On elliptic integrable lattice models

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List of publications

This thesis is based on the following research papers:

[1] S. Brasseur and C. Hagendorf, *Sum rules for the supersymmetric eight-vertex model*, J. Stat. Mech. **2021** (2021) 023102.

[2] S. Brasseur and C. Hagendorf. On the dynamical ten-vertex and nineteen-vertex models with anti-periodic boundary conditions. In preparation.

[3] S. Brasseur and C. Hagendorf. The spin-one XYZ chain with quasi-periodic boundary conditions. In preparation.

Contents

Introduction

Since its introduction in the 19th century, *statistical physics* has been at the origin of many valuable insights on a vast array of topics. These range from thermodynamics to condensed matter and astrophysics, and even include the prediction of new states of matter [4]. Nowadays, its realm of application has even extended past the frontiers of physics, and has reached network systems, computer sciences and socio-economics [5–7]. Statistical physics uses a probabilistic approach to understand the properties of macroscopic systems from the interactions between their microscopic constituents. While this pursuit might appear arduous, given the typical complexity of fundamental interactions, statistical physicists often resort to the use of ***models***. These simplified versions of given systems capture their essential features, while opening the way for exact algebraic or analytic treatment. The most emblematic model of statistical physics is arguably the *Ising model*, proposed in 1920 by Lenz to study ferromagnetism.

The *six-vertex model* is another fascinating model of statistical physics, initially introduced to investigate the thermodynamic properties of ice [8–11]. Let us consider this model on the two-dimensional square ***lattice*** (see the left of Figure 1). Its configurations result from the assignment of arrows to each edge of the lattice (see the left of Figure 3). The arrows can go left or right on horizontal edges, and up or down on vertical ones. The *ice rule* states that, each vertex must possess two ingoing and two outgoing arrows. This allows for only six possibilities, which gave the model its name (see the left of Figure 2). Each allowed vertex is assigned a statistical weight, which quantifies the likelihood of finding it in a configuration. Then, the statistical weight of a configuration is the product of the weights of the vertices within. The six-vertex model garnered considerable attention from the statistical physics community [12], and was later generalised into the *eight-vertex model* [13, 14]. The ice rule was relaxed to a *generalised ice rule*, which led to the addition of two admissible vertices (see the right of Figure 2). See Chapter 1 for detailed definitions.

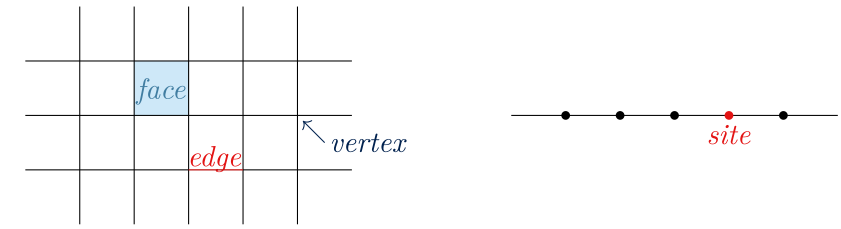


Figure 1: *Left:* A 3×5 two-dimensional square lattice, with examples of *face*, *edge* and *vertex*. The lattice is referred to as 'square' due to the shape of its faces. *Right:* A one-dimensional lattice with 5 *sites*.

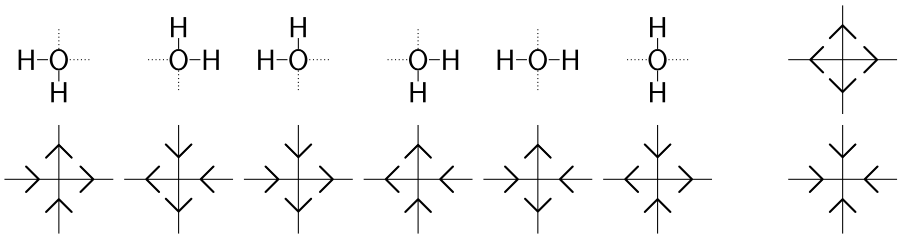


Figure 2: *Left:* Allowed vertices of the six-vertex model (*bottom*) with the corresponding molecular arrangements of H_2O (*top*). *Right:* The two extra allowed vertices of the eight-vertex model.

The *partition function* is a fundamental object in statistical physics, as many thermodynamic quantities can be derived from it. It is defined as the sum over all configurations of their associated statistical weights. The partition function naturally depends on the lattice considered and, particularly, on *boundary conditions*. These specify how extremities of the lattice relate to one another, and can be chosen *open*, *(anti-)periodic*,... We keep their precise definitions for subsequent chapters. Given the importance of the partition function, strategies were developed in order to compute it [15]. Among them is the *transfer-matrix approach*, pioneered by Kramers and Wannier for the Ising model [16, 17]. The method allows one to evaluate a partition function by solving the eigenvalue problem of the transfer matrix instead. Finding expressions for all transfer-matrix eigenvalues and eigenvectors is a major part of the *solution* of a model. The other important part is the computation of *correlation functions*. While these can, in principle, be derived from the spectrum and eigenvectors, in practice they often require the use of specific techniques [18, 19].

A particular feature of the six- and eight-vertex models is that their transfer matrices can be built from *R-matrices* that fulfil the so-called *Yang-Baxter equation* [12]. This emblematic relation, named after the independent works of Yang and Baxter [20, 21], has become central to the theory of *integrability* [12, 22], and to the development of the theory of quantum groups [23–25]. We say that models are ***integrable*** if they can be defined from R-matrices solutions to the Yang-Baxter equation.[1] The R-matrices of the six- and eight-vertex models are such solutions, expressed in terms of trigonometric and elliptic functions, respectively. Thanks to this property, the models are also said to be *trigonometric* and ***elliptic***, respectively.

In the late sixties, the transfer-matrix approach led to the discovery of a relation between vertex models and Heisenberg *spin chains* [9, 10, 12, 13, 27]. The latter are models of quantum magnetism defined on a one-dimensional lattice (see the right of Figure 1). Their degrees of freedom are spins, located at each site, which interact with their nearest neighbours. Depending on the anisotropy of the interactions, the chains might be called XYZ (completely anisotropic), XXZ (partially anisotropic) or XXX (isotropic). Such systems are entirely described by an operator named *Hamiltonian*. Baxter proved that the XXZ and XYZ spin-chain Hamiltonians could be expressed in terms of the transfer matrices of the six- and eight-vertex models, respectively [28, 29]. These Hamiltonians and related transfer matrices can thus be diagonalised simultaneously, which provides an additional motivation for the study of their eigenvalue problems.

Among the techniques developed to solve these problems, the one proposed by Hans Bethe came to be a hallmark in the theory of integrable lattice models [19]. In 1931, he put forward an *Ansatz* (i.e. an educated guess) regarding the expression of the eigenstates of the XXX spin chain [30].

[1] Note, however, that there is no universally accepted definition of quantum integrability [26].

He formulated that they ought to be linear combinations of *plane waves*, with coefficients fixed implicitly by a set of equations. The technique came to be known as the *coordinate Bethe Ansatz*. In the late seventies, the Leningrad School developed a new algebraic framework, the *Quantum Inverse Scattering Method (QISM)* [31, 32]. The method revealed and exploited the existence of operator algebras underlying integrable lattice models, the *Yang-Baxter algebras*. This newfound structure allowed for the systematic derivation of many properties of a given model, and their generalisation to classes of models described by the same algebra [19,25,33]. The Bethe Ansatz was reworked within the QISM framework, and gave rise to the *algebraic Bethe Ansatz* [22]. In this new variant, eigenstates can be built from the action of an element of the Yang-Baxter algebra onto a reference state. This technique proved immensely successful, notably in the exact computation of correlation functions [18, 34–42]. It, however, still exhibits some limitations. Indeed, inherent to its nature as an Ansatz, the completeness of its spectrum description has to be separately proven, which often reveals to be a challenge [43, 44]. Moreover, the existence of an appropriate reference state is not guaranteed [19].

The eight-vertex model and related XYZ spin chain suffered from this last limitation. To overcome it, Baxter related it to another model of two-dimensional statistical physics: the *dynamical six-vertex model*, equivalent to the *eight-vertex solid-on-solid model* [45]. The relation, named *vertex-face correspondence*, connects the R-matrices of both models by means of an operator [45–47]. Baxter managed to solve the eight-vertex solid-on-solid model via a Bethe Ansatz, and transposed his results back to the eight-vertex model [45, 48]. Takhtadzhan and Faddeev later built this idea into a variant of the algebraic Bethe Ansatz, the *generalised Bethe Ansatz* [32].

The Bethe Ansatz has gone through many such generalisations and variants, each meant to cater to models or situations for which the original version somehow fails. One particular variant that shall hold our attention is Sklyanin's *functional Bethe Ansatz* [49, 50]. His method relies on the introduction of a new basis of the space of states, in which the eigenvalue problem of the transfer matrix *separates* into a set of independent lower-dimensional linear systems. Due to this feature, the method is now known as *quantum Separation of Variables*. It has been successfully applied to a host of models, and keeps being an active area of research to this day [51–62].

Some of the results brought by these techniques revealed surprising connections between vertex models, their related spin chains, and *combinatorics*. The first significant evidence of this interplay appeared during the search for the proof of the alternating sign matrix conjecture. *Alternating sign matrices (ASMs)* are square matrices with entries 0,±1 such that *(i)* the entries of each column or line sum up to 1 *(ii)* the non-zero entries alternate in sign along each column and line. See the middle of Figure 3 for an example. These combinatorial objects appeared in the work of Mills, Robbins and Rumsey as they were studying an algorithm for the evaluation of determinants due to Charles L. Dodgson, better known as Lewis Carroll [63]. In 1982, the three authors proposed a now famous conjecture stating that the number $A(n)$ of $n \times n$ ASMs is given by [64, 65]

$$A(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{0.1}$$

Despite the apparent simplicity of its formulation, the conjecture remained an open question until 1995, when Zeilberger presented its first proof [66]. Soon after, Kuperberg realised that this enumeration problem had already been unknowingly studied by statistical physicists in the context of integrable lattice models [67]. Indeed, there exists a one-to-one correspondence between ASMs and configurations of the six-vertex model on a specific domain [67, 68]. In his alternative proof, Kuperberg obtained the number $A(n)$ from the partition function of this model, previously computed by Izergin and Korepin via an induction technique [18, 69]. For a more detailed account of the fascinating story surrounding ASMs and their enumerations, see [63].

In the following years, additional evidence of the interplay between integrable lattice models and combinatorics were brought to light. For instance, Razumov and Stroganov studied the XXZ spin chain with periodic boundary conditions, an odd number of sites, and for a specific value of its anisotropy parameter [70]. In the components of a special eigenstate, they found integer sequences related to the enumeration of ASMs and plane partitions. Motivated by these discoveries, Razumov and Stroganov [71], and Bazhanov and Mangazeev [72, 73], investigated the

XYZ spin chain with periodic boundary conditions and an odd number of sites. They identified a subfamily of its parameters, the *combinatorial line*, for which similar intriguing connections to combinatorics appeared. Over the years, this line was also found to be the stage of exciting relations to topics such as elliptic functional equations [15, 74, 75], the Painlevé VI equation [72, 76, 77], special polynomials [78–83], and supersymmetry [84, 85].

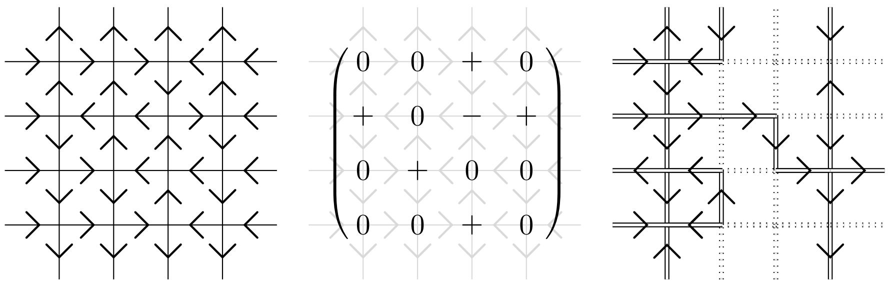


Figure 3: *Left:* A configuration of the six-vertex model on a 4×4 square lattice. *Middle:* A 4×4 ASM with $\pm 1 = \pm$. This ASM corresponds to the configuration of the six-vertex model pictured on the left. *Right:* A configuration of the forty-one-vertex model on a 4×4 square lattice.

Following these developments, one was left to wonder whether other integrable lattice models also showcased combinatorial phenomena. Natural candidates for investigations were *higher-spin generalisations* of vertex models and spin chains. These models are defined by allowing the degrees of freedom to take on a larger range of values. While the six- and eight-vertex models are characterised as spin-1/2, thanks to the 2 types of arrows living on the lattice edges, their spin-$\ell/2$ generalisations possess $\ell + 1$ types of arrows instead, $\ell \in \mathbb{N}_*$. Allowed vertex configurations are then still determined by the generalised ice rule. The spin-1 generalisations of the six- and eight-vertex models are called *nineteen-* and *forty-one-vertex models*, respectively, thanks to their numbers of allowed vertices. (See the right of Figure 3 for an example of configuration.) From an algebraic standpoint, R-matrices that describe these models can be built via a *fusion* procedure from the R-matrices of the spin-1/2 models [86–89]. Through the transfer-matrix approach, one also relates the spin-1 vertex models to spin-1 XXZ and XYZ chains, respectively [90, 91]. Hagendorf and Morin-Duchesne investigated the nineteen-vertex model with quasi-periodic boundary conditions along the horizontal direction [92]. For any system size, and no restriction of anisotropy parameter, they found that components of special eigenvectors were related to weighted enumerations of symmetry classes of ASMs. Moreover, conjectural work on the forty-one-vertex model hints at promising properties as well [93].

The main goal of this thesis is to study the eight-vertex model and related systems such as the dynamical six-vertex model, as well as their higher-spin generalisations. In particular, we investigate special eigenvalues of their transfer matrices with remarkably simple expressions, and their related eigenspaces. Our work naturally leads to results for special eigenstates of the related XYZ spin chains, and connections to combinatorics.

In Chapter 1, we introduce the eight-vertex model and its related XYZ spin chain. We provide a description of these models both on the lattice, and from an algebraic standpoint. Then, we discuss the combinatorial line, where surprising relations to combinatorics are observed. This chapter (significantly) expands the introduction of [1].

In Chapter 2, we consider the eight-vertex model with periodic boundary conditions along the horizontal direction. On the combinatorial line, and for an odd number of vertical lines, its transfer matrix possesses a remarkably simple eigenvalue. Our goal is to investigate the corresponding eigenspace. We compute several scalar products and components of the basis vectors in terms of special polynomials introduced by Rosengren and Zinn-Justin. The proofs of these results rely on the induction technique due to Izergin and Korepin. This chapter contains the analysis initially performed in [1].

In Chapter 3, we consider two higher-spin generalisations of the eight-vertex model: the forty-one- and eighteen-vertex models. After describing them on the lattice, we construct their R-matrices via the fusion procedure. We consider these models with quasi-periodic boundary conditions along the horizontal direction, and define their transfer matrices. Incidentally, these matrices commute, and can thus be diagonalised simultaneously. Simple algebraic arguments allow us to show that they possess remarkably simple eigenvalues. Furthermore, we introduce a spin-1 XYZ chain related to the forty-one-vertex model, and prove that its Hamiltonian possesses the eigenvalue $E = 0$. Lastly, we formulate conjectures regarding the degeneracy of this eigenvalue, and its position within the spectrum. This chapter uses material from [3].

In Chapter 4, in order to solve the eigenvalue problem of the higher-spin models, we follow Baxter's idea and turn to dynamical vertex models instead. In particular, we consider the dynamical ten-vertex model with anti-periodic boundary conditions along the horizontal direction. We build its R-matrix through the fusion procedure, and define its transfer matrix. Following [45, 58, 59, 94], we use the vertex-face correspondence to relate it to that of the eighteen-vertex model with quasi-periodic boundary conditions. This chapter uses material from [2, 3].

In Chapter 5, we investigate the dynamical ten-vertex model with anti-periodic boundary conditions. We apply the quantum Separation of Variables technique in order to solve its eigenvalue problem. This leads us to a complete characterisation of its spectrum and of its eigenvectors. In particular, we compute one eigenvalue explicitly and show that it possesses a remarkably simple form. We find an expression for a quadratic sum rule involving its associated eigenvectors as a determinant formula. Finally, we derive several properties of the eigenvectors that characterise them implicitly. This chapter presents results from [2].

In Chapter 6, we combine the insights developed throughout previous chapters in order to obtain results for the eighteen- and forty-one-vertex models, and the spin-1 XYZ chain. For the vertex models, they follow those obtained for the special eigenvectors of the dynamical ten-vertex model. We find a quadratic sum rule as a determinant formula, and a series of properties of the eigenvectors that characterise them implicitly. For the spin-1 XYZ chain, we find an expression for the square norm of the special eigenstate in terms of new special polynomials, generalisations of Rosengren and Zinn-Justin's. Their definitions follow from the uniformisation procedure of the determinant formula found in the vertex case. They allow us to formulate several conjectures regarding components and sum rules of the special eigenvectors. Moreover, they provide connections to combinatorics. This chapter presents results from [3].

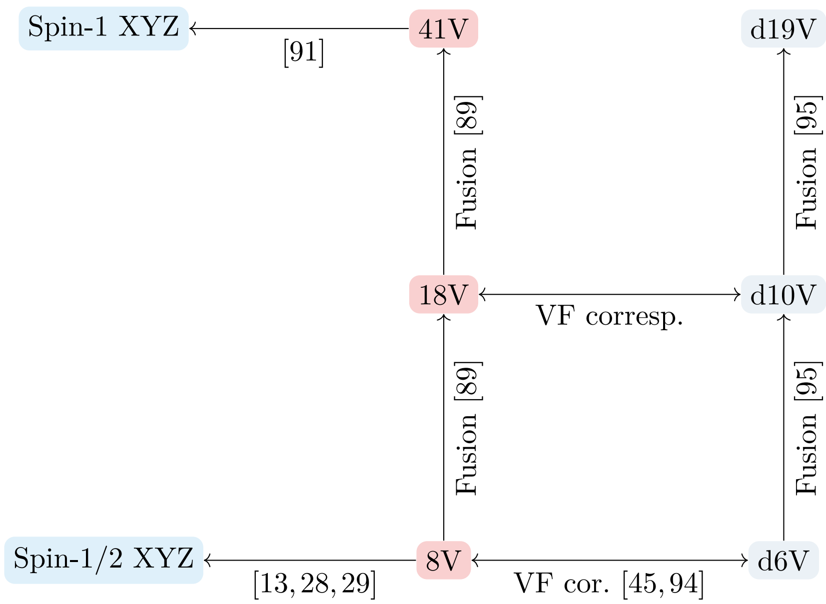


Figure 4: The elliptic integrable models studied in this thesis, and the relations between them. The 41V, 18V and 8V stand for the forty-one-, eighteen- and eight-vertex models, respectively. The d19V, d10V and d6V stand for the dynamical nineteen-, ten- and six-vertex models, respectively.

Chapter 1

The eight-vertex model

Our story starts with a mundane albeit crucially important substance: water. If you freeze water down to low temperatures, its molecules get locked in the ordered structure of an ice crystal. However, many different microscopic molecular arrangements lead to the same macroscopic appearance. As a result, ice is said to still possess degrees of freedom, and thus *residual entropy*. In 1935, Linus Pauling set out to investigate this phenomenon. To this end, he formalised the rules governing the positioning of H_2O molecules within ice crystals [8]. Considering a two-dimensional ice layer, his findings dictate that the oxygen atoms arrange themselves as if on the vertices of a regular square lattice (see the left of Figure 1.1). The hydrogen atoms are located between adjacent oxygen atoms, on the edges of the lattice. Each oxygen atom is thus surrounded by four hydrogen atoms: two attached to it via covalent bound (relatively close), and two that belong to neighbouring H_2O molecules (further away). Incidentally, Slater reached similar conclusions regarding KH_2PO_4 [96]. This structuring rule came to be known as the *ice rule*. Recently, Algara-Siller et al. have reported the observation of a layer of water that fulfilled the ice rule, sandwiched between two sheets of graphene at room temperature [97].[1]

Pauling's physical considerations were later formalised into models of statistical physics: *Square Ice*, then the *six-vertex (6V) model* [9–11, 27].

[1]Although the observation has, since then, been called into question [98].

In turn, these were generalised into the *eight-vertex (8V) model* [13, 14], which shall be our main focus in Chapters 1 and 2.

This chapter is structured as follows: first, we describe the 6V and 8V models on the lattice, starting from the ice rule. Second, we present the algebraic formalism useful to the analysis of the 8V model. In particular, we introduce two important operators, the R-matrix and the transfer matrix. We state their main properties and symmetries. Then, we define the XYZ spin chain, a statistical physics model of quantum magnetism. We explain how this spin chain relates to the 8V model. Finally, we discuss a subfamily of the parameters of the 8V model, for which surprising relations to combinatorics are observed.

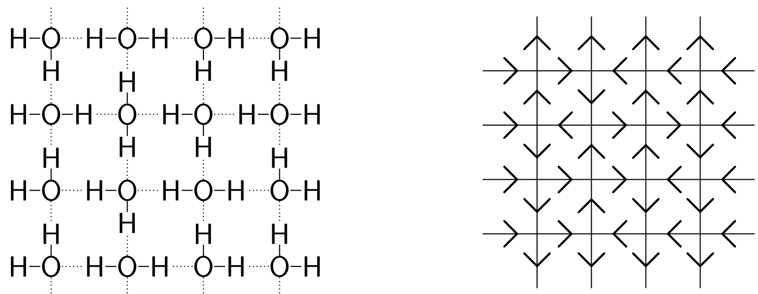


Figure 1.1: *Left:* A configuration of H_2O molecules satisfying the ice rule. Solid lines represent covalent bonds, dotted lines represent hydrogen bonds. *Right:* The corresponding configuration of the 6V model.

1.1 Lattice formulation

1.1.1 The six-vertex model

We define this model on the two-dimensional square lattice. Its microscopic degrees of freedom are *spins* located on each lattice edge. The spins take the values ± 1, and are depicted graphically as arrows through the dictionary

$$1 : \uparrow, \ \rightarrow \ , \qquad\qquad -1 : \downarrow, \ \leftarrow \ . \tag{1.1}$$

The interactions are local, and take place between the four spins surrounding a vertex. They are governed by the *ice rule*. In the arrow picture, this

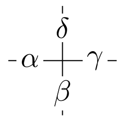


Figure 1.2: Vertex on the square lattice with $\alpha, \beta, \gamma, \delta \in \{\pm 1\}$ on its edges. Throughout our work, we identify $\uparrow = \rightarrow = 1$ and $\downarrow = \leftarrow = -1$, as prescribed by (1.1).

equates to requiring exactly two ingoing and two outgoing arrows around each vertex. In the spin formulation, the ice rule around the vertex of Figure 1.2 simply reads $\alpha + \beta = \gamma + \delta$. The rule results in six allowed vertex configurations, simply refered to as *vertices* (see Figure 1.3). The model takes its name after this feature. We associate a statistical weight to each of the vertices. Then, the statistical weight of a configuration on the lattice is the product of the weights of the vertices within. The Square Ice model corresponds to $w_1 = \ldots = w_6$, where all configurations are equally probable.

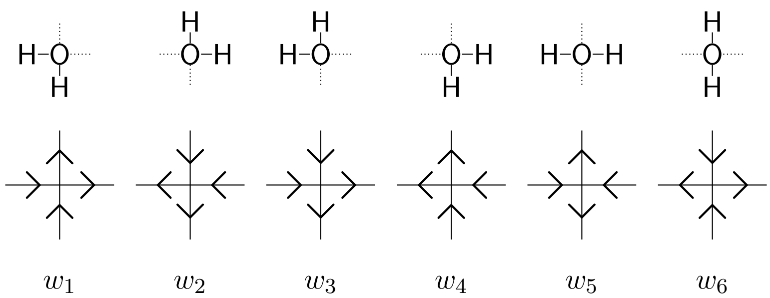


Figure 1.3: *Bottom:* Allowed vertices of the 6V model and their statistical weights. *Top:* Corresponding molecular arrangements. The arrows of the 6V can be seen as electric dipoles pointing from the hydrogen atom toward the oxygen atom to which it is covalently bound.

Over the years, the 6V model has garnered considerable attention [99]. Its study has been connected to the development of the theory of the Yang-Baxter equation [12], Quantum Inverse Scattering Methods [18, 19], quantum groups [24], ...

1.1.2 The eight-vertex model

In 1970, Sutherland [13], and Fan and Wu [14] proposed a generalisation of the 6V model. They relaxed the ice rule to merely require that the number of incoming and outcoming arrows around a vertex be even. In the spin description, this *generalised ice rule* reads

$$\alpha + \beta \equiv \gamma + \delta \pmod 4. \tag{1.2}$$

It results in eight allowed vertex configurations: the six ones of the 6V model, plus an extra *source* and *sink* (see Figure 1.4). We assign to

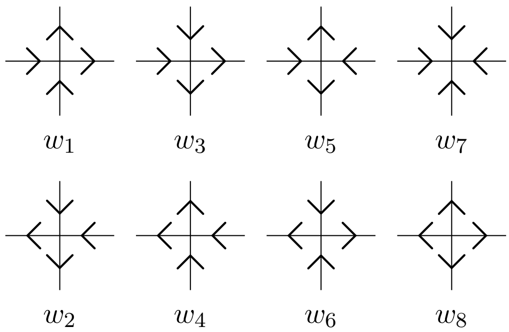


Figure 1.4: Allowed vertices of the 8V model, and their statistical weights. The 7-th and 8-th vertices are respectively a source and a sink.

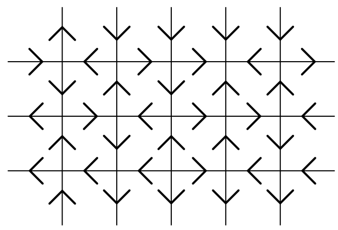


Figure 1.5: Configuration of the zero-field 8V model with weight $a^3b^2c^8d^2$.

each vertex a statistical weight. In this process, we impose *spin-reversal symmetry* by assigning the same weight to vertices related by spin reversal,

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c, \quad w_7 = w_8 = d. \tag{1.3}$$

This choice corresponds to the *zero-field* 8V model [12]. The term originates from the image of the arrows as electric dipoles (see Figure 1.3). As for the 6V model, the weight of a configuration is the product of the weights of the vertices within (see Figure 1.5).

Given a domain of the square lattice, the *partition function* is the weighted sum over all configurations. It is given by

$$Z = \sum_{\mathcal{C}} a^{n_a} b^{n_b} c^{n_c} d^{n_d}, \tag{1.4}$$

where the sum is over all allowed configurations $\mathcal{C}$, and n_j denotes the number of vertices of type j in $\mathcal{C}$, for $j = a, b, c, d$. The partition function is a fundamental object in statistical physics as it gives access to many thermodynamic quantities. For example, we obtain the *free energy per site* in the limit of large systems through the logarithm of the partition function. On a $L \times N$ lattice, this reads

$$f = - \lim_{N,L\to\infty} \frac{1}{NL} \ln Z. \tag{1.5}$$

1.2 Algebraic formulation

1.2.1 Notations and conventions

The Hilbert space of a single spin-1/2 is $\mathbb{C}^2$. Its canonical basis vectors are

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.6}$$

Throughout our text, N denotes a positive integer. We consider the tensor-product space

$$(\mathbb{C}^2)^{\otimes N} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}. \tag{1.7}$$

Its canonical basis vectors are given by

$$|\boldsymbol{\alpha}\rangle = |\alpha_1 \cdots \alpha_N\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle, \tag{1.8}$$

with $\alpha_i \in \{\uparrow, \downarrow\}$, for each $i = 1, \ldots, N$. We refer to the sequences $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_N$ as *spin configurations*. Each vector $|\psi\rangle \in (\mathbb{C}^2)^{\otimes N}$ possesses a decomposition in the basis

$$|\psi\rangle = \sum_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}} |\boldsymbol{\alpha}\rangle. \tag{1.9}$$

We refer to the numbers $\psi_{\boldsymbol{\alpha}}$ in this expansion as the *components* of $|\psi\rangle$. Furthermore, for every vector $|\phi\rangle \in (\mathbb{C}^2)^{\otimes N}$, we define a dual vector by transposition $\langle\phi| = |\phi\rangle^t$. The corresponding dual pairing is

$$\langle\phi|\psi\rangle = \sum_{\boldsymbol{\alpha}} \phi_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}. \tag{1.10}$$

When restricted to $(\mathbb{R}^2)^{\otimes N}$, this dual pairing defines a scalar product. We denote by $||\psi||^2 = \langle\psi|\psi\rangle$ the square norm of a vector in this subspace. For simplicity, we use the term 'scalar product' and 'square norm' even for vectors outside this subspace.

We take the spin operators on $\mathbb{C}^2$ to be the usual Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.11}$$

For each $i = 1, \ldots, N$, we denote by σ_i^a the a-th Pauli matrix acting on the i-th factor $\mathbb{C}^2$ of $(\mathbb{C}^2)^{\otimes N}$,

$$\sigma_i^a = \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{i-1} \otimes \sigma^a \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{I}}_{N-i}, \qquad a = 1, 2, 3, \tag{1.12}$$

where $\mathbb{I}$ is the 2×2 identity matrix. In terms of these operators, we define the *spin-parity* operator P and the *spin-reversal* operator F as

$$P = (-1)^N \prod_{i=1}^N \sigma_i^3, \qquad F = \prod_{i=1}^N \sigma_i^1. \tag{1.13}$$

They obey the relation

$$FP = (-1)^N PF. \tag{1.14}$$

1.2.2 The R-matrix

The R-matrix is a linear operator $R \in \operatorname{End}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ that encodes the weights of the model,

$$\langle\gamma\delta|R|\alpha\beta\rangle = \begin{array}{ccc} & \delta & \\ -\alpha & + & \gamma- \\ & \beta & \end{array}, \tag{1.15}$$

with $\alpha, \beta, \gamma, \delta \in \{\pm 1\}$. By convention, diagrams as in (1.15) stand for their evaluation in terms of statistical weights. In the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle,$ $|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, the R-matrix reads

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \tag{1.16}$$

We use a parameterisation of the vertex weights a, b, c, d in terms of Jacobi theta functions (see Appendix A.1). The parameterisation depends on the *spectral parameter* λ, the *crossing parameter* η and the *elliptic nome* $p = \mathrm{e}^{\mathrm{i}\pi\omega}$, $\operatorname{Im}\omega > 0$. Up to an irrelevant overall factor, the weights are given by

$$a(\lambda) = \vartheta_4(\eta, p^2)\vartheta_1(\lambda + \eta, p^2)\vartheta_4(\lambda, p^2), \tag{1.17a}$$

$$b(\lambda) = \vartheta_4(\eta, p^2)\vartheta_4(\lambda + \eta, p^2)\vartheta_1(\lambda, p^2), \tag{1.17b}$$

$$c(\lambda) = \vartheta_1(\eta, p^2)\vartheta_4(\lambda + \eta, p^2)\vartheta_4(\lambda, p^2), \tag{1.17c}$$

$$d(\lambda) = \vartheta_1(\eta, p^2)\vartheta_1(\lambda + \eta, p^2)\vartheta_1(\lambda, p^2), \tag{1.17d}$$

Here and in the following, we only write out the dependence of the vertex weights on the spectral parameter λ, but not on η and p. For the purposes of our analysis, we allow these parameters to be complex numbers, with the constraint $2\eta \not\equiv 0 \,(\operatorname{mod} \pi, \pi\omega)$.

The parameterisation implies that the R-matrix obeys the so-called *Yang-Baxter equation (YBE)* [20, 100]. On $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, it reads

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu), \tag{1.18}$$

where $R_{ij}(\lambda)$ denotes an R-matrix acting non-trivially on the i-th and j-th factor of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. This equation is at the center of the theory of integrable systems. In this thesis, we consider that a model of statistical physics is *integrable* if it can be defined from an R-matrix solution to the YBE. Moreover, if this solution involves theta or elliptic functions, as in (1.17), the model is characterised as *elliptic*.

In the limit $p \to 0$, after appropriate rescaling, the R-matrix (1.16) with parameterisation (1.17) tends to the R-matrix of the 6V model. The latter fulfils the YBE, and is expressed in terms of trigonometric functions. As a result, both the limit $p \to 0$, and the 6V model are dubbed *trigonometric*.

Furthermore, we note that $R(0) = \vartheta_4(0, p^2)\vartheta_1(\eta, p^2)\vartheta_4(\eta, p^2)\mathcal{P}$, where $\mathcal{P}$ is the permutation operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$, acting on the basis vectors as $\mathcal{P}|\alpha_1\alpha_2\rangle = |\alpha_2\alpha_1\rangle$. We use it to define the $\check{R}$-matrix $\check{R}(\lambda) = \mathcal{P}R(\lambda)$. It obeys the braid form of the Yang-Baxter equation

$$\check{R}_{12}(\lambda - \mu)\check{R}_{23}(\lambda)\check{R}_{12}(\mu) = \check{R}_{23}(\mu)\check{R}_{12}(\lambda)\check{R}_{23}(\lambda - \mu). \tag{1.19}$$

Additionally, although not needed in our work, the YBEs possess a well-known graphical interpretation in terms of scattering of particles [25, 101].

1.2.3 The transfer matrix

We now consider a square lattice with N vertical lines and periodic boundary conditions along the horizontal direction. The *transfer matrix* is an operator $T \in \operatorname{End}(\mathbb{C}^2)^{\otimes N}$ whose entries contain the statistical weights of a row of the lattice. Take two basis vectors $|\boldsymbol{\alpha}\rangle, |\boldsymbol{\beta}\rangle \in (\mathbb{C}^2)^{\otimes N}$, we have

$$\langle \boldsymbol{\alpha}|T|\boldsymbol{\beta}\rangle = \sum_{\gamma_0,\ldots,\gamma_{N-1}=\pm} \begin{array}{ccccccccccc} & \alpha_1 & & \alpha_2 & & \alpha_3 & & & \alpha_N & & \\ -\gamma_0 & + & \gamma_1 & + & \gamma_2 & + & \cdots & & + & \gamma_0 - & \\ & \beta_1 & & \beta_2 & & \beta_3 & & & \beta_N & & \end{array}. \tag{1.20}$$

The transfer matrix can be defined in terms of the R-matrix (1.15) as

$$T = \operatorname{tr}_0 R_{0N} \cdots R_{02} R_{01}. \tag{1.21}$$

In this expression, the R-matrices act on the tensor product $\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes N}$. The first factor $\mathbb{C}^2$ corresponds to the horizontal direction in (1.20), and is called the *auxiliary space*. The trace in (1.21) is taken over this auxiliary space.

In our analysis below, we work with the *inhomogeneous* 8V model. To this end, we allow the weights (1.17) on the i-th vertical line of the lattice to depend on an *inhomogeneity parameter* λ_i, $i = 1, \ldots, N$. The transfer matrix of the inhomogeneous eight-vertex model reads

$$T(\lambda) = \operatorname{tr}_0(R_{0N}(\lambda_N - \lambda) \cdots R_{02}(\lambda_2 - \lambda) R_{01}(\lambda_1 - \lambda)). \tag{1.22}$$

If $\lambda_1 = \cdots = \lambda_N = 0$, then we call the 8V model described by this transfer matrix *homogeneous*.

The transfer matrices form a one-parameter family of commuting matrices. The following statement is a consequence of the YBE (1.18) [12]. We use the standard square bracket notation to denote the commutator of two operators.

Proposition 1.2.1. *For each $\lambda, \mu \in \mathbb{C}$. $[T(\lambda), T(\mu)] = 0$.*

The transfer matrix is diagonalisable for real λ, η, p and $\lambda_1 = \cdots = \lambda_N = 0$ [12]. Moreover, it is likely that this property still holds for $\lambda_1, \ldots, \lambda_N$ sufficiently close to 0. By Proposition 1.2.1, it is possible to choose transfer-matrix eigenvectors independent of the spectral parameter λ [102].

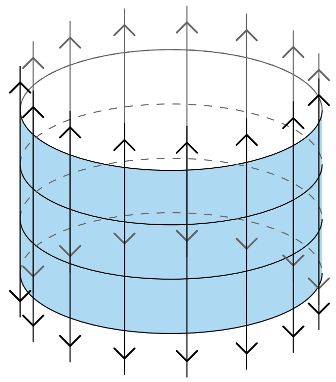


Figure 1.6: A 4×15 square lattice with periodic boundary conditions in the horizontal direction, and vertical extremities encoded in the vectors $|\boldsymbol{\alpha}\rangle = |\uparrow \cdots \uparrow\rangle$ and $|\boldsymbol{\beta}\rangle = |\downarrow \cdots \downarrow\rangle$.

Furthermore, the transfer matrix possesses two elementary symmetries: it is invariant under spin reversal and preserves the spin parity. These symmetries lead to the commutation relations

$$[T(\lambda), F] = [T(\lambda), P] = 0. \tag{1.23}$$

We note that for odd N, the relations (1.14) and (1.23) imply that each transfer-matrix eigenvalue has an even degeneracy.

Based on the interpretation (1.21), the transfer matrix proves to be a useful tool for the evaluation of partition functions [12]. Let us consider a $L \times N$ square lattice with periodic boundary conditions in the horizontal direction, and vertical extremities encoded in the canonical basis vectors $|\boldsymbol{\alpha}\rangle\,, |\boldsymbol{\beta}\rangle \in (\mathbb{C}^2)^{\otimes N}$ (see Figure 1.6). The partition function on this domain simply reads

$$Z = \langle \boldsymbol{\alpha}| T(\lambda)^L |\boldsymbol{\beta}\rangle\,. \tag{1.24}$$

Let $\theta_0, \ldots, \theta_m$ be the eigenvalues of $T(\lambda)$, ordered such that $|\theta_0| > \ldots > |\theta_m|$. We denote by $g_i \geq 1$ the degeneracy of θ_i, and P_i the projector onto its corresponding eigenspace, $i = 0, \ldots, m$. The transfer matrix possesses the spectral decomposition $T(\lambda) = \theta_0 P_0 + \ldots + \theta_m P_m$. This allows us to rewrite the partition function as

$$Z = \sum_{i=0}^{m} \theta_i^L \langle \boldsymbol{\alpha}| P_i |\boldsymbol{\beta}\rangle\,. \tag{1.25}$$

As an example, let us impose periodic boundary conditions in the vertical direction as well. This amounts to working on a torus, and leads to the simplification:

$$Z^{\text{torus}} = \sum_{\boldsymbol{\alpha}} \langle \boldsymbol{\alpha} | T(\lambda)^L | \boldsymbol{\alpha} \rangle = \operatorname{tr}(T(\lambda)^L) = \sum_{i=0}^{m} g_i \theta_i^L. \tag{1.26}$$

In the physical regime, where all vertex weights are real positive numbers, an argument based on the Perron-Frobenius theorem ensures that θ_0 is real and positive [85]. In this case, we find the free energy per site

$$f^{\text{torus}} = - \lim_{N,L\to\infty} \frac{1}{NL} \ln Z = - \lim_{N\to\infty} \frac{1}{N} \ln \theta_0. \tag{1.27}$$

Therefore, one can find expressions for the partition function and the free energy of the 8V model by solving the eigenvalue problem of its transfer matrix.

1.3 The XYZ spin chain

Spin chains are one-dimensional models of quantum magnetism [103]. In particular, the XYZ spin chain consists of a collection of spin-1/2 particles on a line, with nearest-neighbour interactions (see Figure 1.7). Such a chain with N spins is described by the Hamiltonian $H \in \operatorname{End}(\mathbb{C}^2)^{\otimes N}$,

$$H = -\frac{1}{2} \sum_{i=1}^{N} J_4 \sigma_i^1 \sigma_{i+1}^1 + J_3 \sigma_i^2 \sigma_{i+1}^2 + J_2 \sigma_i^3 \sigma_{i+1}^3. \tag{1.28}$$

Here, the real numbers J_i are *anisotropy parameters*, $i = 2, 3, 4$. One fixes boundary conditions by specifying how the first and last spin interact with each other. In our work below, we enforce *periodic boundary conditions* with

$$\sigma_{N+1}^a = \sigma_1^a, \qquad a = 1, 2, 3. \tag{1.29}$$

Other types of boundary conditions include *twisted* or *quasi-periodic boundary conditions*, where σ_{N+1}^a and σ_1^a are related by conjugation with a twist operator, and *open boundary conditions*, where the first and last spin do not interact with each other.

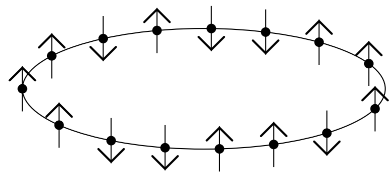


Figure 1.7: A configuration of a chain of 15 spin-1/2 particles with periodic boundary conditions.

We note that the XYZ spin chain is a generalisation of the XXZ and XXX spin chains, where the anisotropy parameters are restricted to $J_4 = J_3 \neq J_2$ and $J_4 = J_3 = J_2$, respectively. A common normalisation for the XXZ model is $J_4 = J_3 = 1$, $J_2 = \Delta$ [12].

As defined in (1.28) and (1.29), the XYZ Hamiltonian is Hermitian, and thus diagonalisable with real eigenvalues. Its diagonalisation is a fundamental step in the analysis of the spin chain, as its energy spectrum and eigenstates characterise the system and its time evolution. Among the eigenstates, the one associated to the lowest energy eigenvalue is called *ground state*, and is of particular interest [104].

Sutherland showed that the 8V transfer matrix commutes with the XYZ Hamiltonian for a certain choice of anisotropy parameters [13]. To state the property, we define

$$\zeta = \frac{c(\lambda)d(\lambda)}{a(\lambda)b(\lambda)} = \left(\frac{\vartheta_1(\eta, p^2)}{\vartheta_4(\eta, p^2)}\right)^2. \tag{1.30}$$

Proposition 1.3.1. *For the anisotropy parameters*[2]

$$J_2 = \frac{a(\lambda)^2 + b(\lambda)^2 - c(\lambda)^2 - d(\lambda)^2}{2a(\lambda)b(\lambda)(1-\zeta^2)}, \quad J_3 = \frac{1}{1+\zeta}, \quad J_4 = \frac{1}{1-\zeta}, \tag{1.31}$$

the XYZ Hamiltonian commutes with the transfer matrix of the homogeneous 8V model,

$$\Big[H, T(\lambda)|_{\lambda_1,\dots,\lambda_N\to 0}\Big] = 0. \tag{1.32}$$

This proposition indicates that the homogeneous 8V transfer matrix and the XYZ Hamiltonian can be diagonalised simultaneously [102]. Furthermore, Baxter showed that the XYZ Hamiltonian with anisotropy

[2]An explicit evaluation shows that the expression of J_2 is independent of λ.

parameters (1.31) can be written as a logarithmic derivative of the 8V transfer matrix [12, 28, 29]. Thus, solving the eigenvalue problem of the 8V transfer matrix provides results for that of the XYZ Hamiltonian as well.

Finally, we note that in the trigonometric limit, Proposition 1.3.1 describes the commutation between a XXZ Hamiltonian and the transfer matrix of the 6V model.

1.4 Combinatorial properties

Vertex models and related spin chains have been found to possess remarkable connections to combinatorial structures. A first observation along this line was made on the XXZ spin chain. It was discovered that, for an odd number of sites and for anisotropy $\Delta = -1/2$, the ground-state eigenvalue has a surprisingly simple expression [105–107]. Razumov and Stroganov examined the ground-state components for periodic boundary conditions [70]. In them, they found integer sequences that enumerate combinatorial objects such as alternating sign matrices (ASMs) and plane partitions. Similar properties were uncovered for the twisted chain with an even number of sites, and for the open chain with boundary magnetic fields [108–110]. Henceforth, the choice $\Delta = -1/2$ was dubbed a *combinatorial point* for the XXZ chain.

Following the developments of the trigonometric case, Razumov and Stroganov [71] and Bazhanov and Mangazeev [72, 73] identified a *combinatorial line* in the space of parameters of the periodic XYZ spin chain, and related 8V model. They analysed the ground-state components along this line for an odd number of sites, and observed polynomials with integer coefficients within. Connections have been made between these polynomials and combinatorial objects [71–73, 111]. Moreover, Hietala recently established a combinatorial interpretation for some of them as partition functions of the three-colour model [82, 83]. Beyond combinatorics, subsequent investigations of the 8V model on the combinatorial line have revealed exciting relations to various topics in mathematical physics. Examples include elliptic functional equations [15, 74, 75], the Painlevé VI equation [72, 76, 77], and special polynomials [78–81]. The last example we mention is due to Hagendorf and Fendley. They discovered that the Hamiltonian of the XYZ spin chain on its combinatorial line is *supersymmetric* [84]. In broad terms, this means that it can be written as the anticommutator of nilpotents operators called supercharges [112].

In this section, we present the main combinatorial objects that shall appear in our work below. We state without proof some results regarding their enumeration.

Alternating sign matrices An *alternating sign matrix* is a square matrix with entries $0, \pm 1$ such that *(i)* the entries of each column or line sum up to 1 *(ii)* the non-zero entries alternate in sign along each column and line. See Figure 1.8 for the 3×3 ASMs.

$$\begin{pmatrix} + & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & + \end{pmatrix}, \quad \begin{pmatrix} + & 0 & 0 \\ 0 & 0 & + \\ 0 & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & + \end{pmatrix}, \quad \begin{pmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{pmatrix}$$

Figure 1.8: The seven 3×3 ASMs, with $\pm = \pm 1$.

These combinatorial objects appeared in the work of Mills, Robbins and Rumsey as a generalisation of permutation matrices [64, 65]. In 1983, the authors formulated a famous conjecture regarding the number of $n \times n$ ASMs. The conjecture generated a wide interest [63], and its proof was finally achieved by Zeilberger in 1995 [66]. Let $A(n)$ denote the number of $n \times n$ ASMs, the now *alternating sign theorem* reads

$$A(n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1.33}$$

Shortly after Zeilberger, Kuperberg proposed an alternative proof of this result [67]. He formulated a bijection between the $n \times n$ ASMs and the configurations of the 6V model on a $n \times n$ lattice with domain-wall boundary conditions (DWBC) [67, 68]. (See Figures 1.9 and 1.10.)

Based on this relation, Kuperberg showed that the partition function of the model, previously computed by Izergin and Korepin [18, 69], simply enumerates ASMs. For a more detailed account of the dawn of ASMs and the story behind the conjecture, see [63].

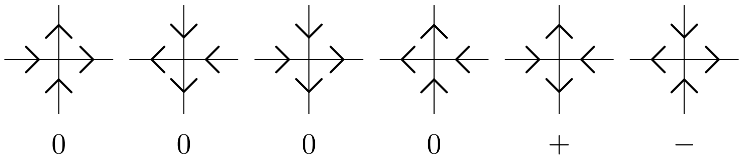


Figure 1.9: Correspondence between vertices of the 6V model and ASM entries.

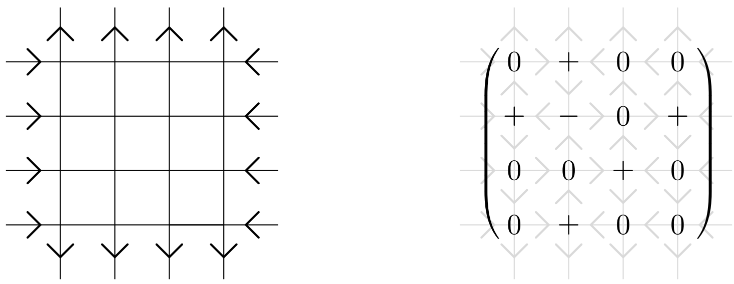


Figure 1.10: *Left:* A 4×4 square lattice with DWBC. *Right:* ASM corresponding to the 6V configuration of Figure 1.1.

Our work will lead us to several more general enumerations of ASMs. Here below are two examples.

The first one is a *refined enumeration*. Following from its definition, an ASM of size n must have a unique $+1$ in its first row. Let $A(n,k)$ denote the number of matrices for which this $+1$ is in column k. This number is given by [113]

$$A(n,k) = \frac{\binom{n+k-2}{n-1}\binom{2n-1-k}{n-1}}{\binom{3n-2}{n-1}} A(n). \tag{1.34}$$

The second one is a *weighted enumeration*. Let $t \in \mathbb{R}_{\geq 0}$, we associate a weight t^k to any ASM with k negative entries. We define the t-*enumerations* of ASMs $A(n;t)$ as the weighted sum over all $n \times n$ ASMs. For the examples of Figure 1.8, we find $A(3;t) = 6 + t$.

Symmetry classes of ASMs Another interesting enumeration problem is that of classes of ASMs invariant under certain symmetries of the square [114, 115]. Eight such symmetry classes exist. Below, we present the ones that appear in our work.

Vertically symmetric ASMs (VSASMs) are ASMs invariant under reflection around the vertical axis (see the left of Figure 1.11). In terms of its entries, the ASM $A = (a_{ij})_{i,j=1}^n$ is vertically symmetric if

$$a_{ij} = a_{i(n+1-j)}, \qquad i, j = 1, \ldots, n. \tag{1.35}$$

VSAMs are exclusively odd-sized, as no ASM of even size can fulfil the symmetry. The number of VSASMs of size $2\ell + 1$ is [114]

$$A_{\mathrm{V}}(2\ell + 1) = \frac{1}{2^\ell} \prod_{i=1}^{\ell} \frac{(6i-2)!(2i-1)!}{(4i-1)!(4i-2)!}. \tag{1.36}$$

Let us now consider their weighted enumeration. The negative entries of symmetric ASMs form orbits under the action of the symmetry group. The weight of such an ASM is then t^k, where k now denotes the number of orbits of negative entries. Practically, one can determine this number by counting the -1s in a fundamental domain of the group action, excluding those forced by symmetry (see the left of Figure 1.11). The t-enumeration of vertically symmetric ASMs reads [92, 116]

$$A_{\mathrm{V}}(2\ell + 1; t) = \det_{i,j=0}^{\ell-1} \left(\sum_{k=0}^{\ell-1} \binom{i+j+1}{i+k+1} \binom{i+k+1}{2k+1} t^k \right). \tag{1.37}$$

Half-turn symmetric ASMs (HTSASMs) are ASMs invariant under a rotation of angle π. In terms of its entries, the ASM $A = (a_{ij})_{i,j=1}^n$ is a HTSASM if

$$a_{ij} = a_{(n+1-i)(n+1-j)}, \qquad i, j = 1, \ldots, n. \tag{1.38}$$

See the right of Figure 1.11 for an example. Kuperberg found a determinant formula for the t-enumeration of HTSASMs $A_{\mathrm{HT}}(n; t)$ for even n [114].

Diagonally and anti-diagonally symmetric ASMs (DADASMs) are ASMs invariant under reflection around both diagonals. In terms of its entries, the ASM $A = (a_{ij})_{i,j=1}^n$ is a DADASM if

$$a_{ij} = a_{ji} = a_{(n+1-j)(n+1-i)}, \qquad i, j = 1, \ldots, n. \tag{1.39}$$

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 \\ + & 0 & - & 0 & + \\ 0 & 0 & + & 0 & 0 \\ 0 & + & - & + & 0 \\ 0 & 0 & + & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 \\ 0 & 0 & + & 0 & - & + \\ + & - & 0 & + & 0 & 0 \\ 0 & + & 0 & - & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{pmatrix}$$

Figure 1.11: *Left:* A 5×5 VSASM with drawn symmetry axis. Its middle column is fixed by symmetry, it thus possesses $k = 0$ orbit of negative entries. *Right:* A 6×6 HTSASM with $k = 2$ orbits of negative entries. The dotted line delimits the fundamental domain of the symmetry action.

Note that each DADASM also fulfils (1.38), and is therefore also half-turn symmetric. See the left of Figure 1.12 for an example. The number of DADASMs of size $2\ell + 1$ is [117]

$$A_{\text{DAD}}(2\ell + 1) = \prod_{i=0}^{\ell} \frac{(3i)!}{(\ell + i)!}. \tag{1.40}$$

No equivalent formula appears to be known for the even size.

Lastly, *ASMs with double U-turn boundary conditions (UUASMs)* are $2\ell \times 2\ell$ matrices in which the successive pairs of rows and columns read in a prescribed way form valid rows and columns of an ASM (see the right of Figure 1.12). Precisely, for each $k = 1, \ldots, \ell$, reading the $(2k-1)$-th row from left to right then the $2k$-th one from right to left yields a valid row of an ASM. The same goes for the $(2k-1)$-th column read from bottom to top followed by the $2k$-th from top to bottom. Although UUASMs are not a class of ASMs invariant under symmetries of the square, they are the generalisation of one [114].

