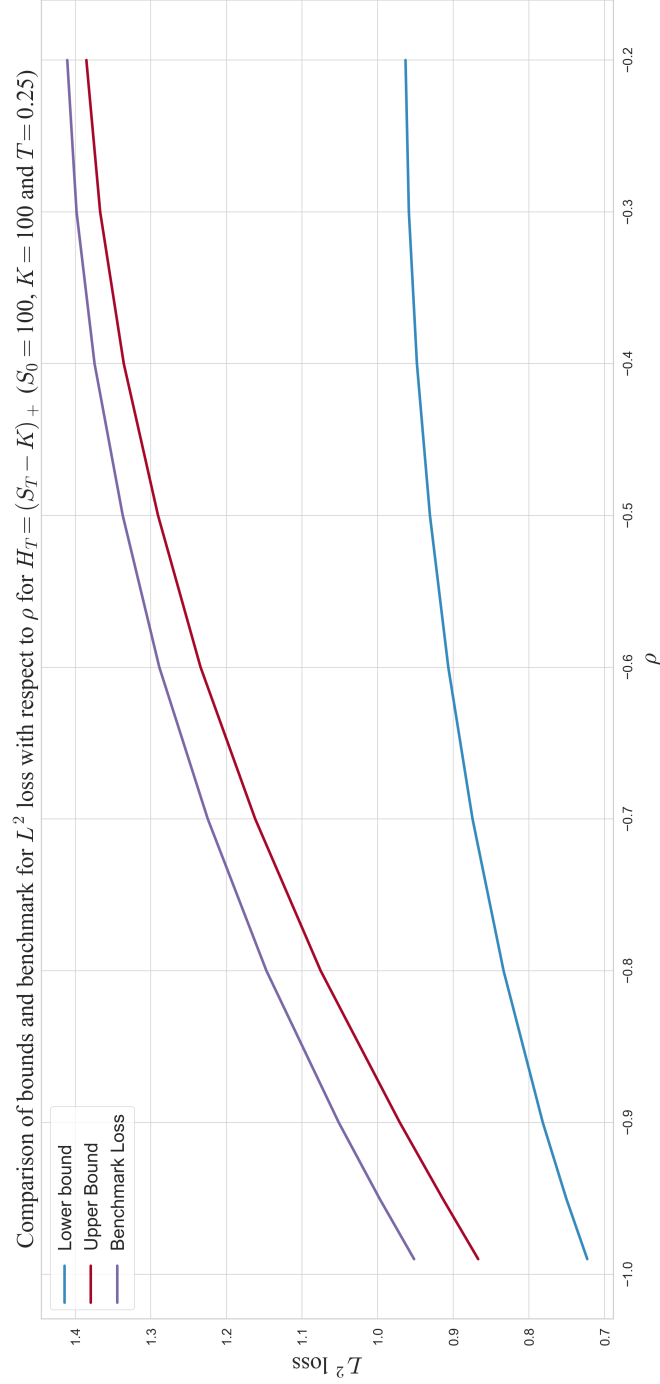


Figure 1.6: Impact of the correlation ρ on bounds and benchmark loss, for ATM Call with $H = 0.1$, $n = 20$, $V_0 = 0.8 \times BS_{call}(\sqrt{\theta})$, $\Delta_t = 1/365$ and $T = 0.25$.

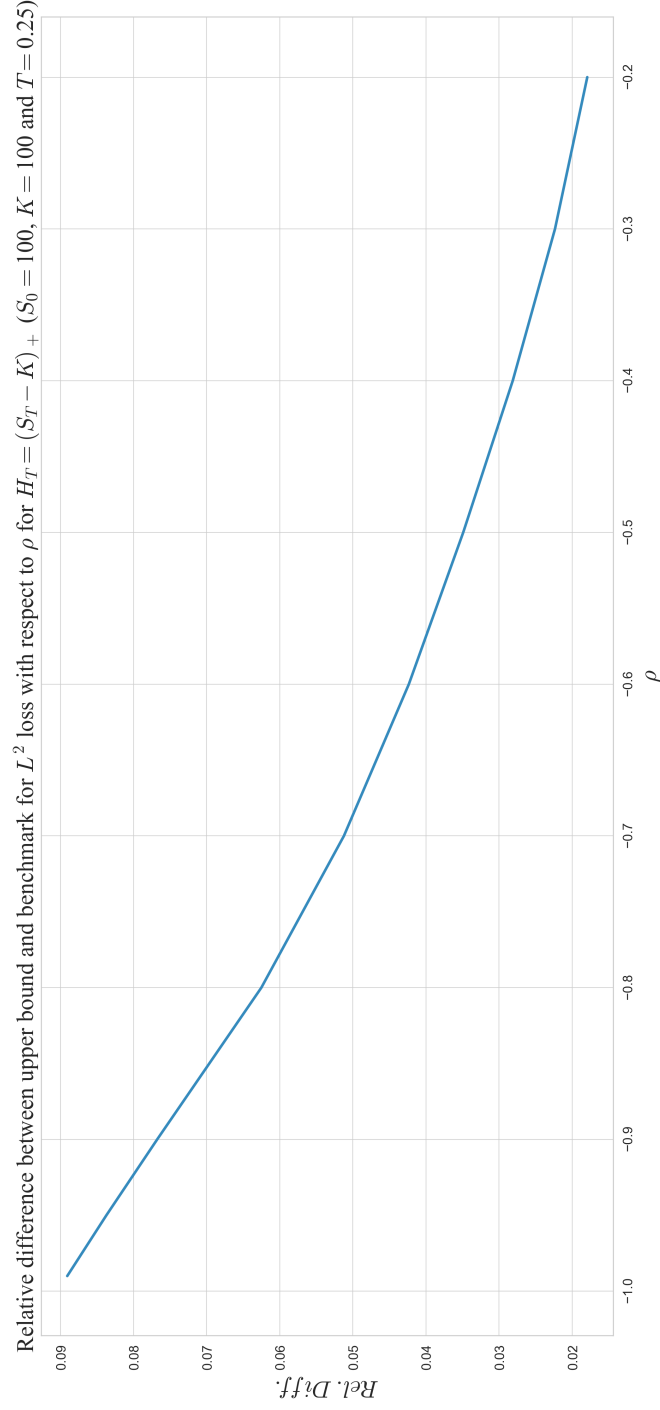


Figure 1.7: Impact of the correlation ρ on the relative difference between the upper bound and the benchmark loss, for ATM Call with $H = 0.1$, $n = 20$, $V_0 = 0.8 \times BS_{call}(\sqrt{\theta})$, $\Delta_t = 1/365$ and $T = 0.25$.

is large enough, we show that the discrepancies between bounds and the optimal solution are quite small. For a particular subset, explicit formulas for lower bound and sub-optimal hedging ratio are deduced in Volterra Heston model with power loss function. Furthermore, in the Volterra Heston model, the sub-optimal hedging ratio exhibits a meaningful interpretation in term of Greeks and can be efficiently computed using a Fourier method for hedging of vanilla options.

Numerical results show satisfying results especially for linear payoffs hedging since errors between bounds and optimal solution are of order $\mathcal{O}(10^{-3})$. For vanilla option hedging, the discrepancy between the bounds is slightly larger, yet remains acceptable. This can be explained by the fact that the nonlinear payoff hedging problem is noisier and that the sub-optimal choices of the dual control have more influence on the hedging strategies than in the linear case.

1.8 Proofs

Proof of Proposition 1.4.4

Proof. First, based on their SDE and provided that $(S_t^n)_{0 \leq t \leq T}$ and $(S_t)_{0 \leq t \leq T}$ have the same initial value $S_0 = S_0^n = s_0$, it can be shown that the processes are solutions of

$$\begin{aligned} S_t &= s_0 \times \exp \left(\int_0^t (\mu_s - \frac{1}{2} \nu_s) ds + \int_0^t \sqrt{\nu_s} dW_S(s) \right) \\ S_t^n &= s_0 \times \exp \left(\int_0^t (\hat{\mu}_s - \frac{1}{2} \hat{\nu}_s) ds + \int_0^t \sqrt{\hat{\nu}_s} dW_S(s) \right), \end{aligned}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. By Theorem (1.4.3) and uniform integrability of $(\hat{\nu}_t)_{n \geq 1}$, for $t \in [0, T]$,

$$E(\hat{\nu}_t) \rightarrow E(\nu_t),$$

as n goes to infinity. By applying Fubini's theorem, we obtain that

$$\lim_{n \rightarrow \infty} E \left(\int_0^t \hat{\nu}_s ds \right) = E \left(\int_0^t \nu_s ds \right).$$

Therefore, by the definition of the stochastic Itô integral, we obtain that

$$\int_0^t \sqrt{\hat{\nu}_s} dW_S(s) \xrightarrow{L^2} \int_0^t \sqrt{\nu_s} dW_S(s)$$

But since L^2 convergence implies convergence in law, we obtain that

$$\int_0^t \sqrt{\hat{\nu}_s} dW_S(s) \xrightarrow{\mathcal{L}} \int_0^t \sqrt{\nu_s} dW_S(s).$$

Moreover, using Theorem (1.4.3), we deduce that

$$\int_0^t (\hat{\mu}_s - \frac{1}{2}\hat{\nu}_s)ds \xrightarrow{\mathcal{L}} \int_0^t (\mu_s - \frac{1}{2}\nu_s)ds.$$

Finally as the exponential function is continuous, we conclude that

$$\exp\left(\int_0^t (\hat{\mu}_s - \frac{1}{2}\hat{\nu}_s)ds + \int_0^t \sqrt{\hat{\nu}_s}dW_S(s)\right) \xrightarrow{\mathbb{M}} \exp\left(\int_0^t (\mu_s - \frac{1}{2}\nu_s)ds + \int_0^t \sqrt{\nu_s}dW_S(s)\right)$$

and then

$$S_t^n \xrightarrow{\mathcal{L}} S_t,$$

as n goes to infinity.

The proof of the convergence of $(V_t^n)_{0 \leq t \leq T}$ is similar. For sake of simplicity, we only consider the case where $r = 0$. In this case, as we assume $\forall t \in [0, T], \xi_t^n = \xi_t$, we have that

$$\begin{aligned} V_t &= v_0 + \int_0^t \xi_s \mu_s S_s ds + \int_0^t \xi_s \sqrt{\nu_s} S_s dW_S(s) \\ V_t^n &= v_0 + \int_0^t \xi_s \hat{\mu}_s S_s^n ds + \int_0^t \xi_s \sqrt{\hat{\nu}_s} S_s^n dW_S(s), \end{aligned}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. By Theorem (1.4.3), and Fubini's theorem, for $t \in [0, T]$,

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}} \left(\int_0^t \xi_s^2 \hat{\nu}_s S_s^n ds \right) = \mathbb{E}^{\mathbb{P}} \left(\int_0^t \xi_s^2 \nu_s S_s ds \right).$$

Therefore, by the definition of the stochastic Itô integral, we obtain that

$$\int_0^t \xi_s \sqrt{\hat{\nu}_s} S_s^n dW_S(s) \xrightarrow{L_2} \int_0^t \xi_s \sqrt{\nu_s} S_s dW_S(s).$$

But since L^2 convergence implies convergence in law, we obtain that

$$\int_0^t \xi_s \sqrt{\hat{\nu}_s} S_s^n dW_S(s) \xrightarrow{\mathcal{L}} \int_0^t \xi_s \sqrt{\nu_s} S_s dW_S(s).$$

Finally, as

$$\int_0^t \xi_s \hat{\mu}_s S_s^n ds \xrightarrow{\mathcal{L}} \int_0^t \xi_s \mu_s S_s ds,$$

we conclude that

$$V_t^n \xrightarrow{\mathcal{L}} V_t,$$

as n goes to infinity. □

1.8.1 Proof of Proposition 1.5.2

Proof. To deduce the PDE of $\hat{l}_n(t, s, \nu, z)$ using the PDE of $l_n(t, s, \nu, v)$, we use the following relations between the primal and dual solution. We can remark that

$$\begin{aligned}\partial_v l_n &= z, \quad \partial_{vv} l_n = -\frac{1}{\partial_{zz} \hat{l}_n} \\ \partial_t l_n &= \partial_t \hat{l}_n, \quad \partial_{sv} l_n = -\frac{\partial_{sz} \hat{l}_n}{\partial_{zz} \hat{l}_n} \\ \partial_s l_n &= \partial_s \hat{l}_n, \quad \partial_{\nu_i v} l_n = -\frac{\partial_{\nu_i z} \hat{l}_n}{\partial_{zz} \hat{l}_n} \\ \partial_{\nu_i} l_n &= \partial_{\nu_i} \hat{l}_n,\end{aligned}$$

and

$$\begin{aligned}\partial_{ss} l_n &= \frac{\partial_{zz} \hat{l}_n \partial_{ss} \hat{l}_n - (\partial_{sz} \hat{l}_n)^2}{\partial_{zz} \hat{l}_n} \\ \partial_{\nu_i \nu_j} l_n &= \frac{\partial_{zz} \hat{l}_n \partial_{\nu_i \nu_j} \hat{l}_n - (\partial_{\nu_i z} \hat{l}_n)(\partial_{\nu_j z} \hat{l}_n)}{\partial_{zz} \hat{l}_n} \\ \partial_{\nu_i s} l_n &= \frac{\partial_{zz} \hat{l}_n \partial_{\nu_i s} \hat{l}_n - (\partial_{sz} \hat{l}_n)(\partial_{\nu_i z} \hat{l}_n)}{\partial_{zz} \hat{l}_n}.\end{aligned}$$

Using now the PDE (1.5.5) satisfied by $l_n(\cdot)$, we can easily derive that the PDE satisfied by $\hat{l}_n(\cdot)$ is given by

$$\begin{aligned}0 &= \partial_t \hat{l}_n + \mathcal{L}_{s, \nu} \hat{l}_n - zr \partial_z \hat{l}_n + \frac{1}{2\hat{\nu} s^2} z^2 (\hat{\mu} - r)^2 s^2 \partial_{zz} \hat{l}_n - z (\hat{\mu} - r) s \partial_{sz} \hat{l}_n \\ &\quad - \frac{1}{\sqrt{\hat{\nu}}} \rho \sum_{i=1}^n \partial_{\nu_i z} \hat{l}_n z (\hat{\mu} - r) \sigma(\hat{\nu}) - \frac{1}{2\partial_{zz} \hat{l}_n} \sigma(\hat{\nu})^2 (1 - \rho^2) \sum_{i=1}^n \sum_{j=1}^n \partial_{\nu_i z} \hat{l}_n \partial_{\nu_j z} \hat{l}_n.\end{aligned}$$

Thus we just have shown that $\hat{l}_n$ satisfies the announced PDE and that concludes the proof. $\square$

1.8.2 Proof of Lemma 1.5.5

Proof. The proof is simple and involves the first order condition since $L(h, v)$ is convex with respect to the variable v . By definition, the dual terminal value is such that

$$\hat{L}(h(s), z) := \inf_v \{L(h(s), v) - zv\},$$

in our case, it reduces to

$$\hat{L}(h(s), z) = \inf_v \left\{ \frac{1}{p} (h(s) - v)^p - zv \right\}. \quad (1.8.1)$$

Using the first order condition, we have that the optimal v is such that

$$-(h(s) - v)^{p-1} - z = 0,$$

we easily deduce that

$$v = h(s) - (-z)^{\frac{1}{p-1}}.$$

Plugging this value in (1.8.1), we have that

$$\begin{aligned} \hat{L}(h(s), z) &= \frac{1}{p}(-z)^{\frac{p}{p-1}} - z(h(s) - (-z)^{\frac{1}{p-1}}) \\ &= \frac{1}{p}z^{\frac{p}{p-1}} - z^{\frac{p}{p-1}} - zh(s) \\ &= -\frac{p-1}{p}z^{\frac{p}{p-1}} - zh(s) \\ &= -\frac{1}{q}z^q - zh(s). \end{aligned}$$

□

1.8.3 Proof of Proposition 1.5.6

Proof. From (1.5.2), we know that $\hat{l}_n$ satisfies the PDE (1.5.6). Let consider the ansatz

$$\hat{l}_n(t, s, \boldsymbol{\nu}, z) = \underbrace{-\frac{z^q}{q} \exp\left(C_t + \sum_{i=1}^n D_t^i \nu^i\right)}_{:=\hat{l}_n^1} - \underbrace{\left(e^{-r(T-t)}\alpha + \beta s\right)}_{:=g(t,s)} z, \quad (1.8.2)$$

with

$$\begin{aligned} C_T &= 0, \\ D_T^i &= 0, \quad i = 1, \dots, n \\ g(t, S_t) &= H_T^{linear} \end{aligned}$$

such that

$$\hat{l}_n(T, s, \boldsymbol{\nu}, z) = \hat{L}(h(s), z) = -\frac{1}{q}z^q - h(s)^{linear} z.$$

Plugging (1.8.2) in the PDE (1.5.6), we obtain

$$\begin{aligned}
0 = & \hat{l}_n^1 \left(\partial_t C_t + \sum_{i=1}^n \partial_t D_t^i \nu_i + \sum_{i=1}^n D_t^i (-x_i \nu_i + \kappa(\theta - \hat{\nu})) + \zeta^2 \hat{\nu} \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \right. \\
& - r q + \frac{1}{2} q(q-1) A^2 \hat{\nu} - q \rho A \hat{\nu} \zeta \sum_{i=1}^n D_t^i - \frac{1}{2} (1 - \rho^2) \zeta^2 \hat{\nu} \frac{q}{q-1} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \Big) \\
& - z \left(\partial_t g + \partial_s g \hat{\mu} s + \frac{1}{2} \partial_{ss} g \hat{\nu} s^2 - r g - (\hat{\mu} - r) s \partial_s g \right),
\end{aligned}$$

equivalently, we have

$$\begin{aligned}
0 = & \hat{l}_n^1 \left(\partial_t C_t + \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) - q \rho A \nu_0 \zeta \right) + \right. \\
& \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \zeta^2 \nu_0 \left(1 - (1 - \rho^2) \frac{q}{q-1} \right) - r q + \frac{1}{2} q(q-1) A^2 \nu_0 \Big) \\
& + \hat{l}_n^1 \sum_{i=1}^n \nu_i \left(\partial_t D_t^i - x_i D_t^i - w_i \left(\kappa + q \rho A \zeta \right) \sum_{j=1}^n D_t^j \right. \\
& + \frac{1}{2} w_i \left(\zeta^2 - (1 - \rho^2) \zeta^2 \frac{q}{q-1} \right) \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \\
& \left. + \frac{1}{2} w_i q(q-1) A^2 \right) \\
& - z \left(\partial_t g + \partial_s g \mu s + \frac{1}{2} \partial_{ss} g \hat{\nu} s^2 - r g - (\hat{\mu} - r) s \partial_s g \right).
\end{aligned}$$

Using the definition of $g(t, s)$, we observe that

$$\partial_t g + \partial_s g \mu s + \frac{1}{2} \partial_{ss} g \nu s^2 - r g - (\hat{\mu} - r) s \partial_s g = 0.$$

In this case, the PDE reduces to

$$\begin{aligned}
0 = & \hat{l}_n^1 \left(\partial_t C_t + \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) - q\rho A\nu_0\zeta \right) \right. \\
& + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \zeta^2 \nu_0 \left(1 - (1 - \rho^2) \frac{q}{q-1} \right) - rq + \frac{1}{2} q(q-1) A^2 \nu_0 \Big) \\
& + \hat{l}_n^1 \sum_{i=1}^n \nu_i \left(\partial_t D_t^i - x_i D_t^i - w_i \left(\kappa + q\rho A\zeta \right) \sum_{j=1}^n D_t^j \right. \\
& + \frac{1}{2} w_i \left(\zeta^2 - (1 - \rho^2) \zeta^2 \frac{q}{q-1} \right) \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \\
& \left. + \frac{1}{2} w_i q(q-1) A^2 \right).
\end{aligned}$$

We obtain that C_t and $(D_t^i)_{i=1, \dots, n}$ solve Riccati type ODEs of the form

$$\begin{aligned}
\partial_t C_t = & rq - \frac{1}{2} q(q-1) A^2 \nu_0 - \sum_{i=1}^n D_t^i \left(\kappa(\theta - \nu_0) - q\rho A\nu_0\zeta \right) \\
& - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n D_t^i D_t^j \zeta^2 \nu_0 \left(1 - (1 - \rho^2) \frac{q}{q-1} \right), \tag{1.8.3} \\
C_T = & 0
\end{aligned}$$

and for $i = 1, \dots, n$,

$$\begin{aligned}
\partial_t D_t^i = & x_i D_t^i + w_i \sum_{j=1}^n D_t^j \left(\kappa + q\rho A\zeta \right) - \frac{1}{2} w_i \left(\zeta^2 - (1 - \rho^2) \zeta^2 \frac{q}{q-1} \right) \sum_{j=1}^n \sum_{k=1}^n D_t^j D_t^k \\
& - \frac{1}{2} w_i q(q-1) A^2, \tag{1.8.4} \\
D_T^i = & 0.
\end{aligned}$$

Moreover, by (1.5.10), we know that the value of the dual process at time $t \in [0, T]$ can be expressed in function of $(t, s, \boldsymbol{\nu}, v)$ such that $z = z(t, s, \boldsymbol{\nu}, v)$ with $z(t, s, \boldsymbol{\nu}, v)$ satisfying

$$\partial_z \hat{l}_n + v = 0.$$

We easily deduce that

$$v = g(t, s) + z^{q-1} \exp(C_t + \sum_{i=1}^n D_t^i \nu^i),$$

and

$$z(t, s, \boldsymbol{\nu}, v) = \exp \left(-\frac{1}{q-1} (C_t + \sum_{i=1}^n D_t^i \nu^i) \right) \times \left(v - g(t, s) \right)^{\frac{1}{q-1}}. \quad (1.8.5)$$

Therefore, using the relation between the primal and the dual, we conclude that the primal solution is given by

$$l_n(t, s, \boldsymbol{\nu}, v) = -\frac{z(t, s, \boldsymbol{\nu}, v)^q}{q} \exp \left(C_t + \sum_{i=1}^n D_t^i \nu^i \right) - z(t, s, \boldsymbol{\nu}, v) \left(e^{-r(T-t)} \alpha + \beta s - v \right),$$

with $z(t, s, \boldsymbol{\nu}, v)$ satisfying (1.8.5). Let now focus on the expression of the optimal hedging ratio. For that, we need to compute partial derivatives of $\hat{l}_n$. Using the closed form of $\hat{l}_n$, we deduce that

$$\partial_z \hat{l}_n = -z^{q-1} \exp(C_t + \sum_{i=1}^n D_t^i \nu^i) - g(t, s),$$

and

$$\partial_{zz} \hat{l}_n = -(q-1) z^{q-2} \exp(C_t + \sum_{i=1}^n D_t^i \nu^i)$$

$$\partial_{zs} \hat{l}_n = -\beta$$

$$\partial_{z\nu_i} \hat{l}_n = -z^{q-1} D_t^i \exp(C_t + \sum_{i=1}^n D_t^i \nu^i),$$

Plugging now these values in the expression of the optimal hedging ratio leads to

$$\begin{aligned} \xi_t^{n,*} &= \frac{1}{\hat{\nu}_t(S_t^n)^2} \left(-(q-1) z(t, S_t^n, \boldsymbol{\nu}_t, V_t^n)^{q-1} \exp(C_t + \sum_{i=1}^n D_t^i \nu_t^i) (\hat{\mu}_t - r) S_t^n \right. \\ &\quad \left. + \beta \hat{\nu}_t(S_t^n)^2 + \rho \zeta S_t^n \hat{\nu}_t \sum_{i=1}^n \left(z(t, S_t^n, \boldsymbol{\nu}_t, V_t^n)^{q-1} D_t^i \exp(C_t + \sum_{i=1}^n D_t^i \nu_t^i) \right) \right). \end{aligned} \quad (1.8.6)$$

Finally, using the fact that $Z_t = z(t, S_t^n, \boldsymbol{\nu}_t, V_t^n)$ with $z(t, s, \boldsymbol{\nu}, v)$ satisfying (1.8.5), we conclude that the optimal hedging ratio is given by

$$\xi_t^{n,*} = \frac{1}{\hat{\nu}_t S_t^n} \left(e^{-r(T-t)} \alpha + \beta S_t^n - V_t^n \right) \left(\frac{1}{p-1} (\hat{\mu}_t - r) - \rho \zeta \hat{\nu}_t \sum_{i=1}^n D_t^i \right) + \beta.$$

That concludes the proof as we have shown all the stated results. $\square$

1.9 Appendix

1.9.1 Feller condition in the multifactor Heston

Proposition. Assume that the approximate volatility process $(\hat{\nu}_t)_{0 \leq t \leq T}$ is modeled by a multifactor Heston model with $b(x) = \kappa(\theta - x)$, $\sigma(x) = \zeta\sqrt{x}$ and suppose that $(w_i)_{i=1, \dots, n}$ and $(x_i)_{i=1, \dots, n}$ satisfy Assumption 1. If $2\kappa\theta > \zeta^2 \sum_{i=1}^n w_i$ and $\nu_0 > 0$ such that for all $t \in [0, T]$, $i \in \{1, \dots, n\}$ and $\varepsilon > 0$,

$$\mathbb{E}^{\mathbb{P}} \left(1_{\{t \leq \tau_\varepsilon\}} w_i \nu_t^i \hat{\nu}_t^{-(m+1)} \right) \leq \mathbb{E}^{\mathbb{P}} \left(1_{\{t \leq \tau_\varepsilon\}} \hat{\nu}_t^{-m} \right),$$

with $\tau_\varepsilon := \min\{t \geq 0 : \hat{\nu}_t \leq \varepsilon\}$ and $m := \frac{2\kappa\theta - \zeta^2 \sum_{i=1}^n w_i}{\zeta^2 \sum_{i=1}^n w_i}$. Then $\forall t \in [0, T]$,

$$\hat{\nu}_t > 0,$$

almost surely.

Proof. Let $\varepsilon > 0$ and, for $t \in [0, T]$, denote $\tau_\varepsilon := \min\{t \geq 0 : \hat{\nu}_t \leq \varepsilon\}$ and $\tau_\varepsilon \wedge t := \min\{\tau_\varepsilon, t\}$. Define a strict positive constant m such that

$$m := \frac{2\kappa\theta - \zeta^2 \sum_{i=1}^n w_i}{\zeta^2 \sum_{i=1}^n w_i} > 0$$

and apply the Ito's lemma to the function $f(x) = x^{-m}$. In this case as

$$\hat{\nu}_t = \nu_0 + \sum_{i=1}^n w_i \nu_t^i,$$

with for $i = 1, \dots, n$

$$d\nu_t^i = \left(-x_i \nu_t^i + \kappa(\theta - \hat{\nu}_t) \right) dt + \zeta \sqrt{\hat{\nu}_t} dW_\nu(t),$$

we deduce that

$$\begin{aligned}
\hat{\nu}_{\tau_\varepsilon \wedge t}^{-m} &= \nu_0^{-m} - \int_0^{\tau_\varepsilon \wedge t} m \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1)} \sum_{i=1}^n w_i \left(-x_i \nu_{\tau_\varepsilon \wedge s}^i + \kappa(\theta - \hat{\nu}_{\tau_\varepsilon \wedge s}) \right) ds \\
&\quad + \frac{1}{2} \int_0^{\tau_\varepsilon \wedge t} m(m+1) \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+2)} \zeta^2 \hat{\nu}_{\tau_\varepsilon \wedge s} \left(\sum_{i=1}^n w_i \right)^2 ds \\
&\quad - \int_0^{\tau_\varepsilon \wedge t} m \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1/2)} \zeta \sum_{i=1}^n w_i dW_\nu(s) \\
&= \nu_0^{-m} + m\kappa \sum_{i=1}^n w_i \int_0^{\tau_\varepsilon \wedge t} \hat{\nu}_{\tau_\varepsilon \wedge s}^{-m} ds \\
&\quad - m\zeta \sum_{i=1}^n w_i \int_0^{\tau_\varepsilon \wedge t} \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1/2)} dW_\nu(s) \\
&\quad + \int_0^{\tau_\varepsilon \wedge t} \left(\frac{m(m+1)}{2} \zeta^2 \sum_{i=1}^n w_i - m\kappa\theta \right) \sum_{i=1}^n w_i \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1)} ds \\
&\quad + \int_0^{\tau_\varepsilon \wedge t} m \sum_{i=1}^n w_i \left(x_i \nu_{\tau_\varepsilon \wedge s}^i \right) \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1)} ds.
\end{aligned}$$

Using the definition of m , we have that

$$\begin{aligned}
\hat{\nu}_{\tau_\varepsilon \wedge t}^{-m} &= \nu_0^{-m} + m\kappa \sum_{i=1}^n w_i \int_0^{\tau_\varepsilon \wedge t} \hat{\nu}_{\tau_\varepsilon \wedge s}^{-m} ds \\
&\quad - m\zeta \sum_{i=1}^n w_i \int_0^{\tau_\varepsilon \wedge t} \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1/2)} dW_\nu(s) \\
&\quad + \int_0^{\tau_\varepsilon \wedge t} m \sum_{i=1}^n w_i \left(x_i \nu_{\tau_\varepsilon \wedge s}^i \right) \hat{\nu}_{\tau_\varepsilon \wedge s}^{-(m+1)} ds.
\end{aligned}$$

Taking now the expectation, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left(\hat{\nu}_{\tau_\varepsilon \wedge t}^{-m}\right) &\leq \nu_0^{-m} + m\kappa \sum_{i=1}^n w_i \int_0^t E\left(\hat{\nu}_{\tau_\varepsilon \wedge s}^{-m}\right) ds \\
&\quad + \int_0^t m \sum_{i=1}^n x_i E\left(w_i 1_{\{s \leq \tau_\varepsilon\}} \nu_s^i \hat{\nu}_s^{-(m+1)}\right) ds \\
&\leq \nu_0^{-m} + m\kappa \sum_{i=1}^n w_i \int_0^t E\left(\hat{\nu}_{\tau_\varepsilon \wedge s}^{-m}\right) ds \\
&\quad + m \sum_{i=1}^n x_i \int_0^t E\left(1_{\{s \leq \tau_\varepsilon\}} \hat{\nu}_s^{-m}\right) ds \\
&\leq \nu_0^{-m} + m\left(\kappa \sum_{i=1}^n w_i + \sum_{i=1}^n x_i\right) \int_0^t E\left(\hat{\nu}_{\tau_\varepsilon \wedge s}^{-m}\right) ds,
\end{aligned}$$

Using the Grönwall's inequality, we deduce that

$$\mathbb{E}^{\mathbb{P}}\left(\hat{\nu}_{\tau_\varepsilon \wedge t}^{-m}\right) \leq \nu_0^{-m} \exp\left(m\left(\kappa \sum_{i=1}^n w_i + \sum_{i=1}^n x_i\right) t\right).$$

Finally, using the Markov's inequality, we obtain that

$$\begin{aligned}
\mathbb{P}(\tau_\varepsilon \leq t) &= \mathbb{P}(\hat{\nu}_{\tau_\varepsilon \wedge t}^m \leq \varepsilon^m) \\
&= \mathbb{P}(\hat{\nu}_{\tau_\varepsilon \wedge t}^{-m} \geq \varepsilon^{-m}) \\
&\leq \varepsilon^m \nu_0^{-m} \exp\left(m\left(\kappa \sum_{i=1}^n w_i + \sum_{i=1}^n x_i\right) t\right),
\end{aligned}$$

with $m > 0$. Therefore we conclude that for $t \in [0, T]$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\tau_\varepsilon \leq t) = 0$$

and $\forall t \in [0, T]$,

$$\hat{\nu}_t > 0,$$

almost surely. □

1.9.2 Characteristic function of the log-price in the multi-factor Heston

Remember that the approximate rough Heston volatility model is given by $\hat{\nu}_t = \nu_0 + \sum_{i=1}^n w_i \nu_t^i$ where $(\nu_t^1, \dots, \nu_t^n)$ is solution of the n -dimensional SDE defined by

$$\begin{aligned}
\nu_t^i &= - \int_0^t x_i \nu_s^i ds + \int_0^t \kappa(\theta - \hat{\nu}_s) ds + \int_0^t \zeta \sqrt{\hat{\nu}_s} dW_\nu(s), \quad i = 1, \dots, n, \\
\nu_0^i &= 0,
\end{aligned} \tag{1.9.1}$$

Proposition. *The characteristic function of the log-price in the multifactor Heston is given by, for $t \in [0, T]$,*

$$\phi_t(T, x) := \mathbb{E}_{t,s,\nu}^{\mathbb{P}} \left(\exp(i x \log(S_T^n)) \right) = \exp \left(C_t + \sum_{i=1}^n D_t^i \nu_t^i + i x \log(S_t^n) \right),$$

with for $0 \leq t \leq T$, C_t and $(D_t^i)_{i=1,\dots,n}$ solve Riccati type ODEs of the form

$$\begin{aligned} \partial_t C_t &= -r i x + \nu_0 \left(\frac{1}{2} x^2 - (A - \frac{1}{2}) i x \right) - \sum_{k=1}^n D_t^k (\rho \zeta \nu_0 i x + \kappa(\theta - \nu_0)) \\ &\quad - \frac{1}{2} \zeta^2 \nu_0 \sum_{k=1}^n \sum_{l=1}^n D_t^k D_t^l, \\ C_T &= 0 \end{aligned}$$

and for $k = 1, \dots, n$,

$$\begin{aligned} \partial_t D_t^k &= x_k D_t^k + \omega_k \left(\frac{1}{2} x^2 - (A - \frac{1}{2}) i x \right) - w_k \sum_{j=1}^n D_t^j \left(\rho \zeta i x - \kappa \right) \\ &\quad - \frac{1}{2} \zeta^2 w_k \sum_{j=1}^n \sum_{l=1}^n D_t^j D_t^l \\ D_T^k &= 0. \end{aligned}$$

Proof. The proof is a direct application of Feynman-Kac theorem with SDEs (1.3.6) and (1.9.1). $\square$

1.9.3 Partial hedging in Black and Scholes framework

Proposition. *Suppose that the risky asset $(S_t)_{0 \leq t \leq T}$ has the following Black-Scholes dynamic under the real measure $\mathbb{P}$*

$$dS_t = \mu S_t dt + \sigma S_t dW_S(t),$$

the optimal hedging ratio associated to a power loss $(\xi_t^{BS})_{0 \leq t \leq T}$ is given by

$$\xi_t^{BS} := \frac{1}{p-1} \left(\mathbb{E}^{\mathbb{Q}}(H_T | \mathcal{F}_t) - V_t \right) \frac{(\mu - r)}{\sigma S_t} + \partial_S \mathbb{E}^{\mathbb{Q}}(H_T | \mathcal{F}_t), \quad (1.9.2)$$

with the “classical” risk-neutral measure $\mathbb{Q}$ such that under this measure

$$dS_t = r S_t dt + \sigma S_t dW_S^{\mathbb{Q}}(t).$$

Proof. In Black and Scholes framework, the dual solution satisfies the PDE (1.5.6) that is linear (the derivatives with respect to the volatility process disappear). Then, for power loss functions, we can easily deduce the explicit form of the dual solution and, using the link between the optimal hedging ratio and the dual solution, we derive the form the optimal hedging ratio (1.9.2). $\square$

CHAPTER 2

The Volterra Stein-Stein model with stochastic interest rates

In this chapter, we introduce the Volterra Stein-Stein model with stochastic interest rates, where both volatility and interest rates are driven by correlated Gaussian Volterra processes. This framework unifies various well-known Markovian and non-Markovian models while preserving analytical tractability for pricing and hedging financial derivatives. We derive explicit formulas for pricing zero-coupon bond and interest rate cap or floor, along with a semi-explicit expression for the characteristic function of the log-forward index using Fredholm resolvents and determinants. This allows for fast and efficient derivative pricing and calibration via Fourier methods. We calibrate our model to market data and observe that our framework is flexible enough to capture key empirical features, such as the humped-shaped term structure of ATM implied volatilities for cap options and the concave ATM implied volatility skew term structure (in log-log scale) of the S&P 500 options. Finally, we establish connections between our characteristic function formula and expressions that depend on infinite-dimensional Riccati equations, thereby making the link with conventional linear-quadratic models.

2.1 Introduction

Modeling the joint dynamics of interest rates and equity is crucial for pricing and hedging hybrid derivatives. These products, which include equity-linked interest rate derivatives, callable hybrid structures, and quanto options, depend on the interaction between stock prices, interest rates, and their volatilities. A realistic model must capture the well-documented stylized facts of each underlying as well as accurately describe their dependency structure to ensure consistent pricing and

risk management across asset classes.

Empirical evidence reveals persistent memory effects in historical time series of interest rates and asset volatility and slow decay of their autocorrelations structures, see Cont (2001); Dai and Singleton (2003); McCarthy et al. (2004).

Beyond historical data, robust and universal patterns emerge in the term structures of option prices across a broad range of maturities. Specifically,

- (i) for interest rates: the term structure of the implied volatilities of cap and floor options is typically hump-shaped, with a steep increase at short maturities followed by a more gradual decay at longer maturities, see for instance Dai and Singleton (2003); De Jong et al. (2004);
- (ii) for stock indices such as the S&P 500: the term structure of the At-The-Money (ATM) skew of the implied volatility of call and put options, when plotted in log-log scale, exhibits typically a concave shape. In addition, the ATM skew displays an approximately linear decrease at long maturities suggesting a power-law decay, though only for sufficiently large maturities, see Bergomi (2015); Bayer et al. (2016); Delemotte et al. (2023); Guyon and El Amrani (2023); Abi Jaber and Li (2024).

Various approaches have been developed to jointly model equity and interest rate dynamics, often by extending stochastic volatility frameworks to include stochastic interest rates. For instance, Boyarchenko and Levendorskii (2007b); Grzelak et al. (2011); Boyarchenko and Levendorskii (2013); Singor et al. (2013) incorporated stochastic rates into the Heston (1993) model, while van Haastrecht et al. (2009); van Haastrecht and Pelsser (2011) adapted the Stein and Stein (1991) and Schöbel and Zhu (1999) volatility models with multi-factor Hull and White (1993) interest rates. However, these frameworks rely on Markovian dynamics with a single characteristic time scale for at least one of the two risk factors, and fail to capture the complex shapes of term structures observed on the market: they lead either to term structures of cap/floor implied volatilities that decrease monotonically - at odds with the empirically observed hump-shaped patterns - or ATM skews of stock indices implied volatility that decay too quickly for longer maturities. While such models offer analytical tractability, their inability to align with empirical data highlights the need for alternative approaches that incorporate more flexibility.

To address these limitations, we propose a flexible yet analytically tractable model based on Volterra processes, capable of accurately reproducing these empirical term structure features while remaining computationally efficient for option pricing and model calibration. Our motivation for incorporating Volterra processes stems from their proven ability to capture key stylized facts of volatility (Bayer et al. (2016); Gatheral et al. (2018); Guyon and Lekeufack (2023); Abi Jaber and Li (2024)) and interest rates (Corcuera et al. (2013); Benth and Rohde (2019); Abi Jaber (2022b); Hainaut (2022a)).

We propose a hybrid equity-rate modeling framework where both volatility and interest rates are driven by (possibly correlated) Gaussian Volterra processes, unifying a broad class of Markovian and non-Markovian models. Building on the Volterra Stein-Stein model with constant interest rates studied by Abi Jaber (2022a), we extend the framework to incorporate stochastic interest rates, capturing key market features more effectively. Our main contributions can be summarized as follows:

- **Mathematical tractability and pricing:** Despite the non-Markovian nature of the processes, we derive explicit pricing formulas for zero-coupon bonds and call and put options on zero-coupon bonds, see Propositions 2.2.1 and 2.2.2. We obtain a semi-explicit expression for the characteristic function of the log-forward index in terms of Fredholm (1903) resolvents and determinants, enabling Fourier-based pricing methods, see Theorem 2.3.1. This result extends the formula derived by Abi Jaber (2022a) for constant interest rates.
- **Flexibility and joint calibration:** We calibrate our model to market data and achieve excellent fits for: (i) the humped-shaped term structure of ATM implied volatilities for cap options by incorporating in interest rates mean reversion as well as long-range memory with a fractional kernel, see Figure 2.4, and (ii) the concave ATM implied volatility skew term structure (in a log-log plot) of S&P 500 options using a shifted fractional kernel, see Figure 2.9. On the selected calibration date, our estimated parameters yield a negative implied correlation between short rates and the index, aligning with historical observations. Furthermore, we compare the impact of a singular power-law kernel for volatility (rough volatility) and demonstrate that rough models underperform our non-rough counterparts in capturing the entire volatility surface, confirming within our framework the findings of Delemotte et al. (2023); Guyon and Lekeufack (2023); Abi Jaber and Li (2024).
- **Link with conventional linear-quadratic models:** We establish connections between our characteristic function formula and expressions that depend on infinite-dimensional Riccati equations, see Proposition 2.5.1 for general kernels and Proposition 2.5.2 for completely monotone convolution kernels. The latter formula establishes the link with conventional linear-quadratic models (Proposition 2.5.3), allows us to recover closed-form solutions in specific cases such as in van Haastrecht et al. (2009) (Corollary 2.5.4), and leads to another numerical approximation method based on multi-factor approximations in the spirit of Abi Jaber and El Euch (2019a), see Section 2.5.3.

The chapter is outlined as follows. In Section 2.2, we introduce the Volterra Stein-Stein model with stochastic rates, derive pricing formulas for zero-coupon bonds, and analyze the forward index dynamics. Section 2.3 provides a semi-explicit expression for the characteristic function of the log-index. In Section 2.4, we validate

the Gaussian Volterra framework by calibrating the interest rate and volatility models to market data. Finally, Section 2.5 establishes connections with conventional linear-quadratic models, derives simplified forms for Markovian volatility models, and introduces a multi-factor approximation for the characteristic function for completely monotone kernels.

2.2 The Volterra Stein-Stein and Hull-White model

We fix a finite horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$ where $\mathbb{Q}$ stands for one risk-neutral probability measure and the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions. We consider a financial market with a financial index denoted by $(I_t)_{0 \leq t \leq T}$ that depends on both stochastic interest rate $(r_t)_{0 \leq t \leq T}$ and volatility $(\nu_t)_{0 \leq t \leq T}$ with dynamic

$$dI_t = r_t I_t dt + \nu_t I_t dW_I^{\mathbb{Q}}(t).$$

In this chapter, the full triplet (I, r, ν) refers to a Volterra Stein-Stein model with stochastic interest rates since, to incorporate memory effects, the two risk factors (r, ν) are modeled by Gaussian Volterra processes:

$$r_t = r_0(t) + \int_0^t G_r(t, s) \kappa_r r_s ds + \int_0^t G_r(t, s) \eta_r dW_r^{\mathbb{Q}}(s), \quad (2.2.1)$$

$$\nu_t = g_0(t) + \int_0^t G_\nu(t, s) \kappa_\nu \nu_s ds + \int_0^t G_\nu(t, s) \eta_\nu dW_\nu^{\mathbb{Q}}(s), \quad (2.2.2)$$

where $(W_I^{\mathbb{Q}}, W_r^{\mathbb{Q}}, W_\nu^{\mathbb{Q}})$ are correlated Brownian motions such that

$$d\langle W_l^{\mathbb{Q}}, W_k^{\mathbb{Q}} \rangle_t = \rho_{lk} dt,$$

with $\rho_{lk} \in [-1, 1]$, $l, k \in \{I, r, \nu\}$ such that the correlation matrix is semi-definite positive, $\kappa_r, \kappa_\nu \in \mathbb{R}$, $\eta_r, \eta_\nu \in \mathbb{R}$, and the kernels $G_r(\cdot)$ and $G_\nu(\cdot)$ satisfy the following definition.

Definition 2.1. A kernel $G : [0, T]^2 \rightarrow \mathbb{R}$ is a Volterra kernel of continuous and bounded type in L^2 if $G(t, s) = 0$ for $s \geq t$ and $u \leq T$

$$\sup_{t \leq T} \int_0^t (|G(t, s)|^2 + |G(s, t)|^2) ds < +\infty, \quad \lim_{h \rightarrow 0} \int_0^T |G(u+h, s) - G(u, s)|^2 ds = 0.$$

Finally, r_0 is a time-dependent function used to perfectly fit the initial term-structure of market bond prices, see Remark 2.1, and g_0 is such that

$$g_0(t) = \nu_0 + \theta_\nu \int_0^t G_\nu(t, s) ds, \quad \nu_0 \in \mathbb{R}_+, \theta_\nu \in \mathbb{R}. \quad (2.2.3)$$

Under Definition 2.1 and for r_0, g_0 locally square integrable, it can be shown there exists a strong solution to (2.2.1) and (2.2.2), such that

$$\sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{Q}} [|r_t|^p + |\nu_t|^p] < +\infty, \quad p \geq 1,$$

this follows from an adaption of (Abi Jaber, 2022a, Theorem A.3). In particular, the joint process (r, ν) is a Gaussian process.

This class of models encompasses well-known Markov and non-Markovian Volterra processes. For an introduction to Volterra processes, we refer among others to (Hainaut, 2022a, Chapter 9). Several kernels satisfy Definition 2.1 such as:

- **the constant kernel:** $G(t, s) = 1_{s < t}$, then we obtain Markovian models such as the classical Stein and Stein (1991) or Schöbel and Zhu (1999) model for volatility and the Hull and White (1993) model for short-term interest rate, such models have been studied by van Haastrecht et al. (2009); Grzelak et al. (2012),
- **the exponential kernel:** $G(t, s) = 1_{s < t} \exp(-\beta(t - s))$ with $\beta \in \mathbb{R}_+$, then we also obtain Markovian models but with modified mean-reversion level and reversion speed,
- **the fractional kernel:** $G(t, s) = 1_{s < t} \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}$ with a Hurst index $H \in (0, 1)$ then we obtain non-semimartingale and non-Markovian long or short memory fractional processes whenever $H \neq 1/2$. For $H < 1/2$, the kernel is singular at $t \rightarrow s$, we get rough models as for example the rough Stein-Stein model studied by Abi Jaber (2022a) and for $H > 1/2$, we obtain long-memory processes as, for example, the fractional version of the Hull and White model, as studied by Jacquier and Oumgari (2024).
- **the shifted fractional kernel:** $G(t, s) = 1_{s < t} \frac{(t-s+\varepsilon)^{H-1/2}}{\Gamma(H+1/2)}$, with $\varepsilon > 0$ and $H \in \mathbb{R}$, then we obtain path-dependent processes that are semi-martingales but non-Markovian processes. Compared to the fractional kernel, the shifted fractional kernel has no singularity as $t \rightarrow s$, and can achieve faster decays than the fractional kernel for larger t with coefficients $H \leq 0$.

2.2.1 Pricing Zero-Coupon bonds and interest rate derivatives

Let us now consider the pricing of zero-coupon bonds as well as interest rate derivatives in our framework. The price at time $t \leq T$ of a T -maturity zero-coupon bond is denoted by $P(t, T)$ and is defined by

$$P(t, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right].$$

We now show that even considering a general Volterra Hull and White model for interest rate dynamics, the zero-coupon bond price admits an analytic expression. To this end, we introduce the concept of resolvent associated to a kernel and give some examples in Table 2.1. Note that we consider the resolvent definition of Abi Jaber (2022a), which may differ from other papers.

Definition 2.2. Let G be a kernel satisfying Definition 2.1. The resolvent of G is the kernel R_G satisfying Definition 2.1 and is such that

$$R_G = G + G \star R_G,$$

where the $\star$ -product is such that, for $(s, w) \in [0, T]^2$,

$$(G \star R_G)(s, w) = \int_0^T G(s, z) R_G(z, w) dz. \quad (2.2.4)$$

Proposition 2.2.1. Let us define, for $t \in [0, T]$, $f_t(\cdot)$ by

$$f_t(s) := r_0(s) + \int_0^t R_{\kappa_r G_r}(s, u) r_0(u) du + \frac{\eta_r}{\kappa_r} \int_0^t R_{\kappa_r G_r}(s, u) dW_r^{\mathbb{Q}}(u), \quad t \leq s, \quad (2.2.5)$$

with $R_{\kappa_r G_r}$ the resolvent associated to $\kappa_r G_r$ with the convention $\frac{R_{\kappa_r G_r}}{\kappa_r} = G_r$ if $\kappa_r = 0^1$. The price of the zero-coupon bond at time $t \in [0, T]$, is given by

$$P(t, T) = A(t, T) \exp \left(- \int_t^T f_t(s) ds \right), \quad (2.2.6)$$

with $A(t, T)$ the time-dependent function given by

$$A(t, T) := \exp \left(- \int_t^T \left(\int_u^T R_{\kappa_r G_r}(s, u) r_0(u) ds + \frac{\eta_r^2}{2} B_{G_r}^2(u, T) \right) du \right).$$

Moreover, the dynamics of $(P(t, T))_{t \in [0, T]}$ are given by

$$\frac{dP(t, T)}{P(t, T)} = r_t dt - \eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}}(t),$$

with

$$B_{G_r}(t, T) := \frac{1}{\kappa_r} \int_t^T R_{\kappa_r G_r}(s, t) ds. \quad (2.2.7)$$

Proof. Using the resolvent associated to the kernel $\kappa_r G_r$ and using (Abi Jaber, 2022a, Theorem A.3), we have that $(r_t)_{0 \leq t \leq T}$ is the strong solution of the following equation

$$r_t = r_0(t) + \int_0^t R_{\kappa_r G_r}(t, s) r_0(s) ds + \frac{1}{\kappa_r} \int_0^t R_{\kappa_r G_r}(t, s) \eta_r dW_r^{\mathbb{Q}}(s), \quad 0 \leq t \leq T,$$

¹This convention follows from $\lim_{\kappa_r \rightarrow 0} R_{\kappa_r G_r} / \kappa_r = G_r$.

Kernel $G(t, s)$	Resolvent $R_G(t, s)$	$\int_t^T R_G(s, t) ds$
$1_{s < t} c$	$1_{s < t} c e^{c(t-s)}$	$e^{c(T-t)} - 1$
$1_{s < t} c \exp(-\beta(t-s))$	$1_{s < t} c e^{(t-s)(c-\beta)}$	$\frac{c}{c-\beta} (e^{(c-\beta)(T-t)} - 1)$
$1_{s < t} c \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$	$1_{s < t} c(t-s)^{\alpha-1} E_{\alpha, \alpha}(c(t-s)^\alpha)$	$c(T-t)^\alpha E_{\alpha, \alpha+1}(c(T-t)^\alpha)$

Table 2.1: Particular kernels, the associated resolvents and the integral of these resolvents where $E_{\beta, \gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\beta + \gamma)}$.

with the convention $\frac{R_{\kappa_r G_r}}{\kappa_r} = G_r$ if $\kappa_r = 0$. In this case, we have that, for $0 \leq t \leq T$,

$$\begin{aligned} \int_t^T r_s ds &= \int_t^T r_0(s) ds + \int_t^T \int_0^s R_{\kappa_r G_r}(s, u) r_0(u) du ds \\ &\quad + \frac{1}{\kappa_r} \int_t^T \int_0^s R_{\kappa_r G_r}(s, u) \eta_r dW_r^{\mathbb{Q}}(u) ds. \end{aligned}$$

Using the definition of f_t given by (2.2.5) together with an application of stochastic Fubini's theorem, we obtain that

$$\begin{aligned} \int_t^T r_s ds &= \int_t^T f_t(s) ds + \int_t^T \int_u^T R_{\kappa_r G_r}(s, u) r_0(u) ds du \\ &\quad + \eta_r \int_t^T \underbrace{\frac{1}{\kappa_r} \int_u^T R_{\kappa_r G_r}(s, u) ds}_{:= B_{G_r}(u, T)} dW_r^{\mathbb{Q}}(u). \end{aligned}$$

Thus, for $t \in [0, T]$, conditional on $\mathcal{F}_t$, the random variable $\int_t^T r_s ds | \mathcal{F}_t$ is Gaussian with mean

$$\mathbb{E}^{\mathbb{Q}} \left[\int_t^T r_s ds \mid \mathcal{F}_t \right] = \int_t^T f_t(s) ds + \int_t^T \int_u^T R_{\kappa_r G_r}(s, u) r_0(u) ds du,$$

and variance

$$\mathbb{V}^{\mathbb{Q}} \left[\int_t^T r_s ds \mid \mathcal{F}_t \right] = \eta_r^2 \int_t^T B_{G_r}^2(u, T) du.$$

Therefore, we readily deduce that

$$P(t, T) = \exp \left(- \int_t^T f_t(s) ds - \int_t^T \left(\int_u^T R_{\kappa_r G_r}(s, u) r_0(u) ds + \frac{\eta_r^2}{2} B_{G_r}^2(u, T) \right) du \right),$$

and we finally obtain that

$$\frac{dP(t, T)}{P(t, T)} = r_t dt - \eta_r \int_t^T \frac{1}{\kappa_r} R_{\kappa_r G_r}(s, t) ds dW_r^{\mathbb{Q}}(t).$$

□

Remark 2.1. Proposition 2.2.1 provides a natural way to estimate the function $r_0(\cdot)$ such that the interest rate model matches perfectly the initial term-structure of market bond prices. In fact from (2.2.6), we deduce that

$$\int_0^t r_0(s) \left(1 + \kappa_r B_{G_r}(s, t) \right) ds = - \ln P(0, t) - \int_0^t \frac{\eta_r^2}{2} B_{G_r}^2(s, t) ds.$$

Thus, if we assume that, for some given maturities $0 = T_0 < T_1 < \dots < T_n$, we have the initial term-structure of market bond prices $P^{Market}(0, T_i)$, and that $r_0(t)$ is piecewise constant such that, for $t \in [T_{i-1}, T_i)$

$$r_0(t) = -\frac{\ln P^{Market}(0, T_i) - \ln P^{Market}(0, T_{i-1}) + \frac{\eta_r^2}{2} \int_{T_{i-1}}^{T_i} B_{G_r}^2(u, T_i) du}{(T_i - T_{i-1}) + \kappa_r \int_{T_{i-1}}^{T_i} B_{G_r}(s, T_i) ds}, \quad (2.2.8)$$

then $P(0, T_i) = P^{Market}(0, T_i)$, $i = 1, \dots, n$.

Based on the expression for the pricing of zero-coupon bonds, we can now easily deduce explicit expressions for the pricing of zero-coupon bond call and put options.

Proposition 2.2.2. *Let us consider T -maturity zero-coupon bond call option $(P(T, S) - K)_+$ and put option $(K - P(T, S))_+$ with $S > T$ and K the strike of the option. The arbitrage-free price at time $t \in [0, T]$ of the call option is given by*

$$Call_{ZC}(t, T, S, K) = P(t, S)\phi(-d_1(t, T)) - KP(t, T)\phi(-d_2(t, T)),$$

and the price of the put option is given by

$$Put_{ZC}(t, T, S, K) = KP(t, T)\phi(d_2(t, T)) - P(t, S)\phi(d_1(t, T)),$$

where $\phi(\cdot)$ is the cumulative function of a normal distribution and

$$d_1(t, T) = d_2(t, T) - v(t, T, S), \quad (2.2.9)$$

$$d_2(t, T) = \frac{1}{v(t, T, S)\sqrt{T-t}} \log \left(\frac{KP(t, T)}{P(t, S)} \right) + \frac{v(t, T, S)\sqrt{T-t}}{2}, \quad (2.2.10)$$

where

$$v(t, T, S)^2 = \frac{\eta_r^2 \int_t^T \left(B_{G_r}(s, T) - B_{G_r}(s, S) \right)^2 ds}{T-t} \quad (2.2.11)$$

Proof. Let us consider the call option. We know that the arbitrage-free price of a call on zero-coupon bond satisfies

$$Call_{ZC}(t, T, S, K) = P(t, T)\mathbb{E}^{\mathbb{Q}^T} [(P(T, S) - K)_+ | \mathcal{F}_t].$$

From Proposition 2.2.1, we know that under the risk-neutral measure $\mathbb{Q}$,

$$\frac{dP(t, S)}{P(t, S)} = r_t dt - \eta_r B_{G_r}(t, S) dW_r(t).$$

Therefore, for $S > T$, we obtain that under the T -forward measure $\mathbb{Q}^{T2}$, the zero-coupon bond dynamics is given by

$$\frac{dP(t, S)}{P(t, S)} = \left(r_t + \eta_r^2 B_{G_r}(t, S) B_{G_r}(t, T) \right) dt - \eta_r B_{G_r}(t, S) dW_r^{\mathbb{Q}^T}(t).$$

²We introduce the forward measure in Section 2.2.2.

Moreover, under $\mathbb{Q}^T$, we have that

$$d\left(\frac{P(t, S)}{P(t, T)}\right) = \frac{P(t, S)}{P(t, T)} \eta_r \left(B_{G_r}(t, T) - B_{G_r}(t, S) \right) dW_r^{\mathbb{Q}^T}(t)$$

and we deduce that

$$\mathbb{E}^{\mathbb{Q}^T} [(P(T, S) - K)_+ | \mathcal{F}_t] = \frac{P(t, S)}{P(t, T)} \phi(-d_1(t, T)) - K \phi(-d_2(t, T)),$$

where $\phi(\cdot)$ is the cumulative function of a normal distribution and d_1, d_2 are given by (2.2.9)-(2.2.10). Finally, using the call/put parity, we easily deduce the form of the put price. $\square$

Based on zero-coupon bonds call or put options, we can price cap and floor options since, as explained in Brigo and Mercurio (2006), cap and floor options can be decomposed into a sum of zero-coupon bonds options. In fact, the cap and floor payoffs are of the form

$$\begin{aligned} \text{Cap} &= \sum_{i=1}^{\beta} (T_i - T_{i-1}) (L(T_{i-1}, T_i) - K)_+, \\ \text{Floor} &= \sum_{i=1}^{\beta} (T_i - T_{i-1}) (K - L(T_{i-1}, T_i))_+ \end{aligned}$$

where L is the reference rate and $(T_1, \dots, T_{\beta})$ are the payment dates. Using Brigo and Mercurio (2006), we have that the price, at time $t < T_0$, of cap and floor options is given by

$$\begin{aligned} \text{Cap}(t, \beta, K) &= \sum_{i=1}^{\beta} (1 + K(T_i - T_{i-1})) \text{Put}_{ZC} \left(t, T_{i-1}, T_i, \frac{1}{1 + K(T_i - T_{i-1})} \right), \\ \text{Floor}(t, \beta, K) &= \sum_{i=1}^{\beta} (1 + K(T_i - T_{i-1})) \text{Call}_{ZC} \left(t, T_{i-1}, T_i, \frac{1}{1 + K(T_i - T_{i-1})} \right). \end{aligned}$$

This enable us to calibrate our Gaussian Volterra model for interest rate to the market data of cap and floor options for any Volterra kernel $G_r(\cdot)$.

2.2.2 Forward measure and pricing of derivatives on the index

Let us now focus on the pricing of financial derivatives on the index I in our framework. Standard arguments imply that the arbitrage-free price of a financial

derivative is obtained by taking the discounted value of a payoff under a risk-neutral measure $\mathbb{Q}$. In this case, if we consider a general T -maturity derivative of the form

$$H_T := h(I_T) \in L^2(\mathbb{Q}),$$

where $h(\cdot)$ is a continuous positive function, the arbitrage-free price at time $t \in [0, T]$, is given by

$$V_t = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} H_T \mid \mathcal{F}_t \right]. \quad (2.2.12)$$

In the presence of stochastic interest rates, except for linear payoffs, it is impossible to obtain a more explicit form of the price of financial derivatives on I if we consider the formulation under the risk-neutral measure. One technique for dealing with this problem is to switch from the risk-neutral measure to the T -forward neutral measure. Thus, let us introduce the T -forward neutral measure denoted by $\mathbb{Q}^T$ which is equivalent to the risk-neutral measure $\mathbb{Q}$. The change of measure from the risk-neutral measure $\mathbb{Q}$ to the T -forward measure $\mathbb{Q}^T$ satisfies

$$\left. \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \right|_t := e^{-\int_0^t r(s) ds} \frac{P(t, T)}{P(0, T)}.$$

We can now introduce the T -forward index denoted by $(I_t^T)_{0 \leq t \leq T}$ and defined as, for $0 \leq t \leq T$,

$$I_t^T := \mathbb{E}^{\mathbb{Q}^T} [I_T \mid \mathcal{F}_t] = \frac{I_t}{P(t, T)}.$$

Since under the forward measure the financial index divided by the zero-coupon bond is a martingale, we obtain that the process $(I_t^T)_{0 \leq t \leq T}$ has the following dynamic

$$\frac{dI_t^T}{I_t^T} = \nu_t dW_I^{\mathbb{Q}^T}(t) + \eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}^T}(t), \quad (2.2.13)$$

with

$$\nu_t = g_0^T(t) + \int_0^t G_\nu(t, s) \kappa_\nu \nu_s ds + \int_0^t G_\nu(t, s) \eta_\nu dW_\nu^{\mathbb{Q}^T}(s), \quad (2.2.14)$$

where $(W_r^{\mathbb{Q}^T}, W_I^{\mathbb{Q}^T}, W_\nu^{\mathbb{Q}^T})$ is a 3-dimensional correlated Brownian motion under the T -forward measure $\mathbb{Q}^T$ and

$$g_0^T(t) = g_0(t) - \eta_\nu \eta_r \rho_{r\nu} \int_0^t G_\nu(t, s) B_{G_r}(s, T) ds.$$

Remark 2.2. By Girsanov's theorem, the 3-dimensional correlated Brownian motion under the T -forward measure $(W_r^{\mathbb{Q}^T}, W_I^{\mathbb{Q}^T}, W_\nu^{\mathbb{Q}^T})$ satisfies, for $t \leq T$,

$$\begin{aligned} dW_r^{\mathbb{Q}^T}(t) &= dW_r^{\mathbb{Q}}(t) + \eta_r B_{G_r}(t, T) dt, \\ dW_I^{\mathbb{Q}^T}(t) &= dW_I^{\mathbb{Q}}(t) + \rho_{Ir} \eta_r B_{G_r}(t, T) dt, \\ dW_\nu^{\mathbb{Q}^T}(t) &= dW_\nu^{\mathbb{Q}}(t) + \rho_{r\nu} \eta_r B_{G_r}(t, T) dt. \end{aligned}$$

By combining the forward measure and the T -forward financial index, we obtain that the arbitrage-free price of a derivative (2.2.12) satisfies

$$V_t = P(t, T) \mathbb{E}^{\mathbb{Q}^T} \left[h(e^{\log I_T^T}) \mid \mathcal{F}_t \right].$$

By switching from the risk-neutral measure to the T -forward measure, we get a simpler framework for pricing financial derivatives. In the following, we will show that the characteristic function of the log-forward index $(\log I_t^T)_{0 \leq t \leq T}$ admits an explicit form. This opens the way to a fast pricing of financial derivatives using Fourier methods. Moreover, using hedging approaches developed in several papers El Euch and Rosenbaum (2018); Abi Jaber and Gérard (2025a) or in Chapter 1, some hedging strategies can also be deduced by Fourier methods for contingent claims that admit a Fourier representation.