We shall encounter weighted enumerations of UUASMs more general than t-enumerations. Let $y, z \in \mathbb{R}_{\geq 0}$, we assign to an UUASM the weight $t^k y^m z^{m'}$, with k the number of negative entries, and m and m' respectively the number of odd rows and columns with an odd number of non-zero entries [92]. See the right of Figure 1.12 for an example. Following Kuperberg's notation, we denote by $A_{\text{UU}}(4\ell; t, y, z)$ the (t, y, z)-

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 & 0 \\ + & - & 0 & + & - & + & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & + & - & + & 0 & - & + \\ 0 & 0 & + & 0 & - & + & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} + & - & 0 & + \\ 0 & + & 0 & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + \end{pmatrix}$$

Figure 1.12: *Left:* A 7×7 DADASM with drawn symmetry axis. *Right:* A 4×4 UUASM with $k = 2$ negative entries and respectively $m = 1$ and $m' = 1$ odd rows and columns with an odd number of non-zero entries. The half-circular bridges represent the U-turns.

enumeration of $2\ell \times 2\ell$ UUASMs. The quantity

$$A^{(2)}_{\mathrm{UU}}(4\ell; t, y, z) = \frac{A_{\mathrm{UU}}(4\ell; t, y, z)}{A_{\mathrm{V}}(2\ell + 1; t)} \tag{1.41}$$

is a polynomial in t, y and z given by [92, 114]

$$\begin{aligned} A^{(2)}_{\mathrm{UU}}(4\ell; t, y, z) = \det_{i,j=0}^{\ell-1} \Bigg(\sum_{k=0}^{\ell-1} (t + (1+y)(1+z)) \binom{i+j}{i+k} \binom{i+k}{2k} t^k \\ \times \left((1+z) \binom{i+j}{i+k} \binom{i+k}{2k+1} + (1+y) \binom{i+j}{i+k+1} \binom{i+k+1}{2k+1} \right) t^{k+1} \Bigg). \end{aligned} \tag{1.42}$$

Moreover, we shall also need

$$\tilde{A}^{(2)}_{\mathrm{UU}}(4\ell; t) = t^{-\ell} A^{(2)}_{\mathrm{UU}}(4\ell; t, y = -1, z = -1), \tag{1.43}$$

a polynomial given by the determinant formula [92]

$$\tilde{A}^{(2)}_{\mathrm{UU}}(4\ell; t) = \det_{i,j=0}^{\ell-1} \left(\sum_{k=0}^{\ell-1} \binom{i+j}{i+k} \binom{i+k}{2k} t^k \right). \tag{1.44}$$

Chapter 2

Sum rules for the supersymmetric eight-vertex model

In this chapter, we pursue an investigation of the square lattice eight-vertex model whose vertex weights a, b, c, d obey the relation

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab). \tag{2.1}$$

This subfamily of weights corresponds to the *combinatorial line* discussed in Section 1.4. We consider the model on a cylinder with $N \geqslant 1$ vertical lines and periodic boundary conditions along the horizontal direction. If the number of vertical lines $N = 2n + 1$, $n \geqslant 0$, is odd, then the transfer matrix of the vertex model possesses the remarkably simple double eigenvalue

$$\Theta_n = (a + b)^{2n+1}. \tag{2.2}$$

The existence of this eigenvalue was first conjectured by Stroganov [107]. Recently, Liénardy and Hagendorf proved this conjecture with the help of supersymmetry techniques [84,85]. Because of the supersymmetry, we refer to the case where the vertex weights obey (2.1) as the *supersymmetric eight-vertex model.*

Our main objective is to investigate the eigenspace of Θ_n. Bazhanov and Mangazeev [73], and Razumov and Stroganov [71] initiated this investigation by exploiting the fact that the eigenvectors of Θ_n are the ground-state vectors of the Hamiltonian of a XYZ quantum spin chain. They computed these vectors for small n with the help of exact diagonalisation methods applied to the spin-chain Hamiltonian. This computation led them to numerous conjectures on the eigenvectors, providing explicit expressions for their components and for scalar products. The latter are often referred to as *sum rules*. Subsequently, a rigorous investigation of the eigenspace for general n was undertaken by Zinn-Justin [111]. He considered a generalisation of the eigenvalue problem for Θ_n to the inhomogeneous eight-vertex model. Based on only one conjecture about its solution, he was able to derive one of the sum rules that had been observed by Bazhanov and Mangazeev. In this chapter, we use Zinn-Justin's conjecture to investigate the eigenspace of Θ_n in further detail. To this end, we apply a well-established induction technique due to Izergin and Korepin [18, 69]. It allows us to derive explicit expressions for several scalar products of an eigenvector of the transfer matrix of the inhomogeneous eight-vertex model. From their homogeneous limits, we deduce several new sum rules and exact expressions for the components of the eigenvectors of Θ_n.

This chapter contains the analysis performed in [1], and is organised as follows: in Section 2.1, we discuss a generalisation of the eigenvalue problem for Θ_n to the inhomogeneous model and the properties of its solution that follow from Zinn-Justin's conjecture. In Section 2.2, we introduce scalar products involving this solution as well as several solutions to the boundary Yang-Baxter equation. We obtain exact expressions for these scalar products in terms of the so-called elliptic Tsuchiya determinant. In Section 2.3, we compute the homogeneous limit of these expressions. We use them to derive and analyse several properties of the eigenvectors of Θ_n. We present our conclusions in Section 2.4.

2.1 Eigenvalue problem

Unless stated otherwise, we consider throughout this chapter the case where $N = 2n + 1$, with $n \geqslant 0$ an integer, and

$$\eta = 2\pi/3. \tag{2.3}$$

For this value of the crossing parameter, the weights (1.17) obey the defining relation (2.1) of the supersymmetric eight-vertex model. Moreover, the transfer matrix of the inhomogeneous model (1.22) is conjectured to possess the double eigenvalue [71]

$$\Theta_n(\lambda) = \prod_{i=1}^{2n+1} r(\lambda_i - \lambda), \tag{2.4}$$

where

$$r(\lambda) = a(\lambda) + b(\lambda) = \frac{1}{2}\vartheta_2(0)\vartheta_4(0,p^2)\vartheta_1(\eta - \lambda). \tag{2.5}$$

For the homogeneous model, where $\lambda_1 = \cdots = \lambda_{2n+1} = 0$, it reduces to the eigenvalue $\Theta_n = (a+b)^{2n+1}$, whose existence has been rigorously established [85].

The eigenvalue problem Since the transfer-matrix eigenvalue has multiplicity 2, it is convenient to lift the degeneracy by imposing the transformation property of the eigenvectors under spin reversal. We consider the following eigenvalue problem

$$T(\lambda)|\Psi\rangle = \Theta_n(\lambda)|\Psi\rangle, \qquad F|\Psi\rangle = (-1)^n|\Psi\rangle, \tag{2.6}$$

where $|\Psi\rangle \in (\mathbb{C}^2)^{\otimes(2n+1)}$. For small n, it is possible to find non-trivial solutions to this system of linear equations through a direct calculation. For arbitrary n, however, their existence has, to our best knowledge, not been rigorously established. The following conjecture postulates this existence and characterises the corresponding space of solutions.

Conjecture 2.1.1 (Zinn-Justin [111])**.** *The space of solutions to the eigenvalue problem* (2.6) *is spanned by a vector*

$$|\Psi_n\rangle = |\Psi_n(\lambda_1,\dots,\lambda_{2n+1})\rangle \in (\mathbb{C}^2)^{\otimes(2n+1)}, \tag{2.7}$$

whose components are entire functions with respect to λ_i, for each $i = 1,\dots,2n+1$. They are generically non-vanishing and do not have a common factor that depends on $\lambda_1,\dots,\lambda_{2n+1}$. Furthermore, the vector obeys the relations

$$\begin{aligned}
|\Psi_n(\dots,\lambda_i + 2\pi\omega,\dots)\rangle &= p^{-4n}\mathrm{e}^{-2\mathrm{i}\sum_{j=1}^{2n+1}(\lambda_i-\lambda_j)}|\Psi_n(\dots,\lambda_i,\dots)\rangle,\\
|\Psi_n(\dots,\lambda_i + \pi,\dots)\rangle &= \prod_{j=1,j\neq i}^{2n+1}\sigma_j^3|\Psi_n(\dots,\lambda_i,\dots)\rangle.
\end{aligned} \tag{2.8}$$

We assume that this conjecture holds and derive the main results of this chapter from it.

Properties of the eigenvector Conjecture 2.1.1 implies several properties of the vector $|\Psi_n\rangle$, which we frequently use in the following. We recall them here in the form of four lemmas, whose proofs can be found in [111]. The first lemma expresses the action of the $\check{R}$-matrix on the vector.

Lemma 2.1.2 (Exchange relations)**.** *For $n \geqslant 1$ and each $i = 1, \dots, 2n$, we have*

$$\check{R}_{i,i+1}(\lambda_{i+1} - \lambda_i)|\Psi_n(\dots, \lambda_i, \lambda_{i+1}, \dots)\rangle \\ = r(\lambda_{i+1} - \lambda_i)|\Psi_n(\dots, \lambda_{i+1}, \lambda_i, \dots)\rangle. \quad (2.9)$$

We note that the compatibility of the exchange relations follows from the braid Yang-Baxter equation (1.19).

The second lemma describes the action of a local spin-reversal operator on the eigenvector.

Lemma 2.1.3 (Local spin reversals)**.** *For each $i = 1, \dots, 2n+1$, we have*

$$\sigma_i^1|\Psi_n(\dots, \lambda_i, \dots)\rangle \\ = (-p)^n \mathrm{e}^{-\mathrm{i}\sum_{j=1}^{2n+1}(\lambda_j - \lambda_i)}|\Psi_n(\dots, \lambda_i + \pi\omega, \dots)\rangle. \quad (2.10)$$

For the third lemma, we consider three inhomogeneity parameters $\lambda_i, \lambda_j, \lambda_k$ with $1 \leqslant i < j < k \leqslant 2n+1$. We say that these parameters form a *wheel* if they can be written as $\lambda_i = x$, $\lambda_j = x + \eta$, $\lambda_k = x + 2\eta$, for some x. The so-called *wheel condition* asserts that the vector vanishes whenever a wheel is formed.

Lemma 2.1.4 (Wheel condition)**.** *For $n \geqslant 1$ and all x, we have*

$$|\Psi_n(\dots, x, \dots, x+\eta, \dots, x+2\eta, \dots)\rangle = 0. \quad (2.11)$$

The fourth lemma provides so-called *reduction relations* for the vector: upon a specialisation of certain inhomogeneity parameters, $|\Psi_n\rangle$ can be expressed in terms of $|\Psi_{n-1}\rangle$. To state the reduction relations, we define for each $N \geqslant 1$ and each $1 \leqslant i \leqslant N+1$ a linear operator

$$\varphi^i : (\mathbb{C}^2)^{\otimes N} \to (\mathbb{C}^2)^{\otimes (N+2)}. \tag{2.12}$$

It acts on the basis vector labelled by $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_N$ according to

$$\varphi^i|\boldsymbol{\alpha}\rangle = |\alpha_1 \cdots \alpha_{i-1}\rangle \otimes |s\rangle \otimes |\alpha_i \cdots \alpha_N\rangle, \tag{2.13}$$

where

$$|s\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle. \tag{2.14}$$

Lemma 2.1.5 (Reduction relations)**.** *For $n \geqslant 1$ and each $i = 1, \ldots, 2n$, we have*

$$\begin{aligned} &|\Psi_n(\ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1} = \lambda_i + \eta, \lambda_{i+2} \ldots)\rangle \\ &\quad = C_n \left(\prod_{j=1,\, j\neq i,i+1}^{2n+1} \vartheta_1(\lambda_i - \lambda_j - \eta) \right) \varphi^i |\Psi_{n-1}(\ldots, \lambda_{i-1}, \lambda_{i+2}, \ldots)\rangle, \end{aligned} \tag{2.15}$$

where C_n is independent of $\lambda_1, \ldots, \lambda_{2n+1}$.

Normalisation The constant C_n in Lemma 2.1.5 depends on the relative normalisation of the vectors. It is not fixed by Conjecture 2.1.1. In the following, we choose

$$C_n = 1, \tag{2.16}$$

for each $n \geqslant 1$. Furthermore, we define

$$|\Psi_0\rangle = |\uparrow\rangle + |\downarrow\rangle. \tag{2.17}$$

These choices and the properties listed in Conjecture 2.1.1 completely fix the vector $|\Psi_n\rangle$ for each $n \geqslant 0$ [111]. For instance, for $n = 1$ its components are

$$\begin{aligned} (\Psi_1)_{\uparrow\uparrow\uparrow} &= \rho\, \vartheta_1(\lambda_1 - \lambda_2 + \eta, p^2)\vartheta_1(\lambda_2 - \lambda_3 + \eta, p^2)\vartheta_1(\lambda_3 - \lambda_1 + \eta, p^2), \\ (\Psi_1)_{\uparrow\downarrow\downarrow} &= \rho\, \vartheta_4(\lambda_1 - \lambda_2 + \eta, p^2)\vartheta_1(\lambda_2 - \lambda_3 + \eta, p^2)\vartheta_4(\lambda_3 - \lambda_1 + \eta, p^2), \\ (\Psi_1)_{\downarrow\uparrow\downarrow} &= \rho\, \vartheta_4(\lambda_1 - \lambda_2 + \eta, p^2)\vartheta_4(\lambda_2 - \lambda_3 + \eta, p^2)\vartheta_1(\lambda_3 - \lambda_1 + \eta, p^2), \\ (\Psi_1)_{\downarrow\downarrow\uparrow} &= \rho\, \vartheta_1(\lambda_1 - \lambda_2 + \eta, p^2)\vartheta_4(\lambda_2 - \lambda_3 + \eta, p^2)\vartheta_4(\lambda_3 - \lambda_1 + \eta, p^2), \end{aligned} \tag{2.18}$$

and

$$(\Psi_1)_{\downarrow\downarrow\downarrow} = -(\Psi_1)_{\uparrow\uparrow\uparrow},\ (\Psi_1)_{\downarrow\uparrow\uparrow} = -(\Psi_1)_{\uparrow\downarrow\downarrow},\ (\Psi_1)_{\uparrow\uparrow\downarrow} = -(\Psi_1)_{\downarrow\downarrow\uparrow}, \tag{2.19}$$

where the overall factor is given by

$$\rho = \frac{2}{\vartheta_2(0)\vartheta_4(0,p^2)}. \tag{2.20}$$

For $n = 2$, Hagendorf recently found an explicit formula for the components by means of elliptic interpolation [118]. For $n \geq 3$, the components are considerably more complicated. An equivalent formula remains to be found.

2.2 Scalar products

In this section, we compute several scalar products that involve the transfer-matrix eigenvector $|\Psi_n\rangle$. In Section 2.2.1, we define the scalar products with the help of two solutions to the boundary Yang-Baxter equation. In Section 2.2.2, we analyse their symmetry and analyticity properties, and show that they obey a set of reduction relations. These properties are sufficient to characterise the scalar products uniquely. In Section 2.2.3, we find determinant formulas in terms of the so-called elliptic Tsuchiya determinant that fulfil the same properties as the scalar products. Using uniqueness and an inductive proof technique, we show that they provide explicit expressions for the scalar products. In Section 2.2.4, we recall a relation between elliptic Tsuchiya determinants and a family of multivariable polynomials introduced by Zinn-Justin and Rosengren.

2.2.1 Definition

In its vector form, the boundary Yang-Baxter equation for the eight-vertex model is given by

$$\begin{aligned} &\check{R}_{12}(x-y)\check{R}_{23}(-x-y)|\chi(x)\rangle \otimes |\chi(y)\rangle \\ &\qquad = \check{R}_{34}(x-y)\check{R}_{23}(-x-y)|\chi(y)\rangle \otimes |\chi(x)\rangle, \end{aligned} \tag{2.21}$$

where $|\chi(x)\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.[1] We consider the following two solutions of this equation:

$$|\chi(x)\rangle = \vartheta_1(x+\kappa, p^2)\vartheta_4(x-\kappa-\eta, p^2)|{\uparrow\downarrow}\rangle \\ + \vartheta_1(x-\kappa-\eta, p^2)\vartheta_4(x+\kappa, p^2)|{\downarrow\uparrow}\rangle, \tag{2.23a}$$

$$|\bar{\chi}(x)\rangle = \vartheta_1(x+\kappa, p^2)\vartheta_1(x-\kappa-\eta, p^2)|{\uparrow\uparrow}\rangle \\ + \vartheta_4(x-\kappa-\eta, p^2)\vartheta_4(x+\kappa, p^2)|{\downarrow\downarrow}\rangle. \tag{2.23b}$$

Here, κ is an arbitrary parameter. Both these solutions are specialisations of the most general solution to the boundary Yang-Baxter equation for the eight-vertex model found by Inami and Konno [120]. In the following two lemmas, we list some useful properties of $|\chi(x)\rangle$ and $|\bar{\chi}(x)\rangle$. They follow from standard identities for Jacobi theta functions. It will be convenient to define

$$g(x) = \frac{\vartheta_4(\eta+2x, p^2)}{\vartheta_4(\eta-2x, p^2)}, \qquad \bar{g}(x) = \frac{\vartheta_1(\eta+2x, p^2)}{\vartheta_1(\eta-2x, p^2)}. \tag{2.24}$$

Lemma 2.2.1. *We have the relations*

$$\check{R}(2x)|\chi(x)\rangle = g(x)r(2x)|\chi(-x)\rangle, \tag{2.25}$$

$$\check{R}(2x)|\bar{\chi}(x)\rangle = \bar{g}(x)r(2x)|\bar{\chi}(-x)\rangle. \tag{2.26}$$

Lemma 2.2.2. *We have the matrix elements*

$$\frac{(\langle\chi(x+\eta)| \otimes \langle\chi(x)|)\,\check{R}_{23}(-2x-\eta)\,(|s\rangle \otimes |s\rangle)}{r(-2x-\eta)} \\ = \frac{\vartheta_2(0)^2\vartheta_1(x-\kappa)\vartheta_1(x+\kappa+\eta)g(x)}{2}, \tag{2.27}$$

$$\frac{(\langle\bar{\chi}(x+\eta)| \otimes \langle\bar{\chi}(x)|)\,\check{R}_{23}(-2x-\eta)\,(|s\rangle \otimes |s\rangle)}{r(-2x-\eta)} \\ = \frac{\vartheta_2(0)^2\vartheta_1(x-\kappa)\vartheta_1(x+\kappa+\eta)\bar{g}(x)}{2}, \tag{2.28}$$

where $|s\rangle$ is the vector defined in (2.14).

[1] Traditionally, the boundary Yang-Baxter equation is presented in a matrix form with solutions $K(x) \in \mathrm{End}(\mathbb{C}^2)$ [119],

$$R_{12}(x-y)K_1(x)R_{12}(x+y)K_2(y) = K_2(y)R_{12}(x+y)K_1(x)R_{12}(x-y). \tag{2.22}$$

To each K-matrix $K(x)$ corresponds a solution $|\chi(x)\rangle$ of the vector form (2.21), defined as $\langle\alpha\beta|\chi(x)\rangle = \langle\alpha|\sigma^2 K(x-\eta/2)|\beta\rangle$, for all $\alpha, \beta \in \{\pm 1\}$.

For each $n \geqslant 1$, we use the two solutions of the boundary Yang-Baxter equation to define the vectors

$$|\xi_n(x_1,\dots,x_n)\rangle = \left(\bigotimes_{i=1}^{n} |\chi(x_i)\rangle\right) \otimes |{\uparrow}\rangle, \tag{2.29}$$

$$|\bar\xi_n^{\pm}(x_1,\dots,x_n)\rangle = \left(\bigotimes_{i=1}^{n} |\bar\chi(x_i)\rangle\right) \otimes \left(|{\uparrow}\rangle \pm |{\downarrow}\rangle\right). \tag{2.30}$$

It will also be convenient to introduce

$$|\xi_0\rangle = |{\uparrow}\rangle, \qquad |\bar\xi_0^{\pm}\rangle = |{\uparrow}\rangle \pm |{\downarrow}\rangle. \tag{2.31}$$

Using these vectors, we define the scalar products

$$Z_n(x_1,\dots,x_n) = \langle \xi_n(x_1,\dots,x_n)|\Psi_n(x_1,-x_1,\dots,x_n,-x_n,0)\rangle, \tag{2.32}$$

$$\bar Z_n^{\pm}(x_1,\dots,x_n) = \langle \bar\xi_n^{\pm}(x_1,\dots,x_n)|\Psi_n(x_1,-x_1,\dots,x_n,-x_n,0)\rangle. \tag{2.33}$$

In the following, we only write out their dependence on the inhomogeneity parameters $x_1,\dots,x_n$ if necessary.

It is straightforward to find Z_n and $\bar Z_n^{\pm}$ for $n=0$ and $n=1$. Using (2.17), we obtain

$$Z_0 = 1, \qquad \bar Z_0^{+} = 2, \qquad \bar Z_0^{-} = 0. \tag{2.34}$$

Furthermore, we use the components (2.18) and (2.19), and find

$$Z_1 = -\frac{\rho}{2}\,\vartheta_2(0)\vartheta_2(\eta-\kappa)\vartheta_4(\eta+2x_1,p^2)\vartheta_1(\eta+x_1)\vartheta_1(\eta-x_1), \tag{2.35a}$$

$$\bar Z_1^{+} = \rho\,\vartheta_4(0)\vartheta_4(\eta-\kappa)\vartheta_1(\eta+2x_1,p^2)\vartheta_3(\eta+x_1)\vartheta_3(\eta-x_1), \tag{2.35b}$$

$$\bar Z_1^{-} = \rho\,\vartheta_3(0)\vartheta_4(\eta-\kappa)\vartheta_1(\eta+2x_1,p^2)\vartheta_4(\eta+x_1)\vartheta_4(\eta-x_1), \tag{2.35c}$$

where ρ is defined in (2.20).

2.2.2 Properties

In this section, we establish the properties of Z_n and $\bar Z_n^{\pm}$ that allow us to find them for $n \geqslant 2$. The proofs of these properties are very similar for Z_n and $\bar Z_n^{\pm}$. Hence, we focus on Z_n. If necessary, we indicate the modifications to be made for $\bar Z_n^{\pm}$.

Symmetry

Proposition 2.2.3. *For $n \geqslant 2$, Z_n and $\bar{Z}_n^{\pm}$ are symmetric functions of $x_1, \dots, x_n$.*

Proof. We sketch the proof of the symmetry of Z_n. It is sufficient to prove that

$$Z_n(\dots, x_i, x_{i+1}, \dots) = Z_n(\dots, x_{i+1}, x_i, \dots), \tag{2.36}$$

for each $i = 1, \dots, n-1$. For $i = 1$, we find

$$\begin{aligned} Z_n(x_1, x_2, \dots) &= \langle \xi_n(x_1, x_2, \dots) | \Psi_n(x_1, -x_1, x_2, -x_2, \dots) \rangle \\ &= \langle \xi_n(x_1, x_2, \dots) | \frac{\check{R}_{23}(-x_2 - x_1)\check{R}_{34}(x_2 - x_1)}{r(-x_2 - x_1) r(x_2 - x_1)} \\ &\qquad\qquad \times |\Psi_n(x_1, x_2, -x_2, -x_1, \dots)\rangle \\ &= \langle \xi_n(x_2, x_1, \dots) | \frac{\check{R}_{23}(-x_2 - x_1)\check{R}_{12}(x_2 - x_1)}{r(-x_2 - x_1) r(x_2 - x_1)} \\ &\qquad\qquad \times |\Psi_n(x_1, x_2, -x_2, -x_1, \dots)\rangle \\ &= \langle \xi_n(x_2, x_1, \dots) | \Psi_n(x_2, -x_2, x_1, -x_1, \dots) \rangle \\ &= Z_n(x_2, x_1, \dots). \end{aligned} \tag{2.37}$$

From the first to the second line, we used Lemma 2.1.2. The third line is obtained by using the symmetry of the $\check{R}$-matrix and the boundary Yang-Baxter equation (2.21). The fourth line is the result of another application of Lemma 2.1.2. This establishes (2.36) for $i = 1$. The cases where $i = 2, \dots, n-1$ are straightforward generalisations. □

Proposition 2.2.4. *For each $i = 1, \dots, n$ we have*

$$\begin{aligned} \vartheta_4(\eta + 2x_i, p^2) Z_n(\dots, -x_i, \dots) &= \vartheta_4(\eta - 2x_i, p^2) Z_n(\dots, x_i, \dots), \\ \vartheta_1(\eta + 2x_i, p^2) \bar{Z}_n^{\pm}(\dots, -x_i, \dots) &= \vartheta_1(\eta - 2x_i, p^2) \bar{Z}_n^{\pm}(\dots, x_i, \dots). \end{aligned}$$

Proof. We present the proof of the first relation. By Proposition 2.2.3, it is sufficient to consider $i = 1$. We compute

$$\begin{aligned} Z_n(-x_1, \dots) &= \langle \xi_n(-x_1, \dots) | \Psi_n(-x_1, x_1, \dots) \rangle \\ &= \langle \xi_n(x_1, \dots) | \frac{\check{R}_{12}(2x_1)}{g(x_1) r(2x_1)} | \Psi_n(-x_1, x_1, \dots) \rangle = g(-x_1) Z_n(x_1, \dots). \end{aligned} \tag{2.38}$$

From the first to the second line, we used Lemma 2.2.1 and the symmetry of the $\check{R}$-matrix. The equality in the second line follows from Lemma 2.1.2 and the explicit expression of $g(x)$, given above.

The proof of the second relation is similar. □

Analyticity In the following, we use the concept of *theta functions*, a class of pseudo-periodic functions of the complex plane. We refer the reader to Appendix A.2, for their definition and main properties.

Proposition 2.2.5. *For $n \geqslant 1$, Z_n and $\bar{Z}_n^{\pm}$ are theta functions of degree $2(n+1)$, nome p and norm $\frac{\pi}{2}+\frac{\eta}{2}$ with respect to x_i for each $i = 1, \dots, n$.*

Proof. We focus on Z_n. By Proposition 2.2.3, it is sufficient to establish the statement for Z_n as a function of x_1. It follows from Conjecture 2.1.1 and from the definition of the vector $|\chi(x)\rangle$ that Z_n is an entire function of x_1. To establish its pseudo-periodicity properties, we note that $|\chi(x)\rangle$ obeys the relations

$$|\chi(x+\pi)\rangle = \sigma_1^3\sigma_1^3|\chi(x)\rangle, \tag{2.39a}$$
$$|\chi(x+\pi\omega)\rangle = -p^{-1}\mathrm{e}^{-\mathrm{i}(2x-\eta)}\sigma_1^1\sigma_2^1|\chi(x)\rangle. \tag{2.39b}$$

Furthermore, it follows from Conjecture 2.1.1 and Lemma 2.1.3 that

$$|\Psi_n(x_1+\pi, -x_1-\pi, \dots)\rangle = \sigma_1^3\sigma_2^3|\Psi_n(x_1, -x_1, \dots)\rangle, \tag{2.40a}$$
$$\begin{aligned}|\Psi_n(x_1+\pi\omega, -x_1-\pi\omega, \dots)\rangle = p^{-(2n+1)}&\mathrm{e}^{-2(2n+1)\mathrm{i}x_1} \\ &\times \sigma_1^1\sigma_2^1|\Psi_n(x_1, -x_1, \dots)\rangle.\end{aligned} \tag{2.40b}$$

We combine (2.39) and (2.40) to obtain

$$Z_n(x_1+\pi, \dots) = \langle\xi_n(x_1+\pi, \dots)|\sigma_1^3\sigma_2^3|\Psi_n(x_1, -x_1, \dots)\rangle \tag{2.41}$$
$$= Z_n(x_1, \dots), \tag{2.42}$$

and

$$\begin{aligned}Z_n(x_1+\pi\omega, \dots) &= p^{-(2n+1)}\mathrm{e}^{-2(2n+1)\mathrm{i}x_1} \\ &\quad\times \langle\xi_n(x_1+\pi\omega, \dots)|\sigma_1^1\sigma_2^1|\Psi_n(x_1, -x_1, \dots)\rangle \end{aligned}\tag{2.43}$$
$$= p^{-2(n+1)}\mathrm{e}^{-2\mathrm{i}(2(n+1)x_1-(\eta/2+\pi/2))}Z_n(x_1, \dots). \tag{2.44}$$

This ends the proof for Z_n. The proof for $\bar{Z}_n^{\pm}$ is similar. □

Zeroes and trivial factors In the next proposition, we identify certain trivial zeroes of the scalar products Z_n and $\bar{Z}_n^{\pm}$. It is practical to introduce the abbreviations

$$\beta_1 = \frac{\eta}{2}, \quad \beta_2 = \frac{\eta}{2} + \frac{\pi}{2}, \quad \beta_3 = \frac{\eta}{2} + \frac{\pi}{2} + \frac{\pi\omega}{2}, \quad \beta_4 = \frac{\eta}{2} + \frac{\pi\omega}{2}. \tag{2.45}$$

Proposition 2.2.6. *For $n \geqslant 1$ and each $i = 1, \dots, n$ we have*

$$Z_n(\dots, x_i = -\beta_1, \dots) = Z_n(\dots, x_i = \beta_1, \dots) = 0, \tag{2.46}$$

$$Z_n(\dots, x_i = -\beta_3, \dots) = Z_n(\dots, x_i = -\beta_4, \dots) = 0, \tag{2.47}$$

$$\bar{Z}_n^{\pm}(\dots, x_i = -\beta_1, \dots) = \bar{Z}_n^{\pm}(\dots, x_i = -\beta_2, \dots) = 0. \tag{2.48}$$

Proof. By Proposition 2.2.3, it is sufficient to prove the proposition for $i = 1$. We focus on the proof of (2.46). We note that

$$\begin{aligned} &Z_n(x_1 = -\beta_1, \dots) \\ &\quad = \langle \xi_n \left(\tfrac{\eta}{2}, x_2, \dots, x_n\right) | \Psi_n \left(-\tfrac{\eta}{2}, \tfrac{\eta}{2}, x_2, -x_2, \dots, x_n, -x_n, 0\right) \rangle \\ &\quad = \langle \xi_n(\tfrac{\eta}{2}, x_2, \dots, x_n) | \prod_{j=1}^{2n} \sigma_j^3 | \Psi_n(-\tfrac{\eta}{2}, \tfrac{\eta}{2}, x_2, -x_2, \dots, x_n, -x_n, \tfrac{3\eta}{2}) \rangle. \end{aligned} \tag{2.49}$$

The first, second and last arguments of $|\Psi_n\rangle$ form a wheel. Lemma 2.1.4 implies that the vector vanishes identically. Hence, $Z_n(x_1 = -\beta_1, \dots) = 0$. Furthermore, it follows from Proposition 2.2.4 that

$$\vartheta_4(2\eta, p^2) Z_n(x_1 = -\beta_1, \dots) = \vartheta_4(0, p^2) Z_n(x_1 = \beta_1, \dots). \tag{2.50}$$

Since $\vartheta_4(0, p^2)$ is non-zero, we find that $Z_n(x_1 = \beta_1, \dots) = 0$. Hence, we obtain (2.46) with $i = 1$.

Finally, we note that (2.47) and (2.48) directly follow from Proposition 2.2.4 and the known zeroes of the Jacobi theta functions. □

We now use the knowledge of the zeroes to find trivial factors of the scalar products. To this end, we use the factorisation property of theta functions of Lemma A.2.2.

Proposition 2.2.7. *For $n \geqslant 1$, we have*

$$Z_n = \left(\prod_{i=1}^{n} \vartheta_4(\eta + 2x_i, p^2) \vartheta_1(\eta + x_i) \vartheta_1(\eta - x_i) \right) X_n, \tag{2.51}$$

$$\bar{Z}_n^{\pm} = \left(\prod_{i=1}^{n} \vartheta_1(\eta + 2x_i, p^2) \right) \bar{X}_n^{\pm}, \tag{2.52}$$

where $X_n = X_n(x_1, \ldots, x_n)$ *and* $\bar{X}_n^{\pm} = \bar{X}_n^{\pm}(x_1, \ldots, x_n)$ *are even theta functions of degree* $2(n-1)$ *and* $2n$*, respectively, nome* p *and norm* 0 *with respect to* x_i *for each* $i = 1, \ldots, n$*. For* $n \geqslant 2$*,* X_n *and* $\bar{X}_n^{\pm}$ *are symmetric functions in* $x_1, \ldots, x_n$*.*

Proof. We present the proof of (2.51). According to Proposition 2.2.5, Z_n is a theta function with respect to x_i with degree $2(n+1)$, nome p and norm $t = \pi/2 + \eta/2$. By Proposition 2.2.6 it vanishes if $x_i = -\beta_1, +\beta_1, -\beta_4$ and $-\beta_3 + \pi + \pi\omega$ (where we used the pseudo-periodicity of the scalar product). Hence, we may apply Lemma A.2.2 to write Z_n as a product of the trivial factors

$$\begin{aligned} &\vartheta_1(x_i+\beta_1)\vartheta_1(x_i-\beta_1)\vartheta_1(x_i+\beta_4)\vartheta_1(x_i+\beta_3-\pi-\pi\omega) \\ &\qquad = B\vartheta_4(2x_i+\eta, p^2)\vartheta_1(\eta+x_i)\vartheta_1(\eta-x_i), \end{aligned} \tag{2.53}$$

where $B = -\mathrm{i}p^{-1/2}\vartheta_4(0,p^2)$, and a theta function with respect to x_i of degree $2(n-1)$, nome p and norm $t = 0$. Since this holds for each $i = 1, \ldots, n$, we obtain the factorisation (2.51), absorbing a power of the constant B into the definition of X_n. The symmetry of X_n for $n \geqslant 2$ follows from Proposition 2.2.3 and the symmetry of the factorised expression. Moreover, it follows from Proposition 2.2.4 that X_n is an even function of each x_i.

The proof of (2.52) is similar. □

Henceforth, we analyse the properties of the functions X_n, $\bar{X}_n^{\pm}$. For coherence, we define

$$X_0 = Z_0 = 1, \qquad \bar{X}_0^+ = \bar{Z}_0^+ = 2, \qquad \bar{X}_0^- = \bar{Z}_0^- = 0. \tag{2.54}$$

We note that for $n = 1$, the expressions (2.35) lead to

$$X_1 = -\frac{\rho}{2}\vartheta_2(0)\vartheta_2(\eta-\kappa), \tag{2.55a}$$

$$\bar{X}_1^+ = \rho\vartheta_4(0)\vartheta_4(\eta-\kappa)\vartheta_3(\eta+x_1)\vartheta_3(\eta-x_1), \tag{2.55b}$$

$$\bar{X}_1^- = \rho\vartheta_3(0)\vartheta_3(\eta-\kappa)\vartheta_4(\eta+x_1)\vartheta_4(\eta-x_1), \tag{2.55c}$$

where ρ is defined in (2.20).

Reduction relations The reduction relations for the vector $|\Psi_n\rangle$, given in Lemma 2.1.5, lead to reduction relations for the functions X_n and $\bar{X}_n^{\pm}$. To state them, we define

$$F(x) = -\frac{\vartheta_2(x)\vartheta_2(x-\eta)}{\vartheta_4(0,p^2)^2}\vartheta_1(x+\kappa)\vartheta_1(\kappa - x+\eta), \tag{2.56}$$

$$\begin{aligned}\bar{F}(x) = &\frac{\vartheta_3(x)\vartheta_3(x-\eta)\vartheta_4(x)\vartheta_4(x-\eta)}{\vartheta_4(0,p^2)^2} \\ &\times \vartheta_1(x+\eta)^2\vartheta_1(x+\kappa)\vartheta_1(\kappa-x+\eta).\end{aligned} \tag{2.57}$$

Proposition 2.2.8. *For $n \geqslant 2$ and for each $2 \leqslant i \leqslant n$, we have the reduction relations*

$$\begin{aligned}&X_n\left(x_1 = x_i + \tfrac{\eta}{2}, \dots, x_i, \dots, x_n\right) \\ &\qquad = F(x_i) \prod_{j=2, j\neq i}^{n} \vartheta_1(x_i - x_j + \eta)^2 \vartheta_1(x_i + x_j+\eta)^2 \\ &\qquad\qquad\qquad \times X_{n-2}(x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n),\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}&\bar{X}_n^{\pm}(x_1 = x_i + \tfrac{\eta}{2}, \dots, x_i, \dots, x_n) \\ &\qquad = \bar{F}(x_i) \prod_{j=2, j\neq i}^{n} \vartheta_1(x_i - x_j + \eta)^2 \vartheta_1(x_i + x_j+\eta)^2 \\ &\qquad\qquad\qquad \times \bar{X}_{n-2}^{\pm}(x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n).\end{aligned} \tag{2.59}$$

Proof. We focus on the proof of the reduction relations for X_n. To this end, we abbreviate $Z_n' = Z_n(x, x+\eta, x_3, \dots, x_n)$. Using the definition of Z_n and Lemma 2.1.2, we may write

$$\begin{aligned}Z_n' = \langle \xi_n(x, x+\eta, \dots)| &\frac{\check{R}_{23}(-2x-\eta)}{r(-2x-\eta)} \\ &\times |\Psi_n(x, x+\eta, -x, -x-\eta, \dots)\rangle.\end{aligned} \tag{2.60}$$

The first two arguments $x, x+\eta$ of the vector $|\Psi_n\rangle$ allow us to apply the reduction relation of Lemma 2.1.5. We obtain

$$\begin{aligned}Z_n' = &\vartheta_1(2x)\vartheta_1(2x-\eta)\vartheta_1(x-\eta)\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)\vartheta_1(x+x_i-\eta) \\ &\times \langle \xi_n(x, x+\eta, \dots)|\frac{\check{R}_{23}(-2x-\eta)}{r(-2x-\eta)}\left(|s\rangle \otimes |\Psi_{n-1}(-x, -x-\eta, \dots)\rangle\right).\end{aligned}$$

Next, we use $\check{R}(-\eta)|s\rangle = -2r(-\eta)|s\rangle$ to write

$$\begin{aligned} Z_n' = {} & \vartheta_1(2x)\vartheta_1(2x-\eta)\vartheta_1(x-\eta)\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)\vartheta_1(x+x_i-\eta) \\ & \times \langle \xi_n(x, x+\eta, \dots)| \frac{\check{R}_{23}(-2x-\eta)\check{R}_{12}(-\eta)}{-2r(-\eta)r(-2x-\eta)} \\ & \times \left(|s\rangle \otimes |\Psi_{n-1}(-x, -x-\eta, \dots)\rangle\right). \end{aligned} \tag{2.61}$$

With the help of the boundary Yang-Baxter equation (2.21) and the symmetry of the $\check{R}$-matrix, we may write

$$\begin{aligned} Z_n' = {} & \vartheta_1(2x)\vartheta_1(2x-\eta)\vartheta_1(x-\eta)\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)\vartheta_1(x+x_i-\eta) \\ & \times \langle \xi_n(x+\eta, x, \dots)| \frac{\check{R}_{23}(-2x-\eta)\check{R}_{34}(-\eta)}{-2r(-\eta)r(-2x-\eta)} \\ & \times \left(|s\rangle \otimes |\Psi_{n-1}(-x, -x-\eta, \dots)\rangle\right) \end{aligned} \tag{2.62}$$

$$\begin{aligned} = {} & \vartheta_1(2x)\vartheta_1(2x-\eta)\vartheta_1(x-\eta)\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)\vartheta_1(x+x_i-\eta) \\ & \times \langle \xi_n(x+\eta, x, \dots)| \frac{\check{R}_{23}(-2x-\eta)}{-2r(-2x-\eta)} \\ & \times \left(|s\rangle \otimes |\Psi_{n-1}(-x-\eta, -x, \dots)\rangle\right). \end{aligned} \tag{2.63}$$

The last equality follows from an application of Lemma 2.1.2. The resulting expression suggests yet another application of Lemma 2.1.5. We obtain

$$\begin{aligned} Z_n' = {} & \vartheta_1(2x)\vartheta_1(2x-\eta)\vartheta_1(x-\eta)^2\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)^2\vartheta_1(x+x_i-\eta)^2 \\ & \times \left(\langle \chi(x+\eta)| \otimes \langle \chi(x)|\right) \frac{\check{R}_{23}(-2x-\eta)}{2r(-2x-\eta)} \left(|s\rangle \otimes |s\rangle\right) Z_{n-2}(x_3, \dots, x_n). \end{aligned}$$

The scalar product in the second line of this equality is given in Lemma 2.2.2. From this scalar product and from the relation between Z_n and X_n, given in Proposition 2.2.7, we infer

$$\begin{aligned} X_n(x, x+\eta, x_3, \dots, x_n) = F(x-\eta/2)\prod_{i=3}^{n}\vartheta_1(x-x_i-\eta)^2\vartheta_1(x+x_i-\eta)^2 \\ \times X_{n-2}(x_3, \dots, x_n). \end{aligned} \tag{2.64}$$

Finally, we set $x = x_2 + \eta/2$ and obtain the reduction relation for X_n with $i = 2$. For $i = 3, \ldots, n$, it follows from the symmetry of X_n.

The proof of the reduction relations for $\bar{X}_n^{\pm}$ is similar. □

Proposition 2.2.9. *For each $n \geqslant 1$, we have the reduction relations*

$$X_n(\beta_2, x_2, \ldots, x_n) = \frac{(-1)^n \vartheta_2(\kappa - \eta)}{\vartheta_4(0, p^2)} \times \left(\prod_{i=2}^{n-1} \vartheta_2(x_i)^2\right) X_{n-1}(x_2, \ldots, x_n), \tag{2.65}$$

$$\bar{X}_n^{\pm}(\beta_3, x_2, \ldots, x_n) = p^{-n/2} \mathrm{e}^{-\mathrm{i}n\eta} \frac{\vartheta_2(\eta)\vartheta_3(0)\vartheta_3(\kappa - \eta)}{\vartheta_4(0, p^2)} \times \left(\prod_{i=2}^{n} \vartheta_3(x_i)^2\right) \bar{X}_{n-1}^{\mp}(x_2, \ldots, x_n), \tag{2.66}$$

$$\bar{X}_n^{\pm}(\beta_4, x_2, \ldots, x_n) = p^{-n/2} \mathrm{e}^{-\mathrm{i}n\eta} \frac{\vartheta_2(\eta)\vartheta_4(0)\vartheta_4(\kappa - \eta)}{\vartheta_4(0, p^2)} \times \left(\prod_{i=2}^{n} \vartheta_4(x_i)^2\right) \bar{X}_{n-1}^{\pm}(x_2, \ldots, x_n). \tag{2.67}$$

Proof. We focus on the reduction relation for X_n. To this end, we consider the vector $|\Psi_n\rangle$ with arguments as set in (2.32). We choose $x_1 = -\beta_2$ and apply Lemma 2.1.5. This leads to

$$|\Psi_n(-\beta_2, \beta_2, x_2, -x_2, \ldots, x_n, -x_n)\rangle = -\vartheta_2(0) \prod_{i=2}^{n} \vartheta_2(x_i)^2 \sigma_1^3 |s\rangle \otimes P|\Psi_{n-1}(x_2, -x_2, \ldots, x_n, -x_n)\rangle, \tag{2.68}$$

where P is the spin-parity operator, defined in (1.13). We use this relation and $\sigma_1^3\sigma_2^3|\chi(x)\rangle = -|\chi(x)\rangle$ to obtain

$$Z_n(-\beta_2, x_2, \ldots, x_n) = (-1)^{n-1} \vartheta_2(0) \prod_{i=2}^{n} \vartheta_2(x_i)^2 \times \Big(\langle \chi(-\beta_2)|\sigma_1^3|s\rangle\Big) Z_{n-1}(x_2, \ldots, x_n). \tag{2.69}$$

Next, we evaluate the matrix element $\langle \chi(-\beta_2)|\sigma_1^3|s\rangle$ with the help of

$$\langle \chi(x)|\sigma_1^3|s\rangle = \langle \chi(x)| \left(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle\right) = \vartheta_2(\kappa - \eta)\vartheta_1(x + \eta). \tag{2.70}$$

We find

$$Z_n(-\beta_2, x_2, \ldots, x_n) = (-1)^{n-1}\vartheta_2(0)\vartheta_2(\eta)\vartheta_2(\kappa-\eta) \times \prod_{i=2}^{n} \vartheta_2(x_i)^2 Z_{n-1}(x_2, \ldots, x_n). \quad (2.71)$$

We obtain (2.65) from this equality and the relation between Z_n and X_n, given in Proposition 2.2.7.

To prove the two reduction relations for $\bar{X}_n^{\pm}$, one needs to use Lemma 2.1.3 and the scalar products

$$\langle \bar{\chi}(x)|(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle) = \vartheta_4(\kappa-\eta)\vartheta_3(x+\eta), \quad (2.72)$$

$$\langle \bar{\chi}(x)|(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) = \vartheta_3(\kappa-\eta)\vartheta_4(x+\eta). \quad (2.73)$$

The calculation is similar to the proof for X_n. $\square$

2.2.3 Determinants

In this section, we present our main results for the inhomogeneous supersymmetric eight-vertex model. They provide explicit expressions for the functions X_n and $\bar{X}_n^{\pm}$, and hence for the scalar products Z_n and $\bar{Z}_n^{\pm}$. These expressions are given in terms of the so-called elliptic Tsuchiya determinant, which originally appeared as a partition function for the SOS model [121].[2]

The elliptic Tsuchiya determinant Let us introduce

$$\mathbb{h}(x,y) = \vartheta_1(x-y+\eta)\vartheta_1(x-y-\eta)\vartheta_1(x+y+\eta)\vartheta_1(x+y-\eta). \quad (2.74)$$

We define $\mathbb{H}_0 = 1$ and, for each $k \geqslant 1$, the function

$$\mathbb{H}_{2k}(x_1, \ldots x_k; x_{k+1}, \ldots, x_{2k}) = \frac{\prod_{i,j=1}^{k} \mathbb{h}(x_i, x_{j+k})}{\mathbb{\Delta}(x_1, \ldots, x_k)\mathbb{\Delta}(x_{k+1}, \ldots, x_{2k})} \det_{i,j=1}^{k}\left(\frac{1}{\mathbb{h}(x_i, x_{j+k})}\right), \quad (2.75)$$

[2] See Chapter 4 for a description of SOS models.

where

$$\mathbb{A}(x_1,\dots,x_k)=\prod_{1\leqslant i<j\leqslant k}\vartheta_1(x_j-x_i)\vartheta_1(x_j+x_i). \tag{2.76}$$

For $k\geqslant 1$, $\mathbb{H}_{2k}(x_1,\dots x_k;x_{k+1},\dots,x_{2k})$ is an even theta function of degree $2(k-1)$ and norm 0 with respect to each x_i, $i=1,\dots,2k$. Moreover, it clearly is separately symmetric in $x_1,\dots,x_k$ and $x_{k+1},\dots,x_{2k}$. Less obvious is that it is symmetric in all its variables [111]. Here, we present a simple proof based on determinant condensation [63].

Proposition 2.2.10. *For each $k\geqslant 1$, $\mathbb{H}_{2k}(x_1,\dots x_k;x_{k+1},\dots,x_{2k})$ is a symmetric function of $x_1,\dots,x_{2k}$.*

Proof. The case $k=1$ is trivial: we have $\mathbb{H}_2(x_1;x_2)=1$, which is obviously symmetric in x_1 and x_2. Therefore, we consider $k\geqslant 2$. Let $A=(a_{ij})_{i,j=1}^k$ be a matrix with $a_{kk}\neq 0$. We have the condensation identity

$$\det A=a_{kk}^{-(k-2)}\det_{i,j=1}^{k-1}\left(\det\begin{pmatrix}a_{ij} & a_{ik}\\ a_{kj} & a_{kk}\end{pmatrix}\right). \tag{2.77}$$

We apply this identity to the determinant in (2.75), and obtain

$$\begin{aligned}\mathbb{H}_{2k}(x_1,\dots,x_k;x_{k+1},\dots,x_{2k})&=\frac{\prod_{i,j=1}^{k-1}\mathbb{h}(x_i,x_{j+k})}{\mathbb{A}(x_1,\dots,x_{k-1})\mathbb{A}(x_{k+1},\dots,x_{2k-1})}\\ &\quad\times\det_{i,j=1}^{k-1}\left(\frac{\mathbb{H}_4(x_i,x_k;x_j,x_{2k})}{\mathbb{h}(x_i,x_{j+k})}\right).\end{aligned} \tag{2.78}$$

We note that

$$\begin{aligned}&\mathbb{H}_4(x_i,x_k;x_j,x_{2k})-\mathbb{H}_4(x_i,x_{2k};x_j,x_k)\\ &\quad=-\frac{\vartheta_1(3\eta)\vartheta_1(x_i-x_j)\vartheta_1(x_i+x_j)\vartheta_1(x_k-x_{2k})\vartheta_1(x_k+x_{2k})}{\vartheta_1(\eta)}.\end{aligned} \tag{2.79}$$

The right-hand side vanishes since $\eta=2\pi/3$. Therefore, we have

$$\mathbb{H}_4(x_i,x_k;x_j,x_{2k})=\mathbb{H}_4(x_i,x_{2k};x_j,x_k). \tag{2.80}$$

Using (2.78), we conclude that $\mathbb{H}_{2k}(x_1,\dots,x_k;x_{k+1},\dots,x_{2k})$ is symmetric in x_k and x_{2k}. Since it is separately symmetric in $x_1,\dots,x_k$ and $x_{k+1},\dots,x_{2k}$, it is symmetric in all its variables. □

To stress the symmetry of the function $\mathbb{H}_{2k}(x_1,\ldots,x_k;x_{k+1},\ldots,x_{2k})$, we omit the semicolon and simply write $\mathbb{H}_{2k}(x_1,\ldots,x_{2k})$. Finally, we note that in [111] the following reduction relation was established:

Lemma 2.2.11. *For each $k\geqslant 1$ and each $i=2,\ldots,2k$, we have*

$$\mathbb{H}_{2k}(x_1=x_i+\tfrac{\eta}{2},\ldots,x_i,\ldots,x_{2k}) = \prod_{j=2,j\neq i}^{2k}\vartheta_1(x_i-x_j+\eta)\vartheta_1(x_i+x_j+\eta)$$
$$\times\,\mathbb{H}_{2(k-1)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots). \quad (2.81)$$

The functions X_n and $\bar{X}_n^{\pm}$ Theorem A.2.5 provides a way to establish the equality between two theta functions. We use this theorem to find an explicit expression for X_n in terms of the elliptic Tsuchiya determinant. To this end, we define for each $k\geqslant 0$ the functions

$$Y_{2k} = \frac{(-1)^k}{\vartheta_4(0,p^2)^{2k}}\mathbb{H}_{2k}(x_1,\ldots,x_{2k}) \quad (2.82)$$
$$\times\,\mathbb{H}_{2(k+1)}(x_1\ldots,x_{2k},\beta_2,\tfrac{\eta}{2}+\kappa),$$
$$Y_{2k+1} = \frac{(-1)^{k+1}\vartheta_2(\kappa-\eta)}{\vartheta_4(0,p^2)^{2k+1}}\mathbb{H}_{2(k+1)}(x_1,\ldots,x_{2k+1},\beta_2) \quad (2.83)$$
$$\times\,\mathbb{H}_{2(k+1)}(x_1\ldots,x_{2k+1},\tfrac{\eta}{2}+\kappa).$$

Lemma 2.2.12. *The function Y_n obeys the reduction relations of X_n given in Propositions 2.2.8 and 2.2.9.*

Proof. The proof is a straightforward calculation using Lemma 2.2.11. □

Theorem 2.2.13. *For each $n\geqslant 0$, we have $X_n=Y_n$.*

Proof. The proof is based on a standard induction argument that goes back to Izergin and Korepin [18, 69]. We start by examining the base cases $n=0$ and $n=1$. From the expressions given above, we find

$$Y_0=1, \qquad Y_1=-\frac{\vartheta_2(\kappa-\eta)}{\vartheta_4(0,p^2)}. \quad (2.84)$$

They are equal to the expressions of X_0 and X_1, given in (2.54) and (2.55), respectively.

We now make the induction hypothesis that $X_n = Y_n$ for $n = m$ and $n = m - 1$, where $m \geqslant 2$ is some integer. For the induction step, we consider X_{m+1} and Y_{m+1} as functions of x_1. By Proposition 2.2.7, X_{m+1} is a theta function of degree $2m$, nome p and norm 0. The same holds for Y_{m+1}, by the properties of the elliptic Tsuchiya determinant. By Lemma 2.2.12 both functions obey the same reduction relations. For $x_1 = \pm(x_j \pm \eta/2)$, $j = 2, \dots, m+1$, they allow us to express X_{m+1} and Y_{m+1} in terms of X_{m-1} and Y_{m-1}, respectively. Likewise, for $x_1 = \pm\beta_2$, they lead to expressions of X_{m+1} and Y_{m+1} in terms of X_m and Y_m respectively. Using these expressions and the induction hypothesis, we conclude that $X_{m+1} = Y_{m+1}$ for $4m + 2$ values of x_1. We choose $x_2, \dots, x_{m+1}$ so that they contain $2m$ independent values. By Theorem A.2.5, we conclude that $X_{m+1} = Y_{m+1}$, and hence $X_n = Y_n$ for $n = m + 1$ and $n = m$, which ends the induction step. The theorem follows. $\square$

Next, we provide explicit expressions for the functions $\bar{X}_n^{\pm}$ in terms of the elliptic Tsuchyia determinant. For each $k \geqslant 0$, we define

$$\begin{aligned}\bar{Y}_{2k}^{\pm} &= \gamma_{2k}^{\pm}\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k}, 0, \beta_4)\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k}, \tfrac{\eta}{2} + \kappa, \beta_3)\\ &\quad + \delta_{2k}^{\pm}\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k}, 0, \beta_3)\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k}, \tfrac{\eta}{2} + \kappa, \beta_4),\end{aligned} \tag{2.85}$$

and

$$\begin{aligned}\bar{Y}_{2k+1}^{\pm} &= \gamma_{2k+1}^{\pm}\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k+1}, 0)\\ &\qquad\qquad \times \mathbb{H}_{2(k+2)}(x_1, \dots, x_{2k+1}, \tfrac{\eta}{2} + \kappa, \beta_3, \beta_4)\\ &\quad + \delta_{2k+1}^{\pm}\mathbb{H}_{2(k+1)}(x_1, \dots, x_{2k+1}, \tfrac{\eta}{2} + \kappa)\\ &\qquad\qquad \times \mathbb{H}_{2(k+2)}(x_1, \dots, x_{2k+1}, 0, \beta_3, \beta_4).\end{aligned} \tag{2.86}$$

Here, the coefficients $\gamma_{2k}^{\pm}$, $\gamma_{2k+1}^{\pm}$ and $\delta_{2k}^{\pm}$, $\delta_{2k+1}^{\pm}$ are given by

$$\gamma_{2k}^{\pm} = \left(\frac{p\mathrm{e}^{-\mathrm{i}\eta}}{\vartheta_4(0,p^2)^2}\right)^k \gamma_0^{\pm}, \qquad \delta_{2k}^{\pm} = \left(\frac{p\mathrm{e}^{-\mathrm{i}\eta}}{\vartheta_4(0,p^2)^2}\right)^k \delta_0^{\pm}, \tag{2.87}$$

$$\gamma_{2k+1}^{\pm} = \left(\frac{p\mathrm{e}^{-\mathrm{i}\eta}}{\vartheta_4(0,p^2)^2}\right)^k \gamma_1^{\pm}, \qquad \delta_{2k+1}^{\pm} = \left(\frac{p\mathrm{e}^{-\mathrm{i}\eta}}{\vartheta_4(0,p^2)^2}\right)^k \delta_1^{\pm}, \tag{2.88}$$

where

$$\gamma_0^{+} = 2\left(\frac{\vartheta_3(0)\vartheta_4(\kappa-\eta)}{\vartheta_2(0)\vartheta_1(\kappa-\eta)}\right)^2, \qquad \delta_0^{+} = -2\left(\frac{\vartheta_4(0)\vartheta_3(\kappa-\eta)}{\vartheta_2(0)\vartheta_1(\kappa-\eta)}\right)^2,$$

$$\gamma_0^{-} = -\frac{2\vartheta_3(0)\vartheta_4(0)\vartheta_3(\kappa-\eta)\vartheta_4(\kappa-\eta)}{\vartheta_2(0)^2\vartheta_1(\kappa-\eta)^2}, \quad \delta_0^{-} = -\gamma_0^{-}.$$

and

$$\gamma_1^{\pm} = \frac{p\mathrm{e}^{-\mathrm{i}\eta}\vartheta_4(0)}{\vartheta_4(0,p^2)\vartheta_4(\kappa-\eta)\vartheta_2(\eta)}\gamma_0^{\pm}, \quad \delta_1^{\pm} = \frac{p\mathrm{e}^{-\mathrm{i}\eta}\vartheta_4(\kappa-\eta)}{\vartheta_4(0,p^2)\vartheta_4(0)\vartheta_2(\eta)}\delta_0^{\pm}.$$

Lemma 2.2.14. *The functions $\bar{Y}_n^{\pm}$ satisfy the reduction relations of $\bar{X}_n^{\pm}$ given in Propositions 2.2.8 and 2.2.9.*

Proof. The proof is a straightforward calculation using Lemma 2.2.11. □

Theorem 2.2.15. *For each $n \geqslant 0$, we have $\bar{Y}_n^{\pm} = \bar{X}_n^{\pm}$.*

Proof. As for Theorem 2.2.13, the proof is based on induction. Let us check the base cases $n = 0$ and $n = 1$, using the explicit expressions given above. For $n = 0$, we find

$$\bar{Y}_0^{+} = 2, \qquad \bar{Y}_0^{-} = 0. \tag{2.89}$$

Furthermore, for $n = 1$, we obtain

$$\bar{Y}_1^{\pm} = \gamma_1^{\pm}\mathbb{H}_4(x_1, \tfrac{\eta}{2}+\kappa, \beta_3, \beta_4) + \delta_1^{\pm}\mathbb{H}_4(x_1, 0, \beta_3, \beta_4). \tag{2.90}$$

We have [111]

$$\begin{aligned}\mathbb{H}_4(x,y,\beta_3,\beta_4) = \frac{p^{-1}\mathrm{e}^{\mathrm{i}\eta}\vartheta_2(\eta)}{\vartheta_2(0)}\Big(&\vartheta_3(x+\eta)\vartheta_3(x-\eta)\vartheta_4(y)^2\\ &+\vartheta_4(x+\eta)\vartheta_4(x-\eta)\vartheta_3(y)^2\Big).\end{aligned} \tag{2.91}$$

We use this expression, together with the definition of $\gamma_1^{\pm}$, $\delta_1^{\pm}$, and find

$$\bar{Y}_1^{+} = \frac{2\vartheta_4(0)\vartheta_4(\kappa-\eta)\vartheta_3(x_1-\eta)\vartheta_3(x_1+\eta)}{\vartheta_2(0)\vartheta_4(0,p^2)}, \tag{2.92}$$

$$\bar{Y}_1^{-} = \frac{2\vartheta_3(0)\vartheta_3(\kappa-\eta)\vartheta_4(x_1+\eta)\vartheta_4(x_1-\eta)}{\vartheta_2(0)\vartheta_4(0,p^2)}. \tag{2.93}$$

We compare our findings with (2.54) and (2.55) and conclude that $\bar{Y}_n^{\pm} = \bar{X}_n^{\pm}$ for $n = 0, 1$.

Next, we make the induction hypothesis that $\bar{X}_n^{\pm} = \bar{Y}_n^{\pm}$ for $n = m$ and $n = m-1$, where $m \geqslant 2$ is some integer. For the induction step, we essentially follow the proof of Theorem 2.2.13. The only differences are *(i)* that we need to use Lemma 2.2.14 and *(ii)* that the degree of the theta functions is now $2(m+1)$. Nonetheless, we may choose $x_2, \dots, x_{m+1}$ to find enough independent points for the application of Theorem A.2.5 to be possible. It allows us to conclude that $\bar{X}^{\pm}_{m+1} = \bar{Y}^{\pm}_{m+1}$. The induction step follows, which ends the proof. □

2.2.4 Polynomials

In this section, we recall the relation between the elliptic Tsuchiya determinant and a family of polynomials introduced by Zinn-Justin and Rosengren. Moreover, we discuss the properties of these polynomials that are relevant to our forthcoming analysis of the scalar products. These properties can either be found in or easily derived from [78–81, 111], but we prefer to include them here for completeness.

Definition and relation to the elliptic Tsuchiya determinant
Let us introduce

$$h(w,w') = 1 - (3+\zeta^2)ww' + (1-\zeta^2)ww'(w+w'). \tag{2.94}$$

We define $H_0 = 1$ and, for each $k \geqslant 1$, the function[3]

$$\begin{aligned} &H_{2k}(w_1,\dots,w_{2k}) \\ &\quad = \frac{\prod_{i,j=1}^{k} h(w_i,w_{j+k})}{\Delta(w_1,\dots,w_k)\Delta(w_{k+1},\dots,w_{2k})} \det_{i,j=1}^{k}\left(\frac{1}{h(w_i,w_{j+k})}\right), \end{aligned} \tag{2.95}$$

where $\Delta(w_1,\dots,w_m) = \prod_{1\leqslant i<j\leqslant m}(w_j - w_i)$ denotes the Vandermonde determinant in m variables. Clearly, $H_{2k}(w_1,\dots,w_{2k})$ is a polynomial in $w_1,\dots,w_{2k}$ and ζ.

In the next proposition, we recall its relation to the elliptic Tsuchiya determinant. To this end, we use the elliptic function

$$w(x) = \frac{\vartheta_4(\eta,p^2)}{\vartheta_4(0,p^2)}\frac{\vartheta_1(x)^2}{\vartheta_1(x-\eta)\vartheta_1(x+\eta)}. \tag{2.96}$$

Moreover, in the following we often use the parameterisation of ζ in terms of the elliptic nome p provided in (1.30).

[3] We use Zinn-Justin's notations. The polynomials $H_{2k}(w_1,\dots,w_{2k})$ are related to Rosengren's polynomials $T(x_1,\dots,x_{2k})$ by a change of variables [78].

Lemma 2.2.16. *Let $k \geqslant 1$ and $w_i = w(x_i)$ for each $i = 1, \ldots, 2k$. If* (1.30) *holds, then we have*

$$\mathbb{H}_{2k}(x_1, \ldots, x_{2k}) = f(x_1, \ldots, x_{2k}) H_{2k}(w_1, \ldots, w_{2k}), \tag{2.97}$$

where

$$f(x_1, \ldots, x_{2k}) = \left(\frac{\vartheta_4(\eta, p^2)}{\vartheta_1(\eta)^2 \vartheta_4(0, p^2)}\right)^{k(k-1)} \times \prod_{i=1}^{2k} \left(\vartheta_1(x_i + \eta)\vartheta_1(x_i - \eta)\right)^{k-1}. \tag{2.98}$$

Proof. The key observations are the two relations

$$w(x) - w(y) = -\frac{\vartheta_4(\eta, p^2)\vartheta_1(\eta)^2 \vartheta_1(x-y)\vartheta_1(x+y)}{\vartheta_4(0, p^2)\vartheta_1(x-\eta)\vartheta_1(x+\eta)\vartheta_1(y-\eta)\vartheta_1(y+\eta)},$$

and

$$h(w(x), w(y)) = \frac{\vartheta_1(\eta)^4 \mathbb{h}(x, y)}{\left(\vartheta_1(x-\eta)\vartheta_1(x+\eta)\vartheta_1(y-\eta)\vartheta_1(y+\eta)\right)^2}. \tag{2.99}$$

They follow from standard identities between Jacobi theta functions. Using them together with the definition of $H_{2k}(w_1, \ldots, w_{2k})$ and $\mathbb{H}_{2k}(x_1, \ldots, x_{2k})$ leads to (2.97) and (2.98). □

The symmetry of the elliptic Tsuchiya determinant implies that $H_{2k}(w_1, \ldots, w_{2k})$ is a symmetric polynomial in $w_1, \ldots, w_{2k}$ for $k \geqslant 1$. In the following, we use for each $k \geqslant 1$ and $j = 1, \ldots, 2k-1$ the abbreviation

$$H_{2k}(w_1, \ldots, w_j) \equiv H_{2k}(w_1, \ldots, w_j, 0, \ldots, 0). \tag{2.100}$$

Clearly, if $j \geqslant 2$ then $H_{2k}(w_1, \ldots, w_j)$ is a symmetric polynomial in $w_1, \ldots, w_j$.

Determinant formulas We combine the determinant formula (2.78) for the elliptic Tsuchiya determinant with Lemma 2.2.16. This leads to

$$H_{2k}(w_1, \ldots, w_{2k}) = \frac{\prod_{i,j=1}^{k-1} h(w_i, w_{j+k})}{\Delta(w_1, \ldots, w_{k-1})\Delta(w_{k+1}, \ldots, w_{2k-1})} \times \det_{i,j=1}^{k-1}\left(\frac{H_4(w_i, w_k, w_{j+k}, w_{2k})}{h(w_i, w_{j+k})}\right), \tag{2.101}$$

for each $k \geqslant 2$, where

$$H_4(w_1, \ldots, w_4) = 3 + \zeta^2 + (\zeta^2 - 1)(w_1 + w_2 + w_3 + w_4 \\ + (\zeta^2 - 1)w_1 w_2 w_3 w_4). \quad (2.102)$$

The determinant formula (2.101) allows us to infer several properties of the symmetric polynomial $H_{2k}(w_1, \ldots, w_{2k})$ for $k > 2$ from $k = 2$. For instance, we easily find the degree of this polynomial with respect to each of its variables.

Lemma 2.2.17. *For each $k \geqslant 1$, $H_{2k}(w_1, \ldots, w_{2k})$ is a polynomial of degree $k - 1$ in w_i for each $i = 1, \ldots, 2k$. Moreover, we have*

$$\lim_{w_{2k-1}\to\infty} w_{2k-1}^{-(k-1)} H_{2k}(w_1, \ldots, w_{2k-1}, 0) \\ = (\zeta^2 - 1)^{k-1} H_{2(k-1)}(w_1, \ldots, w_{2(k-1)}). \quad (2.103)$$

Furthermore, (2.101) allows us to find a useful formula for $H_{2k}(w, w')$ in terms of a determinant of a matrix with polynomial entries in w, w' and ζ. Indeed, from the method of divided differences [122] we obtain:

Lemma 2.2.18. *For each $k \geqslant 2$, we have*

$$H_{2k}(w, w') = \det_{i,j=0}^{k-2} \Big(H_4(w, w')\eta_{i,j} \\ + (\zeta^2 - 1)\big(\eta_{i-1,j} + \eta_{i,j-1} + (\zeta^2 - 1)ww'\eta_{i-1,j-1}\big)\Big), \quad (2.104)$$

where η_{ij} is a polynomial in ζ with integer coefficients, given by

$$\eta_{i,j} = \sum_{n=\lceil \frac{i+j}{3} \rceil}^{\min(i,j)} \frac{n!(3 + \zeta^2)^{3n-(i+j)}(\zeta^2 - 1)^{i+j-2n}}{(i - n)!(j - n)!(3n - (i + j))!}, \quad (2.105)$$

if $i, j \geqslant 0$, and $\eta_{i,j} = 0$ if $i < 0$ or $j < 0$.

The formula given in this lemma is quite practical to explicitly compute the polynomials $H_{2k}(w, w')$ with Mathematica.

Bilinear identities The polynomials $H_{2k}(w_1, \ldots, w_{2k})$ satisfy bilinear identities. They follow from a Plücker relation and the well-known Desnanot-Jacobi identity [63, 123], applied to the determinant in (2.95).

Lemma 2.2.19. *For each $k \geqslant 1$ and all x, y, u, v, we have*

$$\begin{aligned} &(x-y)h(u,v)H_{2(k+1)}(w_1,\ldots,x,y,v)H_{2k}(w_1,\ldots,u)\\ +&(y-u)h(x,v)H_{2(k+1)}(w_1,\ldots,y,u,v)H_{2k}(w_1,\ldots,x)\\ +&(u-x)h(y,v)H_{2(k+1)}(w_1,\ldots,u,x,v)H_{2k}(w_1,\ldots,y)=0,\end{aligned} \tag{2.106}$$

and

$$\begin{aligned} &(x-u)(y-v)H_{2(k+2)}(w_1,\ldots,x,y,u,v)H_{2k}(w_1,\ldots,w_{2k})\\ =\ & h(x,v)h(y,u)H_{2(k+1)}(w_1,\ldots,u,v)H_{2(k+1)}(w_1,\ldots,x,y)\\ &-h(x,y)h(u,v)H_{2(k+1)}(w_1,\ldots,y,u)H_{2(k+1)}(w_1,\ldots,x,v).\end{aligned} \tag{2.107}$$

Fractional linear transformations Finally, the polynomial $H_{2k}(w_1, \ldots, w_{2k})$ has a simple transformation behaviour under the fractional linear transformation $\zeta \to \zeta'$, where

$$\zeta' = \frac{\zeta+3}{\zeta-1}. \tag{2.108}$$

It follows from the property $h(w,w')|_{\zeta\to\zeta'} = h(2w/(\zeta-1), 2w'/(\zeta-1))$:

Lemma 2.2.20. *For each $k \geqslant 0$, we have*

$$H_{2k}\left(w_1,\ldots,w_{2k}\right)|_{\zeta\to\zeta'} = \left(\frac{2}{\zeta-1}\right)^{k(k-1)} H_{2k}\left(\frac{2w_1}{\zeta-1},\ldots,\frac{2w_{2k}}{\zeta-1}\right).$$

2.3 The homogeneous limit

In this section, we consider the case where the inhomogeneity parameters take the values $\lambda_1 = \cdots = \lambda_{2n+1} = 0$. This is commonly referred to as the *homogeneous limit*. To study the homogeneous limit of the vector $|\Psi_n\rangle$, it is convenient to define

$$|\psi_n\rangle = \mathcal{N}_n|\Psi_n(0,\ldots,0)\rangle, \quad |\bar{\psi}_n\rangle = \mathcal{N}_n P|\Psi_n(0,\ldots,0)\rangle. \tag{2.109}$$

Here, $\mathcal{N}_n$ is a normalisation factor. We choose

$$\mathcal{N}_n = \frac{(-1)^{\lfloor\frac{n+1}{2}\rfloor}}{\vartheta_1(\eta)^{n^2}}\left(\frac{\vartheta_4(0,p^2)}{\vartheta_4(\eta,p^2)}\right)^{n(n+1)/2}. \tag{2.110}$$

We shall see below that the vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$ are non-vanishing. Their definition thus implies that they are linearly independent. They constitute a basis of the eigenspace of the eigenvalue $\Theta_n = (a+b)^{2n+1}$ of the transfer matrix of the homogeneous eight-vertex model [85].

The main goal of this section is to study the basis vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$ with the help of the homogeneous limits of the scalar products Z_n and $\bar{Z}_n^{\pm}$. We compute these homogeneous limits in Section 2.3.1 and Section 2.3.2, respectively. This computation allows us to obtain and analyse several components of the basis vectors. In Section 2.3.3, we exploit the relation between the eight-vertex model and the XYZ spin chain to compute the sum of the components of the basis vectors. In Section 2.3.4, we compare our results to several conjectures by Bazhanov and Mangazeev, and Razumov and Stroganov.

2.3.1 The homogeneous limit of Z_n

In this section, we compute the scalar product

$$S_n = \Big((\langle\uparrow\downarrow| + \mu\langle\downarrow\uparrow|)^{\otimes n}\otimes\langle\uparrow|\Big)\,|\psi_n\rangle \tag{2.111}$$

from the homogeneous limit of Z_n. The results of our computations naturally depend on the elliptic nome p. We restrict our considerations to real $0<p<1$ and state our results in terms of the variable ζ, defined in (1.30), and[4]

$$J_2 = -\frac{1}{2}, \qquad J_3 = \frac{1}{1+\zeta}, \qquad J_4 = \frac{1}{1-\zeta}. \tag{2.112}$$

We have $J_k = w(\beta_k)$ for $k=2,3,4$, where $w(x)$ is defined in (2.96). We note that the restriction of the range of the elliptic nome implies that $0<\zeta<1$.

We divide the section into three parts. First, we find a closed-form expression for S_n in terms of the polynomials defined in Section 2.2.4.

[4]The parameters J_i correspond to the evaluation of (1.31) for $\eta = 2\pi/3$, $i=2,3,4$.

Second, we use this expression to compute and analyse the component $(\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}$. Third, we evaluate the scalar product S_n in the so-called trigonometric limit.

Closed-form expression

Theorem 2.3.1. *For each $k \geqslant 0$, S_{2k} and S_{2k+1} are polynomials in μ and ζ, given by*

$$S_{2k} = (2\mu)^k H_{2k} H_{2(k+1)}(J_2, \bar{\mu}), \tag{2.113}$$

$$S_{2k+1} = (2\mu)^k (\mu+1) H_{2(k+1)}(J_2) H_{2(k+1)}(\bar{\mu}), \tag{2.114}$$

where $\bar{\mu} = (\mu-1)^2/(\zeta^2-1)\mu$.

Proof. The proof has two parts. In part 1, we establish the explicit formulas for the scalar products. In part 2, we show that these expressions yield polynomials in both μ and ζ.

Part 1: Explicit formulas. First, we evaluate the scalar product Z_n for $x_1 = \cdots = x_n = 0$. Using (2.32), this evaluation leads to

$$\begin{aligned} Z_n(0,\dots,0) &= (\vartheta_1(\kappa,p^2)\vartheta_4(\kappa+\eta,p^2))^n \\ &\quad \times \Big((\langle\uparrow\downarrow| + \mu\langle\downarrow\uparrow|)^{\otimes n} \otimes \langle\uparrow| \Big) |\Psi_n(0,\dots,0)\rangle, \end{aligned} \tag{2.115}$$

where

$$\mu = -\frac{\vartheta_4(\kappa,p^2)\vartheta_1(\kappa+\eta,p^2)}{\vartheta_1(\kappa,p^2)\vartheta_4(\kappa+\eta,p^2)}. \tag{2.116}$$

We use Proposition 2.2.7 to write the left-hand side of (2.115) in terms of X_n. Moreover, we use (2.109) and (2.111) to rewrite its right-hand side in terms of S_n. Solving for the latter, we obtain

$$S_n = \left(\frac{\vartheta_4(\eta,p^2)\vartheta_1(\eta)^2}{\vartheta_1(\kappa,p^2)\vartheta_4(\kappa+\eta,p^2)}\right)^n \mathcal{N}_n X_n(0,\dots,0). \tag{2.117}$$

Second, we replace $\mathcal{N}_n$ by its definition (2.110), and X_n by its explicit expression in terms of the elliptic Tsuchiya determinant, given in Theorem 2.2.13. This step requires to consider the cases $n = 2k$ and $n = 2k+1$

separately. We focus on the case $n = 2k$, where we obtain

$$S_{2k} = \frac{1}{\vartheta_1(\eta)^{4k(k-1)} \left(\vartheta_1(\kappa,p^2)\vartheta_4(\kappa+\eta,p^2)\right)^{2k}} \left(\frac{\vartheta_4(0,p^2)}{\vartheta_4(\eta,p^2)}\right)^{(2k-1)k} \times \mathbb{H}_{2k}(0,\dots,0)\mathbb{H}_{2(k+1)}(0,\dots,\beta_2,\tfrac{\eta}{2}+\kappa). \quad (2.118)$$

We now use Lemma 2.2.16 to rewrite the Tsuchiya determinants in terms of the polynomials of Section 2.2.4. After some algebra, we obtain

$$S_{2k} = \left(\frac{\vartheta_1(\kappa+\eta)\vartheta_1(\kappa)\vartheta_2(\eta)\vartheta_2(0)\vartheta_4(\eta,p^2)}{\vartheta_1(\kappa,p^2)^2\vartheta_4(\kappa+\eta,p^2)^2\vartheta_4(0,p^2)}\right)^k \times H_{2k}H_{2(k+1)}(w(\beta_2),w(\kappa+\tfrac{\eta}{2})). \quad (2.119)$$

Here, $w = w(x)$ is the elliptic function defined in (2.96). Moreover, the polynomials on the right-hand side of this equality implicitly depend on ζ, which is given in terms of p by (1.30).

Third, we simplify (2.119) with the help of several classical identities between the Jacobi theta functions. For the prefactor, we use

$$\vartheta_1(x)\vartheta_2(0) = 2\vartheta_1(x,p^2)\vartheta_4(x,p^2), \quad (2.120)$$

$$\vartheta_2(\eta)\vartheta_4(\eta,p^2) = 2\vartheta_2(0)\vartheta_4(0,p^2), \quad (2.121)$$

and the relation (2.116). Moreover, the arguments of the polynomial are given by

$$w(\beta_2) = J_2, \qquad w(\kappa+\tfrac{\eta}{2}) = \bar{\mu}. \quad (2.122)$$

Applying these identities, we obtain (2.113).

The proof of (2.114) is similar.

Part 2: Polynomial nature. Lemma 2.2.18 implies that both H_{2k} and $H_{2(k+1)}(J_2)$ are polynomials in ζ. Moreover, one checks that $\mu H_4(w,\bar{\mu})$ is a polynomial in w,μ and ζ. By Lemma 2.2.18, $\mu^k H_{2(k+1)}(w,\bar{\mu})$ is a polynomial in w,μ, and ζ. Hence, both $\mu^k H_{2(k+1)}(\bar{\mu})$ and $\mu^k H_{2(k+1)}(J_2,\bar{\mu})$ are polynomials in μ and ζ. The polynomial nature of S_{2k} and S_{2k+1} follows.

Finally, we comment on the parameterisation (2.116) used in this proof. By the properties of the Jacobi theta functions, μ is a meromorphic function of κ for all $0 < p < 1$. The pole structure of this function implies that each real μ is the image of some $0 < \kappa < \pi$ under this mapping. Hence, the statements of the theorem hold for all real μ. By analytic continuation, they hold for all complex μ. □

Components

Proposition 2.3.2. *For each $k \geqslant 0$, we have the components*

$$(\psi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = 2^k H_{2k} H_{2k}(J_2), \tag{2.123}$$

$$(\psi_{2k+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = 2^k H_{2k} H_{2(k+1)}(J_2). \tag{2.124}$$

Proof. We present the proof of (2.123). For $k = 0$ it is trivial. Hence, we consider $k \geqslant 1$. We have

$$(\psi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = S_{2k}|_{\mu=0} = 2^k H_{2k} \lim_{\mu\to 0} \mu^k H_{2(k+1)}(J_2, \bar{\mu}). \tag{2.125}$$

We compute the limit on the right-hand side with the help of Lemma 2.2.17, and find

$$\lim_{\mu\to 0} \mu^k H_{2(k+1)}(J_2, \bar{\mu}) = (\zeta^2 - 1)^{-k} \lim_{\bar{\mu}\to\infty} \bar{\mu}^{-k} H_{2(k+1)}(J_2, \bar{\mu}) = H_{2k}(J_2).$$

This result leads to (2.123). The derivation of (2.124) is similar. □

This proposition implies that the component $(\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}$ is a polynomial in the variable ζ with integer coefficients. We now compute the coefficients of its lowest-order and highest-order term. To this end, we recall the number $A(n)$ of alternating sign matrices of size n given in (1.33). We also need the number $A_{\mathrm{V}}(2n+1)$ of vertically-symmetric alternating sign matrices of size $2n+1$ given in (1.36). Additionally, in the following, we frequently use the number $N_8(2n)$ of cyclically-symmetric transpose complement plane partitions in a $2n \times 2n \times 2n$ cube [63, 114]:

$$N_8(2n) = \prod_{i=0}^{n-1} \frac{(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!}. \tag{2.126}$$

Proposition 2.3.3. *For each $k \geqslant 0$, we have*

$$(\psi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = A(2k) + \cdots + 2\zeta^{k(2k-1)}, \tag{2.127}$$

$$(\psi_{2k+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = A(2k+1) + \cdots + \zeta^{k(2k+1)}, \tag{2.128}$$

where $\cdots$ denotes intermediate powers of ζ.

Proof. We compute the coefficients of the lowest-order term through the evaluation of the components at $\zeta = 0$. To this end, we use [111]

$$H_{2k}|_{\zeta=0} = A_{\rm V}(2k+1)\,, \quad H_{2k}(J_2)|_{\zeta=0} = 2^{1-k}\frac{A(2k-1)}{A_{\rm V}(2k-1)}. \tag{2.129}$$

Hence, we obtain

$$(\psi_{2k+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}|_{\zeta=0} = 2^k\, H_{2k}|_{\zeta=0}\, H_{2(k+1)}(J_2)\Big|_{\zeta=0} = A(2k+1), \tag{2.130}$$

$$\begin{aligned}(\psi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}|_{\zeta=0} &= 2^k\, H_{2k}|_{\zeta=0}\, H_{2k}(J_2)|_{\zeta=0}\,,\\ &= \frac{2A_{\rm V}(2k+1)A(2k-1)}{A_{\rm V}(2k-1)} = A(2k),\end{aligned} \tag{2.131}$$

where the last equality of the third line follows from Corollary 21 of [124].

To find the coefficients of the highest-order terms of the components, we analyse the highest-order terms of $H_{2k}(w,w')$ for fixed w, w'. For $k=1$ this analysis is trivial, since $H_2(w,w') = 1$. For $k \geqslant 2$, we use the determinant formula of Lemma 2.2.18, and find

$$H_{2k}(w,w') = (1+w+w')(1+w)^{k-2}(1+w')^{k-2}\zeta^{k(k-1)} + \ldots, \tag{2.132}$$

where $\ldots$ denotes lower-order terms in ζ. Hence, we have

$$H_{2k} = \zeta^{k(k-1)} + \ldots, \qquad H_{2k}(J_2) = 2^{1-k}\zeta^{k(k-1)} + \ldots \tag{2.133}$$

Using these results, the highest-order terms of the components follow from the explicit expressions given above. □

It follows from Propositions 2.3.2 and 2.3.3 that the components

$$(\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}, \qquad (\bar\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = (-1)^{n+1}(\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow},$$

do not identically vanish. Hence, the vectors $|\psi_n\rangle$, $|\bar\psi_n\rangle$ do not identically vanish, as was anticipated above. Their linear independence follows from the fact that they are, by construction, eigenvectors of the spin-reversal operator with different eigenvalues: $F|\psi_n\rangle = (-1)^n|\psi_n\rangle$, $F|\bar\psi_n\rangle = (-1)^{n+1}|\bar\psi_n\rangle$.

The trigonometric limit We now evaluate the scalar product S_n for $\zeta \to 0$. This evaluation corresponds to the limit $p \to 0$. After a rescaling, the weights of the eight-vertex model tend, in this limit, to the trigonometric weights of the six-vertex model at $\eta = 2\pi/3$. Hence, we refer to it as the *trigonometric limit*. The trigonometric limit of S_n was rigorously computed in [125] with the help of contour-integral formulas for the components of the ground-state vectors of the XXZ chain at $\Delta = -1/2$. Here, we show that it also follows from Theorem 2.3.1 and a few properties of Schur functions and symplectic characters.

Let $k \geqslant 1$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. We recall that the Schur function s_λ and the symplectic character χ_λ associated with λ are given by

$$s_\lambda(z_1, \ldots, z_k) = \frac{\det_{i,j=1}^k \left(z_i^{\lambda_j+k-j}\right)}{\det_{i,j=1}^k \left(z_i^{k-j}\right)}, \tag{2.134}$$

and

$$\chi_\lambda(z_1, \ldots, z_m) = \frac{\det_{i,j=1}^k \left(z_i^{\lambda_j+k-j+1} - z_i^{-(\lambda_j+k-j+1)}\right)}{\det_{i,j=1}^k \left(z_i^{k-j+1} - z_i^{-(k-j+1)}\right)}, \tag{2.135}$$

respectively. Hereafter, we focus on the case where λ is given by the double-staircase partition $Y_k = (\lfloor (k-i)/2 \rfloor)_{i=1}^k$. For $\zeta = 0$, the polynomial $H_{2k}(w_1, \ldots, w_{2k})$ is related to a symplectic character associated with this partition [111]. Through the parameterisation

$$\bar{w}(z) = \frac{(z-1)^2}{1+z+z^2}, \tag{2.136}$$

one obtains the relation

$$H_{2k}(\bar{w}(z_1), \ldots, \bar{w}(z_{2k}))\big|_{\zeta=0} = 3^{k(k-1)} \prod_{i=1}^{2k} \left(1 + z_i + z_i^{-1}\right)^{-k+1} \times \chi_{Y_{2k}}(z_1, \ldots, z_{2k}), \tag{2.137}$$

for each $k \geqslant 1$.

The next lemma provides particular factorisation properties of Schur functions associated with the double-staircase partitions into symplectic characters. It can be proven through elementary row and column operations in the involved determinants [126].

Lemma 2.3.4. *Let* $\bar{\omega} = \mathrm{e}^{\mathrm{i}\pi/3}$ *and* $k \geqslant 0$ *be an integer, then we have*

$$s_{Y_{4k+2}}(z_1, \dots, z_{2k}, z_1^{-1}, \dots, z_{2k}^{-1}, z, 1) = z^k \prod_{i=1}^{2k} \left(1 + z_i + z_i^{-1}\right) \times \chi_{Y_{2k+2}}(z_1, \dots, z_{2k}, z, \bar{\omega}) \chi_{Y_{2k}}(z_1, \dots, z_{2k}), \quad (2.138)$$

and

$$s_{Y_{4k+4}}(z_1, \dots, z_{2k+1}, z_1^{-1}, \dots, z_{2k+1}^{-1}, z, 1) = z^k (1+z) \prod_{i=1}^{2k+1} \left(1 + z_i + z_i^{-1}\right) \times \chi_{Y_{2k+2}}(z_1, \dots, z_{2k+1}, z) \chi_{Y_{2k+2}}(z_1, \dots, z_{2k+1}, \bar{\omega}). \quad (2.139)$$

We now compute the trigonometric limit of S_n. To this end, we recall that an alternating sign matrix of size n has a unique $+1$ in its first row. In Section 1.4, we denoted by $A(n, k)$ the number of ASMs for which this $+1$ is in column k. This number is given in (1.34).

Proposition 2.3.5. *We have*

$$S_n|_{\zeta=0} = \sum_{k=0}^{n} A(n+1, k+1) \mu^k. \quad (2.140)$$

Proof. We evaluate the expressions of Theorem 2.3.1 for $\zeta = 0$ with the help of (2.137). This evaluation yields

$$\begin{aligned} S_{2k}|_{\zeta=0} &= 3^{-k(2k-1)} (\mu^2 - \mu + 1)^k \\ &\quad \times \chi_{Y_{2k}}(1, \dots, 1) \chi_{Y_{2(k+1)}}(1, \dots, 1, \bar{\omega}, z(\mu)), && (2.141) \\ S_{2k+1}|_{\zeta=0} &= 3^{-k(2k+1)} (\mu + 1)(\mu^2 - \mu + 1)^k \\ &\quad \times \chi_{Y_{2(k+1)}}(1, \dots, 1, \bar{\omega}) \chi_{Y_{2(k+1)}}(1, \dots, 1, z(\mu)), && (2.142) \end{aligned}$$

for $n = 2k$ and $n = 2k + 1$, respectively. Here, $z(\mu) = (\bar{\omega} - \mu)/(\mu\bar{\omega} - 1)$.

Next, we apply Lemma 2.3.4 and rewrite the products of symplectic characters in terms of Schur functions. For both $n = 2k$ and $n = 2k + 1$, we obtain

$$S_n|_{\zeta=0} = 3^{-n(n+1)/2} (\bar{\omega}(1 - \bar{\omega}\mu))^n s_{Y_{2(n+1)}}(1, \dots, 1, z(\mu)). \quad (2.143)$$

Finally, we note that the Schur function on the right-hand side is a specialisation of the partition function of a six-vertex model on an

$(n+1)\times(n+1)$ square grid with domain-wall boundary conditions [115]. The model's configurations are in bijection with the set of alternating sign matrices of size $n+1$. This bijection leads to [63]

$$s_{Y_{2(n+1)}}(1,\dots,1,z(\mu)) = \frac{3^{n(n+1)/2}}{(\bar\omega(1-\bar\omega\mu))^n}\sum_{k=0}^{n} A(n+1,k+1)\mu^k. \quad (2.144)$$

The substitution of this expression into (2.143) ends the proof. □

2.3.2 The homogeneous limit of $\bar Z_n^{\pm}$

In this section, we compute the scalar products

$$\bar S_n^{\pm} = \Big((\langle\uparrow\uparrow| + \nu\langle\downarrow\downarrow|)^{\otimes n}\otimes(\langle\uparrow| \pm \langle\downarrow|)\Big)\,|\psi_n\rangle \quad (2.145)$$

from the homogeneous limit of $\bar Z_n^{\pm}$. As for the previous section, we divide this one into three parts. First, we find closed-form expressions for $\bar S_n^{\pm}$. Second, we use them to compute and analyse the components $(\psi_n)_{\downarrow\cdots\downarrow\downarrow}$ and $(\psi_n)_{\uparrow\cdots\uparrow\downarrow}$. Third, we evaluate the trigonometric limit of $\bar S_n^{\pm}$.

Closed-form expression We use the notation

$$\bar\nu = \frac{(\nu-\zeta)(\nu\zeta-1)}{(\zeta^2-1)\nu}. \quad (2.146)$$

Furthermore, for each $\epsilon,\epsilon' = \pm$, we define

$$c_{\epsilon\epsilon'} = (\zeta+\epsilon)(\nu+\epsilon'), \quad d_\epsilon = \nu(\zeta+\epsilon)^2, \quad \bar d_\epsilon = \zeta(\nu+\epsilon)^2. \quad (2.147)$$

Theorem 2.3.6. *For each $k\geqslant 1$, the scalar products $\bar S_{2k}^{\pm}$ and $\bar S_{2k+1}^{\pm}$ are polynomials in ν and ζ, given by*

$$\bar S_{2k}^{+} = \frac{\nu^k}{2(\nu-\zeta)(\nu\zeta-1)}\Big(c_{+-}^2 H_{2(k+1)}(J_4)H_{2(k+1)}(J_3,\bar\nu) - c_{-+}^2 H_{2(k+1)}(J_3)H_{2(k+1)}(J_4,\bar\nu)\Big), \quad (2.148)$$

$$\bar S_{2k}^{-} = \frac{c_{+-}c_{-+}\nu^k}{2(\nu-\zeta)(\nu\zeta-1)}\Big(H_{2(k+1)}(J_4)H_{2(k+1)}(J_3,\bar\nu) - H_{2(k+1)}(J_3)H_{2(k+1)}(J_4,\bar\nu)\Big), \quad (2.149)$$

and

$$\bar{S}^{+}_{2k+1} = \frac{c_{--}\nu^k}{2\zeta(\nu-\zeta)(\nu\zeta-1)}\Big(d_+ H_{2(k+1)}H_{2(k+2)}(J_3,J_4,\bar{\nu}) \\ - \bar{d}_+ H_{2(k+1)}(\bar{\nu})H_{2(k+2)}(J_3,J_4)\Big), \quad (2.150)$$

$$\bar{S}^{-}_{2k+1} = \frac{c_{++}\nu^k}{2\zeta(\nu-\zeta)(\nu\zeta-1)}\Big(d_- H_{2(k+1)}H_{2(k+2)}(J_3,J_4,\bar{\nu}) \\ - \bar{d}_- H_{2(k+1)}(\bar{\nu})H_{2(k+2)}(J_3,J_4)\Big). \quad (2.151)$$

Proof. The proof is similar to the proof of Theorem 2.3.1. We divide it into two parts. The first part consists of establishing the explicit formulas for the scalar products. In the second part, we show that they define polynomials in ν and ζ.

Part 1: Explicit formulas. First, we evaluate the scalar products $\bar{Z}^{\pm}_n$ for $\lambda_1 = \cdots = \lambda_{2n+1} = 0$. Using (2.33), we find

$$Z^{\pm}_n(0,\dots,0) = (-\vartheta_1(\kappa,p^2)\vartheta_1(\kappa+\eta,p^2))^n \\ \times \left((\langle\uparrow\uparrow| + \nu\langle\downarrow\downarrow|)^{\otimes n}\otimes(\langle\uparrow| \pm \langle\downarrow|)\right)|\Psi_n(0,\dots,0)\rangle, \quad (2.152)$$

where

$$\nu = -\frac{\vartheta_4(\kappa+\eta,p^2)\vartheta_4(\kappa,p^2)}{\vartheta_1(\kappa+\eta,p^2)\vartheta_1(\kappa,p^2)}. \quad (2.153)$$

We use Proposition 2.2.7 to write the left-hand side of (2.152) in terms of $\bar{X}^{\pm}_n$. Moreover, we use (2.109) and (2.145) to write the right-hand side in terms of $\bar{S}^{\pm}_n$. Solving for the latter, we obtain

$$\bar{S}^{\pm}_n = \left(-\frac{\vartheta_1(\eta,p^2)}{\vartheta_1(\kappa,p^2)\vartheta_1(\kappa+\eta,p^2)}\right)^n \mathcal{N}_n\bar{X}^{\pm}_n(0,\dots,0). \quad (2.154)$$

Second, we replace $\mathcal{N}_n$ by its definition and $\bar{X}^{\pm}_n$ by its explicit form in terms of the elliptic Tsuchiya determinant, given in Theorem 2.2.15. This step requires to consider the cases $n = 2k$ and $n = 2k+1$ separately. As in the proof of Theorem 2.3.1, we present the details for $n = 2k$:

$$\bar{S}^{\pm}_{2k} = \frac{p^k e^{-2k\mathrm{i}\eta}}{\vartheta_1(\eta)^{4k^2}}\left(\frac{\vartheta_1(\eta,p^2)}{\vartheta_1(\kappa,p^2)\vartheta_1(\kappa+\eta,p^2)\vartheta_4(0,p^2)}\right)^{2k}\left(\frac{\vartheta_4(0,p^2)}{\vartheta_4(\eta,p^2)}\right)^{k(2k+1)} \\ \times \Big(\gamma^{\pm}_0\mathbb{H}_{2(k+1)}(0,\dots,0,\beta_4)\mathbb{H}_{2(k+1)}(0,\dots,0,w(\kappa+\tfrac{\eta}{2}),\beta_3) \\ + \delta^{\pm}_0\mathbb{H}_{2(k+1)}(0,\dots,0,\beta_3)\mathbb{H}_{2(k+1)}(0,\dots,0,w(\kappa+\tfrac{\eta}{2}),\beta_4)\Big). \quad (2.155)$$

We now rewrite the elliptic Tsuchiya determinants in terms of the polynomials of Section 2.2.4, using Lemma 2.2.16, and find

$$\begin{aligned}\bar{S}^{\pm}_{2k} = \nu^k\big(&\gamma_0^{\pm} H_{2(k+1)}(w(\beta_4))H_{2(k+1)}(w(\beta_3), w(\kappa+\tfrac{\eta}{2}))\\ &+\delta_0^{\pm} H_{2(k+1)}(w(\beta_3))H_{2(k+1)}(w(\beta_4), w(\kappa+\tfrac{\eta}{2}))\big).\end{aligned} \tag{2.156}$$

We note that

$$w(\beta_3) = J_3, \quad w(\beta_4) = J_4, \quad w(\kappa+\tfrac{\eta}{2}) = \frac{(\nu-\zeta)(\zeta\nu-1)}{(\zeta^2-1)\nu} = \bar{\nu}, \tag{2.157}$$

and

$$\gamma_0^{+} = \frac{c_{+-}^2}{2(\nu-\zeta)(\nu\zeta-1)}, \qquad \delta_0^{+} = -\frac{c_{-+}^2}{2(\nu-\zeta)(\nu\zeta-1)}, \tag{2.158}$$

$$\gamma_0^{-} = \frac{c_{+-}c_{-+}}{2(\nu-\zeta)(\nu\zeta-1)}, \qquad \delta_0^{-} = -\frac{c_{+-}c_{-+}}{2(\nu-\zeta)(\nu\zeta-1)}. \tag{2.159}$$

These relations lead to the expressions for $\bar{S}^{\pm}_{2k}$ given above. The derivation of the expressions for $\bar{S}^{\pm}_{2k+1}$ is similar.

Part 2: Polynomial nature. We now show that $S^{\pm}_n$ is a polynomial in ν and ζ, focussing on $n = 2k$. Using Lemma 2.2.18, it is straightforward to show that $H_{2(k+1)}(J_i)$ and $\nu^k H_{2(k+1)}(J_i, \bar{\nu})$ are polynomials in ν and ζ for both $i = 3$ and $i = 4$. Hence, $\bar{S}^{\pm}_{2k}$ is a ratio of polynomials in ν and ζ. Its denominator tends to zero if and only if $\nu \to \zeta$ or $\nu\zeta \to 1$. The numerator tends to zero in these limits, too. This is sufficient to conclude that the ratio is a polynomial in both ν and ζ. The case $n = 2k+1$ is similar albeit slightly more technical.

Finally, we note that this proof relies on the meromorphic parameterisation (2.153) of ν in terms of κ. Similarly to Theorem 2.3.1, one checks that, under this parameterisation, any real ν has a point of its preimage in $0 < \kappa < \pi$. Hence, the theorem holds for all real ν, and, by analytic continuation, for all complex ν. □

Components

Proposition 2.3.7. *For each $k \geqslant 0$, we have*

$$(\psi_{2k})_{\downarrow\cdots\downarrow\downarrow} = \zeta^k H_{2k} H_{2(k+1)}(J_3, J_4), \tag{2.160}$$

$$(\psi_{2k+1})_{\downarrow\cdots\downarrow\downarrow} = \zeta^{k+1} H_{2(k+1)}(J_3) H_{2(k+1)}(J_4), \tag{2.161}$$

and

$$(\psi_{2k})_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2}\zeta^k\Big((1+\zeta)H_{2k}(J_3)H_{2(k+1)}(J_4) \\ + (1-\zeta)H_{2k}(J_4)H_{2(k+1)}(J_3)\Big), \tag{2.162}$$

$$(\psi_{2k+1})_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2}\zeta^k\Big((1-\zeta^2)H_{2(k+1)}H_{2(k+1)}(J_3,J_4) \\ + H_{2k}H_{2(k+2)}(J_3,J_4)\Big). \tag{2.163}$$

Proof. In terms of the scalar products $\bar{S}_n^{\pm}$, we have

$$(\psi_n)_{\downarrow\cdots\downarrow\downarrow} = \frac{(-1)^n}{2}(\bar{S}_n^+\big|_{\nu=0} + \bar{S}_n^-\big|_{\nu=0}), \tag{2.164}$$

$$(\psi_n)_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2}\left(\bar{S}_n^+\big|_{\nu=0} - \bar{S}_n^-\big|_{\nu=0}\right), \tag{2.165}$$

where we used $(\psi_n)_{\downarrow\cdots\downarrow\downarrow} = (-1)^n(\psi_n)_{\uparrow\cdots\uparrow\uparrow}$ to obtain the first expression. We compute the scalar products using the explicit expressions of Theorem 2.3.6. With the help of Lemma 2.2.17, we obtain

$$\bar{S}_{2k}^+\big|_{\nu=0} = \frac{\zeta^{k-1}}{2}\Big((1+\zeta)^2H_{2k}(J_3)H_{2(k+1)}(J_4) \\ - (1-\zeta)^2H_{2k}(J_4)H_{2(k+1)}(J_3)\Big), \tag{2.166}$$

$$\bar{S}_{2k}^-\big|_{\nu=0} = \frac{\zeta^{k-1}}{2}(1-\zeta^2)\Big(H_{2k}(J_3)H_{2(k+1)}(J_4) \\ - H_{2k}(J_4)H_{2(k+1)}(J_3)\Big), \tag{2.167}$$

and

$$\bar{S}_{2k+1}^+\big|_{\nu=0} = \frac{\zeta^{k-1}(1-\zeta)}{2}\Big((1+\zeta)^2H_{2(k+1)}H_{2(k+1)}(J_3,J_4) \\ - H_{2k}H_{2(k+2)}(J_3,J_4)\Big), \tag{2.168}$$

$$\bar{S}_{2k+1}^-\big|_{\nu=0} = \frac{\zeta^{k-1}(1+\zeta)}{2}\Big((1-\zeta)^2H_{2(k+1)}H_{2(k+1)}(J_3,J_4) \\ - H_{2k}H_{2(k+2)}(J_3,J_4)\Big). \tag{2.169}$$

For the components labelled by the polarised spin configuration, we thus find

$$(\psi_{2k})_{\downarrow\cdots\downarrow\downarrow} = \frac{\zeta^{k-1}}{2}\Big((\zeta+1)H_{2k}(J_3)H_{2(k+1)}(J_4) \\ + (\zeta-1)H_{2k}(J_4)H_{2(k+1)}(J_3)\Big), \tag{2.170}$$

$$(\psi_{2k+1})_{\downarrow\cdots\downarrow\downarrow} = \frac{\zeta^{k-1}}{2}\Big(H_{2k}H_{2(k+2)}(J_3,J_4) \\ - (1-\zeta^2)H_{2(k+1)}H_{2(k+1)}(J_3,J_4)\Big), \tag{2.171}$$

for $n = 2k$ and $n = 2k+1$, respectively. We simplify these expressions with the help of the bilinear identities of Lemma 2.2.19. Indeed, we rewrite (2.170) by using (2.106) with $w_1, \dots, w_{2k-1} = 0$ and $x = J_3$, $y = J_4$, $u = v = 0$. To simplify (2.171), we use the identity (2.107), specialised to $w_1, \dots, w_{2k} = 0$ and $x = J_3$, $y = J_4$, $u = v = 0$. These simplifications lead to the expressions (2.160) and (2.161).

For the components labelled by the almost-polarised spin configuration, the substitution yields (2.162) and (2.163). □

It follows that the components $(\psi_n)_{\downarrow\cdots\downarrow\downarrow}$ and $(\psi_n)_{\uparrow\cdots\uparrow\downarrow}$ are polynomials in ζ with integer coefficients. In the next proposition, we provide the coefficients of their lowest-order and highest-order terms. They follow from the evaluations [111]

$$H_{2(k+1)}(J_3,J_4)\Big|_{\zeta=0} = A_{\mathrm{V}}(2k+1), \tag{2.172a}$$

$$H_{2k}(J_3)|_{\zeta=0} = H_{2k}(J_4)|_{\zeta=0} = N_8(2k), \tag{2.172b}$$

and from Lemma 2.2.18, respectively.

Proposition 2.3.8. *For each $k \geqslant 0$, we have*

$$(\psi_{2k})_{\downarrow\cdots\downarrow\downarrow} = A_{\mathrm{V}}(2k+1)^2\zeta^k + \cdots + \zeta^{k(2k+1)}, \tag{2.173}$$

$$(\psi_{2k+1})_{\downarrow\cdots\downarrow\downarrow} = N_8(2k+2)^2\zeta^{k+1} + \cdots + \zeta^{(k+1)(2k+1)}, \tag{2.174}$$

and, where $\cdots$ denotes intermediate powers of ζ, and

$$(\psi_{2k})_{\uparrow\cdots\uparrow\downarrow} = N_8(2k)N_8(2k+2)\zeta^k + \cdots, \tag{2.175}$$

$$(\psi_{2k+1})_{\uparrow\cdots\uparrow\downarrow} = A_{\mathrm{V}}(2k+1)A_{\mathrm{V}}(2k+3)\zeta^k + \cdots, \tag{2.176}$$

where $\cdots$ denotes higher powers of ζ.

The coefficient of the highest-order term of $(\psi_n)_{\uparrow\cdots\uparrow\downarrow}$ is rather difficult to compute. The reason is a non-trivial cancellation between several of the leading powers of the two terms on the right-hand sides of (2.162) and (2.163). For small n, we observe that the coefficient of the highest-order term is given by the n-th Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$, but we have no proof for arbitrary n.

The trigonometric limit Finally, we compute the trigonometric limit of $\bar{S}_n^{\pm}$. It yields simple polynomials in ν, in sharp contrast to the trigonometric limit of S_n discussed above. Despite their simplicity, we have not found them in the literature on the XXZ spin chain at $\Delta = -1/2$.