For convenience of the reader, we end this section by collecting in Table 2.2 the main notation for the objects used in the sequel.

Object	Notation
Kernel	G .
Resolvent defined by (2.2.4)	R .
Integral of the resolvent defined by (2.2.7)	B_G .
Speed of mean reversion	κ .
Risk-factor volatility	η .
Correlation	ρ .

Table 2.2: Summary of notation.

2.3 The characteristic function of the log-forward index

The aim of this section is to derive an analytical form of the characteristic function of the log-forward index $(I_t^T)_{0 \leq t \leq T}$ with dynamics (2.2.13) under $\mathbb{Q}^T$ and apply it for Fourier pricing of derivatives on the index.

We start by recalling some results on operator theory in Hilbert spaces as well as introducing some notations. Let $\mathbf{A}$ be a linear compact operator acting on $L^2([0, T], \mathbb{C})$. Then, $\mathbf{A}$ is a bounded operator i.e. there exists $C > 0$ such that, for all $f \in L^2([0, T], \mathbb{C})$, $\|\mathbf{A}f\| \leq C\|f\|$ and for $B := \{f \in L^2([0, T], \mathbb{C}) : \|f\| \leq 1\}$, the closure of $\mathbf{A}(B)$ is compact in $L^2([0, T], \mathbb{C})$. $\mathbf{A}$ is an integral operator if $\mathbf{A}$ a linear operator induced by a kernel $G \in L^2([0, T]^2, \mathbb{R})$ such that, for $f \in L^2([0, T], \mathbb{C})$ and $s \in [0, T]$,

$$(\mathbf{A}f)(s) = \int_0^T G(s, w) f(w) dw.$$

Moreover, if $\mathbf{A}$ is an integral operator, then $\mathbf{A}$ is a Hilbert-Schmidt operator on $L^2([0, T], \mathbb{C})$ and is in particular compact. The trace of an operator, denoted by $\text{Tr}(\cdot)$, is defined for operators of trace class where a compact operator $\mathbf{A}$ is said to be of trace class if

$$\text{Tr}(\mathbf{A}) = \sum_{n \geq 1} \langle \mathbf{A}v_n, v_n \rangle < \infty,$$

for a given orthonormal basis $(v_n)_{n \geq 1}$. For more details about Hilbert-Schmidt operators and trace class operators, we refer to Gohberg and Krein (1978); Simon (2005); Gohberg et al. (2012), and also (Abi Jaber, 2022a, Section A).

Let us now introduce some notations and properties:

- $\langle \cdot, \cdot \rangle_{L^2}$ denotes the following product on $L^2([0, T], \mathbb{C})$

$$\langle f, g \rangle_{L^2} = \int_0^T f(s)g(s)ds, \quad f, g \in L^2([0, T], \mathbb{C}).$$

Note that $\langle \cdot, \cdot \rangle_{L^2}$ is the inner product in $L^2([0, T], \mathbb{R})$ but not in $L^2([0, T], \mathbb{C})$.

- For any kernels $G_1, G_2 \in L^2([0, T]^2, \mathbb{R})$, the $\star$ -product is defined by

$$(G_1 \star G_2)(s, w) = \int_0^T G_1(s, z)G_2(z, w)dz, \quad (s, w) \in [0, T]^2.$$

- For a kernel G , G^* denotes the adjoint kernel such that, for $(s, w) \in [0, T]^2$,

$$G^*(s, w) = G(w, s),$$

and $\mathbf{G}^*$ is the operator induced by G^* .

- id denotes the identity operator such that $(\text{id}f) = f$, $f \in L^2([0, T], \mathbb{C})$.

2.3.1 Semi-explicit expression

We now derive a semi-explicit form for the characteristic function of the log-forward index in our framework.

To this end, we consider an approach similar to Abi Jaber (2022a) which deduces the explicit form of the log-price characteristic function in equity markets for the Volterra Stein-Stein volatility model. Here, we show that this result can be extended to a framework where interest rates are also stochastic. As in Abi Jaber (2022a), we consider the expression of the adjusted conditional mean of ν_t and then we define a linear operator Ψ_t^u in $L^2([0, T], \mathbb{C})$ that will be useful for the expression

of the characteristic function. The adjusted conditional mean of ν_t denoted by g_t is given by

$$g_t(s) = 1_{t \leq s} \mathbb{E}^{\mathbb{Q}^T} \left[\nu_s - \int_t^T G_\nu(s, w) \kappa_\nu \nu_w dw \mid \mathcal{F}_t \right], \quad s, t \leq T,$$

and we can easily show that g_t reduces to

$$g_t(s) = 1_{t \leq s} \left(g_0^T(s) + \int_0^t G_\nu(s, w) \kappa_\nu \nu_w dw + \int_0^t G_\nu(s, w) \eta_\nu dW_\nu^{\mathbb{Q}^T}(w) \right), \quad s, t \leq T.$$

Definition 2.3. For $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, the operator Ψ_t^u acting on $L^2([0, T], \mathbb{C})$ is defined by

$$\Psi_t^u := (id - b^u \mathbf{G}_\nu^*)^{-1} a^u (id - 2a^u \tilde{\Sigma}_t^u)^{-1} (id - b^u \mathbf{G}_\nu)^{-1}, \quad t \leq T, \quad (2.3.1)$$

where:

- $\mathbf{G}_\nu$ is the integral operator induced by $G_\nu(\cdot)$ and $\mathbf{G}_\nu^*$ the adjoint operator,
- $a^u := \frac{1}{2}(u^2 - u)$ and $b^u := \kappa_\nu + \eta_\nu u \rho_{I\nu}$,
- $\tilde{\Sigma}_t^u$ is the adjusted covariance integral operator such that

$$\tilde{\Sigma}_t^u = (id - b^u \mathbf{G}_\nu)^{-1} \Sigma_t (id - b^u \mathbf{G}_\nu^*)^{-1},$$

with Σ_t the integral operator associated with the covariance kernel given by

$$\Sigma_t(s, w) = \eta_\nu^2 \int_t^T G_\nu(s, z) G_\nu(w, z) dz, \quad t \leq s, w \leq T.$$

Note that using similar arguments to those in the proof of (Abi Jaber et al., 2021a, Lemma 5.6.), we can prove that Ψ_t^u is well defined and is a bounded linear operator acting on $L^2([0, T], \mathbb{C})$.

Based now on g_t and Ψ_t^u , we deduce a semi-explicit expression of the characteristic function. The methodology used in Abi Jaber (2022a) to deduce the analytical expression of the characteristic function can be extended to a framework with stochastic interest rates. Nevertheless, to extend these results, we need to also account for correlation between processes and consider in this sense the process $h_t^u(\cdot)$ defined, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, as, for $s, t \leq T$,

$$h_t^u(s) := g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_r}(s, T) - \int_t^s G_\nu(s, w) (b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(w, T) dw \right),$$

with B_{G_r} given by (2.2.7). In particular when $\rho_{Ir} = \rho_{\nu r} = 0$, $h_t^u(\cdot)$ reduces to $g_t(\cdot)$ and we recover the framework of Abi Jaber (2022a).

Theorem 2.3.1. *Let $g_0(\cdot)$ be given by (2.2.3) and $G_\nu(\cdot)$ a Volterra kernel as in Definition 2.1. Fix $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, then, for all $t \leq T$,*

$$\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] = \exp(\phi_t^u + \chi_t^u + \langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2}), \quad (2.3.2)$$

with:

- Ψ_t^u given by (2.3.1),
- $\phi_t^u = - \int_t^T \text{Tr}(\Psi_s^u \dot{\Sigma}_s) ds$, where $\dot{\Sigma}_t$ is the strong derivative of $t \rightarrow \Sigma_t$ induced by the kernel i.e.

$$\dot{\Sigma}_t(s, u) = -\eta_\nu^2 G_\nu(s, t) G_\nu(u, t), \text{ a.e.}$$

- $\chi_t^u = \frac{1}{2}(u^2 - u) \int_t^T (1 - \rho_{I_r}^2) \eta_r^2 B_{G_r}(s, T)^2 ds$.

Proof. The proof is given in Section 2.6. □

2.3.2 Numerical implementation and Fourier pricing

For a practical perspective, we propose a natural way of approximating the expression of the characteristic function when we consider general Volterra kernels that lead, for example, to non-Markovian models such as in the case of the fractional or shifted fractional kernels. This approximation is based on a natural discretization of the inner product $\langle \cdot, \cdot \rangle_{L^2}$. In this section, for sake of simplicity, we fix $t = 0$.

As proposed by Abi Jaber (2022a), we can approximate the explicit expression of the characteristic function (2.3.2) by an approximate closed form solution using a simple discretization of the operator Ψ_0^u à la Fredholm, which consists of approximating the operators by some matrices. To this end, for fixed $N \in \mathbb{N}$, we consider a uniform partition³ of $[0, T]$ with $t_i = i \frac{T}{N}$ for $i = 0, \dots, N$. Then, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, we can approximate the operator Ψ_0^u by a $N \times N$ matrix denoted $\Psi_0^{u;N}$ and defined by

$$\Psi_0^{u;N} := (I_N - b^u(\mathbf{G}_\nu^N)^T)^{-1} a^u (I_N - 2a^u \tilde{\Sigma}_0^{u;N})^{-1} (I_N - b^u \mathbf{G}_\nu^N)^{-1}, \quad (2.3.3)$$

where I_N is the $N \times N$ identity matrix, $\mathbf{G}_\nu^N$ is the lower triangular matrix such that

$$G_{\nu, ij}^N := 1_{j \leq i-1} \int_{t_{j-1}}^{t_j} G_\nu(t_{i-1}, s) ds, \quad 1 \leq i, j \leq N, \quad (2.3.4)$$

and

$$\tilde{\Sigma}_0^{u;N} := \frac{T}{N} (I_N - b^u \mathbf{G}_\nu^N)^{-1} \Sigma_0^N (I_N - b^u (\mathbf{G}_\nu^N)^T)^{-1},$$

³Other choices of partitions may lead to improved performance; however, in this thesis, we restrict ourselves to the natural choice of a uniform partition.

with Σ_0^N the $N \times N$ discretized covariance matrix given by

$$\Sigma_0^{i,j;N} := \eta_\nu^2 \int_0^T G_\nu(t_{i-1}, s) G_\nu(t_{j-1}, s) ds, \quad 1 \leq i, j \leq N. \quad (2.3.5)$$

When we consider kernels of the form

$$G_\nu(t, s) = \frac{(t - s + \varepsilon)^{H-1/2}}{\Gamma(H + 1/2)}, \quad \varepsilon \geq 0,$$

with $H \in (0, 1)$ for $\varepsilon = 0$ and $H \in \mathbb{R}$ for $\varepsilon > 0$, we observe that the $N \times N$ matrix (2.3.4)-(2.3.5) can be computed in closed form

$$G_{\nu, ij}^N = 1_{j \leq i-1} \frac{(t_{i-1} - t_{j-1} + \varepsilon)^\alpha - (t_{i-1} - t_j + \varepsilon)^\alpha}{\Gamma(\alpha + 1)}, \quad 1 \leq i, j \leq N,$$

$$\Sigma_0^{i,j;N} = \frac{\eta_\nu^2 t_{i-1} (t_{i-1} + \varepsilon)^{\alpha-1} (t_{j-1} + \varepsilon)^{\alpha-1}}{\Gamma(\alpha)^2} F_1(1, 1 - \alpha, 1 - \alpha, 2, \frac{t_{i-1}}{t_{j-1} + \varepsilon}, \frac{t_{i-1}}{t_{i-1} + \varepsilon}),$$

$$\Sigma_0^{i,j;N} = \Sigma_0^{j,i;N}, \quad 1 \leq i \leq j \leq N,$$

where $\alpha := H + 1/2$ and $F_1(a, b, c, d, x, y)$ is the Appell hypergeometric function of the first kind given by

$$F_1(\alpha, \beta, \gamma, \delta, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\gamma)_n}{(\delta)_{m+n} m! n!} x^m y^n$$

with $(\cdot)_n$ the Pochhammer symbol such that

$$(x)_n = \prod_{k=1}^n (x - k + 1).$$

Based on the approximate operator, we now deduce a closed form approximate solution to the characteristic function. Relying on Abi Jaber and Guellil (2026), we have that ϕ_t^u , that appears in (2.3.2), can be rewritten in term of the Fredholm determinant such that

$$\phi_t^u = i\pi k_t(u) - \log \left(\det \left(\text{id} - 2a^u \tilde{\Sigma}_t^u \right)^{1/2} \right),$$

with $k_t(u) \in [-1, 1]$ where the sign is determined by the number of crossings of the nonpositive real line $\mathbb{R}_-$ by the Fredholm determinant. The reason we have rewritten ϕ_t^u is that this enables us to use the Fredholm determinant to calculate ϕ_t^u , instead of having to discretize the trace-dependent expression which would require

the computation of Ψ_t^u for several values of t . Moreover, when approximating operators by matrices, following (Abi Jaber and Guellil, 2026, Proposition 3.4.), we obtain a prefactor-free formula such that

$$e^{-i\pi k_0(u)} \det \left(\Phi_0^{u;N} \right)^{1/2} = \det \left((\Phi_0^{u;N})^{1/2} \right),$$

with $\Phi_0^{u;N} := (I_N - 2a^u \tilde{\Sigma}_0^{u;N})$.

Remark 2.3 (On the use of Fredholm determinant in finance). *The Fredholm determinant, originally introduced by Fredholm (1903), extends the classical notion of a matrix determinant to operators acting on infinite-dimensional spaces. Such objects are not new in mathematical finance. Indeed, related expressions involving Fredholm determinants have already appeared in bond pricing formulas for models in which the short rate is given by the square of a Gaussian process; see, for instance, Grasselli and Hurd (2005); Corcuera et al. (2013); Privault and Wu (2015); Abi Jaber (2022a). This representation in terms of Fredholm determinants is particularly attractive from a numerical perspective, as it significantly reduces computational complexity while improving numerical accuracy; see, for example, Abi Jaber and Guellil (2026).*

Using now the approximate operators, for $N \in \mathbb{N}$ and for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, a natural approximation of the characteristic function is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_0^T} \right) \right] &= \exp(\phi_0^u + \chi_0^u + \langle h_0^u, \Psi_0^u h_0^u \rangle_{L^2}) \\ &\approx \frac{\exp \left(\chi_0^u + \frac{T}{N} (h_0^{u;N})^T \Psi_0^{u;N} h_0^{u;N} \right)}{\det \left((\Phi_0^{u;N})^{1/2} \right)}, \end{aligned} \quad (2.3.6)$$

with:

- $h_0^{u;N} := (h_0^u(t_0), \dots, h_0^u(t_N))$,
- $\Psi_0^{u;N}$ the approximate operator defined by (2.3.3),
- $\chi_0^u = \frac{1}{2}(u^2 - u)(1 - \rho_{I_r}^2)\eta_r^2 \int_0^T B_{G_r}(s, T)^2 ds$.

Remark 2.4. *To compute χ_0^u , we need to compute the integral of the form $\int_0^T B_{G_r}(s, T)^2 ds$, where $B_{G_r}(s, T)$ is defined by (2.2.7) and admits an explicit expression for some well-known kernels, as revealed by Table 2.1. However, except for the indicator or exponential kernels, the integral $\int_0^T B_{G_r}(s, T)^2 ds$ does not have an explicit expression and needs to be computed numerically.*

Remark 2.5 (On the theoretical convergence of the approximate method). *The theoretical convergence of the proposed approximation scheme as $N \rightarrow \infty$ is non-trivial to establish, as it involves the convergence of finite-dimensional matrices toward operators acting on infinite-dimensional spaces. We leave a rigorous treatment of this question for future work. Nevertheless, our numerical experiments*

strongly suggest convergence of the method, together with a convergence rate that appears faster than that of alternative approaches based on multi-factor approximations of the volatility, as discussed in Section 2.5.3.

Let us now discuss the pricing of equity derivatives. From Section 2.2.2, we know that the arbitrage-free price, at time $t \in [0, T]$, of a financial derivative $H_T = h(I_T) \in L^2$ is given by

$$V_t = P(t, T) \mathbb{E}^{\mathbb{Q}^T} \left(h(e^{\log I_T^T}) \mid \mathcal{F}_t \right),$$

where $P(t, T)$ is the price of a zero-coupon bond of maturity T . Exploiting the Fourier link between the density function and the characteristic function, we can deduce how to compute the price of vanilla options such as call and put options using a Fourier method. For $u \in \mathbb{C}$, let us consider φ_t^u defined by

$$\varphi_t^u = \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(iu \log \frac{I_T^T}{I_t^T} \right) \mid \mathcal{F}_t \right],$$

then, from Lewis (2001), the price at time $t \in [0, T]$, of a call option $(I_T - K)_+$ is given by

$$V_t = P(t, T) \left[I_t^T - \frac{K}{\pi} \int_0^{+\infty} \Re \left(e^{(iu+1/2)k_t} \varphi_t^{u-i/2} \right) \frac{du}{u^2 + 1/4} \right], \quad (2.3.7)$$

where $k_t := \log \frac{I_T^T}{K}$. Using a numerical integration of (2.3.7), we can efficiently price call options as well as calibrate the model to equity market data. In this chapter, we use the Gauss-Laguerre quadrature numerical integration which has been demonstrated to be efficient in the context of option pricing (see Abi Jaber and Gérard (2025a)). In practice, for a general kernel function $G_\nu(\cdot)$, such as the fractional or path-dependent kernels, we have to approximate the expression of the characteristic function of $\log \frac{I_T^T}{I_t^T}$ that appears in the integral (2.3.7). To this end, we use the discretization approach detailed above. Note that such approximation procedure is not always necessary. In fact, as we will see later in Section 2.5.2, if we consider the Stein-Stein volatility model i.e. $G_\nu(t, s) = 1_{s < t}$, the analytic expression of the characteristic function is more explicit as shown by Corollary 2.5.4.

The numerical implementation of the Lewis (2001) pricing method with Gauss-Laguerre quadrature, is summarized in Algorithm 1. Care must be taken when selecting the quadrature level L for approximating the integral. As emphasized by Boyarchenko and Levendorskii (2024), the same quadrature level cannot be used uniformly across all maturities. In particular, shorter maturities require a higher number of quadrature points. This is due to the fact that the integrand in (2.3.7) decays to zero more slowly for small maturities than for large ones, as illustrated

Algorithm 1 Lewis pricing with Gauss-Laguerre quadrature

- 1: For a given quadrature level $L \in \mathbb{N}_0$, obtain Gauss-Laguerre quadrature discretization points and weights $(u_j, w_j)_{j=1, \dots, L}$.
- 2: For a given $N \in \mathbb{N}_0$ and for each point u_j , approximate the characteristic function of the log-forward index $\varphi_0^{u_j-i/2}$ by $\tilde{\varphi}_0^{u_j-i/2}$, using the operator discretization approach (2.3.6).
- 3: Approximate the price of call options (2.3.7) such as

$$V_0 \approx P(0, T) \left[I_0^T - \frac{K}{\pi} \sum_{j=1}^L \Re \left(e^{(iu_j+1/2)k_0} \tilde{\varphi}_0^{u_j-i/2} \right) \frac{w_j e^{u_j}}{u_j^2 + 1/4} \right].$$

in Figure 2.13 for the Stein-Stein model⁴. Therefore, in what follows, we adopt $L = 60$ for maturities $T < 0.25$, and $L = 40$ for $T \geq 0.25$. As shown in Figures 2.14, 2.15, and 2.16, the Lewis method with Gauss-Laguerre quadrature produces stable pricing results under the Stein-Stein model with these parameter choices.

The closed form approximate solution of the characteristic function (2.3.6) has been proposed in different papers Abi Jaber (2022a); Abi Jaber (2022b). However, as already noted, a general theoretical convergence result when N tends to infinity is intricate to prove. We nevertheless verify empirically the convergence of this approximation method as $N \rightarrow \infty$. For this purpose, we restrict ourselves to a one-factor Hull and White model for the interest rate i.e. $G_r(t, s) = 1_{s < t}$ and we consider the fractional kernel for the volatility $G_\nu(t, s) = 1_{s < t} \frac{(t-s)^{H_\nu-1/2}}{\Gamma(H_\nu+1/2)}$. Without loss of generality, we fix arbitrarily the model parameters to

$$\begin{aligned} \kappa_r &= -0.03, \eta_r = 0.01, I_0^T = 100, \nu_0 = 0.2, \kappa_\nu = 0, \theta_\nu = 0.1, \sigma_\nu = 0.2, \\ \rho_{I\nu} &= -0.7, \rho_{Ir} = -0.25, \rho_{\nu r} = -0.25. \end{aligned}$$

Table 2.3 compares the error committed when approximating the characteristic function of the log-forward index in the Gauss-Laguerre numerical integration for $H_\nu = 0.5$. We observe that the error rapidly decreases as the number of discretizations N increases. Moreover, Figure 2.1 presents the implied volatility dynamics generated by the operator discretization for $H_\nu = 0.5$ and Figure 2.2 for $H_\nu = 0.3$. As expected, we observe a relative fast convergence as the number of discretization factors N increases.

⁴In this model, the characteristic function of the log-forward index admits an explicit expression.

$k \backslash N$	$T = 0.05$			$T = 0.25$		
	10	40	100	10	40	100
0	1.06×10^{-4}	5.65×10^{-5}	4.71×10^{-5}	8.08×10^{-4}	2.58×10^{-4}	1.52×10^{-4}
0.2	4.36×10^{-6}	1.25×10^{-6}	5.73×10^{-7}	1.34×10^{-3}	3.55×10^{-4}	1.57×10^{-4}
-0.2	1.07×10^{-7}	1.68×10^{-11}	3.91×10^{-12}	2.58×10^{-4}	1.45×10^{-5}	3.25×10^{-6}

Table 2.3: $\left| \sum_{j=1}^L \left[\Re e \left(e^{(iu+1/2)k} \tilde{\varphi}_0^{u_j-i/2} \right) - \Re e \left(e^{(iu+1/2)k} \varphi_0^{u_j-i/2} \right) \right] \frac{w_j e^{u_j}}{u_j^2+1/4} \right|$, for $H_\nu = 0.5$.

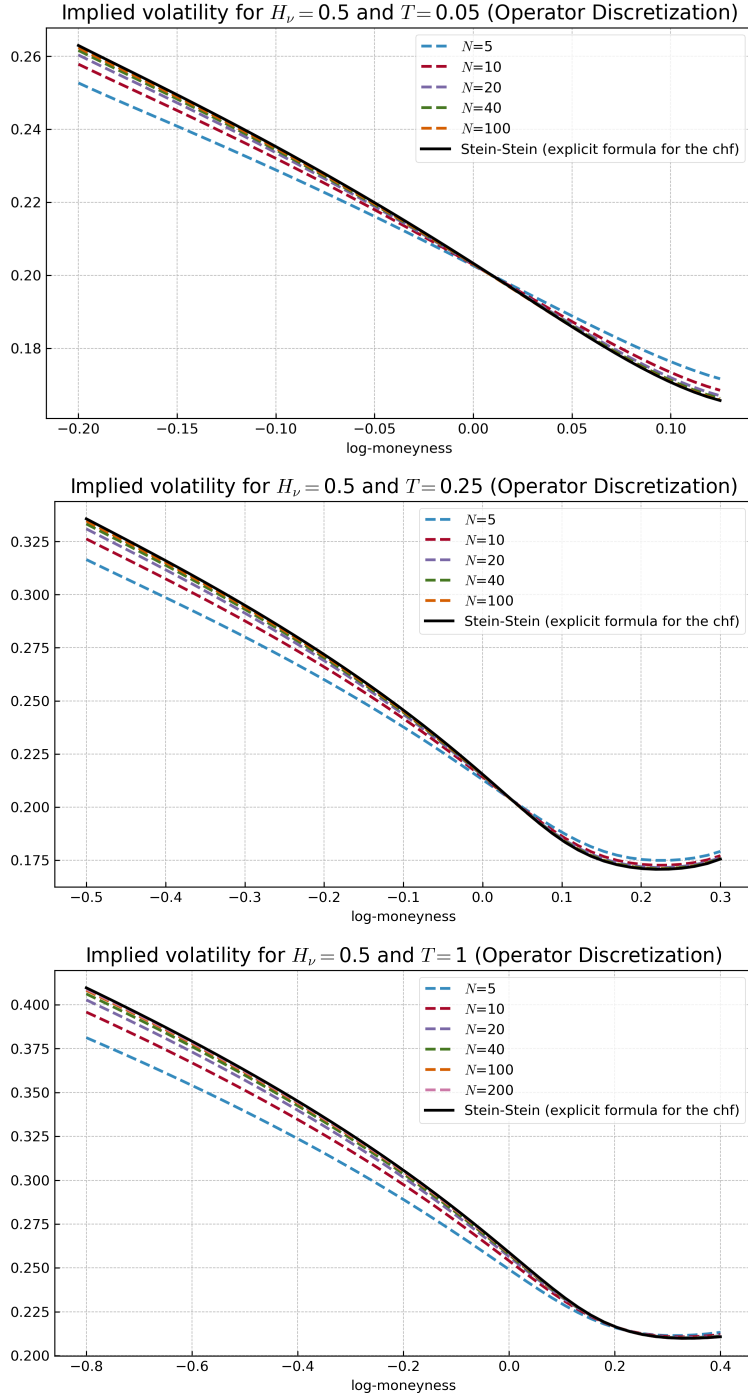


Figure 2.1: Implied volatility dynamics generated by the operator discretization method with $H_\nu = 0.5$ and the Stein-Stein model.

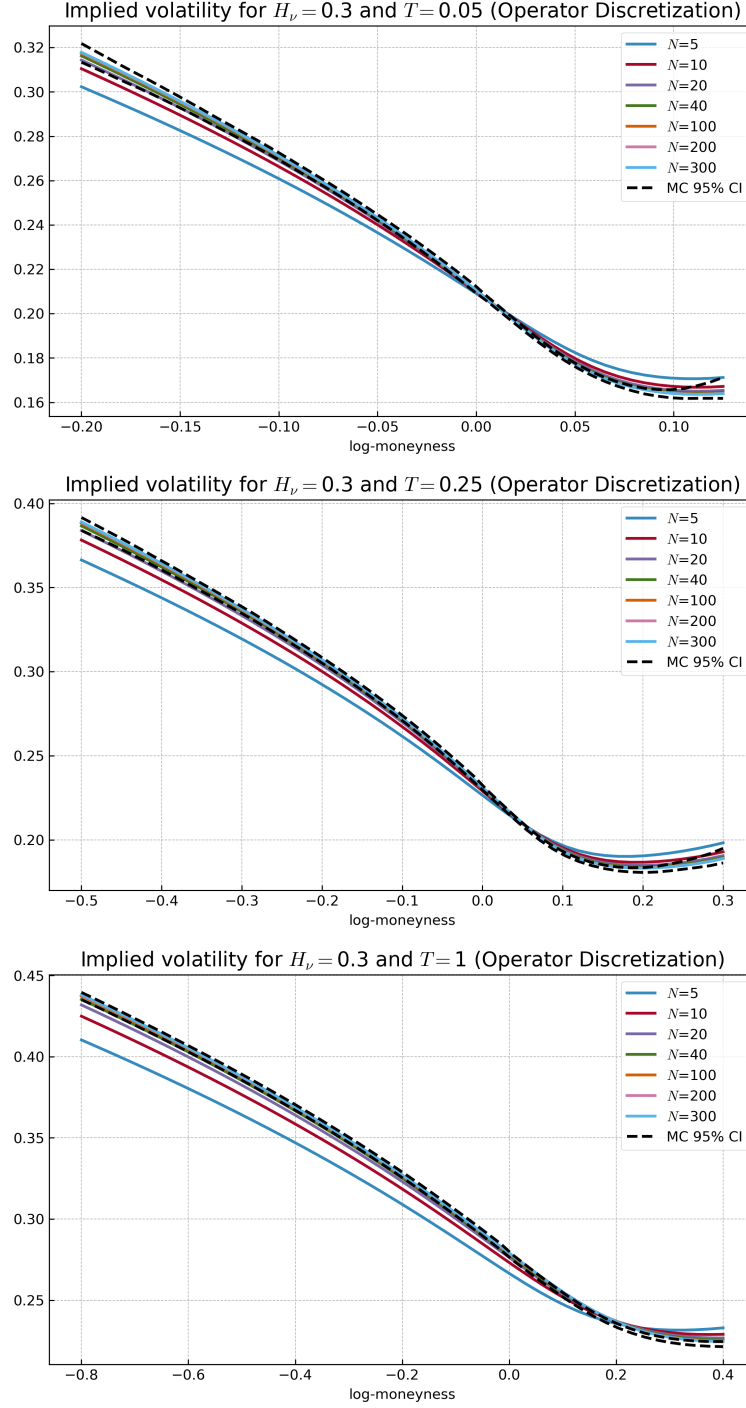


Figure 2.2: Implied volatility dynamics generated by the operator discretization method with $H_\nu = 0.3$. Monte Carlo confidence intervals are generated with 200000 simulations of risk processes using a Euler scheme with 500 as discretization steps.

2.4 Calibration to market data

In this section, we highlight the relevance of our framework for calibration to market data. To this end, we place ourselves in an equity market context where the process $(I_t)_{0 \leq t \leq T}$ represents an equity stock index. Our calibration instruments from market data, are interest rate options, caps and floors, as well as equity vanilla call and put options. Let us now calibrate the models to the market data in order to validate the relevance of the introduced framework. To this end, we consider market data of 25/08/2022 consisting of USD 3M Libor yield curve and ATM USD cap implied volatility data from Bloomberg, and S&P500 implied volatility data purchased from the CBOE website⁵. The calibration procedure takes place in two stages. First we calibrate the interest rate parameters using the USD 3M Libor yield curve and ATM cap data. Then, using these calibrated interest rate parameters, we calibrate the volatility and correlation parameters based on S&P500 implied volatility data. Of course, since we only rely on a give date, there is a over-fitting risk. This risk could in principle be mitigated by considering a calibration on multiples dates. However, we note that the market data observed on 25/08/2022 exhibit the usual shapes encountered in practice for both the yield curve and implied volatility surfaces. As a result, we expect the qualitative conclusions of this calibration to remain valid for other market dates. In the following, the calibration of the model parameters is performed using the standard Python optimisation routine `scipy.optimize.minimize` with the L-BFGS-B algorithm, and several initial guesses within standard ranges were tested in order to reduce the risk of convergence toward local minima.

For the interest model, we consider the fractional kernel of the form $G_r(t, s) = 1_{s < t} \frac{(t-s)^{H_r-1/2}}{\Gamma(H_r+1/2)}$. The data we have for the calibration are the USD 3M Libor yield curve and the ATM USD cap implied volatility for annual maturities ranging from 1 year to 30 years. Firstly, using (2.2.8) and the USD 3M Libor yield curve (Figure 2.3), the input curve $r_0(t)$ can deduced such that $P(0, T) = P^{Market}(0, T)$. Then, since cap options are sum of zero-coupon bond options, we use Proposition 2.2.2 and calibrate the interest rate parameters by minimizing the root square error (RMSE) between model and market ATM cap implied volatility. Results of this calibration are displayed on Figure 2.4 with the corresponding calibrated parameters. The RMSE is 0.3663% and we observe an excellent fit of the data with $\hat{H}_r > 1/2$ as the calibrated model perfectly reproduces the humped shape of the market ATM cap implied volatility term-structure with only three parameters. As revealed by Figure 2.4, this can be explained by the fact that the term-structure of the ZC bond options pricing variance (2.2.11), with $S = T + 1/4$, produces a humped shape⁶

⁵<https://datashop.cboe.com/>.

⁶Due to humped behavior of the resolvent $\frac{R_{\kappa_r G_r}(t, s)}{\kappa_r}$ that drives B_{G_r} and thus the pricing variance.

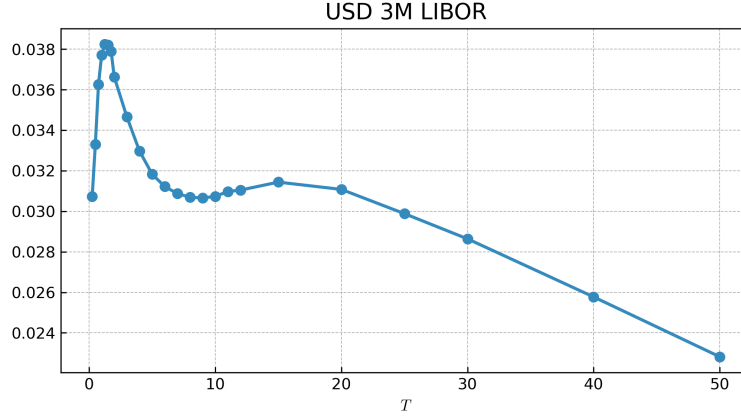


Figure 2.3: USD 3M Libor yield curve of 25/08/2022.

when $\kappa_r < 0$ and $H_r > 1/2$, which is totally not the case when considering the Hull-White model (i.e. $H_r = 1/2$). Moreover, as expected and revealed in Figure 2.5, $H_r > 1/2$ generates a smoother sample path than $H_r = 1/2$, and exhibits long-range dependencies. To check that long-term dependence makes sense, we consider historical data on the USD 3M Libor rate (daily close values) from 01/06/2021 to 30/09/2024, and examine the empirical auto-correlation structure. As we can see in Figure 2.5, the USD rate exhibits strong persistence, which is perfectly in line with previous studies (see Dai and Singleton (2003); McCarthy et al. (2004)), and emphasizes that considering long-range dependence models seems appropriate for modeling the dynamics of short rates. Of course, other interest rate models can also capture the humped shape of the ATM cap implied volatility, see for instance Levendorskii (2005); (Brigo and Mercurio, 2006, Section 3); Boyarchenko and Levendorskii (2007a). However, we want to stress that our model is very parsimonious since we perfectly capture the humped shape with only three parameters. Moreover, it is highly tractable, can account for long-range dependencies, and appears very suitable for a joint equity-rate calibration.

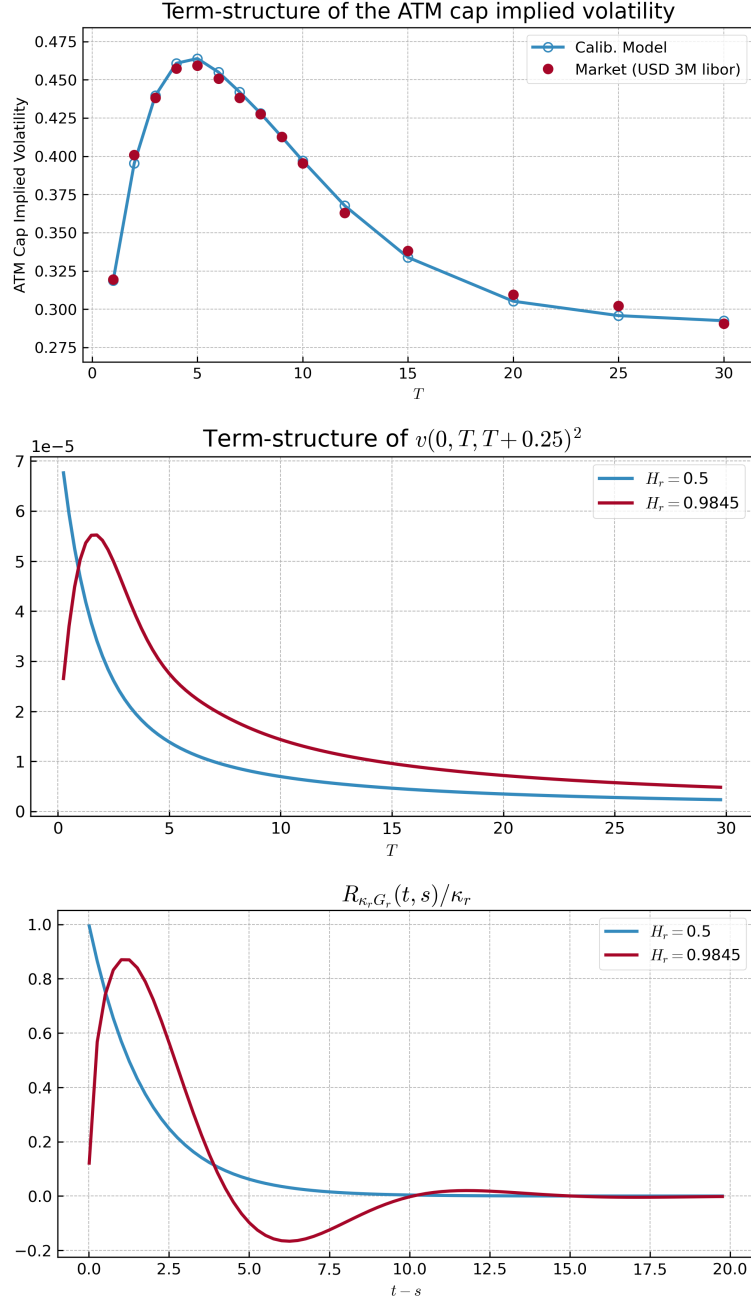


Figure 2.4: ATM cap implied volatility for USD 3M Libor: calibrated fractional model vs market data of 25/08/2022. RMSE: 0.3663%. Calibrated parameters: $\hat{\kappa}_r = -0.5566$, $\hat{\eta}_r = 0.0377$ and $\hat{H}_r = 0.9845$ (top); term-structure of the ZC bond option pricing variance (2.2.11) generated with the calibrated parameters (bottom left); evolution of the resolvent associated to fractional kernel (bottom right).

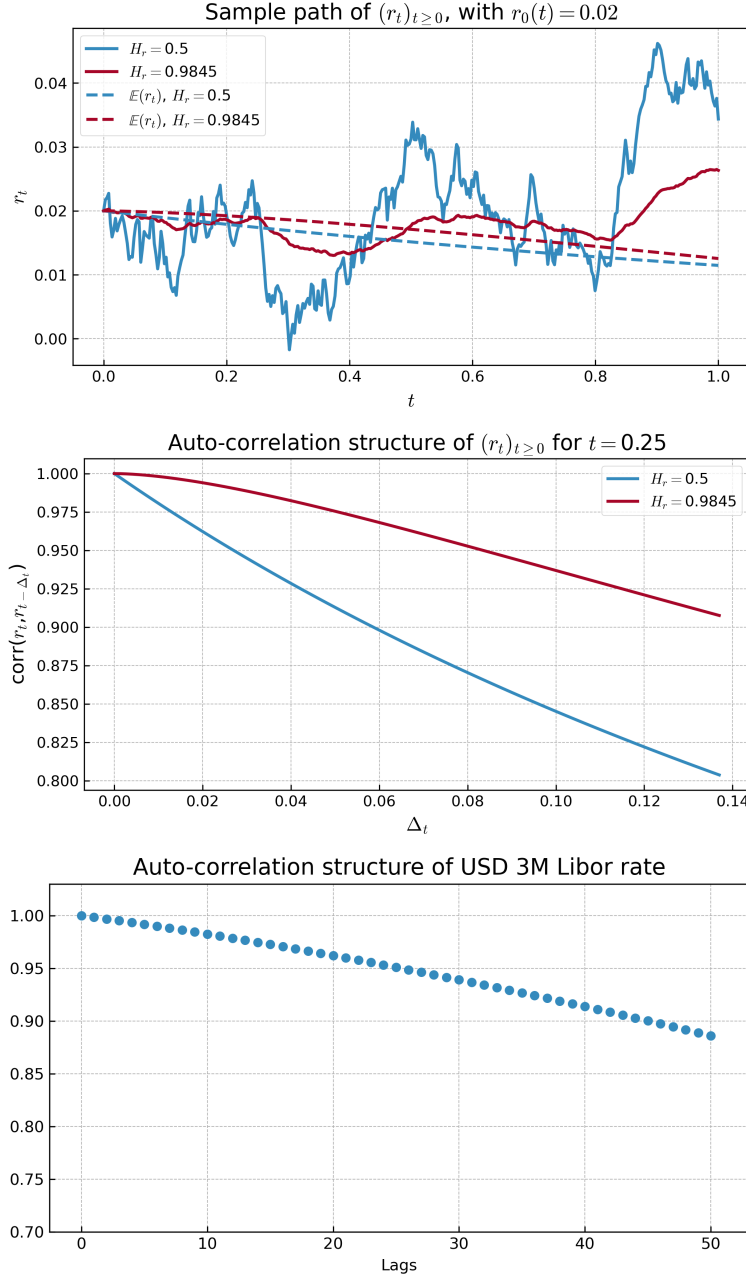


Figure 2.5: Sample path of $(r_t)_{t \geq 0}$ generated with the calibrated parameters with $r_0(t) = 0.02$ and the associated $(\mathbb{E}(r_t))_{t \geq 0}$ (upper left); Theoretical auto-correlation structure of $(r_t)_{t \geq 0}$ with the calibrated parameters (upper right); Empirical auto-correlation structure of USD 3M Libor rates (bottom).

Let us now consider the calibration of the volatility and correlation parameters. To this end, we consider the fractional kernel of the form $G_\nu(t, s) = 1_{s < t} \frac{(t-s)^{H_\nu-1/2}}{\Gamma(H_\nu+1/2)}$ as well as the shifted fractional kernel of the form $G_\nu(t, s) = 1_{s < t} \frac{(t-s+\varepsilon)^{H_\nu-1/2}}{\Gamma(H_\nu+1/2)}$, with $\varepsilon = 1/52$, which corresponds to 1 week and prevents of ATM skew explosion for small maturities. The data we have for the calibration are the S&P500 implied volatility data for different maturities ranging from 1 week to 1.82 years and different strikes. Using the operator discretization method with $N = 40$ to approximate the characteristic function of the log-forward index and a Fourier method, we can price call options efficiently. Therefore, we calibrate the volatility and correlation parameters by minimizing the RMSE between model and S&P500 implied volatility. To have a more parsimonious model, we decide to fix $\kappa_\nu = 0$ and $\rho_{r\nu} = 0$. We prefer to capture the correlation between the index and the interest rate ρ_{Ir} rather than the correlation $\rho_{r\nu}$, but we have a leverage of flexibility if we decide to calibrate $\rho_{r\nu}$ as well.

Results of this calibration are displayed on Figure 2.6, 2.7 and 2.8, we observe that, for both kernels, $\hat{H}_\nu < 1/2$. The RMSE are 0.5321% for the fractional kernel and 0.4602% for the shifted fractional kernel. Overall, the calibration is good for both kernels, especially around the ATM, yet the shifted fractional kernel outperforms the fractional kernel. Moreover, as we can observe on Figure 2.9, the calibrated shifted fractional kernel reproduces the term structure of the ATM skew with a concave shape on the log-log scale for the chosen date, which is not the case for the fractional kernel. The power law decay for low maturities is not compatible with the concave shape of the ATM skew term structure, which is why the fractional kernel fails to reproduce the market term structure. We also observe that $\hat{H}_\nu^{\text{Shifted fractional}} < \hat{H}_\nu^{\text{Fractional}}$, indicating that the shifted kernel exhibits a relative faster decrease for longer maturities while $\varepsilon = 1/52$ prevents it from having an exploding skew for shorter maturities, allowing it to better capture the concave shape of the skew (in log-log scale). Our results demonstrate that the rough model underperforms the non-rough model associated with the shifted fractional kernel in capturing the entire volatility surface, confirming the findings of Delemotte et al. (2023); Guyon and Lekeufack (2023); Abi Jaber and Li (2024). To validate that the shapes of the implied volatilities generated with $N = 40$ for the calibrated parameters are accurate, and to avoid drawing spurious conclusions, we compare them to the Monte Carlo confidence intervals. The results can be seen in Figures 2.17 and 2.18, and as we can observe, the shapes obtained with $N = 40$ are consistent with the confidence intervals.

We also observe that the calibrated correlations between interest rate and index processes $\hat{\rho}_{Ir}$ are significant. To ensure that these estimated values make sense, we looked at the empirical 180 days rolling correlation between the S&P500 index and the USD 3M Libor rate (daily close values). The results are displayed on Figure 2.10. We first observe that the correlation is not constant and almost

never zero. Secondly, we notice that, for the calibration date we have chosen, the empirical rolling correlation is negative. This seems in line with the correlation values obtained during calibration on the implied volatility surface of the S&P500, and thus assuming correlation between processes appears coherent with market data.

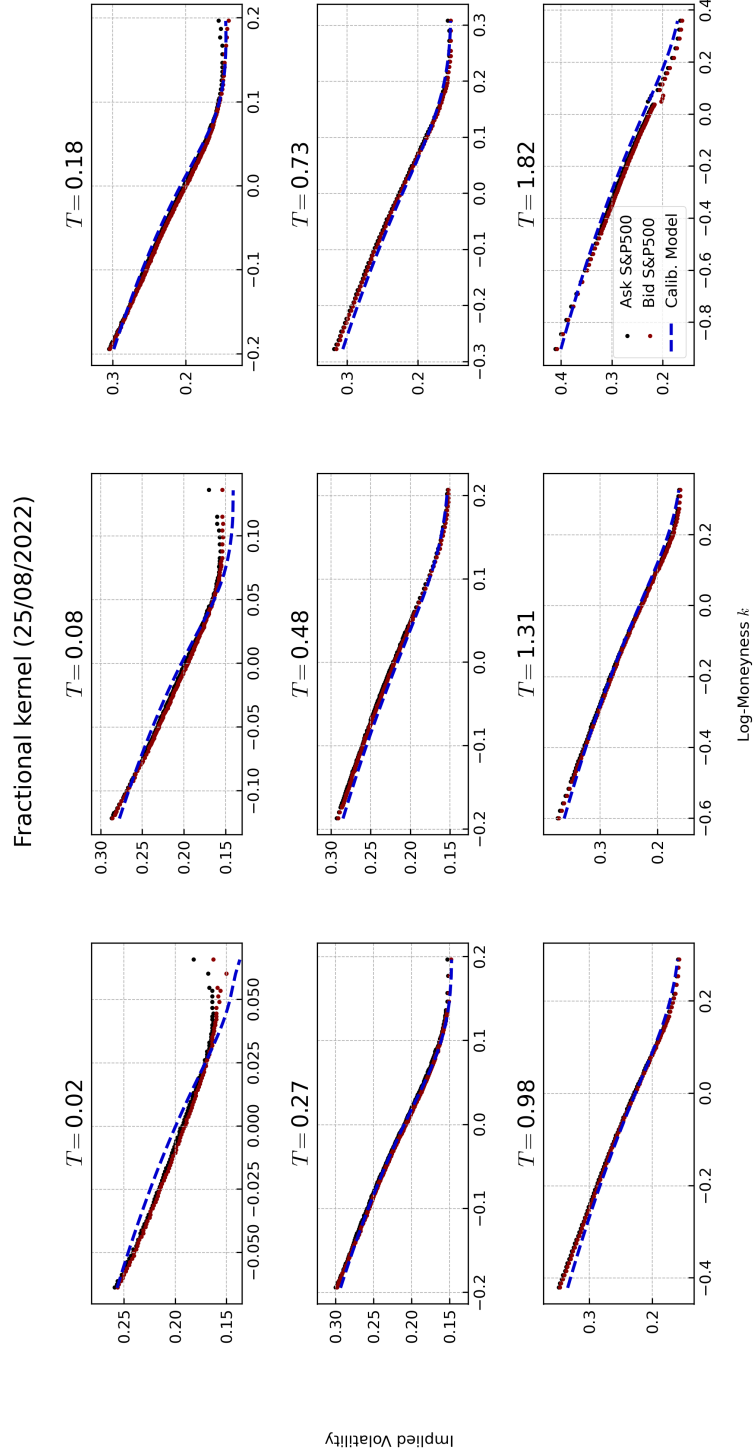


Figure 2.6: S&P500 implied volatility: calibrated fractional model vs market data of 25/08/2022. RMSE: 0.5321%. Calibrated parameters: $\hat{\nu}_0 = 0.1964$, $\hat{\theta}_\nu = -0.0248$, $\hat{\eta}_\nu = 0.2123$, $\hat{\rho}_{I\nu} = -0.7981$, $\hat{\rho}_{Ir} = -0.5971$ and $\hat{H}_\nu = 0.2992$.

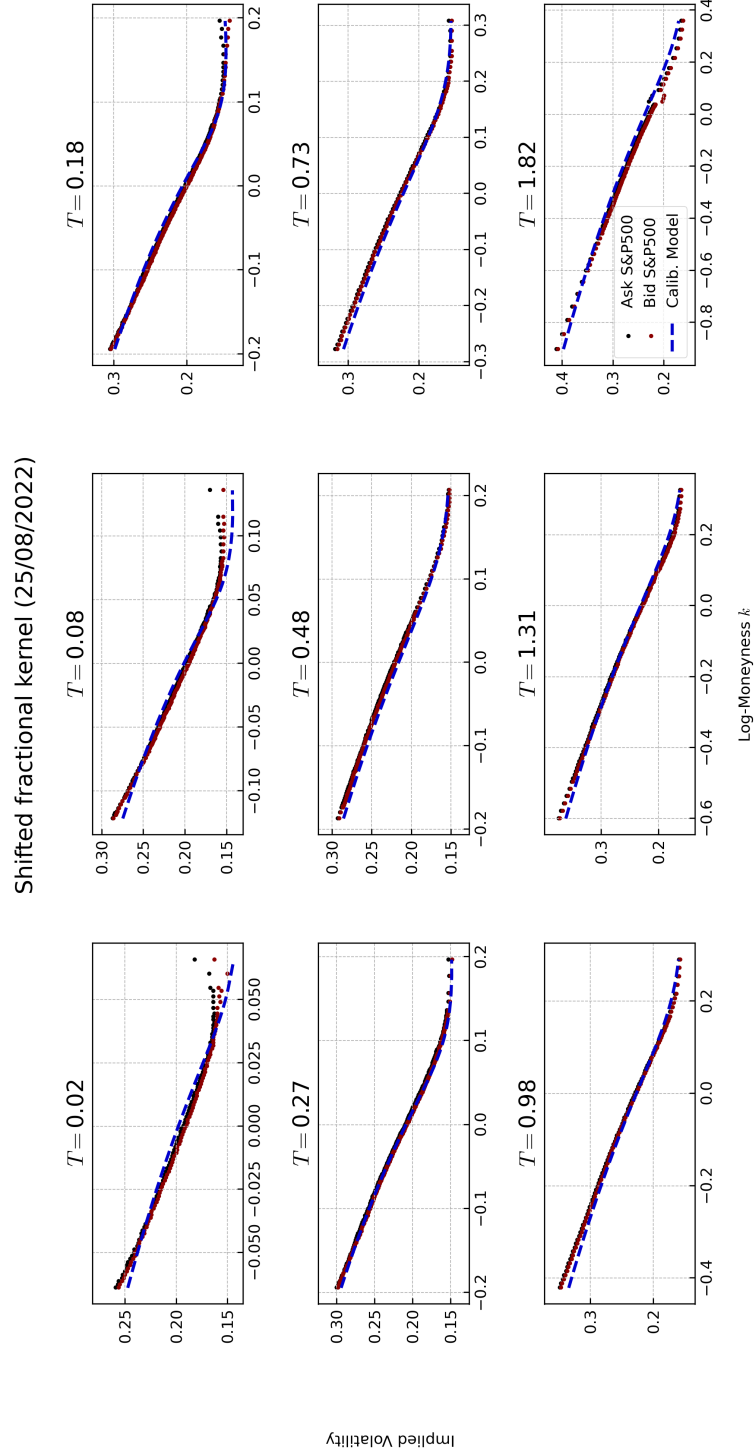


Figure 2.7: S&P500 implied volatility: calibrated shifted fractional model vs market data of 25/08/2022. RMSE: 0.4602%. Calibrated parameters: $\hat{\nu}_0 = 0.1978$, $\hat{\theta}_\nu = -0.0259$, $\hat{\eta}_\nu = 0.2164$, $\hat{\rho}_{I\nu} = -0.7868$, $\hat{\rho}_{Ir} = -0.6107$ and $\hat{H}_\nu = 0.2273$.

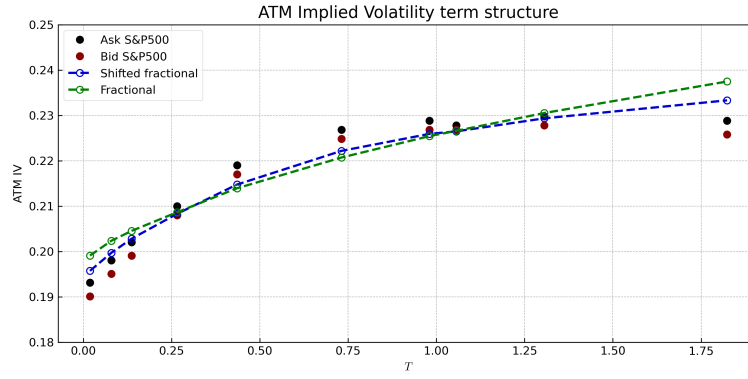


Figure 2.8: ATM implied volatility term structure: calibrated models vs market data of 25/08/2022 with calibrated parameters.

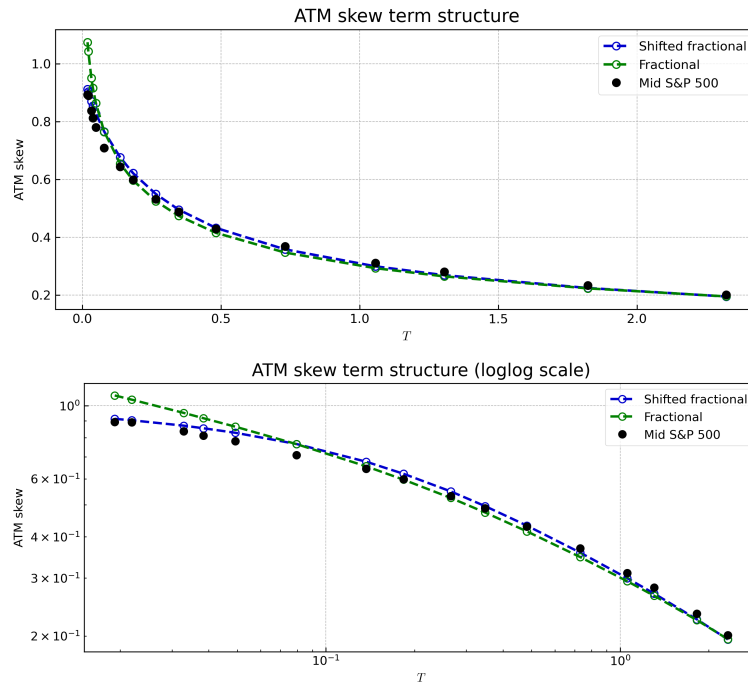


Figure 2.9: ATM skew term structure: calibrated models vs market data of 25/08/2022 with calibrated parameters.

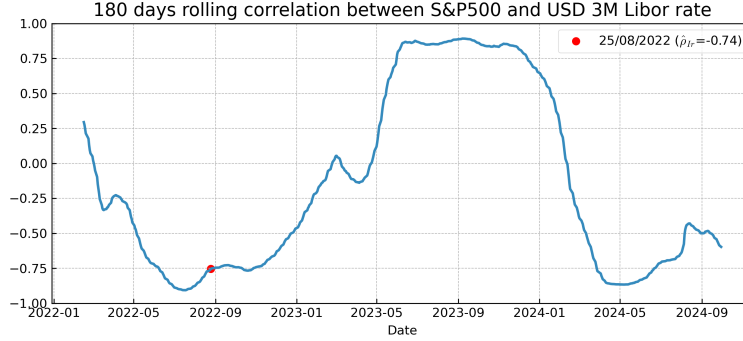


Figure 2.10: 180 days rolling correlation between S&P500 and USD 3M Libor rate.

2.5 Link with conventional quadratic linear models

In this section, we finally establish a connection with classical quadratic linear models, which will ultimately allow us to derive a second approximation scheme applicable to general completely monotone Volterra kernels. While this alternative approach provides additional analytical insight and a useful benchmark, it is typically slower to converge in practice compared to the primary method developed in this chapter.

2.5.1 Link with operator Riccati equations

First, we provide the link between the analytic expression of the characteristic function (2.3.2) and expression that depends on Riccati equations.

Proposition 2.5.1. *Let $g_0(\cdot)$ be given by (2.2.3), $G_\nu(\cdot)$ a Volterra kernel as in Definition 2.1. Fix $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$. Then, for all $t \leq T$,*

$$\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] = \exp \left(\phi_t^u + \chi_t^u + a^u \int_t^T h_t^u(s)^2 ds + \int_t^T \int_t^T h_t^u(s) h_t^u(w) \bar{\psi}_t^u(s, w) ds dw \right), \quad (2.5.1)$$

with χ_t^u as in Theorem 2.3.1, ϕ_t^u such that

$$\begin{aligned} \dot{\phi}_t^u &= -a^u \eta_\nu^2 \int_t^T G_\nu(s, t)^2 ds - \eta_\nu^2 \int_t^T \int_t^T G_\nu(s, t) G_\nu(w, t) \bar{\psi}_t^u(s, w) ds dw, \quad t < T, \\ \phi_T^u &= 0, \end{aligned} \quad (2.5.2)$$

and $\bar{\psi}_t^u(s, w)$ that satisfies a Riccati equation of the form

$$\begin{aligned} \dot{\bar{\psi}}_t^u(s, w) = & 2(a^u)^2 \eta_\nu^2 G_\nu(s, t) G_\nu(w, t) + 2a^u \eta_\nu^2 \left(G_\nu(s, t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(., w))(t) \right. \\ & \left. + G_\nu(w, t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) \right) \\ & + 2\eta_\nu^2 (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(., w))(t), \quad t < T, \quad (s, w) \in (t, T]^2 \text{ a.e.}, \\ \bar{\psi}_t(t, s) = & \bar{\psi}_t(s, t) = b^u \left(a^u G_\nu(s, t) + (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) \right), \quad t \leq s \leq T. \end{aligned}$$

Proof. From Lemma 2.6.1, we have that $\Psi_t^u = a^u \text{id} + \bar{\Psi}_t^u$, where $\bar{\Psi}_t^u$ is an integral operator induced by a symmetric kernel $\bar{\psi}_t^u(s, w)$ such that

$$\begin{aligned} \bar{\psi}_t^u(t, s) = \bar{\psi}_t^u(s, t) = & b^u \int_t^T G_\nu(w, t) (a^u \delta_{w=s} + \bar{\psi}_t^u(s, w)) dw, \quad t \leq s \leq T \\ = & b^u \left(a^u G_\nu(s, t) + (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) \right), \quad t \leq s \leq T. \end{aligned}$$

In this case, we have that

$$\langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2} = a^u \int_t^T h_t^u(s)^2 ds + \int_t^T \int_t^T h_t^u(s) h_t^u(w) \bar{\psi}_t^u(s, w) ds dw.$$

Moreover, we also know from Lemma B.1. in Abi Jaber (2022a) that

$$\begin{aligned} \dot{\Psi}_t^u &= 2\Psi_t^u \dot{\Sigma}_t \Psi_t^u, \quad t < T, \\ \Psi_T^u &= a^u (\text{id} - b^u \mathbf{G}_\nu^*)^{-1} (\text{id} - b^u \mathbf{G}_\nu)^{-1}, \end{aligned}$$

and by the definition, we also have that

$$\begin{aligned} \dot{\bar{\Psi}}_t^u &= 2\Psi_t^u \dot{\Sigma}_t \Psi_t^u, \quad t < T, \\ &= 2(a^u \text{id} + \bar{\Psi}_t^u) \dot{\Sigma}_t (a^u \text{id} + \bar{\Psi}_t^u), \quad t < T, \\ \bar{\Psi}_T^u &= a^u (\text{id} - b^u \mathbf{G}_\nu^*)^{-1} (\text{id} - b^u \mathbf{G}_\nu)^{-1} - a^u \text{id}, \end{aligned}$$

As $\dot{\bar{\Psi}}_t^u$ is composed of integral operators, it is also an integral operator induced by the following kernel

$$2((a^u \delta + \bar{\Psi}_t^u) \star \dot{\Sigma}_t \star (a^u \delta + \bar{\Psi}_t^u))(s, w),$$

with δ the kernel induced by the identity operator such that

$$(\text{id}f)(s) = \int_t^T \delta_{s=w}(ds, dw) f(w) = f(s).$$

Thus, using the dominated convergence theorem, we obtain that $t \rightarrow \bar{\psi}_t^u(s, w)$ solves a Riccati equation given, for $t < s$ and $w \leq T$, by

$$\dot{\bar{\psi}}_t^u(s, w) = 2((a^u \delta + \bar{\Psi}_t^u) \star \dot{\Sigma}_t \star (a^u \delta + \bar{\Psi}_t^u))(s, w).$$

More explicitly, since $G(z, t) = 0$ for $z \leq t$, we also have, for $(s, w) \in (t, T]^2$ a.e.,

$$\begin{aligned} \dot{\bar{\psi}}_t^u(s, w) &= 2(a^u)^2 \eta_\nu^2 G_\nu(s, t) G_\nu(w, t) + 2a^u \eta_\nu^2 G_\nu(s, t) \int_t^T G_\nu(z, t) \bar{\psi}_t^u(z, w) dz \\ &\quad + 2a^u \eta_\nu^2 G_\nu(w, t) \int_t^T G_\nu(z, t) \bar{\psi}_t^u(s, z) dz \\ &\quad + 2\eta_\nu^2 \int_t^T G_\nu(z, t) \bar{\psi}_t^u(s, z) dz \int_t^T G_\nu(z', t) \bar{\psi}_t^u(z', w) dz', \quad t < T. \end{aligned}$$

Thus, using the integral operator $\mathbf{G}_\nu^*$, we obtain that, for $(s, w) \in (t, T]^2$ a.e.,

$$\begin{aligned} \dot{\bar{\psi}}_t^u(s, w) &= 2(a^u)^2 \eta_\nu^2 G_\nu(s, t) G_\nu(w, t) + 2a^u \eta_\nu^2 \left(G_\nu(s, t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(., w))(t) \right. \\ &\quad \left. + G_\nu(w, t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) \right) \\ &\quad + 2\eta_\nu^2 (\mathbf{G}_\nu^* \bar{\psi}_t^u(s, .))(t) (\mathbf{G}_\nu^* \bar{\psi}_t^u(., w))(t), \quad t < T. \end{aligned}$$

In addition, from Theorem 2.3.1, we know that $t \rightarrow \phi_t^u$ satisfies the following ODE

$$\begin{aligned} \dot{\phi}_t^u &= \text{Tr}(\Psi_t^u \dot{\Sigma}_t), \quad t < T \\ &= \text{Tr}((a^u \text{id} + \bar{\Psi}_t^u) \dot{\Sigma}_t), \quad t < T \\ &= a^u \text{Tr}(\dot{\Sigma}_t) + \text{Tr}(\bar{\Psi}_t^u \dot{\Sigma}_t), \quad t < T \\ &= -a^u \eta_\nu^2 \int_t^T G_\nu(s, t)^2 ds - \eta_\nu^2 \int_t^T \int_t^T G_\nu(s, t) G_\nu(w, t) \bar{\psi}_t^u(s, w) ds dw, \quad t < T, \end{aligned}$$

with $\phi_T^u = 0$. □

We go now a step further and deduce more explicit Riccati equations by considering completely monotone Volterra kernels.