Proposition 2.3.9. *For each $k \geqslant 0$, we have*

$$\bar{S}_{2k}^{+}\Big|_{\zeta=0} = 2A_{\mathrm{V}}(2k+1)N_8(2(k+1))\nu^k, \qquad \bar{S}_{2k}^{-}\Big|_{\zeta=0} = 0, \tag{2.177}$$

and

$$\bar{S}_{2k+1}^{\pm}\Big|_{\zeta=0} = -A_{\mathrm{V}}(2k+3)N_8(2(k+1))(\nu \mp 1)\nu^k. \tag{2.178}$$

Proof. We use (2.172) to evaluate the closed-form expressions found in Theorem 2.3.6 for $\zeta = 0$. The evaluation of $\bar{S}_{2k}^{\pm}$ is straightforward and leads to (2.177). The evaluation of $\bar{S}_{2k+1}^{\pm}$ leads, however, to a singular expression. To compute it, we use the bilinear identity (2.106) of Lemma 2.2.19 with $w_1 = \cdots = w_{2k-1} = 0$. Setting $x = J_3, y = J_4, u = \bar{\nu}, v = 0$, we obtain

$$\begin{aligned} H_{2(k+1)}(J_3, J_4)H_{2k}(\bar{\nu}) = \frac{1}{2\nu}\big((1 + (1-\zeta)\nu + \nu^2)H_{2(k+1)}(J_4, \bar{\nu})H_{2k}(J_3) \\ - (1 - (1+\zeta)\nu + \nu^2)H_{2(k+1)}(J_3, \bar{\nu})H_{2k}(J_4)\big). \end{aligned} \tag{2.179}$$

Likewise, setting $x = J_3, y = J_4, u = 0, v = \bar{\nu}$, we find

$$\begin{aligned} &H_{2(k+1)}(J_3, J_4, \bar{\nu})H_{2k} \\ &= \frac{\zeta}{2\nu^2(\zeta^2 - 1)}\big((\nu - 1)^2(1 + (1-\zeta)\nu + \nu^2)H_{2(k+1)}(J_3, \bar{\nu})H_{2k}(J_4) \\ &\qquad - (\nu + 1)^2(1 - (1+\zeta)\nu + \nu^2)H_{2(k+1)}(J_4, \bar{\nu})H_{2k}(J_3)\big). \end{aligned} \tag{2.180}$$

Applying these relations allows us to simplify the scalar products to

$$\bar{S}^{+}_{2k+1} = \frac{\nu^{k-1}(\nu-1)}{2}\big((1+\nu^2)H_{2(k+1)}(J_4)H_{2(k+2)}(J_3,\bar{\nu}) - (1+\nu)^2 H_{2(k+1)}(J_3)H_{2(k+2)}(J_4,\bar{\nu})\big), \quad (2.181)$$

and

$$\bar{S}^{-}_{2k+1} = \frac{\nu^{k-1}(\nu+1)}{2}\big((1-\nu)^2 H_{2(k+1)}(J_4)H_{2(k+2)}(J_3,\bar{\nu}) - (1+\nu^2)H_{2(k+1)}(J_3)H_{2(k+2)}(J_4,\bar{\nu})\big). \quad (2.182)$$

These expressions are non-singular for $\zeta = 0$. We evaluate them with the help of (2.172), which leads to (2.178). □

2.3.3 The sums of components

In this section, we compute the sum of the components of the vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$, given by

$$\Sigma_n = \sum_{\boldsymbol{\alpha}} (\psi_n)_{\boldsymbol{\alpha}}, \qquad \bar{\Sigma}_n = \sum_{\boldsymbol{\alpha}} (\bar{\psi}_n)_{\boldsymbol{\alpha}}, \quad (2.183)$$

respectively. It is possible to find closed-form expressions for these sums from the homogeneous limit of a scalar product for the inhomogeneous supersymmetric eight-vertex model. To find these scalar products, one needs to follow the strategy of Section 2.2, but with a different solution to the boundary Yang-Baxter equation [127]. Here, we present a different approach that exploits the relation between the eight-vertex model and the XYZ spin chain.

XYZ spin chain and parameter range Recall the XYZ Hamiltonian $H \in \mathrm{End}(\mathbb{C}^2)^{\otimes N}$ defined in Section 1.3. For $\eta = 2\pi/3$, its anisotropy parameters J_i are given by (2.112), $i = 2, 3, 4$. Furthermore, we introduce

$$E_0 = -\frac{(2n+1)(3+\zeta^2)}{4(1-\zeta^2)}. \quad (2.184)$$

The vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$ span the one-dimensional spaces of solutions of the eigenvalue problems

$$H|\psi\rangle = E_0|\psi\rangle, \qquad F|\psi\rangle = (-1)^n|\psi\rangle, \quad (2.185)$$

$$H|\psi\rangle = E_0|\psi\rangle, \qquad F|\psi\rangle = (-1)^{n+1}|\psi\rangle, \quad (2.186)$$

where $|\psi\rangle \in (\mathbb{C}^2)^{\otimes(2n+1)}$, respectively [85]. We note that the Hamiltonian's eigenvalue E_0 has multiplicity 2. For $0 < \zeta < 1$ (and even $-1 < \zeta < 1$), it is its ground-state eigenvalue.

We now use the connection to the spin chain to characterise the vectors as functions of ζ:

Lemma 2.3.10. *The vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$ are rational functions of ζ.*

Proof. The space of the solution of the eigenvalue problem (2.185) is one-dimensional. Hence, if one fixes one component of $|\psi\rangle$ then all other components follow from basic linear algebra. For the solution $|\psi_n\rangle$, we have shown in Propositions 2.3.2 and 2.3.3 that the component $(\psi_n)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}$ is a non-vanishing polynomial in ζ. Since both H and E_0 are rational functions of ζ, all the components of $|\psi_n\rangle$ are therefore rational in ζ. The rationality of $|\bar{\psi}_n\rangle$ follows from the relation $|\bar{\psi}_n\rangle = P|\psi_n\rangle$. □

This lemma allows us to consider the vectors $|\psi_n\rangle$ and $|\bar{\psi}_n\rangle$ not only as rational functions on the interval $0 < \zeta < 1$ but on the real line, forgetting about the initial parameterisation (1.30). We adopt this point of view in the following and note that the vectors span the eigenspace of E_0 for all ζ. The following lemma gives the behaviour of $|\psi_n\rangle$ for large ζ:

Lemma 2.3.11. *As $\zeta \to \infty$, we have*

$$|\psi_n\rangle = \zeta^{n(n+1)/2}\left(|\downarrow\cdots\downarrow\rangle + (-1)^n|\uparrow\cdots\uparrow\rangle + o(1)\right). \qquad (2.187)$$

Proof. For $\zeta \to \infty$, the eigenvalue problem (2.185) becomes

$$\frac{1}{4}\left(\sum_{i=1}^{2n+1}\sigma_i^3\sigma_{i+1}^3\right)|\psi\rangle = \frac{2n+1}{4}|\psi\rangle, \qquad F|\psi\rangle = (-1)^n|\psi\rangle. \qquad (2.188)$$

The space of its solutions is spanned by $|\psi\rangle = |\downarrow\cdots\downarrow\rangle + (-1)^n|\uparrow\cdots\uparrow\rangle$. By Lemma 2.3.10, $|\psi_n\rangle$ is a rational function of ζ. We conclude that there are an integer d_n and a complex number a_n such that

$$|\psi_n\rangle = a_n\zeta^{d_n}\left(|\downarrow\cdots\downarrow\rangle + (-1)^n|\uparrow\cdots\uparrow\rangle + o(1)\right), \qquad (2.189)$$

It follows from Proposition 2.3.8 that $a_n = 1$ and $d_n = n(n+1)/2$. □

The sums of components We now compute Σ_n and $\bar{\Sigma}_n$. Using the spin-reversal properties $F|\psi_n\rangle = (-1)^n|\psi_n\rangle$ and $F|\bar{\psi}_n\rangle = (-1)^{n+1}|\bar{\psi}_n\rangle$, we find the trivial results

$$\Sigma_{2k+1} = 0, \qquad \bar{\Sigma}_{2k} = 0, \tag{2.190}$$

for each $k \geqslant 0$. We are going to show that Σ_{2k} and $\bar{\Sigma}_{2k+1}$ are, however, quite non-trivial. To this end, it will be useful to write them as follows:

$$\Sigma_n = 2^{n+1/2}\langle\uparrow\cdots\uparrow|U|\psi_n\rangle, \qquad \bar{\Sigma}_n = 2^{n+1/2}\langle\downarrow\cdots\downarrow|U|\psi_n\rangle. \tag{2.191}$$

Here, $U = 2^{-N/2}\prod_{j=1}^{N}(1+\mathrm{i}\sigma_j^2)$ is an orthogonal operator on $(\mathbb{C}^2)^{\otimes N}$. In the next lemma, we compute the action of U on the vector $|\psi_n\rangle = |\psi_n(\zeta)\rangle$.

Lemma 2.3.12. *For each $n \geqslant 0$, we have*

$$U|\psi_n(\zeta)\rangle = B_n(\zeta)\left(|\psi_n(\zeta')\rangle + (-1)^{n+1}|\bar{\psi}_n(\zeta')\rangle\right), \tag{2.192}$$

where $\zeta' = (\zeta+3)/(\zeta-1)$, and

$$B_n(\zeta)^2 = \frac{\|\psi_n(\zeta)\|^2}{2\|\psi_n(\zeta')\|^2}. \tag{2.193}$$

Proof. First, we write $H = H(\zeta)$ and $E_0 = E_0(\zeta)$ for the XYZ Hamiltonian (1.28) and its special eigenvalue (2.184). One checks that they satisfy the relations

$$H(\zeta')U = \left(\frac{\zeta-1}{2}\right)UH(\zeta), \qquad E_0(\zeta') = \left(\frac{\zeta-1}{2}\right)E_0(\zeta). \tag{2.194}$$

It follows from these relations that $U|\psi_n(\zeta)\rangle$ is an eigenvector of the Hamiltonian $H(\zeta')$ associated to the eigenvalue $E_0(\zeta')$. Since the corresponding eigenspace is spanned by $|\psi_n(\zeta')\rangle$ and $|\bar{\psi}_n(\zeta')\rangle$, we may write

$$U|\psi_n(\zeta)\rangle = B_n(\zeta)|\psi_n(\zeta')\rangle + \bar{B}_n(\zeta)|\bar{\psi}_n(\zeta')\rangle, \tag{2.195}$$

where $B_n(\zeta), \bar{B}_n(\zeta)$ are coefficients.

Second, we compute the scalar product of both sides of this equality with the basis vectors $|\uparrow\cdots\uparrow\rangle$ and $|\downarrow\cdots\downarrow\rangle$. Using (2.191), we find

$$\begin{aligned}
\Sigma_n &= 2^{n+1/2}\left(B_n(\zeta)\psi_n(\zeta')_{\uparrow\cdots\uparrow} + \bar{B}_n(\zeta)\bar{\psi}_n(\zeta')_{\uparrow\cdots\uparrow}\right) && (2.196\text{a})\\
&= 2^{n+1/2}(B_n(\zeta) - \bar{B}_n(\zeta))\psi_n(\zeta')_{\uparrow\cdots\uparrow}, && (2.196\text{b})\\
\bar{\Sigma}_n &= 2^{n+1/2}\left(B_n(\zeta)\psi_n(\zeta')_{\uparrow\cdots\uparrow} + \bar{B}_n(\zeta)\bar{\psi}_n(\zeta')_{\uparrow\cdots\uparrow}\right) && (2.196\text{c})\\
&= 2^{n+1/2}(B_n(\zeta) + \bar{B}_n(\zeta))\psi_n(\zeta')_{\downarrow\cdots\downarrow}. && (2.196\text{d})
\end{aligned}$$

Here, we applied the relations $(\bar{\psi}_n)_{\uparrow\cdots\uparrow} = -(\psi_n)_{\uparrow\cdots\uparrow}$, $(\bar{\psi}_n)_{\downarrow\cdots\downarrow} = (\psi_n)_{\downarrow\cdots\downarrow}$, which straightforwardly follow from $|\bar{\psi}_n\rangle = P|\psi_n\rangle$. We now combine (2.190) with (2.196), and use the fact that the components labelled by the polarised spin configurations do not identically vanish (which follows from Proposition 2.3.8). This leads to

$$\bar{B}_{2k}(\zeta) = -B_{2k}(\zeta), \qquad \bar{B}_{2k+1}(\zeta) = B_{2k+1}(\zeta), \tag{2.197}$$

for each $k \geqslant 0$. Hence, we obtain (2.192).

Third, we compute the scalar product of each side of (2.192) with itself. Using the fact that U is orthogonal, as well as $\langle\bar{\psi}_n(\zeta')|\psi_n(\zeta')\rangle = 0$ and $\|\bar{\psi}_n(\zeta')\|^2 = \|\psi_n(\zeta')\|^2$, we obtain (2.193). □

It is clear from this lemma why we need the continuation of $|\psi_n\rangle$ to values of ζ outside the range $0 < \zeta < 1$. Indeed, if ζ is in this range then $-\infty < \zeta' < -3$. Moreover, the lemma shows that to find an explicit formula for the sums of components we need the square norm of the ground-state vector $|\psi_n\rangle$. One of the main results of [111] are the expressions

$$||\psi_{2k}||^2 = 2^{2k+1}H_{2k}H_{2(k+1)}(J_2,J_3) \tag{2.198a}$$
$$H_{2(k+1)}(J_3,J_4)H_{2(k+1)}(J_2,J_4),$$
$$||\psi_{2k+1}||^2 = 2^{2(k+1)}H_{2(k+1)}(J_2)H_{2(k+1)}(J_3) \tag{2.198b}$$
$$\times H_{2(k+1)}(J_4)H_{2(k+2)}(J_2,J_3,J_4),$$

for each $k \geqslant 0$. We use them to obtain the following result:

Proposition 2.3.13. *For each $k \geqslant 0$, we have*

$$\Sigma_{2k} = 2^{k+1}(\zeta+3)^k H_{2k}H_{2(k+1)}(J_2,J_3), \tag{2.199}$$
$$\bar{\Sigma}_{2k+1} = 2^{k+1}(\zeta+3)^{k+1}H_{2(k+1)}(J_2)H_{2(k+1)}(J_3). \tag{2.200}$$

Proof. The scalar product of (2.192) and the basis vectors $|\uparrow\cdots\uparrow\rangle$ and $|\downarrow\cdots\downarrow\rangle$ implies

$$\Sigma_{2k} = 2^{2k+3/2}B_{2k}(\zeta)\psi_{2k}(\zeta')_{\downarrow\cdots\downarrow}, \tag{2.201}$$
$$\bar{\Sigma}_{2k+1} = 2^{2k+5/2}B_{2k+1}(\zeta)\psi_{2k+1}(\zeta')_{\downarrow\cdots\downarrow}. \tag{2.202}$$

To find these two expressions, we used $(\bar{\psi}_n)_{\uparrow\cdots\uparrow} = -(\psi_n)_{\uparrow\cdots\uparrow}$, $(\bar{\psi}_n)_{\downarrow\cdots\downarrow} = (\psi_n)_{\downarrow\cdots\downarrow}$, as well as $(\psi_{2k})_{\uparrow\cdots\uparrow} = (\psi_{2k})_{\downarrow\cdots\downarrow}$. We compute the coefficients $B_{2k}(\zeta)$ and $B_{2k+1}(\zeta)$ in these expressions by applying the transformation property of Lemma 2.2.20 to the square norms (2.198). We find

$$B_{2k}(\zeta) = \frac{b_{2k}}{2^{1/2}}\left(\frac{\zeta-1}{2}\right)^{k(2k+1)}, \tag{2.203}$$

$$B_{2k+1}(\zeta) = \frac{b_{2k+1}}{2^{1/2}}\left(\frac{\zeta-1}{2}\right)^{(k+1)(2k+1)}, \tag{2.204}$$

where $b_{2k}, b_{2k+1} = \pm 1$ are signs that we fix below. Moreover, using Lemma 2.2.20 and Proposition 2.3.7, we obtain

$$\psi_{2k}(\zeta')_{\downarrow\cdots\downarrow} = \left(\frac{\zeta+3}{\zeta-1}\right)^{k}\left(\frac{2}{\zeta-1}\right)^{2k^2} H_{2k}H_{2(k+1)}(J_2,J_3), \tag{2.205}$$

$$\psi_{2k+1}(\zeta')_{\downarrow\cdots\downarrow} = \left(\frac{\zeta+3}{\zeta-1}\right)^{k+1}\left(\frac{2}{\zeta-1}\right)^{2k(k+1)} H_{2k}H_{2(k+1)}(J_2,J_3).$$

Hence, the sums of components are

$$\Sigma_{2k} = b_{2k}2^{k+1}(\zeta+3)^k H_{2k}H_{2(k+1)}(J_2,J_3), \tag{2.206}$$

$$\bar{\Sigma}_{2k+1} = b_{2k+1}2^{k+1}(\zeta+3)^{k+1} H_{2(k+1)}(J_2)H_{2(k+1)}(J_3). \tag{2.207}$$

To find the signs b_{2k} and b_{2k+1}, we consider the limit $\zeta \to \infty$. On the one hand, Lemma 2.2.18 implies

$$H_{2k} = \zeta^{k(k-1)}(1+o(1)), \qquad H_{2k}(J_2) = 2^{1-k}\zeta^{k(k-1)}(1+o(1)),$$
$$H_{2k}(J_3) = \zeta^{k(k-1)}(1+o(1)), \quad H_{2k}(J_2,J_3) = 2^{1-k}\zeta^{k(k-1)}(1+o(1)),$$

and therefore

$$\Sigma_{2k} = 2b_{2k}\zeta^{k(2k+1)}(1+o(1)), \tag{2.208}$$

$$\bar{\Sigma}_{2k+1} = 2b_{2k+1}\zeta^{(k+1)(2k+1)}(1+o(1)). \tag{2.209}$$

On the other hand, it follows from Lemma 2.3.11 that

$$\Sigma_{2k} = 2\zeta^{k(2k+1)}(1+o(1)), \quad \bar{\Sigma}_{2k+1} = 2\zeta^{(k+1)(2k+1)}(1+o(1)). \tag{2.210}$$

We compare the coefficients of the leading terms and conclude that $b_{2k} = b_{2k+1} = 1$. □

Proposition 2.3.13 implies that the sums of components are polynomials in the parameter ζ with integer coefficients. As for the components studied in Sections 2.3.1 and 2.3.2, we compute the coefficients of its lowest- and highest-order term. For the lowest-order term, we recall the number $A_{\mathrm{DAD}}(2n+1)$ of diagonally and anti-diagonally symmetric alternating sign matrices of size $2n+1$ given in (1.40).

Proposition 2.3.14. *For each $k \geqslant 0$, we have*

$$\Sigma_{2k} = 2A_{\mathrm{DAD}}(4k+1) + \cdots + 2\zeta^{k(2k+1)}, \tag{2.211}$$

$$\bar{\Sigma}_{2k+1} = 2A_{\mathrm{DAD}}(4k+3) + \cdots + 2\zeta^{(k+1)(2k+1)}. \tag{2.212}$$

Proof. Both the coefficient and the exponent of the highest-order term follow from (2.210). To find the coefficient of the lowest-order term, we compute the sums of components for $\zeta = 0$:

$$\Sigma_{2k}|_{\zeta=0} = 2^{k+1}3^k \; H_{2k}|_{\zeta=0} \; H_{2(k+1)}(J_2, J_3)\Big|_{\zeta=0}, \tag{2.213}$$

$$\bar{\Sigma}_{2k+1}\Big|_{\zeta=0} = 2^{k+1}3^{k+1} \; H_{2(k+1)}(J_2)\Big|_{\zeta=0} \; H_{2(k+1)}(J_3)\Big|_{\zeta=0}. \tag{2.214}$$

To evaluate these expressions, we need (2.129) and (2.172), as well as [111]

$$H_{2k}(J_2, J_3)|_{\zeta=0} = 2^{-k} A^{(2)}_{\mathrm{UU}}(4n; 1, 1, 1), \tag{2.215}$$

where the numbers [114]

$$A^{(2)}_{\mathrm{UU}}(4n; 1, 1, 1) = 2^{2n} \prod_{i=1}^{n} \frac{(6i-1)(6i-3)!}{(2(n+i))!} \tag{2.216}$$

appear in the enumeration of alternating sign matrices with two U-turn boundaries (see Section 1.4). The final expression for the lowest-order coefficient is a result of the combinatorial identities

$$A_{\mathrm{DAD}}(4k+1) = 3^k A_{\mathrm{V}}(2k+1) A^{(2)}_{\mathrm{UU}}(4k; 1, 1, 1), \tag{2.217}$$

$$A_{\mathrm{DAD}}(4k+3) = 3^{k+1} N_8(2(k+1)) A(2k+1)/A_{\mathrm{V}}(2k+1). \tag{2.218}$$

They follow, for example, from a factorisation of Schur functions into symplectic and orthogonal characters discussed in [117]. □

We note that the lowest-order term also follows from a rigorous result on the six-vertex model [128].

2.3.4 Discussion

In this section, we consider the eigenvalue problem

$$H|\phi\rangle = E_0|\phi\rangle, \qquad P|\phi\rangle = |\phi\rangle, \qquad |\phi\rangle \in (\mathbb{C}^2)^{\otimes(2n+1)}, \tag{2.219}$$

for the XYZ Hamiltonian (1.28). Its space of solutions is one-dimensional [85]. We work with the solution

$$|\phi_n\rangle = \frac{1}{2}\left(|\psi_n\rangle + |\bar{\psi}_n\rangle\right). \tag{2.220}$$

Our goal is to compare our results for $|\phi_n\rangle$ to several conjectures by Bazhanov and Mangazeev [73], and Razumov and Stroganov [71], on their solutions, which we denote by $|\phi_n^{\text{BM}}\rangle$ and $|\phi_n^{\text{RS}}\rangle$, respectively. We refer to these conjectures as BM-Conjectures and RS-Conjectures. The comparison to our results suggests that $|\phi_n^{\text{BM}}\rangle = |\phi_n^{\text{RS}}\rangle = |\phi_n\rangle$, but this conclusion remains non-rigorous (even if we admit Conjecture 2.1.1). The main reason is that the normalisation conventions for $|\phi_n^{\text{BM}}\rangle$ and $|\phi_n^{\text{RS}}\rangle$ differ from the one for $|\phi_n\rangle$. We recall the normalisation of $|\phi_n\rangle$ is implicitly fixed by (2.16) and (2.17), and by the homogeneous limit (2.109) and (2.110). In the case of Bazhanov and Mangazeev's work, another reason is that the exact link between the families of polynomials they use and ours is still lacking.

The BM-Conjectures The vector $|\phi_n^{\text{BM}}\rangle$ is normalised so that its components are polynomials in ζ without a common polynomial factor. This fixes the vector up to an overall numerical factor. Its value is set by the additional requirement

$$(\phi_{2k}^{\text{BM}})_{\underbrace{\uparrow\cdots\uparrow}_{2k}\underbrace{\downarrow\cdots\downarrow}_{2k+1}}\Big|_{\zeta=0} = 1, \qquad (\phi_{2k+1}^{\text{BM}})_{\underbrace{\uparrow\cdots\uparrow}_{2k+2}\underbrace{\downarrow\cdots\downarrow}_{2k+1}}\Big|_{\zeta=0} = 1. \tag{2.221}$$

Bazhanov and Mangazeev formulate their conjectures on $|\phi_n^{\text{BM}}\rangle$ in terms of two families of polynomials $s_n(z), \bar{s}_n(z)$, where $n \in \mathbb{Z}$, with integer coefficients. These polynomials are solutions to a difference-differential equation [73]. The polynomials $s_n(z)$ are conjectured to possess the factorisation properties

$$s_{2k+1}(y^2) = \bar{c}_{2k+1}p_k(y)p_k(-y), \tag{2.222a}$$

$$s_{2k}(y^2) = \bar{c}_{2k}\bar{p}_{k+1}(y)q_{k-1}(y), \tag{2.222b}$$

for each $k \in \mathbb{Z}$. Here, $p_k(y), q_k(y)$ are polynomials with integer coefficients, and $\bar{c}_n$ are known constants. Moreover, we use the abbreviation

$$\bar{p}_k(y) = \left(\frac{1+3y}{2}\right)^{k(k-1)} p_{-k}\left(\frac{y-1}{1+3y}\right). \tag{2.223}$$

In his investigations of BM-Conjecture 1, Zinn-Justin [111] observed for small k the relations

$$H_{2k} = \zeta^{k(k-1)} q_{k-1}(\zeta^{-1}), \tag{2.224a}$$

$$2^{k-1} H_{2k}(J_2, J_3, J_4) = \zeta^{k(k-1)} q_{-k}(\zeta^{-1}), \tag{2.224b}$$

and

$$H_{2k}(J_2) = \zeta^{k(k-1)} \bar{p}_{-k+1}\left(\zeta^{-1}\right), \tag{2.224c}$$

$$H_{2k}(J_{3/4}) = \zeta^{k(k-1)} p_{k-1}\left(\pm\zeta^{-1}\right), \tag{2.224d}$$

$$2^{k-1} H_{2k}(J_2, J_{3/4}) = \zeta^{k(k-1)} p_{-k}\left(\mp\zeta^{-1}\right), \tag{2.224e}$$

$$2^{k-1} H_{2k}(J_3, J_4) = \zeta^{k(k-1)} \bar{p}_k(\zeta^{-1}). \tag{2.224f}$$

We assume that they hold for arbitrary k, and use them to discuss BM-Conjecture 2, 3 and 4.

BM-Conjecture 3 provides the explicit expression

$$(\phi_n^{\mathrm{BM}})_{\downarrow\cdots\downarrow} = \zeta^{n(n+1)/2} s_n(\zeta^{-2}) \tag{2.225}$$

for the component labelled by a polarised spin configuration. Using Proposition 2.3.7, we obtain the same component for the vector $|\phi_n\rangle$. For each $k \geqslant 0$, we have

$$(\phi_{2k})_{\downarrow\cdots\downarrow} = \zeta^k H_{2k} H_{2(k+1)}(J_3, J_4), \tag{2.226}$$

$$(\phi_{2k+1})_{\downarrow\cdots\downarrow} = \zeta^{k+1} H_{2(k+1)}(J_3) H_{2(k+1)}(J_4). \tag{2.227}$$

From the factorisation properties (2.222), the relations (2.224), and $\bar{c}_{2k+1} = 1$, $\bar{c}_{2k} = 2^{-k}$ for $k \geqslant 0$ [73], we conclude that

$$(\phi_n^{\mathrm{BM}})_{\downarrow\cdots\downarrow} = (\phi_n)_{\downarrow\cdots\downarrow}. \tag{2.228}$$

Likewise, BM-Conjecture 4 gives an explicit expression for the component labelled by an alternating spin configuration: for each $k \geqslant 0$,

$$(\phi_{2k}^{\mathrm{BM}})_{\uparrow\downarrow\cdots\uparrow\downarrow\downarrow} = 2^k \zeta^{2k(k-1)} \bar{p}_{-(k-1)}(\zeta^{-1}) q_{k-1}(\zeta^{-1}), \tag{2.229a}$$

$$(\phi_{2k+1}^{\mathrm{BM}})_{\uparrow\uparrow\downarrow\cdots\uparrow\downarrow} = 2^k \zeta^{2k^2} \bar{p}_{-k}(\zeta^{-1}) q_{k-1}(\zeta^{-1}). \tag{2.229b}$$

Similarly, we infer from Proposition 2.3.2 (and from the invariance of $|\phi_n\rangle$ under translations [85]) the components

$$(\phi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\downarrow} = 2^k H_{2k} H_{2k}(J_2), \tag{2.230}$$

$$(\phi_{2k+1})_{\uparrow\uparrow\downarrow\cdots\uparrow\downarrow} = 2^k H_{2k} H_{2(k+1)}(J_2). \tag{2.231}$$

We use (2.224) to compare these expressions to (2.229), and find

$$(\phi_{2k}^{\mathrm{BM}})_{\uparrow\downarrow\cdots\uparrow\downarrow\downarrow} = (\phi_{2k})_{\uparrow\downarrow\cdots\uparrow\downarrow\downarrow}, \quad (\phi_{2k+1}^{\mathrm{BM}})_{\uparrow\uparrow\downarrow\cdots\uparrow\downarrow} = (\phi_{2k+1})_{\uparrow\uparrow\downarrow\cdots\uparrow\downarrow}. \tag{2.232}$$

Since the space of solutions to the eigenvalue problem (2.219) is one-dimensional, the equalities (2.228) and (2.232) suggest that $|\phi_n^{\mathrm{BM}}\rangle = |\phi_n\rangle$. A proof of this equality would imply that all the components of $|\phi_n\rangle$ are polynomials in ζ. This proof is, however, beyond the scope of this thesis. Nonetheless, it is interesting to explore the consequences of this equality. In BM-Conjecture 2, Bazhanov and Mangazeev claim that

$$\left(\phi_n^{\mathrm{BM}}\right)_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2n+1}\zeta^{n(n-1)/2}\bar{s}_n(\zeta^{-2}). \tag{2.233}$$

We obtain the corresponding component of $|\phi_n\rangle$ from Proposition 2.3.7. For $k \geqslant 0$, we find

$$\begin{aligned}(\phi_{2k})_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2}\zeta^k\Big((1+\zeta)H_{2k}(J_3)H_{2(k+1)}(J_4) \\ + (1-\zeta)H_{2k}(J_4)H_{2(k+1)}(J_3)\Big),\end{aligned} \tag{2.234}$$

$$\begin{aligned}(\phi_{2k+1})_{\uparrow\cdots\uparrow\downarrow} = \frac{1}{2}\zeta^k\Big((1-\zeta^2)H_{2(k+1)}H_{2(k+1)}(J_3,J_4) \\ + H_{2k}H_{2(k+2)}(J_3,J_4)\Big),\end{aligned} \tag{2.235}$$

respectively. We match these expressions to (2.233) and find, for $k \geqslant 0$,

$$\bar{s}_{2k}(y^2) = \frac{(4k+1)}{2y^{2k+1}}((1+y)p_{k-1}(y)p_k(-y) - (1-y)p_{k-1}(-y)p_k(y)),$$

$$\bar{s}_{2k+1}(y^2) = \frac{(4k+3)}{(2y^2)^{k+1}}\left((y^2-1)q_k(y)\bar{p}_{k+1}(y) + \frac{1}{2}q_{k-1}(y)\bar{p}_{k+2}(y)\right).$$

These relations resemble the factorisations (2.222) for $s_n(y^2)$. To our best knowledge, they have not been reported in the literature.

The RS-Conjectures The vector $|\phi_n^{\rm RS}\rangle$ is normalised so that its components are polynomials in ζ without a common polynomial factor, too.[5] Razumov and Stroganov argue that this normalisation convention implies that the component of highest degree is $(\phi_n^{\rm RS})_{\downarrow\cdots\downarrow}$. According to RS-Conjecture 4.2, its highest-order term is

$$(\phi_n^{\rm RS})_{\downarrow\cdots\downarrow} = a_n\zeta^{n(n+1)/2} + \cdots\,, \tag{2.236}$$

where a_n is non-zero and $\cdots$ denotes lower-order terms. Razumov and Stroganov consider $a_n = 1$, which fixes the remaining overall numerical factor in their normalisation. By RS-Conjecture 4.4, this choice implies

$$(\phi_{2k}^{\rm RS})_{\downarrow\cdots\downarrow} = A_{\rm V}(2k+1)^2\zeta^k + \cdots + \zeta^{k(2k+1)}, \tag{2.237}$$

$$(\phi_{2k+1}^{\rm RS})_{\downarrow\cdots\downarrow} = N_8(2k+2)^2\zeta^{k+1} + \cdots + \zeta^{(k+1)(2k+1)}, \tag{2.238}$$

Using Proposition 2.3.8, we find exactly the same expressions for the components $(\phi_{2k})_{\downarrow\cdots\downarrow}$ and $(\phi_{2k+1})_{\downarrow\cdots\downarrow}$, which suggests (but does not prove) the equality $|\phi_n^{\rm RS}\rangle = |\phi_n\rangle$. Another property supporting this equality is the sum rule

$$\sum_{\boldsymbol{\alpha}}(\phi_n^{\rm RS}(\zeta))_{\boldsymbol{\alpha}} = 2^{-n(n-1)/2}(\zeta-1)^{n(n+1)/2}(\phi_n^{\rm RS}(\zeta'))_{\downarrow\cdots\downarrow}, \tag{2.239}$$

where $\zeta' = (\zeta+3)/(\zeta-1)$, which is formulated in RS-Conjecture 5.2. It, indeed, holds for $|\phi_n\rangle$ as well, which straightforwardly follows from Proposition 2.3.13 and its proof.

2.4 Conclusion

In this chapter, we have investigated several scalar products involving a particular eigenvector of the transfer matrix of the inhomogeneous supersymmetric eight-vertex model with periodic boundary conditions. We have found explicit expressions for them in terms of the elliptic Tsuchiya determinant. In the homogeneous limit, they allowed us to compute the scalar products S_n and $\bar{S}_n^{\pm}$ involving the basis vectors $|\psi_n\rangle$, $|\bar{\psi}_n\rangle$ of the eigenspace for the transfer-matrix eigenvalue Θ_n of the

[5] Razumov and Stroganov do not explicitly mention the absence of a common polynomial factor, but this assumption appears to be implicit in their work.

homogeneous eight-vertex model. Our main results are Theorems 2.3.1 and 2.3.6. They provide new and explicit expressions for these scalar products in terms of special polynomials introduced by Zinn-Justin and Rosengren. We have used them to compute several components of the basis vectors, as well as the sum of their components, establish their polynomial nature and compute their trigonometric limit.

We now discuss several open problems and generalisations of the present work. First, our results assume that Conjecture 2.1.1 holds. The proof of this conjecture remains a challenge, which would undoubtedly provide even more insight into the properties of the eigenvector $|\Psi_n\rangle$ and its homogeneous limit. Second, the comparison of our results to the investigations of Bazhanov and Mangazeev [73], and Razumov and Stroganov [71], strongly suggests that all the components of $|\psi_n\rangle$ are polynomials in ζ with integer coefficients. Proving the polynomial nature in ζ would be a step forward to settling most (if not all) of their conjectures. Third, we observe for small n that the integer coefficients of the powers of ζ in a given component of $|\psi_n\rangle$ all have the same sign. This observation suggests that they could have a combinatorial meaning. We note that Hietala has recently found a combinatorial interpretation of the polynomials $H_{2k}, H_{2k}(J_3)$ and $H_{2k}(J_4)$, in terms of a partition function of the three-colour model [82, 83]. Similar results for $H_{2k}(J_2), H_{2k}(J_i, J_j), \ldots$ remain to be found. Fourth, the results of this chapter suggest that an exact finite-size computation of the emptiness or boundary emptiness formation probability for the supersymmetric eight-vertex model could be possible. In the trigonometric limit, these correlation functions are known [128, 129]. Recently, Hagendorf and Rosengren made progress in this regard by computing nearest-neighbour correlation functions [77]. Finally, we mention that there is a conjecture for a simple eigenvalue of the transfer matrix of the inhomogeneous supersymmetric eight-vertex model with open boundary conditions [130]. A characterisation of its eigenspace, similar to Conjecture 2.1.1, is still to be found.

Chapter 3

Higher-spin elliptic models

Recall that the microscopic degrees of freedom of vertex models are labels, or *spins*, located on the lattice edges. So far, these spins have taken the binary values ± 1. It is this feature that characterises the 6V and 8V models as *spin-1/2* models. One natural way to generalise them would be to allow the spins to take a broader range of values. Doing so in a prescribed way leads to *higher-spin vertex models*. Let $\ell \in \mathbb{N}_*$, one describes a spin-$\ell/2$ vertex model on the lattice by considering spins in

$$\{-\ell, -\ell+2, \ldots, \ell-2, \ell\}\,. \tag{3.1}$$

To fully define the model, one then has to assign statistical weights, as prescribed by statistical mechanics.

In the algebraic formulation, the higher-spin vertex models can be constructed from R-matrices, solutions to the Yang-Baxter equation. The *fusion* procedure offers a systematic way to build them, starting from a spin-1/2 R-matrix [86–88].

To this day, higher-spin versions of elliptic integrable models remain less understood than their trigonometric counterparts [54, 92, 93, 131–136]. In this thesis, we shall be particularly interested in the spin-1 generalisation of the 8V model, the *forty-one-vertex (41V) model*, as well as its related *spin-1 XYZ chain.*

Throughout our study of the periodic eight-vertex model, we have come across many intriguing properties. First, we witnessed the existence of a remarkably simple eigenvalue for its transfer matrix [85]. Second, we noted that the related eigenvectors display surprising connections to supersymmetry [84], and special polynomials related to combinatorics [78–81,111]. The catch is: all these elegant features occur only for $\eta = 2\pi/3$, and for odd system sizes $N = 2n + 1$.

The question that naturally arises is: how do these properties generalise to the 41V model? The work of [136] suggests that the quasi-periodic 41V transfer matrix could possess a remarkably simple eigenvalue for any value of its crossing parameter and any system size. Moreover, Hagendorf proved the supersymmetric nature of the associated eigenvector. In its components, he also observed the polynomials $H_{2m}(w_1, \ldots, w_{2m})$, as well as combinatorial properties [136]. Although promising, these initial results are, for the most part, conjectural. Thus, one of our goals for the remainder of this thesis is the rigorous analysis of the remarkable eigenvalue of the quasi-periodic 41V model, and its related eigenspace.

Finally, it is worth pointing out that a similar investigation was successfully performed on the trigonometric limit of the 41V model. Among other results, Hagendorf and Morin-Duchesne established relations between special eigenvectors of the quasi-periodic *nineteen-vertex (19V) model* and generating functions of symmetry classes of ASMs [92].

The chapter is structured as follows: in Section 3.1, we get acquainted with the two higher-spin vertex models that shall be our focus, the forty-one- and eighteen-vertex models. After describing them on the lattice, we move on to the construction of their R-matrices via the fusion procedure. We consider these models with quasi-periodic boundary conditions along the horizontal direction, and define their transfer matrices. Incidentally, the matrices commute, and can thus be diagonalised simultaneously. Through simple algebraic arguments, we show that they possess remarkably simple eigenvalues. In Section 3.2, we introduce a spin-1 XYZ chain. We show how its Hamiltonian relates to the transfer matrix of the 41V model. Furthermore, we prove that it possesses the eigenvalue $E = 0$. Finally, we provide conjectures regarding the degeneracy of this eigenvalue and its position within the spectrum.

3.1 The eighteen- and forty-one-vertex models

Higher-spin elliptic vertex models One can easily generalise the definition of the 8V model of Section 1.1 in order to describe higher-spin elliptic vertex models. Let $\ell \in \mathbb{N}_*$, to define the spin-$\ell/2$ model, we allow the spins located on each lattice edge to take values in

$$\{-\ell, -\ell+2, \dots, \ell-2, \ell\}\,. \tag{3.2}$$

The interactions remain local and governed by the *generalised ice rule*. That means that, around a vertex

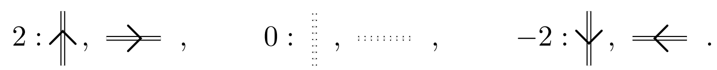
$$\text{-}\alpha \overset{\delta}{\underset{\beta}{+}} \gamma\text{-}, \tag{3.3}$$

the following relation holds:

$$\alpha + \beta \equiv \gamma + \delta \pmod 4, \tag{3.4}$$

with $\alpha, \beta, \gamma, \delta \in \{-\ell, -\ell+2, \dots, \ell-2, \ell\}$.

In the remainder of this thesis, we shall be particularly interested in spin-1 models. For $\ell = 2$, the ice rule allows for forty-one local vertex configurations. Thus, the resulting model is called *forty-one-vertex model*. Similarly to the 8V model, there exists a graphical representation for vertices in terms of arrows. It follows from the dictionary



(3.5)

See Figures 3.1 and 3.4 for examples.

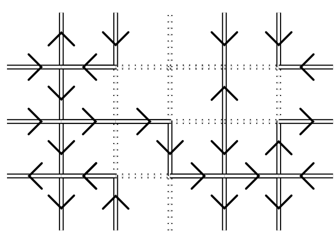


Figure 3.1: A configuration of the 41V model

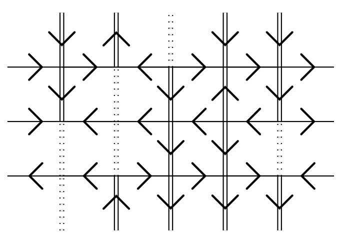


Figure 3.2: A configuration of the 18V model

Mixed-spin vertex models One can generalise even further by considering hybrid models that use two different types of spin. Let $m, n \in \mathbb{N}_*$, we allow the spins on the horizontal, and vertical, edges to take values in

$$\{-m, \ldots, m-2, m\}, \qquad \text{and} \qquad \{-n, \ldots, n-2, n\}, \tag{3.6}$$

respectively. The generalised ice rule still applies. Clearly, for $m = n$, one just gets back the spin-$m/2$ model as described above. In our work below, we shall encounter the mixed-spin model with $(m, n) = (1, 2)$. In this case, eighteen possible local vertex configurations are allowed (see Figure 3.3). The resulting model thus bears the name of *eighteen-vertex model*. Figure 3.2 depicts an example of configuration of the 18V model.

3.1.1 Fusion of the eight-vertex model

The fusion procedure is a way to systematically build a family of R-matrices, $R^{(m,n)}(\lambda)$ acting on $\mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}$, that fulfil the YBE [86, 87]:

$$\begin{aligned} &R_{12}^{(m,n)}(\lambda_{12}) R_{13}^{(m,p)}(\lambda_{13}) R_{23}^{(n,p)}(\lambda_{23}) \\ &\qquad = R_{23}^{(n,p)}(\lambda_{23}) R_{13}^{(m,p)}(\lambda_{13}) R_{12}^{(m,n)}(\lambda_{12}), \end{aligned} \tag{3.7}$$

with $m, n, p \in \mathbb{N}_*$. The construction relies on the existence of an elementary R-matrix, typically $R^{(1,1)}(\lambda)$, with appropriate *degeneration points*, i.e. values of λ for which the matrix does not have maximal rank.

The goal of this section is to review the work of [89]. We choose our elementary R-matrix to be that of the 8V model. We use the fusion procedure to define the matrices $R^{(1,2)}(\lambda)$ and $R^{(2,2)}(\lambda)$ from it. These matrices allow one to assign statistical weights to the vertices of the 18V and 41V models described above.

We start this section with an explanation of our notations and conventions, then we move on to the fusion procedure. Afterwards, we state some useful properties and symmetries of the new R-matrices.

Notations and conventions The Hilbert space of a single spin-1 is $\mathbb{C}^3$. We label the elements of its canonical basis as

$$|\Uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |\Downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.8}$$

We refer to the tensor product space $(\mathbb{C}^3)^{\otimes N}$ as the *spin space*. It possesses the basis vectors

$$|\alpha_1 \cdots \alpha_N\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle\,, \quad \alpha_i \in \{\Uparrow, 0, \Downarrow\}, \quad i = 1, \dots, N. \tag{3.9}$$

To describe the interactions and properties of the models, we use the following spin operators, obtained from the spin-1 representation of $\mathfrak{su}(2)$:

$$s^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We also define the spin-1 *twists*,

$$\Omega^1 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Omega^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Omega^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.10}$$

We refer to $\Omega^{1,2,3}$ as the *anti-diagonal*, *mixed* and *diagonal twist*, respectively. Furthermore, we denote by s_i^a the operator s^a, $a = 1, 2, 3$, acting on the i-th factor $\mathbb{C}^3$ of the spin space,

$$s_i^a = \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{i-1} \otimes s^a \otimes \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{N-i}, \qquad i = 1, \dots, N. \tag{3.11}$$

Finally, we follow the conventions of Appendix A.1 for the Jacobi theta functions. In particular, we use the shorthand notations

$$\vartheta_i(0) = \vartheta_i, \qquad \vartheta_i(z, p^2) = \vartheta_i(z|2\omega) = \hat{\vartheta}_i(z), \qquad \hat{\vartheta}_i(0) = \hat{\vartheta}_i, \tag{3.12}$$

for $i = 1, \dots, 4$. Throughout our work, we often omit the computational steps that rely on the application of identities between Jacobi theta functions.

***R*-matrix of the 8V model** We choose the elementary matrix $R^{(1,1)}(\lambda)$ as the R-matrix of the 8V model. We opt for the following convention:

$$R^{(1,1)}(\lambda) = \rho R(\lambda), \qquad \rho = \frac{2}{\vartheta_2 \hat{\vartheta}_4}, \tag{3.13}$$

where $R(\lambda)$ is as in (1.16). As a preliminary step to fusion, we describe several useful properties of $R^{(1,1)}(\lambda)$.

First, we note that the R-matrix fulfils the YBE (3.7) for $(m, n, p) = (1, 1, 1)$. Second, it obeys the symmetry:

$$[R_{12}^{(1,1)}(\lambda), \sigma_1^a \sigma_2^a] = 0, \qquad a = 1, 2, 3. \tag{3.14}$$

The third property is an essential ingredient for fusion. To state it, we recall that $\mathbb{C}^2 \otimes \mathbb{C}^2$ decomposes into a symmetric and an antisymmetric subspace [137],

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathrm{Sym}(\mathbb{C}^2, \mathbb{C}^2) \oplus (\mathbb{C}^2 \wedge \mathbb{C}^2). \tag{3.15}$$

We denote by $P^{(1)\pm}$ the projectors onto these respective subspaces. They read

$$P^{(1)+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P^{(1)-} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

For $\lambda = \pm\eta$, the matrix $R^{(1,1)}(\lambda)$ becomes one of these projectors, up to multiplication by an operator. The precise statement involves the two parameters

$$X = \frac{\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4(\eta)}{\hat{\vartheta}_1(\eta)\hat{\vartheta}_4(2\eta)}, \qquad Y = \frac{\hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)}{\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)}. \tag{3.17}$$

Proposition 3.1.1 (Degeneration points)**.** *We have*

$$R^{(1,1)}(\eta) = \frac{\vartheta_1(\eta)\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_4} E P^{(1)+}, \quad R^{(1,1)}(-\eta) = -2\vartheta_1(\eta) P^{(1)-}, \tag{3.18}$$

where

$$E = \begin{pmatrix} X & 0 & 0 & Y \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ Y & 0 & 0 & X \end{pmatrix}. \tag{3.19}$$

Fusion of the eight-vertex model We now construct $R^{(1,2)}(\lambda)$, using $R^{(1,1)}(\lambda)$ as a building block. First, we examine the operator

$$P_{23}^{(1)+} R_{13}^{(1,1)}(\lambda+\eta) R_{12}^{(1,1)}(\lambda). \tag{3.20}$$

By the YBE (3.7), we find that it annihilates $\mathbb{C}^2 \otimes (\mathbb{C}^2 \wedge \mathbb{C}^2)$, while it leaves $\mathbb{C}^2 \otimes \mathrm{Sym}(\mathbb{C}^2, \mathbb{C}^2)$ invariant. Therefore, we restrict (3.20) to this last subspace. To this end, we introduce the following matrix:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \alpha & -\alpha & 0 \end{pmatrix}, \qquad \alpha^2 = \frac{\vartheta_1(\eta)}{\vartheta_1(2\eta)}. \tag{3.21}$$

The matrix U is the change of basis between the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, and the basis related to its symmetric and antisymmetric subspaces,

$$\left\{ |\uparrow\uparrow\rangle, \quad \frac{1}{2\alpha}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |\downarrow\downarrow\rangle, \quad \frac{1}{2\alpha}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right\}. \tag{3.22}$$

We choose the value (3.21) for α, as it simplifies some expressions below. We also introduce the restriction matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.23}$$

We define the new R-matrix as

$$\begin{aligned} R^{(1,2)}(\lambda) = \vartheta_1(\lambda+\eta)^{-1}\, Q_{23} U_{23} \\ \times [P_{23}^{(1)+} R_{13}^{(1,1)}(\lambda+\eta) R_{12}^{(1,1)}(\lambda)]\, U_{23}^{-1} Q_{23}^t. \end{aligned} \tag{3.24}$$

In the basis $\{|\uparrow\Uparrow\rangle, |\uparrow 0\rangle, |\uparrow\Downarrow\rangle, |\downarrow\Uparrow\rangle, |\downarrow 0\rangle, |\downarrow\Downarrow\rangle\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^3$, it reads

$$R^{(1,2)} = \frac{\rho}{\hat{\vartheta}_4} \begin{pmatrix} e & 0 & k & 0 & g & 0 \\ 0 & f & 0 & j & 0 & \bar{j} \\ \bar{k} & 0 & \bar{e} & 0 & \bar{g} & 0 \\ 0 & \bar{g} & 0 & \bar{e} & 0 & \bar{k} \\ \bar{j} & 0 & j & 0 & f & 0 \\ 0 & g & 0 & k & 0 & e \end{pmatrix}, \tag{3.25}$$

with

$$e(\lambda) = \hat{\vartheta}_4(\eta)^2 \hat{\vartheta}_1(\lambda + 2\eta)\hat{\vartheta}_4(\lambda), \tag{3.26a}$$

$$f(\lambda) = \hat{\vartheta}_4 \hat{\vartheta}_4(2\eta)\hat{\vartheta}_1(\lambda + \eta)\hat{\vartheta}_4(\lambda + \eta), \tag{3.26b}$$

$$g(\lambda) = \alpha^{-1}\hat{\vartheta}_1(\eta)\hat{\vartheta}_4(\eta)\hat{\vartheta}_1(\lambda)\hat{\vartheta}_1(\lambda + 2\eta), \tag{3.26c}$$

$$j(\lambda) = \alpha \hat{\vartheta}_4 \hat{\vartheta}_1(2\eta)\hat{\vartheta}_4(\lambda + \eta)^2, \tag{3.26d}$$

$$k(\lambda) = \hat{\vartheta}_1(\eta)^2 \hat{\vartheta}_1(\lambda + 2\eta)\hat{\vartheta}_4(\lambda). \tag{3.26e}$$

We obtain $\bar{e}, \bar{g}, \bar{j}, \bar{k}$ from e, g, j, k, repectively, through the transformation $\hat{\vartheta}_1(\cdot) \leftrightarrow \hat{\vartheta}_4(\cdot)$ applied to the λ-dependent Jacobi theta functions. This R-matrix gives rise to a vertex model with eighteen allowed local configurations (see Figure 3.3). We read the corresponding statistical weights in the non-zero entries of $R^{(1,2)}(\lambda)$. One checks that this model is the *18V model* described above. Furthermore, the R-matrix $R^{(1,2)}(\lambda)$ fulfils the YBE (3.7) with $(m, n, p) = (1, 1, 2)$.

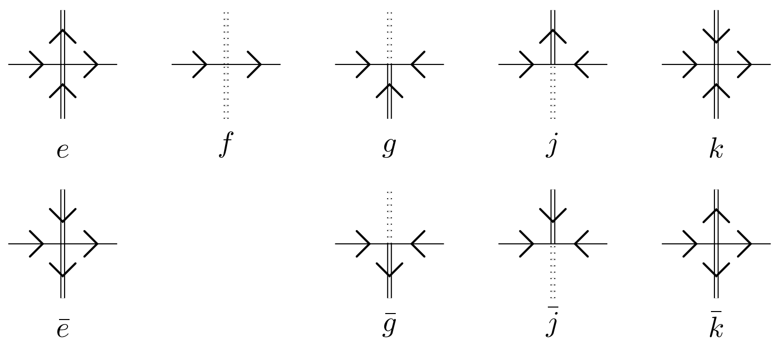


Figure 3.3: Nine of the eighteen allowed vertices of the 18V model, and their statistical weights. Dotted double lines correspond to the 0 spin state of $\mathbb{C}^3$. Reversing the arrows of any depicted vertex gives another allowed vertex of same weight.