Definition 2.4. A kernel $G : [0, T]^2 \rightarrow \mathbb{R}$ is a completely monotone Volterra kernel if it satisfies Definition 2.1 and admits a Laplace representation of the form

$$G(t, s) = 1_{s < t} \int_{\mathbb{R}^+} e^{-(t-s)x} \lambda(dx), \quad (2.5.3)$$

where $\lambda(\cdot)$ is a positive measure.

Proposition 2.5.2. *Assume that the kernel function $G_\nu(t, s)$ is completely monotone and can be represented as (2.5.3). Moreover, assume that $g_0(\cdot)$ is given by (2.2.3) and that the kernel satisfies Definition 2.1. Fix $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$. Then, for all $t \leq T$,*

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_t^T}{I_t} \right) \middle| \mathcal{F}_t \right] &= \exp \left(\Theta_t^u + 2 \int_{\mathbb{R}_+} \Lambda_t^u(x) Y_t(x) \lambda(dx) \right. \\ &\quad \left. + \int_{\mathbb{R}_+^2} \Gamma_t^u(x, y) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy) \right), \end{aligned} \quad (2.5.4)$$

where $t \rightarrow (\Theta_t^u, \Lambda_t^u, \Gamma_t^u)$ solve Riccati equations of the form

$$\begin{aligned} \dot{\Theta}_t^u &= \dot{\chi}_t^u - a^u \left(g_0^T(t) + \rho_{Ir} \eta_r B_{G_r}(t, T) \right)^2 - \eta_\nu^2 \left(2 \left(\int_{\mathbb{R}_+} \Lambda_t^u(x) \lambda(dx) \right)^2 \right. \\ &\quad \left. + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Gamma_t^u(x, y) \lambda(dx) \lambda(dy) \right) \\ &\quad - 2 \int_{\mathbb{R}_+} \left(b^u g_0^T(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \Lambda_t^u(x') \lambda(dx'), \quad t < T, \end{aligned} \quad (2.5.5)$$

$$\begin{aligned} \dot{\Lambda}_t^u(x) &= -a^u \left(g_0^T(t) + \rho_{Ir} \eta_r B_{G_r}(t, T) \right) + x \Lambda_t^u(x) \\ &\quad - 2 \eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \left(\int_{\mathbb{R}_+} \Lambda_t^u(y) \lambda(dy) \right) \\ &\quad - \int_{\mathbb{R}_+} \left(b^u g_0^T(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \Gamma_t^u(x, x') \lambda(dx') \\ &\quad - b^u \int_{\mathbb{R}_+} \Lambda_t^u(x') \lambda(dx'), \quad t < T, \quad x \in \mathbb{R}_+, \end{aligned} \quad (2.5.6)$$

and

$$\begin{aligned} \dot{\Gamma}_t^u(x, y) &= (x + y) \Gamma_t^u(x, y) - a^u - 2 \eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \left(\int_{\mathbb{R}_+} \Gamma_t^u(y, y') \lambda(dy') \right) \\ &\quad - b^u \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') + \int_{\mathbb{R}_+} \Gamma_t^u(y', y) \lambda(dy') \right), \quad t < T, \quad (x, y) \in \mathbb{R}_+^2, \end{aligned} \quad (2.5.7)$$

with $\Theta_T^u = \Lambda_T^u = \Gamma_T^u = 0$.

Proof. The proof is given in Section 2.6. $\square$

Remark 2.6. *A general result about the existence and the uniqueness of solutions to Riccati equations (2.5.5)-(2.5.6)-(2.5.7) is provided in Abi Jaber et al. (2021b).*

2.5.2 Riccati equations for Markovian volatility models

In this section, we consider some particular completely monotone kernels associated to Markovian volatility models. For those models, we derive a simplified version of the characteristic function based on Riccati equations. In particular, for the indicator $G_\nu(t, s) = 1_{s < t}$ and exponential $G_\nu(t, s) = 1_{s < t} \alpha \exp(-\beta(t - s))$ kernels, we obtain closed form expressions similar to the characteristic functions deduced in van Haastrecht et al. (2009).

Proposition 2.5.3. *Suppose that $G_\nu(t, s) = 1_{s < t} \sum_{i=1}^N w_i e^{-x_i(t-s)}$ and $g_0(t) = \nu_0 + \theta_\nu \sum_{i=1}^N \int_0^t w_i \exp(-x_i(t-s)) ds$ with $\nu_0 > 0$ and $\theta_\nu \in \mathbb{R}$, then, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$,*

$$\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] = \exp \left(A_t^{u; N} + 2 \sum_{i=1}^N B_t^{u; i} \nu_t^i + \sum_{i=1}^N \sum_{j=1}^N C_t^{u; ij} \nu_t^i \nu_t^j \right), \quad (2.5.8)$$

with $((\nu_t^i)_{i=1, \dots, N})_{0 \leq t \leq T}$ solution of the following SDEs

$$\begin{aligned} d\nu_t^i &= (-x_i \nu_t^i + \theta(t) + \kappa_\nu \nu_t) dt + \eta_\nu dW_\nu^{\mathbb{Q}^T}(t), \quad t > 0, \quad i = 1, \dots, N, \\ \nu_0^i &= 0, \end{aligned}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{Q}^T)$ with $\theta(t) := \theta_\nu - \eta_\nu \sum_{i=1}^n \eta_i \rho_{i\nu} B_i(t, T)$, such that, almost surely,

$$\nu_t = \nu_0 + \sum_{i=1}^N w_i \nu_t^i, \quad t \leq T,$$

and $A_t^{u; N}$, $(B_t^{u; i})_{i=1, \dots, N}$ and $(C_t^{u; ij})_{i,j=1, \dots, N}$ time-dependent functions satisfying Riccati equations of the form

$$\begin{aligned} \dot{A}_t^{u; N} &= -a^u \left(\nu_0^2 + \eta_r^2 B_{G_r}(t, T)^2 - 2\nu_0 \rho_{Ir} \eta_r B_{G_r}(t, T) \right) \\ &\quad - 2 \sum_{i=1}^N B_t^{u; i} \left(\theta(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) + b^u \nu_0 \right) \\ &\quad - \eta_\nu^2 \sum_{i,j=1}^N (2B_t^{u; i} B_t^{u; j} + C_t^{u; ij}), \quad t < T. \end{aligned} \quad (2.5.9)$$

$$\begin{aligned}
\dot{B}_t^{u;i} &= x_i B_t^{u;i} - a^u w_i \left(\nu_0 + \rho_{Ir} \eta_r B_{G_r}(t, T) \right) - 2\eta_\nu^2 \sum_{j,k=1}^N C_t^{u;ij} B_t^{u;k} \\
&\quad - w_i b^u \sum_{j=1}^N B_t^{u;j} - \sum_{j=1}^N C_t^{u;ij} \left(\theta(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) + b^u \nu_0 \right), \\
t &< T, \quad i = 1, \dots, N,
\end{aligned} \tag{2.5.10}$$

$$\dot{C}_t^{u;ij} = (x_i + x_j) C_t^{u;ij} - w_i w_j a^u - 2\eta_\nu^2 \sum_{k=1}^N \sum_{l=1}^N C_t^{u;ik} C_t^{u;jl} - b^u \sum_{k=1}^N \left(w_j C_t^{u;ik} + w_i C_t^{u;kj} \right) \tag{2.5.11}$$

with $A_T^{u;N} = B_T^{u;i} = C_T^{u;ij} = 0$, for $i, j = 1, \dots, N$.

Proof. The proof is given in Section 2.6. $\square$

Corollary 2.5.4. Suppose that $G_\nu(t, s) = 1_{s < t} \alpha \exp(-\beta(t-s))$ and $g_0(t) = \nu_0 + \theta_\nu \int_0^t \alpha \exp(-\beta(t-s)) ds$, with $\nu_0 > 0$, $\alpha, \beta \in \mathbb{R}_+$ and $\theta_\nu \in \mathbb{R}$. Then, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$,

$$\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] = \exp(A_t^u + 2B_t^u \nu_t + C_t^u \nu_t^2), \tag{2.5.12}$$

with A_t^u , B_t^u and C_t^u time-dependent functions given by

$$\begin{aligned}
A_t^u &= -\frac{1}{2} u(1-u) V_r(t, T) + \int_t^T \left[\left(\alpha \theta_\nu + (u-1) \rho_{\nu r} \alpha \eta_\nu \eta_r B_{G_r}(t, T) \right) B_s^u \right. \\
&\quad \left. + \frac{1}{2} \eta_\nu^2 \left((B_s^u)^2 + C_s^u \right) \right] ds,
\end{aligned} \tag{2.5.13}$$

$$\begin{aligned}
B_t^u &= -\frac{1}{2} \frac{u(1-u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \left[\gamma_0(1 + e^{-2\gamma(T-t)}) \right. \\
&\quad \left. + (\gamma_3 - \gamma_4 e^{-2\gamma(T-t)} - (\gamma_5 e^{-\kappa_r(T-t)} - \gamma_6 e^{-(2\gamma + \kappa_r)(T-t)}) - \gamma_7 e^{-\gamma(T-t)}) \right],
\end{aligned} \tag{2.5.14}$$

$$C_t^u = -\frac{1}{2} u(1-u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{2.5.15}$$

where $V_r(t, T)$ is the integrated variance of the interest rate process (2.2.1) and

$$\begin{aligned}
\gamma &= \sqrt{-(\alpha\kappa_\nu - \beta) - \rho_{I\nu}\alpha\eta_\nu u)^2 - \alpha^2\eta_\nu^2(u^2 - u)}, \quad \gamma_0 = \frac{\alpha\theta_\nu}{\gamma}, \\
\gamma_1 &= \gamma - (\alpha\kappa_\nu - \beta + \rho_{I\nu}\alpha\eta_\nu u), & \gamma_2 &= \gamma((\alpha\kappa_\nu - \beta + \rho_{I\nu}\alpha\eta_\nu u), \\
\gamma_3 &= \frac{\rho_{Ir}\eta_r\gamma_1 + \rho_{\nu I}\eta_r\alpha\eta_\nu(u-1)}{\kappa_r\gamma}, & \gamma_4 &= \frac{\rho_{Ir}\eta_r\gamma_2 - \rho_{\nu r}\eta_r\alpha\eta_\nu(u-1)}{\kappa_r\gamma}, \\
\gamma_5 &= \frac{\rho_{Ir}\eta_r\gamma_1 + \rho_{\nu r}\eta_r\alpha\eta_\nu(u-1)}{\kappa_r(\gamma - \kappa_r)}, & \gamma_6 &= \frac{\rho_{Ir}\eta_r\gamma_2 - \rho_{\nu r}\eta_r\alpha\eta_\nu(u-1)}{\kappa_r(\gamma + \kappa_r)}, \\
\gamma_7 &= (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6).
\end{aligned}$$

Proof. Based on Proposition 2.5.3 with $N = 1$, $w_1 = 1$ and $x_1 = \alpha$, we deduce the Riccati equations satisfied by A_t^u, B_t^u, C_t^u . Thus, using a laborious development as in van Haastrecht et al. (2009); van Haastrecht and Pelsner (2011), we can obtain the explicit forms given by (2.5.13)-(2.5.14)-(2.5.15). $\square$

2.5.3 Multi-factor approximation for completely monotone Volterra kernels

In this section, we propose another alternative to approximate the characteristic function when considering completely monotone Volterra kernels in the sense of Definition 2.1. Using similar arguments as in Chapter 1, the approximate method is based on a Laplace representation of completely monotone kernels. Then, we deduce an expression of the characteristic function in terms of Riccati equations. Unlike the approach proposed in Section 2.3.2, a convergence result is deduced for this multi-factor approximate approach. Using a Laplace transform representation, we have that completely monotone kernels can be rewritten such that

$$G_\nu(t, s) = 1_{s < t} \int_{\mathbb{R}^+} e^{-(t-s)x} \lambda(dx), \quad (2.5.16)$$

with $\lambda(\cdot)$ a positive measure. Hence a natural approximation of the kernel is given by

$$\hat{G}_\nu(t, s) := 1_{s < t} \sum_{i=1}^N \hat{w}_i e^{-(t-s)\hat{x}_i}, \quad (2.5.17)$$

where $(\hat{w}_i)_{i=1, \dots, N}$ are the weights and $(\hat{x}_i)_{i=1, \dots, N}$ the mean reversion terms that should be appropriately defined. Under suitable choice, $\hat{G}_\nu$ converges in $L^2([0, T]^2, \mathbb{R})$ to the completely monotone kernels. Therefore, based on Proposition 2.5.3, we can propose another approximate solution of the characteristic function associated to completely monotone Volterra kernels.

Lemma 2.5.5. *Suppose that for all $N \geq 1$, $(\hat{w}_i)_{i=1,\dots,N}$ and $(\hat{x}_i)_{i=1,\dots,N}$ are such that*

$$\hat{w}_i := \int_{k_{i-1}}^{k_i} \lambda(dk), \quad \hat{x}_i := \frac{1}{\hat{w}_i} \int_{k_{i-1}}^{k_i} k \lambda(dk),$$

such that $k_0 = 0 < k_1 < \dots < k_n$ and

$$k_n \rightarrow \infty, \quad \sum_{i=1}^N \int_{k_{i-1}}^{k_i} (k_i - k)^2 \lambda(dk) \rightarrow 0.$$

as N goes to infinity. For $N \geq 1$ fixed, there exists a function $f_N^{(2)}((k_i)_{i=0,\dots,N})$ such that the following inequality holds

$$\|\hat{G}_\nu - G_\nu\|_{L^2([0,T]^2, \mathbb{R})} \leq f_N^{(2)}((k_i)_{i=1,\dots,N}), \quad (2.5.18)$$

and $\hat{G}_\nu$ converges in $L^2([0,T]^2, \mathbb{R})$ to G_ν when N goes to infinity i.e.

$$\|\hat{G}_\nu - G_\nu\|_{L^2([0,T]^2, \mathbb{R})} \rightarrow 0,$$

as N goes to infinity.

Proof. We refer to the proof of (Abi Jaber and El Euch, 2019a, Proposition 3.3.). $\square$

Remark 2.7. *As stated in Abi Jaber and El Euch (2019a), there are different possible choices of the auxiliary terms $(k_i)_{i=0,\dots,N}$. In this chapter, for fixed $N \geq 1$, we take auxiliary terms $(k_i)_{i=0,\dots,N}$ that minimize the upper bound (2.5.18). Thus, we choose $(k_i)_{i=1,\dots,N}$ solutions of*

$$\inf_{(k_i)_{i=1,\dots,N} \in \mathcal{E}_N} f_N^{(2)}((k_i)_{i=1,\dots,N}),$$

with $\mathcal{E}_N := \{(k_i)_{i=1,\dots,N} : k_0 = 0 < k_1 < \dots < k_n\}$.

Proposition 2.5.6. *Let $(I_t^T)_{0 \leq t \leq T}$ be the solution of (2.2.13) with the completely monotone kernel (2.5.16) and $(\tilde{I}_t^T)_{0 \leq t \leq T}$ solution of (2.2.13) with the approximate kernel (2.5.17). Suppose that the assumptions of Lemma 2.5.5 are satisfied and that, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, the sequence $(\exp(u \log \tilde{I}_t^T))_{N \in \mathbb{N}}$ is uniformly integrable, then*

$$\lim_{N \rightarrow \infty} \exp(A_0^{u;N}) = \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_0^T} \right) \middle| \mathcal{F}_0 \right],$$

where $A_0^{u;N}$ is the solution of Riccati equation (2.5.9) with $w_i = \hat{w}_i$ and $x_i = \hat{x}_i$, for $i = 1, \dots, N$.

Proof. Using arguments similar to those in Chapter 1, we can show that, for $t \in [0, T]$,

$$\tilde{I}_t^T \xrightarrow{\mathcal{L}} I_t^T,$$

as N goes to infinity, where $\xrightarrow{\mathcal{L}}$ stands for weak convergence. Therefore, using the uniform integrability of $\exp(u \log \tilde{I}_t^T)$ and the fact that $I_0^T = \tilde{I}_0^T$, we obtain that

$$\lim_{N \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{\tilde{I}_t^T}{\tilde{I}_0^T} \right) \middle| \mathcal{F}_0 \right] = \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_t^T}{I_0^T} \right) \middle| \mathcal{F}_0 \right],$$

and thus, we conclude the proof using Corollary 2.5.3 and the fact that $\nu_0^i = 0$ for $i = 1, \dots, N$. $\square$

Based on Proposition 2.5.6, we can deduce a second approximate solution of the characteristic function for which a theoretical convergence result is established. However, it is a semi-closed solution in the sense that it requires solving Riccati equations.

Finally, we decide to compare this multi-factor approximate method with the operator discretization method proposed in Section 2.3.2. For this purpose, we consider the same kernels and model parameters as in Section 2.3.2. Figure 2.11 presents the implied volatility generated for different N by the multi-factor method where the system of Riccati $(N^2 + N + 1)$ equations (2.5.9)-(2.5.10)-(2.5.11) is solved numerically using an implicit method. We observe a convergence as the number of factors increases. However, the speed of convergence differs between the methods. As revealed by Figure 2.12, the operator discretization approach converges much faster than the multi-factor approach, either with respect to N or in terms of computation time. Nevertheless, it should be noted that the speed of convergence of the “multi-factor” method depends strongly on the choice of multi-factor parameters $(\hat{w}_i, \hat{x}_i)_{i=1, \dots, N}$, as well as the numerical method used to solve the system of Riccati $(N^2 + N + 1)$ equations and other choices than those used in this chapter could give a faster speed of convergence. The aim of this comparison is above all to illustrate that the operator discretization method converges quickly and is fairly simple to implement compared with the multi-factor method.

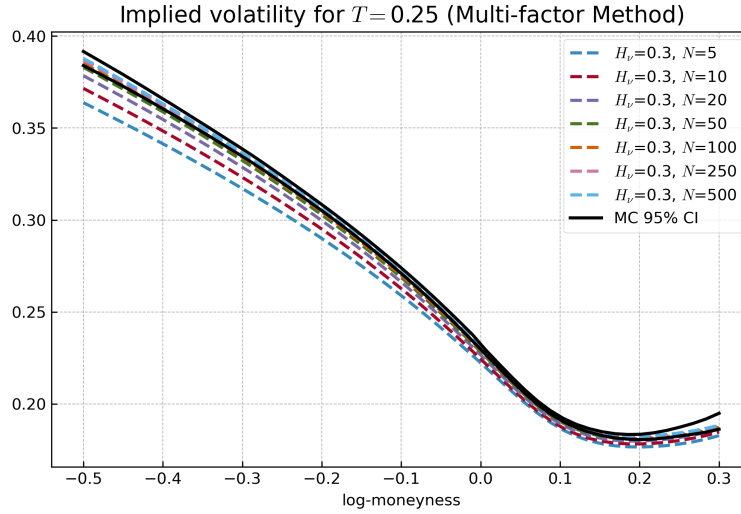


Figure 2.11: Implied volatility dynamics generated by the multi-factor approximation method with $H_v = 0.3$ (right). Monte Carlo confidence intervals are generated with 200 000 simulations of risk processes using a Euler scheme with 500 as discretization steps.

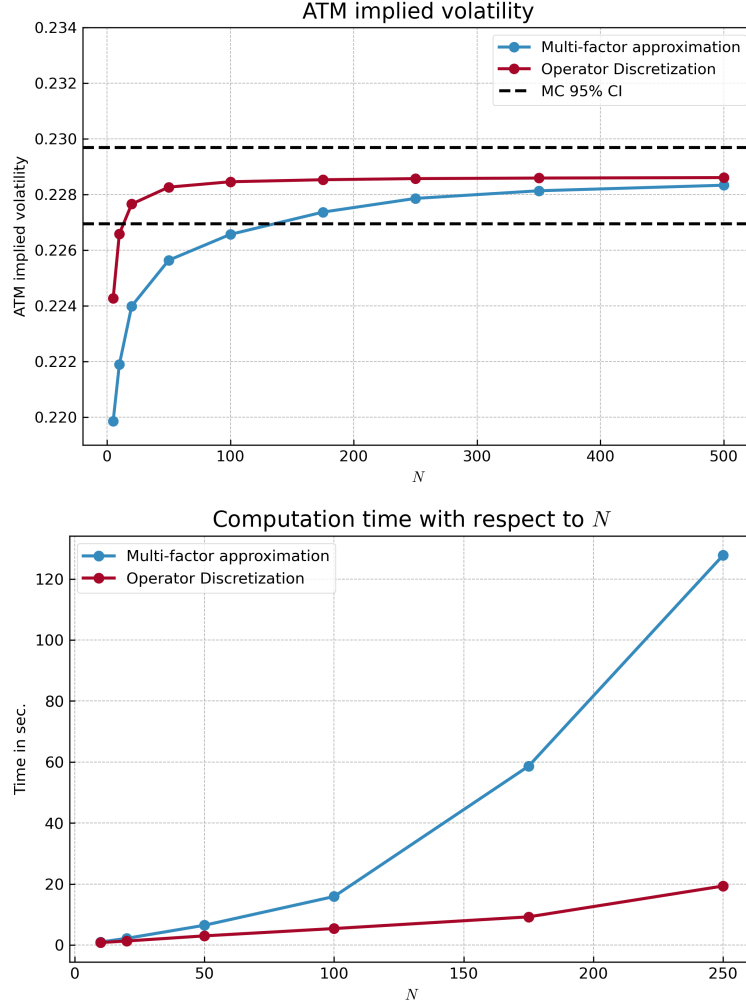


Figure 2.12: ATM forward implied volatility generated by the operator discretization and multi-factor approximation methods with $H_\nu = 0.3$ (left) and the associated computation time (right).

2.6 Proofs

2.6.1 Proof of Theorem 2.3.1

We first consider two useful lemmas before proving Theorem 2.3.1.

Lemma 2.6.1. *For $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$, there exists an integral operator*

$\bar{\Psi}_t^u$ induced by a symmetric kernel $\bar{\psi}_t^u(s, w)$ such that

$$\Psi_t^u = a^u id + \bar{\Psi}_t^u, \quad (2.6.1)$$

and for any $f \in L^2([0, T], \mathbb{C})$,

$$(\Psi_t^u f 1_t)(t) = (a^u id + b^u \mathbf{G}_\nu^* \Psi_t^u)(f 1_t)(t), \quad (2.6.2)$$

with $1_t : s \rightarrow 1_{t \leq s}$. Moreover, the kernel $\bar{\psi}_t(s, w)$ satisfies

$$\bar{\psi}_t^u(t, s) = \bar{\psi}_t^u(s, t) = b^u \int_t^T G_\nu(w, t)(a^u \delta_{w=s} + \bar{\psi}_t^u(s, w))dw, \quad t \leq s \leq T, \text{ a.e.} \quad (2.6.3)$$

Proof. From (Abi Jaber, 2022a, Lemma B.1.), we know that in our setting there exists an integral operator $\bar{\Psi}_t^u$ induced by a symmetric kernel $\bar{\psi}_t(s, w)$ such that (2.6.1) and (2.6.2) are satisfied. Moreover, from (2.6.2), since $\mathbf{G}_\nu^*$ is an integral operator and $\bar{\psi}_t^u(s, w)$ is symmetric, we deduce that, for any $f \in L^2([0, T], \mathbb{C})$,

$$\begin{aligned} (\bar{\Psi}_t^u f 1_t)(t) &= (b^u \mathbf{G}_\nu^* \Psi_t^u)(f 1_t)(t) \\ &= b^u \int_t^T G_\nu^*(t, s) (\Psi_t^u f 1_t)(s) ds \\ &= b^u \int_t^T f(s) \int_t^T G_\nu(w, t)(a^u \delta_{w=s} + \bar{\psi}_t^u(s, w))dw ds \end{aligned}$$

Moreover, since $\bar{\Psi}_t$ is a symmetric integral operator, we also have that

$$\begin{aligned} (\bar{\Psi}_t^u f 1_t)(t) &= \int_t^T \bar{\psi}_t^u(t, s) f(s) ds. \\ &= \int_t^T \bar{\psi}_t^u(s, t) f(s) ds \end{aligned}$$

We conclude that, for any $f \in L^2([0, T], \mathbb{C})$,

$$\begin{aligned} 0 &= \int_t^T f(s) \left[\bar{\psi}_t^u(s, t) - b^u \int_t^T G_\nu(w, t)(a^u \delta_{w=s} + \bar{\psi}_t^u(s, w))dw \right] ds \\ &= \int_t^T f(s) \left[\bar{\psi}_t^u(t, s) - b^u \int_t^T G_\nu(w, t)(a^u \delta_{w=s} + \bar{\psi}_t^u(s, w))dw \right] ds, \end{aligned}$$

and thus, for $t \leq s \leq T$,

$$\bar{\psi}_t^u(t, s) = \bar{\psi}_t^u(s, t) = b^u \int_t^T G_\nu(w, t)(a^u \delta_{w=s} + \bar{\psi}_t^u(s, w))dw,$$

almost everywhere. $\square$

Lemma 2.6.2. For $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$ and $t \in [0, T]$, let us define $h_t^u(\cdot)$ such that, for $s, t \leq T$,

$$h_t^u(s) := g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_r}(s, T) - \int_t^s G_\nu(s, w) (b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(w, T) dw \right).$$

Then the dynamics of $t \rightarrow \langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2}$ is given by

$$\begin{aligned} d \langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2} = & \left(-a^u \left(\nu_t + \rho_{Ir} \eta_r B_{G_r}(t, T) \right)^2 \right. \\ & - 2u \eta_\nu (\rho_{I\nu} \nu_t + \rho_{\nu r} \eta_r B_{G_r}(t, T)) (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) \\ & \left. - Tr(\Psi_t^u \dot{\Sigma}_t) + \langle h_t^u, \dot{\Psi}_t^u h_t^u \rangle_{L^2} \right) dt \\ & + 2\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) dW_\nu^{\mathbb{Q}^T}(t). \end{aligned} \quad (2.6.4)$$

Proof. The proof is obtained directly by adapting the proof of (Abi Jaber, 2022a, Lemma B.2.) with h_t^u instead of g_t . □

Based now of Lemma 2.6.1 and 2.6.2, we are now ready to prove Theorem 2.3.1.

Proof. As explained in Abi Jaber (2022a), it is sufficient to make the proof for $u \in \mathbb{R}$ such $0 \leq u \leq 1$. Fix $u \in [0, 1]$ and consider the processes $(U_t)_{0 \leq t \leq T}$ and $(M_t)_{0 \leq t \leq T}$ defined, for $t \in [0, T]$, by

$$U_t = u \log I_t^T + \phi_t^u + \chi_t^u + \langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2}, \quad (2.6.5)$$

and

$$M_t = \exp(U_t).$$

If we prove that $(M_t)_{0 \leq t \leq T}$ is a martingale under $\mathbb{Q}^T$, then the proof is complete since, in this case, we have that

$$E^{\mathbb{Q}^T}(M_T | \mathcal{F}_t) = M_t,$$

and thus

$$\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] = \exp(\phi_t^u + \chi_t^u + \langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2}).$$

Let us prove that $(M_t)_{0 \leq t \leq T}$ is a (true) martingale under $\mathbb{Q}^T$. First, we can prove that $(M_t)_{0 \leq t \leq T}$ is a local martingale by showing that the drift of the dynamic of $(M_t)_{0 \leq t \leq T}$ is null. Using the dynamic of $(I_t^T)_{0 \leq t \leq T}$ given by (2.2.13), the dynamic

of $(\langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2})_{0 \leq t \leq T}$ given by (2.6.4) and the fact that $a^u = \frac{1}{2}(u^2 - u)$, we observe that

$$\begin{aligned} dU_t &= u d(\log I_t^T) + (\dot{\phi}_t^u + \dot{\chi}_t^u) dt + d\langle h_t^u, \Psi_t^u h_t^u \rangle_{L^2} \\ &= \left(-\frac{u^2}{2} \left(\nu_t^2 + \eta_r^2 B_{G_r}(t, T)^2 + 2\nu_t \eta_r B_{G_r}(t, T) \rho_{I_r} \right) \right. \\ &\quad \left. + \langle h_t^u, \dot{\Psi}_t^u h_t^u \rangle_{L^2} - 2u\eta_\nu (\rho_{I_\nu} \nu_t + \rho_{\nu r} \eta_r B_{G_r}(t, T)) (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) \right) dt \\ &\quad + u\nu_t dW_I^{\mathbb{Q}^T}(t) + u\eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}^T}(t) + 2\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) dW_\nu^{\mathbb{Q}^T}(t). \end{aligned}$$

Moreover, the quadratic variation of $(U_t)_{0 \leq t \leq T}$ satisfies

$$\begin{aligned} d\langle U \rangle_t &= \left((u^2 \nu_t^2 + 4\eta_\nu^2 (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t))^2 + u^2 \eta_r^2 B_{G_r}(t, T)^2 \right. \\ &\quad \left. + 2u^2 \nu_t \rho_{I_r} \eta_r B_{G_r}(t, T) \right. \\ &\quad \left. + 4u\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) \left(\rho_{I_\nu} \nu_t + \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \right) dt. \end{aligned}$$

As the dynamic of $(M_t)_{0 \leq t \leq T}$ can be written such that

$$dM_t = M_t \left(\frac{1}{2} d\langle U \rangle_t + dU_t \right),$$

we easily obtain that

$$\begin{aligned} dM_t &= M_t \left(2\eta_\nu^2 ((\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t))^2 + \langle h_t^u, \dot{\Psi}_t^u h_t^u \rangle_{L^2} \right) dt \\ &\quad + M_t \left(u\nu_t dW_I^{\mathbb{Q}^T}(t) + u\eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}^T}(t) + 2\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) dW_\nu^{\mathbb{Q}^T}(t) \right), \end{aligned}$$

but as

$$2\eta_\nu^2 ((\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t))^2 = -2\langle h_t^u, \Psi_t^u \dot{\Sigma}_t \Psi_t^u h_t^u \rangle_{L^2},$$

we have that

$$\begin{aligned} dM_t &= M_t \left(\langle h_t^u, (\dot{\Psi}_t^u - 2\Psi_t^u \dot{\Sigma}_t \Psi_t^u) h_t^u \rangle_{L^2} \right) dt \\ &\quad + M_t \left(u\nu_t dW_I^{\mathbb{Q}^T}(t) + u\eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}^T}(t) + 2\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) dW_\nu^{\mathbb{Q}^T}(t) \right) \\ &= M_t \left(u\nu_t dW_I^{\mathbb{Q}^T}(t) + u\eta_r B_{G_r}(t, T) dW_r^{\mathbb{Q}^T}(t) + 2\eta_\nu (\mathbf{G}_\nu^* \Psi_t^u h_t^u)(t) dW_\nu^{\mathbb{Q}^T}(t) \right), \end{aligned}$$

and since, from (Abi Jaber, 2022a, Lemma B.1.),

$$\dot{\Psi}_t^u - 2\Psi_t^u \dot{\Sigma}_t \Psi_t^u = 0,$$

we have that $(M_t)_{0 \leq t \leq T}$ is a local martingale. It remains to show that this is a true martingale. Since $\langle h_t^u, \dot{\Psi}_t^u h_t^u \rangle_{L^2} \leq 0$ and $\phi_t^u = - \int_t^T \text{Tr}(\Psi_s^u \dot{\Sigma}_s) ds \leq 0$, from (2.6.5), it follows that

$$U_t \leq u \log I_t^T + \chi_t^u.$$

Furthermore, we observe that

$$\begin{aligned} U_t &\leq u \log I_0^T - \frac{u^2}{2} \int_0^t \left(\left(\nu_s + \rho_{I_r} \eta_r B_{G_r}(s, T) \right)^2 + (1 - \rho_{I_r}^2) \eta_r^2 B_{G_r}(s, T)^2 \right) ds \\ &\quad + \frac{1}{2} (u^2 - u) \left(\int_0^T (1 - \rho_{I_r}^2) \eta_r^2 B_{G_r}(s, T)^2 ds \right) \\ &\quad + \int_0^t u \nu_s dW_I^{\mathbb{Q}^T}(s) + \int_0^t u \eta_r B_{G_r}(s, T) dW_r^{\mathbb{Q}^T}(s), \end{aligned}$$

and, as $u \in [0, 1]$, we deduce that

$$\begin{aligned} U_t &\leq u \log I_0^T - \frac{u^2}{2} \int_0^t \left(\left(\nu_s + \rho_{I_r} \eta_r B_{G_r}(s, T) \right)^2 + (1 - \rho_{I_r}^2) \eta_r^2 B_{G_r}(s, T)^2 \right) ds \\ &\quad + \int_0^t u \nu_s dW_I^{\mathbb{Q}^T}(s) + \int_0^t u \eta_r B_r(s, T) dW_r^{\mathbb{Q}^T}(s). \end{aligned}$$

Then, we define the process $(N_t)_{0 \leq t \leq T}$ such that

$$\begin{aligned} N_t &:= \left(I_0^T \right)^u \exp \left(- \frac{u^2}{2} \int_0^t \left(\left(\nu_s + \rho_{I_r} \eta_r B_{G_r}(s, T) \right)^2 + (1 - \rho_{I_r}^2) \eta_r^2 B_{G_r}(s, T)^2 \right) ds \right. \\ &\quad \left. + \int_0^t u \nu_s dW_I^{\mathbb{Q}^T}(s) + \int_0^t u \eta_r B_{G_r}(s, T) dW_r^{\mathbb{Q}^T}(s) \right), \end{aligned}$$

This process $(N_t)_{0 \leq t \leq T}$ is a true martingale (see arguments in (Abi Jaber et al., 2021b, in Lemma 7.3)). Therefore, we finally have that

$$|M_t| \leq \exp(U_t) \leq N_t$$

and as $(M_t)_{0 \leq t \leq T}$ is a local martingale upper bounded by a true martingale, this a true martingale and it completes the proof. $\square$

2.6.2 Proof of Proposition 2.5.2

We first consider a lemma before proving Proposition 2.5.2.

Lemma 2.6.3. *Assume that the kernel function $G_\nu(t, s)$ is completely monotone and can be represented as (2.5.3). Then,*

$$\nu_t = g_0^T(t) + \int_{\mathbb{R}_+} Y_t(x) \lambda(dx), \quad t \leq T,$$

$$g_t(s) = 1_{t \leq s} \left(g_0^T(s) + \int_{\mathbb{R}_+} e^{-x(s-t)} Y_t(x) \lambda(dx) \right),$$

with

$$Y_t(x) = \int_0^t e^{-(t-w)x} \kappa_\nu \nu_w dw + \int_0^t e^{-(t-w)x} \eta_\nu dW_\nu^{\mathbb{Q}^T}(w), \quad t \leq T, \quad x \in \mathbb{R}_+. \quad (2.6.6)$$

Moreover, for $u \in \mathbb{C}$ such that $0 \leq \Re(u) \leq 1$,

$$h_t^u(s) = g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_t}(s, T) - \int_{\mathbb{R}_+} b_t^u(s, x) \lambda(dx) \right),$$

with

$$b_t^u(s, x) = \int_t^s e^{-(s-w)x} \left((b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(w, T) \right) dw, \quad t \leq s \leq T, \quad x \in \mathbb{R}_+. \quad (2.6.7)$$

Proof. For $t \leq T$, applying Fubini's theorem, we can rewrite ν_t such that

$$\begin{aligned} \nu_t &= g_0^T(s) + \int_0^t G_\nu(t, w) \kappa_\nu \nu_w dw + \int_0^t G_\nu(t, w) \eta_\nu dW_\nu^{\mathbb{Q}^T}(w) \\ &= g_0^T(s) + \int_0^t \int_{\mathbb{R}_+} e^{-(t-w)x} \lambda(dx) \kappa_\nu \nu_w dw + \int_0^t \int_{\mathbb{R}_+} e^{-(t-w)x} \lambda(dx) \eta_\nu dW_\nu^{\mathbb{Q}^T}(w) \\ &= g_0^T(s) + \int_{\mathbb{R}_+} \left(\int_0^t e^{-(t-w)x} \kappa_\nu \nu_w dw + \int_0^t e^{-(t-w)x} \eta_\nu dW_\nu^{\mathbb{Q}^T}(w) \right) \lambda(dx) \\ &= g_0^T(s) + \int_{\mathbb{R}_+} Y_t(x) \lambda(dx), \end{aligned}$$

with $Y_t(x)$ given by (2.6.6). Using the same arguments, we obtain that,

$$g_t(s) = 1_{t \leq s} \left(g_0^T(s) + \int_{\mathbb{R}_+} e^{-x(s-t)} Y_t(x) \lambda(dx) \right).$$

Finally, we have that,

$$\begin{aligned} h_t^u(s) &= g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_r}(s, T) \right. \\ &\quad \left. - \int_t^s G_\nu(s, w) \left((b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(w, T) \right) dw \right) \\ &= g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_r}(s, T) - \int_{\mathbb{R}_+} \left(\int_t^s e^{-(s-w)x} \left((b^u \rho_{Ir} \right. \right. \right. \\ &\quad \left. \left. - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(w, T) \right) dw \right) \lambda(dx) \right) \\ &= g_t(s) + 1_{t \leq s} \left(\rho_{Ir} \eta_r B_{G_r}(s, T) - \int_{\mathbb{R}_+} b_t^u(s, x) \lambda(dx) \right), \end{aligned}$$

with $b_t^u(s, x)$ given by (2.6.7). $\square$

Proof. We divide the proof into two steps. The first step is to show that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] &= \exp \left(\Theta_t^u + 2 \int_{\mathbb{R}_+} \Lambda_t^u(x) Y_t(x) \lambda(dx) \right. \\ &\quad \left. + \int_{\mathbb{R}_+^2} \Gamma_t^u(x, y) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy) \right), \end{aligned}$$

with $t \rightarrow (\Theta_t^u, \Lambda_t^u, \Gamma_t^u)$ such that

$$\Theta_t^u := \phi_t^u + \chi_t^u + \int_t^T a^u \bar{h}_t^u(s)^2 ds + \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \bar{\psi}_t^u(s, w) ds dw, \quad (2.6.8)$$

$$\Lambda_t(x) := \int_t^T a^u \bar{h}_t^u(s) e^{-x(s-t)} ds + \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \bar{\psi}_t^u(s, w) ds dw, \quad x \in \mathbb{R}_+, \quad (2.6.9)$$

and

$$\Gamma_t(x, y) := \int_t^T \int_t^T e^{-x(s-t)} e^{-y(w-t)} (a^u \delta_{s=w} + \bar{\psi}_t^u(s, w)) ds dw, \quad (x, y) \in \mathbb{R}_+^2, \quad (2.6.10)$$

with $\bar{h}_t^u(s)$ a time-dependent function given by

$$\bar{h}_t^u(s) := 1_{t \leq s} \left(g_0^T(s) + \rho_{I_r} \eta_r B_{G_r}(s, T) - \int_{\mathbb{R}_+} b_t^u(s, x) \lambda(dx) \right). \quad (2.6.11)$$

Then, the second step is to deduce the equations satisfied by $(\Theta_t^u, \Lambda_t^u, \Gamma_t^u)$.

Step 1

Using Proposition 2.5.1, we have that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] &= \exp(\phi_t^u + \chi_t^u + a^u \int_t^T h_t^u(s)^2 ds \\ &\quad + \int_t^T \int_t^T h_t^u(s) h_t^u(w) \bar{\psi}_t^u(s, w) ds dw). \end{aligned}$$

Moreover, from Lemma 2.6.3, we have that

$$h_t^u(s) = \bar{h}_t^u(s) + 1_{t \leq s} \int_{\mathbb{R}_+} e^{-x(s-t)} Y_t(x) \lambda(dx),$$

with $\bar{h}_t^u(s)$ given by (2.6.11).

Therefore, using once again Fubini's theorem, we have that

$$\begin{aligned}
\int_t^T h_t^u(s)^2 ds &= \int_t^T \left(\bar{h}_t^u(s) + \int_{\mathbb{R}_+} e^{-x(s-t)} Y_t(x) \lambda(dx) \right) \left(\bar{h}_t^u(s) \right. \\
&\quad \left. + \int_{\mathbb{R}_+} e^{-y(s-t)} Y_t(y) \lambda(dy) \right) ds \\
&= \int_t^T \left(\bar{h}_t^u(s)^2 + 2\bar{h}_t^u(s) \int_{\mathbb{R}_+} e^{-x(s-t)} Y_t(x) \lambda(dx) \right. \\
&\quad \left. + \int_{\mathbb{R}_+^2} e^{-x(s-t)} e^{-y(s-t)} Y_t(x) Y_t(y) \lambda(dx) \lambda(dy) \right) ds \\
&= \int_t^T \bar{h}_t^u(s)^2 ds + \int_{\mathbb{R}_+} \left(\int_t^T 2\bar{h}_t^u(s) e^{-x(s-t)} ds \right) Y_t(x) \lambda(dx) \\
&\quad + \int_{\mathbb{R}_+^2} \left(\int_t^T e^{-x(s-t)} e^{-y(s-t)} ds \right) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy),
\end{aligned}$$

and

$$\begin{aligned}
\int_t^T \int_t^T h_t^u(s) h_t^u(w) \bar{\psi}_t^u(s, w) ds dw &= \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \bar{\psi}_t^u(s, w) ds dw \\
&\quad + \int_{\mathbb{R}_+} \left(\int_t^T \int_t^T 2\bar{h}_t^u(w) e^{-x(s-t)} \bar{\psi}_t^u(s, w) ds dw \right) \\
&\quad Y_t(x) \lambda(dx) + \int_{\mathbb{R}_+^2} \left(\int_t^T \int_t^T e^{-x(s-t)} e^{-y(u-t)} \right. \\
&\quad \left. \bar{\psi}_t^u(s, w) ds dw \right) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy).
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
& a^u \int_t^T h_t^u(s)^2 ds + \int_t^T \int_t^T h_t^u(s) h_t^u(w) \bar{\psi}_t^u(s, w) ds dw \\
&= \int_t^T a^u \bar{h}_t^u(s)^2 ds + \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \bar{\psi}_t^u(s, w) ds dw \\
&+ 2 \int_{\mathbb{R}_+} \left(\int_t^T a^u \bar{h}_t^u(s) e^{-x(s-t)} ds \right. \\
&+ \left. \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \bar{\psi}_t^u(s, w) ds dw \right) Y_t(x) \lambda(dx) \\
&+ \int_{\mathbb{R}_+^2} \left(\int_t^T \int_t^T e^{-x(s-t)} e^{-y(w-t)} (a^u \delta_{s=w} \right. \\
&\quad \left. + \bar{\psi}_t^u(s, w)) ds dw \right) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy).
\end{aligned}$$

By defining $t \rightarrow (\Theta_t^u, \Lambda_t^u, \Gamma_t^u)$ by (2.6.8)-(2.6.9)-(2.6.10), we obtain that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] &= \exp \left(\Theta_t^u + 2 \int_{\mathbb{R}_+} \Lambda_t^u(x) Y_t(x) \lambda(dx) \right. \\
&\quad \left. + \int_{\mathbb{R}_+^2} \Gamma_t^u(x, y) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy) \right).
\end{aligned}$$

Step 2

Let us now deduce the equations satisfied by $(\Theta_t^u, \Lambda_t^u, \Gamma_t^u)$. First, we have that

$$\Gamma_t^u(x, y) = \int_t^T \int_t^T e^{-x(s-t)} e^{-y(w-t)} \left(a^u \delta_{s=w} + \bar{\psi}_t^u(s, w) \right) ds dw.$$

A direct differentiation of $t \rightarrow \Gamma_t^u$ with respect to t leads to

$$\begin{aligned}
\dot{\Gamma}_t^u(x, y) &= (x + y) \Gamma_t^u(x, y) - a^u - \int_t^T \left(e^{-x(s-t)} + e^{-y(s-t)} \right) \bar{\psi}_t^u(s, t) ds \\
&\quad + \int_t^T \int_t^T e^{-x(s-t)} e^{-y(w-t)} \dot{\bar{\psi}}_t^u(s, w) ds dw.
\end{aligned}$$

Using the form of $\dot{\bar{\psi}}_t^u(s, w)$, we obtain that

$$\begin{aligned}
\int_t^T \int_t^T e^{-x(s-t)} e^{-y(w-t)} \dot{\bar{\psi}}_t^u(s, w) ds dw &= -2 \eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \\
&\quad \left(\int_{\mathbb{R}_+} \Gamma_t^u(y, y') \lambda(dy') \right).
\end{aligned}$$

Moreover, using (2.6.3) in Lemma 2.6.1 and once again the Fubini's theorem, we have that

$$\begin{aligned}
\int_t^T e^{-x(s-t)} (\bar{\psi}_t^u(s, t)) ds &= b^u \int_t^T e^{-x(s-t)} \int_t^T G_\nu(w, t) (a^u \delta_{w=s} + \bar{\psi}_t^u(s, w)) dw ds \\
&= b^u \left(\int_t^T e^{-x(s-t)} \int_t^T \left(\int_{\mathbb{R}_+} e^{-x'(w-t)} \lambda(dx') \right) \right. \\
&\quad \left. (a^u \delta_{w=s} + \bar{\psi}_t^u(s, w)) dw ds \right) \\
&= b^u \int_{\mathbb{R}_+} \left(\int_t^T \int_t^T e^{-x'(w-t)} e^{-x(s-t)} \left(a^u \delta_{w=s} \right. \right. \\
&\quad \left. \left. + \bar{\psi}_t^u(s, w) \right) dw ds \right) \lambda(dx') \\
&= b^u \int_{\mathbb{R}_+} \Gamma_t(x, x') \lambda(dx').
\end{aligned}$$

Using the same arguments, we also obtain that

$$\int_t^T e^{-y(s-t)} (a^u \delta_{s=w} + \bar{\psi}_t^u(s, t)) ds = b^u \int_{\mathbb{R}_+} \Gamma_t^u(y', y) \lambda(dy').$$

Thus, we deduce that

$$\begin{aligned}
\dot{\Gamma}_t^u(x, y) &= (x + y) \Gamma_t^u(x, y) - a^u \\
&\quad - 2 \eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \left(\int_{\mathbb{R}_+} \Gamma_t^u(y, y') \lambda(dy') \right) \\
&\quad - b^u \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') + \int_{\mathbb{R}_+} \Gamma_t^u(y', y) \lambda(dy') \right).
\end{aligned}$$

Let us consider now the ODE satisfied by $t \rightarrow \Lambda_t^u$. We have that

$$\begin{aligned}
\Lambda_t^u(x) &= \int_t^T a^u \bar{h}_t^u(s) e^{-x(s-t)} ds + \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \bar{\psi}_t^u(s, w) ds dw \\
&= \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \left(a^u \delta_{s=w} + \bar{\psi}_t^u(s, w) \right) ds dw.
\end{aligned}$$

Using the same argument than for Γ_t^u , we have that

$$\begin{aligned}
\dot{\Lambda}_t^u(x) &= -a^u \bar{h}_t^u(t) + x \Lambda_t^u(x) + \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \dot{\bar{\psi}}_t^u(s, w) ds dw \\
&\quad + \int_t^T \int_t^T \dot{\bar{h}}_t^u(w) e^{-x(s-t)} \left(a^u \delta_{s=w} + \bar{\psi}_t^u(s, w) \right) ds dw \\
&\quad - \int_t^T (\bar{h}_t^u(t) e^{-x(s-t)} + \bar{h}_t^u(s)) \bar{\psi}_t^u(s, t) ds.
\end{aligned}$$

Using the form of $\dot{\bar{\psi}}_t^u(s, w)$, we obtain that

$$\begin{aligned} & \int_t^T \int_t^T \bar{h}_t^u(w) e^{-x(s-t)} \dot{\bar{\psi}}_t^u(s, w) ds dw \\ &= -2\eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \left(\int_{\mathbb{R}_+} \Lambda_t^u(y) \lambda(dy) \right), \end{aligned}$$

and, using similar arguments than previously, we also have that

$$\begin{aligned} & \int_t^T (\bar{h}_t^u(t) e^{-x(s-t)} + \bar{h}_t^u(s)) \bar{\psi}_t^u(s, t) ds \\ &= b^u \int_t^T \int_t^T (\bar{h}_t^u(t) e^{-x(s-t)} + \bar{h}_t^u(s)) G(w, t) \\ & \quad (a^u \delta_{w=s} + \bar{\psi}_t^u(s, w)) dw ds \\ &= b^u \int_{\mathbb{R}_+} \left(\int_t^T \int_t^T e^{-x'(w-t)} (\bar{h}_t^u(t) e^{-x(s-t)} \right. \\ & \quad \left. + \bar{h}_t^u(s)) (a^u \delta_{w=s} + \bar{\psi}_t^u(s, w)) dw ds \right) \lambda(dx') \\ &= b^u \int_{\mathbb{R}_+} \left(\bar{h}_t^u(t) \Gamma_t^u(x, x') + \Lambda_t^u(x') \right) \lambda(dx'). \end{aligned}$$

Moreover, as

$$\dot{\bar{h}}_t^u(s) = (b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(t, T) \int_{\mathbb{R}_+} e^{-(s-t)x'} \lambda(dx'),$$

we have that

$$\begin{aligned} & \int_t^T \int_t^T \dot{\bar{h}}_t^u(w) e^{-x(s-t)} \left(a^u \delta_{s=w} + \bar{\psi}_t^u(s, w) \right) ds dw \\ &= (b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(t, T) \int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx'). \end{aligned}$$

Therefore, as $\bar{h}_t^u(t) = g_0^T(t) + \rho_{Ir}\eta_r B_{G_r}(t, T)$, we obtain that

$$\begin{aligned}
\dot{\Lambda}_t^u(x) &= -a^u \bar{h}_t^u(t) + x \Lambda_t^u(x) - 2\eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \\
&\quad \left(\int_{\mathbb{R}_+} \Lambda_t^u(y) \lambda(dy) \right) \\
&\quad - \int_{\mathbb{R}_+} \left(b^u \bar{h}_t^u(t) - (b^u \rho_{Ir} - u\eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(t, T) \right) \Gamma_t^u(x, x') \lambda(dx') \\
&\quad - b^u \int_{\mathbb{R}_+} \Lambda_t^u(x') \lambda(dx') \\
&= -a^u \left(g_0^T(t) + \rho_{Ir}\eta_r B_{G_r}(t, T) \right) + x \Lambda_t^u(x) \\
&\quad - 2\eta_\nu^2 \left(\int_{\mathbb{R}_+} \Gamma_t^u(x, x') \lambda(dx') \right) \left(\int_{\mathbb{R}_+} \Lambda_t^u(y) \lambda(dy) \right) \\
&\quad - \int_{\mathbb{R}_+} \left(b^u g_0^T(t) + u\eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \Gamma_t^u(x, x') \lambda(dx') \\
&\quad - b^u \int_{\mathbb{R}_+} \Lambda_t^u(x') \lambda(dx').
\end{aligned}$$

Finally, let us consider the equation satisfied by $t \rightarrow \Theta_t^u$. We know that

$$\begin{aligned}
\Theta_t^u &= \phi_t^u + \chi_t^u + \int_t^T a^u \bar{h}_t^u(s)^2 ds + \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \bar{\psi}_t^u(s, w) ds dw \\
&= \phi_t^u + \chi_t^u + \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) (a^u \delta_{s=w} + \bar{\psi}_t^u(s, w)) ds dw.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\dot{\Theta}_t^u &= \dot{\phi}_t^u + \dot{\chi}_t^u - a^u \bar{h}_t^u(t)^2 + \int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \dot{\bar{\psi}}_t^u(s, w) ds dw \\
&\quad - 2\bar{h}_t^u(t) \int_t^T \bar{h}_t^u(s) \bar{\psi}_t^u(s, t) ds \\
&\quad + 2 \int_t^T \int_t^T \dot{\bar{h}}_t^u(s) \bar{h}_t^u(w) (a \delta_{s=w} + \bar{\psi}_t^u(s, w)) ds dw.
\end{aligned}$$

Using once again the form of $\dot{\bar{\psi}}_t^u(s, w)$, we deduce that

$$\begin{aligned}
&\int_t^T \int_t^T \bar{h}_t^u(s) \bar{h}_t^u(w) \dot{\bar{\psi}}_t^u(s, w) ds dw \\
&= -2\eta_\nu^2 \left(\int_{\mathbb{R}_+} \Lambda_t^u(x) \lambda(dx) \right) \left(\int_{\mathbb{R}_+} \Lambda_t^u(y) \lambda(dy) \right).
\end{aligned}$$

Also, we have that

$$-2\bar{h}_t^u(t) \int_t^T \bar{h}_t^u(s) \bar{\psi}_t^u(s, t) ds = -2b^u \bar{h}_t^u(t) \int_{\mathbb{R}_+} \Lambda_t^u(x') \lambda(dx'),$$

and

$$\begin{aligned} \dot{\phi}_t^u &= -\eta_\nu^2 \int_t^T \int_t^T G_\nu(s, t) G_\nu(w, t) \left(a^u \delta_{s=w} + \bar{\psi}_t^u(s, w) \right) ds dw \\ &= -\eta_\nu^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Gamma_t^u(x, y) \lambda(dx) \lambda(dy), \end{aligned}$$

Moreover, as

$$\begin{aligned} &2 \int_t^T \int_t^T \dot{\bar{h}}_t^u(s) \bar{h}_t^u(w) (a^u \delta_{s=w} + \bar{\psi}_t^u(s, w)) ds dw \\ &= 2(b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(t, T) \\ &\quad \int_{\mathbb{R}_+} \left(\int_t^T e^{-(s-t)x'} \bar{h}_t^u(u) (a \delta_{s=u} \right. \\ &\quad \left. + \bar{\psi}_t^u(s, u)) ds du \right) \lambda(dx') \\ &= 2(b^u \rho_{Ir} - u \eta_\nu \rho_{\nu r}) \eta_r B_{G_r}(t, T) \\ &\quad \int_{\mathbb{R}_+} \Lambda_t^u(x') \lambda(dx'), \end{aligned}$$

and $\bar{h}_t^u(t) = g_0^T(t) + \rho_{Ir} \eta_r B_{G_r}(t, T)$, we finally obtain that

$$\begin{aligned} \dot{\Theta}_t^u &= \dot{\chi}_t^u - a^u \left(g_0^T(t) + \rho_{Ir} \eta_r B_{G_r}(t, T) \right)^2 - \eta_\nu^2 \left(2 \left(\int_{\mathbb{R}_+} \Lambda_t^u(x) \lambda(dx) \right)^2 \right. \\ &\quad \left. + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Gamma_t^u(x, y) \lambda(dx) \lambda(dy) \right) \\ &\quad - 2 \int_{\mathbb{R}_+} \left(b^u g_0^T(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \Lambda_t^u(x') \lambda(dx'), \end{aligned}$$

and that concludes the proof. $\square$

2.6.3 Proof of Proposition 2.5.3

Proof. Assuming that $G_\nu(t, s) = 1_{s < t} \sum_{i=1}^N w_i e^{-x_i(t-s)}$ and

$$g_0^T(t) = \nu_0 + \sum_{i=1}^N \int_0^t \theta(s) w_i \exp(-x_i(t-s)) ds,$$

with $\theta(t) := \theta_\nu - \eta_\nu \eta_r \rho_{r\nu} B_{G_r}(t, T)$, the volatility process $(\nu_t)_{0 \leq t \leq T}$ reduces to

$$\begin{aligned} \nu_t = & \nu_0 + \int_0^t \sum_{i=1}^N w_i \exp(-x_i(t-s)) (\theta(s) + \kappa_\nu \nu_s) ds \\ & + \eta_\nu \int_0^t \sum_{i=1}^N w_i \exp(-x_i(t-s)) dW_\nu^{\mathbb{Q}^T}(t). \end{aligned}$$

Let us introduce some auxiliary processes $((\nu_t^i)_{i=1, \dots, N})_{0 \leq t \leq T}$ solution of the following SDEs

$$\begin{aligned} d\nu_t^i &= (-x_i \nu_t^i + \theta(t) + \kappa_\nu \nu_t) dt + \eta_\nu dW_\nu^{\mathbb{Q}^T}(t), \quad t > 0, \quad i = 1, \dots, N, \\ \nu_0^i &= 0. \end{aligned}$$

Then, we can deduce that, almost surely,

$$\nu_t = \nu_0 + \sum_{i=1}^N w_i \nu_t^i, \quad t \leq T.$$

Also, we observe that

$$g_t(s) = 1_{t \leq s} \left(\nu_0 + \sum_{i=1}^N w_i e^{-x_i(s-t)} \nu_t^i \right), \quad s, t \leq T,$$

and

$$G_\nu(t, s) = 1_{s < t} \sum_{i=1}^N w_i e^{-x_i(t-s)} = 1_{s < t} \int_0^{+\infty} e^{-(t-s)x} \lambda(dx),$$

with

$$\lambda(dx) = \sum_{i=1}^N w_i \delta_{x=x_i} dx.$$

We also have that, for $i = 1, \dots, N$,

$$Y_t(x_i) = \nu_t^i - \theta_{x_i}(t).$$

with $\theta_x(t) := \int_0^t e^{-x(t-s)} \theta(s) ds$. Therefore, using Proposition 2.5.2, we have that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T} \left[\exp \left(u \log \frac{I_T^T}{I_t^T} \right) \middle| \mathcal{F}_t \right] &= \exp \left(\Theta_t^u + 2 \int_{\mathbb{R}_+} \Lambda_t^u(x) Y_t(x) \lambda(dx) \right. \\
&\quad \left. + \int_{\mathbb{R}_+^2} \Gamma_t^u(x, y) Y_t(x) Y_t(y) \lambda(dx) \lambda(dy) \right) \\
&= \exp \left(\Theta_t^u + 2 \sum_{i=1}^N w_i \int_{\mathbb{R}_+} \Lambda_t^u(x) Y_t(x) \delta_{x=x_i} dx \right. \\
&\quad \left. + \sum_{i,j=1}^N w_i w_j \int_{\mathbb{R}_+^2} \Gamma_t^u(x, y) Y_t(x) Y_t(y) \delta_{x=x_i} \delta_{y=x_j} dx dy \right) \\
&= \exp \left(\Theta_t^u + 2 \sum_{i=1}^N w_i \Lambda_t^u(x_i) Y_t(x_i) \right. \\
&\quad \left. + \sum_{i,j=1}^N w_i w_j \Gamma_t^u(x_i, x_j) Y_t(x_i) Y_t(x_j) \right) \\
&= \exp \left(\Theta_t^u - 2 \sum_{i=1}^N w_i \Lambda_t^u(x_i) \theta_{x_i}(t) \right. \\
&\quad \left. + \sum_{i=1}^N \sum_{j=1}^N w_i w_j \Gamma_t^u(x_i, x_j) \theta_{x_i}(t) \theta_{x_j}(t) \right. \\
&\quad \left. + 2 \sum_{i=1}^N \nu_t^i w_i \left(\Lambda_t^u(x_i) - \sum_{j=1}^N w_j \Gamma_t^u(x_i, x_j) \theta_{x_j}(t) \right) \right. \\
&\quad \left. + \sum_{i,j=1}^N w_i w_j \Gamma_t^u(x_i, x_j) \nu_t^i \nu_t^j \right) \\
&:= \exp(A_t^{u;N} + 2 \sum_{i=1}^N B_t^{u;i} \nu_t^i + \sum_{i=1}^N \sum_{j=1}^N C_t^{u;ij} \nu_t^i \nu_t^j)
\end{aligned}$$

with

$$\begin{aligned}
A_t^{u;N} &:= \Theta_t^u - 2 \sum_{i=1}^N w_i \Lambda_t^u(x_i) \theta_{x_i}(t) + \sum_{i=1}^N \sum_{j=1}^N w_i w_j \Gamma_t^u(x_i, x_j) \theta_{x_i}(t) \theta_{x_j}(t), \\
B_t^{u;i} &:= w_i \left(\Lambda_t^u(x_i) - \sum_{j=1}^N w_j \Gamma_t^u(x_i, x_j) \theta_{x_j}(t) \right), \quad i = 1, \dots, N, \\
C_t^{u;ij} &:= w_i w_j \Gamma_t^u(x_i, x_j), \quad i, j = 1, \dots, N.
\end{aligned}$$

Using Proposition 2.5.2, we have that

$$\begin{aligned}
\dot{C}_t^{u;ij} &= w_i w_j \dot{\Gamma}_t^u(x_i, x_j) \\
&= w_i w_j \left((x_i + x_j) \Gamma_t^u(x_i, x_j) - a^u - 2\eta_\nu^2 \sum_{k=1}^N \sum_{l=1}^N w_k w_l \Gamma_t^u(x_i, x_k) \Gamma_t^u(x_j, x_l) \right) \\
&\quad - b^u \left(\sum_{k=1}^N w_k \Gamma_t^u(x_i, x_k) + \sum_{k=1}^N w_k \Gamma_t^u(x_k, x_j) \right) \\
&= (x_i + x_j) C_t^{u;ij} - w_i w_j a^u - 2\eta_\nu^2 \sum_{k=1}^N \sum_{l=1}^N C_t^{u;ik} C_t^{u;jl} \\
&\quad - b^u \sum_{k=1}^N \left(w_j C_t^{u;ik} + w_i C_t^{u;kj} \right).
\end{aligned}$$

Moreover, after fastidious calculus, we end up with, for $t < T$, $i = 1, \dots, N$,

$$\begin{aligned}
\dot{B}_t^{u;i} &= x_i B_t^{u;i} - a^u w_i \left(\nu_0 + \rho_{Ir} \eta_r B_{G_r}(t, T) \right) - 2\eta_\nu^2 \sum_{j,k=1}^N C_t^{u;ij} B_t^{u;k} \\
&\quad - b^u \left(\sum_{j=1}^N \left(w_i B_t^{u;j} + \nu_0 C_t^{u;ij} \right) - \sum_{j=1}^N C_t^{u;ij} \left(\theta(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) \right) \right),
\end{aligned}$$

and

$$\begin{aligned}
\dot{A}_t^{u;N} &= -a^u \left(\nu_0^2 + \eta_r^2 B_{G_r}(t, T)^2 - 2\nu_0 \rho_{Ir} \eta_r B_{G_r}(t, T) \right) \\
&\quad - 2 \sum_{i=1}^N B_t^{u;i} \left(\theta(t) + u \eta_\nu \rho_{\nu r} \eta_r B_{G_r}(t, T) + b^u \nu_0 \right) \\
&\quad - \eta_\nu^2 \sum_{i,j=1}^N (2B_t^{u;i} B_t^{u;j} + C_t^{u;ij}).
\end{aligned}$$

□

2.7 Additional figures

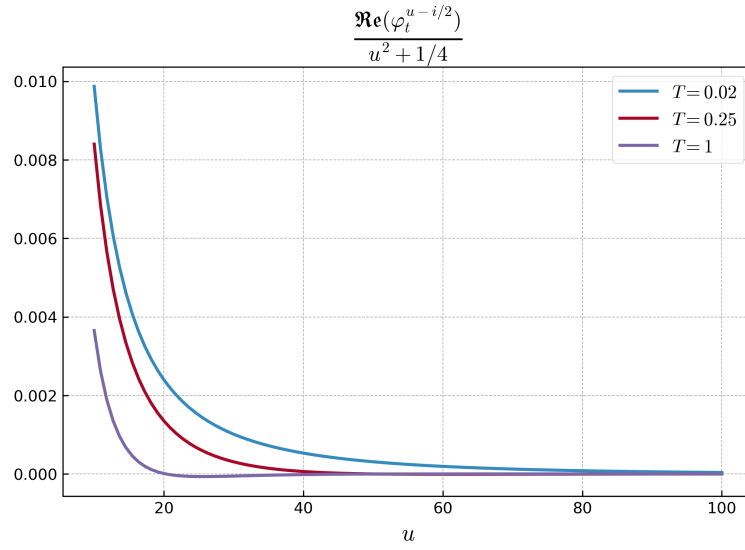


Figure 2.13: Integrand of Lewis pricing formula (2.3.7) for $k_t = 0$. Hull-White and Stein-Stein model with parameters: $\kappa_r = -0.03$, $\eta_r = 0.01$, $I_0^T = 100$, $\nu_0 = 0.1$, $\theta_\nu = 0.1$, $\eta_\nu = 0.125$, $\kappa_\nu = 0$, $\rho_{I\nu} = -0.7$, $\rho_{r\nu} = -0.25$, $\rho_{Ir} = -0.25$.

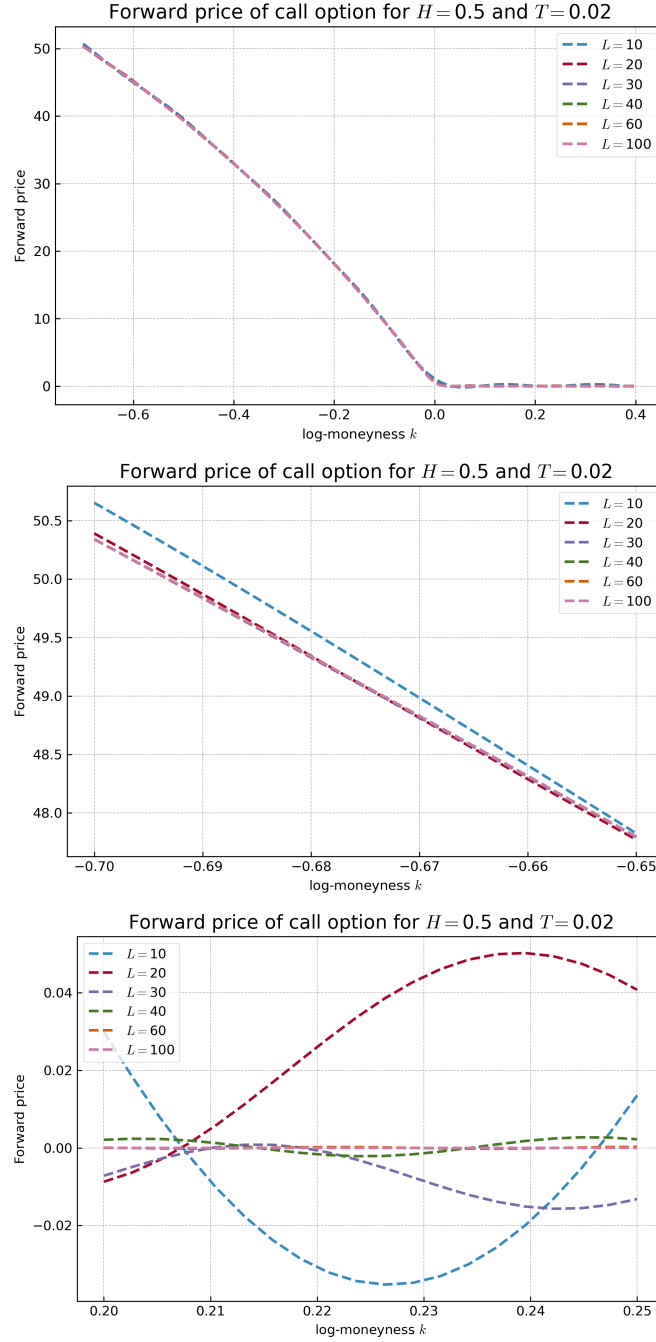


Figure 2.14: Forward price of call option for $T = 0.02$ using Lewis (2001) pricing formula with Gauss-Laguerre quadrature numerical integration by varying L (number of weights). Hull-White and Stein-Stein model with parameters: $\kappa_r = -0.03$, $\eta_r = 0.01$, $I_0^T = 100$, $\nu_0 = 0.1$, $\theta_\nu = 0.1$, $\eta_\nu = 0.125$, $\kappa_\nu = 0$, $\rho_{I\nu} = -0.7$, $\rho_{r\nu} = -0.25$, $\rho_{Ir} = -0.25$.

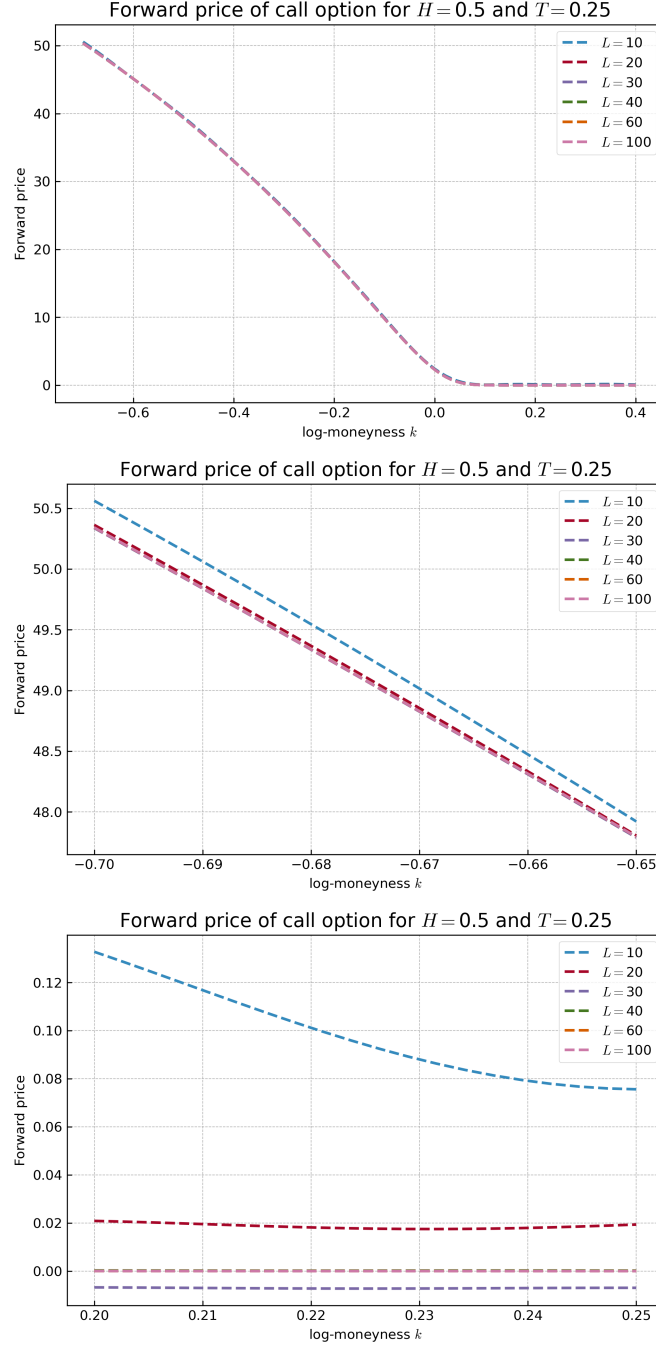


Figure 2.15: Forward price of call option for $T = 0.25$ using Lewis (2001) pricing formula with Gauss-Laguerre quadrature numerical integration by varying L (number of weights). Hull-White and Stein-Stein model with parameters: $\kappa_r = -0.03$, $\eta_r = 0.01$, $I_0^T = 100$, $\nu_0 = 0.1$, $\theta_\nu = 0.1$, $\eta_\nu = 0.125$, $\kappa_\nu = 0$, $\rho_{I\nu} = -0.7$, $\rho_{r\nu} = -0.25$, $\rho_{Ir} = -0.25$.

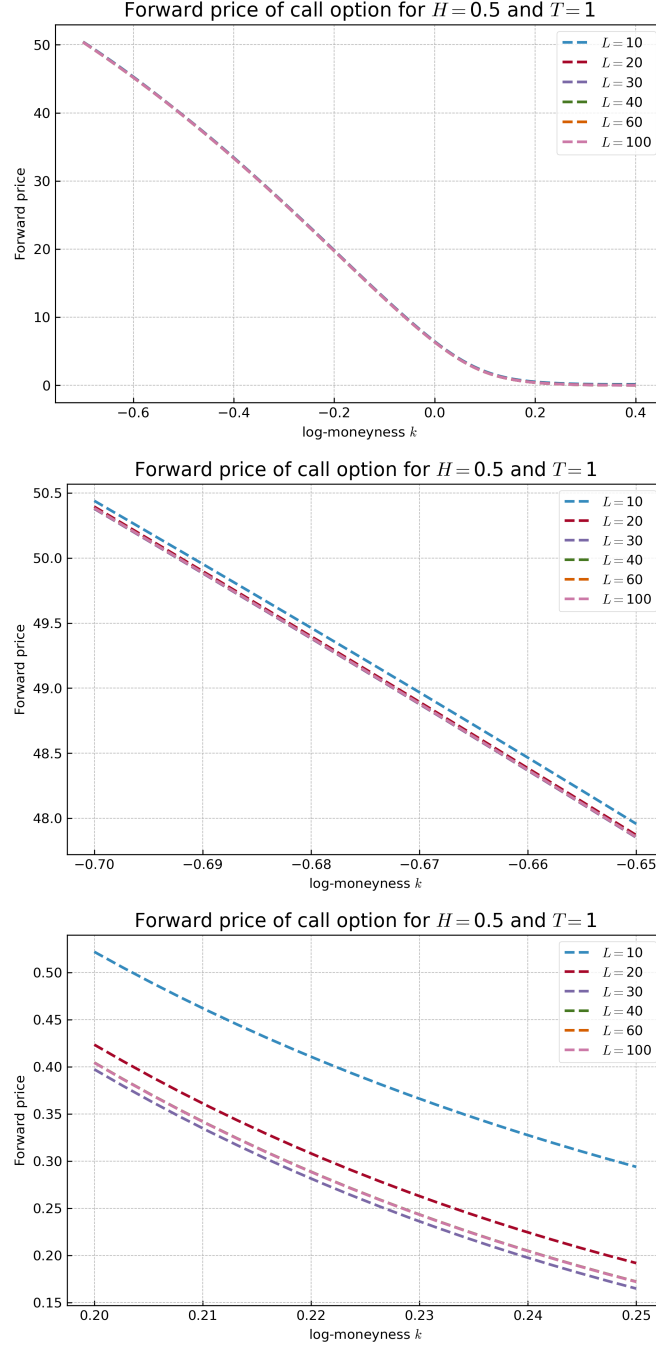


Figure 2.16: Forward price of call option for $T = 1$ using Lewis (2001) pricing formula with Gauss-Laguerre quadrature numerical integration by varying L (number of weights). Hull-White and Stein-Stein model with parameters: $\kappa_r = -0.03$, $\eta_r = 0.01$, $I_0^T = 100$, $\nu_0 = 0.1$, $\theta_\nu = 0.1$, $\eta_\nu = 0.125$, $\kappa_\nu = 0$, $\rho_{I\nu} = -0.7$, $\rho_{r\nu} = -0.25$, $\rho_{Ir} = -0.25$.

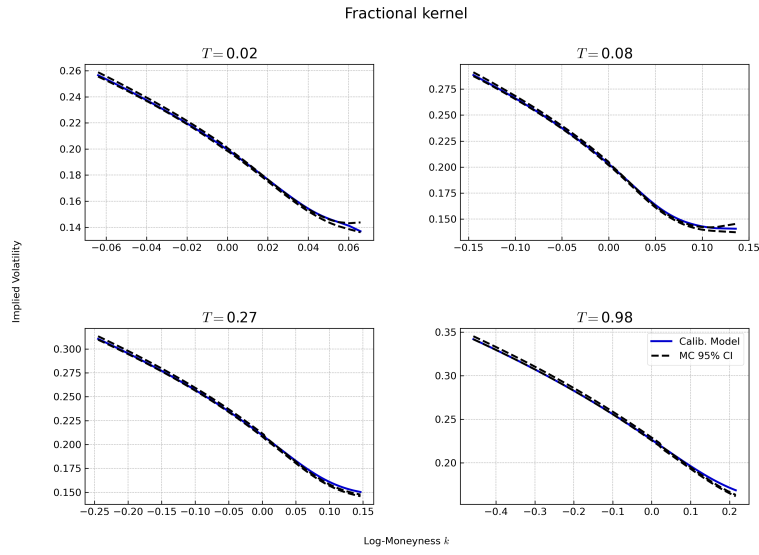


Figure 2.17: Implied volatility for fractional kernel: operator discretization approximation with $N = 40$ vs Monte Carlo confidence intervals. Calibrated parameters: $\hat{\nu}_0 = 0.1964$, $\hat{\theta}_\nu = -0.0248$, $\hat{\eta}_\nu = 0.2123$, $\hat{\rho}_{I\nu} = -0.7981$, $\hat{\rho}_{Ir} = -0.5971$ and $\hat{H}_\nu = 0.2992$.

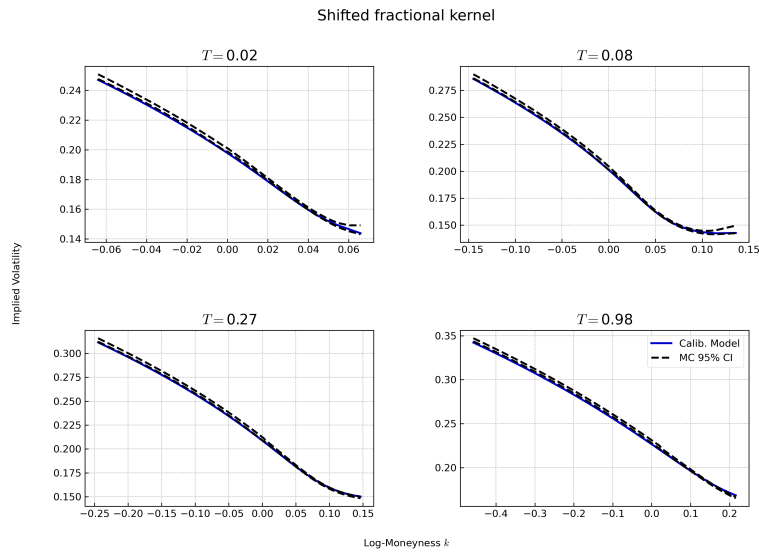


Figure 2.18: Implied volatility for shifted fractional kernel: operator discretization approximation with $N = 40$ vs Monte Carlo confidence intervals. Calibrated parameters: $\hat{\nu}_0 = 0.1978$, $\hat{\theta}_\nu = -0.0259$, $\hat{\eta}_\nu = 0.2164$, $\hat{\rho}_{I\nu} = -0.7868$, $\hat{\rho}_{Ir} = -0.6107$ and $\hat{H}_\nu = 0.2273$.

Part II

Hedging of insurance liabilities via stopping-time-based strategies

CHAPTER 3

Efficient hedging of life insurance portfolio for loss-averse insurers

This chapter investigates the hedging of equity-linked life insurance portfolio for loss-averse insurers. We consider a general arbitrage-free financial market and an actuarial market composed of n -independent policyholders. As the combined market is incomplete, perfect hedging of any actuarial-financial payoff is not possible. Instead, we study the efficient hedging of n -size equity-linked life insurance portfolio for insurers who are only concerned with their losses. To this end, we consider stochastic control problems (under the real-world measure) in order to determine the optimal hedging strategies that either maximize the probability of successful hedge (called quantile hedging) or minimize the expectation for a class of shortfall loss functions (called shortfall hedging). Based on the super-replication theory and a duality approach, we show that the optimal strategies depend both on actuarial and financial risks. Moreover, these strategies adapt not only to the size of the insurance portfolio but also to the risk-aversion of the insurer. The numerical results show that, for loss-averse insurers, the strategies outperform the mean-variance hedging strategy, demonstrating the relevance of adopting the right strategy according to the insurers' risk aversion and portfolio size.

3.1 Introduction

Equity-linked life insurance contracts are characterized by their dependence on both hedgeable and non-hedgeable contingent claims. While the pricing and the fair-valuation of these contracts have been widely discussed in the literature, see among others Bacinello and Ortu (1993); Pelsser and Ghalehjooghi (2016); Dhaene et al. (2017); Deelstra et al. (2020); Dupret and Motte (2026), the question of

hedging, especially for loss-averse insurers, receives less attention. Nevertheless, this question is an essential one and is very important, if not more important than the question of pricing, as the insurance market is by definition incomplete. A perfect hedging strategy for any financial-insurance payoff does not exist, and therefore insurers are exposed to losses. The life insurers' profit and loss (P&L) depends both on the observed mortality of their portfolio and on how they invest the premiums received, which is the only thing they can control. The insurers must therefore think about how they will invest all the premiums received so as to honour their obligations to their policyholders and to minimize their losses, whatever adverse situations arise.

Different hedging approaches for equity-linked life insurance portfolios have been proposed in the literature. Møller (1998); Møller (2001a) first introduced risk-minimizing hedging strategies based on quadratic hedging criteria. These risk-minimizing hedging results were then extended to other models and frameworks: Riesner (2006) considered an incomplete financial market, Dahl and Møller (2006) assumed systematic mortality risk, and Ceci et al. (2017) considered partially observable market models. Other approaches for hedging equity-linked life portfolios were also examined: Møller (2001b) studied hedging strategies associated with financial variance and standard deviation principles; Barigou and Dhaene (2019); Dhaene et al. (2017) introduced convex hedging strategies; Møller (2001b) studied robust hedging strategies for an insurer seeking to maximize its profit; and more recently, Barigou et al. (2022); Chen et al. (2020) investigated loss-averse convex hedging approaches for insurers giving more importance to losses than to gains. Note that, as hedging of insurance portfolios is closely related to asset and liability management (ALM), we can also refer to Hainaut and Devolder (2007); Liang and Ma (2015) that study optimal ALM strategies in expected utility frameworks. Nevertheless, the question of the optimal hedging of life insurance portfolios, considered as a whole, for loss-averse insurers that only focus on losses has not yet been addressed. This chapter, therefore, aims to fill this gap. For this purpose, we will consider insurers who are not interested in maximizing their profits and whose sole aim is to minimize their losses. This framework can be seen as an extension of the frameworks considered in Barigou et al. (2022); Chen et al. (2020), where the coefficient of loss aversion tends towards infinity. For this type of insurer, on the one hand, it is important to consider the global hedging of the portfolio and not individual hedging for each contract composing the portfolio; on the other hand, it is important to determine the optimal strategy by formulating the problem in terms of the real measure $\mathbb{P}$, so as to include all possible situations, even the most severe, that insurers may have to face.

To reflect the profile of insurers who are concerned with the downside only, we consider the partial hedging of equity-linked life insurance portfolios using short-fall loss functions. The partial hedging strategies studied in this chapter therefore

aim to either maximize the probability of successful hedge (called quantile hedging) or minimize a function of the shortfall loss (called shortfall hedging) under the real measure $\mathbb{P}$. These hedging techniques were primarily studied by Föllmer and Leukert (1999); Föllmer and Leukert (2000) and then extended to several fields and settings. In incomplete financial markets, Jonsson and Sircar (2002) extended the strategy in a stochastic volatility environment, and Motte and Hainaut (2024) extended it to rough volatility models. In defaultable markets, such as the insurance market, Nakano (2011); Sekine (2000) considered the partial hedging of defaultable claims, while in actuarial sciences, Melnikov and Nosrati (2017) studied the efficient hedging of life insurance contracts. However, these different approaches, in defaultable markets, essentially consider only one default risk and not multiple default risks. In practice, insurers mutualize their risks by considering sufficiently large portfolios. The study of optimal hedging for portfolios subject to default therefore seems very relevant.

In this chapter, we investigate the optimal hedging of n -size life portfolio for loss-averse insurers that only focus on their losses. To this end, we consider quantile and shortfall hedging approaches applied in a general financial-actuarial mathematical framework. Although Melnikov and Nosrati (2017) discussed the risk-management of a life insurance portfolio in the context of quantile hedging, the optimal strategy is never deduced. Moreover, as the quantile hedging approach does not control the severity of the losses, it is quite interesting to also consider shortfall hedging strategies for insurers seeking to control the size of the loss. The hedging problem considered is thus formulated as a stochastic control problem with a stochastic target that depends both on financial risk (the value of the financial payoff) and actuarial risk (the number of alive policyholders). Based on the theory of super-replication, initially introduced by Cvitanić et al. (1999); El Karoui and Quenez (1995), we show that optimal strategies are equivalent to (super) hedging strategies of modified payoffs that depend on both financial and actuarial risks. Moreover, using a duality approach, we derive (semi)-explicit forms for the optimal strategies, and prove that these strategies adapt to the size of the portfolio as well as to the insurers' aversion. However, apart from restricting ourselves to purely financial hedging strategies, the forms of optimal strategies depend on a dual process that is not explicitly known in general, making it difficult to implement the optimal strategies in practice. In order to consider more explicit strategies admissible in the combined market, we also construct fully explicit hedging strategies adapted to the combined market, that depend on financial risk factors as well as the evolution of the number of deaths through time, naturally updating at each stopping times corresponding to deaths of policyholders. Finally, we demonstrate through numerical results that these strategies outperform the mean-variance hedging strategy, showing the relevance of using the right strategy depending on the size of the portfolio and the risk aversion of the insurers, especially if the size of the portfolio is large.

The chapter is outlined as follows. In Section 3.2, the mathematical framework for financial and actuarial markets is presented, and the hedging problem is posed. Then, Section 3.3 considers the quantile hedging problem and is divided into two sub-sections: Section 3.3.1 considers the hedging problem in a very general case where the hedging strategies are admissible in the combined financial-actuarial market, while in Section 3.3.2, we restrict ourselves to the case where the hedging strategies are only admissible in the financial market. For both sub-sections, we characterize the optimal strategies, and derive the (semi)-explicit form of these strategies. Then, Section 3.4 considers the class of shortfall hedging problems in order to determine hedging strategies that account for the size of the losses. As for the quantile hedging case, we divide this section into two sub-sections depending on whether or not hedging strategies are restricted to purely financial strategies. We characterize the optimal strategies and derive the (semi)-explicit forms of strategies for shortfall and quadratic shortfall losses. In Section 3.5, we show how to construct explicit stopping-time-based hedging strategies adapted to the combined market filtration. Finally, in Section 3.6, we provide numerical applications in a Black and Scholes framework, compare the hedging strategies with the mean-variance hedging strategy and also analyze the information gain by incorporating the actuarial information into hedging strategies.

3.2 Mathematical framework

In this chapter, we work with a financial market composed of two assets: a risky asset $S := (S_t)_{t \geq 0}$ with $S_0 > 0$, and a risk-free asset $B := (B_t)_{t \geq 0}$ with $B_0 = 1$. These assets are described by semi-martingale processes defined on the financial market probability space $(\Omega^f, \mathcal{F}^f, \mathbb{P}^f)$ equipped with a filtration $(\mathcal{F}_t^f)_{t \geq 0}$ satisfying the usual conditions. We make the assumption that the financial market is arbitrage-free. Under this assumption, we know that there exists a set of equivalent measures under which the discounted value of the risky asset is a martingale. This set is denoted by $\mathcal{Q}^f$ such that

$$\mathcal{Q}^f := \{\mathbb{Q}^f : \text{equivalent to } \mathbb{P}^f, \text{ and } \frac{S}{B} \text{ is a martingale under } \mathbb{Q}^f\}.$$

Our aim in the following is to consider the optimal hedging of insurance-financial payoffs by trading in the financial market. In this sense, we consider a self-financing hedging portfolio denoted by $V := (V_t)_{t \geq 0}$ and defined, for $t \geq 0$, by

$$dV_t = (V_t - \xi_t S_t) \frac{dB_t}{B_t} + \xi_t dS_t, \quad (3.2.1)$$

$$V_0 = v_0 \geq 0,$$

on $(\Omega^f, \mathcal{F}^f, \mathbb{P}^f)$, where $\xi := (\xi_t)_{t \geq 0}$ is a dynamic hedging strategy. We say that ξ is an admissible financial hedging strategy if ξ is a $\mathcal{F}_t^f$ -progressively measurable

process such that, for all $t \geq 0$, $\mathbb{E}^{\mathbb{P}^f}(V_t^2) < +\infty$ and $V_t \geq 0$. At this early stage, we can first consider the hedging of purely positive financial payoffs of the form $H_T \in \mathcal{F}_T^f$ with $T > 0$ a fixed maturity. As the financial market can be incomplete, not every positive contingent claim $H_T \in \mathcal{F}_T^f$ can be perfectly replicated by a hedging portfolio, but super-replicating hedging strategies exist. As shown, for example, in Cvitanić et al. (1999); El Karoui and Quenez (1995), for a T -maturity positive given claim $H_T \in \mathcal{F}_T^f$, if

$$v_0 = \sup_{\mathbb{Q}^f \in \mathcal{Q}^f} \mathbb{E}^{\mathbb{Q}^f} \left(\frac{H_T}{B_T} \right),$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A}^f \text{ s.t. } V_T \geq H_T \text{ } P^f - a.s.$$

where $\mathcal{A}^f$ is the set of all admissible financial hedging strategies valued in $\mathbb{R}$. In particular, if in addition the financial market is assumed to be complete, such that the set of equivalent measures reduces to a singleton, i.e. $\mathcal{Q}^f = \left\{ \tilde{\mathbb{Q}}^f \right\}$, then for

$$v_0 = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{H_T}{B_T} \right),$$

a perfect hedging strategy exists i.e.

$$\exists \xi \in \mathcal{A}^f \text{ s.t. } V_T = H_T \text{ } P^f - a.s.$$

and the value at time $t \in [0, T]$ of the perfect hedging portfolio is given by

$$V_t = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_t}{B_T} H_T | \mathcal{F}_t^f \right).$$

We now introduce the life insurance market, described by the probability space $(\Omega^a, \mathcal{F}^a, \mathbb{P}^a)$ equipped with a filtration $(\mathcal{F}_t^a)_{t \geq 0}$. We assume that the remaining lifetime of an x -year-old individual is a random variable denoted by ω_x and defined on $(\Omega^a, \mathcal{F}^a, \mathbb{P}^a)$. The mortality rate is noted $(\kappa_{x+t})_{t \geq 0}$ and the probability of survival from age x till age $x+t$, for $t \geq 0$, is noted ${}_t p_x := \mathbb{E}^{\mathbb{P}^a} \left(\exp \left(- \int_0^t \kappa_{x+s} ds \right) \right)$.

We assume that the insurer has a n -size portfolio with policyholders from the same cohort, such that for $i = 1, \dots, n$, $x_i = x$ and $\kappa_{x_i+t} = \kappa_{x+t}$. In this case, the remaining lifetime of the i^{th} policyholder is denoted by ω_i and $\omega_1, \dots, \omega_n$ are assumed to be independent. If we define, for $i = 1, \dots, n$,

$$N_t^i := 1_{\{\omega_i > t\}},$$

then the total number of policyholders remaining at time t is defined by

$$N_t := \sum_{i=1}^n N_t^i.$$

In the following, we will focus on the combined model denoted $(\Omega, \mathcal{F}, \mathbb{P})$, which is defined as the product space of the two individual spaces, financial and actuarial. We let $\Omega := \Omega^f \times \Omega^a$ and $\mathbb{P} := \mathbb{P}^f \otimes \mathbb{P}^a$. The σ -algebra $\mathcal{F}$ is generated by all subsets of null-sets from $\mathcal{F}^a \otimes \mathcal{F}^f$ and the associated filtration is denoted by $(\mathcal{F}_t)_{t \geq 0}$. We make the standard assumption that actuarial and financial risks are independent.

Remark 3.1 (Independence between actuarial and financial risks.). *The assumption of independence between mortality risk and financial market risk is standard in the literature on hybrid financial-actuarial models. In practice, it is often adopted for tractability. In our setting, we will see that this assumption is not essential for characterizing the optimal hedging strategies in the full market filtration, where both financial and actuarial risks are jointly observed. However, it becomes relevant when restricting attention to strategies that are adapted only to the financial filtration, as it allows us to construct an optimal hedging strategy in a purely financial market. We also note that this independence assumption may be restrictive depending on the asset class considered, as empirical evidence may suggest the existence of dependence between longevity risk and financial variables in some contexts, see for instance Deelstra et al. (2016); Deelstra et al. (2024).*

In this setting, $\mathcal{F}^f$ and $\mathcal{F}^a$ martingales are martingales in $(\Omega, \mathcal{F}, \mathbb{P})$. Even if the financial market is assumed to be complete, the combined market is incomplete since the mortality risk cannot be perfectly hedged. Therefore, we know that the pricing measure is not unique and that there exists a class of equivalent martingale measures denoted by $\mathcal{Q}$ such

$$\mathcal{Q} := \{\mathbb{Q} : \text{equivalent to } \mathbb{P}, \text{ and } \frac{S}{B} \text{ is a martingale under } \mathbb{Q}\}.$$

For any contingent claim of the form $g(S_T, N_T) \in \mathcal{F}_T$, with $T > 0$ a fixed maturity and $g(x, y)$ a positive function increasing in y , we can characterize the super-replication price as the super-replication price of a $\mathcal{F}_T^f$ -measurable payoff. This is the purpose of the next proposition.

Proposition 3.2.1. *Let $g(x, y)$ a positive function increasing in y , then*

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{g(S_T, N_T)}{B_T} \right) = \sup_{\mathbb{Q}^f \in \mathcal{Q}^f} \mathbb{E}^{\mathbb{Q}^f} \left(\frac{g(S_T, n)}{B_T} \right).$$

Proof. The proof relies on Propositions 3.9.1 and 3.9.2 in Appendix 3.9.1. First, by independence between financial and actuarial probability spaces, we observe that

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{g(S_T, N_T)}{B_T} \right) = \sup_{\mathbb{Q}^f \in \mathcal{Q}^f} \sup_{\mathbb{Q}^a \in \mathcal{Q}^a} \mathbb{E}^{\mathbb{Q}^f \otimes \mathbb{Q}^a} \left(\frac{g(S_T, N_T)}{B_T} \right),$$

where $\mathcal{Q}^a$ is the set of measures equivalent to $\mathbb{P}^a$. Then, on the one hand, since $g(x, y)$ is growing in y and $N_T \leq n$, we observe that, for all $\mathbb{Q}^f \in \mathcal{Q}^f$,

$$\sup_{\mathbb{Q}^a \in \mathcal{Q}^a} \mathbb{E}^{\mathbb{Q}^f \otimes \mathbb{Q}^a} \left(\frac{g(S_T, N_T)}{B_T} \right) \leq \mathbb{E}^{\mathbb{Q}^f} \left(\frac{g(S_T, n)}{B_T} \right).$$

On the other hand, using Proposition 3.9.2, for $\varepsilon > -1$, for all $\mathcal{F}_t^a$ -adapted process $(h_t)_{t \geq 0}$ such that for $\forall t \geq 0$, $h_t > -1$ almost surely and for all $\mathbb{Q}^f \in \mathcal{Q}^f$, there exists an equivalent martingale measure $\mathbb{Q}_{h^\varepsilon} \in \mathcal{Q}$ such that

$$\lim_{\varepsilon \rightarrow -1} \mathbb{E}^{\mathbb{Q}_{h^\varepsilon}} \left(\frac{g(S_T, N_T)}{B_T} \right) = \mathbb{E}^{\mathbb{Q}^f} \left(\frac{g(S_T, n)}{B_T} \right).$$

Therefore, we have that, $\forall \mathbb{Q}^f \in \mathcal{Q}^f$,

$$\sup_{\mathbb{Q}^a \in \mathcal{Q}^a} \mathbb{E}^{\mathbb{Q}^f \otimes \mathbb{Q}^a} \left(\frac{g(S_T, N_T)}{B_T} \right) = \mathbb{E}^{\mathbb{Q}^f} \left(\frac{g(S_T, n)}{B_T} \right)$$

and we easily conclude that

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{g(S_T, N_T)}{B_T} \right) = \sup_{\mathbb{Q}^f \in \mathcal{Q}^f} \mathbb{E}^{\mathbb{Q}^f} \left(\frac{g(S_T, n)}{B_T} \right).$$

□

Suppose now that the insurer guarantees to its policyholders a stochastic positive amount $H_T \in \mathcal{F}_T^f$ if they are alive at a predefined maturity $T > 0$. In this case, the total liability of the insurer at maturity is given by

$$H_T^{Total} := N_T \times H_T \in \mathcal{F}_T.$$

Remark 3.2. *In this chapter, we restrict our attention to equity-linked life insurance products with maturity payments only. Extensions to more complex insurance contracts, such as death benefit products, Bermudan-type contracts, or payoffs involving intermediate cash flows, could also be considered. In the concluding Chapter, we discuss some potential starting points for applying the methods developed in this chapter to tackle the associated hedging problems in these more general settings.*

To meet its liabilities, the insurer constructs a hedging portfolio denoted by V that evolves as (3.2.1) on $(\Omega, \mathcal{F}, \mathbb{P})$. The hedging strategy $\xi = (\xi_t)_{0 \leq t \leq T}$ adopted by the insurer is said to be admissible on the combined market if ξ is a $\mathcal{F}_t$ -progressively measurable process such that, for all $t \geq 0$, $\mathbb{E}(V_t^2) < +\infty$ and $V_t \geq 0$. In the best of all possible worlds, the insurer would like to find a strategy that perfectly replicates its liabilities. Due to the incompleteness of the global market, perfect hedging strategy does not exist for any positive contingent claim $H_T^{Total} \in \mathcal{F}_T$. However, if

$$v_0 = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T^{Total}}{B_T} \right) = \sup_{\mathbb{Q}^f \in \mathcal{Q}^f} \mathbb{E}^{\mathbb{Q}^f} \left(n \times \frac{H_T}{B_T} \right), \quad (3.2.2)$$

then there exists a super-replicating hedging strategy i.e.

$$\exists \xi \in \mathcal{A} \text{ s.t. } V_T \geq H_T^{Total} \mathbb{P} - a.s.,$$

where $\mathcal{A}$ is the set of all admissible hedging strategies on the combined market valued in $\mathbb{R}$. The super-replication price of the total liability (3.2.2) is actually too expensive, even if the financial market is complete, to be competitive in practice, which is why the insurer is not going to build the super hedging strategy. In fact, the premium that the insurer has to charge its policyholders for super hedging is the price of the same product without the mortality risk. Policyholders would therefore have no interest in choosing the insurance product over the financial product since the financial product is less risky for them and, consequently, the insurer would not have any business.

In what follows, we will assume that the insurer sells its life product at a total price $\bar{V}_0$ such that

$$\bar{V}_0 \leq \sup_{Q^f \in \mathcal{Q}^f} \mathbb{E}^{Q^f} \left(n \times \frac{H_T}{B_T} \right),$$

and that this agent is loss-averse. So, instead of seeking to maximize the final value of its P&L and therefore its profit, its main aim is to limit the events leading to losses and also to control the size of these losses. Based on its own risk aversion, the insurer can define an efficient hedging strategy.

On one hand, the insurer can seek to maximize the probability of successful hedge. Thus, in this case, the insurer constructs a strategy to maximize the probability (under $\mathbb{P}$) that the terminal value of the hedging portfolio is above the total payoff. From a mathematical point of view, the problem reduces to the following stochastic control problem:

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}} \mathbb{P}(V_T \geq H_T^{Total}), \quad (3.2.3)$$

under the initial constraint

$$v_0 \leq \bar{V}_0.$$

On the other, the insurer can seek to minimize the loss resulting from the hedging operation. In this case, the insurer constructs a strategy to minimize the expectation of a class shortfall loss functions. This problem can be written as stochastic control problem of the form:

$$\min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}} \mathbb{E}^{\mathbb{P}} \left(L \left((H_T^{Total} - V_T)_+ \right) \right), \quad (3.2.4)$$

with $L \in C^1$, under the initial constraint

$$v_0 \leq \bar{V}_0.$$

Such hedging strategies are called quantile hedging or shortfall hedging strategies and were initially studied by Föllmer and Leukert (1999); Föllmer and Leukert (2000). The difference with the Föllmer and Leukert setting is that here we have both financial and actuarial risks. Note that Melnikov and Nosrati (2017) has already studied efficient hedging of equity-linked insurance contracts and has shown that, when restricting ourselves to purely financial hedging strategies, the optimal quantile hedging strategy, when $n = 1$, is the quantile hedging strategy of the purely financial payoff $H_T \in \mathcal{F}_T^f$ (without actuarial risk). However, although the risk management of portfolios is discussed, the optimal hedging strategies are not deduced when $n > 1$.

In the following, we will study quantile and shortfall hedging strategies applied in the context of hedging a n -size life insurance portfolio for $n \geq 1$. Under suitable hypotheses, the stochastic control problems studied in this chapter can be solved using dynamic programming techniques like solving the Hamilton-Jacobi-Bellman (HJB) equation as it was done in Jonsson and Sircar (2002) or in Chapter 1. Nevertheless, instead of applying those techniques, we will use the super-hedging theory to characterize the optimal strategies. In this sense, we will transform stochastic control problems into static optimization problems. This allows us to derive that optimal hedging portfolios are actually the (super) replication hedging portfolios of $\mathcal{F}_T$ -measurable modified payoffs. Moreover, using a duality approach, we can deduce semi-explicit forms for the optimal strategies. The advantage of this approach, compared with dynamic programming techniques, is that it allows to easily characterize the strategies and to derive semi-explicit forms of optimal hedging strategies even if the financial framework is not fully Markovian¹.

Remark 3.3 (Tractability of shortfall losses). *The approach developed in this chapter is primarily tailored to a class of loss functions that focus exclusively on the downside, i.e., shortfall-type criteria. This structural property is crucial, as it allows us to exploit techniques from super-replication theory in order to characterize optimal strategies in a tractable way, within a general actuarial-financial framework. By contrast, more general loss functions that are not purely downside-oriented would typically break this structure and prevent the use of super-replication arguments.*

¹There exist different non-Markovian financial markets, such as, for example, rough volatility or path-dependent volatility models.

3.3 Optimal quantile hedging strategy

3.3.1 Characterization of the optimal strategy

In this section, we characterize the optimal hedging strategy associated with the introduced quantile hedging problem (3.2.3). To this end, for any $\xi \in \mathcal{A}$, we introduce the sets of successful financial hedge $A_1, \dots, A_n \in \mathcal{F}_T$ defined as

$$A_i := \left\{ V_T \geq i H_T \right\}, \quad i = 1, \dots, n, \quad (3.3.1)$$

and we show that the dynamic optimization problem (3.2.3) is equivalent to a static optimization problem that allows to characterize the optimal strategy.

Theorem 3.3.1. *Let $p_0 := \mathbb{P}(N_T = 0)$ and $A_i^* \in \mathcal{F}_T$, $i = 1, \dots, n$ the solution of the following problem*

$$\max_{A_1, \dots, A_n \in \mathcal{F}_T, A_{i+1} \subseteq A_i} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i} \right) + p_0, \quad (3.3.2)$$

under the constraint

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i} \right) \leq \bar{V}_0. \quad (3.3.3)$$

Let ξ^{QH} denote the super-replicating hedging strategy for the modified payoff

$$H_T^{QH} := H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}. \quad (3.3.4)$$

Then $(\bar{V}_0, \xi^{QH})$ solves the stochastic control problem defined by (3.2.3) such that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{N_T=i} \right) + p_0,$$

and $\xi^{QH} \in \mathcal{A}$.

Proof. The proof is given in Section 3.8. □

Remark 3.4. *The existence of a solution to the problem (3.3.2) follows from the Proposition 3.9.3 in Appendix 3.9.2.*

The previous theorem shows that the optimal quantile hedging of a n -size life insurance portfolio is linked to the super replication strategy of a modified payoff that takes into account both financial and actuarial risks. We can now go a step further and characterize the form of the optimal sets A_i^* , $i = 1, \dots, n$. To this end, we

apply a duality approach as proposed, for example, in Kramkov and Schachermayer (1999); Föllmer and Leukert (2000), and consider in this sense the following sets:

$$\mathcal{Y} := \left\{ (Y_t)_{t \in [0, T]} \geq 0 \text{ } \mathbb{P}\text{-a.s.} : Y_0 = 1 \text{ and } \left(\frac{V_t}{B_t} Y_t \right)_{t \in [0, T]} \text{ is a } \mathbb{P}\text{-supermartingale} \right\},$$

and

$$\mathcal{D} := \left\{ Z_T \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq Z_T \leq Y_T \text{ for some } (Y_t)_{t \in [0, T]} \in \mathcal{Y} \right\}.$$

Proposition 3.3.2. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of*

$$\min_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) + \lambda \bar{V}_0 \right), \quad (3.3.5)$$

with

$$A_i^*(\lambda, Z_T) = \left\{ \alpha_i \geq \lambda \times Z_T \frac{H_T}{B_T} \right\}, \quad i = 1, \dots, n,$$

such that:

$$\alpha_n = \min \left(\mathbb{1}_{N_T=n}, \frac{\mathbb{1}_{N_T=n} + \mathbb{1}_{N_T=n-1}}{2}, \dots, \frac{\sum_{j=1}^n \mathbb{1}_{N_T=j}}{n} \right), \quad (3.3.6)$$

and for $i = 1, \dots, n-1$,

$$\alpha_i = \max \left(\alpha_{i+1}, \min_{k=1, \dots, i} \left(\frac{\sum_{j=k}^i \mathbb{1}_{N_T=j}}{i-k+1} \right) \right). \quad (3.3.7)$$

Then, $A_i^* = A_i^*(\lambda^*, Z_T^*)$ is a solution of (3.3.2) such that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{N_T=i} \right) + p_0.$$

Moreover,

$$\bar{V}_0 = E^P \left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

and ξ^{QH} is the admissible strategy associated to the super-replication hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is given in Section 3.8. □

Remark 3.5. In Proposition 3.3.2, we consider the dual problem over the set $\mathcal{D}$ instead of the set of martingale density $\mathcal{L} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}}|_T : \mathbb{Q} \in \mathcal{Q} \right\} \subseteq \mathcal{D}$. We made this choice in order to guarantee the existence of a solution to the dual problem which is not the case if we consider the dual problem over the set $\mathcal{L}$. In addition, we show that there is no duality gap when considering the dual problem over the set $\mathcal{D}$.

From Proposition 3.3.2, we obtain semi-explicit expressions for the optimal sets A_i^* , $i = 1, \dots, n$ that depend on $\lambda^* \in \mathbb{R}_+$ and $Z_T^* \in \mathcal{D}$ solving (3.3.5). We can observe that this quantile hedging strategy generalizes the pure quantile hedging strategy², taking into account the actuarial risk. However, in general, it is difficult to derive the optimal dual process³ associated to the problem (3.3.5) and thus we cannot obtain more explicit expressions for the optimal sets. As shown in the following section, by restricting the set of admissible hedging strategies to the set of financial strategies, we are also able to characterize the solution of the problem and, by considering a complete financial market, the optimal dual process can be explicitly deduced. Moreover, we show in Section 3.5 how to construct an explicit $\mathcal{F}$ -adapted quantile strategy that can be implemented in practice.

3.3.2 Restriction to the set of admissible financial hedging strategies $\mathcal{A}^f \subseteq \mathcal{A}$

In this section, we consider the quantile hedging problem by restricting the set of admissible hedging strategies to the set of financial hedging strategies. This particular setting corresponds, for example, to the case where the insurer has an incomplete information and has no information about the mortality evolution. Note that the developments made in this section will be very useful to construct a more optimal explicit $\mathcal{F}$ -adapted quantile hedging strategy in Section 3.5. When $\xi \in \mathcal{A}^f$, the quantile hedging problem is reduced to

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}), \quad (3.3.8)$$

such that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) \leq \max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}).$$

Let us now solve the quantile hedging problem in the set of financial hedging strategies defined by (3.3.8). To this end, for any $\xi \in \mathcal{A}^f$, we consider the sets of successful financial hedge $A_1, \dots, A_n$ defined by (3.3.1) that are now $\mathcal{F}_T^f$ -measurable, and we show that the dynamic optimization problem (3.3.8) is equivalent to a static optimization problem that allows us to characterize the optimal strategy.

²Strategy associated to the quantile hedging of $H_T \in \mathcal{F}_T^f$

³Except for $n = 1$ in a complete financial market setting, as shown in Nakano (2011).

Theorem 3.3.3. Let $p_i := \mathbb{P}(N_T = i)$ and $A_i^* \in \mathcal{F}_T^f$, $i = 1, \dots, n$ the solution of the following problem

$$\max_{A_1, \dots, A_n \in \mathcal{F}_T^f, A_{i+1} \subseteq A_i} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times p_i \right) + p_0, \quad (3.3.9)$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} E^{Q^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i} \right) \leq \bar{V}_0.$$

Let $\xi^{QH, f}$ denote the (super) replicating hedging strategy for the modified payoff

$$H_T^{QH} := H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}. \quad (3.3.10)$$

Then $(\bar{V}_0, \xi^{QH, f})$ solves the stochastic control problem defined by (3.3.8) such that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times p_i \right) + p_0,$$

and $\xi^{QH, f} \in \mathcal{A}^f$.

Proof. We can easily deduced the announced result by using similar arguments than in the proof of Theorem 3.3.1 combined with the fact that $A_1, \dots, A_n \in \mathcal{F}_T^f$ are independent of N_T . $\square$

The previous theorem shows that the optimal financial quantile hedging of a n -size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i := \mathbb{P}(N_T = i)$, $i = 1, \dots, n$. As in the previous section, we can also go a step further and obtain semi-explicit expressions for the optimal sets A_i^* , $i = 1, \dots, n$ that depends on a dual process $(Z_t^*)_{0 \leq t \leq T}$. Moreover, if the financial market is assumed to be complete i.e. $\mathcal{Q}^f = \{\tilde{\mathbb{Q}}^f\}$, then we can deduce a fully explicit form of the optimal sets A_i^* since the optimal dual process can be derived. To this end, we apply once again duality approach and consider in this sense the following sets:

$$\mathcal{Y}^f := \left\{ (Y_t)_{t \in [0, T]} \geq 0 \text{ } \mathbb{P}^f\text{-a.s.} : Y_0 = 1 \text{ and } \left(\frac{V_t}{B_t} Y_t \right)_{t \in [0, T]} \text{ } \mathbb{P}^f\text{-supermartingale} \right\},$$

and

$$\mathcal{D}^f := \left\{ Z_T \in L^0(\Omega^f, \mathcal{F}^f, \mathbb{P}^f) : 0 \leq Z_T \leq Y_T \text{ for some } (Y_t)_{t \in [0, T]} \in \mathcal{Y}^f \right\}.$$

Proposition 3.3.4. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\min_{\lambda \geq 0, Z_T \in \mathcal{D}^f} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*}(\lambda, Z_T, p) \left(p_i - \lambda Z_T \frac{H_T}{B_T} \right) + \lambda \bar{V}_0 \right), \quad (3.3.11)$$

with

$$A_i^*(\lambda, Z_T, p) = \left\{ \alpha_i(p) \geq \lambda \times Z_T \frac{H_T}{B_T} \right\}, \quad i = 1, \dots, n, \quad (3.3.12)$$

such that:

- $\alpha_n(p) = \min \left(p_n, \frac{p_n + p_{n-1}}{2}, \dots, \frac{\sum_{j=1}^n p_j}{n} \right),$
- for $i = 1, \dots, n-1$, $\alpha_i(p) = \max \left(\alpha_{i+1}(p), \min_{k=1, \dots, i} \left(\frac{\sum_{j=k}^i p_j}{i-k+1} \right) \right).$

Then, $A_i^* = A_i^*(\lambda^*, Z_T^*, p)$ is a solution of (3.3.9) such that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times p_i \right) + p_0.$$

Moreover,

$$\bar{V}_0 = E^{\mathbb{P}} \left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

and the optimal hedging strategy $\xi^{Q^H, f}$ is the admissible financial hedging strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. In particular, if $\mathcal{Q}^f = \{\tilde{\mathbb{Q}}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{P}^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{Q^H, f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is very similar to the proof of Proposition 3.3.2 and it is sufficient to consider p_i instead of $\mathbb{1}_{N_T=i}$ to obtain the announced result. Moreover, if $\mathcal{Q}^f = \{\tilde{\mathbb{Q}}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{P}^f}|_T$, we observe that $\forall (Y_t)_{t \in [0, T]} \in \mathcal{Y}^f$, $\exists (A_t)_{t \in [0, T]} \in \mathcal{F}^f$, an increasing process with $A_0 = 0$ such that, almost surely, $Y_t = \tilde{Z}_t - A_t \leq \tilde{Z}_t$, $\forall t \in [0, T]$. In this case, we have that, for all $\lambda \geq 0$,

$$\min_{Z_T \in \mathcal{D}^f} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*}(\lambda, Z_T, p) \left(p_i - \lambda Z_T \frac{H_T}{B_T} \right) \right) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*}(\lambda, \tilde{Z}_T, p) \left(p_i - \lambda \tilde{Z}_T \frac{H_T}{B_T} \right) \right),$$

and thus $Z_T^* = \tilde{Z}_T$. Finally,

$$\bar{V}_0 = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

and $\xi^{QH,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. $\square$

From Proposition 3.3.4, we deduce that, the optimal financial quantile hedging strategy of insurance-financial payoffs is equivalent to a super-hedging strategy of a sum of knock-out financial options that depend on the actuarial risk through p_i , $i = 1, \dots, n$. In particular, for $n = 1$, we retrieve the results deduced in Melnikov and Nosrati (2017). When $n > 1$, we can observe that the purely financial strategy associated with the quantile hedging of $n \times H_T$ is not optimal and it is preferable for the insurer to adopt the optimal quantile hedging strategy deduced for a n -size portfolio, especially if n is large. For a general incomplete financial market, it is difficult to derive the value at maturity of the optimal dual process Z_T^* solving (3.3.11). However, if the financial market is complete, then the optimal dual process is known and the optimal financial quantile hedging strategy is associated to a perfect hedging strategy of a modified financial payoff that can be readily implemented in practice.

3.4 Optimal shortfall hedging strategy

3.4.1 Characterization of the optimal strategy

In the previous section, we showed how to deduce the optimal quantile hedging strategy associated with the hedging of a life insurance portfolio. However, the main drawback of such hedging strategy is that it does not control the severity of the loss when the final value of the hedging portfolio is not sufficient to totally hedge the payoff H_T^{Total} . This quantile hedging approach therefore suffers from the same issue as optimal dynamic asset allocations under value-at-risk constraint (see Hainaut (2009)), and this can lead to very large infrequent losses. Therefore, in order to control the size of the shortfall loss, the insurer might be motivated to minimize the expectation of a class shortfall loss functions. To this end, we will solve in this section the introduced shortfall hedging problem (3.2.4). As in the previous sections, for any $\xi \in \mathcal{A}$, we consider the auxiliary variables $\phi_1, \dots, \phi_n$ defined such that

$$\phi_i H_T := \begin{cases} 0, & \text{if } V_T \leq (i-1)H_T \\ H_T & \text{if } V_T \geq iH_T \\ V_T - (i-1)H_T & \text{if } (i-1)H_T < V_T < iH_T, \end{cases} \quad (3.4.1)$$

where ϕ_i can be interpreted as the success ratio for the hedging of the i^{th} financial contingent claim H_T . Using these introduced variables, we show that the dynamic optimization problem (3.2.4) is equivalent to a static optimization problem that enables us to characterize the class of optimal shortfall hedging strategies.

Theorem 3.4.1. Let $\phi_i^* \in \mathcal{F}_T$, $i = 1, \dots, n$ the solution of the following problem

$$\min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \phi_0 = 1} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right), \quad (3.4.2)$$

with

$$\mathcal{R} = \left\{ \phi : \Omega \rightarrow [0, 1], \mathcal{F}_T - \text{measurable} \right\}$$

under the constraint

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right) \leq \bar{V}_0.$$

Let ξ^{SH} denote the super replicating hedging strategy of the modified payoff

$$H_T^{SH} = H_T \times \sum_{i=1}^n \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}. \quad (3.4.3)$$

Then $(\bar{V}_0, \xi^{SH})$ solves the stochastic control problem defined by (3.2.4) such that

$$\begin{aligned} \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^{\mathbb{P}} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) = \\ \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right) \end{aligned}$$

and $\xi^{SH} \in \mathcal{A}$.

Proof. The proof is given in Section 3.8. $\square$

Remark 3.6. From Proposition 3.9.4 in Appendix 3.9.2, the existence of a solution to the problem (3.4.2) is guaranteed.

As for the quantile hedging problem, the previous theorem shows that the optimal shortfall hedging of a n -size life insurance portfolio is linked to a super-replication strategy of a modified financial payoff that takes into account both the financial and actuarial risks. Moreover, as for the quantile hedging problem, for some particular loss functions $L(\cdot)$, we can apply a duality approach and characterize the form of the optimal random variables ϕ_i^* , $i = 1, \dots, n$. This is the purpose of the next propositions.

Proposition 3.4.2. Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of

$$\max_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(H_T \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) - \lambda \bar{V}_0, \quad (3.4.4)$$

with

$$A_i^*(\lambda, Z_T) = \left\{ \sum_{j=i}^n \mathbb{1}_{N_T=j} \geq \lambda \frac{Z_T}{B_T} \right\}, \quad i = 1, \dots, n. \quad (3.4.5)$$

Then $\mathbb{1}_{A_i^*} = \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)}$ is a solution to the problem (3.4.2) for $L(x) = x$, such that

$$\min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^{\mathbb{P}} \left((H_T^{Total} - V_T)_+ \right) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \left(\sum_{j=1}^i (1 - \mathbb{1}_{A_j^*}) H_T \right) \right).$$

Moreover,

$$\bar{V}_0 = \mathbb{E}^{\mathbb{P}} \left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is given in Section 3.8. □

Proposition 3.4.3. Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ be the solution of

$$\begin{aligned} \max_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}}) \right)^2 \right. \\ \left. + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \right) - \lambda \bar{V}_0, \end{aligned} \quad (3.4.6)$$

with

$$\phi_n^*(\lambda, Z_T) = \mathbb{1}_{\left\{ \frac{\lambda Z_T}{B_T H_T} \leq \mathbb{1}_{\{N_T=n\}} \right\}} \left(1 - \frac{\lambda}{\mathbb{1}_{\{N_T=n\}}} \frac{Z_T}{B_T H_T} \right) \text{ on } \{H_T > 0\},$$

and, for $i \in \{1, \dots, n-1\}$, on $\{H_T > 0\}$,

$$\begin{aligned} \phi_i^*(\lambda, Z_T) = \mathbb{1}_{A_i(\lambda, Z_T)} \\ + \mathbb{1}_{A_i^c(\lambda, Z_T)} \left(1 - \left(\frac{1}{\sum_{j=i}^n \mathbb{1}_{\{N_T=j\}}} \left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^n \mathbb{1}_{\{N_T=j\}} \right) \wedge 1 \right) \right), \end{aligned} \quad (3.4.7)$$

such that

$$A_i(\lambda, Z_T) = \left\{ \frac{\lambda Z_T}{B_T H_T} \leq \sum_{j=i+1}^n \sum_{k=j}^n \mathbb{1}_{\{N_T=k\}} \right\}.$$

Then $\phi_i^* = \phi_i^*(\lambda^*, Z_T^*)$ is a solution to the problem (3.4.2) for a quadratic function $L(x) = \frac{1}{2}x^2$ such that

$$\min_{v_0 \in \mathbb{R}_+, \xi \in A \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^{\mathbb{P}} \left(\frac{1}{2} \left((H_T^{\text{Total}} - V_T)_+ \right)^2 \right) = \mathbb{E}^{\mathbb{P}} \left(\frac{1}{2} \sum_{i=1}^n \mathbf{1}_{\{N_T=i\}} \left(H_T \sum_{j=1}^i (1 - \phi_j^*) \right)^2 \right).$$

Moreover,

$$\bar{V}_0 = \mathbb{E}^{\mathbb{P}} \left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \phi_i^* \right), \quad (3.4.8)$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$.

Proof. The proof is given in Section 3.8. $\square$

From Proposition 3.4.2 and 3.4.3, we obtain, when $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, an explicit characterization of the optimal variables ϕ_i^* , $i = 1, \dots, n$ that depends on $\lambda^* \in \mathbb{R}_+$ and the T -value of a dual process $Z_T^* \in \mathcal{D}$. As for the quantile hedging problem, we observe that the optimal shortfall strategies of a n -size portfolio for $L(x) = x$ and $L(x) = \frac{1}{2}x^2$, generalizes the pure shortfall hedging strategy deduced in Föllmer and Leukert (2000). In particular, if $n = 1$, we recover the results deduced in Nakano (2011). We prove that the optimal shortfall hedging strategies depend on both financial and actuarial risks. Nevertheless, once again, it is difficult to derive the optimal dual process⁴ associated to the problems (3.4.4) or (3.4.6) and thus, as for the quantile hedging problem, we cannot obtain, in general, more explicit expressions for the optimal variables ϕ_i^* . By restricting the set of admissible hedging strategies to the set of financial strategies, we are still able to characterize the solution of the problem when $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, and, for complete financial markets, the optimal dual process can be explicitly deduced. Furthermore, we show in Section 3.5 how to construct explicit $\mathcal{F}$ -adapted shortfall strategies that can be readily implemented in practice.

3.4.2 Restriction to the set of admissible financial hedging strategy $\mathcal{A}^f \subseteq \mathcal{A}$

In this section, we consider the shortfall hedging problem by restricting the set of admissible hedging strategies to the set of financial hedging strategies. As for the quantile hedging problem, the results of this section will be used to derive

⁴Except for $n = 1$ in a complete financial market setting, as shown in Nakano (2011).

more optimal explicit $\mathcal{F}$ -adapted shortfall hedging strategies in Section 3.5. When $\xi \in \mathcal{A}^f$, the shortfall hedging problem is reduced to

$$\min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right), \quad (3.4.9)$$

such that

$$\begin{aligned} & \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) \geq \\ & \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right). \end{aligned}$$

Let us now solve the shortfall hedging problem in the set of financial hedging strategies defined by (3.4.9). Once again, we consider the auxiliary variables $\phi_1, \dots, \phi_n$ defined by (3.4.1) that are now $\mathcal{F}_T^f$ -measurable since $\xi \in \mathcal{A}^f$. Using these introduced variables, we show that the dynamic optimization problem (3.4.9) is equivalent to a static optimization problem that enables us to characterize the class of optimal shortfall hedging strategies.

Theorem 3.4.4. *Let $p_i := \mathbb{P}(N_T = i)$ and $\phi_i^* \in \mathcal{F}_T^f$, $i = 1, \dots, n$ the solution of the following problem*

$$\min_{\phi_1, \dots, \phi_n \in \mathcal{R}^f, \phi_{i+1} \leq \phi_i, \phi_0 = 1} \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n p_i \times L \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right), \quad (3.4.10)$$

with

$$\mathcal{R}^f = \left\{ \phi : \Omega^f \rightarrow [0, 1], \mathcal{F}_T^f\text{-measurable} \right\}$$

under the constraint

$$\sup_{Q^f \in \mathcal{Q}^f} \mathbb{E}^{Q^f} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right) \leq \bar{V}_0.$$

Let $\xi^{SH, f}$ denote the (super) replicating hedging strategy of the modified payoff

$$H_T^{SH} = H_T \times \sum_{i=1}^n \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}. \quad (3.4.11)$$

Then $(\bar{V}_0, \xi^{SH, f})$ solves the stochastic control problem defined by (3.4.9) such that

$$\begin{aligned} & \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) = \\ & \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n p_i \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right) \end{aligned}$$

and $\xi^{SH} \in \mathcal{A}^f$.

Proof. We can easily deduce the announced result by using similar arguments than in the proof of Theorem 3.4.1 combined with the fact that $\phi_1, \dots, \phi_n \in \mathcal{F}_T^f$ are independent of N_T . $\square$

The previous theorem shows that the optimal financial shortfall hedging of a n -size life insurance portfolio is linked to a (super) replication strategy of a modified financial payoff that takes into account the actuarial risk through the probabilities $p_i := \mathbb{P}(N_T = i)$, $i = 1, \dots, n$. As in the previous section, we can also go a step further and, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, obtain explicit expressions for the optimal variables ϕ_i^* , $i = 1, \dots, n$ depending on a dual process $(Z_t^*)_{0 \leq t \leq T}$. Moreover, if the financial market is assumed to be complete i.e. $\mathcal{Q}^f = \{\tilde{Q}^f\}$, then the optimal dual process is known. This is the purpose of the next propositions.