We apply the fusion procedure a second time, and define

$$R^{(2,2)}(\lambda) = Q_{12}U_{12}[P_{12}^{(1)+} R_{13}^{(1,2)}(\lambda) R_{23}^{(1,2)}(\lambda - \eta)]U_{12}^{-1}Q_{12}^t. \tag{3.27}$$

In the basis $\{|\Uparrow\Uparrow\rangle, |\Uparrow 0\rangle, |\Uparrow\Downarrow\rangle, |0\Uparrow\rangle, |00\rangle, |0\Downarrow\rangle, |\Downarrow\Uparrow\rangle, |\Downarrow 0\rangle, |\Downarrow\Downarrow\rangle\}$ of $\mathbb{C}^3 \otimes \mathbb{C}^3$, it reads

$$R^{(2,2)} = \left(\frac{\rho}{\hat{\vartheta}_4}\right)^2 \begin{pmatrix} G & 0 & A & 0 & B & 0 & A & 0 & H \\ 0 & F & 0 & C & 0 & \bar{C} & 0 & D & 0 \\ \bar{A} & 0 & \bar{G} & 0 & \bar{B} & 0 & \bar{H} & 0 & \bar{A} \\ 0 & C & 0 & F & 0 & D & 0 & \bar{C} & 0 \\ I & 0 & \bar{I} & 0 & E & 0 & \bar{I} & 0 & I \\ 0 & \bar{C} & 0 & D & 0 & F & 0 & C & 0 \\ \bar{A} & 0 & \bar{H} & 0 & \bar{B} & 0 & \bar{G} & 0 & \bar{A} \\ 0 & D & 0 & \bar{C} & 0 & C & 0 & F & 0 \\ H & 0 & A & 0 & B & 0 & A & 0 & G \end{pmatrix}, \tag{3.28}$$

with

$$A(\lambda) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4\hat{\vartheta}_4(\eta)\hat{\vartheta}_1(\lambda)\hat{\vartheta}_4(\lambda)^2\hat{\vartheta}_1(\lambda+2\eta), \tag{3.29a}$$

$$B(\lambda) = \alpha^{-2}\hat{\vartheta}_1(\eta)\hat{\vartheta}_4\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)\hat{\vartheta}_1(\lambda)^2\hat{\vartheta}_4(\lambda)\hat{\vartheta}_1(\lambda+2\eta), \tag{3.29b}$$

$$C(\lambda) = \hat{\vartheta}_1(2\eta)\hat{\vartheta}_4^2\hat{\vartheta}_4(2\eta)\hat{\vartheta}_4(\lambda)^2\hat{\vartheta}_1(\lambda+\eta)\hat{\vartheta}_4(\lambda+\eta), \tag{3.29c}$$

$$D(\lambda) = \hat{\vartheta}_1(2\eta)^2\hat{\vartheta}_4^2\hat{\vartheta}_1(\lambda)\hat{\vartheta}_4(\lambda)\hat{\vartheta}_1(\lambda+\eta)\hat{\vartheta}_4(\lambda+\eta), \tag{3.29d}$$

$$\begin{aligned} E(\lambda) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4\hat{\vartheta}_4(\eta)\Big[&\hat{\vartheta}_1(\lambda-\eta)\hat{\vartheta}_1(\lambda+\eta)^3 \\ &+ \hat{\vartheta}_4(\lambda-\eta)\hat{\vartheta}_4(\lambda+\eta)^3\Big] + F(\lambda), \end{aligned} \tag{3.29e}$$

$$F(\lambda) = \hat{\vartheta}_4^2\hat{\vartheta}_4(2\eta)^2\hat{\vartheta}_1(\lambda)\hat{\vartheta}_4(\lambda)\hat{\vartheta}_1(\lambda+\eta)\hat{\vartheta}_4(\lambda+\eta), \tag{3.29f}$$

$$\begin{aligned} G(\lambda) = \hat{\vartheta}_4(\lambda)\hat{\vartheta}_1(\lambda+2\eta)\Big[&\hat{\vartheta}_1(\eta)^4\hat{\vartheta}_1(\lambda-\eta)\hat{\vartheta}_4(\lambda+\eta) \\ &+ \hat{\vartheta}_4(\eta)^4\hat{\vartheta}_1(\lambda+\eta)\hat{\vartheta}_4(\lambda-\eta)\Big], \end{aligned} \tag{3.29g}$$

$$H(\lambda) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4\hat{\vartheta}_4(\eta)\hat{\vartheta}_1(\lambda)^3\hat{\vartheta}_1(\lambda+2\eta), \tag{3.29h}$$

$$I(\lambda) = \alpha^2[\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4]^2\frac{\vartheta_1(\lambda)}{\vartheta_1(\eta)}\Big[\hat{\vartheta}_1(\eta)^2\hat{\vartheta}_4(\lambda+\eta)^2 + \hat{\vartheta}_4(\eta)^2\hat{\vartheta}_1(\lambda+\eta)^2\Big].$$

Similarly to the previous case, we obtain $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{G}, \bar{H}, \bar{I}$ from A, B, C, D, G, H, I through the transformation $\hat{\vartheta}_1(\cdot) \leftrightarrow \hat{\vartheta}_4(\cdot)$ applied to the λ-dependent Jacobi theta functions. This R-matrix gives rise to a vertex model with forty-one allowed local configurations (see Figure 3.4). We read the corresponding statistical weights in the non-zero entries of

$R^{(2,2)}(\lambda)$. This model is the *41V model* described above. We check that the R-matrix $R^{(2,2)}(\lambda)$ fulfils the YBE (3.7) with $(m, n, p) = (1, 2, 2)$ and $(2, 2, 2)$.

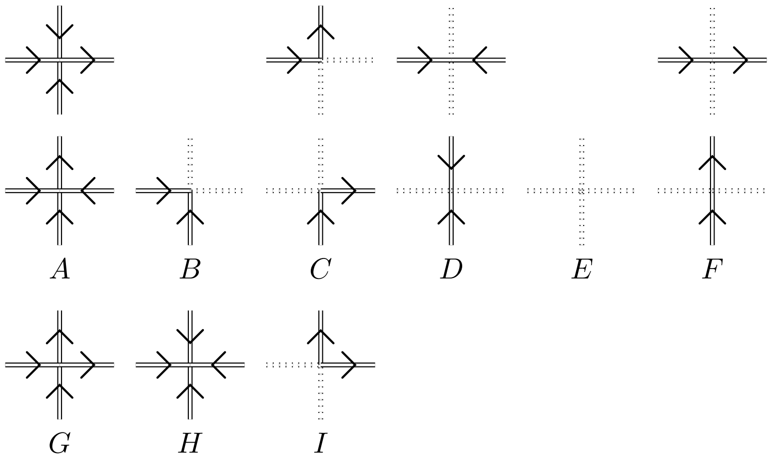


Figure 3.4: Thirteen of the forty-one allowed vertices of the 41V model with their statistical weights. Reversing all the arrows of any depicted vertex yields another allowed vertex of same weight. Reversing the vertical arrows of a vertex of weight A, B, C, D, G, H, I yields another allowed vertex of weight $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{G}, \bar{H}, \bar{I}$, respectively.

Properties of the fused *R*-matrices The R-matrix of the 18V model obeys the following symmetry:

$$[R_{12}^{(1,2)}(\lambda), \sigma_1^a \Omega_2^a] = 0, \qquad a = 1, 2, 3. \tag{3.30}$$

It is possible to relate $R^{(1,2)}(\lambda)$ to its partial transposes. The property is called *crossing symmetry*. To state it, we introduce the following matrix:

$$M = \begin{pmatrix} -\alpha^2 Y & 0 & -\alpha^2 X \\ 0 & 1 & 0 \\ -\alpha^2 X & 0 & -\alpha^2 Y \end{pmatrix}, \tag{3.31}$$

where X and Y are defined in (3.17).

Proposition 3.1.2. *We have*

$$R_{12}^{(1,2)}(\lambda)^{t_1} = -\sigma_1^2 R_{12}^{(1,2)}(-\lambda - 2\eta)\sigma_1^2, \tag{3.32}$$

$$R_{12}^{(1,2)}(\lambda)^{t_2} = -M_2 R_{12}^{(1,2)}(-\lambda - 2\eta)M_2^{-1}, \tag{3.33}$$

where ^{t_j} *denotes the transposition with respect to the* j*-th factor of* $\mathbb{C}^2 \otimes \mathbb{C}^3$, $j = 1, 2$.

Proposition 3.1.2 hints at the possibility of symmetrising $R^{(1,2)}(\lambda)$. Let us define $W \in \mathrm{End}(\mathbb{C}^3)$ as

$$W = \begin{pmatrix} \beta_+ & 0 & \beta_- \\ 0 & 1 & 0 \\ \beta_- & 0 & \beta_+ \end{pmatrix}, \tag{3.34}$$

with

$$\beta_\pm = \alpha \sqrt{\frac{\vartheta_2(\eta)\hat{\vartheta}_4}{2\vartheta_2\hat{\vartheta}_4(2\eta)}} \left(\sqrt{\frac{\vartheta_4(\eta)}{\vartheta_4}} \pm \sqrt{\frac{\vartheta_3(\eta)}{\vartheta_3}} \right). \tag{3.35}$$

This matrix fulfils the relation

$$W^2 = -M\Omega^2. \tag{3.36}$$

Proposition 3.1.3. *The matrix* $W_2 R^{(1,2)}_{12}(\lambda) W_2^{-1}$ *is symmetric.*

Proof. The proposition follows from Proposition 3.1.2, (3.30), (3.36). □

The product of two 18V R-matrices of specific arguments is proportional to the identity operator.

Proposition 3.1.4 (Unitarity relation)**.** *We have*[1]

$$R^{(1,2)}(\lambda)R^{(1,2)}(-\lambda-\eta) = -\vartheta_1(\lambda-\eta)\vartheta_1(\lambda+2\eta). \tag{3.37}$$

The unitarity relation indicates that $R^{(1,2)}(\lambda)$ is invertible except at $\lambda \equiv -2\eta, \eta \pmod{\pi, \pi\omega}$.

The following lemma proves useful for establishing a relation between the transfer matrices of the 18V and 41V models:

Lemma 3.1.5. *We have*

$$R^{(1,2)}_{23}(\lambda)R^{(1,2)}_{13}(\lambda-\eta)P^{(1)-}_{12} = \vartheta_1(\lambda-\eta)\vartheta_1(\lambda+2\eta)P^{(1)-}_{12}. \tag{3.38}$$

[1] The right-hand side of this equality is a multiple of the identity matrix on $\mathbb{C}^2 \otimes \mathbb{C}^3$. We often omit such identity matrices to lighten the notation.

Proof. Let us set $\lambda_{12} = -\eta$ in the YBE (3.7) with $(m, n, p) = (1, 1, 2)$. By Proposition 3.1.1, $R^{(1,2)}_{23}(\lambda)R^{(1,2)}_{13}(\lambda - \eta)$ leaves $(\mathbb{C}^2 \wedge \mathbb{C}^2) \otimes \mathbb{C}^3$ invariant. We determine the action on $\mathbb{C}^3$ through a direct computation, and obtain (3.38). □

We move on to the properties of the R-matrix of the 41V model. The matrix fulfils the following symmetry:

$$[R^{(2,2)}_{12}(\lambda), \Omega^a_1\Omega^a_2] = 0, \qquad a = 1, 2, 3. \tag{3.39}$$

The R-matrix also obeys a crossing symmetry.

Proposition 3.1.6. *For each $j = 1, 2$, we have*

$$R^{(2,2)}_{12}(\lambda)^{t_j} = M_j R^{(2,2)}_{12}(-\lambda - \eta)M_j^{-1}, \tag{3.40}$$

where M is defined in (3.31).

Moreover, it can be symmetrised by a similarity transformation.

Proposition 3.1.7. *The matrix $W_1W_2R^{(2,2)}_{12}(\lambda)W_1^{-1}W_2^{-1}$ is symmetric.*

Proof. The proposition follows from Proposition 3.1.6, (3.36), (3.39). □

Let $\mathcal{P}^{(2)}$ denote the permutation operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$, i.e.

$$\mathcal{P}^{(2)} |\alpha\beta\rangle = |\beta\alpha\rangle, \qquad \forall\, \alpha, \beta \in \{\Uparrow, 0, \Downarrow\}. \tag{3.41}$$

For $\lambda = 0$, $R^{(2,2)}(\lambda)$ is proportional to this permutation operator.

Proposition 3.1.8. *We have*

$$R^{(2,2)}(0) = \vartheta_1(\eta)\vartheta_1(2\eta)\mathcal{P}^{(2)}. \tag{3.42}$$

Similarly to the previous case, the product of two 41V R-matrices of specific arguments is proportional to the identity operator.

Lemma 3.1.9 (Unitarity relation)**.** *We have*

$$R^{(2,2)}(\lambda)R^{(2,2)}(-\lambda) = \vartheta_1(\lambda + \eta)\vartheta_1(\lambda - \eta)\vartheta_1(\lambda + 2\eta)\vartheta_1(\lambda - 2\eta). \tag{3.43}$$

The unitarity relation implies that $R^{(2,2)}(\lambda)$ is invertible, except at $\lambda \equiv \pm\eta, \pm 2\eta \pmod{\pi, \pi\omega}$.

Let $P^{(2)-}$ denote the projector onto the subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ spanned by the *singlet*

$$|s\rangle = |{\Uparrow}{\Downarrow}\rangle - |00\rangle + |{\Downarrow}{\Uparrow}\rangle\,. \tag{3.44}$$

For $\lambda = -\eta$, the symmetrised $R^{(2,2)}(\lambda)$ is proportional to this projector.

Proposition 3.1.10. *We have*

$$W_1 W_2 R_{12}^{(2,2)}(-\eta) W_1^{-1} W_2^{-1} = 3\vartheta_1(\eta)\vartheta_1(2\eta) P_{12}^{(2)-}\,. \tag{3.45}$$

3.1.2 The transfer matrices

In our pursuits below, we work with inhomogeneous vertex models. To this end, we introduce *inhomogeneity parameters* λ_i, $i = 1, \ldots, N$. We associate one to each factor $\mathbb{C}^3$ of the spin space. We refer to the limit $\lambda_1 = \ldots = \lambda_N = 0$ as the *homogeneous limit.*

We define the monodromy matrix of the 18V model as

$$M_0(\lambda_0) = \prod_{j=1}^{\overset{\curvearrowleft}{N}} R_{0j}^{(1,2)}(\lambda_{0j}) = \begin{pmatrix} A(\lambda_0) & B(\lambda_0) \\ C(\lambda_0) & D(\lambda_0) \end{pmatrix}_0\,. \tag{3.46}$$

Here, 0 labels the auxiliary space $\mathbb{C}^2$, first factor of the spin space. Moreover, the $\curvearrowleft$ over the product indicates a reversal of the product order. The transfer matrix of the 18V model with quasi-periodic boundary conditions is

$$T^{(1)}(\lambda) = \operatorname{tr}_0\left(\mathrm{i}\sigma_0 M_0(\lambda)\right), \tag{3.47}$$

where tr_0 denotes the trace over the auxiliary space, and $\sigma = \sigma^\kappa$ are the Pauli matrices, $\kappa = 1, 2, 3$.

Similarly, the transfer matrix of the 41V model with quasi-periodic boundary conditions is

$$T^{(2)}(\lambda_0) = \operatorname{tr}_0\left(\Omega_0 \prod_{j=1}^{\overset{\curvearrowleft}{N}} R_{0j}^{(2,2)}(\lambda_{0j})\right), \tag{3.48}$$

where the auxiliary space is now $\mathbb{C}^3$, and $\Omega = \Omega^\kappa$ are the spin-1 twists defined in (3.10), $\kappa = 1, 2, 3$.

In this section, we present a few properties of the transfer matrices, mostly related to their diagonalisability. We follow with a short argument proving the existence of remarkably simple eigenvalues for $T^{(1)}(\lambda)$ and $T^{(2)}(\lambda)$.

Properties We define the functions

$$\mathrm{D}(\lambda) = \prod_{j=1}^{N} \vartheta_1(\lambda - \lambda_j), \qquad \mathrm{A}(\lambda) = \mathrm{D}(\lambda + 2\eta). \tag{3.49}$$

Proposition 3.1.11. *We have*

$$M_0(\lambda)\sigma_0^2 M_0(\lambda - \eta)^{t_0}\sigma_0^2 = \mathrm{A}(\lambda)\mathrm{D}(\lambda - \eta), \tag{3.50}$$

where t_0 denotes the transposition with respect to the auxiliary space.

The 18V transfer matrix fulfils the following symmetry, where we use the standard curly bracket notation for the anticommutator of two operators:

Proposition 3.1.12. *Let $a = 1, 2, 3$, $a \neq \kappa$, we have*

$$\left[T^{(1)}(\lambda), (\Omega^\kappa)^{\otimes N}\right] = 0, \qquad \left\{T^{(1)}(\lambda), (\Omega^a)^{\otimes N}\right\} = 0. \tag{3.51}$$

Proof. The symmetry follows from (3.30), the cyclic property of the trace, and $\sigma^\kappa \sigma^a \sigma^\kappa = -\sigma^a$. □

We deduce the equivalent symmetry for $T^{(2)}(\lambda)$ from (3.39).

Proposition 3.1.13. *For each $a = 1, 2, 3$, we have*

$$\left[T^{(2)}(\lambda), (\Omega^a)^{\otimes N}\right] = 0. \tag{3.52}$$

The two transfer matrices form a family of commuting matrices.

Proposition 3.1.14. *For each $m, n = 1, 2$, we have*

$$[T^{(m)}(\lambda), T^{(n)}(\mu)] = 0. \tag{3.53}$$

Proof. We show (3.53) by using the YBE, the invertibility of $R^{(m,n)}(\lambda)$, and the similarity invariance of the trace [138, 139]. $\square$

In the case where the transfer matrices are diagonalisable, Proposition 3.1.14 ensures that we can find a common eigenbasis of $T^{(1)}(\lambda)$ and $T^{(2)}(\lambda)$, independent of their spectral parameter [102].

Next, we relate the transfer matrices to their transposes via a similarity transformation. To this end, we highlight their dependence on the inhomogeneity parameters $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ by writing them as arguments.

Proposition 3.1.15. *For each $m, n = 1, 2$, $m \neq n$, we have*

$$T^{(m)}(\lambda; \boldsymbol{\lambda})^t = (-1)^{mN} M^{\otimes N} T^{(m)}(-\lambda - n\eta; -\boldsymbol{\lambda})(M^{-1})^{\otimes N}, \qquad (3.54)$$

with M as defined in (3.31)*.*

Proof. The property (3.54) follows from Propositions 3.1.2 and 3.1.6. $\square$

From Proposition 3.1.15, we establish the diagonalisability of the transfer matrices in a certain regime.

Proposition 3.1.16. *For each $m = 1, 2$, real λ, η, p and $\lambda_1 = \cdots = \lambda_N = 0$, the transfer matrix $T^{(m)}(\lambda)$ is unitarily diagonalisable.*

Proof. In linear algebra, a matrix with complex-valued entries is said to be *normal* if it commutes with its (complex) conjugate transpose. Normal matrices are diagonalisable by means of a unitary transformation [102]. The idea of this proof is to show that, for $\lambda_1 = \cdots = \lambda_N = 0$, the symmetrised version of $T^{(m)}(\lambda)$ is normal. We make the reasoning explicit for $m = 1$, the other case is similar.

First, we consider Propositions 3.1.12 and 3.1.15, and the relation (3.36) to obtain

$$\begin{aligned}(W^{\otimes N} T^{(1)}(\lambda; \boldsymbol{\lambda})(W^{-1})^{\otimes N})^t \\ = (-1)^{\kappa+N} W^{\otimes N} T^{(1)}(-\lambda - 2\eta; -\boldsymbol{\lambda})(W^{-1})^{\otimes N}. \qquad (3.55)\end{aligned}$$

Using (3.55) with $\lambda_1 = \cdots = \lambda_N = 0$ and Proposition 3.1.14, we compute the following commutation relation:

$$\begin{aligned}
&[W^{\otimes N}T^{(1)}(\lambda;\mathbf{0})(W^{-1})^{\otimes N},(W^{\otimes N}T^{(1)}(\lambda;\mathbf{0})(W^{-1})^{\otimes N})^t] \\
&\quad = (-1)^{\kappa+N}W^{\otimes N}[T^{(1)}(\lambda;\mathbf{0}),T^{(1)}(-\lambda-2\eta;\mathbf{0})](W^{-1})^{\otimes N} \\
&\quad = 0.
\end{aligned} \tag{3.56}$$

This relation establishes that the symmetrised transfer matrix commutes with its transpose for $\lambda_1 = \cdots = \lambda_N = 0$.

Second, for real λ, η, p and λ_i, $i = 1, \ldots, N$, the entries of the symmetrised transfer matrix are either real numbers ($\kappa = 1, 3$), or pure imaginary numbers ($\kappa = 2$). Moreover, from their definition in (3.35), one deduces that $\beta_\pm$ are either real numbers, pure imaginary numbers, complex conjugates, or opposite thereof. Respectively, $W^* = W$, $W^* = W\Omega^3$, $W^* = -W\Omega^1$, or $W^* = -W\Omega^1\Omega^3$. By Proposition 3.1.12, we deduce that the symmetrised transfer matrix is equal to its complex conjugate, up to a sign. Therefore, for $\lambda_1 = \cdots = \lambda_N = 0$, it is normal, and unitarily diagonalisable. □

It is likely that Proposition 3.1.16 still holds for $\lambda_1, \ldots, \lambda_N$ sufficiently close to zero.

We now relate the two transfer matrices via the *fusion relation*.

Proposition 3.1.17. *We have*

$$T^{(1)}(\lambda)T^{(1)}(\lambda-\eta) = T^{(2)}(\lambda) + \mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta). \tag{3.57}$$

Proof. Let us consider

$$T^{(1)}(\lambda_0)T^{(1)}(\lambda_0-\eta) = \mathrm{tr}_{00'}\left[\mathrm{i}\sigma_0^\kappa \mathrm{i}\sigma_{0'}^\kappa \prod_{i=1}^{\overset{\curvearrowleft}{N}} R_{0i}^{(1,2)}(\lambda_{0i})R_{0'i}^{(1,2)}(\lambda_{0i}-\eta)\right].$$

We insert the identity operator $\mathbb{I}_{00'} = P_{00'}^{(1)-} + P_{00'}^{(1)+}$ inside the trace on the left. The linearity of the trace leads to two contributions, one for each projector. First, we transform the term associated to $P_{00'}^{(1)-}$. The successive use of the cyclic property of the trace, Lemma 3.1.5, the definitions (3.49), and the equality

$$\mathrm{tr}_{00'}\left[\mathrm{i}\sigma_0^\kappa \mathrm{i}\sigma_{0'}^\kappa P_{00'}^{(1)-}\right] = 1, \tag{3.58}$$

yields the equality

$$\mathrm{tr}_{00'}\left[P^{(1)-}_{00'}\mathrm{i}\sigma^\kappa_0\mathrm{i}\sigma^\kappa_{0'}\prod_{i=1}^{\overset{\curvearrowleft}{N}} R^{(1,2)}_{0i}(\lambda_{0i})R^{(1,2)}_{0'i}(\lambda_{0i}-\eta)\right] \\ = \mathrm{A}(\lambda_0)\mathrm{D}(\lambda_0-\eta). \quad (3.59)$$

Second, we work on the term associated to $P^{(1)+}_{00'}$. We observe that $[P^{(1)+}_{00'},\sigma^\kappa_0\sigma^\kappa_{0'}]=0$. Moreover, by the YBE,

$$P^{(1)+}_{00'}R^{(1,2)}_{0i}(\lambda)R^{(1,2)}_{0'i}(\lambda-\eta) = P^{(1)+}_{00'}R^{(1,2)}_{0i}(\lambda)R^{(1,2)}_{0'i}(\lambda-\eta)P^{(1)+}_{00'}. \quad (3.60)$$

We use this equation, along with the similarity invariance of the trace, to rewrite

$$\begin{aligned}&\mathrm{tr}_{00'}\left[P^{(1)+}_{00'}\mathrm{i}\sigma^\kappa_0\mathrm{i}\sigma^\kappa_{0'}\prod_{i=1}^{\overset{\curvearrowleft}{N}} R^{(1,2)}_{0i}(\lambda_{0i})R^{(1,2)}_{0'i}(\lambda_{0i}-\eta)\right] \quad (3.61)\\ &= \mathrm{tr}_{00'}\left[Q_{00'}U_{00'}\mathrm{i}\sigma^\kappa_0\mathrm{i}\sigma^\kappa_{0'}\prod_{i=1}^{\overset{\curvearrowleft}{N}} P^{(1)+}_{00'}R^{(1,2)}_{0i}(\lambda_{0i})R^{(1,2)}_{0'i}(\lambda_{0i}-\eta)U^{-1}_{00'}Q^t_{00'}\right],\end{aligned}$$

where Q and U are defined in (3.23) and (3.21), respectively. Next, the definition of $R^{(2,2)}(\lambda)$ and the relation

$$QU\,(\mathrm{i}\sigma^\kappa\otimes\mathrm{i}\sigma^\kappa)\,U^{-1}Q^t=\Omega^\kappa, \quad (3.62)$$

lead to the equality

$$\mathrm{tr}_{00'}\left[P^{(1)+}_{00'}\mathrm{i}\sigma^\kappa_0\mathrm{i}\sigma^\kappa_{0'}\prod_{i=1}^{\overset{\curvearrowleft}{N}} R^{(1,2)}_{0i}(\lambda_{0i})R^{(1,2)}_{0'i}(\lambda_{0i}-\eta)\right] = T^{(2)}(\lambda_0). \quad (3.63)$$

Combining (3.59) with (3.63) establishes the proposition. □

Proposition 3.1.17 indicates that one can solve the eigenvalue problem of $T^{(2)}(\lambda)$ by working on that of $T^{(1)}(\lambda)$ instead.

Simple eigenvalues We now use the properties derived above to prove the existence of remarkably simple eigenvalues for the transfer matrices.

Proposition 3.1.18. *When diagonalisable, the transfer matrices $T^{(1)}(\lambda)$ and $T^{(2)}(\lambda)$ possess the respective eigenvalues*

$$\theta^{(1)}(\lambda) = 0, \qquad \textit{and} \qquad \theta^{(2)}(\lambda) = -\mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta). \tag{3.64}$$

Proof. Let $|\psi\rangle$ be an eigenvector of $T^{(1)}(\lambda)$ with eigenvalue $\theta(\lambda)$. By Proposition 3.1.12, we have

$$T^{(1)}(\lambda)(\Omega^a)^{\otimes N}|\psi\rangle = -\theta(\lambda)(\Omega^a)^{\otimes N}|\psi\rangle\,, \tag{3.65}$$

for each $a = 1, 2, 3$, $a \neq \kappa$. Hence, $-\theta(\lambda)$ is an eigenvalue of the transfer matrix as well. The number of non-zero eigenvalues is thus even. However, $\dim(\mathbb{C}^3)^{\otimes N} = 3^N$ is always odd. Therefore, $\theta^{(1)}(\lambda) = 0$ must be an eigenvalue of $T^{(1)}(\lambda)$ with odd degeneracy. From Proposition 3.1.17, we deduce that $\theta^{(2)}(\lambda) = -\mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta)$ must be an eigenvalue of $T^{(2)}(\lambda)$. □

Proposition 3.1.19. *For each $a = 1, 2, 3$, $a \neq \kappa$, and diagonalisable transfer matrices, the non-zero part of the spectrum of $T^{(2)}(\lambda)$ restricted to the subsectors where $(\Omega^a)^{\otimes N} \equiv 1$ and $(\Omega^a)^{\otimes N} \equiv -1$ coincide, including degeneracies.*

Proof. Let $|\psi\rangle$ be an eigenvector of $T^{(1)}(\lambda)$ with eigenvalue $\theta(\lambda) \neq 0$. By Proposition 3.1.12, $(\Omega^a)^{\otimes N}|\psi\rangle$ is an eigenvector of $T^{(1)}(\lambda)$ with eigenvalue $-\theta(\lambda) \neq 0$. From Proposition 3.1.17, we have that $|\psi\rangle$ and $(\Omega^a)^{\otimes N}|\psi\rangle$ are both eigenvectors of $T^{(2)}(\lambda)$ with same eigenvalue. This is also the case for the linear combination $|\psi_\pm\rangle = (1 \pm (\Omega^a)^{\otimes N})|\psi\rangle/2$. Moreover, the $|\psi_\pm\rangle$ belong to the eigenspaces of $(\Omega^a)^{\otimes N}$ of eigenvalue ± 1, respectively. □

3.2 The spin-1 XYZ chain

Like its lower-spin counterpart, the 41V model is related to a one-dimensional integrable quantum spin chain. In this case, it is the spin-1 XYZ chain, first described by Fateev [91]. In this section, we are interested in solving the eigenvalue problem of its Hamiltonian. We present preliminary our results in this regard. We prove the existence of the remarkably simple eigenvalue $E = 0$. Furthermore, we examine the related eigenvector, which we conjecture to be the ground-state eigenvalue for most system parameters.

In Section 3.2.1, we define the spin-1 XYZ Hamiltonian. We present its relation to the 41V transfer matrix, and some of its symmetries. In Section 3.2.2, we point out properties of its spectrum, including a conjecture regarding the degeneracy of the eigenvalue $E = 0$ and its position within the spectrum.

3.2.1 The Hamiltonian

Definition We study the Hamiltonian of a spin-1 XYZ chain [90, 91]:

$$H = \sum_{i=1}^{N} \sum_{b=1,2,3} J_b \left(s_i^b s_{i+1}^b + \left(s_i^b\right)^2 + \left(s_{i+1}^b\right)^2 \right) - \sum_{b,c=1,2,3} A_{bc} s_i^b s_i^c s_{i+1}^b s_{i+1}^c. \quad (3.66)$$

The anisotropy parameters J_b and A_{bc} obey the relations

$$A_{bc} = A_{cb}, \qquad A_{bb} = J_b, \qquad b, c = 1, 2, 3. \quad (3.67)$$

We parameterise them in terms of two real parameters, denoted by X and Y,

$$J_1 = 1 + XY, \qquad A_{12} = 1 - Y^2, \quad (3.68a)$$
$$J_2 = 1 - XY, \qquad A_{13} = (X-1)(1+Y), \quad (3.68b)$$
$$J_3 = -1 + (X^2 + Y^2)/2, \qquad A_{23} = (X-1)(1-Y). \quad (3.68c)$$

The choices (3.67) and (3.68) make the spin chain integrable. We note that the limit $Y = 0$ recovers the spin-1 XXZ chain [90, 92].

We are interested in a spin chain with quasi-periodic boundary conditions. We impose these by specifying how s_{N+1}^b relates to s_1^b,

$$s_{N+1}^b = \Omega_1 \, s_1^b \, \Omega_1, \qquad b = 1, 2, 3, \quad (3.69)$$

with $\Omega = \Omega^\kappa$, the spin-1 twists defined in (3.10), $\kappa = 1, 2, 3$.

We note that for real X, Y, and for each twist, the Hamiltonian is Hermitian. It is thus diagonalisable with real eigenvalues.

Relation to the 41V model In this paragraph, we express the spin-1 XYZ Hamiltonian as a logarithmic derivative of the transfer matrix of the 41V model.

As a first step, we write the logarithmic derivative of the 41V transfer matrix as a sum of local operators. We recall the definition $\check{R}^{(2,2)}(\lambda) = \mathcal{P}^{(2)} R^{(2,2)}(\lambda)$, with $\mathcal{P}^{(2)}$ the permutation operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Lemma 3.2.1. *We have*

$$\left(T^{(2)\prime}(\lambda) T^{(2)}(\lambda)^{-1}\right)\Big|_{\substack{\lambda=0\\ \lambda_i=0}} = (\vartheta_1(\eta)\vartheta_1(2\eta))^{-1} \left(\sum_{i=1}^{N-1} \check{R}^{(2,2)\prime}_{ii+1}(0) + \Omega_1^\kappa \check{R}^{(2,2)\prime}_{N1}(0)\Omega_1^\kappa\right), \quad (3.70)$$

where the $'$ denotes the derivation with respect to λ.

Proof. First, we use Proposition 3.1.8 and the relation $\mathrm{tr}_0\, \mathcal{P}^{(2)}_{0j} = \mathbb{I}_j$ to compute

$$\left(T^{(2)}(\lambda)^{-1}\right)\Big|_{\substack{\lambda=0\\ \lambda_i=0}} = (\vartheta_1(\eta)\vartheta_1(2\eta))^{-N} \mathcal{P}^{(2)}_{12}\mathcal{P}^{(2)}_{13}\dots\mathcal{P}^{(2)}_{1N}\Omega_1^\kappa. \quad (3.71)$$

Second, we use the product rule to derivate the transfer matrix. We obtain

$$\left(T^{(2)\prime}(\lambda)\right)\Big|_{\substack{\lambda=0\\ \lambda_i=0}} = (\vartheta_1(\eta)\vartheta_1(2\eta))^{N-1} \Bigg(\Omega_1^\kappa R^{(2,2)\prime}_{1N}(0)\mathcal{P}^{(2)}_{1N-1}\dots\mathcal{P}^{(2)}_{13}\mathcal{P}^{(2)}_{12} + \sum_{i=1}^{N-1} R^{(2,2)\prime}_{ii+1}(0)\Omega_1^\kappa \mathcal{P}^{(2)}_{1N}\dots\mathcal{P}^{(2)}_{1i+1}\mathcal{P}^{(2)}_{1i-1}\dots\mathcal{P}^{(2)}_{12}\Bigg). \quad (3.72)$$

We multiply (3.71) by (3.72), which yields (3.70). □

As a second step, we rewrite the Hamiltonian as

$$H = \sum_{i=1}^{N-1} h_{ii+1} + \Omega_1^\kappa h_{N1} \Omega_1^\kappa, \quad (3.73)$$

where h_{ii+1} is the *Hamiltonian density*

$$h = \sum_{b=1,2,3} J_b \left(s^b \otimes s^b + \left(s^b \otimes \mathbb{I}\right)^2 + \left(\mathbb{I} \otimes s^b\right)^2\right) - \sum_{b,c=1,2,3} A_{bc} s^b s^c \otimes s^b s^c, \quad (3.74)$$

acting on sites i, $i+1$. We then show that the symmetrised $\check{R}^{(2,2)\prime}(0)$ commutes with the Hamiltonian density.

Lemma 3.2.2. *For real η, p, and X, Y as defined in* (3.17), *we have,*

$$(\vartheta_1(\eta)\vartheta_1(2\eta))^{-1} W_1 W_2 \check{R}^{(2,2)\prime}_{12}(0) W_1^{-1} W_2^{-1} = \nu h_{12}(X,Y) - (\nu - \mu),$$

where

$$\nu = \frac{\hat{\vartheta}_1'}{\hat{\vartheta}_4} \frac{\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_1(2\eta)}, \qquad \mu = \frac{\hat{\vartheta}_4'(2\eta)}{\hat{\vartheta}_4(2\eta)}. \tag{3.75}$$

Proof. The proof relies on the explicit computation of the left- and right-hand sides. We provide the entries of $R^{(2,2)\prime}(0)$ as a substep:

$$A'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)^2\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_1', \qquad \bar{A}'(0) = 0, \tag{3.76a}$$

$$B'(0) = 0, \qquad \bar{B}'(0) = \alpha^{-2}\hat{\vartheta}_1(\eta)\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)^2\hat{\vartheta}_1', \tag{3.76b}$$

$$C'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4^4\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)\left(\frac{\hat{\vartheta}_1'(\eta)}{\hat{\vartheta}_1(\eta)} + \frac{\hat{\vartheta}_4'(\eta)}{\hat{\vartheta}_4(\eta)}\right), \tag{3.76c}$$

$$\bar{C}'(0) = 0, \qquad D'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)^2\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_1', \tag{3.76d}$$

$$E'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4^4\hat{\vartheta}_4(\eta)\hat{\vartheta}_4'(2\eta) + F'(0), \tag{3.76e}$$

$$F'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)^2\hat{\vartheta}_1', \tag{3.76f}$$

$$\begin{aligned} G'(0) = {} & \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4\hat{\vartheta}_4(\eta)[\hat{\vartheta}_1(\eta)^4 + \hat{\vartheta}_4(\eta)^4]\left(\frac{\hat{\vartheta}_1'(\eta)}{\hat{\vartheta}_1(\eta)} - \frac{\hat{\vartheta}_4'(\eta)}{\hat{\vartheta}_4(\eta)}\right) \\ & + \hat{\vartheta}_1(\eta)\hat{\vartheta}_4^4\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)\hat{\vartheta}_1'(2\eta), \end{aligned} \tag{3.76g}$$

$$\bar{G}'(0) = -\hat{\vartheta}_1(\eta)\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_4(2\eta)^2\hat{\vartheta}_1', \tag{3.76h}$$

$$H'(0) = 0, \qquad \bar{H}'(0) = \hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)\hat{\vartheta}_4^4\hat{\vartheta}_4(\eta)\hat{\vartheta}_4'(2\eta), \tag{3.76i}$$

$$I'(0) = 2\alpha^2\hat{\vartheta}_1(\eta)\hat{\vartheta}_1(2\eta)^2\hat{\vartheta}_4^3\hat{\vartheta}_4(\eta)\hat{\vartheta}_1', \tag{3.76j}$$

$$\bar{I}'(0) = \alpha^2\frac{\hat{\vartheta}_1(2\eta)^2\hat{\vartheta}_4^3}{\hat{\vartheta}_1(\eta)\hat{\vartheta}_4(\eta)}\left[\hat{\vartheta}_1(\eta)^4 + \hat{\vartheta}_4(\eta)^4\right]\hat{\vartheta}_1'. \tag{3.76k}$$

The rest of the computation is straightforward albeit tedious. □

Finally, Lemmas 3.2.1 and 3.2.2 lead to the following:

Proposition 3.2.3. *For real η, p, and X, Y as defined in* (3.17)*, we have,*

$$W^{\otimes N}\left(T^{(2)\prime}(\lambda)T^{(2)}(\lambda)^{-1}\right)\Big|_{\substack{\lambda=0\\ \lambda_i=0}}(W^{-1})^{\otimes N} = \nu H - N(\nu-\mu), \tag{3.77}$$

where W and ν, μ are defined in (3.34) *and* (3.75)*, respectively.*

Proposition 3.2.3 indicates that it is possible to solve the eigenvalue problem of the spin-1 XYZ Hamiltonian by considering that of the 41V (or 18V) transfer matrix instead.

Symmetries The XYZ Hamiltonian enjoys several symmetry properties useful to our work. We detail them in this paragraph. The first one is a $\mathbb{Z}_2$ symmetry.

Proposition 3.2.4. *For each $a = 1, 2, 3$, we have*

$$[H, (\Omega^a)^{\otimes N}] = 0. \tag{3.78}$$

To state the second symmetry, we define $E^a = \exp(\mathrm{i}\pi s^a/2)$, $a = 1, 2, 3$. This unitary operator implements a rotation of angle $\pi/2$ in the single spin space $\mathbb{C}^3$. We now show that the conjugation of the Hamiltonian with $(E^a)^{\otimes N}$ changes its twist and its parameters X, Y. The transformations on the parameters are

$$\varphi_1(X,Y) = \left(\frac{2+X-Y}{X+Y}, \frac{2-X+Y}{X+Y}\right), \tag{3.79a}$$

$$\varphi_2(X,Y) = \left(\frac{2+X+Y}{X-Y}, \frac{-2+X+Y}{X-Y}\right), \tag{3.79b}$$

$$\varphi_3(X,Y) = (X, -Y). \tag{3.79c}$$

We also define the functions

$$f_1(X,Y) = \frac{(X+Y)^2}{4}, \quad f_2(X,Y) = \frac{(X-Y)^2}{4}, \quad f_3(X,Y) = 1. \tag{3.80}$$

We highlight the dependence of the Hamiltonian on the parameters X, Y and the twist Ω^κ, by writing $H = H^\kappa(X,Y)$, $\kappa = 1, 2, 3$.

Proposition 3.2.5. *For each $a, b = 1, 2, 3$, we have*

$$(E^b)^{\otimes N} H^a(X,Y)((E^b)^{-1})^{\otimes N} = f_b(X,Y) H^c(\varphi_b(X,Y)), \tag{3.81}$$

where either $a = b = c$, or $\{a, b, c\} = \{1, 2, 3\}$.

Proof. We recall the rewriting of the Hamiltonian in terms of the Hamiltonian density $h = h(X, Y)$ (3.73). First, we check through direct computation the transformation:

$$(E^b)^{\otimes 2} h(X,Y)((E^b)^{-1})^{\otimes 2} = f_b(X,Y) h(\varphi_b(X,Y)). \tag{3.82}$$

Second, we consider the commutation relation $E^b \Omega^a (E^b)^{-1} = \Omega^c$, where c is defined as in the statement of the proposition. With these two considerations, we establish:

$$\begin{aligned}(E^b)^{\otimes 2}(\mathbb{I} \otimes \Omega^a) h(X,Y)(\mathbb{I} \otimes \Omega^a)((E^b)^{-1})^{\otimes 2}& \\ = f_b(X,Y)(\mathbb{I} \otimes \Omega^c) h(\varphi_b(X,Y))(\mathbb{I} \otimes \Omega^c).&\end{aligned} \tag{3.83}$$

We combine the decomposition (3.73) with (3.82) and (3.83) to obtain (3.81). □

We introduce a fourth unitary transformation

$$E^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\pi/4} & 0 & \mathrm{e}^{-\mathrm{i}\pi/4} \\ 0 & \sqrt{2} & 0 \\ \mathrm{e}^{-\mathrm{i}\pi/4} & 0 & \mathrm{e}^{\mathrm{i}\pi/4} \end{pmatrix}. \tag{3.84}$$

In the next proposition, we show that the conjugation of the Hamiltonian with $(E^0)^{\otimes N}$ exchanges the X, Y parameters, with $\varphi_0(X, Y) = (Y, X)$.

Proposition 3.2.6. *For each $a = 1, 2, 3$, we have*

$$(E^0)^{\otimes N} H^a(X,Y)((E^0)^{-1})^{\otimes N} = H^a(\varphi_0(X,Y)). \tag{3.85}$$

Proof. The transformation (3.85) follows from

$$(E^0)^{\otimes 2} h(X,Y)((E^0)^{-1})^{\otimes 2} = h(\varphi_0(X,Y)), \tag{3.86}$$

and the commutation relation $\Omega^a E^0 = E^0 \Omega^a$, for $a = 1, 2, 3$. □

The set $\{\varphi_i\}_{i=0}^3$ generates a group of point transformations on the XY-plane. We define the fundamental domain

$$\Delta = \left\{(X,Y) \in \mathbb{R}^2 \middle| 0 \leq X \leq 2,\, 0 \leq Y \leq \min(X, 2-X)\right\}. \tag{3.87}$$

Elementary geometric arguments lead to:

Lemma 3.2.7. *The image of Δ under the group generated by $\{\varphi_i\}_{i=0}^3$ is $\overline{\mathbb{R}}^2$.*

Therefore, Lemma 3.2.7 presents us with two options to simplify our investigation of the eigenvalue problem of H. First, we can study the Hamiltonian with $(X,Y) \in \Delta$ for the three twists. We obtain any result outside this domain from Propositions 3.2.5 and 3.2.6. Second, we can consider the Hamiltonian for all values of $X, Y \in \mathbb{R}$ but focus on a single twist. We obtain results for the other twists from Proposition 3.2.5.

Moreover, we observe that the fundamental domain Δ is in the image of the parameterisation of X, Y (3.17).

Conjecture 3.2.8. *The application*

$$[0, \pi/2] \times [0, 1] \to \Delta, \qquad (\eta, p) \mapsto (X, Y), \tag{3.88}$$

is a surjection. Here, Δ is (3.87) *and X, Y are as in* (3.17)*.*

3.2.2 The spectrum

We start the analysis of the spectrum with the existence of a remarkably simple eigenvalue.

Theorem 3.2.9. *The Hamiltonian H possesses the eigenvalue $E = 0$.*

Proof. We exploit the relation between the Hamiltonian and the 41V transfer matrix. By Propositions 3.1.14, 3.1.16 and 3.1.18, for values of λ, η, p and $\lambda_1, \ldots, \lambda_N$ for which the transfer matrix is diagonalisable, there exists a non-zero $|\psi^R\rangle$ independent of λ such that

$$T^{(2)}(\lambda)\,|\psi^R\rangle = \theta^{(2)}(\lambda)\,|\psi^R\rangle\,. \tag{3.89}$$

For $\lambda \not\equiv -2\eta, \eta \pmod{\pi, \pi\omega}$, $|\psi^R\rangle$ is also an eigenvector of $T^{(2)\prime}(\lambda)$ and $T^{(2)}(\lambda)^{-1}$ with eigenvalue $\theta^{(2)\prime}(\lambda)$ and $\theta^{(2)}(\lambda)^{-1}$, respectively. We compute

$$\left.\frac{\theta^{(2)\prime}(\lambda)}{\theta^{(2)}(\lambda)}\right|_{\substack{\lambda=0\\ \lambda_i=0}} = -N(\nu-\mu). \tag{3.90}$$

By Proposition 3.2.3, we deduce that $W^{\otimes N}\,|\psi^R\rangle\big|_{\lambda_i=0}$ is an eigenvector of H with eigenvalue $E=0$, for X, Y as in (3.17). By Conjecture 3.2.8, this holds for all $X, Y \in \mathbb{R}$. □

Now, we turn to the non-zero part of the spectrum. The $\mathbb{Z}_2$ symmetries of Proposition 3.2.4 indicate that it is possible to diagonalise the Hamiltonian in eigenspaces of the $\mathbb{Z}_2$ operators.

Proposition 3.2.10. *For each $a = 1, 2, 3$, with $a \neq \kappa$, the non-zero part of the spectrum of H restricted to the subsectors where $(\Omega^a)^{\otimes N} \equiv 1$ and $(\Omega^a)^{\otimes N} \equiv -1$ coincide, including degeneracies.*

Proof. The proposition follows from Propositions 3.1.19 and 3.2.3 and the commutation relation $[W, \Omega^b] = 0$, $b = 1, 2, 3$. □

Next, using MATHEMATICA, we investigated the spectrum of the Hamiltonian with diagonal twist. We considered system sizes $N = 1, 2, \ldots, 5$, for several fixed values of $Y \in [0, 10]$. In Figure 3.5, see examples for $N = 3$ and $Y = 1/2, 3/2$. Based on our observations, we formulate the next conjecture, which generalises a conjecture issued for the spin-1 XXZ chain by Hagendorf and Morin-Duchesne [92].

Conjecture 3.2.11. *For each $N = 2n$, and $X, Y \in \mathbb{R}$, $E = 0$ is the ground-state eigenvalue of the Hamiltonian with diagonal twist. It is non-degenerate except for $(X, Y) = (0, 0)$, where it has degeneracy $N + 1$.*

For each $N = 2n + 1$, $n \geq 1$, and $X^2 + Y^2 \geq 2$ or $X^2 = Y^2$, $E = 0$ is the ground-state eigenvalue of the Hamiltonian with diagonal twist. In the rest of the XY-plane, it is the energy of the first excited state. The eigenvalue is non-degenerate except in the two following cases:

(i) for $(X, Y) = (0, 0)$, its degeneracy is $2N + 1$;
(ii) for $X^2 = Y^2$ or $X^2 + Y^2 = 2$, its degeneracy is 3.

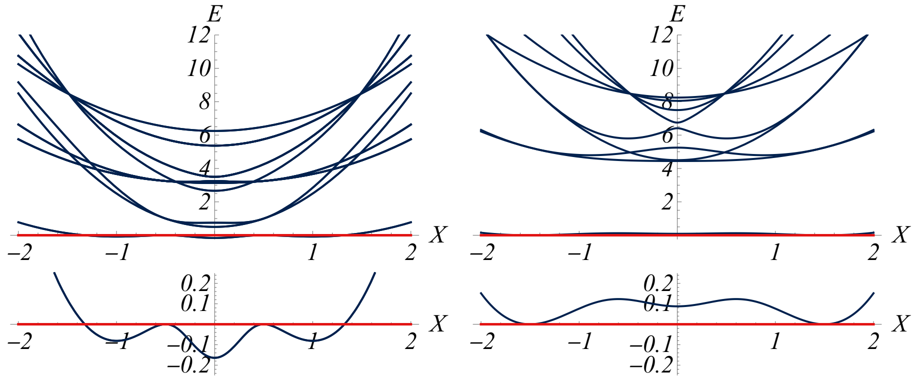


Figure 3.5: Spectrum of the Hamiltonian with diagonal twist for size $N = 3$, with fixed values of Y. On the left, $Y = 1/2$, on the right, $Y = 3/2$. On the bottom, magnification of the spectrum for $E \in [-0.2, 0.2]$.

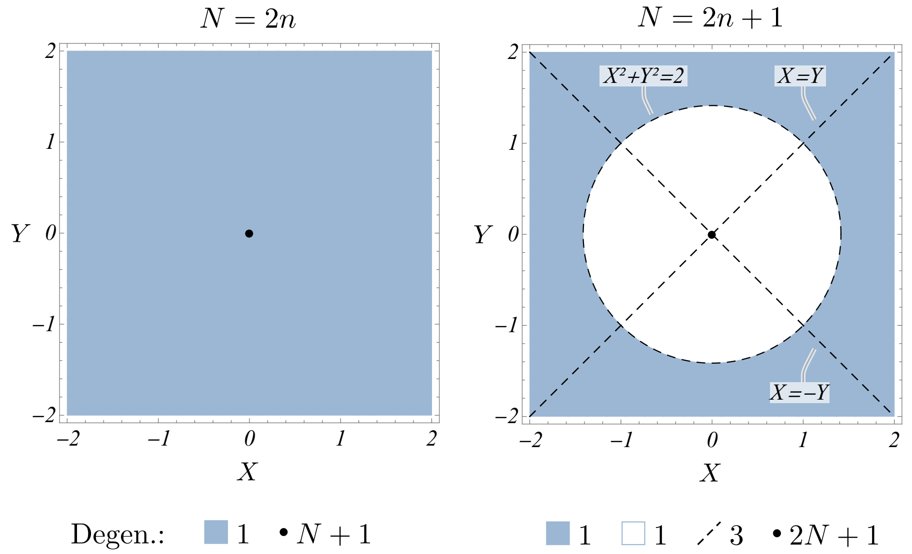


Figure 3.6: Degeneracies observed for the $E = 0$ eigenvalue of the Hamiltonian with diagonal twist in the XY-plane. In the coloured region, $E = 0$ appears to be the ground-state eigenvalue. In the white region, it appears to be the energy of the first excited state.

Figure 3.6 provides a graphical summary of Conjecture 3.2.11.

The equivalent of Conjecture 3.2.11 for the anti-diagonal and mixed twists follows from Proposition 3.2.5.

Chapter 4

Dynamical vertex models

Historically, the elliptic integrable models that we have described so far have been more difficult to treat within the QISM framework than their trigonometric counterparts. For instance, the 8V model has notoriously eluded the algebraic Bethe Ansatz, as it lacks the global reference state necessary for its application [19].

To overcome this issue, Baxter related the 8V model to another model of statistical physics, the *eight-vertex solid-on-solid (8VSOS) model* [45, 48]. The latter belongs to the class of *interaction-round-a-face (IRF)*, or *face models* [101]. Its configurations are the assignment of real numbers, called *heights*, to each face of the square lattice. Its interactions are local, and take place between the four faces surrounding each vertex. The 8VSOS model has the advantage of being more readily accessible to the standard techniques of integrability. The relation between 8V and 8VSOS models was dubbed *vertex-IRF* or *vertex-face correspondence*. Thanks to this correspondence, Baxter managed to compute the eigenvectors of the periodic 8V model, for an even number of vertical lines $N = 2n$. His proofs relied on the Bethe Ansatz analysis of the 8VSOS model [45, 48].

Many investigations around the 8V and 8VSOS models followed in the wake of Baxter's work [131, 132, 135, 140], most often under the need of some constraints. For instance, Takhtadzhan and Faddeev developped the *generalised Bethe Ansatz* [32, 133, 134]. This Bethe Ansatz variant integrated the idea of the vertex-face correspondence as a built-in feature to systematically find a family of valid reference states. Their technique has been effective on the 8V model for periodic boundary conditions and $N = 2n$. Another example of constrained cases for which exact computations could be performed are height-restricted SOS models. These include the *restricted SOS (RSOS)* or *ABF models*, and *cyclic SOS (CSOS) models*, which have notably found connections to conformal field theory [141–146].

Furthermore, the 8VSOS model was shown to be equivalent to a *dynamical six-vertex (d6V) model* [45–47]. It could therefore be studied using the algebraic formalism of vertex models [12, 18, 19, 147]. Namely, it was associated to an R-matrix, solution to a *dynamical Yang-Baxter equation* [46, 148]. Since we use this vertex formalism in our work below, we favour the terminology *dynamical six-vertex model*.

Recently, Niccoli, Terras and Levy-Bencheton have successfully applied Sklyanin's *quantum Separation of Variables (qSoV)* method to analyse the d6V model with anti-periodic boundary conditions [58, 59, 94]. Using the vertex-face correspondence, they managed to characterise the complete set of eigenvalues and eigenvectors of the 8V model under several (quasi-) periodic boundary conditions. In particular, their results included the case of periodic boundary conditions with odd system sizes $N = 2n + 1$, which had long resisted other methods.

In Chapters 4 and 5, we investigate a *dynamical nineteen-vertex (d19V) model*, with the objective of transposing our findings back to the 41V and 18V models. The vertex weights of the d19V model are defined through a solution to the dynamical Yang-Baxter equation. This solution can be obtained by applying the fusion procedure to the d6V model [89]. The fusion procedure naturally leads to an intermediate *dynamical ten-vertex (d10V) model*, which will be at the core of our investigation. We consider the transfer matrices of both models with *anti-periodic boundary conditions* along the horizontal direction. For these boundary conditions, the two transfer matrices mutually commute. In Chapter 5, we generalise Niccoli and Terras' qSoV approach in order to characterise their spectra and compute their common eigenvectors [58, 59, 94].