Proposition 3.4.5. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\max_{\lambda \geq 0, Z_T \in \mathcal{D}^f} \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T, p)} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n p_j \right) \right) - \lambda \bar{V}_0,$$

with

$$A_i^*(\lambda, Z_T, p) = \left\{ \sum_{j=i}^n p_j \geq \lambda \frac{Z_T}{B_T} \right\}, \quad i = 1, \dots, n. \quad (3.4.12)$$

Then $\mathbb{1}_{A_i^*} = \mathbb{1}_{A_i^*(\lambda^*, Z_T^*, p)}$ is a solution to the problem (3.4.10) for $L(x) = x$, such that

$$\min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left((H_T^{\text{Total}} - V_T)_+ \right) = \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n p_i \left(\sum_{j=1}^i (1 - \mathbb{1}_{A_j^*}) H_T \right) \right).$$

Moreover,

$$\bar{V}_0 = \mathbb{E}^\mathbb{P} \left(Z_T^* \frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

and the optimal hedging strategy $\xi^{SH, f}$ is the admissible financial strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. In particular, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{d\mathbb{P}^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{SH, f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$.

Proof. The proof is very similar to the proof of Proposition 3.4.2 and it is sufficient to consider p_i instead of $\mathbb{1}_{N_T=i}$ to obtain the announced result. Moreover, if $\mathcal{Q}^f =$

$\{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{d\mathbb{P}^f}|_T$, we have that, for all $\lambda \geq 0$,

$$\begin{aligned} \max_{Z_T \in \mathcal{D}^f} \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T, p)} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n p_j \right) \right) = \\ \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, \tilde{Z}_T, p)} \left(\lambda \frac{\tilde{Z}_T}{B_T} - \sum_{j=i}^n p_j \right) \right), \end{aligned}$$

and thus $Z_T^* = \tilde{Z}_T$. Finally,

$$\bar{V}_0 = \mathbb{E}^{\tilde{Q}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right),$$

$\xi^{SH, f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. $\square$

Proposition 3.4.6. *Let $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}^f$ be the solution of*

$$\begin{aligned} \max_{\lambda \geq 0, Z_T \in \mathcal{D}^f} \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n p_i \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}}) \right)^2 \right. \\ \left. + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \right) - \lambda \bar{V}_0, \end{aligned}$$

with

$$\phi_n^*(\lambda, Z_T, p) = \left(1 - \frac{\lambda}{p_n} \frac{Z_T}{B_T H_T} \right)_+ \text{ on } \{H_T > 0\},$$

and, for $i \in [1, \dots, n-1]$, on $\{H_T > 0\}$,

$$\phi_i^*(\lambda, Z_T, p) = \mathbb{1}_{A_i(\lambda, Z_T, p)} + \mathbb{1}_{A_i^c(\lambda, Z_T, p)} \left(1 - \left(\frac{1}{\sum_{j=i}^n p_j} \left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^n p_j \right) \wedge 1 \right) \right), \quad (3.4.13)$$

such that

$$A_i(\lambda, Z_T, p) = \left\{ \frac{\lambda Z_T}{B_T H_T} \leq \sum_{j=i+1}^n \sum_{k=j}^n p_k \right\}.$$

Then $\phi_i^* = \phi_i^*(\lambda^*, Z_T^*)$ is a solution to the problem (3.4.10) for a quadratic function $L(x) = \frac{1}{2}x^2$ such that

$$\min_{v_0 \in \mathbb{R}_+, \xi \in A^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(\frac{1}{2} \left((H_T^{Total} - V_T)_+ \right)^2 \right) = \mathbb{E}^\mathbb{P} \left(\frac{1}{2} \sum_{i=1}^n p_i \left(H_T \sum_{j=1}^i (1 - \phi_j^*) \right)^2 \right).$$

Moreover,

$$\bar{V}_0 = E^{\tilde{Q}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \phi_i^* \right),$$

and the optimal hedging strategy $\xi^{SH,f}$ is the admissible financial strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$. In particular, if $\mathcal{Q}^f = \{\tilde{Q}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{Q}^f}{d\mathbb{P}^f}|_T$, then $Z_T^* = \tilde{Z}_T$ and $\xi^{SH,f}$ is the admissible financial hedging strategy associated to the perfect hedging of the modified payoff $H_T \times \sum_{i=1}^n \phi_i^*$.

Proof. The result is deduced using the same similar arguments as for the proof of Proposition 3.4.3 and 3.4.5. $\square$

From Proposition 3.4.5 and 3.4.6, we deduce that, the optimal financial shortfall hedging strategies of insurance-financial payoffs, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, are equivalent to (super)-hedging strategies of a sum of knock-out financial options that depend on a dual process $(Z_t^*)_{0 \leq t \leq T}$. If the financial market is complete, the optimal dual process is known and then, the optimal financial shortfall hedging strategies, for $L(x) = x$ or $L(x) = \frac{1}{2}x^2$, are associated to a perfect hedging strategies of a modified financial payoff that can be readily implemented in practice.

3.5 Enlarging the optimal $\mathcal{F}^f$ –adapted strategies with actuarial information: stopping-time based strategies

In the previous sections, we succeeded in characterizing optimal strategies by considering either $\xi \in \mathcal{A}$ or $\xi \in \mathcal{A}^f$. Moreover, when the financial market is assumed to be complete and $\xi \in \mathcal{A}^f$, fully explicit expressions for the optimal strategies are deduced. However, by considering only $\xi \in \mathcal{A}^f$, we have a non-realistic situation where the insurer cannot measure the evolution of the number of survivors through time. In practice, the insurer can measure the number of survivors through time and can adapt its hedging strategy taking into account this actuarial information. In this case, we observe that it is, in general, difficult to derive explicitly the optimal strategies. In order to obtain explicit $\mathcal{F}$ –adapted hedging strategies, we propose to enlarge the optimal $\mathcal{F}^f$ –adapted strategies with actuarial information. The idea is to adjust the optimal strategies for $\xi \in \mathcal{A}^f$ each time a death occurs. In this case, we obtain $\mathcal{F}$ –adapted hedging strategies which are more optimal than the optimal $\mathcal{F}^f$ –adapted hedging strategies. This is summarized by the following propositions, for which the proofs are easily deduced by recursive procedures. Note that for the sake of consistency, we have only included the proof for the quantile strategy, but the proof for the shortfall strategies can be directly deduced using the same arguments.

Definition 3.1. For $i = 0, \dots, n$, let us define τ_i as the $\mathcal{F}^a$ -stopping time associated to the i^{th} death with the convention $\tau_0 := 0$, and $p^{\tau_i} := (p_0^{\tau_i}, \dots, p_n^{\tau_i})$ such that

$$p_j^{\tau_i} := \mathbb{P}(N_T = j | \mathcal{F}_{\tau_i}), \quad i, j = 0, \dots, n.$$

Proposition 3.5.1. Let us assume that $\mathcal{Q}^f = \{\tilde{\mathbb{Q}}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{P}^f}|_T$. Let $(\tilde{\xi}_t)_{t \in [0, T]}$ be a hedging strategy associated to a hedging portfolio $(\tilde{V}_t)_{t \in [0, T]}$ with $\tilde{V}_0 = \bar{V}_0$ and let us consider τ , the $\mathcal{F}$ -stopping time defined by

$$\tau := \inf \left\{ \tau \in \{\tau_0 \wedge T, \dots, \tau_n \wedge T\} : \tilde{V}_\tau 1_{\tau < T} \geq 1_{\tau < T} \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_\tau}{B_T} H_T \times N_T | \mathcal{F}_\tau \right) \right\}.$$

Assume that for $i = 0, \dots, n-1$, and $\tau_i < \tau$, $(\tilde{\xi}_t)_{t \in [\tau_i, \tau_{i+1} \wedge \tau]}$ is the perfect hedging strategy of

$$H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})}, \quad (3.5.1)$$

where $A_i^*(.)$ is given by (3.3.12) and λ_{τ_i} satisfied

$$\tilde{V}_{\tau_i-} = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_{\tau_i}}{B_T} H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})} | \mathcal{F}_{\tau_i} \right),$$

and $(\tilde{\xi}_t)_{t \in [\tau, T]}$ is the perfect hedging strategy of

$$H_T \times N_\tau.$$

Then,

$$\tilde{V}_T = \begin{cases} H_T \times \sum_{j=1}^{N_T} \mathbb{1}_{A_j^*(\lambda_{\tau_n-N_T}, \tilde{Z}_T, p^{\tau_n-N_T})} & \text{if } \tau = T, \\ H_T \times N_\tau + \frac{B_\tau}{B_T} \left(\tilde{V}_{\tau-} - \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_\tau}{B_T} H_T \times N_\tau | \mathcal{F}_\tau \right) \right) & \text{if } \tau < T, \end{cases}$$

and

$$\begin{aligned} \max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) &\leq \mathbb{P}(\tilde{V}_T \geq H_T^{Total}) \\ &\leq \max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}). \end{aligned} \quad (3.5.2)$$

Proof. The proof is given in Section 3.8. $\square$

Proposition 3.5.2. *Let us assume that $\mathcal{Q}^f = \{\tilde{\mathbb{Q}}^f\}$ such that $\tilde{Z}_T = \frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{P}^f}|_T$. Suppose that $L(x) = x$ or $L(x) = \frac{1}{2}x^2$ and let $(\tilde{\xi}_t)_{t \in [0, T]}$ be a hedging strategy associated to a hedging portfolio $(\tilde{V}_t)_{t \in [0, T]}$ with $\tilde{V}_0 = \bar{V}_0$ and let us consider τ , the $\mathcal{F}$ -stopping time defined by*

$$\tau := \inf \left\{ \tau \in \{\tau_0 \wedge T, \dots, \tau_n \wedge T\} : \tilde{V}_\tau 1_{\tau < T} \geq 1_{\tau < T} E^{\tilde{\mathbb{Q}}^f} \left(\frac{B_\tau}{B_T} H_T \times N_T | \mathcal{F}_\tau \right) \right\}.$$

Assume that for $i = 0, \dots, n-1$, and $\tau_i < \tau$, $(\tilde{\xi}_t)_{t \in [\tau_i, \tau_{i+1} \wedge \tau]}$ is the perfect hedging strategy of

$$H_T^{\tau_i} := \begin{cases} H_T \times \sum_{j=1}^{N_{\tau_i}} \mathbb{1}_{A_j^*(\lambda_{\tau_i}, \tilde{Z}_T, p^{\tau_i})} & \text{if } L(x) = x, \\ H_T \times \sum_{j=1}^{N_{\tau_i}} \phi_j^*(\lambda_{\tau_i}, \tilde{Z}_{T-\tau_i}, p^{\tau_i}) & \text{if } L(x) = \frac{1}{2}x^2, \end{cases} \quad (3.5.3)$$

where $A_i^*(\cdot)$ is given by (3.4.12), $\phi_i^*(\cdot)$ is given by (3.4.13), and λ_{τ_i} satisfied

$$\tilde{V}_{\tau_i-} = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_{\tau_i}}{B_T} H_T^{\tau_i} | \mathcal{F}_{\tau_i} \right), \quad (3.5.4)$$

and $(\tilde{\xi}_t)_{t \in [\tau, T]}$ is the perfect hedging strategy of

$$H_T \times N_\tau.$$

Then,

$$\tilde{V}_T = \begin{cases} H_T^{\tau_n - N_T} & \text{if } \tau = T, \\ H_T \times N_\tau + \frac{B_T}{B_\tau} \left(\tilde{V}_{\tau-} - \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_\tau}{B_T} H_T \times N_\tau | \mathcal{F}_\tau \right) \right) & \text{if } \tau < T, \end{cases}$$

and

$$\begin{aligned} \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L((H_T^{\text{Total}} - V_T)_+) \right) &\leq \mathbb{E}^\mathbb{P} \left(L((H_T^{\text{Total}} - \tilde{V}_T)_+) \right) \\ &\leq \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L((H_T^{\text{Total}} - V_T)_+) \right). \end{aligned}$$

Remark 3.7 (Implementation of the stopping-time based strategies). *Let $H_T^{\tau_i}(\lambda_{\tau_i})$ be the modified quantile or shortfall hedging payoffs defined by (3.5.1) and (3.5.3). Here is the procedure to follow to implement the stopping-time based strategies:*

1. *At time $t = 0$, deduce λ_0 by solving a zero-search problem⁵:*

$$\mathbb{E}^{\tilde{\mathbb{Q}}^f} (H_T^{\tau_0}(\lambda_0)) = \bar{V}_0,$$

and follow the perfect dynamic hedging strategy of $H_T^{\tau_0}(\lambda_0)$ until $\tau_1 \wedge T$.

⁵In practice, λ_0 can be deduced either with a deterministic root-finding method when a closed-form expression of $\mathbb{E}^{\tilde{\mathbb{Q}}^f} (H_T^{\tau_0}(\lambda_0))$ is available, or with a stochastic root-finding procedure otherwise; see Appendix 3.9.3 for details.

2. At time $\tau_i < T$, for $i = 1, \dots, n$,

- If $\tilde{V}_{\tau_i}^- \geq \mathbb{E}^{\tilde{\mathbb{Q}}^f}(H_T \times N_{\tau_i} | \mathcal{F}_{\tau_i})$, then follow the perfect hedging strategy $H_T \times N_{\tau_i}$ until the maturity T ;
- If $\tilde{V}_{\tau_i}^- < \mathbb{E}^{\tilde{\mathbb{Q}}^f}(H_T \times N_{\tau_i} | \mathcal{F}_{\tau_i})$, Update the modified payoff by deducing λ_{τ_i} such that

$$\mathbb{E}^{\tilde{\mathbb{Q}}^f}(H_T^{\tau_i}(\lambda_i) | \mathcal{F}_{\tau_i}) = \tilde{V}_{\tau_i}^-,$$

and follow the perfect hedging strategy of $H_T^{\tau_i}(\lambda_{\tau_i})$ until $\tau_{i+1} \wedge T$.

3.6 Application in Black and Scholes framework

The various optimal strategies studied in this chapter, admit explicit characterizations that depend on the T -value of a dual process Z_T^* . We observe that, when we restrict ourselves to the set of admissible financial hedging strategies $\mathcal{A}^f$ and consider complete financial markets, then the T -value of the optimal dual process Z_T^* is known and the hedging strategies can therefore be applied in practice. Moreover, we show in Section 3.5, how to enlarge these optimal $\mathcal{F}^f$ -adapted strategies with actuarial information in order to obtain explicit $\mathcal{F}$ -adapted hedging strategies. In this section, we illustrate these different strategies through numerical results by considering a complete financial market. We observe that the strategies are very relevant for hedging equity-linked life insurance portfolio when we consider loss-averse insurers. For the sake of simplicity, we consider a simple Black and Scholes model for the financial market, but the strategies can be applied in more sophisticated models, such as, for example, local volatility models, fully-endogenous (rough) stochastic volatility models, or path-dependent volatility models. Thus, in this section, we consider a financial market composed of the two assets that satisfy, for $t \in [0, T]$, the following dynamics:

$$dB_t = rB_t dt,$$

and

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

on $(\Omega^f, \mathcal{F}^f, \mathbb{P}^f)$, with $r \in \mathbb{R}$ the risk-free interest rate, $\mu > r$, $\sigma \in \mathbb{R}_+$ respectively the return and the volatility of the risky asset, and $(W_t)_{0 \leq t \leq T}$ is a standard Brownian motion defined on $\mathbb{P}^f$. This financial market is arbitrage-free and complete. Therefore, we know that there exists a unique equivalent martingale measure

denoted by $\tilde{\mathbb{Q}}^f$ and characterized by the following change of measure, for $t \in [0, T]$,

$$\begin{aligned}\tilde{Z}_t &:= \frac{d\tilde{\mathbb{Q}}^f}{d\mathbb{P}^f}|_t \\ &= \exp\left(-\frac{(\mu-r)}{\sigma}W_t - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}t\right). \\ &= K_t \times S_t^{-(\mu-r)/\sigma^2}, \quad K_t \in \mathbb{R}.\end{aligned}$$

As the Black and Scholes market is complete, we also know that for any contingent claim of the form $H_T = f(S_T) \in \mathcal{F}_T^f$, we can construct a hedging portfolio $(V_t^{BS})_{0 \leq t \leq T}$ driven by a hedging strategy $(\xi_t^{BS})_{0 \leq t \leq T}$ such, $\mathbb{P}^f$ -almost surely,

$$V_T^{BS} = f(S_T)$$

with for $t \in [0, T]$,

$$V_t^{BS} = E^{\tilde{\mathbb{Q}}^f}\left(e^{-r(T-t)}H_T|\mathcal{F}_t^f\right)$$

and

$$\xi_t^{BS} = \frac{\partial V_t^{BS}}{\partial S_t}.$$

For the insurance market, we consider a constant mortality rate $\kappa_{x+t} = \kappa \in \mathbb{R}_+$ such that ${}_{T-t}p_{x+t} = \exp(-\kappa(T-t))$, $N_T|\mathcal{F}_t \sim \text{Bin}(N_t, {}_{T-t}p_{x+t})$ and for $i = 0, \dots, N_t$, $p_i^t = \mathbb{P}(N_T = i|\mathcal{F}_t) = \binom{N_t}{i} ({}_{T-t}p_{x+t})^i \times (1-{}_{T-t}p_{x+t})^{N_t-i}$. In the numerical results, we will vary the initial survival probability ${}_Tp_x$ ⁶ in order to analyze its effect on the hedging strategies.

We assume that the insurer has sold a life insurance product guaranteeing a stochastic positive amount at maturity given by a call payoff

$$H_T = (S_T - K_{call})_+,$$

and at an initial price⁷

$${}_Tp_x \times \mathbb{E}^{\tilde{\mathbb{Q}}^f}(e^{-rT}(S_T - K_{call})_+),$$

In this case, the initial constraint on the value of the hedging portfolio is given by

$$\bar{V}_0 = n \times {}_Tp_x \times \mathbb{E}^{\tilde{\mathbb{Q}}^f}(e^{-rT}(S_T - K_{call})_+).$$

We now construct the efficient hedging strategies for this type of n -size portfolio in a Black and Scholes setting. To this end, we consider three different loss-averse insurers:

⁶Which is equivalent to vary the mortality rate $\kappa \in \mathbb{R}_+$.

⁷Without loss of generality, we assume that the insurer does not consider safety-loading for the mortality risk.

1. A risk-averse insurer who seeks to avoid as much as possible the events of losses, but who does not particularly intend to control the size of those losses. For this agent, we consider the quantile hedging approach.
2. An insurer who controls the size of the losses and minimizes the expected shortfall loss. For this agent, we consider the shortfall hedging approach.
3. An insurer who minimizes the expected shortfall loss while also controlling the variance of this loss. For this agent, we consider the quadratic shortfall hedging approach.

Note that for the numerical results, without loss of generality, we consider the following parameters for the Black and Scholes model:

$$S_0 = 100, r = 0.02, \mu = 0.06, \sigma = 0.2.$$

3.6.1 Quantile hedging

Let us consider the quantile hedging of a n -size life insurance portfolio with call payoff as stochastic maturity value. First, we can observe that in the particular setting considered for numerical illustration, the modified payoff associated to the optimal quantile hedging strategy when $\xi \in \mathcal{A}^f$ can be more explicitly decomposed which allows us to more easily interpret this hedging strategy. The same holds for the $\mathcal{F}$ -adapted quantile hedging strategy constructed in Section 3.5. In fact, if we restrict ourselves to $\mathcal{A}^f$, from Proposition 3.3.4, we know that the success sets A_i^* , $i = 1, \dots, n$, are given by

$$\begin{aligned} A_i^* &= \left\{ \alpha_i(p) \geq \lambda^* \times \tilde{Z}_T \frac{H_T}{B_T} \right\} \\ &= \left\{ e^{rT} \times \alpha_i(p) \times \frac{S_T^{(\mu-r)/\sigma^2}}{K_T} \geq \lambda^* (S_T - K_{call})_+ \right\}, \end{aligned}$$

In general, we cannot obtain a more explicit expression for the modified payoff $H_T \sum_{i=1}^n \mathbb{1}_{A_i^*}$. Nevertheless, assuming that $\mu - r = \sigma^2$ holds, we can rewrite A_i^* , $i = 1, \dots, n$ such that

$$A_i^* = \left\{ \alpha_i(p) \times \frac{e^{rT}}{K_T \lambda^*} \times S_T \geq (S_T - K_{call})_+ \right\}$$

with λ^* satisfying

$$\bar{V}_0 = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \right).$$

In this particular case, the modified payoff becomes

$$H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*} = \sum_{i=1}^n (S_T - K_{call})_+ \times \left(\mathbb{1}_{\left\{ S_T \leq \frac{K_{call}}{1 - \frac{e^{rT}}{K_T \lambda^*} \times \alpha_i(p)} \right\} \cap \left\{ 1 \geq \frac{e^{rT}}{K_T \lambda^*} \times \alpha_i(p) \right\}} + \mathbb{1}_{\left\{ 1 < \frac{e^{rT}}{K_T \lambda^*} \times \alpha_i(p) \right\}} \right).$$

We therefore observe that the modified payoff associated with the quantile hedging of the n -size portfolio with call option is equal to a sum of up-and-out call options with out strikes larger than K_{call} . Moreover, as in Black and Scholes model, the price of up-and-out call options admits a closed form, the initial value of the modified payoff as well as the optimal quantile hedging loss are obtained by closed formula, and we can therefore easily deduce the optimal constant λ^* . Note that, when $\mu - r \neq \sigma^2$, we can use a stochastic algorithm (see Appendix 3.9.3) to deduce the optimal constant λ^* . When we enlarge the optimal $\mathcal{F}^f$ -adapted quantile strategy with actuarial information and consider in this sense the $\mathcal{F}$ -adapted quantile strategy proposed in Section 3.5, using similar reasoning and Proposition 3.5.1, we observe that, the modified payoff associated to this hedging strategy is also a combination of up-and-out call options where the out-strikes are updated at each jump in the process $(N_t)_{t \in [0, T]}$. Nevertheless, the quantile hedging loss associated to this strategy cannot be computed directly with closed formula. It can be computed through Monte Carlo simulations, where the point process is simulated and the strategy is updated at each jump as explained in Proposition 3.5.1. More precisely, we have to adjust the out-strikes i.e. the constant λ_{τ_i} at each stopping time $\tau_i < T$, $i = 1, \dots, n$. Note that, when Monte Carlo simulations are employed, the number of simulated paths is fixed to $N^{MC} = 25,000$.

To illustrate the efficiency of the quantile hedging strategies with numerical results, we will benchmark them against the mean-variance financial hedging strategy that is well known in actuarial sciences (see for example Barigou and Dhaene (2019); Dhaene et al. (2017)). This benchmark strategy consists of a perfect hedge of the financial payoff given by $n \times {}_T p_x \times H_T$. Note that the probability of successful hedge for the benchmark strategy can be easily deduced and is given by

$$\mathbb{P}(V_T^{Bench} \geq H_T^{Total}) = \mathbb{P}(N_T \leq n \times {}_T p_x) \mathbb{P}(H_T > 0) + \mathbb{P}(H_T = 0).$$

Figure 3.1 and Table 3.1 compare the probability of successful hedge with respect to the survival probability ${}_T p_x$ for ATM call. We observe that, in terms of success probability, the quantile hedging strategies outperform the mean-variance hedging strategy. We observe that it is therefore in the insurer's interest to implement these strategies instead of the benchmark strategy if it wants to maximize its probability of successful hedge. The quantile strategies actually offer a better trade-off between actuarial and financial risks. Moreover, by comparing the two quantile

	$\mathbb{P}(V_T \geq H_T^{total})$			
τp_x	$\mathcal{F}$ -adapted quantile hedging	Opt. $\mathcal{F}^f$ -adapted quantile hedging	MV hedging	
0.99	0.9993	0.9964	0.7304	
0.95	0.9909	0.9815	0.6789	
0.9	0.9841	0.9712	0.6779	

Table 3.1: Comparison of success probability for the quantile hedging strategies and the benchmark strategy for ATM call payoff with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times \tau p_x \times E^{\bar{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

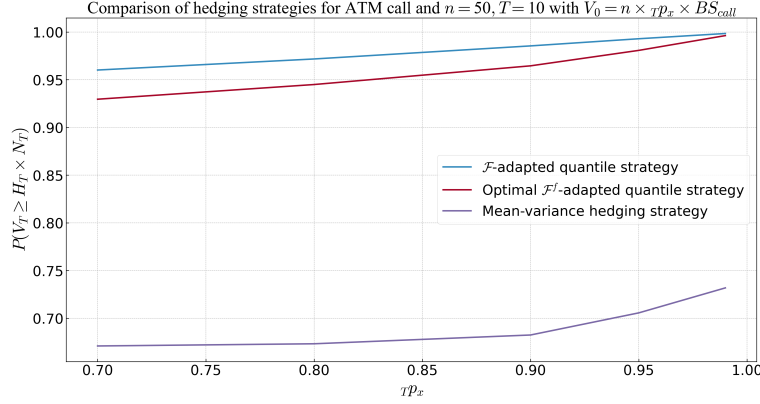


Figure 3.1: Success probability for the optimal quantile hedging strategy and the benchmark strategy for ATM call payoff with respect to tp_x with $n = 50$, $T = 10$ and $\bar{V}_0 = n \times tp_x \times \mathbb{E}^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

hedging strategies proposed, we observe that there are a non-negligible gain from incorporating the actuarial information inside the trading strategy, especially when we consider low survival probabilities. The gain is less significant when higher survival probabilities are considered, but it is not zero and the insurer has an incentive to incorporate actuarial information into its strategy even for high survival probabilities.

We can also analyze the effect of the portfolio size n . Table 3.2 compares the effect of the size of the portfolio on the hedging strategy for a fixed survival probability $tp_x = 0.95$. Once again, we clearly observe that quantile hedging strategies always outperform the benchmark strategy in term of success probability. We also observe that by increasing the number of policyholders in the portfolio, the probability of successful hedge from the quantile strategies increases. We also notice that the gap between the $\mathcal{F}$ -adapted strategy and the optimal strategy for $\xi \in \mathcal{A}^f$ slightly increases with the size of the portfolio. It is therefore in the insurer's interest to opt for the $\mathcal{F}$ -adapted strategy when it has a large portfolio.

3.6.2 Shortfall hedging

Let us now consider that the insurer minimizes the expected shortfall of its n -size portfolio with call payoff. As for the quantile strategy, we can observe that, in the particular setting considered, the modified payoff associated to the optimal shortfall hedging strategy when $\xi \in \mathcal{A}^f$ can be more explicitly decomposed which also allows us to more easily interpret this hedging strategy. When we restrict

n	$\mathbb{P}(V_T \geq H_T^{Total})$		
	$\mathcal{F}$ -adapted quantile hedging	Opt. $\mathcal{F}^f$ -adapted quantile hedging	MV hedging
50	0.9892	0.9808	0.6019
100	0.9902	0.9814	0.6751
300	0.9958	0.9871	0.6701

Table 3.2: Success probability for ATM call payoff with respect to n with $tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times tp_x \times \mathbb{E}^{\bar{\mathbb{Q}}^f}(e^{-rT}(S_T - K_{call})_+)$

$\mathcal{A}$ to $\mathcal{A}^f$, from Proposition 3.4.5, the optimal sets A_i^* , for $i = 1, \dots, n$, are given by

$$\begin{aligned} A_i^* &= \left\{ \sum_{j=i}^n p_j \geq \lambda^* \frac{\tilde{Z}_T}{B_T} \right\} \\ &= \left\{ \sum_{j=i}^n p_j \geq \frac{\lambda^* K_T}{e^{rT}} S_T^{-(\mu-r)/\sigma^2} \right\} \\ &= \left\{ S_T \geq K_i \right\}, \end{aligned}$$

with $m := \frac{\mu-r}{\sigma^2}$, $K_i := \frac{\lambda^* K_T e^{-rT}}{\left(\sum_{j=i}^n p_j \right)^{1/m}}$ for $i = 1, \dots, n$ and λ^* satisfying

$$\bar{V}_0 = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*} \right).$$

Thus, the modified payoff becomes

$$\sum_{i=1}^n H_T \mathbb{1}_{A_i^*} = \sum_{i=1}^n (S_T - K_{call})_+ \times \mathbb{1}_{\{S_T \geq K_i\}}.$$

We observe that the modified payoff associated with the optimal expected shortfall hedging is decomposed as a sum of up-and-in call options. If we compare the modified payoff with that for quantile hedging, we can see that they are very similar. Instead of sum of up-and-out call options for the quantile hedging strategy, we have a sum of up-and-in call options for the shortfall hedging strategy. This is simply explained by the fact that having up-and-in options instead of up-and-out makes it possible to limit losses in the event of default, which is the aim of the shortfall strategy. When we consider the $\mathcal{F}$ -adapted shortfall hedging strategy constructed in Section 3.5, using similar reasoning, the modified payoff can also be decomposed as a sum of up-and-in call options, where the in-strikes are adjusted at each jump in the process $(N_t)_{t \in [0, T]}$. Thus, in this case, the shortfall loss associated to this strategy needs, once again, to be computed through Monte Carlo simulations, where the point process is simulated and the strategy is adapted at each jump as explained in Proposition 3.5.2 and as it was done for the quantile hedging strategy.

Let us now illustrate, with numerical results, the shortfall hedging strategies for a n -size life insurance portfolio with $H_T = (S_T - K_{call})_+$. We consider the same parameters as for the quantile hedging results. Figure 3.2 and Table 3.3 compare the expected shortfall with respect to the survival probability ${}_T p_x$ for ATM call. We observe that, in terms of expected shortfall, the shortfall hedging strategies also outperform the benchmark strategy (mean-variance approach), and the gap is

	$\mathbb{E}^{\mathbb{P}} \left((H_T^{Total} - V_T)_+ \right)$			
τp_x	$\mathcal{F}$ -adapted shortfall hedging	Opt. $\mathcal{F}^f$ -adapted shortfall hedging	MV hedging	
0.99	23.07	26.62	32.87	
0.95	57.23	63.81	76.79	
0.9	64.35	89.10	106.58	

Table 3.3: Comparison of the expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 100$, $T = 10$ and $\bar{V}_0 = n \times \tau p_x \times \mathbb{E}^{\tilde{\mathbb{Q}}^f} (e^{-rT} (S_T - K_{call})_+)$

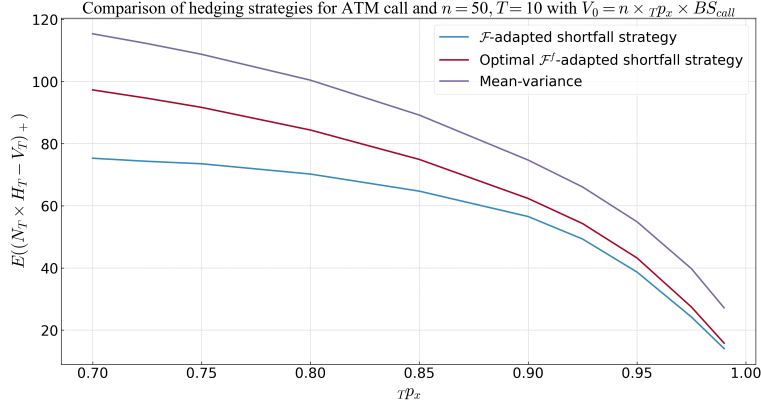


Figure 3.2: Expected shortfall for the optimal shortfall hedging strategy and the benchmark strategy for ATM call payoff with respect to Tp_x with $n = 50$, $T = 10$ and $\bar{V}_0 = n \times Tp_x \times E^{\tilde{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

more pronounced for low values of Tp_x . We conclude that it is therefore in the insurer's interest to choose the shortfall hedging strategies instead of the benchmark strategy if it wants to minimize the expected shortfall loss. In addition, we once again notice an information gain by incorporating actuarial information in the shortfall strategy. This information gain is even more pronounced than in the case of quantile strategies.

We can also analyze the effect of the portfolio size n . Table 3.4 compares the effect of the size n of the portfolio on the hedging strategy for a fixed survival probability $Tp_x = 0.95$ in terms of expected shortfall loss. We observe that by increasing the number of policyholders in the portfolio, the gap between the hedging strategies increases. An insurer with an aversion to shortfall losses should prefer the $\mathcal{F}$ -adapted shortfall strategy, over the two other strategies, especially if it has a large portfolio.

3.6.3 Quadratic shortfall hedging

Finally, let us consider that the insurer minimizes the expected quadratic shortfall loss of its n -size portfolio. Unlike the two other strategies, it is not possible to obtain a more explicit form of modified payoffs when we consider quadratic shortfall hedging. To deduce the optimal constant λ^* that satisfies (3.4.8) (or λ_{τ_i} at each stopping time $\tau_i < T$, for the $\mathcal{F}$ -adapted strategy that satisfies (3.5.4)), we use a stochastic algorithm (see Appendix 3.9.3), and the quadratic shortfall losses are computed via Monte Carlo method. This makes computation much more time consuming, especially when we simulate the P&L at maturity of the $\mathcal{F}$ -adapted

	$\mathbb{E}^{\mathbb{P}} \left((H_T^{Total} - V_T)_+ \right)$		
n	$\mathcal{F}$ -adapted shortfall hedging	Optimal $\mathcal{F}^I$ -adapted shortfall hedging	Mean-variance hedging
50	39.68	44.22	56.27
100	57.77	63.81	76.79
300	88.56	112.25	134.49

Table 3.4: Expected shortfall for ATM call payoff with respect to n with $tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times tp_x \times \mathbb{E}^{\tilde{\mathbb{Q}}^f} (e^{-rT} (S_T - K_{call})_+)$

strategy, since at each jump time $\tau_i < T$, we have to perform a stochastic root finding to obtain the optimal value of λ_{τ_i} . Moreover, the smaller the ${}_T p_x$ and the larger the portfolio size n , the more jumps there will be and, therefore, the higher the computation time. This is why, unlike the two other hedging strategies, we decide to only compare the quadratic shortfall strategies for high initial survival probabilities ${}_T p_x > 0.9$ and relative low portfolio sizes $n < 50$. Let us now compare the quadratic shortfall hedging strategies with the mean-variance hedging strategy. To this end, we consider the same setting as before. Table 3.5 compares the expected quadratic shortfall with respect to the survival probability ${}_T p_x$ for ATM call for fixed $n = 20$ and Table 3.6 shows the loss with respect to the portfolio size for fixed ${}_T p_x = 0.95$. We observe that, in terms of quadratic shortfall loss, the quadratic shortfall strategies outperform the benchmark strategy. As for the two other hedging strategies, we conclude that it is therefore in the insurer's interest to implement the optimal strategies instead of the benchmark strategy if it wants to minimize the expected quadratic shortfall loss. Once gain, we also notice a gain in the quadratic loss when some actuarial information is incorporated into the hedging strategy, even if this gain decreases as ${}_T p_x$ increases. Similarly, as the size of the portfolio increases, the gap between the optimal $\mathcal{F}^f$ -adapted strategy and the $\mathcal{F}$ -adapted strategy increases. Thus, the larger the size of the portfolio, the greater the insurer's interest in using the actuarial information to build its quadratic shortfall hedging strategy.

3.6.4 Comparison of hedging strategies

Finally, we compare the different hedging strategies discussed in this chapter using different metrics. To better analyze the quadratic shortfall hedging strategy, we can split the expected quadratic shortfall loss in terms of the quadratic mean and variance of the loss. Table 3.7 compares the different strategies using several metrics for fixed portfolio size $n = 25$ and survival probability ${}_T p_x = 0.95$. Firstly, we clearly observe that even if the optimal quantile hedging strategy maximizes the probability of successful hedge, this strategy does not control the size of losses at all, and by following this strategy, the insurer is exposed to infrequent but very large losses, as revealed by comparing the expected shortfall and quadratic shortfall against other strategies. Next, the optimal shortfall strategy is effectively the one that minimizes the expected shortfall, but we can see that the variance of this shortfall can be significant. At last, we observe that, as expected, the quadratic hedging strategy appears to be the strategy leading to the best low mean-variance trade-off for the shortfall loss.

To get more insights about the distribution of the P&Ls generated by the hedging strategies, we can also analyze other metrics such as quantiles, kurtosis and skewness of the distributions. These metrics are provided at Table 3.8. First, we observe that quantile and shortfall hedging strategies lead to more asymmetric P&L distributions (left-skewed for the quantile strategy and right-skewed for the shortfall

	$\mathbb{E}^{\mathbb{P}} \left(\frac{1}{2} (H_T^{Total} - V_T)^2 \right)_+$			
τp_x	$\mathcal{F}$ -adapt. quad. shortfall hedging	Opt. $\mathcal{F}^f$ -adapt. quad. shortfall hedging	Mean-variance hedging	
0.99	49.12	55.12	371.22	
0.95	851.62	1121.69	4067.61	
0.9	2102.21	2643.55	8583.47	

Table 3.5: Comparison of the expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with $n = 20$, $T = 10$ and $\bar{V}_0 = n \times \tau p_x \times \mathbb{E}^{\bar{\mathbb{Q}}^f} (e^{-rT} (S_T - K_{call})_+)$

	$\mathbb{E}^{\mathbb{P}} \left(\frac{1}{2} (H_T^{Total} - V_T)^2 \right)_+$			
n	$\mathcal{F}$ -adapt. quad. shortfall hedging	Opt. $\mathcal{F}^f$ -adapt. quad. shortfall hedging	Mean-variance hedging	
50	3212.51	3619.01		11366.80
25	1284.86	1419.28		5182.42
10	327.53	372.24		1698.41

Table 3.6: Comparison of the expected quadratic shortfall for the optimal quadratic shortfall hedging strategy and the benchmark strategy for ATM call payoff with $tp_x = 0.95$, $T = 10$ and $\bar{V}_0 = n \times tp_x \times \mathbb{E}^{\mathbb{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

Hedging strategy	$\mathbb{P}(\pi_T \geq 0)$	$\mathbb{E}^{\mathbb{P}}\left((- \pi_T)_+\right)$	$V^P\left((- \pi_T)_+\right)$	$\mathbb{E}^{\mathbb{P}}\left((- \pi_T)_+^2\right)$
$\mathcal{F}$ -adapted quantile	0.9886	177.01	2 914 157.74	2 945 487.82
$\mathcal{F}$ -adapted quad. shortfall	0.5643	28.88	1735.32	2569.72
$\mathcal{F}$ -adapted shortfall	0.6465	27.86	3303.55	4079.72
Mean-variance	0.5286	39.25	8519.55	10 064.84

Table 3.7: Comparison of the hedging strategies for ATM call payoff with $\pi_T := V_T - H_T^{Total}$, $_{Tp_x} = 0.95$, $T = 10$, $n = 25$ and $\bar{V}_0 = n \times _{Tp_x} \times \mathbb{E}^{\hat{Q}^f}(e^{-rT}(S_T - K_{call})_+)$

strategies) with fatter tails than the mean-variance hedging strategy. The P&L's distribution associated the quantile strategy has left fat tail indicating, once again, that although this strategy is the one that maximizes the probability of successful hedge, it leads to a risk of very large losses in the rare event of default. Regarding shortfall and quadratic shortfall hedging strategies, we observe that the P&L's distributions appear very similar which seems logical since they share the common goal of limiting the size of losses in the event of default and therefore explains the more pronounced right-skewed nature of their distribution.

3.7 Concluding remarks

This chapter investigates the efficient hedging of equity-linked life insurance portfolio. To this end, we consider loss-averse insurers and a general financial-actuarial framework. As the insurers are assumed to be loss-averse, their main goal is to minimize either the events leading to losses or the size of these losses. Therefore, we focus on optimal quantile and shortfall hedging strategies and posed the hedging problems as stochastic control problems. Based on the theory of super-replication and on a duality approach, we are able to characterize the optimal strategies with (semi)-explicit expressions that depend on a dual process. We show that these strategies depend not only on financial and actuarial risks but also on the size of the portfolio as well as the risk aversion of the insurers. When restricting ourselves to purely financial hedging strategies and complete financial markets, the optimal dual process is known and the forms of the strategies are fully explicit. Furthermore, in order to obtain hedging strategies in the combined market that could be easily implemented, we also construct $\mathcal{F}$ -adapted strategies by enlarging the optimal $\mathcal{F}^f$ -adapted strategies with actuarial information. Numerical results show satisfying results since, depending on the risk-aversion of the insurers, the strategies outperform the mean-variance hedging strategy. Hence, it demonstrates the relevance of using the right strategy for a given risk-aversion and portfolio size. We also observe that the $\mathcal{F}$ -adapted strategies, by taking into account actuarial filtration, outperform the optimal $\mathcal{F}^f$ -adapted strategies strategies, illustrating that there is a non-negligible gain to be obtained from incorporating actuarial information into hedging strategies.

3.8 Proofs

3.8.1 Theorem 3.3.1

Proof. 1) Let us denote a solution to the Problem (3.3.2) by $A_i^* \in \mathcal{F}_T$, $i = 1, \dots, n$ such that $A_{i+1}^* \subseteq A_i^*$. For any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we first

Hedging strategy	$Q_{1\%}(\pi_T)$	$Q_{25\%}(\pi_T)$	$Q_{50\%}(\pi_T)$	$Q_{75\%}(\pi_T)$	$Q_{99\%}(\pi_T)$	Kurtosis	Skewness
$\mathcal{F}$ -adapted quantile	-11799.19	0	0	52.31	485.92	118.43	-10.37
$\mathcal{F}$ -adapted quad. shortfall	-133.04	-54.51	0	15.88	595.99	46.01	4.96
$\mathcal{F}$ -adapted shortfall	-249.66	-32.11	0	12.53	580.02	30.83	3.97
Mean-variance	-464.35	-31.64	0	8.11	558.54	22.01	1.70

Table 3.8: Comparison of quantiles, kurtosis and skewness of P&Ls distributions for ATM call payoff with $\pi_T := V_T - H_T^{Total}$, $tp_x = 0.95$, $T = 10$, $n = 25$ and $\bar{V}_0 = n \times tp_x \times \mathbb{E}^{\bar{\mathbb{Q}}^f}(e^{-rT}(S_T - K_{call})_+)$, where $Q_{\alpha\%}(\cdot)$ is the quantile at level $\alpha\%$ under the probability measure P . $Kurtosis = \mathbb{E}^P \left[\left(\frac{\pi_T - \mathbb{E}^P(\pi_T)}{V^P(\pi_T)^{1/2}} \right)^4 \right]$, measure of the tailedness of a distribution. $Skewness = \mathbb{E}^P \left[\left(\frac{\pi_T - \mathbb{E}^P(\pi_T)}{V^P(\pi_T)^{1/2}} \right)^3 \right]$, measure of the asymmetry of a distribution.

observe that the associated success set is given by

$$\mathbb{1}_{\{V_T \geq H_T^{Total}\}} = \mathbb{1}_{N_T=0} + \sum_{i=1}^n \mathbb{1}_{\{V_T \geq i H_T\}} \mathbb{1}_{\{N_T=i\}}.$$

Moreover, we have that, $\mathbb{P}$ -almost surely,

$$V_T \geq H_T \times \sum_{i=1}^n \mathbb{1}_{\{V_T \geq i H_T\}}.$$

Therefore, for $A_i = \left\{ V_T \geq i H_T \right\} \in \mathcal{F}_T$ such that $A_{i+1} \subseteq A_i$, we observe that,

$$\bar{V}_0 \geq v_0 \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{V_T}{B_T} \right) \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i} \right),$$

for any $\mathbb{Q} \in \mathcal{Q}$, since non-negative local $\mathbb{Q}$ -local martingale $\left(\frac{V_t}{B_t} \right)_{t \in [0, T]}$ is super martingale, this implies

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{V_T \geq H_T^{Total}\}} \right) &= \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{\{N_T=i\}} + \mathbb{1}_{\{N_T=0\}} \right) \\ &\leq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}} \right) + p_0 \end{aligned} \quad (3.8.1)$$

and thus

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) \leq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}} \right) + p_0. \quad (3.8.2)$$

2) Let now consider the hedging strategy $\xi^{QH} \in \mathcal{A}$ associated the (super) hedging for the modified payoff $H_T^{QH} = H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$ such that $\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T^{QH}}{B_T} \right) \leq \bar{V}_0$.

In this case, if we denote $\bar{V}_T$ the portfolio with initial value $\bar{V}_0$ that evolves according to the hedging strategy ξ^{QH} , we have that

$$\bar{V}_T \geq H_T^{QH},$$

P -almost surely.

The corresponding maximal success set satisfies

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}} \right) &\geq \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{H_T^{QH} \geq H_T^{Total}\}} \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=0\}} + \mathbb{1}_{\{N_T \geq 1\}} \mathbb{1}_{\{H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*} \geq N_T H_T\}} \right) \\ &= p_0 + \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times \mathbb{1}_{\{H_T \mathbb{1}_{A_i^*} \geq H_T\}} \right) \end{aligned}$$

For $i = 1, \dots, n$, let define $\tilde{A}_i \in \mathcal{F}_T$ such that

$$\tilde{A}_i := \left\{ H_T \mathbb{1}_{A_i^*} \geq H_T \right\} \supset A_i^*$$

and $\tilde{A}_{i+1} \subseteq \tilde{A}_i$.

Thus, we deduce that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}} \right) &\geq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{\tilde{A}_i} \times \mathbb{1}_{\{N_T=i\}} \right) + p_0 \\ &\geq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}} \right) + p_0. \end{aligned} \tag{3.8.3}$$

Combining (3.8.1), (3.8.2) and (3.8.3), we have that

$$\begin{aligned} \max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) &= \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*} \times \mathbb{1}_{\{N_T=i\}} \right) + p_0 \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{\bar{V}_T \geq H_T^{Total}\}} \right), \end{aligned}$$

and the optimality of the strategy $(\bar{V}_0, \xi^{QH})$ in Problem (3.2.3) follows. Finally, as ξ^{QH} is the hedging strategy associated to the super-replication of a positive payoff, we have that $\xi^{QH} \in \mathcal{A}$. $\square$

3.8.2 Proposition 3.3.2

Proof. To prove this proposition, we use a duality approach as proposed, for example, in Kramkov and Schachermayer (1999); Föllmer and Leukert (2000). Let us define $\mathcal{C}$ by

$$\mathcal{C} := \left\{ (A_1, \dots, A_n) \in \mathcal{F}_T : A_{i+1} \subseteq A_i, \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i} \right) \leq \bar{V}_0 \right\}.$$

We observe that, for any $(A_1, \dots, A_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$\mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i} \right) + p_0 \leq p_0 + \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) + \lambda \bar{V}_0. \quad (3.8.4)$$

Moreover, for all $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, by taking the sets $(\tilde{A}_1(\lambda, Z_T), \dots, \tilde{A}_n(\lambda, Z_T))$ such that

$$\tilde{A}_n(\lambda, Z_T) = \left\{ \mathbb{1}_{N_T=n} \geq \lambda Z_T \frac{H_T}{B_T} \right\}, \quad (3.8.5)$$

and

$$\tilde{A}_i(\lambda, Z_T) = \left\{ \mathbb{1}_{N_T=i} + \sum_{j=i+1}^n \mathbb{1}_{\tilde{A}_j(\lambda, Z_T)} \mathbb{1}_{N_T=j} \geq \lambda Z_T \frac{H_T}{B_T} \left(1 + \sum_{j=i+1}^n \mathbb{1}_{\tilde{A}_j(\lambda, Z_T)} \right) \right\}, \quad (3.8.6)$$

we observe that, for all $(A_1, \dots, A_n) \in \mathcal{F}_T$,

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) \\ & \leq \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\tilde{A}_1(\lambda, Z_T)} \left(\mathbb{1}_{N_T=1} - \lambda Z_T \frac{H_T}{B_T} \right. \right. \\ & \quad \left. \left. + \mathbb{1}_{\tilde{A}_2(\lambda, Z_T)} \left(\mathbb{1}_{N_T=2} - \lambda Z_T \frac{H_T}{B_T} + \dots + \mathbb{1}_{\tilde{A}_n(\lambda, Z_T)} \left(\mathbb{1}_{N_T=n} - \lambda Z_T \frac{H_T}{B_T} \right) \dots \right) \right) \right). \end{aligned}$$

Thus by taking $A_1^*(\lambda, Z_T), \dots, A_n^*(\lambda, Z_T) \in \mathcal{F}_T$ such that

$$A_i^*(\lambda, Z_T) := \bigcap_{j=1}^i \tilde{A}_j(\lambda, Z_T), \quad (3.8.7)$$

we obtain that $A_{i+1}^*(\lambda, Z_T) \subseteq A_i^*(\lambda, Z_T)$ and for any $(A_1, \dots, A_n) \in \mathcal{F}_T$, $A_{i+1} \subseteq A_i$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) & \leq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \mathbb{1}_{N_T=i} \right) \\ & \quad - \lambda \mathbb{E}^{\mathbb{P}} \left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \right). \end{aligned}$$

In this case, using inequality (3.8.4), we obtain that

$$\begin{aligned}
& \max_{(A_1, \dots, A_n) \in \mathcal{C}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i} \right) \\
& \leq \inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} \max_{(A_1, \dots, A_n) \in \mathcal{F}_T, A_{i+1} \subseteq A_i} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) \\
& \quad + \lambda \bar{V}_0 \\
& = \inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) + \lambda \bar{V}_0.
\end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see (Föllmer and Leukert, 2000, Theorem 7.2.)), following similar arguments as in (Kramkov and Schachermayer, 1999, Theorem 3.2.) and (Föllmer and Leukert, 2000, Theorem 7.3.), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\begin{aligned}
& \inf_{\lambda \in \mathbb{R}_+, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda, Z_T)} \times \left(\mathbb{1}_{N_T=i} - \lambda Z_T \frac{H_T}{B_T} \right) \right) + \lambda \bar{V}_0 \\
& = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \times \left(\mathbb{1}_{N_T=i} - \lambda^* Z_T^* \frac{H_T}{B_T} \right) \right) + \lambda^* \bar{V}_0,
\end{aligned}$$

and⁸

$$\bar{V}_0 = \mathbb{E}^{\mathbb{P}} \left(Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \right) = \sup_{Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \right). \quad (3.8.8)$$

The previous equality (3.8.8) implies that

$$\bar{V}_0 \geq \sup_{Q \in \mathcal{Q}} E^Q \left(\frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \right),$$

and as $\left(A_i^*(\lambda^*, Z_T^*) \right)_{i=1, \dots, n} \in \mathcal{F}_T$ such that $A_{i+1}^*(\lambda^*, Z_T^*) \subseteq A_i^*(\lambda^*, Z_T^*)$, we have that $\left(A_i^*(\lambda^*, Z_T^*) \right)_{i=1, \dots, n} \in \mathcal{C}$. We finally obtain that there is no duality gap since

$$\max_{(A_1, \dots, A_n) \in \mathcal{C}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i} \right) + p_0 = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i^*(\lambda^*, Z_T^*)} \times \mathbb{1}_{N_T=i} \right) + p_0.$$

⁸The constraint $\bar{V}_0 \leq E^{\bar{Q}^f} \left(n \times \frac{H_T}{B_T} \right)$ ensures that $\lambda^* \in \mathbb{R}_+$.

Therefore, $(A_1^*, \dots, A_n^*)$ defined as, for $i = 1, \dots, n$,

$$A_i^* := A_i^*(\lambda^*, Z_T^*),$$

is solution of (3.3.2). Furthermore, using together (3.8.5), (3.8.6) and (3.8.7), we can easily deduce that, for $i = 1, \dots, n$,

$$A_i^* = \left\{ \alpha_i \geq \lambda^* \times Z_T^* \frac{H_T}{B_T} \right\},$$

where $(\alpha_1, \dots, \alpha_n)$ are defined by (3.3.6) and (3.3.7). In addition, the optimal hedging strategy ξ^{QH} is the admissible strategy associated to the super-replication hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. $\square$

3.8.3 Theorem 3.4.1

Proof. 1) Let us denote a solution to the Problem (3.4.2) by $\phi_i^* \in \mathcal{R}$, $i = 1, \dots, n$ such that $\phi_{i+1}^* \leq \phi_i^*$. Firstly, for any admissible hedging strategy with $v_0 \leq \bar{V}_0$, we can observe that

$$\begin{aligned} (H_T^{Total} - V_T)_+ &= H_T N_T - \min \left\{ V_T, N_T H_T \right\} \\ &= \mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \min \left\{ V_T, i H_T \right\} \mathbb{1}_{\{N_T = i\}}. \end{aligned}$$

For $i = 1, \dots, n$, we postulate that

$$\Phi_i H_T := \min \left\{ V_T, i H_T \right\},$$

with

$$\Phi_i \in \left\{ \phi : \Omega \rightarrow [0, i], \mathcal{F}_T - \text{measurable} \right\}$$

such that P -almost surely

$$\Phi_{i+1} \geq \Phi_i,$$

and then

$$(H_T^{Total} - V_T)_+ = \mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \Phi_i H_T \mathbb{1}_{\{N_T = i\}}.$$

Moreover, for $i = 1, \dots, n$, we observe that, for ϕ_i defined by (3.4.1),

$$(\Phi_i - \Phi_{i-1}) H_T = \phi_i H_T$$

such that

$$\phi_i \in \left\{ \phi : \Omega \rightarrow [0, 1], \mathcal{F}_T - \text{measurable} \right\}$$

and P -almost surely

$$\phi_{i+1} \leq \phi_i.$$

Thus, for $i = 1, \dots, n$, we can rewrite Φ_i as

$$\Phi_i = \sum_{j=1}^i \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}},$$

with $\phi_0 := 1$. In this case, we deduce that

$$\begin{aligned} L\left((H_T^{Total} - V_T)_+\right) &= L\left(\mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T - \sum_{i=1}^n \Phi_i H_T \mathbb{1}_{\{N_T=i\}}\right) \\ &= L\left(\mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T (1 - \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}})\right). \end{aligned}$$

Moreover, as we have that

$$\min \left\{ V_T, H_T N_T \right\} = \mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}},$$

then

$$V_T \geq \sum_{i=1}^n H_T \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}},$$

and we observe that

$$\bar{V}_0 \geq v_0 \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{V_T}{B_T} \right) \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \right),$$

for any $\mathbb{Q} \in \mathcal{Q}$, since non-negative local $\mathbb{Q}$ -local martingale $\left(\frac{V_t}{B_t} \right)_{t \in [0, T]}$ is super martingale. This implies that

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) \\ &\geq \mathbb{E}^{\mathbb{P}} \left(L \left(\mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T (1 - \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}) \right) \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right), \end{aligned}$$

and thus

$$\begin{aligned} & \min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) \\ & \geq \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right). \end{aligned} \quad (3.8.9)$$

2) Let now consider the hedging strategy $\xi^{SH} \in \mathcal{A}$ associated to the (super) replicating hedging of the modified payoff $H_T^{SH} = H_T \times \sum_{i=1}^n \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}$ such that $\sup_{Q \in \mathcal{Q}} \mathbb{E}^Q \left(\frac{H_T^{SH}}{B_T} \right) \leq \bar{V}_0$. In this case, if we denote $\bar{V}_T$ the portfolio with initial value $\bar{V}_0$ that evolves according to the hedging strategy ξ^{SH} , we have that

$$\bar{V}_T \geq H_T^{SH},$$

P -almost surely.

The expected shortfall loss of this hedging strategy satisfies

$$\mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - \bar{V}_T)_+ \right) \right) \quad (3.8.10)$$

$$\leq \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - H_T^{SH})_+ \right) \right) \quad (3.8.11)$$

$$\begin{aligned} & = \mathbb{E}^\mathbb{P} \left(L \left(\mathbb{1}_{\{N_T \geq 1\}} \sum_{i=1}^{N_T} H_T (1 - \phi_i^* \mathbb{1}_{\{\phi_{i-1}^*=1\}}) \right) \right) \\ & = \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right) \end{aligned}$$

Combining (3.8.9) and (3.8.11), we have that

$$\begin{aligned} & \min_{v_0, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - V_T)_+ \right) \right) \\ & = \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n \mathbb{1}_{\{N_T=i\}} \times L \left(\sum_{j=1}^i H_T (1 - \phi_j^* \mathbb{1}_{\{\phi_{j-1}^*=1\}}) \right) \right) \\ & = \mathbb{E}^\mathbb{P} \left(L \left((H_T^{Total} - \bar{V}_T)_+ \right) \right), \end{aligned}$$

and the optimality of the strategy $(\bar{V}_0, \xi^{SH})$ in Problem (3.2.4) follows. Finally, as ξ^{SH} is the hedging strategy associated to the super-replication of a positive payoff, we conclude that $\xi^{SH} \in \mathcal{A}$ $\square$

3.8.4 Proposition 3.4.2

Proof. To prove this proposition, we use once again a duality approach. Let us define $\mathcal{C}$ by

$$\mathcal{C} := \left\{ \phi_1, \dots, \phi_n \in \mathcal{R} : \phi_{i+1} \leq \phi_i, \phi_0 = 1, \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) \leq \bar{V}_0 \right\}.$$

We observe that, for $L(x) = x$, for any $(\phi_1, \dots, \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) \\ & \geq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times i H_T \right) + \mathbb{E}^{\mathbb{P}} \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) - \lambda \bar{V}_0. \end{aligned} \quad (3.8.12)$$

Moreover, for $\phi_1, \dots, \phi_n \in \mathcal{R}$ with $\phi_0 = 1$ and $\phi_{i+1} \leq \phi_i$, for any $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, we have that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) \\ & \geq \mathbb{E}^{\mathbb{P}} \left(H_T \left(\mathbb{1}_{A_1(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \sum_{i=1}^n \mathbb{1}_{N_T=i} \right. \right. \right. \\ & \quad \left. \left. \left. + \mathbb{1}_{A_2(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \sum_{i=2}^n \mathbb{1}_{N_T=i} + \dots + \mathbb{1}_{A_n(\lambda, Z_T)} \left(\lambda \frac{Z_T}{B_T} - \mathbb{1}_{N_T=n} \right) \dots \right) \right) \right) \right) \\ & = \mathbb{E}^{\mathbb{P}} \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right), \end{aligned}$$

with

$$A_n(\lambda, Z_T) = \left\{ \mathbb{1}_{N_T=n} \geq \lambda \frac{Z_T}{B_T} \right\}, \quad (3.8.13)$$

for $i = 1, \dots, n-1$,

$$A_i(\lambda, Z_T) = \left\{ \sum_{j=i}^n \mathbb{1}_{N_T=j} + \sum_{j=i+1}^n \mathbb{1}_{A_j(\lambda)} \left(\sum_{k=j}^n \mathbb{1}_{N_T=k} \right) \geq \lambda \left(1 + \sum_{j=i+1}^n \mathbb{1}_{A_j(\lambda)} \right) \frac{Z_T}{B_T} \right\}, \quad (3.8.14)$$

such that

$$A_i(\lambda, Z_T) \subseteq A_{i-1}(\lambda, Z_T),$$

and

$$\phi_i^*(\lambda, Z_T) = \prod_{j=1}^i \mathbb{1}_{A_j}(\lambda, Z_T) \quad (3.8.15)$$

Therefore, for $\lambda \in \mathbb{R}_+$, and $Z_T \in \mathcal{D}$ since $\phi_i^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$, we easily deduce that

$$\begin{aligned} \min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \phi_0=1} \mathbb{E}^\mathbb{P} \left(H_T \left(\sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}} \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) \right) \\ = \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right). \end{aligned}$$

Thus, we obtain the following inequality

$$\begin{aligned} \min_{(\phi_1, \dots, \phi_n) \in \mathcal{C}} \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) \\ \geq \mathbb{E}^\mathbb{P} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times i H_T \right) \\ + \sup_{\lambda \geq 0, Z_T \in \mathcal{D}} \left(\mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) - \lambda \bar{V}_0 \right). \end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see (Föllmer and Leukert, 2000, Theorem 7.2.)), following similar arguments as in (Kramkov and Schachermayer, 1999, Theorem 3.2.) and (Föllmer and Leukert, 2000, Theorem 7.3.), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\begin{aligned} \sup_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \left(\lambda \frac{Z_T}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) - \lambda \bar{V}_0 \\ = \mathbb{E}^\mathbb{P} \left(H_T \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \left(\lambda^* \frac{Z_T^*}{B_T} - \sum_{j=i}^n \mathbb{1}_{N_T=j} \right) \right) - \lambda^* \bar{V}_0, \end{aligned}$$

and⁹

$$\bar{V}_0 = \mathbb{E}^\mathbb{P} \left(Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right) = \sup_{Z_T \in \mathcal{D}} \mathbb{E}^\mathbb{P} \left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right). \quad (3.8.16)$$

⁹The constraint $\bar{V}_0 \leq E^{\tilde{\mathbb{Q}}^f} \left(n \times \frac{H_T}{B_T} \right)$ ensures that $\lambda^* \in \mathbb{R}_+$.