The goal of this chapter is twofold. On the one hand, we introduce the models, and lay the foundations necessary to our qSoV analysis. On the other hand, we describe the vertex-face correspondence between the d10V and the 18V models. Here is the structure we follow: we begin Section 4.1

with a review of the d6V model and its algebraic description. Moreover, we recall how to apply the fusion procedure to this model, which allows us to construct the vertex weights of the d10V and d19V models. In particular, we discuss in detail the properties of the monodromy matrix of the d10V as an operator on the *spin-dynamical space*. The latter is an infinite-dimensional $\mathbb{Z}$-graded space. On one of its subspaces, we construct a family of commuting transfer matrices for both the d10V and the d19V model with anti-periodic boundary conditions. In Section 4.2, we present the vertex-face correspondence between the 8V and d6V models. We apply the fusion procedure to find its equivalent for the higher-spin models of Chapters 3 and 4. Lastly, we use it to relate the transfer matrices of the d10V and 18V models.

4.1 Dynamical ten- and nineteen-vertex models

The purpose of this section is to recall the definition of the dynamical ten and nineteen-vertex models. Section 4.1.1 is a reminder of the dynamical six-vertex model. In particular, its algebraic description allows us to introduce key notations and conventions we use throughout the chapter. In Section 4.1.2, we apply the fusion procedure to the d6V model to construct the R-matrices of the d10V and d19V models, whose properties we analyse in some detail. Section 4.1.3 introduces the so-called dynamical space, which allows us to define the monodromy matrix of the ten-vertex model. We devote much of the section to the study of their properties, which will be crucial to our qSoV analysis of Chapter 5. We define these transfer matrices and analyse their properties in Section 4.1.4.

4.1.1 The dynamical six-vertex model

Lattice formulation The dynamical six-vertex model is equivalent to the eight-vertex solid-on-solid model [45–47]. The latter belongs to the class of *interaction-round-a-face (IRF)*, or *face models*.[1] We consider this face model on the square lattice. Its configurations are the assignment of real numbers, called *heights*, to each face of the lattice. The interactions

[1]See [101] for a very pedagogical introduction.

are local. They are controlled by the rule that heights on two adjacent faces t, t' differ by a fixed step value $\eta > 0$,

$$|t - t'| = \eta. \tag{4.1}$$

We refer to the collection of heights on the four adjacent faces surrounding any given vertex as a *local height configuration*. Fixing the height in the upper left corner to t, the rule (4.1) allows for six local height configurations (see Figure 4.1). We associate a statistical weight to each of them, which typically depends on t [45, 101]. The weight of a configuration is the product of the statistical weights of the local height configurations within it.

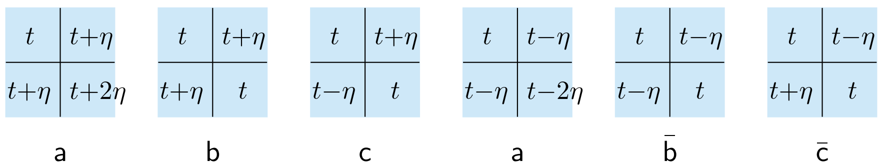
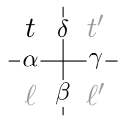


Figure 4.1: Local height configurations of the 8VSOS model and their statistical weights.

Baxter noticed that the condition (4.1) allows one to map each local configuration of the 8VSOS to a vertex of the six-vertex (6V) model [45]. The mapping is constructed by moving clockwise around a vertex. If the height increases from one face to the next, the edge in between is assigned an outgoing arrow. Conversely, if the height decreases, the edge is assigned an ingoing arrow. See Figure 4.2 with $\alpha, \beta, \gamma, \delta \in \{\pm 1\}$ for a graphical depiction. The resulting vertex configurations can be read on Figure 4.3. This correspondence is a bijection, provided that we keep track of one height, which we choose on the upper-left face. The resulting six-vertex model is called *dynamical* because of the possible height-dependence of the vertex weights. We depict an example of 8VSOS configuration and its d6V image in Figure 4.4.

Finally, note that (4.1) implies that all heights belong in $\ell_0 + \eta\mathbb{Z}$. Here, ℓ_0 is a reference height, which we allow to be complex. We will set its value below.



$$
\begin{aligned}
t' &= t + \eta\delta,\\
\ell' &= t + \eta(\delta+\gamma) = t + \eta(\alpha+\beta),\\
\ell &= t + \eta\alpha.
\end{aligned}
$$

Figure 4.2: Correspondence between local height configurations and vertices. The coherence between the two values for ℓ' enforces the *ice rule*, $\alpha + \beta = \delta + \gamma$.

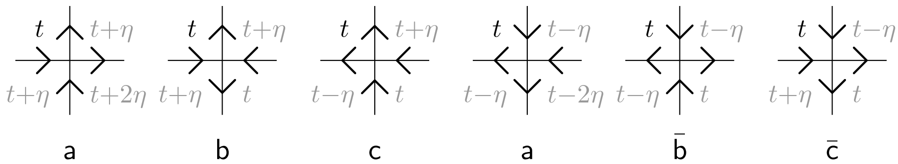


Figure 4.3: Local configurations of the d6V model and their statistical weights. In grey, equivalent local height configurations of the 8VSOS.

0	1	2	3	2	1
1	2	1	2	1	0
2	1	2	1	0	−1
3	2	1	0	−1	−2

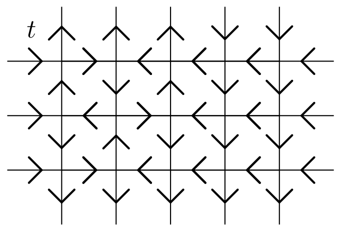


Figure 4.4: *Left:* A configuration of the 8VSOS model in terms of face labels. The height associated to a label a is $t + a\eta$. *Right:* The corresponding configuration of the d6V model.

Algebraic description The correspondence between the local height configurations and the vertex configurations of the six-vertex model, decorated by a height, allows us to work with the algebraic description that is customary to vertex models [12, 18, 19, 147]. To this end, we follow the notations and conventions presented in Section 1.2.1.

We encode the statistical weights of the d6V model in an R-matrix. In the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$, it reads

$$\mathsf{R}^{(1,1)} = \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 \\ 0 & \mathsf{b} & \mathsf{c} & 0 \\ 0 & \bar{\mathsf{c}} & \bar{\mathsf{b}} & 0 \\ 0 & 0 & 0 & \mathsf{a} \end{pmatrix}. \tag{4.2}$$

We express the weights $\mathsf{a}, \mathsf{b}, \bar{\mathsf{b}}, \mathsf{c}, \bar{\mathsf{c}}$ in terms of the Jacobi theta function $\vartheta_1(\lambda) = \vartheta_1(\lambda, p)$.[2] Our parameterisation depends on the *spectral parameter* λ, the *height parameter* t, the *crossing parameter* η, the *elliptic nome* $p = \mathrm{e}^{\mathrm{i}\pi\omega}$, and $\mathsf{y} \in \{0, 1\}$. Throughout, we allow all the parameters to be complex numbers, with the constraints

$$t \not\equiv 0\,(\operatorname{mod} \pi, \pi\omega), \qquad 2\eta \not\equiv 0\,(\operatorname{mod} \pi, \pi\omega), \qquad \operatorname{Im}\omega > 0. \tag{4.3}$$

The parameterisation is

$$\mathsf{a}(\lambda) = \vartheta_1(\lambda + \eta), \qquad \bar{\mathsf{a}}(\lambda) = \mathsf{a}(\lambda), \tag{4.4a}$$

$$\mathsf{b}(\lambda|t) = \mathrm{e}^{\mathrm{i}\mathsf{y}\eta}\vartheta_1(\lambda)\frac{\vartheta_1(t+\eta)}{\vartheta_1(t)}, \quad \bar{\mathsf{b}}(\lambda|t) = \mathrm{e}^{-\mathrm{i}\mathsf{y}\eta}\vartheta_1(\lambda)\frac{\vartheta_1(t-\eta)}{\vartheta_1(t)}, \tag{4.4b}$$

$$\mathsf{c}(\lambda|t) = \mathrm{e}^{\mathrm{i}\mathsf{y}\lambda}\vartheta_1(\eta)\frac{\vartheta_1(t+\lambda)}{\vartheta_1(t)}, \quad \bar{\mathsf{c}}(\lambda|t) = \mathrm{e}^{-\mathrm{i}\mathsf{y}\lambda}\vartheta_1(\eta)\frac{\vartheta_1(t-\lambda)}{\vartheta_1(t)}, \tag{4.4c}$$

Note that, by the parity of the Jacobi theta functions, the weights $\bar{\mathsf{b}}(\lambda|t)$, $\bar{\mathsf{c}}(\lambda|t)$ follow from $\mathsf{b}(\lambda|t)$, $\mathsf{c}(\lambda|t)$ through the substitution $t \to -t, \mathsf{y} \to -\mathsf{y}$. Throughout, we write $\mathsf{R}^{(1,1)} = \mathsf{R}^{(1,1)}(\lambda|t)$.

Properties of the R-matrix In this section, we provide several useful properties of the R-matrix. To this end, we introduce some notations. First, recall that, in the standard tensor-leg notation, we write

$$\mathsf{R}^{(1,1)}_{i,i+1}(\lambda|t) = \mathbb{1}^{\otimes(i-1)} \otimes \mathsf{R}^{(1,1)}(\lambda|t) \otimes \mathbb{1}^{\otimes(N-i-1)}, \tag{4.5}$$

for the action of $\mathsf{R}^{(1,1)}(\lambda|t)$ on the factors i and $i+1$ of $(\mathbb{C}^2)^{\otimes N}$, $i = 1, \dots, N-1$. Second, let $p^{\pm} = (\mathbb{1} \pm \sigma^3)/2$ be the projectors on the canonical basis vectors of $\mathbb{C}^2$. We write

$$\mathsf{R}^{(1,1)}_{12}(\lambda|t + \eta\sigma^3_3) = \mathsf{R}^{(1,1)}_{12}(\lambda|t+\eta)p^+_3 + \mathsf{R}^{(1,1)}_{12}(\lambda|t-\eta)p^-_3 \tag{4.6}$$

[2]See Appendix A.1 for definitions and useful properties. Throughout our text, we often omit substeps of computations that involve identities between Jacobi theta functions.

for the action of the R-matrix on the first two factors of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, with a *dynamical shift* of the height parameter that depends on the third factor.

The R-matrix $\mathsf{R}^{(1,1)}(\lambda|t)$ fulfils the *dynamical Yang-Baxter equation (dYBE)* on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ [45, 46, 148].

Proposition 4.1.1 (Dynamical Yang-Baxter equation)**.** *We have,*

$$\begin{aligned}\mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t+\eta\sigma_3^3)\mathsf{R}^{(1,1)}_{13}(\lambda_{13}|t)\mathsf{R}^{(1,1)}_{23}(\lambda_{23}|t+\eta\sigma_1^3)\\ = \mathsf{R}^{(1,1)}_{23}(\lambda_{23}|t)\mathsf{R}^{(1,1)}_{13}(\lambda_{13}|t+\eta\sigma_2^3)\mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t),\end{aligned} \tag{4.7}$$

where $\lambda_{ij} = \lambda_i - \lambda_j$*, for all* $i, j = 1, 2, 3$.

The dynamical Yang-Baxter relation is one of the key properties of the parameterisation (4.4), and plays a central role in the analysis of the d6V model. In addition, the R-matrix possesses several important properties. We state them without proof and refer the reader to [58, 59, 101] for details.

Proposition 4.1.2. *The* R*-matrix commutes with the total spin operator of* $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$[\mathsf{R}^{(1,1)}_{12}(\lambda|t), \sigma_1^3 + \sigma_2^3] = 0. \tag{4.8}$$

Moreover, it possesses the following pseudo-periodicity properties :

$$\mathsf{R}^{(1,1)}_{12}(\lambda|t+\pi) = \mathsf{R}^{(1,1)}_{12}(\lambda|t), \tag{4.9a}$$

$$\mathsf{R}^{(1,1)}_{12}(\lambda|t+\mathsf{y}\pi\omega) = \sigma_1^1\sigma_2^1\mathsf{R}^{(1,1)}_{12}(\lambda|-t)\sigma_1^1\sigma_2^1. \tag{4.9b}$$

We denote by $\mathcal{P}^{(1)}$ the permutation operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$, i.e.

$$\mathcal{P}^{(1)}\,|\alpha\beta\rangle = |\beta\alpha\rangle \qquad\qquad \forall\,\alpha,\beta \in \{\uparrow,\downarrow\}. \tag{4.10}$$

It allows us to define

$$\mathsf{R}^{(1,1)}_{21}(\lambda|t) = \mathcal{P}^{(1)}_{12}\mathsf{R}^{(1,1)}_{12}(\lambda|t)\mathcal{P}^{(1)}_{12}. \tag{4.11}$$

Proposition 4.1.3 (Unitarity relation)**.** *We have*[3]

$$\mathsf{R}^{(1,1)}_{12}(\lambda|t)\mathsf{R}^{(1,1)}_{21}(-\lambda|t) = \vartheta_1(\eta-\lambda)\vartheta_1(\eta+\lambda). \tag{4.12}$$

[3]The right-hand side of this equality is a multiple of the identity matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$. We often omit such identity matrices to lighten the notation.

The unitarity relation implies that the R-matrix is invertible, except at $\lambda \equiv \pm\eta \pmod{\pi, \pi\omega}$. We examine its behaviour at these values. To this end, we recall the decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^2$ into symmetric and antisymmetric subspaces (3.15). For $\lambda = \pm\eta$, the matrix $\mathsf{R}^{(1,1)}(\lambda|t)$ becomes one of the projectors $P^{(1)\pm}$ (3.16), up to a multiplication by a diagonal operator.

Proposition 4.1.4. *We have*

$$\mathsf{R}^{(1,1)}(0|t) = \vartheta_1(\eta)\mathcal{P}^{(1)}, \tag{4.13}$$

and

$$\mathsf{R}^{(1,1)}(-\eta|t) = -2P^{(1)-}\mathsf{c}(\eta\sigma_1^3|t), \qquad \mathsf{R}^{(1,1)}(\eta|t) = G(t)P^{(1)+}, \tag{4.14}$$

where $\mathsf{c}(\lambda|t)$ *is defined in* (4.4)*, and*

$$G(t) = \operatorname{diag}\left(\vartheta_1(2\eta), 2\mathsf{b}(\eta|t), 2\bar{\mathsf{b}}(\eta|t), \vartheta_1(2\eta)\right). \tag{4.15}$$

4.1.2 Fusion of the dynamical six-vertex model

In this section, we apply the fusion procedure to the R-matrix of the d6V model [86, 88, 89, 95, 135], which allows us to define the R-matrices of the d10V and d19V models. We discuss some of their key properties, such as the dynamical Yang-Baxter equations, symmetry, and pseudo-periodicity. They will be of great importance in our forthcoming analysis. We conclude the section with a short discussion on the trigonometric limits of the R-matrices, establishing a link with previous work.

The dynamical ten-vertex model We construct the R-matrix of the d10V model through the fusion of the R-matrix of the d6V model [95]. Let us recall some of its key steps. We specialise the dynamical Yang-Baxter equation of Proposition 4.1.1 to $\lambda_{12} = \lambda$, $\lambda_{13} = \lambda + \eta$, $\lambda_{23} = \eta$ and use Proposition 4.1.4, which allows us to write

$$\begin{aligned} &P_{23}^{(1)+}\mathsf{R}_{13}^{(1,1)}(\lambda+\eta|t+\eta\sigma_2^3)\mathsf{R}_{12}^{(1,1)}(\lambda|t) \\ &\quad = G_{23}(t)^{-1}\mathsf{R}_{12}^{(1,1)}(\lambda|t+\eta\sigma_3^3)\mathsf{R}_{13}^{(1,1)}(\lambda+\eta|t)G_{23}(t+\eta\sigma_1^3)P_{23}^{(1)+}. \end{aligned} \tag{4.16}$$

It follows that the projector $P_{23}^{(1)-}$ annihilates the operator

$$\mathcal{R}_{123}(\lambda|t) = P_{23}^{(1)+}\mathsf{R}_{13}^{(1,1)}(\lambda+\eta|t+\eta\sigma_2^3)\mathsf{R}_{12}^{(1,1)}(\lambda|t), \tag{4.17}$$

both from the right and the left. Hence, it can be seen as an operator on $\mathbb{C}^2 \otimes \mathrm{Sym}(\mathbb{C}^2, \mathbb{C}^2)$. As a function of λ, its entries are theta functions that vanish at $\lambda = -\eta$ (see Appendix A.2). Indeed, by Proposition 4.1.4, we have

$$\begin{aligned}\mathcal{R}_{123}(-\eta|t) &= -2\vartheta_1(\eta) P_{23}^{(1)+} \mathcal{P}_{13} P_{12}^{(1)-} \mathsf{c}(\eta\sigma_1^3|t) \\ &= 2\vartheta_1(\eta) P_{23}^{(1)+} P_{23}^{(1)-} \mathcal{P}_{13} \mathsf{c}(\eta\sigma_1^3|t) = 0 \end{aligned} \tag{4.18}$$

Hence, by the factorisation properties of theta functions, $\mathcal{R}_{123}(\lambda|t)$ is the product of $\vartheta_1(\lambda+\eta)$ and an operator on $\mathbb{C}^2 \otimes \mathrm{Sym}(\mathbb{C}^2, \mathbb{C}^2)$, whose entries are theta functions, too. Motivated by this observation, we define

$$\mathsf{R}_{1(23)}^{(1,2)}(\lambda|t) = \frac{U_{23}\mathcal{R}_{123}(\lambda|t)U_{23}^{-1}}{\vartheta_1(\lambda+\eta)}, \tag{4.19}$$

where U is the change of basis defined in (3.21). We may identify the basis vectors of the symmetric subspace with the canonical basis vectors of $\mathbb{C}^3$, and thus consider $\mathsf{R}_{1(23)}^{(1,2)}(\lambda|t)$ as an operator on $\mathbb{C}^2 \otimes \mathbb{C}^3$. Its matrix representation with respect to the canonical basis follows from

$$\mathsf{R}^{(1,2)}(\lambda|t) = Q_{23}\mathsf{R}_{1(23)}^{(1,2)}(\lambda|t)Q_{23}^t, \tag{4.20}$$

where Q is the matrix defined in (3.23).

Explicitly, in the basis $\{|{\uparrow}{\Uparrow}\rangle, |{\uparrow}0\rangle, |{\uparrow}{\Downarrow}\rangle, |{\downarrow}{\Uparrow}\rangle, |{\downarrow}0\rangle, |{\downarrow}{\Downarrow}\rangle\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^3$, the fused R-matrix reads

$$\mathsf{R}^{(1,2)} = \begin{pmatrix} \mathsf{e} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{f} & 0 & \mathsf{j} & 0 & 0 \\ 0 & 0 & \mathsf{h} & 0 & \mathsf{g} & 0 \\ 0 & \bar{\mathsf{g}} & 0 & \bar{\mathsf{h}} & 0 & 0 \\ 0 & 0 & \bar{\mathsf{j}} & 0 & \bar{\mathsf{f}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathsf{e} \end{pmatrix}, \tag{4.21}$$

with

$$\mathsf{e}(\lambda) = \vartheta_1(2\eta+\lambda), \tag{4.22a}$$

$$\mathsf{f}(\lambda|t) = \mathrm{e}^{\mathrm{i}\eta\mathsf{y}} \frac{\vartheta_1(\lambda+\eta)\vartheta_1(t+2\eta)}{\vartheta_1(t+\eta)}, \tag{4.22b}$$

$$\mathsf{g}(\lambda|t) = \mathrm{e}^{\mathrm{i}(\eta+\lambda)\mathsf{y}} \frac{\vartheta_1(\eta)}{\alpha} \frac{\vartheta_1(\lambda+t)}{\vartheta_1(t-\eta)}, \tag{4.22c}$$

$$\mathsf{h}(\lambda|t) = \mathrm{e}^{2\mathrm{i}\eta\mathsf{y}} \frac{\vartheta_1(\lambda)\vartheta_1(t+\eta)}{\vartheta_1(t-\eta)}, \tag{4.22d}$$

$$\mathsf{j}(\lambda|t) = \mathrm{e}^{\mathrm{i}\lambda\mathsf{y}} \alpha\, \vartheta_1(2\eta) \frac{\vartheta_1(t+\lambda+\eta)}{\vartheta_1(t+\eta)}. \tag{4.22e}$$

As previously, $\bar{\mathsf{g}}(\lambda|t), \bar{\mathsf{h}}(\lambda|t), \bar{\mathsf{j}}(\lambda|t)$ follow from $\mathsf{g}(\lambda|t), \mathsf{h}(\lambda|t), \mathsf{j}(\lambda|t)$, respectively, through the transformations $t \to -t$ and $\mathsf{y} \to -\mathsf{y}$.

The R-matrix $\mathsf{R}^{(1,2)}(\lambda|t)$ possesses ten entries that are not identically zero. We interpret them as the weights of the vertex configurations, decorated with a height t, of the *dynamical ten-vertex model*. In these configurations, the horizontal edges carry spin-1/2 labels, whereas the vertical edges carry spin-1 labels. Around the generic vertex of Figure 4.2, that means $\alpha, \gamma \in \{\pm 1\}$, and $\beta, \delta \in \{0, \pm 2\}$. Graphical representations in terms of arrows follow from the dictionaries (1.1) and (3.5), and can be read on Figure 4.5.

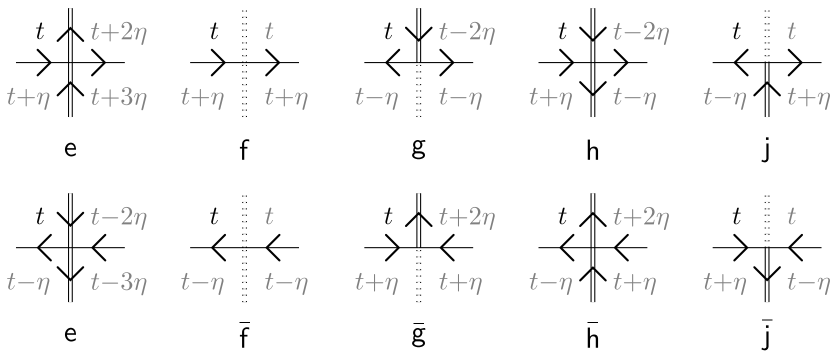


Figure 4.5: Local configurations of the d10V model and their statistical weights. In grey, equivalent formulation as local height configurations of a face model.

We note that this model still fulfils the ice rule. Indeed, for each decorated vertex of the d10V, the relation $\alpha + \beta = \gamma + \delta$ holds. Furthermore, the decorated vertices are in bijection with the local height configurations of a face model (see Figure 4.2). For this model, the possible height differences between two vertically adjacent faces are $\pm\eta$. For two horizontally-adjacent faces, they are $0, \pm 2\eta$. Figure 4.6 illustrates the correspondence between a height configuration of this face model and a dynamical vertex configuration.

0	−2	0	0	−2	−4
1	−1	−1	−1	−3	−5
2	0	−2	−2	−4	−6
3	1	−1	−3	−5	−7

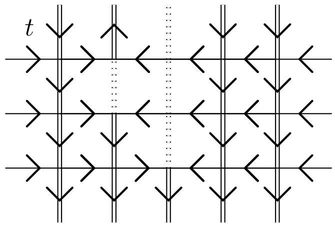


Figure 4.6: *Left:* A configuration of the face model in terms of face labels. The height associated to a label a is $t + a\eta$. *Right:* The corresponding configuration of the d10V model.

The dynamical nineteen-vertex model A second application of the fusion procedure allows us to construct another R-matrix, which is a linear operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$,

$$\begin{aligned} \mathsf{R}^{(2,2)}(\lambda|t) = {} & Q_{12}U_{12} \\ & \times [P_{12}^{(1)+}\mathsf{R}_{13}^{(1,2)}(\lambda|t)\mathsf{R}_{23}^{(1,2)}(\lambda-\eta|t+\eta\sigma_1^3)]\, U_{12}^{-1}Q_{12}^t. \end{aligned} \tag{4.23}$$

Explicitly, in the basis $\{|{\Uparrow\Uparrow}\rangle, |{\Uparrow}0\rangle, |{\Uparrow\Downarrow}\rangle, |0{\Uparrow}\rangle, |00\rangle, |0{\Downarrow}\rangle, |{\Downarrow\Uparrow}\rangle, |{\Downarrow}0\rangle, |{\Downarrow\Downarrow}\rangle\}$ of $\mathbb{C}^3 \otimes \mathbb{C}^3$, the fused R-matrix reads

$$\mathsf{R}^{(2,2)} = \begin{pmatrix} \mathsf{p}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{p}_2 & 0 & \mathsf{p}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{p}_4 & 0 & \mathsf{p}_5 & 0 & \mathsf{p}_6 & 0 & 0 \\ 0 & \mathsf{p}_7 & 0 & \mathsf{p}_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{p}_9 & 0 & \mathsf{p}_{10} & 0 & \bar{\mathsf{p}}_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\mathsf{p}}_8 & 0 & \bar{\mathsf{p}}_7 & 0 \\ 0 & 0 & \bar{\mathsf{p}}_6 & 0 & \bar{\mathsf{p}}_5 & 0 & \bar{\mathsf{p}}_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\mathsf{p}}_3 & 0 & \bar{\mathsf{p}}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{p}_1 \end{pmatrix}, \tag{4.24}$$

with

$$\mathsf{p}_1(\lambda) = \vartheta_1(\lambda+\eta)\vartheta_1(\lambda+2\eta), \tag{4.25a}$$

$$\mathsf{p}_2(\lambda|t) = \mathrm{e}^{2\mathrm{i}y\eta}\frac{\vartheta_1(\lambda)\vartheta_1(\lambda+\eta)\vartheta_1(t+3\eta)}{\vartheta_1(t+\eta)}, \tag{4.25b}$$

$$\mathsf{p}_3(\lambda|t) = \mathrm{e}^{\mathrm{i}y\lambda}\vartheta_1(2\eta)\frac{\vartheta_1(\lambda+\eta)\vartheta_1(t+\lambda+\eta)}{\vartheta_1(t+\eta)}, \tag{4.25c}$$

$$\mathsf{p}_4(\lambda|t) = \mathrm{e}^{4\mathrm{i}y\eta}\frac{\vartheta_1(\lambda)\vartheta_1(\lambda-\eta)\vartheta_1(t+\eta)\vartheta_1(t+2\eta)}{\vartheta_1(t)\vartheta_1(t-\eta)}, \tag{4.25d}$$

$$\mathsf{p}_5(\lambda|t) = \mathrm{e}^{\mathrm{iy}(\lambda+2\eta)}\frac{\vartheta_1(\eta)}{\alpha^2}\frac{\vartheta_1(\lambda)\vartheta_1(t+\lambda)\vartheta_1(t+\eta)}{\vartheta_1(t)\vartheta_1(t-\eta)}, \tag{4.25e}$$

$$\mathsf{p}_6(\lambda|t) = \mathrm{e}^{2\mathrm{iy}\lambda}\vartheta_1(\eta)\vartheta_1(2\eta)\frac{\vartheta_1(t+\lambda)\vartheta_1(t+\lambda-\eta)}{\vartheta_1(t)\vartheta_1(t-\eta)}, \tag{4.25f}$$

$$\mathsf{p}_7(\lambda|t) = \mathrm{e}^{-\mathrm{iy}\lambda}\vartheta_1(2\eta)\frac{\vartheta_1(\lambda+\eta)\vartheta_1(t-\lambda+\eta)}{\vartheta_1(t+\eta)}, \tag{4.25g}$$

$$\mathsf{p}_8\lambda|t) = \mathrm{e}^{-2\mathrm{iy}\eta}\frac{\vartheta_1(\lambda)\vartheta_1(\lambda+\eta)\vartheta_1(t-\eta)}{\vartheta_1(t+\eta)}, \tag{4.25h}$$

$$\mathsf{p}_9(\lambda|t) = \mathrm{e}^{-\mathrm{iy}(\lambda-2\eta)}\alpha^2\frac{\vartheta_1(2\eta)^2}{\vartheta_1(\eta)}\frac{\vartheta_1(\lambda)\vartheta_1(t-\lambda)\vartheta_1(t+2\eta)}{\vartheta_1(t+\eta)\vartheta_1(t-\eta)}, \tag{4.25i}$$

and lastly,

$$\begin{aligned}\mathsf{p}_{10}(\lambda|t) = {} & \frac{1}{2\vartheta_1(t)\vartheta_1(t+\eta)\vartheta_1(t-\eta)} \\ & \times (\vartheta_1(\eta)\vartheta_1(2\eta)[\vartheta_1(t+\lambda)\vartheta_1(t+\eta)\vartheta_1(t-\lambda-\eta) \\ & + \vartheta_1(t-\lambda)\vartheta_1(t-\eta)\vartheta_1(t+\lambda+\eta)] + \vartheta_1(\lambda)\vartheta_1(\lambda+\eta) \\ & \times [\vartheta_1(t-2\eta)\vartheta_1(t+\eta)^2 + \vartheta_1(t+2\eta)\vartheta_1(t-\eta)^2]).\end{aligned} \tag{4.25j}$$

As previously, the expression for $\bar{\mathsf{p}}_i(\lambda|t)$ follows from $\mathsf{p}_\mathsf{i}(\lambda|t)$ through the transformations $t \to -t$ and $\mathsf{y} \to -\mathsf{y}$, for each $i = 1, \ldots, 10$.

The R-matrix $\mathsf{R}^{(2,2)}(\lambda|t)$ possesses nineteen entries that are not identically zero. Their interpretation as local vertex configurations, decorated with a height t, naturally leads to the *dynamical nineteen-vertex model* on the square lattice. For this model, the horizontal and vertical edges carry the spin-1 labels $0, \pm 2$. We represent them graphically as oriented double lines or unoriented dotted double lines according to (3.5), see Figure 4.7.

From Figure 4.2, with $\alpha, \beta, \gamma, \delta \in \{0, \pm 2\}$, we see that the ice rule $\alpha + \beta = \gamma + \delta$ holds for the decorated vertices of the d19V. Moreover, it allows one to identify the dynamical nineteen-vertex model with a face model in which the possible height differences between adjacent faces are $0, \pm 2\eta$.

Properties of the fused R-matrices We now provide several properties of the R-matrices $\mathsf{R}^{(1,2)}(\lambda|t)$ and $\mathsf{R}^{(2,2)}(\lambda|t)$ that will be important in our analysis of the d10V and d19V models. To this end, we generalise

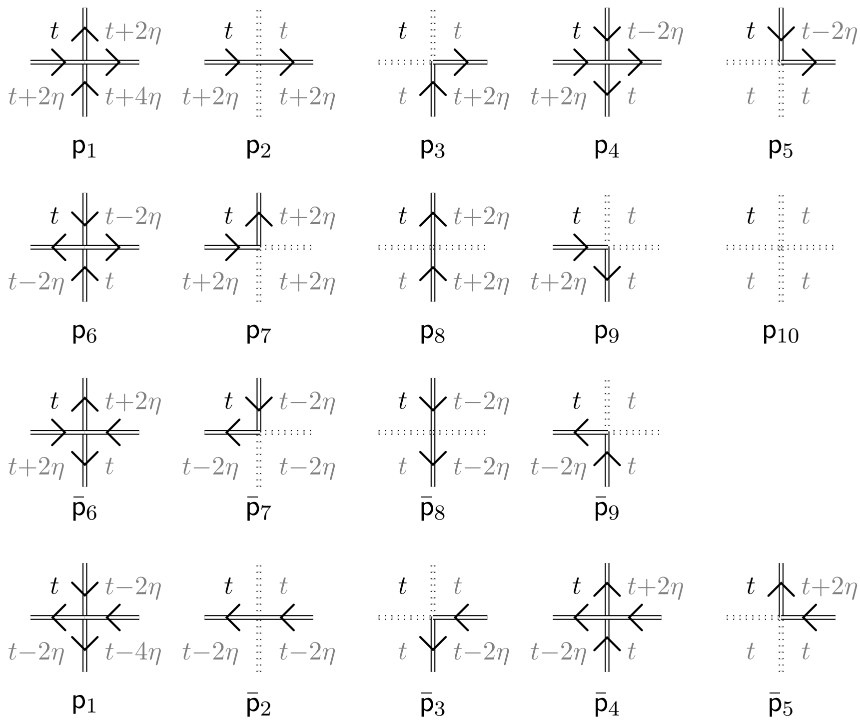


Figure 4.7: Local configurations of the d19V model and their statistical weights. In grey, equivalent formulation as local height configurations of a face model.

the notation (4.6) for the action of R-matrices with a *dynamical shift* of their height parameter t. On $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, we write

$$\mathsf{R}^{(1,2)}_{12}(\lambda|t+2\eta s^3_3) = \mathsf{R}^{(1,2)}_{12}(\lambda|t+2\eta)\bar{p}^+_3 + \mathsf{R}^{(1,2)}_{12}(\lambda|t)\bar{p}^0_3 + \mathsf{R}^{(1,2)}_{12}(\lambda|t-2\eta)\bar{p}^-_3\,, \quad (4.26)$$

where

$$\bar{p}^+ = \mathrm{diag}(1,0,0), \quad \bar{p}^0 = \mathrm{diag}(0,1,0), \quad \bar{p}^- = \mathrm{diag}(0,0,1). \quad (4.27)$$

We use the same notation for the action of $\mathsf{R}^{(2,2)}(\lambda|t)$ with a dynamical shift, as well as for other operators.

Proposition 4.1.5 (Dynamical Yang-Baxter equations)**.** *We have*

$$\mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t+2\eta s^3_3)\mathsf{R}^{(1,2)}_{13}(\lambda_{13}|t)\mathsf{R}^{(1,2)}_{23}(\lambda_{23}|t+\eta\sigma^3_1) = \mathsf{R}^{(1,2)}_{23}(\lambda_{23}|t)\mathsf{R}^{(1,2)}_{13}(\lambda_{13}|t+\eta\sigma^3_2)\mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t), \quad (4.28)$$

on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, *and*

$$\begin{aligned}\mathsf{R}_{12}^{(1,2)}(\lambda_{12}|t+2\eta s_3^3)\mathsf{R}_{13}^{(1,2)}(\lambda_{13}|t)\mathsf{R}_{23}^{(2,2)}(\lambda_{23}|t+\eta\sigma_1^3)\\ = \mathsf{R}_{23}^{(2,2)}(\lambda_{23}|t)\mathsf{R}_{13}^{(1,2)}(\lambda_{13}|t+2\eta s_2^3)\mathsf{R}_{12}^{(1,2)}(\lambda_{12}|t),\end{aligned} \tag{4.29}$$

on $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Proof. The proof of both (4.28) and (4.29) follows from the dynamical Yang-Baxter equation for the d6V model, given in Proposition 4.1.1, and the fusion procedure. We only sketch the proof of (4.28).

First, by Propositions 4.1.1 and 4.1.2, we have the following relation on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{aligned}&\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t+\eta(\sigma_3^3+\sigma_4^3))\mathsf{R}_{13}^{(1,1)}(\lambda_{13}+\eta|t+\eta\sigma_4^3)\mathsf{R}_{14}^{(1,1)}(\lambda_{14}|t)\\ &\qquad\times \mathsf{R}_{23}^{(1,1)}(\lambda_{23}+\eta|t+\eta(\sigma_1^3+\sigma_4^3))\mathsf{R}_{24}^{(1,1)}(\lambda_{24}|t+\eta\sigma_1^3)\\ &=\mathsf{R}_{23}^{(1,1)}(\lambda_{23}+\eta|t+\eta\sigma_4^3)\mathsf{R}_{24}^{(1,1)}(\lambda_{24}|t)\\ &\qquad\times \mathsf{R}_{13}^{(1,1)}(\lambda_{13}+\eta|t+\eta(\sigma_2^3+\sigma_4^3))\mathsf{R}_{14}^{(1,1)}(\lambda_{14}|t+\eta\sigma_2^3)\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t).\end{aligned} \tag{4.30}$$

We set $\lambda_4 = \lambda_3$ in this equality, multiply both sides from the left by $P_{34}^{(1)+}$, and use (4.19) to obtain the following relation on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathrm{Sym}(\mathbb{C}^2, \mathbb{C}^2)$:

$$\begin{aligned}\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t+\eta(\sigma_3^3+\sigma_4^3))\mathsf{R}_{1(34)}^{(1,2)}(\lambda_{13}|t)\mathsf{R}_{2(34)}^{(1,2)}(\lambda_{23}|t+\eta\sigma_1^3)\\ = \mathsf{R}_{2(34)}^{(1,2)}(\lambda_{23}|t)\mathsf{R}_{1(34)}^{(1,2)}(\lambda_{13}|t+\eta\sigma_2^3)\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t).\end{aligned} \tag{4.31}$$

It remains to use (4.20), and

$$Q_{34}U_{34}(\sigma_3^3+\sigma_4^3)U_{34}^{-1}Q_{34}^t = 2s^3, \tag{4.32}$$

to express this equality as a relation on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. □

The following commutation relations follow immediately from the matrix representations of $\mathsf{R}^{(1,2)}(\lambda|t)$ and $\mathsf{R}^{(2,2)}(\lambda|t)$, given in (4.22) and (4.25):

Proposition 4.1.6. *We have*

$$[\mathsf{R}_{12}^{(1,2)}(\lambda|t), \sigma_1^3+2s_2^3]=0, \qquad [\mathsf{R}_{12}^{(1,2)}(\lambda|t), \sigma_1^3\Omega_2^3]=0, \tag{4.33a}$$

$$[\mathsf{R}_{12}^{(2,2)}(\lambda|t), s_1^3+s_2^3]=0, \qquad [\mathsf{R}_{12}^{(2,2)}(\lambda|t), \Omega_1^3\Omega_2^3]=0, \tag{4.33b}$$

and the pseudo-periodicity properties

$$\mathsf{R}^{(1,2)}(\lambda|t+\pi) = \mathsf{R}^{(1,2)}(\lambda|t), \tag{4.34a}$$

$$\mathsf{R}^{(1,2)}(\lambda|t+\mathsf{y}\pi\omega) = \sigma_1^1\Omega_2^1\mathsf{R}^{(1,2)}(\lambda|-t)\sigma_1^1\Omega_2^1. \tag{4.34b}$$

We now address the analyticity and pseudo-periodicity properties of $\mathsf{R}^{(1,2)}(\lambda|t)$. Both are a straightforward consequence of the properties of the Jacobi theta functions (see Appendix A.1, in particular Proposition A.1.2).

Proposition 4.1.7 (Pseudo-periodicity and analyticity)**.** *The matrix* $\mathsf{R}^{(1,2)}(\lambda|t)$ *is an entire function of* λ.[4] *Moreover, it fulfils*

$$\mathsf{R}_{12}^{(1,2)}(\lambda+\pi|t) = -(\sigma_1^3)^{\mathsf{y}}\,\mathsf{R}_{12}^{(1,2)}(\lambda|t)\,(\sigma_1^3)^{\mathsf{y}}, \tag{4.35}$$

$$\begin{aligned}\mathsf{R}_{12}^{(1,2)}(\lambda+\pi\omega|t) = {}& -p^{-1}\mathrm{e}^{-2\mathrm{i}\lambda}\mathrm{e}^{-2\mathrm{i}\eta(1+\sigma_1^3 s_2^3)}p^{\mathsf{y}\sigma_1^3/2}\mathrm{e}^{-\mathrm{i}t\sigma_1^3} \\ & \times \mathsf{R}_{12}^{(1,2)}(\lambda|t)\,\mathrm{e}^{\mathrm{i}t\sigma_1^3}p^{-\mathsf{y}\sigma_1^3/2}.\end{aligned} \tag{4.36}$$

We use the permutation operator $\mathcal{P}^{(2)}$ of $\mathbb{C}^3\otimes\mathbb{C}^3$ to define

$$\mathsf{R}_{21}^{(2,2)}(\lambda|t) = \mathcal{P}_{12}^{(2)}\mathsf{R}_{12}^{(2,2)}(\lambda|t)\mathcal{P}_{12}^{(2)}. \tag{4.37}$$

Proposition 4.1.8 (Unitarity relation)**.** *We have*

$$\mathsf{R}_{12}^{(2,2)}(\lambda|t)\mathsf{R}_{21}^{(2,2)}(-\lambda|t) = \vartheta_1(\lambda+2\eta)\vartheta_1(\lambda+\eta)\vartheta_1(\lambda-\eta)\vartheta_1(\lambda-2\eta). \tag{4.38}$$

The unitarity relation implies that $\mathsf{R}^{(2,2)}(\lambda|t)$ is invertible, except at $\lambda \equiv \pm\eta, \pm 2\eta \pmod{\pi, \pi\omega}$. As for the d6V model, it is interesting to examine the behaviour of the R-matrix at these values. We focus on $\lambda=-\eta$, where the R-matrix has rank one. To this end, recall $P^{(2)-}$, the projector onto the subspace of $\mathbb{C}^3\otimes\mathbb{C}^3$ spanned by the singlet state

$$|s\rangle = |{\Uparrow\Downarrow}\rangle - |00\rangle + |{\Downarrow\Uparrow}\rangle\,. \tag{4.39}$$

The following proposition straightforwardly follows from (4.25).

Proposition 4.1.9. *We have*

$$\mathsf{R}^{(2,2)}(-\eta|t) = 3\vartheta_1(2\eta)\,\mathsf{c}(\eta s_1^3|t)P^{(2)-}\mathsf{d}(\eta s_1^3|t), \tag{4.40}$$

where $\mathsf{c}(\lambda|t)$ *is defined in* (4.4)*, and*

$$\mathsf{d}(\lambda|t) = \mathrm{e}^{3\mathrm{i}\mathsf{y}\lambda}\frac{\vartheta_1(t+2\lambda)}{\vartheta_1(t-\lambda)}. \tag{4.41}$$

[4]Throughout, the analyticity properties stated for matrices or vectors are understood to hold for each entry individually.

The trigonometric limit The dynamical six-, ten- and nineteen-vertex models defined above are commonly referred to as *elliptic*, as their weights are defined in terms of Jacobi theta functions (which can be thought of as the building blocks of elliptic functions).

For an appropriate choice of reference height ℓ_0, which we will specify in Section 4.1.4, taking the limit $\omega \to \mathrm{i}\infty$ yields *trigonometric* models [59]. For $\mathsf{y} = 0$, the limit yields the trigonometric dynamical six-, ten- and nineteen-vertex models. Conversely, for $\mathsf{y} = 1$, we obtain the corresponding trigonometric non-dynamical models, which allows us to connect the present work with [92, 93]. In the following proposition, we provide the limit for the weights of the dynamical ten-vertex model:

Proposition 4.1.10. *Let* $\mathsf{y} = 1$ *and* $t = t' + \pi\omega/2$*, where* t' *is a fixed complex number, then*

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\mathsf{e}(\lambda) = \sin(\lambda + 2\eta) \tag{4.42a}$$

and

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\mathsf{f}(\lambda|t) = \lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\bar{\mathsf{f}}(\lambda|t) = \sin(\lambda + \eta), \tag{4.42b}$$

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\mathsf{g}(\lambda|t) = \lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\bar{\mathsf{g}}(\lambda|t) = \frac{1}{\beta}\sin\eta, \tag{4.42c}$$

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\mathsf{h}(\lambda|t) = \lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\bar{\mathsf{h}}(\lambda|t) = \sin\lambda, \tag{4.42d}$$

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\mathsf{j}(\lambda|t) = \lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\bar{\mathsf{j}}(\lambda|t) = \beta\sin 2\eta, \tag{4.42e}$$

where

$$\beta^2 = \frac{\sin\eta}{\sin 2\eta}. \tag{4.43}$$

Proof. From Proposition A.1.3, we have the identity

$$\vartheta_1(\lambda + \pi\omega/2) = \mathrm{i}\mathrm{e}^{-\mathrm{i}(\lambda+\pi\omega/4)}\vartheta_4(\lambda), \tag{4.44}$$

and from Definition A.1.1, we have the limits

$$\lim_{\omega\to\mathrm{i}\infty} \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi\omega/4}\vartheta_1(\lambda) = \sin\lambda, \qquad \lim_{\omega\to\mathrm{i}\infty} \vartheta_4(\lambda) = 1. \tag{4.45}$$

The proposition follows from applying them to the vertex weights. □

4.1.3 The monodromy matrix

In this section, we define and analyse the monodromy matrix of the inhomogeneous d10V model, which is one of the focal points of this chapter and the next. The monodromy matrix takes the height dependence of the vertex weights into account. We closely follow [58, 59, 94], and implement this dependence through the definition of the monodromy matrix as an operator on the *spin-dynamical space*, which is an infinite-dimensional vector space. We study the properties of the monodromy matrix in great detail. First, we establish several algebraic relations between its operator entries, issued from the Yang-Baxter algebra associated with the d6V R-matrix. Second, we analyse their behaviour under the exchange of inhomogeneity parameters, examine their analyticity and pseudo-periodicity properties, and discuss a recursive structure. We conclude the section with a note on graphical calculus.

The spin-dynamical spaces In Section 4.1.2, we have been considering the R-matrices only as operators on tensor products of $\mathbb{C}^2$ or $\mathbb{C}^3$. We now extend our algebraic description to take the model heights into account. We introduce *height states* $|t\rangle$, for all $t \in \ell_0 + \eta\mathbb{Z}$, with ℓ_0 the reference height discussed in Section 4.1.1. The dual height states are given by $\langle k|$, for all $k \in \ell_0 + \eta\mathbb{Z}$. We name the spaces that they span the *dynamical space* and its dual, respectively. They are formally defined as the following infinite-dimensional vector spaces:

$$\mathbb{D}_0 = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}|\ell_0 + j\eta\rangle, \qquad \mathbb{D}_0^* = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}\langle\ell_0 + j\eta|. \tag{4.46}$$

The dual pairing of $\mathbb{D}_0^*$ and $\mathbb{D}_0$ is the bilinear form defined through

$$\langle k|t\rangle = \delta_{k,t}. \tag{4.47}$$

We introduce dynamical operators, τ and $T_\tau^\pm$. The operator τ is diagonal on the height states,

$$\tau\,|t\rangle = t\,|t\rangle\,, \qquad \langle k|\,\tau = k\,\langle k|\,. \tag{4.48}$$

The operators $T_\tau^\pm$ act as ladder operators on the height states,

$$T_\tau^\pm\,|t\rangle = |t \mp \eta\rangle\,, \qquad \langle k|\,T_\tau^\pm = \langle k \pm \eta|\,. \tag{4.49}$$

From (4.48) and (4.49), we deduce the following commutation relation:

$$T_\tau^\pm \tau = (\tau \pm \eta) T_\tau^\pm . \tag{4.50}$$

Moreover, the dynamical operators commute with all spin operators. Below, we consider the *spin-dynamical space* and its dual,

$$\mathbb{D}_N = (\mathbb{C}^3)^{\otimes N} \otimes \mathbb{D}_0, \qquad \mathbb{D}_N^* = (\mathbb{C}^3)^{\otimes N *} \otimes \mathbb{D}_0^*. \tag{4.51}$$

The canonical basis elements of these spaces are respectively given by the tensor products between canonical basis elements of (the dual of) $(\mathbb{C}^3)^{\otimes N}$, and (dual) height states.

Finally, we note that the actions of operators given in this chapter naturally generalise to the spin-dynamical space. As an example, we show how using this space allows us to implement the heights in our evaluation of the vertex weights of the d10V model. We consider the vertex at the right of figure 4.2, with $\alpha, \gamma \in \{\pm 1\}$ and $\beta, \delta \in \{0, \pm 2\}$. Algebraically, the corresponding vertex weight is

$$\begin{aligned} &\langle \gamma\delta | \otimes \langle t| \; \mathsf{R}_{12}^{(1,2)}(\lambda|\tau) T_\tau^{\sigma_1^3} \; |\alpha\beta\rangle \otimes |\ell\rangle \\ &\qquad\qquad = \langle \gamma\delta | \otimes \langle t| \; \mathsf{R}_{12}^{(1,2)}(\lambda|t) \; |\alpha\beta\rangle \otimes |\ell - \eta\alpha\rangle \,, \end{aligned} \tag{4.52}$$

where we used (4.49) and (4.48). We thus see that (4.47) now enforces $\ell = t + \eta\alpha$.

The monodromy matrix Let us introduce the operators

$$S^j = \sum_{k=1}^{j} s_k^3, \qquad j = 1, \dots, N, \tag{4.53}$$

and $S^0 = 0$. In particular, $S = S^N$ denotes the *total spin operator* on $(\mathbb{C}^3)^{\otimes N}$.

In our analysis below, we work with the inhomogeneous d10V model. For this purpose, we introduce *inhomogeneity parameters* $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, each associated to a factor of the tensor product $(\mathbb{C}^3)^{\otimes N}$. For all $\lambda_0 \in \mathbb{C}$, we define the following operator on $\mathbb{C}^2 \otimes (\mathbb{C}^3)^{\otimes N}$:

$$\mathsf{M}_0(\lambda_0|t) = \prod_{j=1}^{\overset{\curvearrowleft}{N}} \mathsf{R}_{0j}^{(1,2)} \left(\lambda_{0j} \,\middle|\, t + 2\eta S^{j-1} \right) , \tag{4.54}$$

where $\lambda_{0j} = \lambda_0 - \lambda_j$, $j = 1, \ldots, N$, and the curved arrow over the product indicates a reversal of the product order. Moreover, the 0 subscripts label the *auxiliary space* $\mathbb{C}^2$.

It is common to consider $\mathsf{M}_0(\lambda|t)$ as a 2×2 matrix in the auxiliary space, with entries acting on $(\mathbb{C}^3)^{\otimes N}$. We write this matrix representation as follows:

$$\mathsf{M}_0(\lambda|t) = \begin{pmatrix} \mathsf{A}(\lambda|t) & \mathsf{B}(\lambda|t) \\ \mathsf{C}(\lambda|t) & \mathsf{D}(\lambda|t) \end{pmatrix}_0 . \tag{4.55}$$

The *monodromy matrix* of the inhomogeneous d10V model is the linear operator $\mathcal{M}_0(\lambda)$ on $\mathbb{C}^2 \otimes \mathbb{D}_N$, defined as

$$\mathcal{M}_0(\lambda) = \mathsf{M}_0(\lambda|\tau) T_\tau^{\sigma_0^3}. \tag{4.56}$$

As for $\mathsf{M}_0(\lambda|t)$, it will be useful to consider the matrix representation with respect to the auxiliary space,

$$\mathcal{M}_0(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}_0 = \begin{pmatrix} \mathsf{A}(\lambda|\tau) T_\tau^+ & \mathsf{B}(\lambda|\tau) T_\tau^- \\ \mathsf{C}(\lambda|\tau) T_\tau^+ & \mathsf{D}(\lambda|\tau) T_\tau^- \end{pmatrix}_0 . \tag{4.57}$$

The entries $\mathcal{A}(\lambda), \mathcal{B}(\lambda), \mathcal{C}(\lambda)$ and $\mathcal{D}(\lambda)$ are operators on $\mathbb{D}_N$ and $\mathbb{D}_N^*$. We refer to them as *monodromy-matrix entries*. Together, they constitute a representation of the *dynamical Yang-Baxter algebra*, an operator algebra associated to the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_2)$ [47,149]. We devote the remainder of this section to the study of the properties of the monodromy matrix and its entries.

Algebraic relations We now derive commutations relations between the monodromy-matrix entries. These algebraic relations follow from an application of the dYBE (4.28) to a product of monodromy matrices.

Proposition 4.1.11. *We have*

$$\begin{aligned} &\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau + 2\eta S) \mathcal{M}_0(\lambda_0) \mathcal{M}_{0'}(\lambda_{0'}) \\ &\qquad = \mathcal{M}_{0'}(\lambda_{0'}) \mathcal{M}_0(\lambda_0) \mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau) \end{aligned} \tag{4.58}$$

on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{D}_N$, where 0 and $0'$ are the labels of the first and the second factor $\mathbb{C}^2$, respectively.

Proof. By (4.28) and (4.55), we have

$$\begin{aligned}\mathsf{R}^{(1,1)}_{00'}(\lambda_{00'}|\tau+2\eta S)\mathsf{M}_0(\lambda_0|\tau+\eta\sigma^3_{0'})\mathsf{M}_{0'}(\lambda_{0'}|\tau)\\ =\mathsf{M}_{0'}(\lambda_{0'}|\tau)\mathsf{M}_0(\lambda_0|\tau+\eta\sigma^3_0)\mathsf{R}^{(1,1)}_{00'}(\lambda_{00'}|\tau).\end{aligned}\tag{4.59}$$

We multiply both sides of this equality on the right by $T_\tau^{\sigma_0^3}T_\tau^{\sigma_{0'}^3}$. Using the fact that

$$\mathsf{R}^{(1,1)}_{00'}(\lambda|\tau)T_\tau^{\sigma_0^3}T_\tau^{\sigma_{0'}^3}=T_\tau^{\sigma_0^3}T_\tau^{\sigma_{0'}^3}\mathsf{R}^{(1,1)}_{00'}(\lambda|\tau),\tag{4.60}$$

which, crucially, follows from the height-independence of the weight $\mathsf{a}(\lambda)$, we obtain (4.58). □

Writing out the matrix representations of both sides of (4.59), with respect to the factor $\mathbb{C}^2\otimes\mathbb{C}^2$ of the two auxiliary spaces, straightforwardly leads to sixteen relations between the monodromy-matrix entries. In the next proposition, we gather the relations that will be important in the following.

Proposition 4.1.12. *Let μ_1,μ_2 be complex numbers and $\mu_{12}=\mu_1-\mu_2$, then*

$$[\mathcal{A}(\mu_1),\mathcal{A}(\mu_2)]=[\mathcal{B}(\mu_1),\mathcal{B}(\mu_2)]=0,\tag{4.61}$$

$$[\mathcal{C}(\mu_1),\mathcal{C}(\mu_2)]=[\mathcal{D}(\mu_1),\mathcal{D}(\mu_2)]=0,\tag{4.62}$$

and

$$\begin{aligned}\bar{\mathsf{b}}(\mu_{12}|\tau+2\eta S)\mathcal{C}(\mu_1)\mathcal{A}(\mu_2)&=\mathcal{A}(\mu_2)\mathcal{C}(\mu_1)\mathsf{a}(\mu_{12})\\&\quad-\bar{\mathsf{c}}(\mu_{12}|\tau+2\eta S)\mathcal{A}(\mu_1)\mathcal{C}(\mu_2),\end{aligned}\tag{4.63}$$

$$\begin{aligned}\mathsf{b}(\mu_{12}|\tau+2\eta S)\mathcal{A}(\mu_1)\mathcal{C}(\mu_2)&=\mathcal{C}(\mu_2)\mathcal{A}(\mu_1)\mathsf{a}(\mu_{12})\\&\quad-\mathsf{c}(\mu_{12}|\tau+2\eta S)\mathcal{C}(\mu_1)\mathcal{A}(\mu_2),\end{aligned}\tag{4.64}$$

$$\begin{aligned}\mathcal{D}(\mu_2)\mathcal{C}(\mu_1)\mathsf{b}(\mu_{12}|\tau)&=\mathsf{a}(\mu_{12})\mathcal{C}(\mu_1)\mathcal{D}(\mu_2)\\&\quad-\mathcal{C}(\mu_2)\mathcal{D}(\mu_1)\bar{\mathsf{c}}(\mu_{12}|\tau),\end{aligned}\tag{4.65}$$

$$\begin{aligned}\mathcal{C}(\mu_2)\mathcal{D}(\mu_1)\bar{\mathsf{b}}(\mu_{12}|\tau)&=\mathsf{a}(\mu_{12})\mathcal{D}(\mu_1)\mathcal{C}(\mu_2)\\&\quad-\mathcal{D}(\mu_2)\mathcal{C}(\mu_1)\mathsf{c}(\mu_{12}|\tau),\end{aligned}\tag{4.66}$$

Furthermore, we have

$$\begin{aligned}\mathsf{b}(\mu_{12}|\tau)\mathcal{B}(\mu_2)\mathcal{C}(\mu_1)&=\bar{\mathsf{b}}(\mu_{12}|\tau+2\eta S)\mathcal{C}(\mu_1)\mathcal{B}(\mu_2)\\&\quad+\bar{\mathsf{c}}(\mu_{12}|\tau+2\eta S)\mathcal{A}(\mu_1)\mathcal{D}(\mu_2)-\mathcal{A}(\mu_2)\mathcal{D}(\mu_1)\bar{\mathsf{c}}(\mu_{12}|\tau),\end{aligned}\tag{4.67}$$

and

$$\begin{aligned}\mathcal{C}(\mu_2)\mathcal{B}(\mu_1)\bar{\mathsf{b}}(\mu_{12}|\tau) &= \mathsf{b}(\mu_{12}|\tau+2\eta S)\mathcal{B}(\mu_1)\mathcal{C}(\mu_2)\\ &\quad + \mathsf{c}(\mu_{12}|\tau+2\eta S)\mathcal{D}(\mu_1)\mathcal{A}(\mu_2) - \mathcal{D}(\mu_2)\mathcal{A}(\mu_1)\mathsf{c}(\mu_{12}|\tau). \quad (4.68)\end{aligned}$$

Since we study a particular representation of the dynamical Yang-Baxter algebra, additional relations hold that do not immediately follow from Proposition 4.1.11. Some of them follow from the so-called *inversion formula*, which relates the monodromy matrix and its transpose (with respect to the auxiliary space).

Proposition 4.1.13 (Inversion formula)**.** *We have*

$$\mathcal{M}_0(\lambda)\,\sigma_0^2\mathcal{M}_0(\lambda-\eta)^{t_0}\sigma_0^2 = \frac{\mathrm{e}^{-2\mathrm{i}y\eta S}\vartheta_1(\tau)}{\vartheta_1(\tau+2\eta S)}\mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta), \tag{4.69}$$

where $\mathrm{A}(\lambda)$, $\mathrm{D}(\lambda)$ *are defined in* (3.49)*, and* t_0 *denotes the transposition with respect to the auxiliary space.*

Proof. We relegate the proof of this formula to Section 4.A. It is based on a similar formula for the monodromy matrix of the dynamical six-vertex model, whose combination with the fusion procedure allows us to establish (4.69). □

Writing out the inversion formula for the entries of the monodromy matrix allows us to compute the so-called *quantum determinant* $\det_q \mathcal{M}_0(\lambda)$ of the dynamical Yang-Baxter algebra [47, 149] for the representation that we consider. This quantum determinant is defined as

$$\begin{aligned}&\det_q \mathcal{M}(\lambda)\\ &\quad = \frac{\vartheta_1(\tau+2\eta S)}{\vartheta_1(\tau)}\mathrm{e}^{2\mathrm{i}y\eta S}\left(\mathcal{A}(\lambda)\mathcal{D}(\lambda-\eta) - \mathcal{B}(\lambda)\mathcal{C}(\lambda-\eta)\right), \quad (4.70\mathrm{a})\\ &\quad = \frac{\vartheta_1(\tau+2\eta S)}{\vartheta_1(\tau)}\mathrm{e}^{2\mathrm{i}y\eta S}\left(\mathcal{D}(\lambda)\mathcal{A}(\lambda-\eta) - \mathcal{C}(\lambda)\mathcal{B}(\lambda-\eta)\right). \quad (4.70\mathrm{b})\end{aligned}$$

The equality of the two expressions for the quantum determinant follows from the algebraic relations of Proposition 4.1.11. They also imply that the quantum determinant belongs to the center of the dynamical Yang-Baxter algebra. In the representation that we consider, it acts as a multiple of the identity matrix.

Corollary 4.1.14 (Quantum determinant relation)**.** *We have*

$$\det_q \mathcal{M}(\lambda) = \mathrm{A}(\lambda)\mathrm{D}(\lambda - \eta). \tag{4.71}$$

Exchange relation We will need an additional algebraic relation that involves the monodromy-matrix entry $\mathcal{C}(\lambda)$. To this end, we define the $\check{\mathsf{R}}$-matrix of the d19V model

$$\check{\mathsf{R}}^{(2,2)}(\lambda|t) = \mathcal{P}^{(2)}\mathsf{R}^{(2,2)}(\lambda|t). \tag{4.72}$$

The multiplication of the monodromy matrix by this operator allows one to swap two of its inhomogeneity parameters. We sometimes need to exclude certain singular choices for these parameters, or treat them separately. To this end, the following terminology will be useful:

Definition 4.1.15. *We call the inhomogeneity parameters $\lambda_1, \ldots, \lambda_N$* generic *if*

$$\lambda_{ij} - \eta h \not\equiv 0 \ (mod\, \pi, \pi\omega), \tag{4.73}$$

for all $1 \leqslant i < j \leqslant N$ and $h \in \mathbb{Z}$.

Throughout, the inhomogeneity parameters are generic unless stated otherwise.

We now prove the *exchange relations* for the monodromy matrix. To this end, we write $\mathcal{M}_0(\lambda_0) = \mathcal{M}_0(\lambda_0; \lambda_1, \ldots, \lambda_N)$ to stress the dependence on the inhomogeneity parameters.

Proposition 4.1.16 (Exchange relations)**.** *For each $i = 1, \ldots, N$, we have*

$$\begin{aligned} &\check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1} \,\middle|\, \tau + 2\eta S^{i-1}\right) \mathcal{M}_0(\lambda_0; \cdots \lambda_i, \lambda_{i+1} \cdots) \\ &\qquad = \mathcal{M}_0(\lambda_0; \cdots \lambda_{i+1}, \lambda_i \cdots)\check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1} \,\middle|\, \tau + 2\eta S^{i-1}\right). \end{aligned} \tag{4.74}$$

Proof. We consider the left-hand side of (4.74) and use (4.54) and (4.56) to replace the monodromy matrix by a product of R-matrices, multiplied on the right by the dynamical shift operator. The $\check{\mathsf{R}}$-matrix commutes with the $N-i-1$ leftmost factors of the product of R-matrices. Hence, we move it to the right of their product and encounter the term

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1}\middle|\tau+2\eta S^{i-1}\right)\mathsf{R}^{(1,2)}_{0,i+1}\left(\lambda_{0,i+1}\middle|\tau+2\eta S^{i}\right) \times \mathsf{R}^{(1,2)}_{0,i}\left(\lambda_{0,i}\middle|\tau+2\eta S^{i-1}\right). \quad (4.75)$$

By Proposition 4.1.5, this term is equal to

$$\mathsf{R}^{(1,2)}_{0,i+1}\left(\lambda_{0,i}\middle|\tau+2\eta S^{i}\right)\mathsf{R}^{(1,2)}_{0,i}\left(\lambda_{0,i+1}\middle|\tau+2\eta S^{i-1}\right) \times \check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1}\middle|\tau+2\eta S^{i-1}+\eta\sigma_0^3\right). \quad (4.76)$$

The remaining $i-1$ factors of the product of R-matrices commute with the $\check{\mathsf{R}}$-matrix, too, thanks to Proposition 4.1.6. Moving the $\check{\mathsf{R}}$-matrix to their right, we encounter the term

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1}\middle|\tau+2\eta S^{i-1}+\eta\sigma_0^3\right)T_\tau^{\sigma_0^3} = T_\tau^{\sigma_0^3}\check{\mathsf{R}}^{(2,2)}_{i,i+1}\left(\lambda_{i,i+1}\middle|\tau+2\eta S^{i-1}\right),$$

where we applied (4.50). Using again (4.54) and (4.56), we note that commuting the $\check{\mathsf{R}}$-matrix to the right yields the monodromy matrix with λ_i and λ_{i+1} exchanged, which leads to the right-hand side of (4.74). □

The exchange relations are instrumental in proving a host of properties that involve the inhomogeneity parameters. As a first application, we use them to establish the vanishing of a triple-product of monodromy-matrix entries.

Proposition 4.1.17. *For each $j=1,\dots,N$, we have*

$$\mathcal{C}(\lambda_j)\mathcal{C}(\lambda_j-\eta)\mathcal{C}(\lambda_j-2\eta)=0. \quad (4.77)$$

Proof. First, we examine the case $j=N$. We expand the product of operators $\mathcal{C}$ according to their definition. The resulting expression contains the factor

$$(\langle\downarrow\downarrow\downarrow|_{00'0''}\otimes\mathbb{I})\mathsf{R}^{(1,2)}_{0N}\left(0\middle|\tau+2\eta S^{N-1}\right)\mathsf{R}^{(1,2)}_{0'N}\left(-\eta\middle|\tau+2\eta S^{N-1}+\eta\right) \times \mathsf{R}^{(1,2)}_{0''N}\left(-2\eta\middle|\tau+2\eta S^{N-1}+2\eta\right)=0, \quad (4.78)$$

which follows from a direct computation.

Second, we combine this result with the exchange relations to prove (4.77) for $j = 1, \dots, N-1$. We write $\mathcal{C}(\lambda_0) = \mathcal{C}(\lambda_0; \lambda_1, \dots, \lambda_N)$ to stress the dependence on the inhomogeneity parameters. Using Proposition 4.1.16, we find

$$\begin{aligned}
\overset{\curvearrowleft}{\prod_{k=j}^{N-1}} \check{\mathsf{R}}^{(2,2)}_{k,k+1}\left(\lambda_{j,k+1} \,\middle|\, \tau + 2\eta S^{k-1}\right) \mathcal{C}(\lambda_j)\mathcal{C}(\lambda_j - \eta)\mathcal{C}(\lambda_j - 2\eta) \\
= \mathcal{C}(\lambda_j; \cdots \lambda_N, \lambda_j)\mathcal{C}(\lambda_j - \eta; \cdots \lambda_N, \lambda_j)\mathcal{C}(\lambda_j - 2\eta; \cdots \lambda_N, \lambda_j) \\
\times \overset{\curvearrowleft}{\prod_{k=j}^{N-1}} \check{\mathsf{R}}^{(2,2)}_{k,k+1}\left(\lambda_{j,k+1} \,\middle|\, \tau + 2\eta S^{k-1}\right). \quad (4.79)
\end{aligned}$$

From the case $j = N$ studied above, the right-hand side vanishes. Hence, we obtain

$$\overset{\curvearrowleft}{\prod_{j=i}^{N-1}} \check{\mathsf{R}}^{(2,2)}_{j,j+1}\left(\lambda_{i,j+1} \,\middle|\, \tau + 2\eta S^{j-1}\right) \mathcal{C}(\lambda_i)\mathcal{C}(\lambda_i - \eta)\mathcal{C}(\lambda_i - 2\eta) = 0 \quad (4.80)$$

The product of $\check{\mathsf{R}}$-matrices on the left-hand side is invertible by Proposition 4.1.8, for generic inhomogeneity parameters. By continuity, this implies (4.77) for $j = 1, \dots, N-1$ and arbitrary inhomogeneity parameters. □

Analyticity and pseudo-periodicity We now investigate the analyticity and pseudo-periodicity properties of the monodromy matrix. First, we characterise the monodromy matrix as a function of its spectral parameter. To simplify our presentation, we will often attribute a property to the monodromy matrix (or its entries) if this property holds for all its matrix elements with respect to the canonical basis of $\mathbb{C}^2 \otimes \mathbb{D}_N$ (or $\mathbb{D}_N$).

Proposition 4.1.18. *The monodromy matrix $\mathcal{M}(\lambda)$ is an entire function of the spectral parameter λ. Moreover, it fulfils*

$$\mathcal{M}_0(\lambda + \pi) = (-1)^N (\sigma_0^3)^{\mathsf{y}} \mathcal{M}_0(\lambda)(\sigma_0^3)^{\mathsf{y}}, \quad (4.81)$$

$$\begin{aligned}
\mathcal{M}_0(\lambda + \pi\omega) = \left(-p^{-1}\mathrm{e}^{-2\mathrm{i}(\lambda+\eta)}\right)^N \mathrm{e}^{2\mathrm{i}\sum_{j=1}^N \lambda_j} p^{\mathsf{y}\sigma_0^3/2} \mathrm{e}^{-\mathrm{i}(\tau+2\eta S)\sigma_0^3} \quad (4.82) \\
\times \mathcal{M}_0(\lambda)\mathrm{e}^{\mathrm{i}\tau\sigma_0^3} p^{-\mathsf{y}\sigma_0^3/2} \mathrm{e}^{-\mathrm{i}\eta}.
\end{aligned}$$

Proof. The analyticity follows from Proposition 4.1.7 and the definition of $\mathcal{M}(\lambda)$. For the pseudo-periodicity, we apply Proposition 4.1.7 to each R-matrix in the definition (4.54). It yields

$$\mathsf{M}_0(\lambda+\pi|t) = (-1)^N(\sigma_0^3)^{\mathsf{y}}\mathsf{M}_0(\lambda|t)(\sigma_0^3)^{\mathsf{y}}, \tag{4.83}$$

$$\mathsf{M}_0(\lambda+\pi\omega|t) = (-p^{-1}\mathrm{e}^{-2\mathrm{i}(\lambda+\eta)})^N \mathrm{e}^{2\mathrm{i}\sum_{j=1}^N \lambda_j} p^{\mathsf{y}\sigma_0^3/2}\mathrm{e}^{-\mathrm{i}(t+2\eta S)\sigma_0^3} \times \mathsf{M}_0(\lambda|t)\mathrm{e}^{\mathrm{i}t\sigma_0^3}p^{-\mathsf{y}\sigma_0^3/2}. \tag{4.84}$$

The proposition follows from the commutation relation (4.50). □

Corollary 4.1.19. *The monodromy-matrix entries $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$ are entire functions of the spectral parameter λ. Moreover, they fulfil*

$$\mathcal{B}(\lambda+\pi) = (-1)^{N+\mathsf{y}}\mathcal{B}(\lambda), \tag{4.85}$$

$$\mathcal{B}(\lambda+\pi\omega) = p^{\mathsf{y}}(-p^{-1}\mathrm{e}^{-2\mathrm{i}(\lambda+\eta)})^N \mathrm{e}^{2\mathrm{i}\sum_{j=1}^N \lambda_j}\mathrm{e}^{-2\mathrm{i}S_\tau}\mathcal{B}(\lambda), \tag{4.86}$$

$$\mathcal{C}(\lambda+\pi) = (-1)^{N+\mathsf{y}}\mathcal{C}(\lambda), \tag{4.87}$$

$$\mathcal{C}(\lambda+\pi\omega) = p^{-\mathsf{y}}(-p^{-1}\mathrm{e}^{-2\mathrm{i}(\lambda+\eta)})^N \mathrm{e}^{2\mathrm{i}\sum_{j=1}^N \lambda_j}\mathrm{e}^{2\mathrm{i}S_\tau}\mathcal{C}(\lambda). \tag{4.88}$$

Furthermore, we characterise $\mathcal{M}(\lambda)$ as a function of the inhomogeneity parameters. To this end, we introduce the following operators:

$$\Gamma^j = \mathrm{e}^{2\mathrm{i}(\tau+2\eta S^{j-1})s_j^3+2\mathrm{i}\eta(s_j^3)^2}p^{-\mathsf{y}s_j^3}, \qquad j=1,\ldots,N. \tag{4.89}$$

Proposition 4.1.20. *The monodromy matrix $\mathcal{M}(\lambda)$ is an entire function of each inhomogeneity parameter. Moreover, for each $i=1,\ldots,N$, it fulfils*

$$\mathcal{M}_0(\lambda_0)|_{\lambda_i\to\lambda_i+\pi} = -(\Omega_i^3)^{\mathsf{y}}\mathcal{M}_0(\lambda_0)(\Omega_i^3)^{\mathsf{y}}, \tag{4.90a}$$

$$\mathcal{M}_0(\lambda_0)|_{\lambda_i\to\lambda_i+\pi\omega} = -p^{-1}\mathrm{e}^{2\mathrm{i}(\lambda_{0i}+\eta)}\,(\Gamma^i)^{-1}\mathcal{M}_0(\lambda_0)\,\Gamma^i. \tag{4.90b}$$

Proof. The only dependence of the monodromy matrix on the inhomogeneity parameter λ_i appears in the $\mathsf{R}_{0i}^{(1,2)}(\lambda_{0i}|\tau+2\eta S^{i-1})$ in (4.54). The analyticity follows directly from Proposition 4.1.7. For the pseudo-periodicity, we focus on the proof of (4.90b). The reasoning behind (4.90a) is similar. We consider

$$\mathcal{M}_0(\lambda_0)|_{\lambda_i \to \lambda_i + \pi\omega} = \prod_{j=i+1}^{\overset{\curvearrowleft}{N}} \mathsf{R}^{(1,2)}_{0j}\left(\lambda_{0j}\,\middle|\tau + 2\eta S^{j-1}\right)$$

$$\times \mathsf{R}^{(1,2)}_{0i}(\lambda_{0i} - \pi\omega|\tau + 2\eta S^{i-1}) \prod_{j=1}^{\overset{\curvearrowleft}{i-1}} \mathsf{R}^{(1,2)}_{0j}\left(\lambda_{0j}\,\middle|\tau + 2\eta S^{j-1}\right) T_\tau^{\sigma_0^3}. \quad (4.91)$$

Propositions 4.1.6 and 4.1.7 allow us to rewrite the factor involving $\mathsf{R}^{(1,2)}_{0i}(\lambda|t)$ as

$$\mathsf{R}^{(1,2)}_{0i}(\lambda_{0j} - \pi\omega|\tau')$$
$$= -p^{-1}\mathrm{e}^{2\mathrm{i}(\lambda_{0i}+\eta)}(\Gamma^i)^{-1}\,\mathsf{R}^{(1,2)}_{0i}(\lambda_{0i}|\tau')\,\Gamma^i \mathrm{e}^{2i\eta\sigma_0^3 s_i^3}, \quad (4.92)$$

where $\tau' = \tau + \eta S^{i-1}$. Next, we commute $(\Gamma^i)^{-1}$ and $\Gamma^i \mathrm{e}^{2i\eta\sigma_0^3 s_i^3}$ to the left and right of the products of R-matrices in (4.91), respectively. Finally, using (4.50), we obtain (4.90b). $\square$

Recursive structure In (4.54) and (4.56), we defined $\mathsf{M}_0(\lambda|t)$ and $\mathcal{M}_0(\lambda)$ as operators acting on $\mathbb{C}^2 \otimes (\mathbb{C}^3)^{\otimes N}$ and $\mathbb{C}^2 \otimes \mathbb{D}_N$, respectively. Let us highlight the size of these spaces by writing N as superscript on the operators and their entries. Moreover, we denote by $\mathcal{M}_0^{N-1}(\lambda) \otimes \mathbb{I}$ the natural embedding of $\mathcal{M}_0^{N-1}(\lambda)$ into $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{D}_N)$ that acts like the identity on the N-th factor of $(\mathbb{C}^3)^{\otimes N}$. This notation naturally carries over to the monodromy-matrix entries. The following proposition expresses $\mathcal{A}^N(\lambda) = \mathcal{A}(\lambda)$ and $\mathcal{C}^N(\lambda) = \mathcal{C}(\lambda)$ in terms of $\mathcal{A}^{N-1}(\lambda) \otimes \mathbb{I}$ and $\mathcal{C}^{N-1}(\lambda) \otimes \mathbb{I}$:

Lemma 4.1.21. *For each $N \geq 2$, we have the following decompositions:*

$$\mathcal{A}^N(\lambda_0) = \mathsf{A}^1_N(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{A}^{N-1}(\lambda_0) \otimes \mathbb{I}) \quad (4.93)$$
$$+ \mathsf{B}^1_N(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{C}^{N-1}(\lambda_0) \otimes \mathbb{I}),$$
$$\mathcal{C}^N(\lambda_0) = \mathsf{C}^1_N(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{A}^{N-1}(\lambda_0) \otimes \mathbb{I}) \quad (4.94)$$
$$+ \mathsf{D}^1_N(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{C}^{N-1}(\lambda_0) \otimes \mathbb{I}),$$

Proof. We focus on the proof of (4.93). The same logic applies for (4.94). Using (4.54) and (4.56), we write

$$\mathcal{M}_0^N(\lambda_0) = \mathsf{R}^{(1,2)}_{0N}(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{M}_0^{N-1}(\lambda_0) \otimes \mathbb{I}). \quad (4.95)$$

We consider the monodromy-matrix entry

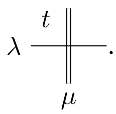
$$\mathcal{A}^N(\lambda_0) = \langle\uparrow|_0\, \mathcal{M}_0^N(\lambda_0)\, |\uparrow\rangle_0 \tag{4.96}$$
$$= \langle\uparrow|_0\, \mathsf{R}_{0N}^{(1,2)}(\lambda_{0N}|\tau + 2\eta S^{N-1})(\mathcal{M}_0^{N-1}(\lambda_0) \otimes \mathbb{I})\, |\uparrow\rangle_0\,. \tag{4.97}$$

Inserting $\mathbb{I} = |\uparrow\rangle_0 \langle\uparrow|_0 + |\downarrow\rangle_0 \langle\downarrow|_0$ right after the R-matrix in (4.97), we get

$$\mathcal{A}^N(\lambda_0) = \langle\uparrow|_0\, \mathsf{R}_{0N}^{(1,2)}(\lambda_{0N}|\tau + 2\eta S^{N-1})\, |\uparrow\rangle_0\, (\langle\uparrow|_0\, \mathcal{M}_0^{N-1}(\lambda_0)\, |\uparrow\rangle_0 \otimes \mathbb{I})$$
$$+ \langle\uparrow|_0\, \mathsf{R}_{0N}^{(1,2)}(\lambda_{0N}|\tau + 2\eta S^{N-1})\, |\downarrow\rangle_0\, (\langle\downarrow|_0\, \mathcal{M}_0^{N-1}(\lambda_0)\, |\uparrow\rangle_0 \otimes \mathbb{I}). \tag{4.98}$$

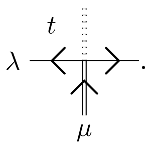
Finally, we evaluate the various matrix elements of the right-hand side with the help of (4.54) and (4.56). □

Diagrammatic calculus Below, we sometimes resort to graphical arguments. This common technique in the theory of planar lattice models provides a convenient way to visualise and manipulate algebraic expressions. The building blocks of our graphical expressions are the R-matrices. Recall that the d10V R-matrix is an operator on $\mathbb{C}^2 \otimes \mathbb{C}^3$. We represent it as

$\mathsf{R}^{(1,2)}(\lambda - \mu|t) =$ 

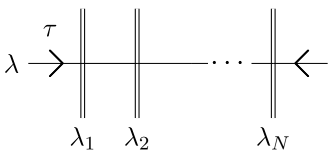
. (4.99)

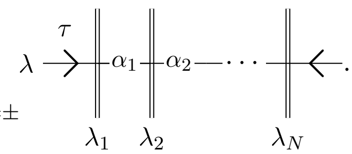
The single line represents the space $\mathbb{C}^2$, and the double line the space $\mathbb{C}^3$. Assigning orientations to vertex edges corresponds to evaluating a matrix element of the R-matrix. For instance, we have

$\mathsf{j}(\lambda - \mu|t) = \langle\uparrow 0|\ \mathsf{R}^{(1,2)}(\lambda - \mu|t)\ |\downarrow\Uparrow\rangle =$ 

. (4.100)

When combining R-matrices, any inner edge without an arrow obeys an implicit summation rule: the diagram corresponds to the sum over all orientations of this edge. Let us take the graphical representation of the operator $\mathcal{C}$ as an example,

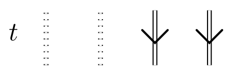
$\mathcal{C}(\lambda) =$ 

$= \sum_{\substack{\alpha_1,\ldots,\\ \alpha_{N-1}=\pm}}$ 

.

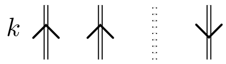
The elements of the spin-dynamical space and its dual are respectively represented as

$$|00{\Downarrow}{\Downarrow}\rangle \otimes |t\rangle = \tag{4.101}$$



,

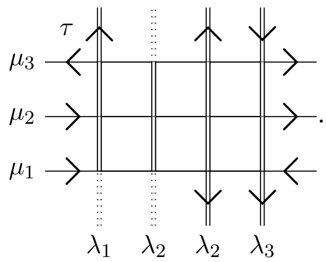
$$\langle{\Uparrow}{\Uparrow}0{\Downarrow}| \otimes \langle k| = \tag{4.102}$$



.

Lastly, the right-to-left order of application in dual pairings translates to diagrams drawn from bottom to top. Here is an example:

$$\langle{\Uparrow}0{\Uparrow}{\Downarrow}|\, \mathcal{B}(\mu_3)\mathcal{A}(\mu_2)\mathcal{C}(\mu_1)\, |00{\Downarrow}{\Downarrow}\rangle = \tag{4.103}$$



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4.1.4 The transfer matrix

In this section, we define the transfer matrices of the inhomogeneous d10V and d19V models with anti-periodic boundary conditions. For this purpose, we view the spin-dynamical space and its dual as $\mathbb{Z}$-*graded* vector spaces. We analyse the transfer matrices on the subspaces of *homogeneous elements of degree* 0. In particular, on these subspaces, we establish the existence of an infinite family of commuting transfer matrices. Moreover, we establish the so-called fusion relations, which connects the transfer matrices of the two models.

Spin-dynamical space decomposition Recall that the definition of the dynamical vertex models involves a reference height ℓ_0, that we have not used so far. Below, we focus on the case where

$$\ell_0 = \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi\omega}{2}, \tag{4.104}$$

with

$$(\mathsf{x},\mathsf{y}) \in \{(1,0),(0,1),(1,1)\}. \tag{4.105}$$

The integer y explicitly appears in the weights of the R-matrices of the dynamical ten- and nineteen-vertex models. Conversely, x only occurs through the action of certain operators, such as the entries of the monodromy matrix, on the spin-dynamical space or its dual.[5] The limitation to (4.105) guarantees that the height parameter of any vertex weight always respects the constraint (4.3), which implies that the vertex weights are always well-defined.

Following [58, 59, 94], we introduce the operator

$$S_\tau = \tau + \eta S. \tag{4.106}$$

on $\mathbb{D}_N$ and $\mathbb{D}_N^*$. This operator is diagonalisable, with eigenvalues

$$r\eta + \mathsf{x}\tfrac{\pi}{2} + \mathsf{y}\tfrac{\pi\omega}{2}, \qquad \forall\, r \in \mathbb{Z}. \tag{4.107}$$

We denote by $\mathbb{D}_N^{(r)}$ the corresponding eigenspace, and by $\mathbb{D}_N^{(r)*}$ its dual. The elements of their canonical basis are respectively given by

$$|\boldsymbol{\alpha}\rangle \otimes |t_{\boldsymbol{\alpha},r}\rangle\,, \qquad \langle\boldsymbol{\alpha}| \otimes \langle t_{\boldsymbol{\alpha},r}|\,, \qquad \boldsymbol{\alpha} \in \{\Uparrow, 0, \Downarrow\}^N, \tag{4.108}$$

where

$$t_{\boldsymbol{\alpha},r} = \eta(r - s_{\boldsymbol{\alpha}}) + \mathsf{x}\tfrac{\pi}{2} + \mathsf{y}\tfrac{\pi\omega}{2}, \tag{4.109}$$

and $s_{\boldsymbol{\alpha}}$ is the magnetisation of $\boldsymbol{\alpha}$, defined through $S|\boldsymbol{\alpha}\rangle = s_{\boldsymbol{\alpha}}|\boldsymbol{\alpha}\rangle$. The space $\mathbb{D}_N$ is a direct sum of the eigenspaces of S_τ,

$$\mathbb{D}_N = \bigoplus_{r\in\mathbb{Z}} \mathbb{D}_N^{(r)} \qquad\qquad \mathbb{D}_N^* = \bigoplus_{r\in\mathbb{Z}} \mathbb{D}_N^{(r)*}. \tag{4.110}$$

Thus, we can think of $\mathbb{D}_N$ and its dual as $\mathbb{Z}$-graded vector spaces, with the integer r playing the role of the grading of the homogeneous subspaces. We now study the action of the monodromy-matrix entries on these subspaces. To this end, we use the following lemma:

[5] Studying this action reveals that the pair (x,y) can be thought of as the characteristics of certain theta functions. We will, however, not develop this point of view in this chapter.

Lemma 4.1.22. *We have the commutation relations*

$$[\mathcal{A}(\lambda), S] = 0, \qquad [\mathcal{B}(\lambda), S] = \mathcal{B}(\lambda), \tag{4.111a}$$

$$[\mathcal{C}(\lambda), S] = -\mathcal{C}(\lambda), \qquad [\mathcal{D}(\lambda), S] = 0, \tag{4.111b}$$

and

$$[\mathcal{A}(\lambda), S_\tau] = \eta\mathcal{A}(\lambda), \qquad [\mathcal{B}(\lambda), S_\tau] = 0, \tag{4.112a}$$

$$[\mathcal{C}(\lambda), S_\tau] = 0, \qquad [\mathcal{D}(\lambda), S_\tau] = -\eta\mathcal{D}(\lambda). \tag{4.112b}$$

Proof. By Proposition 4.1.6, (4.54) and (4.56), we have the commutation relation

$$[\mathcal{M}_0(\lambda), \sigma_0^3 + 2S] = 0. \tag{4.113}$$

We obtain (4.111) by writing out this relation for the monodromy-matrix entries. Moreover, the relations (4.112) follow from (4.111) and

$$[\mathcal{A}(\lambda), \tau] = \eta\mathcal{A}(\lambda), \qquad [\mathcal{B}(\lambda), \tau] = -\eta\mathcal{B}(\lambda), \tag{4.114}$$

$$[\mathcal{C}(\lambda), \tau] = \eta\mathcal{C}(\lambda), \qquad [\mathcal{D}(\lambda), S] = -\eta\mathcal{D}(\lambda), \tag{4.115}$$

which are a straightforward consequence of (4.50). □

An immediate consequence of this lemma is:

Corollary 4.1.23. *For each $r \in \mathbb{Z}$, we have*

$$\mathcal{A}(\lambda)\mathbb{D}_N^{(r)} \subset \mathbb{D}_N^{(r+1)}, \qquad \mathcal{B}(\lambda)\mathbb{D}_N^{(r)} \subset \mathbb{D}_N^{(r)}, \tag{4.116}$$

$$\mathcal{C}(\lambda)\mathbb{D}_N^{(r)} \subset \mathbb{D}_N^{(r)}, \qquad \mathcal{D}(\lambda)\mathbb{D}_N^{(r)} \subset \mathbb{D}_N^{(r-1)}, \tag{4.117}$$

and

$$\mathbb{D}_N^{(r)*}\mathcal{A}(\lambda) \subset \mathbb{D}_N^{(r-1)*}, \qquad \mathbb{D}_N^{(r)*}\mathcal{B}(\lambda) \subset \mathbb{D}_N^{(r)*}, \tag{4.118}$$

$$\mathbb{D}_N^{(r)*}\mathcal{C}(\lambda) \subset \mathbb{D}_N^{(r)*}, \qquad \mathbb{D}_N^{(r)*}\mathcal{D}(\lambda) \subset \mathbb{D}_N^{(r+1)*}. \tag{4.119}$$

The transfer matrix The transfer matrix of the inhomogeneous dynamical ten-vertex model with anti-periodic boundary conditions is the linear operator $\mathcal{T}(\lambda)$ on $\mathbb{D}_N$ defined by

$$\mathcal{T}(\lambda) = \mathrm{tr}_0\left(\sigma_0^1\mathcal{M}_0(\lambda)\right) = \mathcal{B}(\lambda) + \mathcal{C}(\lambda), \tag{4.120}$$

where tr_0 denotes the trace over the auxiliary space. By Corollary 4.1.23, the transfer matrix preserves the homogeneous subspaces of $\mathbb{D}_N$ and $\mathbb{D}_N^*$. Its action on $\mathbb{D}_N^{(0)}$ and $\mathbb{D}_N^{(0)*}$ is particularly interesting. We now discuss several important properties of the transfer matrix on these subspaces. First, we establish that the d10V transfer matrix defines a one-parameter family of commuting operators on these subspaces.

Proposition 4.1.24. *For all λ, λ', we have the commutation relation*

$$[\mathcal{T}(\lambda), \mathcal{T}(\lambda')] = 0 \tag{4.121}$$

on $\mathbb{D}_N^{(0)}$ and $\mathbb{D}_N^{(0)*}$.

Proof. By (4.120), we have

$$\mathcal{T}(\lambda_0)\mathcal{T}(\lambda_{0'}) = \mathrm{tr}_{00'}\left(\sigma_0^1\sigma_{0'}^1\mathcal{M}_0(\lambda_0)\mathcal{M}_{0'}(\lambda_{0'})\right). \tag{4.122}$$

Let $\lambda_{00'} \not\equiv \pm\eta \,(\mathrm{mod}\, \pi,\, \pi\omega)$. We use Proposition 4.1.11, as well as the invertibility of the R-matrix of the d6V model, inferred from Proposition 4.1.3, to get

$$\mathcal{T}(\lambda_0)\mathcal{T}(\lambda_{0'}) = \mathrm{tr}_{00'}\Big(\sigma_0^1\sigma_{0'}^1\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau + 2\eta S)^{-1} \\ \times \mathcal{M}_{0'}(\lambda_{0'})\mathcal{M}_0(\lambda_0)\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau)\Big). \tag{4.123}$$

The cyclic property of the trace allows us to move the rightmost R-matrix to the left of the trace. By (4.60), its height parameter stays unchanged,

$$\mathcal{T}(\lambda_0)\mathcal{T}(\lambda_{0'}) = \mathrm{tr}_{00'}\Big(\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau)\sigma_0^1\sigma_{0'}^1\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau + 2\eta S)^{-1} \\ \times \mathcal{M}_{0'}(\lambda_{0'})\mathcal{M}_0(\lambda_0)\Big). \tag{4.124}$$

Using the symmetries of Proposition 4.1.2, we obtain

$$\mathcal{T}(\lambda_0)\mathcal{T}(\lambda_{0'}) = \mathrm{tr}_{00'}\Big(\sigma_0^1\sigma_{0'}^1\mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}| -\tau + \mathsf{x}\pi + \mathsf{y}\pi\omega) \\ \times \mathsf{R}_{00'}^{(1,1)}(\lambda_{00'}|\tau + 2\eta S)^{-1}\mathcal{M}_{0'}(\lambda_{0'})\mathcal{M}_0(\lambda_0)\Big). \tag{4.125}$$

Note that on $\mathbb{D}_N^{(0)*}$, we may replace $\tau + 2\eta S = -\tau + \mathsf{x}\pi + \mathsf{y}\pi\omega$. Therefore, we have $\mathcal{T}(\lambda_0)\mathcal{T}(\lambda_{0'}) = \mathcal{T}(\lambda_{0'})\mathcal{T}(\lambda_0)$ on $\mathbb{D}_N^{(0)*}$. By continuity, the commutation relation holds for $\lambda_{00'} \equiv \pm\eta \,(\mathrm{mod}\, \pi,\, \pi\omega)$, as well. Moreover, it also holds on $\mathbb{D}_N^{(0)}$ by duality. □

Second, we characterise $\mathcal{T}(\lambda)$ as a theta function of the spectral parameter.

Lemma 4.1.25. *The operator* $\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda)$*, restricted to* $\mathbb{D}_N^{(0)}$ *or* $\mathbb{D}_N^{(0)*}$*, is a theta function in* λ *of order* N*, nome* p*, and norm* $\sum_{j=1}^{N}(\lambda_j-\eta)+\mathsf{x}\frac{\pi}{2}+\mathsf{y}\frac{\pi\omega}{2}$.

Proof. By Corollary 4.1.19, the transfer matrix is an entire function of λ. Moreover, we obtain the following pseudo-periodicity relations:

$$\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda)\Big|_{\lambda\to\lambda+\pi} = (-1)^N\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda), \tag{4.126}$$

$$\begin{aligned}\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda)\Big|_{\lambda\to\lambda+\pi\omega} &= (-1)^N\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}p^{-N+\mathsf{y}}\mathrm{e}^{-2\mathsf{i}\sum_{j=1}^{N}(\lambda+\eta-\lambda_j)} \\ &\qquad\times\left(p^{\mathsf{y}}\mathrm{e}^{-2\mathrm{i}S_\tau}\mathcal{B}(\lambda)+p^{-\mathsf{y}}\mathrm{e}^{2\mathrm{i}S_\tau}\mathcal{C}(\lambda)\right).\end{aligned} \tag{4.127}$$

To simplify the second equality, we note that S_τ commutes with $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$ by Lemma 4.1.22. Hence, on $\mathbb{D}_N^{(0)}$, we may replace S_τ by $\mathsf{x}\frac{\pi}{2}+\mathsf{y}\frac{\pi\omega}{2}$. The second pseudo-periodicity (4.127) simplifies to

$$\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda)\Big|_{\lambda\to\lambda+\pi\omega} = (-1)^{N+\mathsf{x}}p^{-N+\mathsf{y}}\mathrm{e}^{-2\mathsf{i}\sum_{j=1}^{N}(\lambda+\eta-\lambda_j)}\mathrm{e}^{\mathsf{i}\mathsf{y}\lambda}\mathcal{T}(\lambda). \tag{4.128}$$

The proposition follows from (4.126), (4.128) and Definition A.2.1. □

Third, we examine a spin-reversal symmetry of the transfer matrix. The *spin-reversal operator* F on $(\mathbb{C}^3)^{\otimes N}$ is

$$F = (-1)^N(\Omega^1)^{\otimes N}. \tag{4.129}$$

Its generalisation to the spin-dynamical space is

$$F_\tau = FT_\tau^{-2S}. \tag{4.130}$$

We find that F_τ preserves $\mathbb{D}_N^{(0)}$ and $\mathbb{D}_N^{(0)*}$.

Lemma 4.1.26. *We have*

$$F_\tau\mathcal{B}(\lambda) = \mathcal{C}(\lambda)F_\tau, \qquad F_\tau\mathcal{C}(\lambda) = \mathcal{B}(\lambda)F_\tau, \tag{4.131}$$

on $\mathbb{D}_N^{(0)}$ *and* $\mathbb{D}_N^{(0)*}$.

Proof. Since $F_\tau^2 = \mathbb{I}$ on $\mathbb{D}_N^{(0)}$, it is sufficient to prove the first equality.

First, we compute

$$\mathcal{C}(\lambda)F = \mathsf{C}(\lambda|\tau)T_\tau^+ F = \langle\downarrow|_0\, \mathsf{M}_0(\lambda|\tau)\,|\uparrow\rangle_0\, FT_\tau^+ \tag{4.132}$$

$$= \langle\uparrow|_0\, \sigma_0^1 \mathsf{M}_0(\lambda|\tau)\sigma_0^1 F\,|\downarrow\rangle_0\, T_\tau^+ . \tag{4.133}$$

Using the symmetry properties of Proposition 4.1.6 and the definition (4.54), it is straightforward to establish the commutation relation $\mathsf{M}_0(\lambda|\tau)\sigma_0^1 F = \sigma_0^1 F\mathsf{M}_0(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)$, which yields

$$\mathcal{C}(\lambda)F = F\,\langle\uparrow|_0\, \mathsf{M}_0(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)\,|\downarrow\rangle_0\, T_\tau^+ \tag{4.134}$$

$$= F\mathsf{B}(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)T_\tau^+ . \tag{4.135}$$

Second, let $\boldsymbol{\alpha} \in \{\Uparrow, 0, \Downarrow\}^N$. We use the following shorthand notation for the elements of the canonical basis of $\mathbb{D}_N^{(0)}$:

$$|e_{\boldsymbol{\alpha}}^{(0)}\rangle = |\boldsymbol{\alpha}\rangle \otimes |t_{\boldsymbol{\alpha},0}\rangle\,. \tag{4.136}$$

We examine the action

$$\mathcal{C}(\lambda)F_\tau\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle = \mathsf{C}(\lambda|\tau)FT_\tau^{-2S+1}\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle \tag{4.137}$$

$$= F\mathsf{B}(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)T_\tau^{-2S+1}\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\,, \tag{4.138}$$

where the last equality follows from (4.135). Moreover, by (4.50), we have

$$\begin{aligned}\mathcal{C}(\lambda)F_\tau\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle &= F\mathsf{B}(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)T_\tau^{-2s_{\boldsymbol{\alpha}}+1}\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\\ &= FT_\tau^{-2(s_{\boldsymbol{\alpha}}-1)}\mathsf{B}(\lambda|-\tau-2\eta(s_{\boldsymbol{\alpha}}-1)+\mathsf{x}\pi+\mathsf{y}\pi\omega)T_\tau^-\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\,. \end{aligned}\tag{4.139}$$

Since $\tau T_\tau^-\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle = (t_{\boldsymbol{\alpha},0}+\eta)T_\tau^-\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle$ and, crucially, $t_{\boldsymbol{\alpha},0} = -\eta s_{\boldsymbol{\alpha}} + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi\omega}{2}$, we may rewrite this expression as

$$\begin{aligned}\mathcal{C}(\lambda)F_\tau\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle &= FT_\tau^{-2(s_{\boldsymbol{\alpha}}-1)}\mathsf{B}(\lambda|t_{\boldsymbol{\alpha},0}+\eta)T_\tau^-\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\\ &= FT_\tau^{-2(s_{\boldsymbol{\alpha}}-1)}\mathsf{B}(\lambda|\tau)T_\tau^-\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle = FT_\tau^{-2(s_{\boldsymbol{\alpha}}-1)}\mathcal{B}(\lambda)\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\,. \end{aligned}\tag{4.140}$$

By Lemma 4.1.22, we may replace the action of $T_\tau^{-2(s_{\boldsymbol{\alpha}}-1)}$ by T_τ^{-2S}, which yields

$$\mathcal{C}(\lambda)F_\tau\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle = F_\tau\mathcal{B}(\lambda)\,|e_{\boldsymbol{\alpha}}^{(0)}\rangle\,. \tag{4.141}$$

This equality holds for any spin configuration $\boldsymbol{\alpha}$, which proves $F_\tau\mathcal{B}(\lambda) = \mathcal{C}(\lambda)F_\tau$ on $\mathbb{D}_N^{(0)}$. On $\mathbb{D}_N^{(0)*}$, it follows by duality. □

The spin-reversal symmetry of the transfer matrix follows straightforwardly from the previous lemma.

Proposition 4.1.27. *We have* $[\mathcal{T}(\lambda), F_\tau] = 0$, *on* $\mathbb{D}_N^{(0)}$ *and* $\mathbb{D}_N^{(0)*}$.

The fusion relation We now show how to obtain the transfer matrix of the d19V model from that of the d10V. This property allows us to focus our analysis on the d10V model. We use the superscripts (1) and (2) to distinguish between quantities of the d10V and d19V models, respectively.

By analogy with (4.54) and (4.56), we define the monodromy matrix of the inhomogeneous d19V model as the following operator on $\mathbb{C}^3 \otimes \mathbb{D}_N$:

$$\mathcal{M}^{(2)}(\lambda) = \left(\prod_{j=1}^{\overset{\curvearrowleft}{N}} \mathsf{R}_{0j}^{(2,2)} \left(\lambda_{0j} \,\middle|\, \tau + 2\eta S^{j-1} \right) \right) T_\tau^{2s_0^3}. \tag{4.142}$$

Here, the auxiliary space labelled by 0 is the space $\mathbb{C}^3$. The transfer matrix of the inhomogeneous d19V model with anti-periodic boundary conditions is defined as

$$\mathcal{T}^{(2)}(\lambda) = -\operatorname{tr}_0 \left(\Omega_0^1 \mathcal{M}_0^{(2)}(\lambda) \right). \tag{4.143}$$

We now write this transfer matrix in terms of the $\mathcal{T}(\lambda) = \mathcal{T}^{(1)}(\lambda)$.

Proposition 4.1.28 (Fusion relation)**.** *We have the identity*

$$\mathcal{T}^{(2)}(\lambda) = \mathcal{T}^{(1)}(\lambda)\mathcal{T}^{(1)}(\lambda - \eta) + \mathrm{A}(\lambda)\mathrm{D}(\lambda - \eta) \frac{\mathrm{e}^{-2\mathrm{i}y\eta S} \vartheta_1(\tau)}{\vartheta_1(\tau + 2\eta S)}. \tag{4.144}$$

Proof. Using the identity $\mathbb{I} = P^{(1)+} + P^{(1)-}$, we may write

$$\mathcal{T}^{(1)}(\lambda_0)\mathcal{T}^{(1)}(\lambda_0 - \eta) = \mathcal{O}^+(\lambda_0) + \mathcal{O}^-(\lambda_0) \tag{4.145}$$

with the two terms

$$\mathcal{O}^\pm(\lambda_0) = \operatorname{tr}_{00'} \left(\sigma_0^1 \sigma_{0'}^1 P_{00'}^{(1)\pm} \mathcal{M}_0^{(1)}(\lambda_0) \mathcal{M}_{0'}^{(1)}(\lambda_0 - \eta) \right). \tag{4.146}$$

We treat these two terms separately.

First, it follows from (4.50), (4.54), (4.56), and the symmetry properties of Proposition 4.1.6, that

$$\begin{aligned} \mathcal{O}^+(\lambda_0) = \operatorname{tr}_{00'} \Bigg(\sigma_0^1 \sigma_{0'}^1 P_{00'}^{(1)+} \prod_{j=1}^{\overset{\curvearrowleft}{N}} \Big(& \mathsf{R}_{0j}^{(1,2)}(\lambda_{0j} | \tau + 2\eta S^{j-1}) \\ & \times \mathsf{R}_{0'j}^{(1,2)}(\lambda_{0j} - \eta | \tau + 2\eta S^{j-1} + \sigma_0^3) \Big) T_\tau^{\sigma_0^3 + \sigma_{0'}^3} \Bigg). \end{aligned} \tag{4.147}$$

As a straightforward consequence of Propositions 4.1.4 and 4.1.5, we have, for each $j = 1, \dots, N$,

$$\begin{aligned} &P^{(1)+}_{00'} \mathsf{R}^{(1,2)}_{0j}(\lambda_{0j}|t) \mathsf{R}^{(1,2)}_{0'j}(\lambda_{0j} - \eta | t + \eta\sigma^3_0) \\ &\qquad = P^{(1)+}_{00'} \mathsf{R}^{(1,2)}_{0j}(\lambda_{0j}|t) \mathsf{R}^{(1,2)}_{0'j}(\lambda_{0j} - \eta | t + \eta\sigma^3_0) P^{(1)+}_{00'}. \end{aligned} \tag{4.148}$$

Using this identity, we rewrite

$$\begin{aligned} \mathcal{O}^+(\lambda_0) = \operatorname{tr}_{00'}\Bigg(\sigma^1_0\sigma^1_{0'} \prod_{j=1}^{\overset{\curvearrowleft}{N}} \Bigg(& P^{(1)+}_{00'} \mathsf{R}^{(1,2)}_{0j}(\lambda_{0j}|\tau + 2\eta S^{j-1}) \\ & \times \mathsf{R}^{(1,2)}_{0'j}(\lambda_{0j} - \eta|\tau + 2\eta S^{j-1} + \sigma^3_0)\Bigg) T_\tau^{\sigma^3_0 + \sigma^3_{0'}}\Bigg). \end{aligned} \tag{4.149}$$

Moreover, from (4.148), and the commutation $[\sigma^1_0\sigma^1_{0'}, P^{(1)+}_{00'}] = 0$, the operator inside the trace in (4.149) trivially vanishes on $(\mathbb{C}^2 \wedge \mathbb{C}^2) \otimes \mathbb{D}_N$ and its dual. This property and the similarity invariance of the trace allows us to write

$$\begin{aligned} \mathcal{O}^+(\lambda_0) = \operatorname{tr}_{00'}\Bigg(& Q_{00'} U_{00'} \sigma^1_0 \sigma^1_{0'} U^{-1}_{00'} Q^t_{00'} \\ & \times \prod_{j=1}^{\overset{\curvearrowleft}{N}} \Bigg(Q_{00'} U_{00'} P^{(1)+}_{00'} \mathsf{R}^{(1,2)}_{0j}(\lambda_{0j}|\tau + 2\eta S^{j-1}) \\ & \times \mathsf{R}^{(1,2)}_{0'j}(\lambda_{0j} - \eta|\tau + 2\eta S^{j-1} + \sigma^3_0) U^{-1}_{00'} Q^t_{00'}\Bigg) \\ & \times Q_{00'} U_{00'} T_\tau^{\sigma^3_0+\sigma^3_{0'}} U^{-1}_{00'} Q^t_{00'}\Bigg), \end{aligned} \tag{4.150}$$

where Q and U are defined in (3.23) and (3.21), respectively. Using (4.23) and the identities

$$Q_{00'} U_{00'} \sigma^1_0 \sigma^1_{0'} U^{-1}_{00'} Q^t_{00'} = -\Omega^1, \tag{4.151}$$

$$Q_{00'} U_{00'} (\sigma^3_0 + \sigma^3_{0'}) U^{-1}_{00'} Q^t_{00'} = 2s^3, \tag{4.152}$$

leads to

$$\mathcal{O}^+(\lambda_0) = \mathcal{T}^{(2)}(\lambda_0). \tag{4.153}$$

Second, a direct computation reveals

$$\begin{aligned} \mathcal{O}^-(\lambda_0) = -\frac{1}{2}(&\mathcal{A}(\lambda_0)\mathcal{D}(\lambda_0 - \eta) - \mathcal{B}(\lambda)\mathcal{C}(\lambda_0 - \eta) \\ & + \mathcal{D}(\lambda_0)\mathcal{A}(\lambda_0 - \eta) - \mathcal{C}(\lambda_0)\mathcal{B}(\lambda_0 - \eta)). \end{aligned}$$

On the right-hand side, we recognise, up to a factor, the quantum determinant (4.70) of the dynamical Yang-Baxter algebra. Using Corollary 4.1.14, we find

$$\mathcal{O}^-(\lambda_0) = -\mathrm{A}(\lambda_0)\mathrm{D}(\lambda_0-\eta)\frac{\mathrm{e}^{-2\mathrm{i}y\eta S}\vartheta_1(\tau)}{\vartheta_1(\tau+2\eta S)}, \tag{4.154}$$

which ends the proof. □

4.2 Vertex-face correspondence

In order to compute the partition function of the 8V model, Baxter related it to the 8VSOS model by means of *intertwining vectors* [45]. This relation later came to be known as the *vertex-IRF transformation* or *vertex-face correspondence* [47, 89]. In the description of the 8VSOS as a dynamical vertex model, the transformation relies on an operator to connect the R-matrices of the 8V and d6V models [32, 46, 47]. In this section, we present this vertex-face correspondence. Then, we apply the fusion procedure to generalise it to the higher-spin models of Chapters 3 and 4. Finally, we use it to relate the anti-periodic d10V transfer matrix to the quasi-periodic 18V transfer matrix.