The previous equality (3.8.16) implies that

$$\bar{V}_0 \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \mathbb{1}_{\phi_i^*(\lambda^*, Z_T^*)} \right),$$

and as $\phi_i^*(\lambda^*, Z_T^*) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda^*, Z_T^*) \leq \phi_i^*(\lambda^*, Z_T^*)$, we have that

$$\left(\phi_i^*(\lambda^*, Z_T^*) \right)_{i=1, \dots, n} \in \mathcal{C}.$$

We finally obtain that there is no duality gap since

$$\begin{aligned} & \min_{(\phi_1, \dots, \phi_n) \in \mathcal{C}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right) \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda^*, Z_T^*)) \right) \right). \end{aligned}$$

Therefore, $(\phi_1^*, \dots, \phi_n^*)$ defined as, for $i = 1, \dots, n$,

$$\phi_i^* := \phi_i^*(\lambda^*, Z_T^*),$$

is solution of (3.4.2) for $L(x) = x$. Finally, using (3.8.13), (3.8.14) and (3.8.15), we can deduce that, for $i = 1, \dots, n$, $\phi_i^* = \mathbb{1}_{A_i^*}$ with A_i^* defined by (3.4.5) for $\lambda = \lambda^*$ and $Z_T = Z_T^*$. Therefore, we have that

$$\min_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{E}^{\mathbb{P}} \left((H_T^{Total} - V_T)_+ \right) = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \left(\sum_{j=1}^i (1 - \mathbb{1}_{A_j^*}) H_T \right) \right),$$

and the optimal hedging strategy ξ^{SH} is the admissible strategy associated to the super-hedging of the modified payoff $H_T \times \sum_{i=1}^n \mathbb{1}_{A_i^*}$. $\square$

3.8.5 Proposition 3.4.3

Proof. Once again, let us take $\mathcal{C}$ defined by

$$\mathcal{C} := \left\{ \phi_1, \dots, \phi_n \in \mathcal{R} : \phi_{i+1} \leq \phi_i, \phi_0 = 1, \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) \leq \bar{V}_0 \right\}.$$

We observe that, for $L(x) = \frac{1}{2}x^2$, for any $(\phi_1, \dots, \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, the following inequality holds

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 \right) \\ & \geq \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) - \lambda \bar{V}_0. \end{aligned} \quad (3.8.17)$$

Moreover, for $(\phi_1, \dots, \phi_n) \in \mathcal{C}$, $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, we have that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) \\ & \geq \min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 \right. \\ & \quad \left. + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i \right) \\ & = \min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i} \mathcal{L}(\phi_1, \dots, \phi_n; \lambda, Z_T). \end{aligned}$$

For any $\lambda \in \mathbb{R}_+$ and $Z_T \in \mathcal{D}$, since $\mathcal{L}$ is convex, using a first order condition, if there exists $\phi_1^*(\lambda, Z_T), \dots, \phi_n^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$ that satisfy

$$\frac{\partial \mathcal{L}(\phi_1^*(\lambda, Z_T), \dots, \phi_n^*(\lambda, Z_T); \lambda, Z_T)}{\partial \phi_i} = 0, \quad i = 1, \dots, n, \quad (3.8.18)$$

then

$$\min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \phi_0=1} \mathcal{L}(\phi_1, \dots, \phi_n; \lambda, Z_T) = \mathcal{L}(\phi_1^*(\lambda, Z_T), \dots, \phi_n^*(\lambda, Z_T); \lambda, Z_T).$$

We observe that, for $\phi_1^*(\lambda, Z_T), \dots, \phi_n^*(\lambda, Z_T) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda, Z_T) \leq \phi_i^*(\lambda, Z_T)$, the condition (3.8.18) is equivalent, for $i = 1, \dots, n$, to

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left(H_T \mathbb{1}_{\{\phi_{i-1}^*(\lambda, Z_T)=1\}} \mathbb{1}_{\{H_T \phi_i^*(\lambda, Z_T) > 0\}} \right. \\ & \quad \left. \left(\sum_{j=i}^n \mathbb{1}_{N_T=j} \times \left(H_T \sum_{k=1}^j (1 - \phi_k^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{k-1}^*(\lambda, Z_T)=1\}}) \right) - \lambda \frac{Z_T}{B_T} \right) \right) = 0. \end{aligned} \quad (3.8.19)$$

We can check that $\phi_1^*(\lambda, Z_T), \dots, \phi_n^*(\lambda, Z_T)$ defined such as, on $\{H_T > 0\}$,

$$\phi_n^*(\lambda, Z_T) := \mathbb{1}_{\left\{ \frac{\lambda Z_T}{B_T H_T} \leq \mathbb{1}_{\{N_T=n\}} \right\}} \left(1 - \frac{\lambda}{\mathbb{1}_{\{N_T=n\}} B_T H_T} Z_T \right),$$

and for $i \in [1, \dots, n-1]$, on $\{H_T > 0\}$

$$\phi_i^*(\lambda, Z_T) := 1 - \frac{1}{\sum_{j=i}^n \mathbb{1}_{N_T=j}} \left(\frac{\lambda Z_T}{B_T H_T} - \sum_{j=i+1}^n \left((1 - \phi_j^*(\lambda, Z_T)) \times \sum_{k=j}^n \mathbb{1}_{N_T=k} \right) \right) \wedge 1, \quad (3.8.20)$$

solve the system of n -equations (3.8.19) with $\phi_0^* = 1$. From the inequality (3.8.17), we thus have that

$$\begin{aligned} & \min_{(\phi_1, \dots, \phi_n) \in \mathcal{C}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 \right) \\ & \geq \sup_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}}) \right)^2 \right. \\ & \quad \left. + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \right) - \lambda \bar{V}_0. \end{aligned}$$

Furthermore, since $\mathcal{D}$ is convex and closed under almost sure convergence (see (Föllmer and Leukert, 2000, Theorem 7.2.)), following similar arguments as in (Kramkov and Schachermayer, 1999, Theorem 3.2.) and (Föllmer and Leukert, 2000, Theorem 7.3.), we know that there exists a solution $(\lambda^*, Z_T^*) \in \mathbb{R}_+ \times \mathcal{D}$ such that

$$\begin{aligned} & \sup_{\lambda \geq 0, Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda, Z_T) \mathbb{1}_{\{\phi_{j-1}^*(\lambda, Z_T)=1\}}) \right)^2 \right. \\ & \quad \left. + \lambda Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda, Z_T) \right) - \lambda \bar{V}_0 \\ & = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j^*(\lambda^*, Z_T^*) \mathbb{1}_{\{\phi_{j-1}^*(\lambda^*, Z_T^*)=1\}}) \right)^2 \right. \\ & \quad \left. + \lambda^* Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right) - \lambda^* \bar{V}_0 \end{aligned}$$

and

$$\bar{V}_0 = \mathbb{E}^{\mathbb{P}} \left(Z_T^* \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right) = \sup_{Z_T \in \mathcal{D}} \mathbb{E}^{\mathbb{P}} \left(Z_T \frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right). \quad (3.8.21)$$

The previous equality (3.8.21) implies that

$$\bar{V}_0 \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i^*(\lambda^*, Z_T^*) \right),$$

and as $\phi_1^*(\lambda^*, Z_T^*), \dots, \phi_n^*(\lambda^*, Z_T^*) \in \mathcal{R}$ such that $\phi_{i+1}^*(\lambda^*, Z_T^*) \leq \phi_i^*(\lambda^*, Z_T^*)$, we have that $\left(\phi_1^*(\lambda^*, Z_T^*), \dots, \phi_n^*(\lambda^*, Z_T^*)\right) \in \mathcal{C}$. We finally obtain that there is no duality gap and

$$\begin{aligned} & \min_{(\phi_1, \dots, \phi_n) \in \mathcal{C}} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}}) \right)^2 \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times \frac{1}{2} \left(\sum_{j=1}^i H_T (1 - \phi_{j-1}^*(\lambda^*, Z_T^*) \mathbb{1}_{\{\phi_{j-1}^*(\lambda^*, Z_T^*)=1\}}) \right)^2 \right). \end{aligned}$$

Therefore, $(\phi_1^*, \dots, \phi_n^*)$ defined as, for $i = 1, \dots, n$,

$$\phi_i^* := \phi_i^*(\lambda^*, Z_T^*),$$

is solution of (3.4.2) for $L(x) = \frac{1}{2}x^2$. Finally, by using the recursive relation (3.8.20) and the fact that, for $i = 1, \dots, n-1$:

- if $\phi_i^* = 1$, then $\phi_{i-1}^* = 1$,
- if $0 < \phi_i^* < 1$ then $\phi_{i-1}^* = 1$ and $\phi_j^* = 0$ for $j = i+1, \dots, n$,
- if $\phi_i^* = 0$ then $\phi_{i-1}^* < 1$,

we can deduce that for $i = 1, \dots, n-1$, ϕ_i^* satisfy (3.4.7) for $\lambda = \lambda^*$ and $Z_T = Z_T^*$. $\square$

3.8.6 Proposition 3.5.1

Proof. First, it is easy to see that $(\tilde{\xi}_t)_{t \in [0, T]} \in \mathcal{A}$, and then we obviously observe that

$$\mathbb{P}(\tilde{V}_T \geq H_T^{Total}) \leq \max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A} \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}).$$

Let us now prove that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \bar{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) \leq \mathbb{P}(\tilde{V}_T \geq H_T^{Total}).$$

Using similar arguments that used in Section 3.3.2 and as done for example in Föllmer and Leukert (1999); Föllmer and Leukert (2000), we can deduce that, for any $t \in [0, T]$ and $n \in \mathbb{N}_0$ such that $N_t = n$,

$$\max_{V_t \in \mathbb{R}_+, \xi \in \mathcal{A}_t^f \text{ s.t. } V_t \leq \bar{V}_t} \mathbb{P}(V_T \geq H_T^{Total} | \mathcal{F}_t) = \mathbb{P}(V_T^{f,t} \geq H_T^{Total} | \mathcal{F}_t), \quad (3.8.22)$$

where $\mathcal{A}_t^f$ is the set of all admissible financial hedging strategies valued from t to T in $\mathbb{R}$, and $(V_s^{f,t})_{s \in [t, T]}$, such that $V_t^{f,t} = \bar{V}_t$, is the hedging portfolio associated

to the perfect hedging of the modified payoff

$$H_T \times \sum_{j=1}^n \mathbb{1}_{A_j^*(\lambda_t, \tilde{Z}_T, p^t)},$$

where $p^t = (p_1^t, \dots, p_n^t)$ and λ_t satisfies

$$V_t^{f,t} \wedge \mathbb{E}^{\tilde{Q}_f}(H_T \times n | \mathcal{F}_t) = \mathbb{E}^{\tilde{Q}_f} \left(\frac{B_t}{B_T} H_T \times \sum_{j=1}^n \mathbb{1}_{A_j^*(\lambda_t, \tilde{Z}_T, p^t)} | \mathcal{F}_t \right).$$

We can now prove the statement using a recursive approach. If $n = 1$, then we observe that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \tilde{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{P}(\tilde{V}_T \geq H_T^{Total}).$$

If $n = 2$, we have that

$$\begin{aligned} \mathbb{P}(V_T^f \geq H_T^{Total}) &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{N_T=2\right\}} \mathbb{1}_{\left\{V_T^f \geq 2H_T\right\}} \right) + \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{N_T < 2\right\}} \mathbb{1}_{\left\{V_T^f \geq N_T H_T\right\}} \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{N_T=2\right\}} \mathbb{1}_{\left\{\tilde{V}_T \geq 2H_T\right\}} \right) \\ &\quad + \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{N_T < 2\right\}} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{V_T^f \geq N_T H_T\right\}} | \mathcal{F}_{\tau_1} \right) \right). \end{aligned}$$

Using (3.8.22) and the definition of the hedging strategy $\tilde{\xi}$ in Proposition 3.5.1 with $n = 2$, we observe that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{V_T^f \geq N_T H_T\right\}} | \mathcal{F}_{\tau_1} \right) &\leq \max_{V_{\tau_1} \in \mathbb{R}_+, \xi \in \mathcal{A}_{\tau_1}^f \text{ s.t. } V_{\tau_1} \leq \tilde{V}_{\tau_1}} \mathbb{P}(V_T \geq H_T^{Total} | \mathcal{F}_{\tau_1}) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{\tilde{V}_T \geq N_T H_T\right\}} | \mathcal{F}_{\tau_1} \right), \text{ on } \{N_T < 2\}, \end{aligned}$$

and thus

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \tilde{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{P}(V_T^f \geq H_T^{Total}) \leq \mathbb{P}(\tilde{V}_T \geq H_T^{Total}).$$

Let us assume now that the inequality (3.5.2) holds for $n = k \in \mathbb{N}_0$ and prove that it is still holds for $n = k + 1$. For $n = k$, we have that

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \tilde{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) \leq \mathbb{P}(\tilde{V}_T \geq H_T^{Total}).$$

For any $t \in [0, T]$ such that $N_t = k$, let us consider, for $i = 1, \dots, k$, $\tau_i^{k,t}$ as the $\mathcal{F}^a$ -stopping time associated to the i^{th} jump in the process $(N_s)_{s \in [t, T]}$ with the convention $\tau_0^{k,t} := t$. Then, as done before, we can also prove that, for any $t \in [0, T]$ such that $N_t = k$,

$$\max_{V_t \in \mathbb{R}_+, \xi \in \mathcal{A}_t^f \text{ s.t. } V_t \leq \bar{V}_t} \mathbb{P}(V_T \geq H_T^{Total} | \mathcal{F}_t) \leq \mathbb{P}(\tilde{V}_T^t \geq H_T^{Total} | \mathcal{F}_t), \quad (3.8.23)$$

where $(\tilde{V}_s^t)_{s \in [t, T]}$ such that $\tilde{V}_t^t = \bar{V}_t$ is the hedging portfolio associated to $(\tilde{\xi}_s^t)_{s \in [t, T]}$ defined, for $i = 0, \dots, k-1$, with $\tau_i^{k,t} < \tau^{k,t}$, $(\tilde{\xi}_s^t)_{s \in [\tau_i^{k,t}, \tau_{i+1}^{k,t} \wedge \tau^{k,t})}$ is the perfect hedging strategy of

$$H_T \times \sum_{j=1}^{N_{\tau_i^{k,t}}} \mathbb{1}_{A_j^*(\lambda_{\tau_i^{k,t}}, \bar{Z}_T, p^{\tau_i^{k,t}})},$$

where $\lambda_{\tau_i^{k,t}}$ satisfied

$$\tilde{V}_{\tau_i^{k,t}}^t = \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_{\tau_i^{k,t}}}{B_T} H_T \times \sum_{j=1}^{N_{\tau_i^{k,t}}} \mathbb{1}_{A_j^*(\lambda_{\tau_i^{k,t}}, \bar{Z}_T, p^{\tau_i^{k,t}})} | \mathcal{F}_{\tau_i^{k,t}} \right).$$

and $(\tilde{\xi}_s^t)_{s \in [\tau^{k,t}, T]}$ is the perfect hedging strategy of

$$H_T \times N_{\tau^{k,t}},$$

where

$$\tau^{k,t} := \inf \left\{ \tau \in \{\tau_0^{k,t} \wedge T, \dots, \tau_n^{k,t} \wedge T\} : \tilde{V}_\tau^t 1_{\tau < T} \geq 1_{\tau < T} \mathbb{E}^{\tilde{\mathbb{Q}}^f} \left(\frac{B_\tau}{B_T} H_T \times N_T | \mathcal{F}_\tau \right) \right\}.$$

Therefore, for $n = k+1$, we have that

$$\begin{aligned} \mathbb{P}(V_T^f \geq H_T^{Total}) &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{ N_T = k+1 \right\}} \mathbb{1}_{\left\{ V_T^f \geq (k+1)H_T \right\}} \right) \\ &\quad + \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{ N_T \leq k \right\}} \mathbb{1}_{\left\{ V_T^f \geq N_T H_T \right\}} \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{ N_T = k+1 \right\}} \mathbb{1}_{\left\{ \tilde{V}_T \geq (k+1)H_T \right\}} \right) \\ &\quad + \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{ N_T \leq k \right\}} \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\left\{ V_T^f \geq N_T H_T \right\}} | \mathcal{F}_{\tau_1} \right) \right), \end{aligned}$$

and using (3.8.23) with $t = \tau_1 = \tau_0^{k, \tau_1}$, as well as the definition of the strategy $\tilde{\xi}$ in Proposition 3.5.1 with $n = k + 1$, we deduce that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\left\{ V_T^f \geq N_T H_T \right\}} \middle| \mathcal{F}_{\tau_1} \right) &\leq \max_{V_{\tau_1} \in \mathbb{R}_+, \xi \in \mathcal{A}_{\tau_1}^f \text{ s.t. } V_{\tau_1} \leq \tilde{V}_{\tau_1}} \mathbb{P}(V_T \geq H_T^{Total} | \mathcal{F}_{\tau_1}) \\ &\leq \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\left\{ \tilde{V}_T \geq N_T H_T \right\}} \middle| \mathcal{F}_{\tau_1} \right), \text{ on } \{N_T \leq k\}, \end{aligned}$$

and finally,

$$\max_{v_0 \in \mathbb{R}_+, \xi \in \mathcal{A}^f \text{ s.t. } v_0 \leq \tilde{V}_0} \mathbb{P}(V_T \geq H_T^{Total}) = \mathbb{P}(V_T^f \geq H_T^{Total}) \leq \mathbb{P}(\tilde{V}_T \geq H_T^{Total}).$$

□

3.9 Appendix

3.9.1 Equivalent Martingale Measures in combined market

Proposition 3.9.1. *For $i = 1, \dots, n$, suppose that*

$$M_t^i = (1 - N_t^i) - \int_0^t N_u^i \kappa_{x+u} du,$$

$$M_t := \sum_{i=1}^n M_t^i.$$

and $(L_t^h)_{t \geq 0}$ such that

$$\begin{aligned} dL_t^h &= L_{t-}^h h_t dM_t, \\ L_0^h &= 1, \end{aligned} \tag{3.9.1}$$

for a given $\mathcal{F}_t^a$ -adapted process $(h_t)_{t \geq 0}$ such that for all $t \geq 0$, $h_t > -1$ almost surely. Then, for all $\mathcal{F}_t^a$ -adapted processes $(h_t)_{t \geq 0}$ and $\mathbb{Q}^f \in \mathcal{Q}^f$ there exists a probability measure denoted by $\mathbb{Q}^h$ such that

$$\frac{d\mathbb{Q}^h}{d\mathbb{P}} \Big|_t = L_t^h Z_t^f, \tag{3.9.2}$$

with $Z_t^f = \frac{d\mathbb{Q}^f}{d\mathbb{P}} \Big|_t$ and

$$\mathbb{Q}^h \in \mathcal{Q}.$$

Proof. Let $(h_t)_{t \geq 0}$ be a $\mathcal{F}_t^a$ -adapted process such that for all $t \geq 0$, $h_t > -1$ almost surely and $\mathbb{Q}^f \in \mathcal{Q}^f$ such that $Z_t^f = \frac{d\mathbb{Q}^f}{d\mathbb{P}}|_t$. For $i = 1, \dots, n$, using the Doob-Meyer decomposition of $(N_t^i)_{t \geq 0}$, we observe that $(M_t^i)_{t \geq 0}$ is a $\mathcal{F}_t^a$ -martingale and, as $M_t = \sum_{i=1}^n M_t^i$, M_t is also a $\mathcal{F}_t^a$ -martingale. Since, $(L_t^h)_{t \geq 0}$ with $L_0^h = 1$ is a strictly positive $\mathcal{F}_t^a$ -martingale and $(Z_t^f)_{t \geq 0}$ with $Z_0^f = 1$ is a strictly positive $\mathcal{F}^f$ -martingale, we have that, for $0 \leq s \leq t$,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}(L_t^h Z_t^f | \mathcal{F}_s) &= \mathbb{E}^{\mathbb{P}^a}(L_t^h | \mathcal{F}_s^a) E^{P^f}(Z_t^f | \mathcal{F}_s^f) \\ &= L_s^h Z_s^f, \end{aligned}$$

and in particular,

$$\mathbb{E}^{\mathbb{P}}(L_t^h Z_t^f) = 1.$$

Hence the change of measure (3.9.2) is well defined and as $(L_t^h Z_t^f)_{t \geq 0}$ is a strictly positive process, the probability measure $\mathbb{Q}^h$ is equivalent to $\mathbb{P}$. Moreover, we observe that, for $0 \leq s \leq t$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^h} \left(\frac{S_t}{B_t} | \mathcal{F}_s \right) &= \frac{\mathbb{E}^{\mathbb{P}} \left(L_t^h Z_t^f \frac{S_t}{B_t} | \mathcal{F}_s \right)}{L_s^h Z_s^f} \\ &= \frac{\mathbb{E}^{\mathbb{P}^a}(L_t^h | \mathcal{F}_s^a)}{L_s^h} \frac{\mathbb{E}^{\mathbb{P}^f} \left(Z_t^f \frac{S_t}{B_t} | \mathcal{F}_s^f \right)}{Z_s^f} \\ &= \mathbb{E}^{\mathbb{Q}^f} \left(\frac{S_t}{B_t} | \mathcal{F}_s^f \right) \\ &= \frac{S_s}{B_s}. \end{aligned}$$

We conclude that $\left(\frac{S_t}{B_t} \right)_{t \geq 0}$ is a martingale under $\mathbb{Q}^h$ and then, as $\mathbb{Q}^h \sim \mathbb{P}$, $\mathbb{Q}^h \in \mathcal{Q}$. $\square$

Proposition 3.9.2. *For $\epsilon > -1$, for all $\mathcal{F}_t^a$ -adapted processes $(h_t)_{t \geq 0}$ such that $\forall t \geq 0$, $h_t > -1$ almost surely and for all financial martingale measures $\mathbb{Q}^f \in \mathcal{Q}^f$, consider the probability measure $\mathbb{Q}_{h^\epsilon}$ such that*

$$\frac{d\mathbb{Q}_{h^\epsilon}}{d\mathbb{P}}|_t := L_t^{h^\epsilon} Z_t^f,$$

with $Z_t^f := \frac{d\mathbb{Q}^f}{d\mathbb{P}}|_t$, $(L_t^{h^\epsilon})_{t \geq 0}$ defined by (3.9.1) and $(h_t^\epsilon)_{t \geq 0} := ((1 + \epsilon)h_t + \epsilon)_{t \geq 0}$. Then,

$$\mathbb{Q}_{h^\epsilon} \in \mathcal{Q},$$

and

$$\lim_{\epsilon \rightarrow -1} \mathbb{E}^{\mathbb{Q}_{h^\epsilon}} \left(f(S_T, N_T) \right) = \mathbb{E}^{\mathbb{Q}^f} \left(f(S_T, n) \right).$$

Proof. Let $\varepsilon > -1$, consider a given $\mathcal{F}_t^a$ -adapted process $(h_t)_{t \geq 0}$ such that for all $t \geq 0$, $h_t > -1$ almost surely and a financial martingale measure $\mathbb{Q}^f \in \mathcal{Q}^f$. As $(h_t^\varepsilon)_{t \geq 0} = ((1 + \varepsilon)h_t + \varepsilon)_{t \geq 0}$ is $\mathcal{F}_t^a$ -adapted such that for all $t \geq 0$, $h_t^\varepsilon > -1$, using Proposition 3.9.1, we have that

$$\mathbb{Q}_{h^\varepsilon} \in \mathcal{Q}.$$

We observe that, by independence between actuarial and financial risks,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_{h^\varepsilon}} \left(f(S_T, N_T) \right) &= \sum_{i=0}^n \mathbb{E}^{\mathbb{Q}_{h^\varepsilon}} \left(f(S_T, i) \right) \times \mathbb{Q}_{h^\varepsilon}(N_T = i) \\ &= \sum_{i=0}^n \mathbb{E}^{\mathbb{Q}^f} \left(f(S_T, i) \right) \times \mathbb{Q}_{h^\varepsilon}(N_T = i). \end{aligned}$$

Using Ito's lemma, we obtain that

$$L_t^{h^\varepsilon} = \prod_{i=1}^n (1 + h_{\tau_i}^\varepsilon)^{1-N_t^i} \exp \left(- \int_0^t h_u^\varepsilon \kappa_{x+u} N_u du \right).$$

Therefore, under $\mathbb{Q}_{h^\varepsilon}$, for $i = 0, \dots, n$, we have

$$\begin{aligned} \mathbb{Q}_{h^\varepsilon}(N_T = i) &= \mathbb{E}^{\mathbb{Q}_{h^\varepsilon}} (\mathbb{1}_{\{N_T=i\}}) \\ &= \mathbb{E}^{\mathbb{P}} (L_T^{h^\varepsilon} Z_T \mathbb{1}_{\{N_T=i\}}) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=i\}} \prod_{j=1}^n (1 + h_{\tau_j}^\varepsilon)^{1-N_T^j} \exp \left(- \int_0^T h_u^\varepsilon \kappa_{x+u} N_u du \right) \right). \end{aligned}$$

In particular, for $i = n$, we obtain

$$\begin{aligned} \mathbb{Q}_{h^\varepsilon}(N_T = n) &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=n\}} \prod_{j=1}^n (1 + h_{\tau_j}^\varepsilon)^{1-N_T^j} \exp \left(- \int_0^T h_u^\varepsilon \kappa_{x+u} N_u du \right) \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=n\}} \exp \left(- n \int_0^T \kappa_{x+u} h_u^\varepsilon du \right) \right) \\ &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=n\}} \exp \left(- n \int_0^T \kappa_{x+u} ((1 + \varepsilon)h_u + \varepsilon) du \right) \right). \end{aligned}$$

By monotone convergence, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow -1} \mathbb{Q}_{h^\varepsilon}(N_T = n) &= \mathbb{E}^{\mathbb{P}} \left(\mathbb{1}_{\{N_T=n\}} \exp \left(n \int_0^T \kappa_{x+u} du \right) \right) \\ &= 1, \end{aligned}$$

and thus for $i = 0, \dots, n-1$,

$$\lim_{\varepsilon \rightarrow -1} \mathbb{Q}_{h^\varepsilon}(N_T = i) = 0,$$

We easily conclude that

$$\begin{aligned} \lim_{\varepsilon \rightarrow -1} \mathbb{E}^{\mathbb{Q}_{h^\varepsilon}} \left(f(S_T, N_T) \right) &= \sum_{i=0}^n \mathbb{E}^{\mathbb{Q}_f} \left(f(S_T, i) \right) \times \lim_{\varepsilon \rightarrow -1} \mathbb{Q}_{h^\varepsilon}(N_T = i) \\ &= \mathbb{E}^{\mathbb{Q}_f} \left(f(S_T, n) \right). \end{aligned}$$

□

3.9.2 Existence results for solutions to static optimization problems

Proposition 3.9.3. *There exists a solution $\tilde{A} = (\tilde{A}_1, \dots, \tilde{A}_n) \in \mathcal{F}_T$ such that $\tilde{A}_{i+1} \subseteq \tilde{A}_i$, $i = 1, \dots, n-1$ to the problem*

$$\max_{A_1, \dots, A_n \in \mathcal{F}_T, A_{i+1} \subseteq A_i} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \mathbb{1}_{A_i} \times \mathbb{1}_{N_T=i} \right) + p_0, \quad (3.9.3)$$

under the constraint

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(\frac{H_T}{B_T} \times \sum_{i=1}^n \mathbb{1}_{A_i} \right) \leq \bar{V}_0. \quad (3.9.4)$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in Föllmer and Leukert (2000). Let define $\mathcal{R}^n := \left\{ (\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}) \mid A_i \in \mathcal{F}_T, A_{i+1} \subseteq A_i, i = 1, \dots, n-1 \right\}$. In this case, we can rewrite the problem as

$$\max_{g \in \mathcal{R}^n} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n g^i \times \mathbb{1}_{N_T=i} \right) + p_0,$$

under the constraint

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}^Q \left(\frac{H_T}{B_T} \times \sum_{i=1}^n g^i \right) \leq \bar{V}_0. \quad (3.9.5)$$

Let $\mathcal{R}_0^n$ the elements of $\mathcal{R}^n$ satisfying (3.9.5) and consider $(g_m)_{m \geq 1}$ a maximizing sequence for (3.9.3) in $\mathcal{R}_0^n$. Using Lemma A.1.1. in Delbaen and Schachermayer (1994), we can choose $\tilde{g}_m \in \text{conv}(g_m, g_{m+1}, \dots)$ such that $(\tilde{g}_m)_{m \geq 1}$ converges $\mathbb{P}$ -almost surely to $\tilde{g} \in \mathcal{R}^n$. Therefore, we have that, by dominated convergence theorem¹⁰,

$$\mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \tilde{g}_m^i \times \mathbb{1}_{N_T=i} \right) + p_0 \rightarrow \max_{g \in \mathcal{R}^n} \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n g^i \times \mathbb{1}_{N_T=i} \right) + p_0 = \mathbb{E}^{\mathbb{P}} \left(\sum_{i=1}^n \tilde{g}^i \times \mathbb{1}_{N_T=i} \right) + p_0$$

¹⁰Since $\forall m \geq 1, \forall i = 1, \dots, n, 0 \leq \tilde{g}_m^i \leq 1$

as m tends to infinity. Moreover, by Fatou's lemma, for all $\mathbb{Q} \in \mathcal{Q}$, we observe that

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \tilde{g}^i\right) \leq \liminf \mathbb{E}^{\mathbb{Q}}\left(\frac{H_T}{B_T} \times \sum_{i=1}^n \tilde{g}_m^i\right) \leq \bar{V}_0,$$

then

$$\tilde{g} \in \mathcal{R}_0^n$$

and the existence of a solution to the problem (3.9.3) with constraint (3.9.4) follows. $\square$

Proposition 3.9.4. *There exists a solution $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ such that $\tilde{\phi}_i \in \mathcal{R}$ and $\tilde{\phi}_i \geq \tilde{\phi}_{i-1}$, $i = 1, \dots, n$, to the problem*

$$\min_{\phi_1, \dots, \phi_n \in \mathcal{R}, \phi_{i+1} \leq \phi_i, \phi_0=1} \mathbb{E}^{\mathbb{P}}\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times L\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right), \quad (3.9.6)$$

with

$$\mathcal{R} = \left\{ \phi : \Omega \rightarrow [0, 1], \mathcal{F}_T - \text{measurable} \right\}$$

under the constraint

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \quad (3.9.7)$$

Proof. The proof is just an adaptation of the proof of Proposition 3.1. in Föllmer and Leukert (2000). Let define $\mathcal{R}^n := \left\{ \phi = (\phi_1, \dots, \phi_n) \mid \phi_i \in \mathcal{R}, \phi_i \leq \phi_{i-1}, i = 1, \dots, n \right\}$. In this case, the problem is equivalent to

$$\min_{\phi \in \mathcal{R}^n} \mathbb{E}^{\mathbb{P}}\left(\sum_{i=1}^n \mathbb{1}_{N_T=i} \times L\left(\sum_{j=1}^i H_T(1 - \phi_j \mathbb{1}_{\{\phi_{j-1}=1\}})\right)\right),$$

under the constraint

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left(\frac{H_T}{B_T} \sum_{i=1}^n \phi_i \mathbb{1}_{\{\phi_{i-1}=1\}}\right) \leq \bar{V}_0. \quad (3.9.8)$$

Let $\mathcal{R}_0^n$ the elements of $\mathcal{R}^n$ satisfying (3.9.8) and consider $(g_m)_{m \geq 1}$ a minimizing sequence for (3.9.6) in $\mathcal{R}_0^n$. By using similar arguments than in the proof of Proposition 3.9.3, we can easily deduce the existence of a solution to the problem (3.9.6) with constraint (3.9.7). $\square$

3.9.3 Stochastic algorithm for root finding

Assume that we want to find the root for a function of the form $h(\lambda) = \mathbb{E}^P(H(\lambda, Z))$ with $\lambda \in \mathbb{R}$ and Z a random variable defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. To this end, we can determine a stochastic algorithm that will converge (from any starting point) to λ^* . The stochastic algorithm can be seen as an extension of the Newton-Raphson algorithm. Let $(Z_n)_{n \geq 1}$ simulations of Z and $(\gamma_n)_{n \geq 1}$ a deterministic sequence such that $\sum_{n \geq 1} \gamma_n = +\infty$, $\sum_{n \geq 1} \gamma_n^2 < +\infty$. We consider a stochastic sequence $(\lambda_n)_{n \geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ and propose the following algorithm for root finding of $h(\lambda) = E^P(H(\lambda, Z))$:

$$\begin{aligned} \lambda_{n+1} &= \lambda_n - \gamma_{n+1} H(\lambda_n, Z_{n+1}), \\ \lambda_0 &\in L^2(P). \end{aligned} \tag{3.9.9}$$

Proposition 3.9.5. (*Robbins-Monro algorithm*, see Corollary 6.1 in Pagès (2018))
Assume that the mean function $h(\lambda) = E^P(H(\lambda, Z))$ is continuous and satisfies

$$\forall \lambda \in \mathbb{R}, \lambda \neq \lambda^*, (\lambda - \lambda^*)h(\lambda) > 0$$

Suppose furthermore that $\lambda_0 \in L^2(\mathbb{P})$ and that H satisfies

$$\forall \lambda \in \mathbb{R}, \|H(\lambda, Z)\|_{2,P} \leq C(1 + |\lambda|).$$

Assume that $(\gamma_n)_{n \geq 1}$ satisfies

$$\sum \gamma_n = +\infty, \sum \gamma_n^2 < +\infty.$$

Then $(\lambda_n)_{n \geq 1}$ defined in (3.9.9) is such that

$$\lambda_n \xrightarrow{a.s.} \lambda^*,$$

and in every $L^p(\mathbb{P})$, $p \in (0, 2)$.

Part III

Path-signatures for risk-management

CHAPTER 4

Signature approach for pricing and hedging path-dependent options with frictions

In the chapter, we introduce a novel signature approach for pricing and hedging path-dependent options with instantaneous and permanent market impact under a mean-quadratic variation criterion. Leveraging the expressive power of signatures, we recast an inherently nonlinear and non-Markovian stochastic control problem into a tractable form, yielding hedging strategies in (possibly infinite) linear feedback form in the time-augmented signature of the control variables, with coefficients characterized by non-standard infinite-dimensional Riccati equations on the extended tensor algebra. Numerical experiments demonstrate the effectiveness of these signature-based strategies for pricing and hedging general path-dependent payoffs in the presence of frictions. In particular, market impact naturally smooths optimal trading strategies, making low-truncated signature approximations highly accurate and robust in frictional markets, contrary to the frictionless case.

4.1 Introduction

As already highlighted in this thesis, pricing and hedging derivatives is a central problem in quantitative finance. In classical complete models, perfect replication is achieved by trading continuously in the underlying asset, assuming frictionless markets. In practice, however, large trades affect prices. Two types of market impact are typically distinguished: temporary impact, which affects the execution price but not future mid-prices, and permanent impact, which shifts the future mid-price due to supply/demand imbalances. Accounting for these frictions transforms option hedging into a stochastic control problem, where the objective boils down to tracking certain option's sensitivities (such as the delta) while balancing replication

accuracy, trading costs, and market risk. The control problem usually falls outside the tractable linear-quadratic framework.

The majority of existing literature on pricing and hedging with frictions focuses on European payoffs. Solutions are typically characterized via Hamilton-Jacobi-Bellman (HJB) equations as in Almgren and Li (2016); Becherer and Bilarev (2018); Cetin et al. (2010); Ekren and Nadtochiy (2022); Rogers and Singh (2010); Bouchard et al. (2017); Said (2019); Guéant and Pu (2017); Lions and Lasry (2007); Loeper (2018). While these frameworks provide rigorous insights for European options, they generally do not extend easily to path-dependent derivatives, such as Asian, barrier, or look-back options, whose payoffs depend on the entire trajectory of the underlying. Yet these are among the most common contracts in practice, especially in equity, FX, and commodity markets, making the gap between theory and applications particularly striking. A notable exception is the case of Accelerated Share Repurchase contracts with an Asian-type component studied by Guéant et al. (2015); Jaimungal et al. (2017), where, after approximations and dimension reduction, the problem can be made Markovian in three dimensions. Another exception is the work of Bank et al. (2017), which considers general non-Markovian payoffs, with an instantaneous market impact component but without incorporating transient or permanent impact. In that setting, the problem falls back into the linear-quadratic class (with stochastic coefficients) and can be solved explicitly.

We develop a novel signature approach for hedging path-dependent options under both temporary and permanent impact within a mean-quadratic variation framework as in Almgren and Li (2016). Almgren and Li (2016) studied the problem of hedging European options, and relate the optimal hedging strategies to an HJB equation. They obtain explicit solutions for European options with a constant Gamma. We extend their setting to path-dependent options, namely with signature payoffs. Because the problem is inherently non-Markovian, the classical HJB approach is no longer applicable, which is the first major challenge. To overcome this, we lift the hedging problem into the space of path-signatures, which restores Markovianity. In principle, this leads to an infinite-dimensional HJB equation, but such an equation is numerically intractable, especially since permanent impact makes the control problem highly nonlinear and far beyond the linear-quadratic class.

The key insight is that, for polynomial European payoffs, the HJB equation decouples into a system of non-standard infinite-dimensional Riccati ordinary differential equations (ODE), thanks to the algebraic structure of polynomials, see Section 4.2.2. Crucially, this algebraic structure is inherited by signatures, since signatures extend polynomials to path space. We leverage this to transform the nonlinear, path-dependent control problem into an infinite-dimensional yet numerically tractable Markovian framework. We relate the solution to an infinite-dimensional Riccati ODE system on the extended tensor algebra with an optimal hedging strategy given in linear feedback form in the time-extended signature of

the control variables, see Theorem 4.5.7. In short, the time-extended signatures of the control variables serve not only as the state variables of the control problem but also effectively linearize it leading to Riccati-type structures familiar from classical linear-quadratic control.

Our contribution falls naturally within the recent line of work on rough paths and signature methods, which provide hierarchical, finite-dimensional representations for encoding path-dependence. In mathematical finance, path-signatures, introduced by Chen (1957); Chen (1977), were recently used to solve different problems such as:

- Pricing and hedging: Abi Jaber and Gérard (2025a); Abi Jaber and Gérard (2025b); Bayer et al. (2025b); Cuchiero et al. (2025a); Lyons et al. (2020),
- Optimal stopping: Bayer et al. (2023); Bayer et al. (2025a),
- Portfolio allocation and trading: Cuchiero and Möller (2025); Fütter et al. (2023); Kalsi et al. (2020).

Moreover, Bank et al. (2025) have recently used signatures to tackle stochastic control problems. They considered two signature-based classes of controls to approximate the class of admissible controls, showed that those classes are dense in the set of admissible controls, and proposed numerical methods to solve the problem. Compared to Bank et al. (2025), we propose a distinct approach. Rather than restricting the optimization to a possibly smaller subset from the outset, we consider a general set of admissible progressively measurable controls. Using the standard martingale optimality principle, we then derive a verification result expressed in terms of an infinite-dimensional system of Riccati equations, which yields the optimal control in feedback form as a (possibly infinite) linear combination of time-extended signature elements, with the time-dependent coefficients given by the solutions of the Riccati equations. In the literature, for instance in Lyons et al. (2020); Bank et al. (2025), only time-independent coefficients have been considered so far when parameterizing the solutions in terms of time-extended signatures. This time-independence implies an approximation by controls that are analytic in the time variable, a property that the true optimal control generally does not satisfy. This provides one explanation for why our approach achieves greater accuracy and numerical stability.

For the first time in the context of stochastic optimal control, we establish this connection between non-Markovian control problems and infinite-dimensional Riccati equations on the extended tensor algebra. We note, however, that similar infinite-dimensional systems of Riccati equations have previously appeared in the literature for solving uncontrolled mathematical finance problems using signatures, to compute the characteristic function associated with signature volatility models and signature SDEs, see Abi Jaber and Gérard (2025a); Abi Jaber et al. (2024b); Cuchiero and Möller (2025). A general existence and uniqueness theory for such

Riccati equations and the associated convergence of the power expansions appears to be intricate and remains an open problem, with certain partial results available for the Riccati equations in Abi Jaber et al. (2024b); Cuchiero and Möller (2025) and for the power series in Abi Jaber et al. (2024a); Arous (1989).

We show that our signature approach offers a flexible, systematic, and computationally feasible route to manage path-dependent payoffs in markets with frictions. In particular, our contributions can be summarized as follows:

- **Perfect hedging of signature payoffs in a frictionless market:** We derive an explicit solution for the fair price and the perfect hedging strategy for signature payoffs using the Fawcett (2003) formula, as stated in Theorem 4.4.1.
- **Existence and uniqueness for the stochastic control problem with frictions:** We use a variational approach to derive the existence and uniqueness of a solution to the stochastic control problem of hedging signature payoffs with market frictions in a mean-variation quadratic framework. More precisely, Theorem 4.5.4 states that under a monotonicity condition (4.5.16) on the Gâteaux derivative of the gain functional, there exists a unique maximizer of the problem, which is also the unique solution of the first-order condition (4.5.17). We also provide two examples (see Proposition 4.5.5 and Proposition 4.5.6) where the monotonicity condition is fulfilled.
- **Verification result for the stochastic control problem with frictions:** By combining the Itô formula for signatures (Theorem 4.3.2) and a square completion approach, we derive a general verification theorem using the martingale optimality principle, see Theorem 4.5.7. Under some assumptions, including the existence of a suitable solution to an infinite-dimensional system of Riccati equations given by (4.5.20) and the convergence of the related infinite series, this theorem explicitly characterizes the optimal solution of the stochastic control problem in a feedback form as a linear combination of time-extended signature elements with time-dependent coefficients, see (4.5.22). Moreover, we provide two concrete examples for which we can establish the existence of an explicit solution to the system of Riccati equations:
 - (i) Proposition 4.5.9 considers the case without permanent impact and allows us to recover the result of (Bank et al., 2017, Theorem 3.1), but applied to signature payoffs.
 - (ii) Proposition 4.5.10 considers European quadratic payoff, and we retrieve the result of (Almgren and Li, 2016, Eq. (25)).
- **Numerical illustrations:** In Section 4.6, we illustrate numerically our signature-based hedging strategies, given by Theorem 4.5.7 by solving the system of infinite-dimensional Riccati equations (4.5.20), demonstrating that our approach can be effectively implemented even in highly path-dependent

settings. First, we consider exact signature payoffs, such as European or Asian quadratic options, using sanity checks against explicit solutions to validate both the assumptions of Theorem 4.5.7 and our numerical implementation. Second, we consider more general path-dependent payoffs, which we approximate by signature payoffs relying on the universal approximation theorem. In the frictional market, we show that the signature-based strategy computed via the Riccati system for the approximate signature payoffs provides a sufficiently accurate approximation of the true optimal strategy, demonstrating a striking improvement compared to signature-based strategies in the frictionless setting. This improvement can be attributed to the fact that market impact naturally smooths the optimal trading strategies, making polynomial approximations more accurate and efficient. We also evaluate the impact of the permanent market impact component on both the fair price and the hedging strategies of several path-dependent options.

The chapter is outlined as follows. In Section 4.2, we introduce the Bachelier framework with both temporary and permanent market impact. Then, Section 4.3 recalls key results on path-signatures. In Section 4.4, we first address the hedging of a signature payoff in frictionless market and we deduce the perfect hedging strategy using the Fawcett (2003) formula. Section 4.5 is devoted to the stochastic control problem of hedging a signature payoff under market impact within a mean-quadratic variation framework, where we establish an existence and uniqueness result (Section 4.5.1) and provide a verification theorem (Section 4.5.2). Finally, Section 4.6 presents numerical illustrations.

4.2 Bachelier model with temporary and permanent impact

We fix a finite horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ that satisfies the usual conditions. We consider a financial market with a tradable asset $(S_t)_{t \in [0, T]}$ with dynamic

$$dS_t = \mu dt + \sigma dW_t, \quad t \leq T, \quad (4.2.1)$$

with $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$ and W a standard Brownian motion. In this financial market, a trader shorts an option on S that she aims to hedge. We assume that by trading the asset, the trader incurs both temporary and permanent market impacts which affect her profit and loss (P&L). More precisely, if $(X_t^\theta)_{t \in [0, T]}$ denotes the number of shares held by the trader through time defined as

$$X_t^\theta = X_0 + \int_0^t \theta_s ds, \quad (4.2.2)$$

with $X_0 \in \mathbb{R}^1$, and $(\theta_t)_{t \in [0, T]}$ the trading speed, the price of the asset, taking account of permanent impact, is given by

$$P_t^\theta = S_t + \nu(X_t^\theta - X_0), \quad t \leq T, \quad (4.2.3)$$

with $\nu \geq 0$, the permanent market impact parameter. Moreover, as the trader incurs transaction costs (temporary impact), the effective traded asset price is

$$\tilde{P}_t^\theta = P_t^\theta + \eta\theta_t, \quad t \leq T, \quad (4.2.4)$$

with $\eta \geq 0$, the temporary price impact parameter.

Remark 4.1. *In this chapter, we restrict ourselves to a simple diffusion model for the risky asset S , in contrast with the more general frameworks considered in Chapters 1 and 2. This modelling choice is motivated by the fact that the resulting stochastic control problem associated with hedging under market impact is already highly non-trivial, even in the simplest Bachelier setting, as already emphasized by Almgren and Li (2016). Extensions to more complex dynamics, such as signature-SDEs or the Volterra-type models studied in earlier chapters, could in principle be considered. However, in such settings, the characterisation of the solution via an infinite-dimensional Riccati equation, obtained through the optimal martingality principle, would take place in a functional space of the form $T((\mathbb{R}^d))$ with $d \gg 1$. This high-dimensional structure makes any direct numerical discretisation scheme computationally intractable in practice. We briefly discuss these extensions and related perspectives in the concluding Chapter.* ■

4.2.1 The Almgren and Li setup

Before addressing the problem of hedging signature payoffs with market impacts, and to get more insight about our motivation to use a signature approach in this context, let us start from the Markovian framework of Almgren and Li (2016). For this, we fix a European option whose frictionless Bachelier delta at time t is $\Delta(t, S_t)$ and we consider the problem of hedging this option with temporary and permanent market impact. This essentially boils down to tracking the Bachelier delta $\Delta(t, P_t^\theta)$ evaluated with the impacted price P^θ given by (4.2.3). Almgren and Li (2016) consider a mean-quadratic variation criteria on the P&L $^\theta$ generated by the trading strategy θ of the form²

$$\mathbb{E} \left(\text{P\&L}_T^\theta - \frac{\lambda}{2} [\text{P\&L}^\theta, \text{P\&L}^\theta]_T \right),$$

where $\lambda > 0$ is the risk aversion of the trader. The aim is to maximize the criterion over the trading strategies θ . Using a dynamic programming approach, they characterized the solution in terms of the solution to a Hamilton-Jacobi-Bellman (HJB)

¹As explained in Guéant and Pu (2017), X_0 corresponds to the number of shares in the hedging portfolio at inception. In illiquid markets, the client that buys the option may provide an initial number of shares (see Section 4.5). Therefore, we could consider $X_0 = 0$ or the case where X_0 is equal to the Bachelier Δ of the option sold, depending on the initial agreement with the client.

²The mean-quadratic variation hedging problem is introduced in detail in Section 4.5

equation. More precisely, they showed that the value function $J(t, p, x)$ depends, at time $t < T$, on (P_t^θ, X_t^θ) and satisfies a nonlinear HJB equation of the following form³⁴:

$$\begin{aligned} J_t &= -\frac{\lambda}{2}\sigma^2 [x - \Delta(t, p)]^2 - \frac{1}{2}\sigma^2 J_{pp} + \frac{1}{4\eta} [\nu(x - \Delta(t, p)) - J_x - \nu J_p]^2 \\ &\quad + \mu [(x - \Delta(t, p)) - J_p], \\ J(T, p, x) &= 0. \end{aligned} \tag{4.2.5}$$

When the Bachelier Gamma $\Gamma(t, p)$ is constant, i.e. $\Gamma(t, p) = \Gamma \in \mathbb{R}$, for all $(t, p) \in [0, T] \times \mathbb{R}^+$, and $\mu = 0$, they deduced that $J(t, p, x)$ has an explicit form. This explicit solution follows from the fact that, when Γ is constant, the problem becomes linear-quadratic: a constant gamma implies that the delta is linear in p , and therefore that the payoff is quadratic. A natural idea is thus to extend this observation to more general polynomial payoffs.

4.2.2 Key observation for Markovian polynomial payoffs

In fact, we can go further and obtain a semi-explicit characterization of the solution in terms of infinite dimensional Riccati ordinary differential equations when we consider polynomial payoffs of the form

$$H_T^\theta = \sum_{i=0}^M \alpha_i (P_T^\theta)^i, \quad (\alpha_i)_{i=1, \dots, M} \in \mathbb{R}^M, \quad M \in \mathbb{N}. \tag{4.2.6}$$

Then, there exists $(\delta_i(t))_{i=0, \dots, M} \in \mathbb{R}^M$ such that

$$\Delta(t, p) = \sum_{i=0}^{M-1} \delta_i(t) p^i.$$

Making an ansatz on the value function of the form of a power series in p and x :

$$J(t, p, x) = \sum_{i,j=0}^{+\infty} \psi_t^{i,j} p^i x^j, \tag{4.2.7}$$

³The HJB equation differs slightly from that in Almgren and Li (2016) because, in this chapter, we consider a short position to be hedged and X^θ as a state variable instead of $Y^\theta := X^\theta - \Delta(t, P^\theta)$. Moreover, we also assume no overnight risk and $\mu \neq 0$.

⁴Almgren and Li (2016) do not provide existence or well-posedness results for the HJB equation (4.2.5). For a rigorous analysis of related HJB equations, see Ekren and Naddtochiy (2022).

and plugging this ansatz in (4.2.5), the HJB equation decouples into a system of infinite-dimensional Riccati equations given, for $i, j \in \mathbb{N}$, by

$$\begin{aligned} \dot{\psi}_t^{i,j} &= -\frac{1}{2}\sigma^2\psi_t^{i+2,j} - \frac{\lambda}{2}\sigma^2(\beta_t * \beta_t)^{i,j} + \frac{1}{4\eta}(\tilde{\psi}_t * \tilde{\psi}_t)^{i,j} \\ &\quad + \mu\left[\beta_t^{i,j} - (i+1)\psi_t^{i+1,j}\right], \\ \psi_T^{i,j} &= 0, \end{aligned} \tag{4.2.8}$$

with, $\beta_t = (\beta_t^{i,j})_{i,j \geq 0}$ and $\tilde{\psi}_t = (\tilde{\psi}_t^{i,j})_{i,j \geq 0}$ given by

$$\begin{aligned} \beta_t^{i,j} &:= \mathbb{1}_{\{i=0\} \cap \{j=1\}} - \mathbb{1}_{\{i \leq M-1\} \cap \{j=0\}} \delta_i(t), \\ \tilde{\psi}_t^{i,j} &:= \nu\beta_t^{i,j} - \nu\psi_t^{i+1,j}(i+1) - \psi_t^{i,j+1}(j+1), \end{aligned}$$

and the $2D$ Cauchy product defined, for $\mathbf{a} = (a^{i,j})_{i,j \geq 0}$, $\mathbf{b} = (b^{i,j})_{i,j \geq 0}$, by

$$(\mathbf{a} * \mathbf{b})^{i,j} := \sum_{m=0}^i \sum_{n=0}^j a^{mn} b^{i-m,j-n}.$$

Therefore, in the Markovian case (European payoffs), by restricting ourselves to polynomial payoffs, we can semi-explicitly express the optimal solution as a power series in the state variables with deterministic coefficients ψ coming from an infinite-dimensional system of Riccati equations. Since path-signatures play a similar role to polynomials on the path space, by considering signatures payoffs, we can naturally expect to be able to derive the same kind of characterization for the control problem in the non-Markovian case, and this is precisely what is established in our main result, Theorem 4.5.7.

4.3 Reminder on signatures

In this section, we provide a reminder about the path-signature theory. We also refer to the first sections in Lyons et al. (2020); Bayer et al. (2023); Cuchiero and Möller (2025); Abi Jaber et al. (2024a).

4.3.1 Tensor algebra

Let $d \in \mathbb{N}$ and denote by $\otimes$ the tensor product over $\mathbb{R}^d$, e.g. $(x \otimes y \otimes z)_{ijk} = x_i y_j z_k$, for $i, j, k = 1, \dots, d$, for $x, y, z \in \mathbb{R}^d$. For $n \geq 1$, we denote by $(\mathbb{R}^d)^{\otimes n}$ the space of tensors of order n and by $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. In this chapter, we consider path-signatures that are mathematical objects that live on the extended tensor algebra space $T((\mathbb{R}^d))$ over $\mathbb{R}^d$, that is the space of (infinite) sequences of tensors defined by

$$T((\mathbb{R}^d)) := \{\ell = (\ell^n)_{n=0}^\infty : \ell^n \in (\mathbb{R}^d)^{\otimes n}\}.$$

For $M \geq 0$, we define the truncated tensor algebra $T^M(\mathbb{R}^d)$ as the space of sequences of tensors of order at most M defined by

$$T^M(\mathbb{R}^d) := \{ \ell \in T((\mathbb{R}^d)) : \ell^n = 0, \text{ for all } n > M \},$$

and the tensor algebra $T(\mathbb{R}^d)$ as the space of all finite sequences of tensors defined by

$$T(\mathbb{R}^d) := \bigcup_{M \in \mathbb{N}} T^M(\mathbb{R}^d).$$

For $\ell = (\ell^n)_{n \in \mathbb{N}}$, $\mathbf{p} = (\mathbf{p}^n)_{n \in \mathbb{N}} \in T((\mathbb{R}^d))$ and $\lambda \in \mathbb{R}$, we define the following operations:

$$\begin{aligned} \ell + \mathbf{p} &:= (\ell^n + \mathbf{p}^n)_{n \in \mathbb{N}} \\ \ell \otimes \mathbf{p} &:= \left(\sum_{k=0}^n \ell^k \otimes \mathbf{p}^{n-k} \right)_{n \in \mathbb{N}} \\ \lambda \ell &:= (\lambda \ell^n)_{n \in \mathbb{N}}. \end{aligned}$$

These operations induce analogous operations on $T^M(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$.

Let $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ be the canonical basis of $\mathbb{R}^d$ and $A_d = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$ be the corresponding alphabet. To ease reading, for $i \in \{1, \dots, d\}$, we write e_i as the blue letter $\mathbf{i}$ and for $n \geq 1, i_1, \dots, i_n \in \{1, \dots, d\}$, we write $e_{i_1} \otimes \dots \otimes e_{i_n}$ as the concatenation of letters $\mathbf{i}_1 \dots \mathbf{i}_n$, that we call a word of length n . We note that $(e_{i_1} \otimes \dots \otimes e_{i_n})_{(i_1, \dots, i_n) \in \{1, \dots, d\}^n}$ is a basis of $(\mathbb{R}^d)^{\otimes n}$ that can be identified with the set of words of length n defined by

$$V_n := \{\mathbf{i}_1 \dots \mathbf{i}_n : \mathbf{i}_k \in A_d \text{ for } k = 1, 2, \dots, n\}. \quad (4.3.1)$$

Moreover, $\emptyset$ denotes the empty word and $V_0 = \{\emptyset\}$ serves as a basis for $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. It follows that $V := \bigcup_{n \geq 0} V_n$ represents the standard basis of $T((\mathbb{R}^d))$. In this case, every $\ell \in T((\mathbb{R}^d))$, can be decomposed as

$$\ell = \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \ell^{\mathbf{v}} \mathbf{v}, \quad (4.3.2)$$

where $\ell^{\mathbf{v}}$ is the real coefficient of ℓ at coordinate $\mathbf{v}$.

We now introduce two important concepts such as the concatenation and the projection. The concatenation $\ell \mathbf{v}$ of elements $\ell \in T((\mathbb{R}^d))$ and the word $\mathbf{v} = \mathbf{i}_1 \cdots \mathbf{i}_n$ means $\ell \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}$. The projection $\ell|_{\mathbf{u}} \in T((\mathbb{R}^d))$ is defined as

$$\ell|_{\mathbf{u}} := \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \ell^{\mathbf{v}\mathbf{u}} \mathbf{v} \quad (4.3.3)$$

for all $\mathbf{u} \in V$. Moreover, if we consider $\xi \in T((\mathbb{R}^d))$, then we define the projection $\ell|_{\xi} \in T((\mathbb{R}^d))$ by

$$\ell|_{\xi} := \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \left[\sum_{m=0}^{\infty} \sum_{\mathbf{u} \in V_m} \ell^{\mathbf{v}\mathbf{u}} \xi^{\mathbf{u}} \right] \mathbf{v}. \quad (4.3.4)$$

The projection plays an important role in the space of iterated integrals as it is closely linked to partial differentiation, in contrast with the concatenation that relates to integration. We now define the bracket between $\ell \in T(\mathbb{R}^d)$, and $\mathbf{p} \in T((\mathbb{R}^d))$ by

$$\langle \ell, \mathbf{p} \rangle := \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \ell^{\mathbf{v}} \mathbf{p}^{\mathbf{v}}. \quad (4.3.5)$$

We observe that the bracket is well defined since $\ell \in T(\mathbb{R}^d)$ and has therefore finitely many non-zero terms. For $\ell \in T((\mathbb{R}^d))$, the series in (4.3.5) involves infinitely many terms and requires special care to be well defined. We discuss it in Subsection 4.3.3. Finally, we consider another operation on the space of words which is the shuffle product. The shuffle product plays a crucial role for an integration by parts formula on the space of iterated integrals, see Proposition 4.3.1 below.

Definition 4.1 (Shuffle product). *The shuffle product $\sqcup\sqcup: V \times V \rightarrow T(\mathbb{R}^d)$ is defined inductively for all words $\mathbf{v}$ and $\mathbf{w}$ and all letters $\mathbf{i}$ and $\mathbf{j}$ in A_d by*

$$\begin{aligned} (\mathbf{vi}) \sqcup\sqcup (\mathbf{wj}) &= (\mathbf{v} \sqcup\sqcup (\mathbf{wj}))\mathbf{i} + ((\mathbf{vi}) \sqcup\sqcup \mathbf{w})\mathbf{j}, \\ \mathbf{w} \sqcup\sqcup \emptyset &= \emptyset \sqcup\sqcup \mathbf{w} = \mathbf{w}. \end{aligned}$$

With some abuse of notation, the shuffle product on $T((\mathbb{R}^d))$ induced by the shuffle product on V is also denoted by $\sqcup\sqcup$.

4.3.2 Signatures

We define the (path) signature of a semimartingale process as the sequence of iterated stochastic integrals in the sense of Stratonovich. Notice that, in the chapter, we denote the Itô integral by $\int_0^\cdot Y_t dX_t$ and the Stratonovich integral by $\int_0^\cdot Y_t \circ dX_t$.

Definition 4.2. Fix $T > 0$. Let $(X_t)_{t \geq 0}$ be a continuous semimartingale in $\mathbb{R}^d$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The signature of X is defined by

$$\begin{aligned} \mathbb{X} : \Omega \times [0, T] &\rightarrow T((\mathbb{R}^d)) \\ (\omega, t) &\mapsto \mathbb{X}_t(\omega) := (1, \mathbb{X}_t^1(\omega), \dots, \mathbb{X}_t^n(\omega), \dots), \end{aligned}$$

where

$$\mathbb{X}_t^n := \int_{0 < u_1 < \dots < u_n < t} \circ dX_{u_1} \otimes \dots \otimes \circ dX_{u_n}$$

takes value in $(\mathbb{R}^d)^{\otimes n}$, $n \geq 0$. Similarly, the truncated signature of order $M \in \mathbb{N}$ is defined by

$$\mathbb{X}^{\leq M} : [0, T] \rightarrow T^M(\mathbb{R}^d) \quad (4.3.6)$$

$$(\omega, t) \mapsto \mathbb{X}_t^{\leq M}(\omega) := (1, \mathbb{X}_t^1(\omega), \dots, \mathbb{X}_t^M(\omega), 0, \dots, 0, \dots). \quad (4.3.7)$$

The signature plays a similar role to polynomials on path-space. Indeed, in dimension $d = 1$, the signature of X is the sequence of monomials $(\frac{1}{n!}(X_t - X_0)^n)_{n \in \mathbb{N}}$. In particular, any finite combination of elements of the signature $\langle \ell, \mathbb{X}_t \rangle$, defined in (4.3.5) for $\ell \in T^M(\mathbb{R}^d)$, is a polynomial of degree M in X_t . For example, if $d = 2$ and $X_t = \widehat{W}_t := (t, W_t)$ where W is a 1-dimensional Brownian motion, the first few signature orders are given by

$$\widehat{\mathbb{W}}_t^0 = 1, \quad \widehat{\mathbb{W}}_t^1 = \begin{pmatrix} t \\ W_t \end{pmatrix}, \quad \widehat{\mathbb{W}}_t^2 = \begin{pmatrix} \frac{t^2}{2!} & \int_0^t s dW_s \\ \int_0^t W_s ds & \frac{W_t^2}{2!} \end{pmatrix} \quad (4.3.8)$$

and

$$\widehat{\mathbb{W}}_t^3 = \begin{pmatrix} \frac{t^3}{3!} & \int_0^t \frac{s^2}{2!} dW_s & \\ \int_0^t \int_0^s u dW_u ds & \int_0^t \int_0^s u dW_u \circ dW_s & \\ \int_0^t \int_0^s W_u du ds & \int_0^t \int_0^s W_u du \circ dW_s & \\ \int_0^t \frac{W_s^2}{2!} ds & \frac{W_t^3}{3!} & \end{pmatrix}. \quad (4.3.9)$$

4.3.3 Infinite linear combinations of signature elements

In this section, we recall some results on infinite linear combinations $\langle \ell, \mathbb{X}_t \rangle$ for certain admissible $\ell \in T((\mathbb{R}^d))$ for which the infinite series makes sense. The crucial ingredients for this chapter are the shuffle product (Proposition 4.3.1) and an Itô's formula (Theorem 4.3.2). Those results are based on Abi Jaber et al. (2024a), and we refer to this paper for more details.

We first consider the space $\mathcal{A}(\mathbb{X})$ of admissible elements ℓ below, using the associated semi-norm:

$$\|\ell\|_t^{\mathcal{A}(\mathbb{X})} := \sum_{n=0}^{\infty} \left| \sum_{\mathbf{v} \in V_n} \ell^{\mathbf{v}} \mathbb{X}_t^{\mathbf{v}} \right|, \quad t \geq 0,$$

recall the definition of V_n in (4.3.1) and the decomposition (4.3.2). Whenever, $\|\ell\|_t^{\mathcal{A}(\mathbb{X})} < \infty$ a.s., the infinite linear combination

$$\langle \ell, \mathbb{X}_t \rangle = \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \ell^{\mathbf{v}} \mathbb{X}_t^{\mathbf{v}}$$

is well defined. This leads to the following definition for the admissible set $\mathcal{A}(\mathbb{X})$:

$$\mathcal{A}(\mathbb{X}) := \left\{ \ell \in T((\mathbb{R}^d)) : \|\ell\|_t^{\mathcal{A}(\mathbb{X})} < \infty \text{ for all } t \in [0, T] \text{ a.s.} \right\}.$$

Note that $T(\mathbb{R}^d) \subset \mathcal{A}(\mathbb{X})$ and that $\langle \ell, \mathbb{X}_t \rangle$ is an extension of (4.3.5), as the two bracket operations $\langle \cdot, \cdot \rangle$ coincide whenever $\ell \in T(\mathbb{R}^d)$. The admissible set $\mathcal{A}(\mathbb{X})$ enables us to linearize polynomials on infinite linear combination of the signature. This is what is commonly referred to in the literature as the linearization power of the signature.

Proposition 4.3.1 (Shuffle property). *If $\ell_1, \ell_2 \in \mathcal{A}(\mathbb{X})$, then $\ell_1 \sqcup \ell_2 \in \mathcal{A}(\mathbb{X})$ and*

$$\langle \ell_1, \mathbb{X}_t \rangle \langle \ell_2, \mathbb{X}_t \rangle = \langle \ell_1 \sqcup \ell_2, \mathbb{X}_t \rangle.$$

By the definition of the admissible set $\mathcal{A}(\mathbb{X})$, we know that, for elements $\ell \in \mathcal{A}(\mathbb{X})$, the process $(\langle \ell, \mathbb{X}_t \rangle)_{t \leq T}$ is well defined. This naturally raises the question of when it is a semimartingale and how to obtain its Itô decomposition. For elements ℓ in the set $\mathcal{I}(\mathbb{X})$ defined by

$$\mathcal{I}(\mathbb{X}) := \left\{ \begin{array}{l} \ell \in \mathcal{A}(\mathbb{X}) : \\ \text{for every } \mathbf{i} \in A_d \text{ and } \mathbf{j} \in \mathcal{Q}_i(X), \ell|_{\mathbf{i}}, \ell|_{\mathbf{j}} \in \mathcal{A}(\mathbb{X}) \text{ and} \\ \int_0^T \left(\|\ell|_{\mathbf{i}}\|_t^{\mathcal{A}(\mathbb{X})} d|X^{\mathbf{i}}|_t + \|\ell|_{\mathbf{j}}\|_t^{\mathcal{A}(\mathbb{X})} d[X^{\mathbf{i}}, X^{\mathbf{j}}]_t \right. \\ \left. + \left(\|\ell|_{\mathbf{i}}\|_t^{\mathcal{A}(\mathbb{X})} \right)^2 d[X^{\mathbf{i}}, X^{\mathbf{j}}]_t \right) < \infty \text{ a.s.} \end{array} \right\}, \quad (4.3.10)$$

where $\mathcal{Q}_i(X)$ defines the set of coordinates of X that have a non-zero quadratic covariation with $X^{\mathbf{i}}$

$$\mathcal{Q}_i(X) := \{ \mathbf{j} \in A_d : [X^{\mathbf{j}}, X^{\mathbf{i}}]_t \neq 0, \text{ on a set of non-zero } dt \otimes d\mathbb{P} \text{ measure} \},$$

then, we can prove that $(\langle \ell, \mathbb{X}_t \rangle)_{t \leq T}$ is a semimartingale and we can compute its Itô decomposition. Notice that, for $M > 0$, we have $T^M(\mathbb{R}^d) \in \mathcal{I}(\mathbb{X})$, i.e. showing

that finite linear combinations of the signature are always semimartingales. More generally, we recall the result for time-dependent linear combinations $(\langle \ell_t, \mathbb{X}_t \rangle)_{t \leq T}$ with $\ell : [0, T] \rightarrow \mathcal{A}(\mathbb{X})$ in the set

$$\mathcal{I}'(\mathbb{X}) := \left\{ \begin{array}{l} \ell : [0, T] \rightarrow \mathcal{A}(\mathbb{X}) : \\ \text{for every } \mathbf{v} \in V, \ell^{\mathbf{v}} \in C^1([0, T], \mathbb{R}), \\ \text{for every } \mathbf{i} \in A_d \text{ and } \mathbf{j} \in \mathcal{Q}_i(X) \\ \text{and for all } t \in [0, T], \ell_t|_{\mathbf{i}}, \ell_t|_{\mathbf{j}}, \dot{\ell}_t \in \mathcal{A}(\mathbb{X}) \\ \text{and } \int_0^T \left(\|\ell_t|_{\mathbf{i}}\|_t^{A(\mathbb{X})} d|X^{\mathbf{i}}|_t \right. \\ \left. + \|\ell_t|_{\mathbf{j}}\|_t^{A(\mathbb{X})} d[X^{\mathbf{i}}, X^{\mathbf{j}}]_t + \left(\|\ell_t|_{\mathbf{i}}\|_t^{A(\mathbb{X})} \right)^2 d[X^{\mathbf{i}}, X^{\mathbf{j}}]_t \right) < \infty \end{array} \right\}, \quad (4.3.11)$$

where $\dot{\ell}_t := \sum_{n=0}^{\infty} \sum_{\mathbf{v} \in V_n} \frac{d}{dt} \ell_t^{\mathbf{v}} \mathbf{v}$ for all $t \in [0, T]$.

Theorem 4.3.2 (Itô's formula). *Let $\ell \in \mathcal{I}(\mathbb{X})$, then $\langle \ell, \mathbb{X}_t \rangle$ is an Itô process such that*

$$\langle \ell, \mathbb{X}_t \rangle = \ell^{\mathbf{o}} + \sum_{i \in A_d} \int_0^t \langle \ell_s|_{\mathbf{i}}, \mathbb{X}_s \rangle dX_s^i + \frac{1}{2} \sum_{i \in A_d} \sum_{j \in A_d} \int_0^t \langle \ell_s|_{\mathbf{j}}, \mathbb{X}_s \rangle d[X^j, X^i]_s, \quad t \leq T. \quad (4.3.12)$$

Moreover, if $\ell \in \mathcal{I}'(\mathbb{X})$, then

$$\begin{aligned} \langle \ell_t, \mathbb{X}_t \rangle &= \ell_t^{\mathbf{o}} + \sum_{i \in A_d} \int_0^t \langle \ell_s|_{\mathbf{i}}, \mathbb{X}_s \rangle dX_s^i + \frac{1}{2} \sum_{i \in A_d} \sum_{j \in A_d} \int_0^t \langle \ell_s|_{\mathbf{j}}, \mathbb{X}_s \rangle d[X^j, X^i]_s \\ &\quad + \int_0^t \langle \dot{\ell}_s, \mathbb{X}_s \rangle ds, \quad t \leq T. \end{aligned} \quad (4.3.13)$$

Proof. We refer to (Abi Jaber et al., 2024a, Theorem 3.3. and Corollary 3.4.) for the proof. $\square$

4.4 Pricing and perfect hedging in frictionless market

In this section, we first show how to price and perfectly replicate a signature payoff in a frictionless market. This perfect hedging strategy will enter in the hedging problem with market impact in Section 4.5. In the absence of market impact, i.e. $\eta = \nu = 0$, the asset traded price is S_t . In this context, we consider that the trader wants to hedge a given signature payoff

$$H_T = \langle \xi, \hat{S}_T \rangle, \quad (4.4.1)$$

for a certain admissible $\xi \in T(\mathbb{R}^2)$, hence ξ has a finite number of non-zero terms. We start by giving some examples of path-dependent payoffs that can be exactly represented as signature payoffs.

Example 1. *Signature payoffs include:*

- *European polynomial payoffs of the form $H_T = \sum_{k=0}^N \alpha_k (S_T - K)^k$, for given $N \in \mathbb{N}$, $K \in \mathbb{R}$ and $(\alpha_k)_{k \leq N} \in \mathbb{R}^N$, since*

$$H_T = \left\langle \sum_{k=0}^N \alpha_k (\mathbf{2} + \phi(S_0 - K))^{\sqcup k}, \hat{S}_T \right\rangle;$$

- *Asian polynomial payoffs of the form $H_T = \sum_{k=0}^N \alpha_k \left(\frac{1}{T} \int_0^T S_t dt - K \right)^k$, for given $N \in \mathbb{N}$, $K \in \mathbb{R}$ and $(\alpha_k)_{k \leq N} \in \mathbb{R}^N$, since*

$$H_T = \left\langle \sum_{k=0}^N \alpha_k \left(\frac{1}{T} \mathbf{21} + \phi(S_0 - K) \right)^{\sqcup k}, \hat{S}_T \right\rangle.$$

■

Let $(V_t^X)_{t \in [0, T]}$ denote the hedging portfolio of the trader such that

$$dV_t^X = X_t dS_t, \quad V_0^X = V_0 \in \mathbb{R}^+.$$

where X is an admissible hedging in

$$\mathcal{L}^2 := \left\{ X : [0, T] \times \Omega \rightarrow \mathbb{R} \text{ prog. meas. process such that } \mathbb{E} \left(\int_0^T X_t^2 dt \right) < \infty \right\}.$$

The fair pricing of the option corresponds to V_0 . For $\ell \in T((\mathbb{R}^d))$, let us define the tensor product exponential as

$$\exp_{\otimes}(\ell) := \sum_{k \geq 0} \frac{\ell^{\otimes k}}{k!}.$$

In the frictionless market, a perfect hedging strategy can be deduced using the quantity

$$\xi_t := \xi|_{\mathcal{E}_{T-t}}, \quad t \leq T, \quad (4.4.2)$$

recall the projection defined by (4.3.4), where

$$\mathcal{E}_t = \exp_{\otimes} \left(\left(\mathbf{1} + \frac{\sigma^2}{2} \mathbf{22} \right) t \right) \quad (4.4.3)$$

corresponds to the expectation of the signature of the time-augmented stock price S under the risk-neutral probability measure thanks to the Fawcett (2003) formula.

Theorem 4.4.1. *Let $\xi \in T(\mathbb{R}^2)$. Then, $(\xi_t)_{t \leq T}$ defined by (4.4.2) is well defined for all $t \leq T$ and belongs to $\mathcal{I}'(\hat{\mathbb{S}})$. Furthermore, the option (4.4.1) can be perfectly hedged using the portfolio with initial wealth $V_0 = \xi_0^\circ$ and strategy $X_t = \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle$, that is*

$$V_T^X = H_T = \langle \xi, \hat{\mathbb{S}}_T \rangle, \quad (4.4.4a)$$

In particular,

$$V_t^X = \langle \xi_t, \hat{\mathbb{S}}_t \rangle, \quad t \leq T. \quad (4.4.4b)$$

Proof. The proof follows from two following lemmas. By combining Lemma 4.4.2 and Lemma 4.4.3, we deduce that the signature payoff $\langle \xi, \hat{\mathbb{S}}_T \rangle$ can be perfectly replicated using the portfolio with initial wealth $V_0 = \xi_0^\circ$ and strategy $X_t = \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle$. $\square$

Lemma 4.4.2. *Let $\xi \in T(\mathbb{R}^2)$. Assume that there exists $(\xi_t)_{t \in [0, T]} \in \mathcal{I}'(\hat{\mathbb{S}})$ satisfying the following equation*

$$\dot{\xi}_t = -\xi_t|_1 - \frac{1}{2}\sigma^2 \xi_t|_{22}, \quad \xi_T = \xi. \quad (4.4.5)$$

Then, the option (4.4.1) can be perfectly hedged using the portfolio with initial wealth $V_0 = \xi_0^\circ$ and strategy $X_t = \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle$, that is (4.4.4a) and (4.4.4b) are satisfied.

Proof. First, by applying the Itô's formula in Theorem 4.3.2 to $(\langle \xi_t, \hat{\mathbb{S}}_t \rangle)_{t \leq T}$ and using (4.4.5), we have that

$$\begin{aligned} \langle \xi_T, \hat{\mathbb{S}}_T \rangle &= \xi_0^\circ + \int_0^T \langle \dot{\xi}_t + \xi_t|_1 + \mu \xi_t|_2 + \frac{1}{2}\sigma^2 \xi_t|_{22}, \hat{\mathbb{S}}_t \rangle dt + \int_0^T \sigma \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle dW_t, \\ &= \xi_0^\circ + \int_0^T \langle \mu \xi_t|_2, \hat{\mathbb{S}}_t \rangle dt + \int_0^T \sigma \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle dW_t. \end{aligned}$$

But as

$$V_T^X = V_0 + \int_0^T \mu X_t dt + \sigma \int_0^T X_t dW_t,$$

we observe that by taking

$$X_t = \langle \xi_t|_2, \hat{\mathbb{S}}_t \rangle \text{ and } V_0 = \xi_0^\circ,$$

then, we directly obtain (4.4.4b) and (4.4.4a). $\square$

Lemma 4.4.3. *Let $\xi \in T(\mathbb{R}^2)$. Then, $(\xi_t)_{t \leq T}$ given by (4.4.2) is well defined, belongs to $\mathcal{I}'(\hat{\mathbb{S}})$ and satisfies equation (4.4.5).*

Proof. First, since $\xi \in T(\mathbb{R}^2)$, we know that there exists $M > 0$ such that $\xi \in T^M(\mathbb{R}^2)$. Thus, we deduce that ξ_t has finitely many non-zero terms and is well defined, such that $\xi_t \in \mathcal{I}'(\hat{\mathbb{S}})$. Then, as $\xi_t = \xi|_{\mathcal{E}_{T-t}}$, we have that

$$\xi_t = \sum_{n=0}^M \sum_{\mathbf{v} \in V_n} \xi_t^{\mathbf{v}} \mathbf{v},$$

with, for each $\mathbf{v} \in V_n$, $n \leq M$,

$$\xi_t^{\mathbf{v}} = \sum_{m=0}^{M-n} \sum_{\mathbf{w} \in V_m} \xi^{\mathbf{vw}} \mathcal{E}_{T-t}^{\mathbf{w}}.$$

Since

$$\dot{\mathcal{E}}_{T-t} = - \left(\mathbf{1} + \frac{1}{2} \mathbf{22} \right) \otimes \mathcal{E}_{T-t},$$

we observe that, for each $\mathbf{v} \in V_n$, $n \leq M$,

$$\begin{aligned} \dot{\xi}_t^{\mathbf{v}} &= - \sum_{m=0}^{M-n} \sum_{\mathbf{w} \in V_m} \xi^{\mathbf{vw}} \left(\left(\mathbf{1} + \frac{\sigma^2}{2} \mathbf{22} \right) \otimes \mathcal{E}_{T-t} \right)^{\mathbf{w}} \\ &= - \sum_{m=0}^{M-n} \sum_{\mathbf{w} \in V_m} \xi^{\mathbf{vw}} \sum_{l=0}^m \left(\sum_{\mathbf{w}' \in V_l, \mathbf{1w}'=\mathbf{w}} \mathcal{E}_{T-t}^{\mathbf{w}'} + \frac{\sigma^2}{2} \sum_{\mathbf{w}' \in V_l, \mathbf{22w}'=\mathbf{w}} \mathcal{E}_{T-t}^{\mathbf{w}'} \right) \\ &= - \sum_{l=0}^{M-n} \sum_{\mathbf{w} \in V_l} \left(\xi^{\mathbf{v1w}} + \frac{\sigma^2}{2} \xi^{\mathbf{v22w}} \right) \mathcal{E}_{T-t}^{\mathbf{w}} \\ &= - \left(\xi_t^{\mathbf{v1}} + \frac{\sigma^2}{2} \xi_t^{\mathbf{v22}} \right). \end{aligned}$$

Therefore, we infer that

$$\dot{\xi}_t = \sum_{n=0}^M \sum_{\mathbf{v} \in V_n} \dot{\xi}_t^{\mathbf{v}} \mathbf{v} = - \sum_{n=0}^M \sum_{\mathbf{v} \in V_n} \left(\xi_t^{\mathbf{v1}} + \frac{\sigma^2}{2} \xi_t^{\mathbf{v22}} \right) \mathbf{v} = -\xi_t|_{\mathbf{1}} - \frac{\sigma^2}{2} \xi_t|_{\mathbf{22}},$$

which gives (4.4.5). $\square$

4.5 Mean-quadratic variation hedging with market impact

Let us now consider the hedging problem of signature payoffs by assuming both temporary and permanent market impact as in (4.2.3) and (4.2.4). In this case, the trader aims to optimally hedge a signature payoff given by

$$H_T^\theta = \langle \xi, \hat{\mathbb{P}}_T^\theta \rangle, \quad (4.5.1)$$

with $\xi \in T^M(\mathbb{R}^2)$, for some $M > 0$. Since we assume permanent price impact, the signature payoff (4.5.1) depends now on the time-augmented signature of P^θ given by (4.2.3) instead of the time-augmented signature of S (compare with (4.4.1)). In addition, the hedging strategy will be of finite variation in the form (4.2.2), thanks to the regularizing effect of the temporary impact component with coefficient $\eta > 0$ present in (4.2.4). In this frictional market, we follow Almgren and Li (2016) and we assume that the trader's P&L is marked to market using the Bachelier option price without friction such that the P&L at time $t \leq T$ resulting from the hedging strategy denoted by R_t^θ is given by

$$R_t^\theta = (V_0 - X_0 S_0) + X_t^\theta P_t^\theta - \int_0^t \tilde{P}_s^\theta \theta_s ds - \langle \xi_t, \hat{\mathbb{P}}_t^\theta \rangle, \quad t \leq T, \quad (4.5.2)$$

with V_0 the initial value of the hedging portfolio composed of cash and holdings X_0 in the traded asset, and $(\xi_t)_{t \in [0, T]}$ defined by (4.4.2). In particular,

$$R_T^\theta = (V_0 - X_0 S_0) + X_T^\theta P_T^\theta - \int_0^T \tilde{P}_t^\theta \theta_t dt - H_T^\theta, \quad (4.5.3)$$

Moreover, we assume that the trader optimally hedges her short position using a mean-quadratic variation criteria in the form

$$\sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right), \quad (4.5.4)$$

with $\lambda > 0$, the risk-aversion parameter, and $\mathcal{A}$ the set of admissible trading speed defined by

$$\mathcal{A} := \left\{ \theta : [0, T] \times \Omega \rightarrow \mathbb{R} \text{ prog. meas. process} \right. \\ \left. \text{such that } \mathbb{E} \left(\int_0^T \theta_t^p dt \right) < \infty, \text{ for } p := 2 \vee 2\tilde{M}\mathbb{1}_{\{\nu > 0\}} \right\}, \quad (4.5.5)$$

$\tilde{M}$ being the truncation order of $\xi_t|_2$.

Remark 4.2 (Choice of loss functional). *We adopt a mean-quadratic variation loss functional for reasons of tractability and time consistency. In practice, traders often*

seek to control the evolution of their P&L along the entire trajectory rather than only at maturity, which makes this criterion particularly relevant in our setting. More sophisticated loss functionals, such as quantile- or shortfall-based criteria (studied in Chapter 3) would lead to significantly more difficult control problems. ■

Using the optimal strategy from the frictionless market derived in Section 4.4, we can rewrite the dynamic of R^θ given by (4.5.2), which facilitates the computation of its quadratic variation and simplifies the expression of the objective criterion.

Lemma 4.5.1. *Let $(\xi_t)_{t \in [0, T]}$ be defined by (4.4.2) and $\theta \in \mathcal{A}$, then*

$$\begin{aligned} R_T^\theta &= (V_0 - \xi_0^\theta) + \int_0^T (\mu + \nu\theta_t) \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt - \int_0^T \eta\theta_t^2 dt \\ &\quad + \sigma \int_0^T \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dW_t, \end{aligned} \quad (4.5.6)$$

and its quadratic variation is given by

$$\mathbb{E}([R^\theta, R^\theta]_T) = \sigma^2 \mathbb{E} \left(\int_0^T \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 dt \right).$$

The stochastic control problem can be rewritten as

$$\begin{aligned} \sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right) &= (V_0 - \xi_0^\theta) \\ &\quad + \sup_{\theta \in \mathcal{A}} \left[\mathbb{E} \left(\int_0^T (\mu + \nu\theta_t) \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt \right. \right. \\ &\quad \left. \left. - \int_0^T \eta\theta_t^2 dt \right) - \frac{\lambda}{2} \sigma^2 \mathbb{E} \left(\int_0^T \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 dt \right) \right]. \end{aligned} \quad (4.5.7)$$

Proof. An application of Itô formula in Theorem 4.3.2 to $\langle \xi_t, \hat{\mathbb{P}}_t^\theta \rangle$ and (4.4.5) yields

$$H_T^\theta = \langle \xi_T, \hat{\mathbb{P}}_T^\theta \rangle = \xi_0^\theta + \int_0^T (\mu + \nu\theta_t) \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle dt + \sigma \int_0^T \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle dW_t,$$

with ξ_t defined by (4.4.2) and thus, we can rewrite R_T^θ in the form

$$\begin{aligned} R_T^\theta &= V_0 + \int_0^T ((\mu + \nu\theta_t)X_t^\theta - \eta\theta_t^2) dt + \sigma \int_0^T X_t^\theta dW_t - \langle \xi_T, \hat{\mathbb{P}}_T^\theta \rangle \\ &= (V_0 - \xi_0^\theta) + \int_0^T (\mu + \nu\theta_t) \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt - \int_0^T \eta\theta_t^2 dt \\ &\quad + \sigma \int_0^T \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dW_t. \end{aligned}$$

The quadratic-variation of R^θ follows from this representation. $\square$

Remark 4.3. From the reformulation of the stochastic control problem (4.5.7), we see that the problem is not Linear-Quadratic, except for the case of $\xi_t|_{\mathbf{22}} = \Gamma \phi$, $\Gamma \in \mathbb{R}$, which corresponds to the case of payoffs with constant Gamma as studied by Almgren and Li (2016). ■

Before tackling the control problem (4.5.4), it is worth first asking at what price the trader should sell the option in this market with frictions. Indeed, there is no reason for the trader to sell the option at the frictionless Bachelier price. We follow Guéant and Pu (2017) and consider an indifference pricing approach. More precisely, we assume that at time $t = 0^-$ (just before the deal), the trader only holds a cash position $C \in \mathbb{R}$ and has no incentive to take a position on the risky asset. Then, the deal between the client and the trader is done as follows at $t = 0$:

- the trader writes the option with payoff $\langle \xi, \hat{\mathbb{P}}_T^\theta \rangle$ and the client pays a price π ;
- the client gives X_0 shares to the trader and receives $X_0 S_0$ in cash from the trader.

As the trader considers a mean-quadratic variation criteria, the value she gives is the solution to the control problem (4.5.4) with $V_0 = C + \pi$. Moreover, as at inception (before the deal), the trader only holds cash C , her utility is simply C . As a consequence, the trader is indifferent to make the deal if the price π is such that

$$\sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right) = C,$$

where $V_0 = C + \pi$. Finally, using (4.5.7), we deduce that the indifference price π is given by

$$\begin{aligned} \pi = \xi_0^\phi - \sup_{\theta \in \mathcal{A}} \left[\mathbb{E} \left(\int_0^T (\mu + \nu \theta_t) \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt - \int_0^T \eta \theta_t^2 dt \right) \right. \\ \left. - \frac{\lambda}{2} \sigma^2 \mathbb{E} \left(\int_0^T \left(X_t^\theta - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 dt \right) \right]. \end{aligned} \quad (4.5.8)$$

The indifference price is composed of two terms: the frictionless Bachelier price ξ_0^ϕ , and a spread that depends on market impact parameters, so that the indifference price π includes the additional risk incurred by the trader due to market frictions. In particular, when $\mu = 0$ and $\nu = 0$, as in Guéant and Pu (2017), it follows from (4.5.8) that $\pi \geq \xi_0^\phi$ for all $\eta \geq 0$, with $\pi = \xi_0^\phi$ for $\eta = 0$. However, when $\mu > 0$ or $\nu > 0$, there is no reason why this inequality should hold in all cases, as the trader could exploit a signal or an arbitrage to maximize her risk-adjusted P&L, and therefore require an indifference price that could be lower than ξ_0^ϕ .

4.5.1 Existence and uniqueness result

In this section, we study existence and uniqueness result of the stochastic control problem (4.5.7). We consider the functional $J : \mathcal{A} \rightarrow \mathbb{R}$ by

$$J(\theta) := \mathbb{E} \left(\int_0^T \left[(\mu + \nu\theta_t) \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \eta\theta_t^2 - \frac{\lambda}{2} \sigma^2 \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 \right] dt \right). \quad (4.5.9)$$

Following (4.5.7), the stochastic control problem can be rewritten as

$$\sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right) = \sup_{\theta \in \mathcal{A}} J(\theta) + (V_0 - \xi_0^\theta). \quad (4.5.10)$$

We are interested in studying the concavity of this functional J to deduce an existence and uniqueness result for the control problem. To this end, we define the inner product $\langle \cdot, \cdot \rangle_{L_2}$ by

$$\langle f, g \rangle_{L_2} := \mathbb{E} \left(\int_0^T f_t g_t dt \right), \quad f, g \in \mathcal{A}.$$

We first prove that J is Gâteaux differentiable. Then, we deduce an existence and uniqueness theorem under a monotonicity condition on the Gâteaux derivative of the functional J .

Lemma 4.5.2. *For any $\theta, \phi, v \in \mathcal{A}$ and $\alpha_t \in T^p(\mathbb{R}^2)$ with $p = 2 \vee 2\tilde{M}\mathbb{1}_{\{\nu > 0\}}$, we have that*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E} \left(v_t \langle \alpha_t, \hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} - \hat{\mathbb{P}}_t^\theta \rangle \right) = \nu \int_0^t \mathbb{E} \left(\phi_s v_t \langle \alpha_t, \hat{\mathbb{P}}_s^\theta \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{s,t}^\theta \rangle \right) ds, \quad t \leq T. \quad (4.5.11)$$

Proof. The proof is given in Section 4.7. □

Lemma 4.5.3. *For any $\theta \in \mathcal{A}$, the functional J is Gâteaux differentiable. For any $h \in \mathcal{A}$,*

$$\lim_{\varepsilon \rightarrow 0} \frac{J(\theta + \varepsilon h) - J(\theta)}{\varepsilon} = \langle \nabla J(\theta), h \rangle_{L_2},$$

with the Gâteaux differential given by

$$\nabla J(\theta) = -2\eta\theta + \nu\mathbf{A}\theta - \lambda\sigma^2\mathbf{B}\theta + \mu\mathbf{C}\theta,$$

where the operators $\mathbf{A} : \mathcal{A} \rightarrow \mathcal{A}$, $\mathbf{B} : \mathcal{A} \rightarrow \mathcal{A}$ and $\mathbf{C} : \mathcal{A} \rightarrow \mathcal{A}$ are defined, for $\theta \in \mathcal{A}$, by

$$(\mathbf{A}\theta)_t = X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle + \int_t^T \mathbb{E}(\theta_s (1 - \nu \Gamma_{t,s}^\theta) | \mathcal{F}_t) ds, \quad t \leq T, \quad (4.5.12)$$

$$(\mathbf{B}\theta)_t = \int_t^T \mathbb{E}(X_s^\theta (1 - \nu \Gamma_{t,s}^\theta) | \mathcal{F}_t) ds + \int_t^T \mathbb{E}\left(\frac{1}{2} \nu \tilde{\Gamma}_{t,s}^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle | \mathcal{F}_t\right) ds, \quad t \leq T, \quad (4.5.13)$$

$$(\mathbf{C}\theta)_t = (T - t) - \int_t^T \mathbb{E}(\nu \Gamma_{t,s}^\theta | \mathcal{F}_t) ds, \quad t \leq T, \quad (4.5.14)$$

with

$$\Gamma_{t,s}^\theta := \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{t,s}^\theta \rangle, \quad \tilde{\Gamma}_{t,s}^\theta := \langle (\xi_s | \mathbf{2})^{\sqcup \mathbf{2}}, \hat{\mathbb{P}}_t^\theta \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{t,s}^\theta \rangle, \quad t < s \leq T.$$

Proof. The proof is given in Section 4.7. $\square$

We are now ready to prove the existence and uniqueness result. To this end, we use general results of convex optimization in infinite dimensional spaces and we show that under suitable monotonicity conditions, the functional J is strongly concave. We recall that, given a positive constant $\beta > 0$, a map $\mathcal{T} : \mathcal{A} \rightarrow \mathbb{R}$ is β -strongly concave if, for every $\theta, \phi \in \mathcal{A}$, the following inequality holds

$$\mathcal{T}(\alpha\theta + (1 - \alpha)\phi) \geq \alpha\mathcal{T}(\theta) + (1 - \alpha)\mathcal{T}(\phi) + \frac{\beta\alpha(1 - \alpha)}{2} \|\theta - \phi\|_{\mathcal{A}}^2, \quad \alpha \in [0, 1], \quad (4.5.15)$$

with $\|\theta\|_{\mathcal{A}} := \mathbb{E}\left(\int_0^T \theta_s^p ds\right)^{1/p}$, $p := 2 \vee 2\tilde{M}\mathbb{1}_{\{\nu > 0\}}$.

Theorem 4.5.4. *For a given $\varepsilon > 0$, assume that the operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, defined by (4.5.12), (4.5.13) and (4.5.14), satisfy the following monotonicity condition, for $\theta, \phi \in \mathcal{A}$,*

$$\left\langle -2\eta(\theta - \phi) + \varepsilon \left(\|\theta\|_{\mathcal{A}}^{2-p} \theta^{p-1} - \|\phi\|_{\mathcal{A}}^{2-p} \phi^{p-1} \right) + \nu(\mathbf{A}(\theta) - \mathbf{A}(\phi)) - \lambda\sigma^2(\mathbf{B}(\theta) - \mathbf{B}(\phi)) + \mu(\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \right\rangle_{L_2} \leq 0. \quad (4.5.16)$$

Then, J is ε -strongly concave in the sense of (4.5.15), and there exists a unique admissible optimal control $\theta^* \in \mathcal{A}$ satisfying (4.5.10), which is also the unique solution of

$$\nabla J(\theta^*) = 0. \quad (4.5.17)$$

Proof. The proof of the theorem statements are consequences of well-known results of convex analysis in Banach spaces that can be found in Ekeland and Temam (1999). First, let us prove that J is ε -strongly concave. To this end, we define the functional $\tilde{J}$ by

$$\tilde{J}(\theta) := -J(\theta) - \frac{\varepsilon}{2} \|\theta\|_{\mathcal{A}}^2.$$

If we now prove that $\tilde{J}$ is convex then, we deduce that J is ε -strongly concave. By Lemma 4.5.3, we know that J is Gâteaux differentiable, therefore $\tilde{J}$ is also Gâteaux differentiable. In this case, (Ekeland and Temam, 1999, Proposition 5.5. of Chapter I) gives an equivalence between the convexity of $\tilde{J}$ and the monotonicity of the Gâteaux differential $\nabla \tilde{J}(\theta)$ defined as

$$\langle \nabla \tilde{J}(\theta), h \rangle_{L_2} = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\theta + \varepsilon h) - \tilde{J}(\theta)}{\varepsilon}, \quad h \in \mathcal{A}.$$

In fact, $\tilde{J}$ is convex if, for $\theta, \phi \in \mathcal{A}$,

$$\langle \nabla \tilde{J}(\theta) - \nabla \tilde{J}(\phi), \theta - \phi \rangle_{L_2} \geq 0. \quad (4.5.18)$$

Using Lemma 4.5.3, under our monotonicity assumption given by (4.5.16), we deduce that (4.5.18) is satisfied, and therefore, we obtain that $\tilde{J}$ is convex as well as J is ε -strongly concave. As we prove that J is ε -strongly concave, we have that J is strictly concave and coercive, thus by relying on (Ekeland and Temam, 1999, Proposition 1.2. of Chapter II), we immediately have that there exists a unique admissible optimal control $\theta^* \in \mathcal{A}$ satisfying (4.5.10). Finally, since J is Gâteaux differentiable, from (Ekeland and Temam, 1999, Proposition 2.1. of Chapter II), we know that the set of optimal strategies satisfying (4.5.10) coincides with the set of solutions to (4.5.17) and therefore, we deduce that the unique optimal control $\theta^* \in \mathcal{A}$, is also the unique solution of (4.5.17). $\square$

Proposition 4.5.5. *If $\nu = 0$, then the monotonicity condition (4.5.16) is satisfied.*

Proof. The proof is given in Section 4.7. $\square$

Proposition 4.5.6. *Let us consider that $\nu > 0$ and for all $t \leq T$, $\xi_t|_2 \in T^{\leq 1}(\mathbb{R}^2)$. Moreover, let us assume that, for a given $0 < \varepsilon < 2\eta$,*

$$\frac{1}{\nu} \left(1 - \frac{2\eta - \varepsilon}{T\nu} \right) \leq \xi_t^{22} \leq \frac{1}{\nu}, \quad t \leq T. \quad (4.5.19)$$

Then, the monotonicity condition (4.5.16) is satisfied.

Proof. The proof is given in Section 4.7. $\square$

Remark 4.4. The class of payoffs concerned by Proposition 4.5.6 (i.e. $\xi_t|_2 \in T^{\leq 1}(\mathbb{R}^2)$) includes all signature payoffs with $\xi \in T^{\leq 2}(\mathbb{R}^2)$, but also some signature payoffs with $\xi \in T^{>2}(\mathbb{R}^2)$ such as, for example, the signature payoff of the form $H_T^\theta = \frac{1}{T} \int_0^T \frac{1}{2} (P_t^\theta - S_0)^2 dt = \frac{1}{T} \langle \mathbf{221}, \hat{\mathbb{P}}_T^\theta \rangle$. ■

Theorem 4.5.4 gives a general existence and uniqueness result, and characterizes the optimal control through the first order condition (4.5.17). However, this is intricate to deduce the optimal control using this condition. In the next section, we provide a more tractable verification result to derive the optimal control.

4.5.2 Verification result using infinite-dimensional Riccati equations

In this section, we consider a verification result for the stochastic control problem. We first give a general verification theorem such that, under given assumptions, including the existence of a solution to an infinite-dimensional system of Riccati equations, the optimal control associated with the mean-quadratic variation hedging problem can be written in a feedback form, as a linear combination of elements of the signature with time-dependent coefficients. Then, we provide two examples for which the assumptions of the verification theorem are satisfied and for which we have an explicit solution to the system of infinite-dimensional Riccati equations.

General verification theorem

The following theorem provides a verification result, with a semi-explicit solution for the mean-quadratic variation hedging problem of signature payoffs when the trader incurs both temporary and permanent market impact, in terms of the infinite-dimensional system of Riccati equations on the extended tensor algebra space $T((\mathbb{R}^3))$:

$$\begin{aligned} \dot{\psi}_t &= -\psi_t|_1 - \frac{1}{2}\sigma^2\psi_t|_{22} - \frac{\lambda}{2}\sigma^2(\mathbf{3} + \phi X_0 - \tilde{\xi}_t)^{\sqcup\sqcup 2} + \mu \left[(\mathbf{3} + \phi X_0 - \tilde{\xi}_t) - \psi_t|_2 \right] \\ &\quad + \frac{1}{4\eta} \left[\nu(\mathbf{3} + \phi X_0 - \tilde{\xi}_t) - (\nu\psi_t|_2 + \psi_t|_3) \right]^{\sqcup\sqcup 2}, \\ \psi_T &= 0, \end{aligned} \tag{4.5.20}$$

where $\tilde{\xi}_t$ is defined, for all $\mathbf{w} = \mathbf{i}_1 \dots \mathbf{i}_n$ with $n \in \mathbb{N}$, by

$$\tilde{\xi}_t^{\mathbf{w}} := \begin{cases} 0, & \text{if } \exists k \in \{1, \dots, n\} \text{ such that } \mathbf{i}_k = \mathbf{3}, \\ (\xi_t|_2)^{\mathbf{w}}, & \text{if } \nexists k \in \{1, \dots, n\} \text{ such that } \mathbf{i}_k = \mathbf{3}, \end{cases}$$

with ξ_t given by (4.4.2).

Theorem 4.5.7. Let us define $\hat{Z}_t^\theta := (t, P_t^\theta, X_t^\theta)$ and suppose that there exists $(\psi_t)_{t \in [0, T]}$ solution of the infinite-dimensional system of Riccati equations (4.5.20).

Moreover, let us assume that, for all $\theta \in \mathcal{A}$, $(\psi_t)_{t \in [0, T]} \in \mathcal{I}'(\hat{\mathbb{Z}}^\theta)$,

$$\mathbb{E} \left(\int_0^T \langle \psi_t | \mathbf{2}, \hat{\mathbb{Z}}_t^\theta \rangle^2 dt \right) < \infty, \quad (4.5.21)$$

and that there exists $\theta^* \in \mathcal{A}$ satisfying the feedback equation

$$\theta_t^* = \frac{1}{2\eta} \left[\nu \left(X_t^{\theta^*} - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta^*} \rangle \right) - \langle \nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}, \hat{\mathbb{Z}}_t^{\theta^*} \rangle \right]. \quad (4.5.22)$$

Then, the value of the mean-quadratic variation hedging problem is given by

$$\sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right) = (V_0 - \xi_0^\theta) - \psi_0^\theta, \quad (4.5.23)$$

and the optimum is attained for θ^* .

Proof. Assume that there exists $(\psi_t)_{t \in [0, T]} \in \mathcal{I}'(\hat{\mathbb{Z}}^\theta)$ solution of the infinite-dimensional system of Riccati equations (4.5.20) and define, for any $\theta \in \mathcal{A}$, the process $(U_t^\theta)_{t \in [0, T]}$ by

$$\begin{aligned} U_t^\theta = \int_0^t & \left[(\mu + \nu \theta_s) \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right) - \eta \theta_s^2 - \frac{\lambda}{2} \sigma^2 \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right)^2 \right] ds \\ & - \langle \psi_t, \hat{\mathbb{Z}}_t^\theta \rangle, \quad t \leq T. \end{aligned} \quad (4.5.24)$$

Since, $\psi \in \mathcal{I}'(\hat{\mathbb{Z}}_t^\theta)$, an application of Ito's lemma in Theorem 4.3.2 leads to

$$\begin{aligned} dU_t^\theta = & \left[(\mu + \nu \theta_t) \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \eta \theta_t^2 - \frac{\lambda}{2} \sigma^2 \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 \right] dt \\ & - d \langle \psi_t, \hat{\mathbb{Z}}_t^\theta \rangle \\ = & \left[\nu \theta_t \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \eta \theta_t^2 - \frac{\lambda}{2} \sigma^2 \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 \right. \\ & - \left\langle \dot{\psi}_t + \psi_t | \mathbf{1} + \mu \psi_t | \mathbf{2} + \theta_t (\nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}) + \frac{1}{2} \sigma^2 \psi_t | \mathbf{22}, \hat{\mathbb{Z}}_t^\theta \right\rangle \\ & \left. - \sigma \langle \psi_t | \mathbf{2}, \hat{\mathbb{Z}}_t^\theta \rangle dW_t \right] dt. \end{aligned}$$

By a square completion, we observe that

$$\begin{aligned} dU_t^\theta = & \left[-\eta \left(\theta_t - \frac{1}{2\eta} \left[\nu \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \langle \nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}, \hat{\mathbb{Z}}_t^\theta \rangle \right] \right)^2 \right. \\ & + \frac{1}{4\eta} \left(\nu \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \langle \nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}, \hat{\mathbb{Z}}_t^\theta \rangle \right)^2 \\ & - \frac{\lambda}{2} \sigma^2 \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 - \left\langle \dot{\psi}_t + \psi_t | \mathbf{1} + \mu \psi_t | \mathbf{2} + \frac{1}{2} \sigma^2 \psi_t | \mathbf{22}, \hat{\mathbb{Z}}_t^\theta \right\rangle \\ & \left. + \mu \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) \right] dt - \sigma \langle \psi_t | \mathbf{2}, \hat{\mathbb{Z}}_t^\theta \rangle dW_t. \end{aligned}$$

As, for $t \leq T$, $\tilde{\xi}_t \in \mathcal{A}(\hat{\mathbb{Z}}^\theta)$ and $(\nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}) \in \mathcal{A}(\hat{\mathbb{Z}}^\theta)$, since $\tilde{\xi} \in \mathcal{I}'(\hat{\mathbb{Z}}^\theta)$ and $\psi \in \mathcal{I}'(\hat{\mathbb{Z}}^\theta)$, using the shuffle property of Proposition 4.3.1, we deduce that

$$\left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right)^2 = \left\langle \left(\mathbf{3} + \mathbf{0} X_0 - \tilde{\xi}_t \right)^{\sqcup \mathbf{2}}, \hat{\mathbb{Z}}_t^\theta \right\rangle,$$

and

$$\begin{aligned} & \left(\nu \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \langle \nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}, \hat{\mathbb{Z}}_t^\theta \rangle \right)^2 \\ & = \left\langle \left(\nu(\mathbf{3} + \mathbf{0} X_0 - \tilde{\xi}_t) - (\nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}) \right)^{\sqcup \mathbf{2}}, \hat{\mathbb{Z}}_t^\theta \right\rangle. \end{aligned}$$

Moreover, using the Riccati equation (4.5.20) satisfied by ψ_t , we get that

$$dU_t^\theta = -\eta(\theta_t - \mathcal{T}_t(\theta))^2 dt - \sigma \langle \psi_t | \mathbf{2}, \hat{\mathbb{Z}}_t^\theta \rangle dW_t,$$

with

$$\mathcal{T}_t(\theta) := \frac{1}{2\eta} \left[\nu \left(X_t^\theta - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) - \langle \nu \psi_t | \mathbf{2} + \psi_t | \mathbf{3}, \hat{\mathbb{Z}}_t^\theta \rangle \right].$$

Let us now define $(M_t^\theta)_{t \in [0, T]}$ as

$$M_t^\theta := U_t^\theta + \eta \int_0^t (\theta_s - \mathcal{T}_s(\theta))^2 ds = -\langle \psi_0, \hat{\mathbb{Z}}_0^\theta \rangle - \sigma \int_0^t \langle \psi_s | \mathbf{2}, \hat{\mathbb{Z}}_s^\theta \rangle dW_s.$$

Since we assume (4.5.21), we have that $(M_t^\theta)_{t \in [0, T]}$ is a true martingale. In this

case, since $\mathbb{E}(M_T^\theta | \mathcal{F}_t) = M_t^\theta$ and $\langle \psi_T, \hat{Z}_T^\theta \rangle = 0$, we observe that

$$\begin{aligned} \langle \psi_t, \hat{Z}_t^\theta \rangle &= -\mathbb{E}(M_T^\theta | \mathcal{F}_t) \\ &\quad + \int_0^t \left[\nu \theta_s \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right) - \eta \theta_s^2 - \frac{\lambda}{2} \sigma^2 \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right)^2 \right] ds \\ &\quad + \eta \int_0^t (\theta_s - \mathcal{T}_s(\theta))^2 ds. \\ &= -\mathbb{E} \left(\int_t^T \left[\nu \theta_s \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right) - \eta \theta_s^2 \right. \right. \\ &\quad \left. \left. - \frac{\lambda}{2} \sigma^2 \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right)^2 \right] ds \middle| \mathcal{F}_t \right) \\ &\quad - \eta \mathbb{E} \left(\int_t^T (\theta_s - \mathcal{T}_s(\theta))^2 ds \middle| \mathcal{F}_t \right). \end{aligned}$$

By defining $J_t(\theta)$ as

$$J_t(\theta) := \mathbb{E} \left(\int_t^T \left[\nu \theta_s \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right) - \eta \theta_s^2 - \frac{\lambda}{2} \sigma^2 \left(X_s^\theta - \langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle \right)^2 \right] ds \middle| \mathcal{F}_t \right),$$

we obtain that

$$J_t(\theta) + \langle \psi_t, \hat{Z}_t^\theta \rangle = -\eta \mathbb{E} \left(\int_t^T (\theta_s - \mathcal{T}_s(\theta))^2 ds \middle| \mathcal{F}_t \right). \quad (4.5.25)$$

We observe that the right hand side of (4.5.25) is always nonpositive and vanishes for $\theta_t = \mathcal{T}_t(\theta)$, i.e. $\theta_t = \theta_t^*$ with θ_t^* given by (4.5.22). As we assume that there exists an admissible control process $\theta^* \in \mathcal{A}$ such that (4.5.22) holds, then the proof is complete. In fact, in this case, for fixed $t \leq T$, we have that, $\langle \psi_t, \hat{Z}_t^{\theta^*} \rangle = \langle \psi_t, \hat{Z}_t^{\theta'} \rangle$, for all $\theta' \in \mathcal{A}_t(\theta^*) := \{\theta \in \mathcal{A} : \theta_s = \theta_s^*, \text{ for } s \leq t\}$. Therefore, from (4.5.25), we deduce that

$$\sup_{\theta \in \mathcal{A}_t(\theta^*)} J_t(\theta) = J_t(\theta^*) = -\langle \psi_t, \hat{Z}_t^{\theta^*} \rangle,$$

which shows that θ^* is an optimal control. In particular, for $t = 0$, we have that

$$\sup_{\theta \in \mathcal{A}} J_0(\theta) = J_0(\theta^*) = -\psi_0^\theta,$$

and as

$$\sup_{\theta \in \mathcal{A}} \mathbb{E} \left(R_T^\theta - \frac{\lambda}{2} [R^\theta, R^\theta]_T \right) = \left(V_0 - \langle \xi_0, \hat{\mathbb{P}}_0^\theta \rangle \right) + \sup_{\theta \in \mathcal{A}} J_0(\theta),$$

we finally obtain (4.5.23). $\square$

Remark 4.5 (Interpretation of θ^*). θ^* has a feedback form in $\hat{\mathbb{Z}}_t^{\theta^*}$ that can be decomposed into two terms:

- The first term $\frac{1}{2\eta}\nu \left(X_t^{\theta^*} - \left\langle \xi_t|_2, \hat{\mathbb{P}}_t^{\theta^*} \right\rangle \right)$ corresponds to an arbitrage opportunity. Since $P\mathcal{B}L$ is marked-to-market, when $\nu > 0$, the trader benefits from pushing the price upward by buying more, when her inventory X_t^θ is larger than the perfect hedging strategy $\langle \xi_t|_2, \hat{\mathbb{P}}_t^\theta \rangle$. Similarly, she benefits from pushing the price downward by selling if it is smaller. This can be clearly seen by examining the expression of R_T^θ given by (4.5.6). When $\nu > 0$, the trader can deliberately move the market to make her current position more valuable.
- The second term $-\frac{1}{2\eta} \left\langle \nu \psi_t|_2 + \psi_t|_3, \hat{\mathbb{Z}}_t^{\theta^*} \right\rangle$ expresses the trader's incentive to adjust both her inventory X^θ and, when $\nu > 0$, the public traded price P^θ in the direction that increases her utility at time $t < T$.

Remark 4.6. Assumptions of Theorem 4.5.7 are intricate to prove in general, especially concerning the existence of a solution to the infinite-dimensional system of Riccati equations. Nevertheless, in the next section, we provide two concrete examples for which the existence of an explicit solution to the infinite-dimensional system of Riccati equations can be established. ■

Remark 4.7 (Polynomial payoffs). If $\xi = (\mathbf{2}^{\otimes m})_{m=0,\dots,M}$, then we can show that the infinite-dimensional system of Riccati equations reduces to (4.2.8). ■

Remark 4.8 (Numerical implementation). Care should be taken regarding the numerical resolution of the system of Riccati equations and the choice of the truncation order associated to ψ . In fact, the shuffle product $\sqcup\sqcup$ cannot be exact since at each discretization steps of the ODE, it will double the truncation order of ψ . If we assume that $\xi_t|_2 \in T^{\tilde{M}}(\mathbb{R}^2)$, then, as in Abi Jaber and Gérard (2025a), we decide to fix, at each step, the truncation order of ψ to $2\tilde{M}$, and to consider the shuffle product projected on $T^{2\tilde{M}}(\mathbb{R}^3)$ defined as $\widetilde{\sqcup\sqcup} : (T^{2\tilde{M}}(\mathbb{R}^3))^2 \rightarrow T^{2\tilde{M}}(\mathbb{R}^3)$. Hence, given $\xi_t|_2 \in T^{\tilde{M}}(\mathbb{R}^2)$, ψ is an element of $T^{2\tilde{M}}(\mathbb{R}^3)$. We refer to (Abi Jaber and Gérard, 2025a, Section 5.1.) for a more in-depth discussion, and an analysis of the quality of convergence given different truncation orders for ψ . ■

Explicit solution to the infinite-dimensional system of Riccati equations: two examples

We now consider two examples for which the infinite-dimensional system of Riccati equations admits an explicit solution and for which the assumptions of the verification theorem are satisfied. The first example is the case without permanent impact i.e. $\nu = 0$. In this case, we fall back into the Bank et al. (2017) framework, so it is not surprising to obtain an existence result for the system of Riccati equations. The second example is the case where we restrict ourselves to European quadratic payoff such that $\xi = \Gamma \mathbf{22}$ with $\Gamma \in \mathbb{R}$, and we retrieve the result of Almgren and Li (2016). Furthermore, for both examples, we also deduce the explicit form of the

optimal trading speed θ^* .

Before discussing the two examples, we consider a lemma that gives us a sufficient condition for the existence and admissibility of θ^* .

Lemma 4.5.8. *Assume that there exists $(\psi_t)_{t \in [0, T]} \in \mathcal{I}'(\hat{\mathbb{Z}}^\theta)$ solution of (4.5.20) such that (4.5.21) holds. Moreover, assume there exists a constant $C > 0$ such that, for $\theta, \phi \in \mathcal{A}$,*

$$\left| \frac{1}{2\eta} \left\langle \left(\nu(\mathbf{3} + \phi X_0 - \tilde{\xi}_t) - (\nu\psi_t|_2 + \psi_t|_3) \right), \hat{\mathbb{Z}}_s^\theta \otimes (\nu\mathbf{2} + \mathbf{3}) \otimes \hat{\mathbb{Z}}_{s,t}^\phi \right\rangle \right| \leq C, \quad s < t \leq T. \quad (4.5.26)$$

Then, there exists $\theta^* \in \mathcal{A}$ satisfying (4.5.22).

Proof. First, for a given $\alpha > 0$, we define the Banach space $\mathcal{L}^p(\alpha)$ by

$$\mathcal{L}^p(\alpha) := \left\{ \theta : [0, T] \times \Omega \rightarrow \mathbb{R} \text{ prog. measurable process} \right. \\ \left. \text{such that } \|\theta\|_{\mathcal{L}^p(\alpha)}^p := \mathbb{E} \left(\int_0^T e^{-\alpha t} |\theta_t|^p dt \right) < \infty \right\},$$

with $p = 2 \vee 2\tilde{M}\mathbb{1}_{\{\nu > 0\}}$. To prove the existence of θ^* , we use a fixed point approach. Namely, we prove that, for $\theta \in \mathcal{L}^p(\alpha)$, the functional F defined such as, for $t \leq T$,

$$F(\theta)_t := \frac{1}{2\eta} \langle \nu(\mathbf{3} + \phi X_0 - \tilde{\xi}_t) - (\nu\psi_t|_2 + \psi_t|_3), \hat{\mathbb{Z}}_t^\theta \rangle,$$

is a contraction. As done in the proof of Lemma 4.5.2, using a variation of constant approach, we observe that, for $\theta, \phi \in \mathcal{L}^p(\alpha)$ and $t \leq T$,

$$\hat{\mathbb{Z}}_t^\theta - \hat{\mathbb{Z}}_t^\phi = \int_0^t \left[\hat{\mathbb{Z}}_s^\theta \otimes (\nu\mathbf{2} + \mathbf{3}) \otimes \hat{\mathbb{Z}}_{s,t}^\phi \right] (\theta_s - \phi_s) ds.$$

Therefore, we have that

$$\begin{aligned} & |F(\theta)_t - F(\phi)_t| \\ &= \left| \int_0^t \frac{1}{2\eta} \langle \nu(\mathbf{3} + \phi X_0 - \tilde{\xi}_t) - (\nu\psi_t|_2 + \psi_t|_3), \hat{\mathbb{Z}}_s^\theta \otimes (\nu\mathbf{2} + \mathbf{3}) \otimes \hat{\mathbb{Z}}_{s,t}^\phi \rangle (\theta_s - \phi_s) ds \right|, \end{aligned}$$

and as we assume (4.5.26), using Hölder's inequality, we observe that, for $t \leq T$,

$$|F(\theta)_t - F(\phi)_t|^p \leq C^p t^{p-1} \int_0^t (\theta_s - \phi_s)^p ds.$$

In this case, we infer that

$$\begin{aligned} \|F(\theta) - F(\phi)\|_{\mathcal{L}^p(\alpha)}^p &= \mathbb{E} \left(\int_0^T e^{-\alpha t} |F(\theta)_t - F(\phi)_t|^p dt \right) \\ &\leq C^p \mathbb{E} \left(\int_0^T e^{-\alpha t} t^{p-1} \int_0^t (\theta_s - \phi_s)^p ds dt \right) \\ &\leq \frac{C^p T^{p-1}}{\alpha} \|\theta - \phi\|_{\mathcal{L}^p(\alpha)}^p. \end{aligned}$$

Then, there exists $\alpha > 0$, such that F is a contraction for $\theta \in \mathcal{L}^p(\alpha)$. Therefore, using the Banach fixed-point theorem, there exists a unique θ^* that satisfies (4.5.22). Since there exists $\alpha > 0$ such that $\theta^* \in \mathcal{L}^p(\alpha)$, we deduce that

$$\mathbb{E} \left(\int_0^T |\theta_t^*|^p dt \right) \leq e^{\alpha T} \|\theta^*\|_{\mathcal{L}^p(\alpha)}^p < \infty,$$

and then $\theta^* \in \mathcal{A}$. $\square$

Proposition 4.5.9 (No permanent market impact). *Let us assume that $\nu = 0$ and $\mu = 0$. Let us define the time-dependent functions $f(t)$ and $K(t, s)$ as, for $t < T$,*

$$f(t) := c \tanh \left(\frac{c(T-t)}{\eta} \right), \quad (4.5.27)$$

with $c := \sqrt{\frac{\lambda \sigma^2 \eta}{2}}$ and, for $t < s < T$,

$$K(t, s) := \frac{c}{\eta} \frac{\cosh \left(\frac{c(T-s)}{\eta} \right)}{\sinh \left(\frac{c(T-t)}{\eta} \right)}. \quad (4.5.28)$$

Moreover, let us define $\tilde{\mathcal{E}}_t$ such that for all $\mathbf{w} = \mathbf{i}_1 \dots \mathbf{i}_n$ with $n \in \mathbb{N}$,

$$\tilde{\mathcal{E}}_t^{\mathbf{w}} := \begin{cases} 0, & \text{if } \exists k \in \{1, \dots, n\} \text{ such that } \mathbf{i}_k = \mathbf{3}, \\ \mathcal{E}_t^{\mathbf{w}}, & \text{if } \nexists k \in \{1, \dots, n\} \text{ such that } \mathbf{i}_k = \mathbf{3}, \end{cases}$$

with $\mathcal{E}_t$ given by (4.4.3), and $\hat{\xi}_t$ such as, for $t < T$,

$$\hat{\xi}_t := \int_t^T K(t, s) \tilde{\xi}_s|_{\tilde{\mathcal{E}}_{s-t}} ds. \quad (4.5.29)$$

Then, ψ_t defined, for $t < T$, by

$$\begin{aligned} \psi_t &= f(t) \left[\mathbf{3} + \phi X_0 - \hat{\xi}_t \right]^{\sqcup 2} + \frac{\lambda \sigma^2}{2} \int_t^T \left(\tilde{\xi}_s - \hat{\xi}_s \right)^{\sqcup 2} \Big|_{\tilde{\mathcal{E}}_{s-t}} ds \\ &\quad + \sigma^2 \int_t^T f(s) \left(\hat{\xi}_s|_{\mathbf{2}} \right)^{\sqcup 2} \Big|_{\tilde{\mathcal{E}}_{s-t}} ds, \end{aligned} \quad (4.5.30)$$

is a solution to the infinite-dimensional system of Riccati equation (4.5.20) belongs to $\mathcal{I}'(\hat{\mathbb{Z}}^\theta)$, and satisfies (4.5.21). Moreover, there exists $\theta^* \in \mathcal{A}$ solution to

$$\theta_t^* = \frac{1}{\eta} f(t) \left(\langle \hat{\xi}_t, \hat{\mathbb{Z}}_t^{\theta^*} \rangle - X_t^{\theta^*} \right), \quad t \leq T, \quad (4.5.31)$$

and θ^* is the optimal control.

Proof. First, let us define, for $t < s < T$, $\ell_{t,s}^1 := \left(\hat{\xi}_s|_{\mathbf{2}} \right)^{\sqcup \sqcup 2} \Big|_{\tilde{\mathcal{E}}_{s-t}}$ and $\ell_{t,s}^2 := \left(\tilde{\xi}_s - \hat{\xi}_s \right)^{\sqcup \sqcup 2} \Big|_{\tilde{\mathcal{E}}_{s-t}}$. Then, we observe that

$$\begin{aligned} \dot{\psi}_t &= \dot{f}(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right]^{\sqcup \sqcup 2} - 2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right] \sqcup \dot{\xi}_t - \sigma^2 f(t) \left(\hat{\xi}_t|_{\mathbf{2}} \right)^{\sqcup \sqcup 2} \\ &\quad + \sigma^2 \int_t^T f(s) \dot{\ell}_{t,s}^1 ds - \frac{\lambda \sigma^2}{2} \left(\tilde{\xi}_t - \hat{\xi}_t \right)^{\sqcup \sqcup 2} + \frac{\lambda \sigma^2}{2} \int_t^T \dot{\ell}_{t,s}^2 ds \\ \psi_t|_{\mathbf{1}} &= -2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right] \sqcup \hat{\xi}_t|_{\mathbf{1}} + \sigma^2 \int_t^T f(s) \ell_{t,s}^1|_{\mathbf{1}} ds + \frac{\lambda \sigma^2}{2} \int_t^T \ell_{t,s}^2|_{\mathbf{1}} ds \\ \psi_t|_{\mathbf{22}} &= -2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right] \sqcup \hat{\xi}_t|_{\mathbf{22}} + 2f(t) \left(\hat{\xi}_t|_{\mathbf{2}} \right)^{\sqcup \sqcup 2} + \sigma^2 \int_t^T f(s) \ell_{t,s}^1|_{\mathbf{22}} ds \\ &\quad + \frac{\lambda \sigma^2}{2} \int_t^T \ell_{t,s}^2|_{\mathbf{22}} ds \\ \psi_t|_{\mathbf{3}} &= 2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right], \end{aligned}$$

with

$$\dot{\xi}_t = -K(t, t) \tilde{\xi}_t + \int_t^T \left(\dot{K}(t, s) \tilde{\xi}_s|_{\tilde{\mathcal{E}}_{s-t}} + K(t, s) \dot{\xi}_s|_{\tilde{\mathcal{E}}_{s,t}} \right) ds.$$

Using similar arguments as in the proof of Lemma 4.4.3, we have that for a given $\ell \in T(\mathbb{R}^3)$ then, for $t < s$,

$$\dot{\ell}|_{\tilde{\mathcal{E}}_{s-t}} = - \left(\dot{\ell}|_{\tilde{\mathcal{E}}_{s-t}} \right)|_{\mathbf{1}} - \frac{1}{2} \sigma^2 \left(\dot{\ell}|_{\tilde{\mathcal{E}}_{s-t}} \right)|_{\mathbf{22}}. \quad (4.5.32)$$

Therefore, we observe that

$$\begin{aligned} -\psi_t|_{\mathbf{1}} - \frac{1}{2} \sigma^2 \psi_t|_{\mathbf{22}} &= -2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right] \sqcup \int_t^T K(t, s) \dot{\xi}_s|_{\tilde{\mathcal{E}}_{s,t}} ds \\ &\quad - \sigma^2 f(t) \left(\hat{\xi}_t|_{\mathbf{2}} \right)^{\sqcup \sqcup 2} + \sigma^2 \int_t^T f(s) \dot{\ell}_{t,s}^1 ds + \frac{\lambda \sigma^2}{2} \int_t^T \dot{\ell}_{t,s}^2 ds. \end{aligned} \quad (4.5.33)$$

Moreover, we also have that

$$\dot{f}(t) = \frac{1}{\eta} f(t)^2 - \frac{\lambda}{2} \sigma^2, \quad \dot{K}(t, s) = K(t, s) K(t, t),$$

with $K(t, t) = \frac{c^2}{\eta f(t)} = \frac{\lambda \sigma^2}{2} \frac{1}{f(t)}$. Thus, we infer that

$$\begin{aligned} & \dot{f}(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right]^{\sqcup \sqcup 2} - 2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right] \\ & \sqcup \sqcup \left[-K(t, t) \tilde{\xi}_t + \int_t^T \dot{K}(t, s) \tilde{\xi}_s |_{\tilde{\mathcal{E}}_{s-t}} ds \right] - \frac{\lambda \sigma^2}{2} (\tilde{\xi}_t - \hat{\xi}_t)^{\sqcup \sqcup 2} \\ & = \dot{f}(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right]^{\sqcup \sqcup 2} - \frac{\lambda}{2} \sigma^2 \left[(\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t)^{\sqcup \sqcup 2} - (\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t)^{\sqcup \sqcup 2} \right] \\ & = -\frac{\lambda}{2} \sigma^2 (\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t)^{\sqcup \sqcup 2} + \frac{1}{4\eta} (\psi_t | \mathbf{3})^{\sqcup \sqcup 2}. \end{aligned} \tag{4.5.34}$$

Putting together (4.5.33) and (4.5.34), we obtain that ψ_t satisfies the system of Riccati equations (4.5.20). As $K(t, s)$ is bounded for $t < s < T$, $\hat{\xi}_t \in T(\mathbb{R}^3)$ for all $t < T$, then, since $f(t)$ is also bounded for all $t < T$, ψ_t is well defined, has finitely many non-zero terms and belongs to $\mathcal{I}'(\hat{\mathbb{Z}}^\theta)$. We also observe that, as $\hat{\xi}_t \in T(\mathbb{R}^3)$ and $f(t)$ is bounded, there exists a positive bounded time-dependent function $C(t)$ such that, for $\theta \in \mathcal{A}$,

$$\begin{aligned} \mathbb{E} \left(\int_0^T \langle \psi_t | \mathbf{2}, \hat{\mathbb{Z}}_t^\theta \rangle^2 dt \right) &= \mathbb{E} \left(\int_0^T \langle 2f(t) [\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t] \sqcup \hat{\xi}_t | \mathbf{2} - \sigma^2 \int_t^T f(s) \ell_{t,s}^1 | \mathbf{2} ds \right. \\ &\quad \left. - \frac{\lambda \sigma^2}{2} \int_t^T \ell_{t,s}^2 | \mathbf{2} ds, \hat{\mathbb{Z}}_t^\theta \rangle^2 dt \right) \\ &\leq \mathbb{E} \left(\int_0^T C(t) \left(1 + |X_t^\theta|^2 + |S_t|^{4\tilde{M}} \right) dt \right) < \infty, \end{aligned}$$

and hence (4.5.21) is satisfied. Furthermore, using the explicit form of ψ_t , the feedback equation (4.5.22) reduces

$$\theta_t^* = -\frac{f(t)}{\eta} \langle \mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t, \hat{\mathbb{Z}}_t^{\theta^*} \rangle = \frac{f(t)}{\eta} \left(\langle \hat{\xi}_t, \hat{\mathbb{Z}}_t^{\theta^*} \rangle - X_t^{\theta^*} \right). \tag{4.5.35}$$

It remains to show that there exists θ^* satisfying (4.5.35) and belonging to $\mathcal{A}$. For that, it is sufficient to show that the assumption (4.5.26) of Lemma 4.5.8 is satisfied. But as we have that

$$\psi_t | \mathbf{3} = 2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \hat{\xi}_t \right],$$

with $f(t)$ bounded, then we observe that

$$\left| \frac{1}{2\eta} \left\langle \psi_t|_{\mathbf{3}}, \hat{\mathbb{Z}}_s^\theta \otimes \mathbf{3} \otimes \hat{\mathbb{Z}}_{s,t}^\phi \right\rangle \right| = \frac{1}{\eta} |f(t)| \leq \frac{1}{\eta} \sup_{t \leq T} |f(t)|,$$

and thus the assumption (4.5.26) is satisfied. $\square$

Proposition 4.5.10 (European quadratic payoff). *Let us assume that $\mu = 0$, $\nu < \sqrt{2\eta\lambda\sigma^2}$, and $\xi = \Gamma\mathbf{22}$ with $\Gamma \in \mathbb{R}$ such that $|\nu\Gamma| < 1$. Moreover, let us consider the time-dependent function $f(t)$ given by*

$$f(t) = \frac{1}{2(1-\nu\Gamma)} \left[\nu + c \tanh \left(\frac{c(1-\nu\Gamma)}{2\eta} (T-t) - \frac{1}{2} \ln \left(\frac{c+\nu}{c-\nu} \right) \right) \right], \quad (4.5.36)$$

with $c := \sqrt{2\eta\lambda\sigma^2}$. Then, ψ_t defined as, for $t < T$,

$$\psi_t = f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t \right]^{\sqcup 2} + \sigma^2 \boldsymbol{\phi} \int_t^T f(s) \Gamma^2 ds \quad (4.5.37)$$

is a solution to the infinite-dimensional system of Riccati equation (4.5.20), belongs to $\mathcal{I}'(\hat{\mathbb{Z}}^\theta)$, and satisfies (4.5.21). Moreover, there exists $\theta^* \in \mathcal{A}$ solution to

$$\theta_t^* = \frac{\nu - 2f(t)(1-\nu\Gamma)}{2\eta} \left(X_t^{\theta^*} - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^{\theta^*} \rangle \right), \quad (4.5.38)$$

and θ^* is the optimal control.

Proof. First, we observe that, since $\xi_t|_{\mathbf{2}} = \Gamma\mathbf{2}$, then $\dot{\xi}_t = 0$, $\tilde{\xi}_t|_{\mathbf{2}} = \Gamma\boldsymbol{\phi}$ and $\tilde{\xi}_t|_{\mathbf{1}} = \tilde{\xi}_t|_{\mathbf{22}} = 0$. In this case,

$$\begin{aligned} \dot{\psi}_t &= \dot{f}(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t \right]^{\sqcup 2} - \sigma^2 f(t) \Gamma^2 \boldsymbol{\phi} \\ \psi_t|_{\mathbf{1}} &= 0 \\ \psi_t|_{\mathbf{2}} &= -2\Gamma f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t \right] \\ \psi_t|_{\mathbf{22}} &= 2f(t) \Gamma^2 \boldsymbol{\phi} \\ \psi_t|_{\mathbf{3}} &= 2f(t) \left[\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t \right]. \end{aligned}$$

We deduce that

$$-\sigma^2 f(t) \Gamma^2 \boldsymbol{\phi} = -\psi_t|_{\mathbf{1}} - \frac{1}{2} \sigma^2 \psi_t|_{\mathbf{22}}. \quad (4.5.39)$$

Moreover, we also observe that

$$\frac{1}{4\eta} \left[\nu(\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t) - (\nu\psi_t|_{\mathbf{2}} + \psi_t|_{\mathbf{3}}) \right]^{\sqcup 2} = \frac{1}{4\eta} (\nu - 2f(t)(1-\nu\Gamma))^2 (\mathbf{3} + \boldsymbol{\phi} X_0 - \tilde{\xi}_t)^{\sqcup 2}.$$

Thus, using the form of $f(t)$ given by (4.5.36), we observe that

$$\dot{f}(t) = \frac{1}{4\eta} (\nu - 2f(t)(1 - \nu\Gamma))^2 - \frac{\lambda}{2}\sigma^2, \quad t < T \text{ and } f(T) = 0,$$

and we infer that

$$\begin{aligned} \dot{f}(t) \left[\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t \right]^{\sqcup\sqcup 2} &= -\frac{\lambda}{2}\sigma^2 (\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t)^{\sqcup\sqcup 2} \\ &\quad + \frac{1}{4\eta} \left[\nu(\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t) - (\nu\psi_t|_{\mathbf{2}} + \psi_t|_{\mathbf{3}}) \right]^{\sqcup\sqcup 2}. \end{aligned}$$

Putting all together, we deduce that ψ_t satisfies the system of Riccati equations (4.5.20). Since $f(t)$ is bounded for all $t < T$, ψ_t is well defined, has finitely many non-zero terms and belongs to $\mathcal{I}'(\hat{\mathbb{Z}}^\theta)$. We also have that (4.5.21) is satisfied since, for $\theta \in \mathcal{A}$,

$$\mathbb{E} \left(\int_0^T \langle \psi_t|_{\mathbf{2}}, \hat{\mathbb{Z}}_t^\theta \rangle^2 dt \right) = 4\Gamma^2 \mathbb{E} \left(\int_0^T f(t)^2 (X_t^\theta - \Gamma P_t^\theta)^2 dt \right) < \infty.$$

Furthermore, using the explicit form of ψ_t , the feedback equation (4.5.22) reduces

$$\theta_t^* = \frac{\nu - 2f(t)(1 - \nu\Gamma)}{2\eta} \left(X_t^{\theta^*} - \langle \xi_t|_{\mathbf{2}}, \hat{\mathbb{P}}_t^{\theta^*} \rangle \right). \quad (4.5.40)$$

It remains to show that there exists θ^* satisfying (4.5.40) and belonging to $\mathcal{A}$. For that, it is sufficient to show that the assumption (4.5.26) of Lemma 4.5.8 is satisfied. As

$$\nu(\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t) - (\nu\psi_t|_{\mathbf{2}} + \psi_t|_{\mathbf{3}}) = (\nu - 2f(t)(1 - \nu\Gamma)) \left[\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t \right],$$

with $f(t)$ bounded, $\tilde{\xi}_t|_{\mathbf{2}} = \Gamma\boldsymbol{\phi}$, then we get that

$$\begin{aligned} &\left| \frac{1}{2\eta} \left\langle \left(\nu(\mathbf{3} + \boldsymbol{\phi}X_0 - \tilde{\xi}_t) - (\nu\psi_t|_{\mathbf{2}} + \psi_t|_{\mathbf{3}}) \right), \hat{\mathbb{Z}}_s^\theta \otimes (\nu\mathbf{2} + \mathbf{3}) \otimes \hat{\mathbb{Z}}_{s,t}^\phi \right\rangle \right| \\ &= \frac{1}{2\eta} |(\nu - 2f(t)(1 - \nu\Gamma))(1 - \nu\Gamma)| \\ &\leq \frac{1}{2\eta} \sup_{t \leq T} |(\nu - 2f(t)(1 - \nu\Gamma))(1 - \nu\Gamma)|, \end{aligned}$$

and the assumption (4.5.26) is satisfied. $\square$

4.6 Numerical illustration

In this section, we illustrate numerically the signature-based hedging strategies given by Theorem 4.5.7 by solving numerically the system of infinite-dimensional

Riccati equations (4.5.20). First we consider exact signature payoffs such as European or Asian quadratic options and then we take more general path-dependent payoffs that we approximate by signature payoffs relying on the universal approximation theorem.

Moreover, we compare the signature-based strategy taking into account both temporary and permanent market impact with the optimal hedging strategy that only accounts for temporary market impact deduced for general path-dependent payoffs in Bank et al. (2017).

For our numerical results, we set the initial inventory X_0 equal to the Bachelier Δ . For signature payoffs, this reduces to $X_0 = (\xi_0|_2)^\theta$. Moreover, we assume that the initial portfolio value V_0 is equal to this indifference price π defined in Section 4.5. Therefore, the initial portfolio value will vary depending on the choice of market impact parameters η and ν , which will impact the final value of the P&L but not the trading speed. It reduces to $V_0 = \xi_0^\theta - \psi_0$ for signature payoffs. Finally, recall that, as explained in Remark 4.8, when we solve numerically the Riccati equation (4.5.20), we consider the shuffle product projected on $T^{2\tilde{M}}(\mathbb{R}^3)$ defined as $\widetilde{\sqcup} : (T^{2\tilde{M}}(\mathbb{R}^3))^2 \rightarrow T^{2\tilde{M}}(\mathbb{R}^3)$, such that $\psi_t \in T^{2\tilde{M}}(\mathbb{R}^3)$, $\tilde{M}$ being the truncation order $\xi_t|_2 \in T^{\tilde{M}}(\mathbb{R}^2)$. Unless otherwise stated, we consider the following parameters for our numerical results:

$$S_0 = 10, \mu = 0, \sigma = 2, T = 0.2, \eta = 0.001, \nu = 0.001, \lambda = 0.01.$$

4.6.1 Signature payoffs

We consider two particular signature payoffs:

- European quadratic payoff: $H_T^\theta = N \times (P_T^\theta - K)^2$,
- Asian quadratic payoff: $H_T^\theta = N \times \left(\frac{1}{T} \int_0^T P_t^\theta dt - K \right)^2$,

where N corresponds to a given nominal.

First, since the European quadratic payoff has a constant Gamma, an application of Proposition 4.5.10 yields an explicit expression for the optimal trading speed, in the spirit of Almgren and Li (2016). Thus, this allows us to perform a sanity check on the stability of the numerical solution of the Riccati equation (4.5.20). Figure 4.1 compares a sample path of trading speed obtained for a given European quadratic option. As expected, we see that the optimal strategy obtained by numerically solving the Riccati equation perfectly matches the closed form expression deduced in Proposition 4.5.10. This suggests that the numerical resolution of the equation appears to be stable.

Moving to Asian quadratic options, we no longer have an explicit expression for the optimal trading speed, except for $\nu = 0$, as stated by Proposition 4.5.9, where

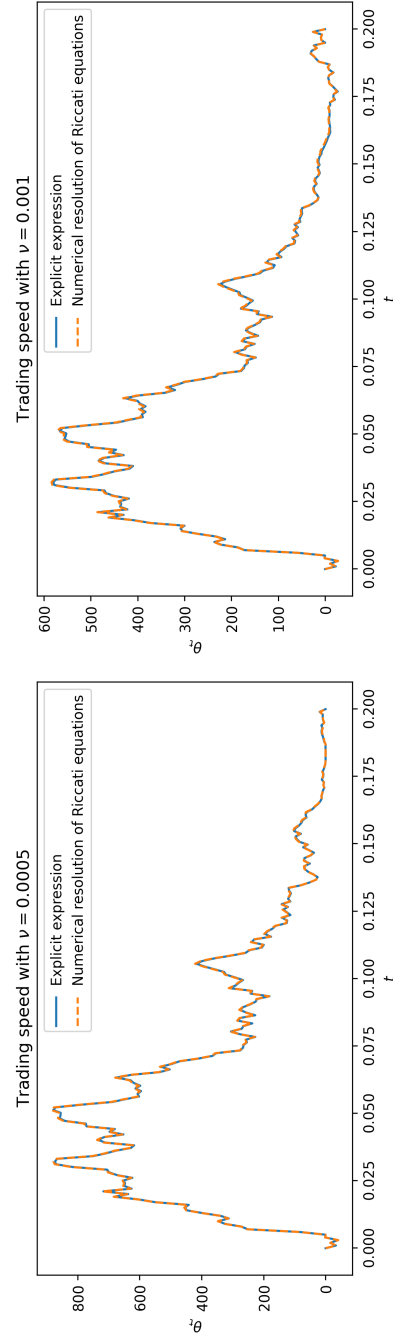


Figure 4.1: Sample path of trading speed for the European quadratic payoff: explicit expression given by Proposition 4.5.10 vs numerical resolution of Riccati equation with truncated shuffle product. Parameters: $M = 2$, $K = S_0$ and $N = 200$.

we retrieve the same strategy as in Bank et al. (2017). Therefore, for $\nu > 0$, numerically solving the Riccati equation becomes essential to determine the optimal strategy. As a sanity check, as $\nu \rightarrow 0$, the solution obtained from the Riccati system is expected to converge to the explicit solution. This is illustrated in Figure 4.2, which shows a sample path of the optimal trading speed and inventory for a given Asian quadratic option under different permanent market impact parameters. We clearly observe the convergence phenomenon as $\nu \rightarrow 0$, providing an additional validation of our numerical implementation of the Riccati equations. When permanent market impact is introduced ($\nu > 0$), a gap appears whose magnitude increases with the value of the permanent impact parameter ν . As expected, the larger ν is, the slower and flatter the resulting trading speed becomes.

Moreover, to highlight the benefits of incorporating the permanent market impact information into the trading strategy, we compare the P&L generated by two traders, when $\nu > 0$. The first one is aware of the permanent market impact, incorporates this information into her strategy and follows the optimal trading speed for $\nu > 0$. The second one ignores the permanent impact and follows the explicit trading strategy assuming $\nu = 0$ given by Proposition 4.5.9. We assume that both traders have the same risk aversion and ask an indifference price to hedge the option. Therefore, the trader assuming $\nu > 0$ asks a price $\pi(\nu > 0)$ while the second trader, ignoring the permanent impact, asks $\pi(\nu = 0)$. Both traders' strategies are evaluated using the same criterion with $\nu > 0$: the first trader's strategy is optimal, whereas the second trader's is not. Results are shown in Figure 4.3. We can see that, when there is a permanent market impact, there is an advantage to be gained by incorporating this information into the hedging strategy and following the signature-based strategy assuming $\nu > 0$, rather than the strategy assuming only a temporary impact. Obviously, this information gain depends on the size of the permanent impact parameter ν and will be higher when the permanent impact is higher. The gap between the two P&Ls can be attributed to two things:

- the trader omitting the permanent impact will misprice the option, underestimates the risk she takes (see Table 4.1). This will affect the final value of her P&L,
- the trader who overlooks the permanent impact will trade too aggressively (see Figure 4.2), and push the price further in the wrong direction, which will adversely impact her P&L over time. The optimal strategy with $\nu > 0$, taking into account this impact, attempts to push the price in the right direction (see Remark 4.5).

Bachelier price	$\nu = 0$	$\nu = 0.0005$	$\nu = 0.001$	$\nu = 0.0015$
53.333	79.559	80.559	81.218	81.617

Table 4.1: Indifference prices π for Asian quadratic payoff with respect to ν . Other parameters: $M = 4$, $K = S_0$, and $N = 200$.

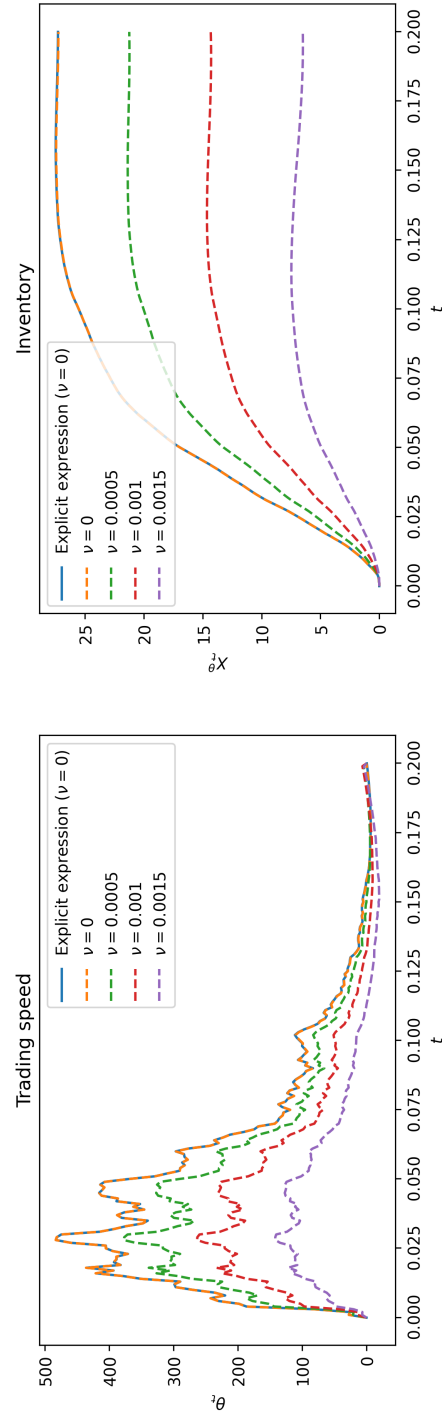


Figure 4.2: Sample path of trading speed and inventory for Asian quadratic payoff. Solid line: explicit hedging strategy assuming $\nu = 0$ given by Proposition 4.5.9; dashed lines: signature-based strategies obtained by numerically solving the Riccati equations with projected shuffle product. Parameters: $M = 4$, $K = S_0$, and $N = 200$.

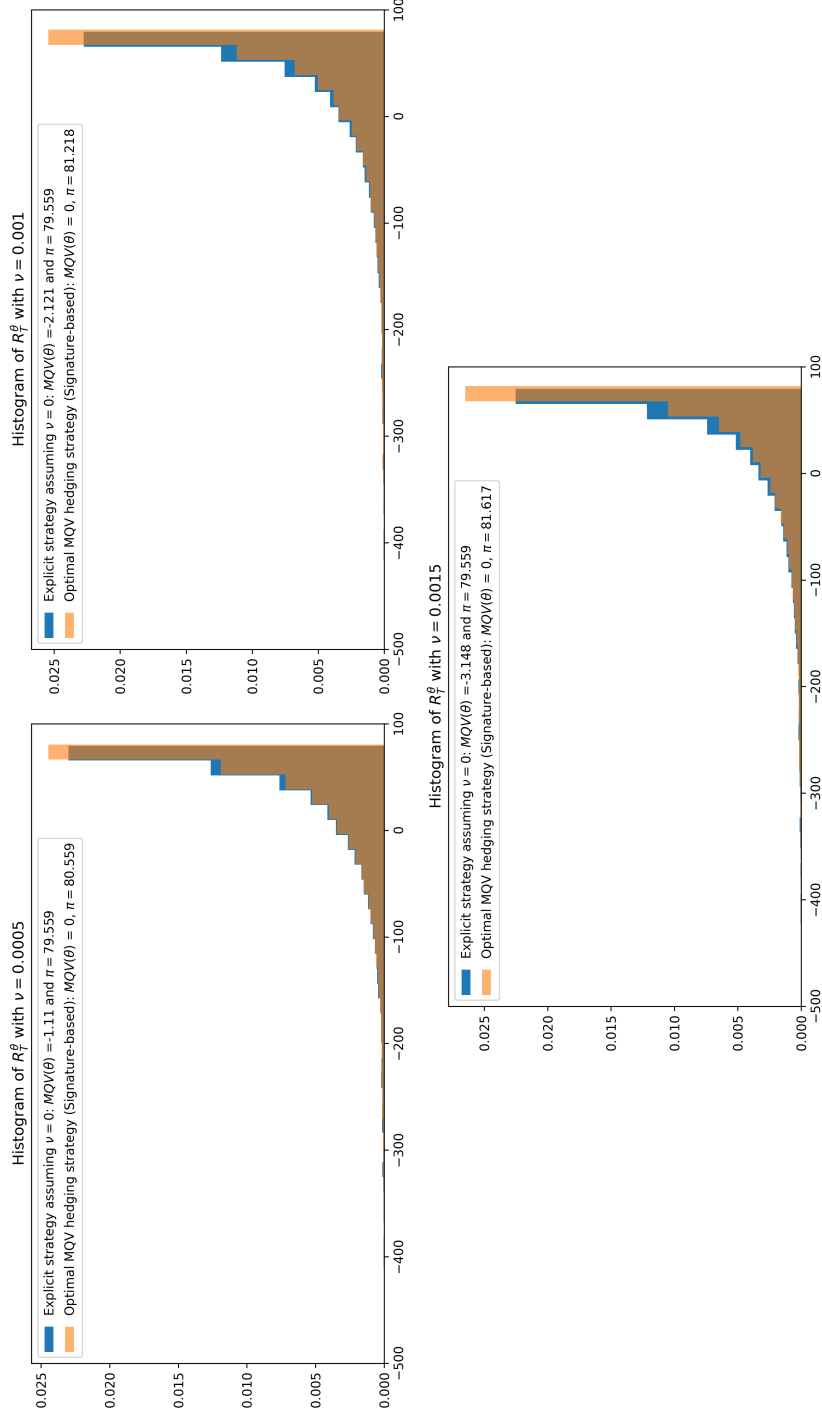


Figure 4.3: Histogram of R_T^θ for Asian quadratic payoff with $\nu > 0$: signature-based strategy vs explicit hedging strategy assuming $\nu = 0$ given by Proposition 4.5.9. Other parameters: $M = 4$, $K = S_0$, and $N = 200$. $MQV(\theta) = \mathbb{E}\left(R_T^\theta - \frac{\lambda}{2}[R^\theta, R^\theta]_T\right)$.

4.6.2 Non-signature path-dependent payoffs

We consider now payoffs that cannot be exactly represented as signature payoffs as it is the case for European, Asian or barrier call options. By relying on the universal linearization property of signatures, see Lyons et al. (2020); Cuchiero et al. (2025b), we can approximate those path-dependent payoffs by signature payoffs, i.e. by linear combination of time-augmented signature elements. More precisely, for a continuous functional F , a continuous semimartingale $(Y_t)_{t \in [0, T]}$ valued in $\mathbb{R}^d$, and a given truncation order $M > 0$, there exists $\ell \in T^M(\mathbb{R}^{d+1})$ such that

$$F((t, Y_t)_{t \leq T}) \approx \langle \ell, \hat{Y}_t \rangle.$$

This means that the semi-explicit signature-based hedging strategy can also be used to hedge general (non-signature) path-dependent payoffs. However, to apply the signature-based hedging strategy, we first have to regress those path-dependent payoffs against truncated signature payoffs to determine ℓ , and then solve the infinite-dimensional Riccati equation associated with the signature approximate payoff $\langle \ell, \hat{Y}_t \rangle$. Algorithm 2 explicitly states how to get a linear signature representation for a given path-dependent payoff. Furthermore, as we consider non-signature payoffs, the indifference price π does not have an explicit form and should also be approximated. In this sense, we approximate the indifference price by $\tilde{\pi}$ such that it satisfies the following equation

$$\mathbb{E} \left(R_T^{\theta_{sig}^*} - \frac{\lambda}{2} [R^{\theta_{sig}^*}, R^{\theta_{sig}^*}]_T \right) = 0, \quad (4.6.1)$$

where

$$R_t^{\theta_{sig}^*} = (\tilde{\pi} - X_0 S_0) + X_t^{\theta_{sig}^*} P_t^{\theta_{sig}^*} - \int_0^t \tilde{P}_s^{\theta_{sig}^*} (\theta_{sig}^*)_s ds - h \left(\left(s, P_s^{\theta_{sig}^*} \right)_{s \leq t} \right), \quad t \leq T,$$

with $h \left(\left(s, P_s^{\theta_{sig}^*} \right)_{s \leq t} \right)$ the Bachelier price of the path-dependent payoff at time $t < T$, and θ_{sig}^* the optimal trading speed associated to the regressed signature payoff.

Algorithm 2 Regression of path-dependent payoffs against truncated signature payoffs

Consider a payoff of the following form

$$H_T = F((t, Y_t)_{t \leq T}),$$

with $(Y_t)_{t \in [0, T]}$ a given stochastic process valued in $\mathbb{R}$.

Input:

- Fix $J > 0$ the number of points in the discretization of $[0, T]$,
- Fix $L > 0$ the number of realisations of trajectory of Y ,
- Fix $M \geq 0$ the truncation order of ℓ ,

Online:

1. Generate L realizations of the Y , denoted by $Y^{(1)}, \dots, Y^{(L)}$,
2. For each realization $l = 1, \dots, L$, compute $H_T^{(l)}$ and the truncated signature $\hat{Y}_T^{(l), \leq M}$ up to order M for $t \in [0, T]$,
3. Regress $(H_T^{(l)})_{1 \leq l \leq L}$ against $(\hat{Y}_T^{(l), \leq M})_{1 \leq l \leq L}$ to learn the coefficients of $\ell \in T^M(\mathbb{R}^2)$ that minimize

$$\mathcal{L} = \frac{1}{L} \sum_{l=1}^L \left| H_T^{(l)} - \langle \ell, \hat{Y}_T^{(l), \leq M} \rangle \right|^2.$$

Let us move on to some numerical illustrations. We consider different types of payoffs:

- European call: $H_T^\theta = N \times (P_T^\theta - K)_+$,
- Asian call: $H_T^\theta = N \times (\frac{1}{T} \int_0^T P_s^\theta ds - K)_+$,
- One-touch max: $H_T^\theta = N \times \mathbb{1}_{\{\max_{t < T} P_t^\theta \geq H\}}$,
- Look-back call with floating strike: $H_T^\theta = N \times (P_T^\theta - \min_{\{t < T\}} P_t^\theta)$,

where N corresponds to a given nominal.

Remark 4.9 (Choice of truncation order M). *To approximate general path-dependent payoffs, we apply a truncation order $M = 5$. This choice is primarily driven by computational considerations, as the complexity and the number of terms increase exponentially with M . While higher truncation orders would in principle reduce the approximation error, they also lead to a significant increase in computational cost. We further observe that, in our market impact setting, relatively low truncation orders (typically $M \leq 5$) already provide accurate and numerically stable results, as discussed in more detail below. ■*

First, recall that, for general European options, Almgren and Li (2016) characterized the optimal hedging strategy as a solution to the HJB equation given by (4.2.5). Using this information, for an European call payoff, we compare the signature-based strategy associated to the regressed signature payoff with the optimal strategy obtained by numerically solving the corresponding HJB equation given by (4.2.5). This allows us to see whether, in this context of market impact, the signature-based strategies are able to well approximate the optimal strategy when payoffs do not admit an exact signature representation.

We consider two types of market: a frictionless market with $\eta = \nu = 0$ and a frictional market with $\eta > 0$ and $\nu > 0$. In the frictionless market, see Figure 4.4, the signature-based strategy provides a close approximation of the perfect hedging strategy for small times t , but it fails to recover the correct behavior as $t \rightarrow T$. The discrepancy is primarily due to the strong nonlinearity of the perfect hedging strategy for call options near maturity. This observation mitigates the conclusion of (Lyons et al., 2020, Section 7.1) regarding the ability of signature-based strategies to accurately recover the perfect hedging strategy in a frictionless setting. By contrast, in the presence of market frictions, see Figure 4.5, the signature approximation performs significantly better. In this case, the sample paths of trading speeds and inventories generated by the signature-based approach are much closer to those of the optimal strategy. This improvement can be attributed to two effects: first, market impact naturally smoothens the optimal trading speeds; second, the signature-approximated payoff leads to a smoother (target) perfect hedging strategy compared to the call payoff. Together, these effects mitigate approximation errors and enhance the accuracy of the signature-based strategy. Thus, in the frictional market, the signature-based strategy can be considered a sufficiently accurate approximation of the optimal strategy, showing a striking improvement compared to the frictionless setting (compare the error on the inventory in Figures 4.4 and 4.5). Clearly, in the absence of frictions, approximating the payoff of a call option by a fifth-order polynomial is not sufficiently accurate for pricing and hedging. However, once market frictions are introduced, the fifth-order polynomial approximation becomes significantly more accurate, even though it additionally requires truncating and numerically solving infinite-dimensional Riccati equations.

We now emphasize the relevance of signature-based strategies to hedge non-signature path-dependent payoffs when $\nu > 0$. To this end, we consider different types of options: European call, Asian call, one-touch and look-back call with floating strike. As in the previous subsection, we compare the P&L generated by two types of traders: the first one is aware of the permanent market impact and follows the signature-based strategy for $\nu > 0$, while the second one ignores the permanent impact and follows the trading strategy of Bank et al. (2017), which consists of tracking the Bachelier Delta of the non-signature payoffs. Both traders' strategies are evaluated using the same criterion with $\nu > 0$.

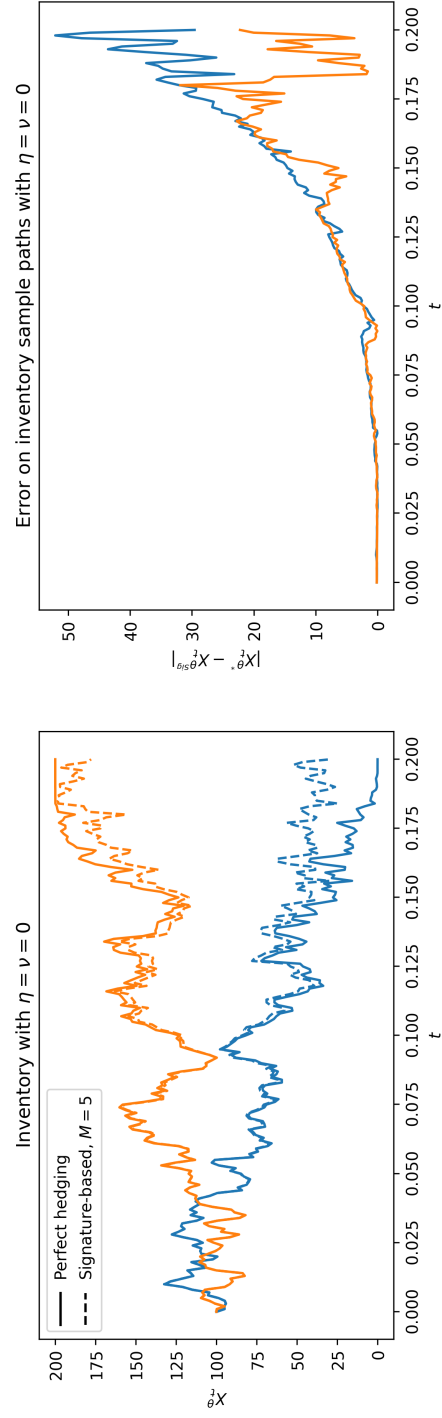


Figure 4.4: Sample paths of inventory X_t^θ for European call payoff with $\eta = \nu = 0$: signature-based approximated strategy vs Bachelier Delta hedging strategy. Parameters: $M = 5$, $K = S_0$, and $N = 200$.

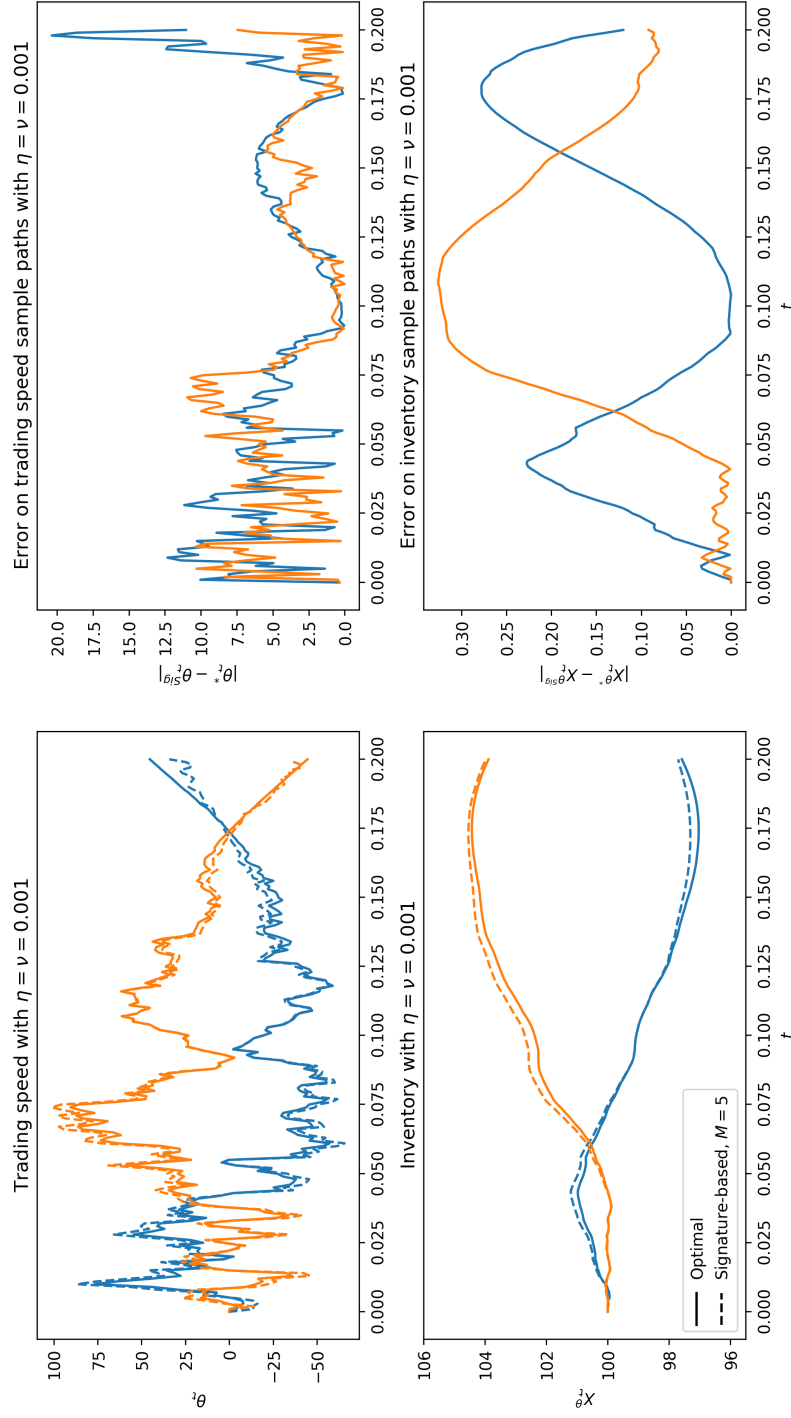


Figure 4.5: Sample paths of trading speed θ_t and inventory X_t for European call payoff with $\eta = \nu = 0.001$: signature-based approximated strategy vs optimal hedging strategy by solving the HJB equation (4.2.5). Parameters: $M = 5$, $K = S_0$, and $N = 200$.

Figure 4.6 shows the histograms of the P&L at maturity generated by two strategies for different types of payoffs. Looking at the histograms, we remark that even though the signature-based strategy is an approximation of the optimal strategy, it seems very relevant for hedging non-signature payoffs in the presence of market frictions. Furthermore, if we compare the mean-quadratic variation functional $MQV(\theta)$, we observe that the signature-based strategy outperforms the Bank et al. (2017) strategy that does not take into account the permanent impact. Thus, we see that there is therefore a substantial advantage in favoring the signature-based hedging strategy. The discrepancy between the P&Ls can be attributed to the same factors as in the case of Asian quadratic payoff. At inception, the trader that follows Bank et al. (2017) strategy misprices the options regarding the risk she takes when $\nu > 0$ (see Table 4.2). Moreover, during the trading period, the trader that does not take into account the permanent impact will trade too aggressively, and this will more adversely impact her P&L over time.

4.7 Proofs

4.7.1 Proof of Lemma 4.5.2

Proof. First, for given $\theta, \phi \in \mathcal{A}$, we have that

$$d\hat{\mathbb{P}}_t^\theta = \hat{\mathbb{P}}_t^\theta \otimes d\hat{P}_t^\theta, \quad d\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} = \hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} \otimes d\hat{P}_t^\theta + \nu\varepsilon\phi_t\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} \otimes \mathbf{2} \, dt.$$

Therefore, we infer that

$$d\left[\frac{\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} - \hat{\mathbb{P}}_t^\theta}{\varepsilon}\right] = \frac{\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} - \hat{\mathbb{P}}_t^\theta}{\varepsilon} \otimes d\hat{P}_t^\theta + \nu\phi_t\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} \otimes \mathbf{2} \, dt.$$

Let us now define $\mathbb{Y}_t$ and β_t by

$$\mathbb{Y}_t := \frac{\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} - \hat{\mathbb{P}}_t^\theta}{\varepsilon}, \quad \beta_t = \nu\phi_t\hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} \otimes \mathbf{2}.$$

Then, we have that

$$d\mathbb{Y}_t = \mathbb{Y}_t \, d\hat{P}_t^\theta + \beta_t dt.$$

In this case, we observe that

$$d\left(\mathbb{Y}_t \otimes (\hat{\mathbb{P}}_t^\theta)^{-1}\right) = d\mathbb{Y}_t \otimes (\hat{\mathbb{P}}_t^\theta)^{-1} + \mathbb{Y}_t \otimes d(\hat{\mathbb{P}}_t^\theta)^{-1},$$

and thus

$$\mathbb{Y}_t \otimes (\hat{\mathbb{P}}_t^\theta)^{-1} = \int_0^t \beta_s \otimes (\hat{\mathbb{P}}_s^\theta)^{-1} \, ds.$$

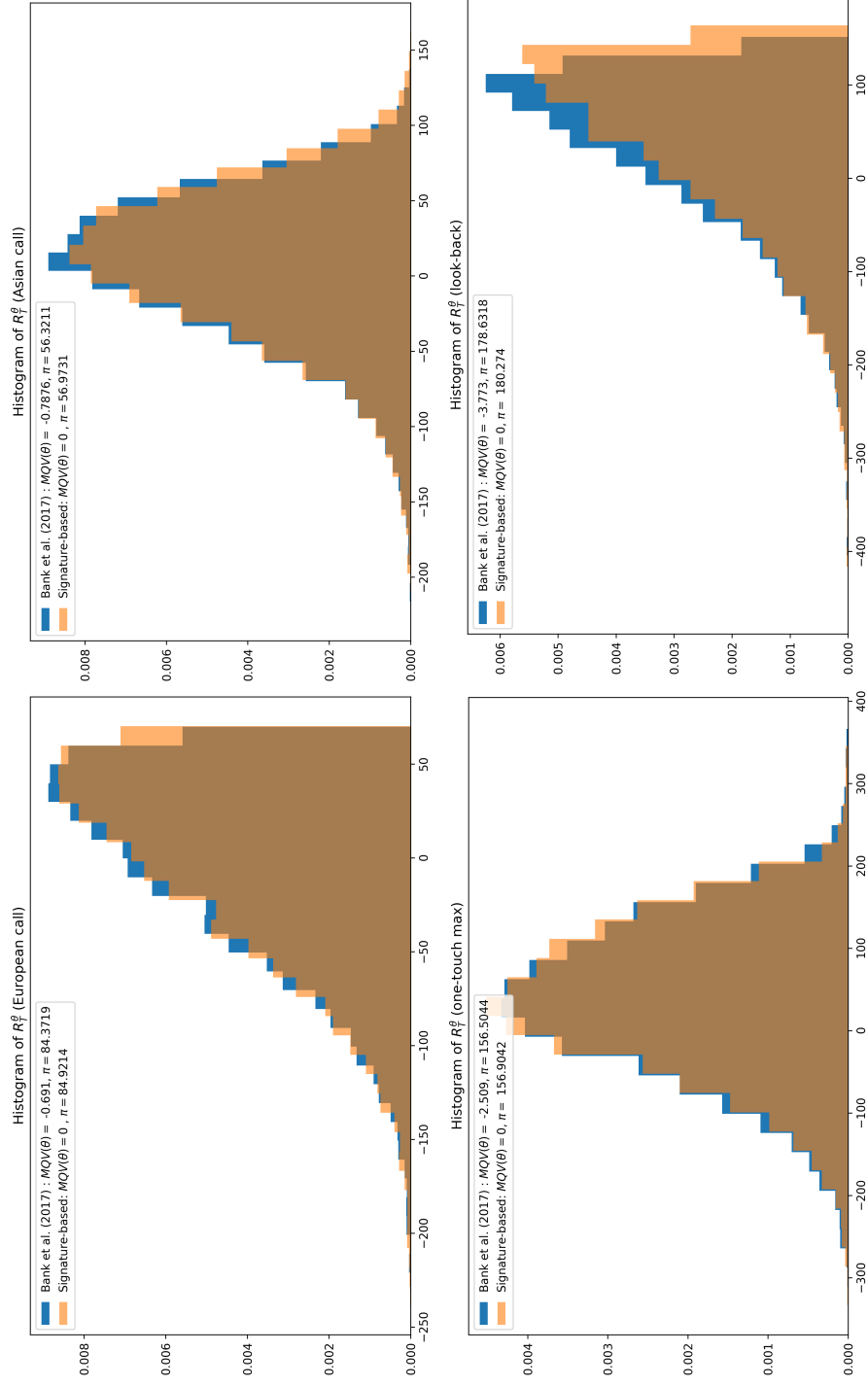


Figure 4.6: Histogram of R_T^θ for different payoffs: signature-based hedging strategy vs Bank et al. (2017) hedging strategy ($\nu = 0$). Parameters: $M = 5$, $K = S_0$, $H = 1.05 \times S_0$ and $N = 200$. $MQV(\theta) = \mathbb{E}(R_T^\theta - \frac{\lambda}{2}[R^\theta, R^\theta]_T)$.

Payoff	Bachelier price	Bank et al. (2017) ($\nu = 0$)	Sig-approx. ($\nu = 0.001$)
European call	71.3649	84.3719	84.9214
Asian call	41.2025	56.3211	56.9731
One-touch	115.2300	156.5044	156.9042
Look-back	142.7299	178.6318	180.2740

Table 4.2: Indifference prices for different payoffs: Bank et al. (2017) indifference price is obtained by using the Bank et al. (2017) strategy with Bachelier Delta as tracking strategy; the indifference price for $\nu = 0.001$ is approximated by $\hat{\pi}$ using (4.6.1). Parameters: $M = 5$, $K = S_0$, $H = 1.05 \times S_0$, and $N = 200$.

Thus, we have that

$$\mathbb{Y}_t = \int_0^t \beta_s \otimes (\hat{\mathbb{P}}_s^\theta)^{-1} \otimes (\hat{\mathbb{P}}_t^\theta) ds = \int_0^t \beta_s \otimes \hat{\mathbb{P}}_{s,t}^\theta ds = \nu \int_0^t (\phi_s \hat{\mathbb{P}}_s^{\theta+\varepsilon\phi} \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{s,t}^\theta) ds.$$

By the Fubini's theorem and dominated convergence theorem, since $\theta, \phi, \nu \in \mathcal{A}$ and $\alpha_t \in T^p(\mathbb{R}^2)$, we have that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E} \left(\nu_t \langle \alpha_t, \hat{\mathbb{P}}_t^{\theta+\varepsilon\phi} - \hat{\mathbb{P}}_t^\theta \rangle \right) = \nu \int_0^t \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left(\phi_s \nu_t \langle \alpha_t, \hat{\mathbb{P}}_s^{\theta+\varepsilon\phi} \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{s,t}^\theta \rangle \right) ds.$$

Finally, as $\theta, \phi, \nu \in \mathcal{A}$ and $\alpha_t \in T^p(\mathbb{R}^2)$, we observe that there exists $\tilde{p} > 1$ such that, for all $s < t < T$,

$$\sup_{\varepsilon > 0} \mathbb{E} \left(\left| \phi_s \nu_t \langle \alpha_t, \hat{\mathbb{P}}_s^{\theta+\varepsilon\phi} \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{s,t}^\theta \rangle \right|^{\tilde{p}} \right) < \infty,$$

hence, $\left(\phi_s \nu_t \langle \alpha_t, \hat{\mathbb{P}}_s^{\theta+\varepsilon\phi} \otimes \mathbf{2} \otimes \hat{\mathbb{P}}_{s,t}^\theta \rangle \right)_{\varepsilon > 0}$ is $L^{\tilde{p}}$ bounded with $\tilde{p} > 1$, and then uniformly integrable. Therefore, we deduce (4.5.11). $\square$

4.7.2 Proof of Lemma 4.5.3

Proof. First, we can rewrite the functional J as

$$\begin{aligned} J(\theta) &= -\eta \langle \theta, \theta \rangle_{L_2} + \langle \mu + \nu \theta, X^\theta - \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} - \frac{\lambda}{2} \sigma^2 \langle X^\theta - \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle, X^\theta - \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} \\ &= -\eta \langle \theta, \theta \rangle_{L_2} + \nu \langle \theta, X^\theta \rangle_{L_2} - \nu \langle \theta, \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} - \frac{\lambda}{2} \sigma^2 \langle X^\theta, X^\theta \rangle_{L_2} \\ &\quad - \frac{\lambda}{2} \sigma^2 \langle \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle, \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} + \lambda \sigma^2 \langle X^\theta, \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} + \langle \mu, X^\theta - \langle \xi, \mathbf{2}, \hat{\mathbb{P}}^\theta \rangle \rangle_{L_2} \\ &=: \sum_{i=1}^7 J_i(\theta). \end{aligned}$$

For $i = 1, \dots, 7$, let us compute the Gâteaux derivatives of J_i for $h \in \mathcal{A}$, such that

$$\langle \nabla J_i(\theta), h \rangle_{L_2} := \lim_{\varepsilon \rightarrow 0} \frac{J_i(\theta + \varepsilon h) - J_i(\theta)}{\varepsilon}.$$

First, we easily observe that

$$\begin{aligned} \langle \nabla J_1(\theta), h \rangle_{L_2} &= -2\eta \langle \theta, h \rangle_{L_2}, \\ \langle \nabla J_4(\theta), h \rangle_{L_2} &= -\lambda \sigma^2 \left\langle \int_0^T \mathbb{E}(X_s^\theta | \mathcal{F}_t) ds, h \right\rangle_{L_2}. \end{aligned}$$

Then, for $i = 2$, we get that

$$\begin{aligned} \frac{J_2(\theta + \varepsilon h) - J_2(\theta)}{\varepsilon} &= \frac{1}{\varepsilon} \left[\nu \mathbb{E} \left(\int_0^T ((\theta_t + \varepsilon h_t) X_t^{\theta + \varepsilon h}) dt \right) - \nu \mathbb{E} \int_0^T (\theta_t X_t^\theta dt) \right] \\ &= \frac{1}{\varepsilon} \nu \left[\mathbb{E} \left(\int_0^T \varepsilon h_t X_t^{\theta + \varepsilon h} dt \right) + \mathbb{E} \left(\int_0^T \theta_t (X_t^{\theta + \varepsilon h} - X_t^\theta) dt \right) \right], \end{aligned}$$

thus, by the tower property of the conditional expectation and Fubini's theorem, taking $\lim_{\varepsilon \rightarrow 0}$ gives,

$$\langle \nabla J_2(\theta), h \rangle_{L_2} = \nu \langle X_t^\theta + \int_{\cdot}^T \mathbb{E}(\theta_s | \mathcal{F}_{\cdot}) ds, h \rangle_{L_2}.$$

For $i = 3$,

$$\begin{aligned} \frac{J_3(\theta + \varepsilon h) - J_3(\theta)}{\varepsilon} &= \frac{1}{\varepsilon} \nu \left[- \mathbb{E} \left(\int_0^T (\theta_t + \varepsilon h_t) \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle dt \right) \right. \\ &\quad \left. + \mathbb{E} \left(\int_0^T \theta_t \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle dt \right) \right] \\ &= -\frac{1}{\varepsilon} \nu \mathbb{E} \left(\int_0^T \theta_t \left(\langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt \right) \\ &\quad - \nu \langle \langle \xi_{\cdot} | \mathbf{2}, \hat{\mathbb{P}}_{\cdot}^{\theta + \varepsilon h} \rangle, h \rangle_{L_2}. \end{aligned}$$

Since $\theta, h \in \mathcal{A}$, using Lemma 4.5.2, the dominated convergence theorem and Fubini's theorem, we obtain that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E} \left(\int_0^T \theta_t \left(\langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt \right) = \mathbb{E} \left(\int_0^T h_t \int_t^T \mathbb{E}(\theta_s \nu \Gamma_{t,s}^\theta | \mathcal{F}_t) ds dt \right).$$

Therefore, we infer that

$$\langle \nabla J_3(\theta), h \rangle_{L_2} = -\nu \left[\left\langle \int_{\cdot}^T \mathbb{E}(\theta_s \nu \Gamma_{\cdot,s}^\theta | \mathcal{F}_{\cdot}) ds + \langle \xi_{\cdot} | \mathbf{2}, \hat{\mathbb{P}}_{\cdot}^\theta \rangle, h \right\rangle_{L_2} \right].$$

Using similar arguments, we have that

$$\frac{J_5(\theta + \varepsilon h) - J_5(\theta)}{\varepsilon} = -\frac{1}{\varepsilon} \frac{\lambda}{2} \sigma^2 \mathbb{E} \left(\int_0^T \left(\langle (\xi_t | \mathbf{2})^{\perp \perp 2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle - \langle (\xi_t | \mathbf{2})^{\perp \perp 2}, \hat{\mathbb{P}}_t^\theta \rangle \right) dt \right),$$

taking $\lim_{\varepsilon \rightarrow 0}$ leads to

$$\lim_{\varepsilon \rightarrow 0} \frac{J_5(\theta + \varepsilon h) - J_5(\theta)}{\varepsilon} = -\frac{\lambda}{2} \sigma^2 \left\langle \int_{\cdot}^T \mathbb{E}(\nu \tilde{\Gamma}_{\cdot,s}^\theta | \mathcal{F}_{\cdot}) ds, h \right\rangle_{L_2}.$$

For $i = 6$, we observe that

$$\begin{aligned} \frac{J_6(\theta + \varepsilon h) - J_6(\theta)}{\varepsilon} = & \lambda \sigma^2 \left[\mathbb{E} \left(\int_0^T \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle \int_0^t h_s ds dt \right) \right. \\ & \left. + \frac{1}{\varepsilon} \mathbb{E} \left(\int_0^T \left(\langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle \right) X_t^\theta dt \right) \right] \end{aligned}$$

taking the limit $\lim_{\varepsilon \rightarrow 0}$ leads to

$$\lim_{\varepsilon \rightarrow 0} \frac{J_6(\theta + \varepsilon h) - J_6(\theta)}{\varepsilon} = \lambda \sigma^2 \left\langle \int_0^T \mathbb{E} \left(\langle \xi_s | \mathbf{2}, \hat{\mathbb{P}}_s^\theta \rangle + X_s^\theta \nu \Gamma_{\cdot, s}^\theta | \mathcal{F}_\cdot \right) ds, h \right\rangle_{L_2}.$$

Finally, for $i = 7$, we have that

$$\frac{J_7(\theta + \varepsilon h) - J_7(\theta)}{\varepsilon} = \frac{1}{\varepsilon} \mu \left[\mathbb{E} \int_0^T \left((X_t^{\theta + \varepsilon h} - X_t^\theta) - (\langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^{\theta + \varepsilon h} \rangle - \langle \xi_t | \mathbf{2}, \hat{\mathbb{P}}_t^\theta \rangle) \right) dt \right],$$

and using similar arguments, we obtain that

$$\langle \nabla J_7(\theta), h \rangle_{L_2} = \mu \langle (T - \cdot) - \int_0^T \mathbb{E} (\nu \Gamma_{\cdot, s}^\theta | \mathcal{F}_\cdot) ds, h \rangle_{L_2}.$$

Taking all together, we finally get that

$$\langle \nabla J(\theta), h \rangle_{L_2} = \sum_{i=1}^7 \langle \nabla J_i(\theta), h \rangle_{L_2} = \langle -2\eta\theta + \nu \mathbf{A}\theta - \lambda \sigma^2 \mathbf{B}\theta + \mu \mathbf{C}\theta, h \rangle_{L_2},$$

with the operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ defined by (4.5.12) and (4.5.13). $\square$

4.7.3 Proof of Proposition 4.5.5

Since $\nu = 0$, then $p = 2$ and then the monotonicity condition reduces to

$$\langle -(2\eta - \varepsilon)(\theta - \phi) - \lambda \sigma^2 (\mathbf{B}(\theta) - \mathbf{B}(\phi)) + \mu (\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \rangle_{L_2} \leq 0.$$

Moreover, we have that

$$\langle (\mathbf{B}(\theta) - \mathbf{B}(\phi)), \theta - \phi \rangle_{L_2} = \mathbb{E} \left(\int_0^T (X_t^\theta - X_t^\phi)^2 dt \right),$$

and

$$\langle \mu (\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \rangle_{L_2} = 0.$$

Hence, for any $\varepsilon > 0$ such that $0 < \varepsilon < 2\eta$, we infer that

$$\begin{aligned} & \left\langle -(2\eta - \varepsilon)(\theta - \phi) - \lambda\sigma^2 (\mathbf{B}(\theta) - \mathbf{B}(\phi)) + \mu (\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \right\rangle_{L_2} \\ & \leq (-(2\eta - \varepsilon)) \mathbb{E} \left(\int_0^T (\theta_t - \phi_t)^2 dt \right) - \lambda\sigma^2 \mathbb{E} \left(\int_0^T (X_t^\theta - X_t^\phi)^2 dt \right) \\ & \leq 0 \end{aligned}$$

and the monotonicity condition (4.5.16) is satisfied.

4.7.4 Proof of Proposition 4.5.6

Proof. First, since $\tilde{M} \leq 1$, then $p = 2$ and the monotonicity condition reduces to

$$\begin{aligned} & \left\langle -(2\eta - \varepsilon)(\theta - \phi) + \nu (\mathbf{A}(\theta) - \mathbf{A}(\phi)) - \lambda\sigma^2 (\mathbf{B}(\theta) - \mathbf{B}(\phi)) \right. \\ & \quad \left. + \mu (\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \right\rangle_{L_2} \leq 0. \end{aligned}$$

Moreover, since $\xi_t|_{\mathbf{2}} \in T^{\leq 1}(\mathbb{R}^2)$, we observe that

$$\Gamma_{t,s}^\theta = \xi_s^{\mathbf{22}}, \quad \tilde{\Gamma}_{t,s}^\theta = 2\xi_s^{\mathbf{2}}\xi_s^{\mathbf{22}} + 2P_s^\theta (\xi_s^{\mathbf{22}})^2 + 2\xi_s^{\mathbf{12}}\xi_s^{\mathbf{21}}s.$$

In this case, by Cauchy-Schwarz inequality, as we assume (4.5.19), we infer that

$$\langle \nu(\mathbf{A}(\theta) - \mathbf{A}(\phi)), \theta - \phi \rangle_{L_2} \leq \frac{(2\eta - \varepsilon)}{\nu} \mathbb{E} \left(\int_0^T (\theta_t - \phi_t)^2 dt \right),$$

as well as

$$\langle (\mathbf{B}(\theta) - \mathbf{B}(\phi)), \theta - \phi \rangle_{L_2} = \mathbb{E} \left(\int_0^T (1 - \nu\xi_t^{\mathbf{22}})^2 (X_t^\theta - X_t^\phi)^2 dt \right),$$

and finally

$$\langle \mu(\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \rangle_{L_2} = 0.$$

Hence, we readily infer that

$$\begin{aligned} & \left\langle -(2\eta - \varepsilon)(\theta - \phi) + \nu (\mathbf{A}(\theta) - \mathbf{A}(\phi)) - \lambda\sigma^2 (\mathbf{B}(\theta) - \mathbf{B}(\phi)) \right. \\ & \quad \left. + \mu (\mathbf{C}(\theta) - \mathbf{C}(\phi)), \theta - \phi \right\rangle_{L_2} \\ & \leq -\lambda\sigma^2 \mathbb{E} \left(\int_0^T (1 - \nu\xi_t^{\mathbf{22}})^2 (X_t^\theta - X_t^\phi)^2 dt \right) \\ & \leq 0, \end{aligned}$$

and therefore the monotonicity condition (4.5.16) is satisfied. $\square$

Conclusion and perspectives

This thesis investigates various path-dependent problems arising in incomplete financial and actuarial markets, which are particularly challenging, notably because classical Markovian methods are often not directly applicable. To address these challenges, we have developed and applied a range of novel tools, providing both practical solutions and theoretical insights. In Chapters 1 and 4, we lift non-Markovian problems to high-dimensional but Markovian frameworks, either through multifactor Markovian approximations or path-signatures method. In Chapters 1 and 3, a duality approach is employed to solve nonlinear control problems. Finally, in Chapter 2, we utilize non-standard analytical tools, such as Fredholm resolvent and determinant, to tackle pricing problems. These diverse methodologies illustrate the breadth of techniques necessary to handle the intricacies of path-dependence in financial and actuarial modeling.

First, in Chapter 1, we address the path-dependent problem of hedging European options within a rough volatility framework, where the volatility is modeled by an affine Volterra process. To tackle the non-Markovian nature of the problem, we lift it into a high-dimensional Markovian setting using a multifactor approximation of the volatility process. We then establish the convergence of the approximate solution toward that of the original non-Markovian problem. Focusing next on the resulting nonlinear Markovian control problem, we introduce a dual control approach to derive upper and lower bounds for the value function and to construct sub-optimal hedging strategies. In the specific case of the Volterra Heston model, this approach also yields a semi-explicit expression for the partial hedging strategies.

Then, in Chapter 2, we introduce the Volterra Stein-Stein model with stochastic interest rates, a hybrid equity-rate framework where both the volatility of a financial index and the interest rate are modeled by correlated Gaussian Volterra processes. We demonstrate that this model is tractable for pricing and hedging derivatives, deriving explicit formulas for zero-coupon bonds, caps, and floors, as well as a

semi-explicit formula for the characteristic function of the log-forward index using Fredholm resolvent and determinant. We also explore joint calibration to equity-rate market data, showing that the model is flexible and consistent with observed market behavior, capturing key empirical stylized facts such as the humped term structure of interest rate cap implied volatilities and the concave term structure (in log-log scale) of the ATM skew for the equity index.

In Chapter 3, we turn to the insurance market and study the problem of hedging an equity-linked life insurance portfolio. Hedging is considered at the portfolio level in a general model-free financial-actuarial market, with insurers assumed to be loss-averse in the sense that they either consider quantile or shortfall hedging strategies. We analyze two sets of admissible controls: financial-only controls, which do not incorporate actuarial information, and combined financial-actuarial controls, which can exploit all available financial and actuarial information. Using a duality approach, we theoretically characterize the optimal quantile and shortfall hedging strategies under both control sets. In the case of financial-only controls within complete financial markets, we obtain explicit expressions for the optimal strategies. These expressions are then leveraged to construct efficient explicit strategies for the combined market, updated at each stopping time corresponding to a death and depending on the path of the survival process. Numerical results highlight the significant impact of path-dependent actuarial information on the efficiency of hedging strategies.

Finally, in Chapter 4, we introduce a novel signature-based approach to tackle a non-Markovian and nonlinear control problem arising in the pricing and hedging of path-dependent derivatives with temporary and permanent market impact. Working in a Bachelier model with market frictions, we first consider signature payoffs, i.e., finite linear combinations of time-augmented signature elements of the stock index. Using the Fawcett (2003) formula, we derive perfect pricing and hedging in the frictionless case. We then focus on mean-quadratic variation hedging with frictions, proving existence and uniqueness of the optimal solution under a monotonicity condition on the Gateaux derivative of the gain functional. By applying the Itô formula for signatures together with the optimal martingality principle, we obtain a verification result: the optimal trading rate can be expressed in feedback form as a linear combination of time-augmented signature elements, with time-dependent coefficients characterized by non-standard infinite-dimensional Riccati equations on the extended tensor algebra. Furthermore, leveraging the universal approximation property of path signatures, the optimal signature-based strategy can be applied to hedge any general path-dependent payoff. Numerical results demonstrate the efficiency of this approach for hedging path-dependent payoffs in the presence of market frictions.

Several research topics related to path-dependent problems in finance and insurance have been explored throughout this thesis. Yet, many avenues remain open for further investigation, particularly concerning the application of path-signatures in this context.

In Chapter 3, we investigate quantile and shortfall hedging for insurance portfolios containing equity-linked life products. A natural extension of this work would be to examine whether these portfolio-level hedging techniques can also be applied to other classes of insurance contracts, such as death benefit products, Bermudian-type contracts or payoffs involving intermediate financial payments prior to maturity. Possible starting points for such an investigation are (Melnikov and Nosrati, 2017, Chapter 7), where the quantile hedging problem for guaranteed minimum death benefit (GMDB) contracts is studied at the contract level, or Bouchard et al. (2016); Bouveret and Chassagneux (2018), which studied the quantile hedging of Bermudan options.

In Chapter 4, we establish a connection between non-Markovian control problems and systems of infinite-dimensional Riccati equations. However, Theorem 4.5.7 provides only a verification result, and existence results for the associated system of infinite-dimensional Riccati equations have been proven only in specific examples. Investigating general existence and uniqueness results for such systems of Riccati equations would be highly valuable, as these equations arise not only in our hedging framework but also in other pricing and hedging problems, as discussed in Abi Jaber and Gérard (2025a). First existence results can be found in Abi Jaber et al. (2026a); Abi Jaber et al. (2026b).

Building on the results of Chapter 4, the signature approach could be extended to more general non-Markovian control problems involving multiple risk factors and controlled variables in both finance and insurance. For example, one could consider frameworks where the risk factors follow linear signature-SDEs, with coefficients expressed as linear functions of the time-augmented signatures of the state variables. This framework encompasses classical models such as Bachelier or Black-Scholes, as well as more advanced models, like the signature volatility models introduced in Abi Jaber and Gérard (2025a), which include some particular Volterra stochastic volatility models studied in Chapters 1 and 2. Under a suitable utility framework, the martingale optimality approach of Chapter 4 could be applied to characterize the optimal control in feedback form, expressed as a linear combination of the time-augmented signatures of the state variable Z , with time-dependent coefficients characterized by infinite-dimensional Riccati equations on the extended tensor algebra $T((\mathbb{R}^{d+1}))$, d being the dimension of the state variables, such that

$$\theta_t^* = \langle \psi_t, \hat{Z}_t^{\theta^*} \rangle, \quad t \leq T.$$

Extensions to multi-agent control problems could also be investigated. Generalizing the signature approach to solve more general non-Markovian control problems will

probably raise a significant empirical challenge regarding the numerical resolution of the system of infinite-dimensional Riccati equations. In Chapter 4, we follow the method of Abi Jaber and Gérard (2025a), using a projected shuffle product combined with an Euler scheme to approximate the solution of the Riccati equations. However, this approach may become computationally infeasible when approximating time-dependent coefficients solution of the Riccati equations on the extended tensor algebra $T((\mathbb{R}^{d+1}))$, for high-dimensional state spaces ($d \gg 1$). Building on the known structure of the optimal control, an alternative research direction would be to approximate the time-dependent coefficients using a well-designed neural network, which could then be used to efficiently approximate the optimal control in high dimensions as

$$\theta_t^* \approx \tilde{\theta}_t := \langle NN_\phi(T-t), \hat{\mathbb{Z}}_t^{\tilde{\theta}} \rangle, \quad t \leq T,$$

with ϕ the parameters of the neural network. A comparison with the approaches proposed in Bank et al. (2025) would be very interesting. Moreover, drawing on recent papers Abi Jaber and Gérard (2025b); Cirone and Salvi (2025); Futter et al. (2025), investigating data-driven approaches for pricing and hedging path-dependent options, potentially in the presence of market frictions, using signature features could also constitute a promising research avenue.

Finally, path-signatures may also be employed for the analysis and forecasting of financial and actuarial time series. Since signatures are inherently not time-invariant, studying forecasting requires first considering time-invariant transformations, such as sliding-window signatures or the exponentially fading memory signatures introduced in Abi Jaber and Sotnikov (2025). A promising research direction is to explore these time-invariant signature constructions for forecasting financial and actuarial time series, including equity returns or volatility, electricity demand, and mortality rates, by incorporating suitable signature transforms of covariates (e.g. liquidity, macroeconomic, or climate risk factors) to capture path-dependent effects. A first attempt on forecasting electricity demand with climate risk factors using sliding-window signatures has been conducted by Drobac et al. (2025).

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