Let us define the matrix [94]

$$V^{(1)}(\lambda|t) = \begin{pmatrix} \hat{\vartheta}_2(-\lambda+t) & \hat{\vartheta}_2(\lambda+t) \\ \hat{\vartheta}_3(-\lambda+t) & \hat{\vartheta}_3(\lambda+t) \end{pmatrix} \mathrm{e}^{\mathrm{i}\mathsf{y}(t-\lambda\sigma^3)/2}. \tag{4.155}$$

This matrix allows us to formulate the *vertex-face correspondence* [45–47]:

Proposition 4.2.1. *We have*

$$\begin{aligned} R^{(1,1)}_{12}(\lambda_{12})V^{(1)}_1(\lambda_1|t)V^{(1)}_2(\lambda_2|t+\eta\sigma^3_1) \\ = V^{(1)}_2(\lambda_2|t)V^{(1)}_1(\lambda_1|t+\eta\sigma^3_2)\mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t). \end{aligned} \tag{4.156}$$

We show that for generic values of its arguments, $V^{(1)}(\lambda|t)$ is invertible.

Proposition 4.2.2. *For each $\lambda, t \not\equiv 0$ (mod $\pi, \pi\omega$), the matrix $V^{(1)}(\lambda|t)$ is invertible.*

Proof. By the addition formula of the Jacobi theta functions (A.12), we compute $\det V^{(1)}(\lambda|t) = \mathrm{e}^{\mathrm{i}\mathsf{y}t}\vartheta_1(\lambda)\vartheta_1(t)$, which is non-zero for all $\lambda, t \not\equiv 0$ $(\mathrm{mod}\,\pi, \pi\omega)$. □

Given the parity and pseudo-periodicity properties of Jacobi theta functions, the matrix $V^{(1)}(\lambda|t)$ obeys:

Lemma 4.2.3. *We have*

$$V^{(1)}(\lambda|-t) = V^{(1)}(\lambda|t)\sigma^1 \mathrm{e}^{-\mathrm{i}\mathsf{y}(t+\lambda\sigma^3)}, \tag{4.157}$$

$$V^{(1)}(\lambda|t+\mathsf{x}\pi+\mathsf{y}\pi\omega) = (-1)^{\mathsf{y}}\mathrm{i}(\mathrm{i}\sigma^1)^{\mathsf{y}}(\mathrm{i}\sigma^3)^{\mathsf{x}}V^{(1)}(\lambda|t)\mathrm{e}^{-\mathrm{i}\mathsf{y}(t-\lambda\sigma^3)}. \tag{4.158}$$

To derive the vertex-face correspondence relating the R-matrices of the 18V and d10V models, we need the following result, obtained from two consecutive applications of Proposition 4.2.1:

Lemma 4.2.4. *We have*

$$\begin{aligned} R^{(1,1)}_{12}(\lambda_{12})R^{(1,1)}_{13}(\lambda_{13})V^{(1)}_1(\lambda_1|t)V^{(1)}_3(\lambda_3|t+\eta\sigma^3_1)V^{(1)}_2(\lambda_2|t+\eta(\sigma^3_1+\sigma^3_3)) \\ = V^{(1)}_3(\lambda_3|t)V^{(1)}_2(\lambda_2|t+\eta\sigma^3_3)V^{(1)}_1(\lambda_1|t+\eta(\sigma^3_2+\sigma^3_3)) \\ \times \mathsf{R}^{(1,1)}_{12}(\lambda_{12}|t+\eta\sigma^3_3)\mathsf{R}^{(1,1)}_{13}(\lambda_{13}|t). \end{aligned} \tag{4.159}$$

4.2.1 Fusion of the vertex-face correspondence

We apply the fusion procedure described in Section 4.1.2 to $V^{(1)}(\lambda|t)$ in order to build its spin-1 generalisation. It reads

$$V^{(2)}(\lambda|t) = Q_{12}U_{12}[P^{(1)+}_{12}V^{(1)}_1(\lambda|t)V^{(1)}_2(\lambda-\eta|t+\eta\sigma^3_1)]U^{-1}_{12}Q^t_{12}, \tag{4.160}$$

with U and Q as in (3.21) and (3.23). A direct computation involving identities between Jacobi theta functions leads to the explicit expression:

$$\begin{aligned} &V^{(2)}(\lambda|t) = \\ &\begin{pmatrix} \hat\vartheta_2(t-\lambda)\hat\vartheta_2(t-\lambda+2\eta) & \alpha^{-1}\hat\vartheta_2(t-\lambda)\hat\vartheta_2(t+\lambda) & \hat\vartheta_2(t+\lambda)\hat\vartheta_2(t+\lambda-2\eta) \\ \alpha\vartheta_2(\eta)\vartheta_2(t-\lambda+\eta) & \vartheta_2(t)\vartheta_2(\lambda) & \alpha\vartheta_2(\eta)\vartheta_2(t+\lambda-\eta) \\ \hat\vartheta_3(t-\lambda)\hat\vartheta_3(t-\lambda+2\eta) & \alpha^{-1}\hat\vartheta_3(t-\lambda)\hat\vartheta_3(t+\lambda) & \hat\vartheta_3(t+\lambda)\hat\vartheta_3(t+\lambda-2\eta) \end{pmatrix} \\ &\times \mathrm{e}^{\mathrm{i}\mathsf{y}(t+(\eta-\lambda)s^3)}. \end{aligned} \tag{4.161}$$

This matrix allows us to formulate the vertex-face correspondence between the R-matrices of the 18V and d10V models.

Proposition 4.2.5. *We have*

$$R_{12}^{(1,2)}(\lambda_{12})V_1^{(1)}(\lambda_1|t)V_2^{(2)}(\lambda_2|t+\eta\sigma_1^3) \\ = V_2^{(2)}(\lambda_2|t)V_1^{(1)}(\lambda_1|t+2\eta s_2^3)\mathsf{R}_{12}^{(1,2)}(\lambda_{12}|t). \quad (4.162)$$

Proof. To prove the statement, we apply transformations to (4.159) until the definitions of $R^{(1,2)}(\lambda_{12})$ and $\mathsf{R}^{(1,2)}(\lambda_{12}|t)$ appear. We specialise (4.159) to $\lambda_3 = \lambda_2 - \eta$, and multiply it by $\mathsf{R}_{23}^{(1,1)}(\eta|t+\eta\sigma_1^z)$ on the right,

$$R_{12}^{(1,1)}(\lambda_{12})R_{13}^{(1,1)}(\lambda_{12}+\eta)\, V_1(\lambda_1|t) \\ \times [V_3^{(1)}(\lambda_2-\eta|t+\eta\sigma_1^3)V_2^{(1)}(\lambda_2|t+\eta(\sigma_1^3+\sigma_3^3))\mathsf{R}_{23}^{(1,1)}(\eta|t+\eta\sigma_1^3)] \\ = V_3^{(1)}(\lambda_3|t)V_2^{(1)}(\lambda_2|t+\eta\sigma_3^3)V_1^{(1)}(\lambda_1|t+\eta(\sigma_2^3+\sigma_3^3)) \\ \times [\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t+\eta\sigma_3^3)\mathsf{R}_{13}^{(1,1)}(\lambda_{13}|t)\mathsf{R}_{23}^{(1,1)}(\eta|t+\eta\sigma_1^3)]. \quad (4.163)$$

We denote by X_{L} and X_{R} the left- and right-hand sides of (4.163), respectively. First, we work on X_{L}. We apply the vertex-face correspondence (4.156) on the bracketed product, then the YBE. We obtain

$$X_{\mathrm{L}} = [R_{23}^{(1,1)}(\eta)R_{13}^{(1,1)}(\lambda_{12}+\eta)R_{12}^{(1,1)}(\lambda_{12})]V_1^{(1)}(\lambda_1|t) \\ \times [P_{23}^{(1)+}V_2^{(1)}(\lambda_2|t+\eta\sigma_1^3)V_3^{(1)}(\lambda_2-\eta|t+\eta(\sigma_1^3+\sigma_2^3))]. \quad (4.164)$$

Second, we work on X_{R}. We apply the dYBE on the bracketed product. Then, we use (4.8) to commmute $\mathsf{R}_{23}^{(1,1)}(\eta|t)$ to the left of $V_1^{(1)}(\lambda_1|t + \eta(\sigma_2^3+\sigma_3^3))$. Lastly, we use the vertex-face correspondence (4.156) to get

$$X_{\mathrm{R}} = [R_{23}^{(1,1)}(\eta)V_2^{(1)}(\lambda_2|t)V_3^{(1)}(\lambda_2-\eta|t+\eta\sigma_2^3)]V_1^{(1)}(\lambda_1|t+\eta(\sigma_2^3+\sigma_3^3)) \\ \times [P_{23}^{(1)+}\mathsf{R}_{13}^{(1,1)}(\lambda_{12}+\eta|t+\eta\sigma_2^3)\mathsf{R}_{12}^{(1,1)}(\lambda_{12}|t)]. \quad (4.165)$$

By Proposition 3.1.1, and the definitions (3.25) and (4.160), we find

$$\frac{\hat{\vartheta}_4}{\vartheta_1(\eta)\hat{\vartheta}_4(2\eta)}Q_{23}U_{23}E_{23}^{-1}X_{\mathrm{L}}U_{23}^{-1}Q_{23}^t \\ = R_{12}^{(1,2)}(\lambda_{12})V_1^{(1)}(\lambda_1|t)V_2^{(2)}(\lambda_2|t+\eta\sigma_1^3). \quad (4.166)$$

We apply the same transformations on the right-hand side. We use Proposition 3.1.1, the definitions (4.21) and (4.160) and the relation

$$Q_{12}U_{12}(\sigma_1^3+\sigma_2^3)U_{12}^{-1}Q_{12}^t = 2s^3. \quad (4.167)$$

We obtain

$$\frac{\hat{\vartheta}_4}{\vartheta_1(\eta)\hat{\vartheta}_4(2\eta)} Q_{23}U_{23}E_{23}^{-1}X_{\mathrm{R}}U_{23}^{-1}Q_{23}^t \\ = V_2^{(2)}(\lambda_2|t)V_1^{(1)}(\lambda_1|t+2\eta s_2^3)\mathsf{R}_{12}^{(1,2)}(\lambda_{12}|t). \quad (4.168)$$

Finally, equating (4.166) and (4.168) yields (4.162). □

We define

$$V_q^{(2)}(t) = \prod_{j=1}^{N} V_j^{(2)}(\lambda_j|t+2\eta S^{j-1}). \quad (4.169)$$

This operator $V_q^{(2)}$ allows us to formulate the vertex-face correspondence relating the monodromy matrices of the 18V and d10V models.

Proposition 4.2.6. *We have*

$$M_0(\lambda)V_0^{(1)}(\lambda|t)V_q^{(2)}(t+\eta\sigma_0^3) = V_q^{(2)}(t)V_0^{(1)}(\lambda|t+2\eta S)\mathsf{M}_0(\lambda|t). \quad (4.170)$$

Proof. The relation follows from N successive applications of Proposition 4.2.5. □

One can reproduce the reasoning of Proposition 4.2.5, starting from (4.170) for $N=2$, to derive the *fused vertex-face correspondence.*

Proposition 4.2.7. *We have*

$$R_{12}^{(2,2)}(\lambda_{12})V_1^{(2)}(\lambda_1|t)V_2^{(2)}(\lambda_2|t+2\eta s_1^3) \\ = V_2^{(2)}(\lambda_2|t)V_1^{(2)}(\lambda_1|t+2\eta s_2^3)\mathsf{R}_{12}^{(2,2)}(\lambda_{12}|t). \quad (4.171)$$

To conclude this subsection, we state analyticity and pseudo-periodicity properties for $V^{(2)}(\lambda|t)$ and $V_q^{(2)}(t)$.

Proposition 4.2.8. *The matrix $V^{(2)}(\lambda|t)$ is an entire function of λ. Moreover, it obeys the following pseudo-periodicity relations:*

$$V^{(2)}(\lambda+\pi|t) = -\Omega^3 V^{(2)}(\lambda|t)(\Omega^3)^{\mathsf{y}}, \quad (4.172a)$$

$$V^{(2)}(\lambda+\pi\omega|t) = -p^{-1}\mathrm{e}^{-2\mathrm{i}\lambda}\Omega^1 V^{(2)}(\lambda|t)\mathrm{e}^{2\mathrm{i}ts^3+2\mathrm{i}\eta(s^3)^2}p^{-\mathsf{y}s^3}. \quad (4.172b)$$

From the previous proposition, we deduce analyticity and pseudo-periodicity properties for $V_q^{(2)}(t)$.

Proposition 4.2.9. *The matrix $V_q^{(2)}(t)$ is an entire function of each inhomogeneity parameter. Moreover, for $i = 1, \dots, N$, it obeys the following pseudo-periodicity relations:*

$$V_q^{(2)}(t)\Big|_{\lambda_i=\lambda_i+\pi} = -\Omega_i^3 V_q^{(2)}(t)(\Omega_i^3)^{\mathsf{y}},$$
$$V_q^{(2)}(t)\Big|_{\lambda_i=\lambda_i+\pi\omega} = -p^{-1}\mathrm{e}^{-2\mathrm{i}\lambda_i}\Omega_i^1 V_q^{(2)}(t)\mathrm{e}^{2\mathrm{i}(t+2\eta S^{i-1})s_i^3+2\mathrm{i}\eta(s_i^3)^2}p^{-\mathsf{y}s_i^3}.$$

Moreover, $V_q^{(2)}(t)$ possesses an exchange relation, deduced from Proposition 4.2.7:

Proposition 4.2.10. *For each $i = 1, \dots, N-1$, we have*

$$\check{R}_{i,i+1}^{(2,2)}(\lambda_{i,i+1})V_q^{(2)}(t) = V_q^{(2)}(t)\Big|_{\lambda_i\leftrightarrow\lambda_{i+1}} \check{\mathsf{R}}_{i,i+1}^{(2,2)}(\lambda_{i,i+1}|t+2\eta S^{i-1}). \quad (4.174)$$

4.2.2 Relation between transfer matrices

Let us introduce a projector $\mathbf{P} : \mathbb{D}_N \to (\mathbb{C}^3)^{\otimes N}$ such that for each $|\boldsymbol{\alpha}\rangle \otimes |t\rangle \in \mathbb{D}_N$,

$$\mathbf{P}\,|\boldsymbol{\alpha}\rangle \otimes |t\rangle = |\boldsymbol{\alpha}\rangle\,. \quad (4.175)$$

Naturally, we have

$$\mathbf{P}T_\tau^{\pm} = \mathbf{P}. \quad (4.176)$$

Moreover, when restricted to $\mathbb{D}_N^{(0)}$, $\mathbf{P}$ is an isomorphism. We define an operator $\mathbf{V}_q \in \mathrm{End}(\mathbb{C}^3)^{\otimes N}$ such that, restricted to the subspace $\mathbb{D}_N^{(0)}$, we have

$$\mathbf{P}V_q^{(2)}(\tau) = \mathbf{V}_q\mathbf{P}. \quad (4.177)$$

On each element $|\boldsymbol{\alpha}\rangle$ of the canonical basis of $(\mathbb{C}^3)^{\otimes N}$, the operator acts as

$$\mathbf{V}_q\,|\boldsymbol{\alpha}\rangle = V_q^{(2)}(t_{\boldsymbol{\alpha},0})\,|\boldsymbol{\alpha}\rangle\,, \quad (4.178)$$

with $t_{\boldsymbol{\alpha},0}$ as defined in (4.109).

The following assumption is essential to establish that the restriction of $\mathbf{V}_q\mathbf{P}$ to $\mathbb{D}_N^{(0)}$ is invertible. In turn, this operator proves useful in relating the eigenvalue problems of the transfer matrices of the anti-periodic d10V and the quasi-periodic 18V models.

Assumption 4.2.11. *The operator* $\mathbf{V}_q$ *is an automorphism of* $(\mathbb{C}^3)^{\otimes N}$.

The main obstacle to proving this assumption resides in the non-local nature of the operator $\mathbf{V}_q$, brought on by the S^{j-1} in (4.169) and by the dependence on $t_{\boldsymbol{\alpha},0}$ in (4.178). Note that an inductive argument for the spin-1/2 equivalent of Assumption 4.2.11 was presented in [94]. However, its rigorous adaptation to the present case has revealed to be a challenge. We assume that Assumption 4.2.11 holds throughout the remainder of our work.

The operator $\mathbf{V}_q\mathbf{P}$ relates the anti-periodic d10V transfer matrix $\mathcal{T}(\lambda)$ to the quasi-periodic 18V transfer matrix $T^{(1)}(\lambda)$. Each choice of the parameters (x,y) corresponds to a choice of boundary conditions for the vertex model. To reflect this relation, we re-label the twists of (3.47) and (3.48) in terms of (x,y) as

$$\mathrm{i}\sigma = (\mathrm{i}\sigma^1)^{\mathsf{y}}(\mathrm{i}\sigma^3)^{\mathsf{x}}, \qquad \Omega = (\Omega^1)^{\mathsf{y}}(\Omega^3)^{\mathsf{x}}. \tag{4.179}$$

The twists (4.179) for $(\mathsf{x},\mathsf{y}) = (0,1),(1,1),(1,0)$ are equal to $\mathrm{i}\sigma^\kappa$, Ω^κ, with $\kappa = 1,2,3$, respectively.

Proposition 4.2.12. *In the subspace* $\mathbb{D}_N^{(0)}$, *we have*

$$T^{(1)}(\lambda)\mathbf{V}_q\mathbf{P} = (-1)^{1+\mathsf{y}}\,\mathrm{i}\,\mathbf{V}_q\mathbf{P}\mathcal{T}(\lambda). \tag{4.180}$$

Proof. We use Propositions 3.1.11 and 4.1.13 to rewrite Proposition 4.2.6 as

$$\begin{aligned} V_0^{(1)}(\lambda+\eta|\tau)V_q^{(2)}(\tau+\eta\sigma_0^3)T_\tau^{\sigma_0^3}\sigma_0^2\mathcal{M}_0(\lambda)^{t_0}\sigma_0^2\frac{\vartheta_1(\tau+2\eta S)}{\vartheta_1(\tau)}\mathrm{e}^{2\mathrm{i}\eta S} \\ = \sigma_0^2 M_0(\lambda)^{t_0}\sigma_0^2 V_q^{(2)}(\tau)V_0^{(1)}(\lambda+\eta|\tau+2\eta S). \end{aligned} \tag{4.181}$$

The idea behind this proof is to apply algebraic operations to this equation, until the definitions of $T^{(1)}(\lambda)$ and $\mathcal{T}(\lambda)$ appear. First, due to (A.4), we note that on $\mathbb{D}_N^{(0)}$,

$$\frac{\vartheta_1(\tau+2\eta S)}{\vartheta_1(\tau)}\mathrm{e}^{2\mathrm{i}\eta S} = 1. \tag{4.182}$$

We thus remove this operator from (4.181). Second, by Lemma 4.2.3, we have

$$V^{(1)}(\lambda|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega) = (-1)^{\mathsf{y}}\mathrm{i}(\mathrm{i}\sigma^1)^{\mathsf{y}}(\mathrm{i}\sigma^3)^{\mathsf{x}}V^{(1)}(\lambda|\tau)\sigma^1. \tag{4.183}$$

We use this relation on the matrix $V_0^{(1)}(\lambda+\eta|\tau)$ in the left-hand side of (4.181), and obtain

$$\begin{aligned}&V_0^{(1)}(\lambda+\eta|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)\sigma_0^1 V_q^{(2)}(\tau+\eta\sigma_0^3)T_\tau^{\sigma_0^3}\sigma_0^2\mathcal{M}_0(\lambda)^{t_0}\sigma_0^2\\&=(-1)^{\mathsf{y}}\mathrm{i}(\mathrm{i}\sigma^1)^{\mathsf{y}}(\mathrm{i}\sigma^3)^{\mathsf{x}}\sigma_0^2 M_0(\lambda)^{t_0}\sigma_0^2 V_q^{(2)}(\tau)V_0^{(1)}(\lambda+\eta|\tau+2\eta S).\end{aligned}\tag{4.184}$$

By Proposition 4.2.2, the matrix $V_0^{(1)}(\lambda+\eta|\tau+2\eta S)$ is invertible in $\mathbb{C}^2\otimes\mathbb{D}_N^{(0)}$ for $\lambda\not\equiv-\eta \pmod{\pi,\pi\omega}$. Indeed, the canonical basis of $\mathbb{D}_N^{(0)}$ is an eigenbasis of $\tau+2\eta S$ with eigenvalues in $\mathsf{x}\frac{\pi}{2}+\mathsf{y}\frac{\pi\omega}{2}+[-N,N]\eta$. Therefore, the evaluation of $\tau+2\eta S$ on the basis is never congruent to $0 \pmod{\pi,\pi\omega}$. We multiply (4.184) on the right by $V_0^{(1)}(\lambda+\eta|\tau+2\eta S)^{-1}$, then take its trace with respect to the auxiliary space,

$$\begin{aligned}&\mathrm{tr}_0\Big(V_0^{(1)}(\lambda+\eta|-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega)\sigma_0^1V_q^{(2)}(\tau+\eta\sigma_0^3)\\&\qquad\times T_\tau^{\sigma_0^3}\sigma_0^2\mathcal{M}_0(\lambda)^{t_0}\sigma_0^2V_0^{(1)}(\lambda+\eta|\tau+2\eta S)^{-1}\Big)\\&\qquad=(-1)^{\mathsf{y}}\mathrm{i}\,\mathrm{tr}_0\left((\mathrm{i}\sigma_0^1)^{\mathsf{y}}(\mathrm{i}\sigma_0^3)^{\mathsf{x}}\sigma_0^2M_0(\lambda)^{t_0}\sigma_0^2\right)V_q^{(2)}(\tau).\end{aligned}\tag{4.185}$$

Let us denote by X_{L} and X_{R} the left- and right-hand sides of this equation, respectively. We work on them separately, starting with X_{R}. Using the similarity and transposition invariance of the trace, and the commutation relations of Pauli matrices, we find

$$X_{\mathrm{R}}=(-1)^{\mathsf{y}}\mathrm{i}\,\mathrm{tr}_0\left(\sigma_0^2(\mathrm{i}\sigma_0^1)^{\mathsf{y}}(\mathrm{i}\sigma_0^3)^{\mathsf{x}}\sigma_0^2M_0(\lambda)^{t_0}\right)V_q^{(2)}(\tau),\tag{4.186}$$

$$=(-1)^{\mathsf{x}}\mathrm{i}\,\mathrm{tr}_0\left(((\mathrm{i}\sigma_0^1)^{\mathsf{y}}(\mathrm{i}\sigma_0^3)^{\mathsf{x}})^{t_0}M_0(\lambda)\right)V_q^{(2)}(\tau),\tag{4.187}$$

$$=(-1)^{1+\mathsf{y}}\mathrm{i}T^{(1)}(\lambda)V_q^{(2)}(\tau).\tag{4.188}$$

We apply $\mathbf{P}$ to the left of the expression. By (4.177), this yields

$$\mathbf{P}X_{\mathrm{R}}=(-1)^{1+\mathsf{y}}\mathrm{i}T^{(1)}(\lambda)\mathbf{V}_q\mathbf{P}.\tag{4.189}$$

We now turn our attention to X_{L}. A direct computation reveals that the quantity

$$T_\tau^{\sigma_0^3}\sigma_0^2\mathcal{M}_0(\lambda)^{t_0}\sigma_0^2=\begin{pmatrix}\mathsf{D}(\lambda|\tau+\eta)&-\mathsf{B}(\lambda|\tau+\eta)\\-\mathsf{C}(\lambda|\tau-\eta)&\mathsf{A}(\lambda|\tau-\eta)\end{pmatrix}_0\tag{4.190}$$

does not involve the operators $T_\tau^{\pm}$. Moreover, in $\mathbb{D}_N^{(0)}$, $\tau+2\eta S=-\tau+\mathsf{x}\pi+\mathsf{y}\pi\omega$. These remarks, along with the cyclic property of the trace, allow us to simplify the operators $V^{(1)}$ in X_{L}. It becomes

$$X_{\mathrm{L}} = \mathrm{tr}_0 \left(\sigma_0^1 T_\tau^{\sigma_0^3} V_q^{(2)}(\tau) \sigma_0^2 \mathcal{M}_0(\lambda)^{t_0} \sigma_0^2 \right), \tag{4.191}$$

where we used (4.50). We apply $\mathbf{P}$ to the left of the expression. The properties (4.176) and (4.177), the similarity and transposition invariance of the trace, and the commutation relations of Pauli matrices lead to

$$\mathbf{P} X_{\mathrm{L}} = \mathbf{V}_q \mathbf{P}\, \mathrm{tr}_0 \left((\sigma_0^2 \sigma_0^1 \sigma_0^2)^{t_0} \mathcal{M}_0(\lambda) \right) = -\mathbf{V}_q \mathbf{P} \mathcal{T}(\lambda). \tag{4.192}$$

Equating $\mathbf{P} X_{\mathrm{L}}$ to $\mathbf{P} X_{\mathrm{R}}$ establishes (4.180) for $\lambda \not\equiv -\eta \pmod{\pi, \pi\omega}$. By continuity, the relation holds for all $\lambda \in \mathbb{C}$. □

By Assumption 4.2.11, we have the following result:

Proposition 4.2.13. *We have*

$$T^{(1)}(\lambda) = (-1)^{1+\mathrm{y}}\, \mathrm{i}\, \mathbf{V}_q \mathbf{P} \mathcal{T}(\lambda) [\mathbf{V}_q \mathbf{P}]^{-1}. \tag{4.193}$$

This last proposition indicates that we can solve the eigenvalue problem of the quasi-periodic 18V transfer matrix by considering that of the anti-periodic d10V transfer matrix instead.

4.A Inversion formulas

In this appendix, we provide a proof for the inversion formula of Proposition 4.1.13. First, we state and prove a similar inversion formula for the d6V model. Then, we apply the fusion procedure to it in order to obtain that of the d10V model.

For the dynamical six-vertex model Let us introduce the operators Λ^j as

$$\Lambda^j = \sum_{k=1}^{j} \sigma_k^3, \qquad j = 1, \ldots, N, \tag{4.194}$$

and $\Lambda^0 = 0$. In particular, $\Lambda = \Lambda^N$ is the *total spin operator* on $(\mathbb{C}^2)^{\otimes N}$.

We define the monodromy matrix of the d6V model as

$$\tilde{\mathcal{M}}_0(\lambda) = \tilde{\mathsf{M}}_0(\lambda|\tau)T_\tau^{\sigma_0^3}, \qquad \tilde{\mathsf{M}}_0(\lambda_0|t) = \prod_{j=1}^{\curvearrowleft N} \mathsf{R}_{0j}^{(1,1)}\left(\lambda_{0j}\left|t + \eta\Lambda^{j-1}\right.\right). \tag{4.195}$$

This matrix possesses an *inversion formula* [58, 59], which implicitely relates its inverse to its transpose with respect to the auxiliary space.

Proposition 4.A.1. *We have*

$$\tilde{\mathcal{M}}_0(\lambda)\,\sigma_0^2\tilde{\mathcal{M}}_0(\lambda-\eta)^{t_0}\sigma_0^2 = \frac{\mathrm{e}^{-\mathrm{i}y\eta\Lambda}\vartheta_1(\tau)}{\vartheta_1(\tau+\eta\Lambda)}\mathrm{A}(\lambda-\eta)\mathrm{D}(\lambda-\eta). \tag{4.196}$$

Elements of its proof can be found in [58, 101]. In this appendix, we expose the full reasoning. The first intermediate result is the *crossing symmetry* of the R-matrix of the d6V model, which we obtain through direct computation. Recall the notation

$$\mathsf{R}_{10}^{(1,1)}(\lambda|t) = \mathcal{P}_{01}^{(1)}\mathsf{R}_{01}^{(1,1)}(\lambda|t)\mathcal{P}_{01}^{(1)}. \tag{4.197}$$

Proposition 4.A.2. *We have*

$$\mathsf{R}_{10}^{(1,1)}(\lambda|\tau) = -T_\tau^{\sigma_0^3}\sigma_0^2\left(\mathsf{R}_{01}^{(1,1)}(-\lambda-\eta|\tau)T^{\sigma_0^3}\right)^{t_0}\sigma_0^2 \times \frac{\mathrm{e}^{\mathrm{i}y\eta\sigma_1^3}\vartheta_1(\tau+\eta\sigma_1^3)}{\vartheta_1(\tau)}. \tag{4.198}$$

Another useful equality relates the transpose of the monodromy matrix to a product of transposed R-matrices.

Lemma 4.A.3. *We have*

$$T_\tau^{-\sigma_0^3}\tilde{\mathcal{M}}_0(\lambda_0)^{t_0} = \prod_{j=1}^{N} T_\tau^{-\sigma_0^3}\left(\mathsf{R}_{0j}^{(1,1)}\left(\lambda_{0j}\left|\tau+\eta\Lambda^{j-1}\right.\right)T_\tau^{\sigma_0^3}\right)^{t_0}. \tag{4.199}$$

Proof. We proceed by induction on system size N. We start the proof with two preliminary considerations. First, we recall the projectors on the canonical basis vectors of $\mathbb{C}^2$,

$$p^\pm = \frac{1}{2}(\mathbb{I} \pm \sigma^3). \tag{4.200}$$

For any $A \in \mathrm{End}(\mathbb{C}^2)$, the decomposition of the identity as $\mathbb{I} = p^+ + p^-$, and (4.50), allow us to write

$$A_0(\tau + \eta\sigma_1^3) = \textstyle\sum_{\varepsilon=\pm} p_1^\varepsilon A_0(\tau + \eta\varepsilon), \tag{4.201a}$$

$$T_\tau^{-\sigma_0^3}(A_0(\tau)T_\tau^{\sigma_0^3})^{t_0} = \textstyle\sum_{\varepsilon=\pm} p_0^\varepsilon A_0(\tau - \eta\varepsilon)^{t_0}. \tag{4.201b}$$

Second, we recall some useful commutation relations. One checks that the following equalities hold:

$$[(\mathsf{R}_{12}^{(1,1)}(\lambda|t))^{t_1}, \sigma_1^3 - \sigma_2^3] = 0, \qquad [(\mathsf{R}_{12}^{(1,1)}(\lambda|t))^{t_1}, \sigma_1^3\sigma_2^3] = 0, \tag{4.202}$$

where the second one follows from the first one. From these, we deduce that

$$[(\mathsf{R}_{12}^{(1,1)}(\lambda|t))^{t_1},\, p_1^\delta p_2^\varepsilon] = 0, \qquad \forall\, \delta, \varepsilon \in \{+,-\},\, \delta \neq \varepsilon, \tag{4.203a}$$

$$[(\mathsf{R}_{12}^{(1,1)}(\lambda|t))^{t_1},\, p_1^+ p_2^+ + p_1^- p_2^-] = 0. \tag{4.203b}$$

We now move on to the inductive proof. For $N = 1$, the definition of $\tilde{\mathcal{M}}_0(\lambda)$ ensures that (4.199) holds. Let us assume that the equality holds for $N = L - 1$, with $L \geq 2$. We decompose

$$\begin{aligned} &\prod_{j=1}^{L} T_\tau^{-\sigma_0^3}\left(\mathsf{R}_{0j}^{(1,1)}\left(\lambda_{0j}\,\middle|\tau + \eta\Lambda^{j-1}\right)T_\tau^{\sigma_0^3}\right)^{t_0} \\ &\quad = T_\tau^{-\sigma_0^3}\left(\mathsf{R}_{01}^{(1,1)}(\lambda_{01}|\tau)T_\tau^{\sigma_0^3}\right)^{t_0} T_\tau^{-\sigma_0^3}\left(\tilde{\mathsf{M}}_{0,2\cdots L}(\lambda_0|\tau + \eta\sigma_1^3)T_\tau^{\sigma_0^3}\right)^{t_0}, \end{aligned} \tag{4.204}$$

where $\tilde{\mathsf{M}}_{0,2\cdots L}(\lambda|t)$ is the size-$(L-1)$ monodromy matrix that acts non-trivially on the auxiliary space 0, and the factors of $\mathbb{C}^2$ of the spin space labelled by $2, 3, \cdots, L$.

We apply the decompositions (4.201) to the right-hand side of (4.204). This yields

$$\sum_{\gamma,\delta,\varepsilon=\pm} p_0^\gamma\, \mathsf{R}_{01}^{(1,1)}(\lambda_{01}|\tau - \eta\gamma)^{t_0}\, p_0^\delta p_1^\varepsilon\, \tilde{\mathsf{M}}_{0,2\cdots L}(\lambda_0|\tau + \eta(\varepsilon - \delta))^{t_0}. \tag{4.205}$$

We note that for $\delta = \varepsilon = \pm$, the summands multiplying the projectors $p_0^\delta p_1^\varepsilon$ have the same expression. By virtue of (4.203), we can therefore commute the projectors $p_0^\delta p_1^\varepsilon$ to the left of the R-matrix for all $\delta, \varepsilon \in \{+,-\}$. After simplication, (4.205) becomes

$$\sum_{\delta,\varepsilon=\pm} p_0^\delta p_1^\varepsilon \, \mathsf{R}_{01}^{(1,1)}(\lambda_{01}|\tau-\eta\delta)^{t_0}\tilde{\mathsf{M}}_{0,2\cdots L}(\lambda_0|\tau+\eta(\varepsilon-\delta))^{t_0}. \tag{4.206}$$

By property of the transposition, and (4.201), this expression equals $T_\tau^{-\sigma_0^3}\tilde{\mathcal{M}}_0(\lambda_0)^{t_0}$, which concludes the proof. □

Proof of Proposition 4.A.1. Proposition 4.1.3 establishes

$$\prod_{j=1}^{\overset{\curvearrowleft}{N}} \mathsf{R}_{0j}^{(1,1)}\left(\lambda_{0j}\,\middle|\,\tau+\eta\Lambda^{j-1}\right)\prod_{j=1}^{N}\mathsf{R}_{j0}^{(1,1)}\left(-\lambda_{0j}\,\middle|\,\tau+\eta\Lambda^{j-1}\right) \\ = (-1)^N \mathrm{A}(\lambda-\eta)\mathrm{D}(\lambda-\eta). \tag{4.207}$$

We recognise the definition (4.195) and use Proposition 4.A.2 on the elements of the rightmost product. We obtain

$$\tilde{\mathcal{M}}_0(\lambda_0)T_\tau^{-\sigma_0^3}\prod_{j=1}^{N}T_\tau^{\sigma_0^3}\sigma_0^2\left(\mathsf{R}_{0j}^{(1,1)}\left(\lambda_{0j}-\eta\,\middle|\,\tau+\eta\Lambda^{j-1}\right)T_\tau^{\sigma_0^3}\right)^{t_0}\sigma_0^2 \\ = \frac{\mathrm{e}^{-\mathrm{i}y\eta\Lambda}\vartheta_1(\tau)}{\vartheta_1(\tau+\eta\Lambda)}\mathrm{A}(\lambda-\eta)\mathrm{D}(\lambda-\eta). \tag{4.208}$$

Finally, the property $\sigma_0^2\, T_\tau^{\sigma_0^3}\,\sigma_0^2 = T_\tau^{-\sigma_0^3}$, along with Lemma 4.A.3, lead to (4.199). □

For the dynamical ten-vertex model We now prove the inversion formula for the d10V model. To this end, we apply the fusion procedure to the inversion formula of the d6V model.

Proof of Proposition 4.1.13. Let us consider (4.196) for system size $2N$, and with spectral and inhomogeneity parameters specialised to

$$\lambda = \lambda_0', \qquad \lambda_{2i-1} = \lambda_i', \qquad \lambda_{2i} = \lambda_i' - \eta, \qquad i = 1, \ldots, N. \tag{4.209}$$

We denote by X_{L} and X_{R} the left- and right-hand sides of the equation, respectively. The left-hand side is

$$\begin{aligned}X_{\mathrm{L}} = \prod_{j=1}^{\overset{\curvearrowleft}{N}} \mathsf{R}^{(1,1)}_{0,2j}\left(\lambda'_{0j}+\eta \,\middle|\, \tau+\eta\Lambda^{2j-1}\right) \mathsf{R}^{(1,1)}_{0,2j-1}\left(\lambda'_{0j} \,\middle|\, \tau+\eta\Lambda^{2(j-1)}\right) T_{\tau}^{\sigma_0^3} \\ \times \sigma_0^2 \Bigg(\prod_{j=1}^{\overset{\curvearrowleft}{N}} \mathsf{R}^{(1,1)}_{0,2j}\left(\lambda'_{0j} \,\middle|\, \tau+\eta\Lambda^{2j-1}\right) \\ \times \mathsf{R}^{(1,1)}_{0,2j-1}\left(\lambda'_{0j}-\eta \,\middle|\, \tau+\eta\Lambda^{2(j-1)}\right) T_{\tau}^{\sigma_0^3} \Bigg)^{t_0} \sigma_0^2. \quad (4.210)\end{aligned}$$

We multiply it on the left by $P^{(1)+}_{2N,2N-1}\cdots P^{(1)+}_{21}$. We observe that, for all $t\in\mathbb{C}$, the operator

$$P^{(1)+}_{2i,2i-1}\mathsf{R}^{(1,1)}_{0,2i}\left(\lambda'_{0i}+\eta \,\middle|\, t+\eta\sigma^3_{2i-1}\right) \mathsf{R}^{(1,1)}_{0,2i-1}\left(\lambda'_{0i} \,\middle|\, t\right) \quad (4.211)$$

leaves invariant the left symmetric subspace of the $2i$ and $(2i-1)$-th factors $\mathbb{C}^2$ in the spin space. Moreover, the projector obeys the commutation relation

$$[P^{(1)+}_{2i,2i-1}, \sigma^3_{2i}+\sigma^3_{2i-1}] = 0. \quad (4.212)$$

Therefore, the expression of $P^{(1)+}_{2N,2N-1}\cdots P^{(1)+}_{21}X_{\mathrm{L}}$ corresponds to (4.210), with an extra operator $P^{(1)+}_{2j,2j-1}$ on the left of the j-th factor of both products. We conjugate with the appropriate change of basis (3.21) and restriction operator (3.23), and find

$$\begin{aligned}\left[\prod_{j=1}^{\overset{\curvearrowleft}{N}} Q_{2j,2j-1}U_{2j,2j-1}P^{(1)+}_{2j,2j-1}\right] X_{\mathrm{L}} \left[\prod_{j=1}^{N} U^{-1}_{2j,2j-1}Q^{t}_{2j,2j-1}\right] \\ = \mathrm{D}(\lambda'_0)\mathrm{D}(\lambda'_0+\eta)\mathcal{M}_0(\lambda'_0)\,\sigma_0^2\mathcal{M}_0(\lambda'_0-\eta)^{t_0}\sigma_0^2, \quad (4.213)\end{aligned}$$

where we used (4.167). We reproduce all these operations on the right-hand side. We obtain

$$\begin{aligned}\left[\prod_{j=1}^{\overset{\curvearrowleft}{N}} Q_{2j,2j-1}U_{2j,2j-1}P^{(1)+}_{2j,2j-1}\right] X_{\mathrm{R}} \left[\prod_{j=1}^{N} U^{-1}_{2j,2j-1}Q^{t}_{2j,2j-1}\right] \\ = \frac{\mathrm{e}^{-2\mathrm{i}y\eta S}\vartheta_1(\tau)}{\vartheta_1(\tau+2\eta S)}\mathrm{A}(\lambda'_0)\mathrm{D}(\lambda'_0)\mathrm{D}(\lambda'_0+\eta)\mathrm{D}(\lambda'_0-\eta). \quad (4.214)\end{aligned}$$

The simplication of (4.213) and (4.214) allows us to reach (4.69). □

Chapter 5

Dynamical ten-vertex model via quantum Separation of Variables

Separation of Variables (SoV) is a solving technique that has kept its relevance throughout the ages [150, 151]. At its core is a simple concept: the *reduction of a multidimensional problem to a series of one-dimensional ones* [50]. One of its noteworthy successes occurred in the context of Liouville-classical integrability, where SoV was used to greatly simplify the Hamilton-Jacobi equations of certain systems [152]. Ultimately, this inspired Gutzwiller [153], and then notably Sklyanin to apply the idea to the realm of quantum integrability [49, 50]. This new version was originally devised as a type of Bethe Ansatz, the *functional Bethe Ansatz*. The technique later came to be known as *quantum Separation of Variables (qSoV)*, and has stayed an active topic of research ever since [51–59].

The strategy behind the qSoV approach[1] relies on the introduction of a new basis of the space of states, the *qSoV basis*. In this basis, the eigenvalue problem of the transfer matrix reduces to a set of lower-dimensional linear systems. Solving them leads to a complete (albeit implicit) description of the spectrum and eigenvectors. On the one hand, eigenvalues are characterised as a certain class of functions that fulfil a set of discrete

[1]We provide here a brief symptomatic description, for a more complete review, see for example [154].

equations. On the other hand, the entries of the eigenvectors in the qSoV basis are shown to possess a factorised form.

The qSoV approach presents some advantages compared to the *algebraic Bethe Ansatz (ABA)*. The first advantage is that qSoV has been successfully applied to models where the ABA could not, notably because of the lack of an appropriate reference state. Examples include the anti-periodic spin-1/2 XXX and XXZ spin chains [51, 52], and higher-spin generalisations [53, 54]. Additionally, Maillet and Niccoli recently proposed a variant of qSoV, meant to broaden its scope of application to higher-rank models [55]. The second advantage is that qSoV provides a complete characterisation of the spectrum and eigenvectors as an inherent feature. In contrast, in the ABA, the proof of the completeness of the spectrum description is often an issue [19]. Finally, the qSoV approach has proven effective at obtaining determinant formulas for scalar products of *separate states*[2], and for *form factors*[3] [51–53, 59]. These quantities have been used as stepping stones in the computation of correlation functions [155].

Another type of models where qSoV was successfully applied are solid-on-solid, or dynamical vertex models [56–59]. In this chapter, we follow [58,59], and exploit qSoV to investigate the eigenvalue problem of the transfer matrix of the anti-periodic d10V model. We now detail the steps that we take in order to solve this problem.

In Section 5.1, we introduce a set of vectors, defined from the repeated action of the monodromy-matrix entry $\mathcal{C}(\lambda)$ on reference states.[4] We call them *qSoV states*. We derive a number of properties for these states, notably the action of monodromy-matrix entries on them, and the dual pairing between any two states. This last result allows us to prove that the qSoV states form a basis of the space of states.

In Section 5.2, we consider the eigenvalue problem of the anti-periodic d10V transfer matrix. Its solution follows the qSoV program described above. In particular, we compute one eigenvalue explicitly and show

[2]Separate states are a class of vectors with factorised components in the qSoV basis, which includes the eigenvectors of the transfer matrix.

[3]Form factors are matrix elements of local operators in the eigenbasis of the transfer matrix.

[4]The qSoV approach also makes use of *reference states*, not to be confused with those required by the ABA.

that it possesses a remarkably simple form. We find an expression for a quadratic sum rule involving its associated eigenvectors as a determinant formula. Finally, we exploit the qSoV formulation and derive a series of properties of these special eigenvectors that characterise them implicitly.

5.1 The qSoV basis

We begin our investigation in Section 5.1.1 with the definition of the *left* and *right qSoV states*. In Section 5.1.2, we examine the actions of the monodromy-matrix entries on the qSoV states. These are relatively simple compared to the actions on the canonical basis vectors. In Section 5.1.3, we compute a formula for the dual pairing between any left and right qSoV states. From it we deduce the main result of this section: for each integer r, certain subsets of the right and left qSoV states form a basis of the subspace $\mathbb{D}_N^{(r)}$ and its dual, respectively. The purpose of Section 5.1.4 is to analyse the qSoV states as functions of the inhomogeneity parameters. In particular, we discuss their pseudo-periodicity and analyticity properties, and establish exchange and reduction relations.

5.1.1 Left and right qSoV states

Bold letters of the Latin alphabet $\boldsymbol{h}, \boldsymbol{k}$ stand for elements of $\{0,1,2\}^N$, unless stated otherwise. We write $\boldsymbol{h} = (h_1, \cdots, h_N)$, and use the conventions $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{2} = (2, \cdots, 2)$.

We consider the following right and left *reference states*:

$$\|\mathbf{2}, r\rangle\rangle = |\Downarrow \cdots \Downarrow\rangle \otimes |t_{\mathbf{2},r}\rangle\,, \tag{5.1a}$$

$$\langle\langle \mathbf{0}, r\| = \langle \Uparrow \cdots \Uparrow| \otimes \langle t_{\mathbf{0},r}|\,, \tag{5.1b}$$

with

$$t_{\boldsymbol{h},r} = \eta(r - s_{\boldsymbol{h}}) + \mathsf{x}\tfrac{\pi}{2} + \mathsf{y}\tfrac{\pi\omega}{2}, \qquad s_{\boldsymbol{h}} = \sum_{j=1}^{N}(1-h_j). \tag{5.2}$$

They belong in $\mathbb{D}_N^{(r)}$ and its dual, respectively.

We build the qSoV states through the repeated action of the monodromy-matrix entry $\mathcal{C}(\lambda)$ on the two reference states. In order for them to be well-defined, it is crucial that the inhomogeneity parameters be generic, as defined in Definition 4.1.15.

Definition 5.1.1. *For each $\boldsymbol{h} \in \{0,1,2\}^N$ and $r \in \mathbb{Z}$, we define the* right *and* left qSoV states *respectively as*[5]

$$\|\boldsymbol{h}, r\rangle\rangle = \left[\prod_{j=1}^{N}\prod_{h=h_j}^{1}\frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))}\right]\|\boldsymbol{2}, r\rangle\rangle, \tag{5.3}$$

$$\langle\langle \boldsymbol{h}, r\| = \langle\langle \boldsymbol{0}, r\|\left[\prod_{j=1}^{N}\prod_{h=0}^{h_j-1}\frac{\mathcal{C}(\lambda_j - \eta h)}{\mathrm{D}(\lambda_j - \eta(h+1))}\right], \tag{5.4}$$

where $\mathrm{D}(\lambda)$ *is defined in* (3.49).

The qSoV states are eigenvectors of both S and τ. This property follows from the definition of the monodromy matrix and the reference states, as well as from Lemma 4.1.22.

Proposition 5.1.2. *We have*

$$\tau\|\boldsymbol{h}, r\rangle\rangle = t_{\boldsymbol{h},r}\|\boldsymbol{h}, r\rangle\rangle, \qquad \langle\langle \boldsymbol{h}, r\|\tau = t_{\boldsymbol{h},r}\langle\langle \boldsymbol{h}, r\|, \tag{5.5a}$$

$$S\|\boldsymbol{h}, r\rangle\rangle = s_{\boldsymbol{h}}\|\boldsymbol{h}, r\rangle\rangle, \qquad \langle\langle \boldsymbol{h}, r\|S = s_{\boldsymbol{h}}\langle\langle \boldsymbol{h}, r\|. \tag{5.5b}$$

Combining this proposition with the definition of S_τ, we immediately obtain:

Corollary 5.1.3. *We have $\|\boldsymbol{h}, r\rangle\rangle \in \mathbb{D}_N^{(r)}$ and $\langle\langle \boldsymbol{h}, r\| \in \mathbb{D}_N^{(r)*}$.*

5.1.2 Action of the monodromy-matrix entries

We now examine the actions of the monodromy-matrix entries $\mathcal{B}(\lambda), \mathcal{C}(\lambda)$ and $\mathcal{D}(\lambda)$ on the left and right qSoV states $\langle\langle \boldsymbol{h}, r\|$ and $\|\boldsymbol{h}, r\rangle\rangle$. We find that $\mathcal{D}(\lambda)$ acts as a shift operator on the r, while it leaves the $\boldsymbol{h}$ unchanged. Conversely, $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$ act as weighted sums of shift operators on the $\boldsymbol{h}$, while the r remains unchanged. The proofs of these results are often similar for the left and right qSoV states. Hence, we only present them for the right qSoV states. If necessary, we point out the modifications for the left qSoV states.

[5] We follow the convention that products over empty sets evaluate to 1, i.e. $\prod_{h=2}^{1} f(h) = \prod_{h=0}^{-1} f(h) = 1$.

Action on the reference states Before addressing the general case, we start with preliminary results regarding the reference states. Recall that we write N as superscript on the monodromy-matrix entries to stress that they are operators on $\mathbb{D}_N$. In the next proposition, we evaluate the actions of $\mathcal{A}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}$, $\mathcal{B}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}$, and $\mathcal{D}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}$ on the reference states, $L = 1, \ldots, N$. These operators are the respective natural embeddings of $\mathcal{A}^L(\lambda)$, $\mathcal{B}^L(\lambda)$, $\mathcal{D}^L(\lambda)$ into $\mathrm{End}(\mathbb{D}_N)$ that act like the identity on the last $(N-L)$ factors of $(\mathbb{C}^3)^N$. It will also be convenient to introduce

$$\mathrm{D}^L(\lambda) = \prod_{j=1}^{L} \vartheta_1(\lambda - \lambda_j), \qquad \mathrm{A}^L(\lambda) = \mathrm{D}^L(\lambda + 2\eta). \tag{5.6}$$

Note that $\mathrm{D}^N(\lambda) = \mathrm{D}(\lambda)$ and $\mathrm{A}^N(\lambda) = \mathrm{A}(\lambda)$, as defined in (3.49).

Proposition 5.1.4. *For each $L = 1, \ldots, N$, and $t \in \ell_0 + \eta\mathbb{Z}$, we have*

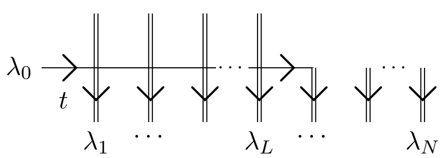
$$\begin{aligned}
\left(\mathcal{A}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) |\Downarrow \cdots \Downarrow\rangle \otimes |t\rangle &= \mathrm{e}^{2\mathrm{i}y\eta L} \frac{\vartheta_1(t)}{\vartheta_1(t - 2\eta L)} \\
&\quad \times \mathrm{D}^L(\lambda)\, |\Downarrow \cdots \Downarrow\rangle \otimes |t - \eta\rangle\,, & (5.7)\\
\left(\mathcal{B}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) |\Downarrow \cdots \Downarrow\rangle \otimes |t\rangle &= 0, & (5.8)\\
\left(\mathcal{D}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) |\Downarrow \cdots \Downarrow\rangle \otimes |t\rangle &= \mathrm{A}^L(\lambda)\, |\Downarrow \cdots \Downarrow\rangle \otimes |t + \eta\rangle\,, & (5.9)
\end{aligned}$$

and

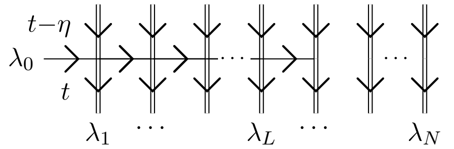
$$\begin{aligned}
\langle\Uparrow \cdots \Uparrow| \otimes \langle t| \left(\mathcal{A}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) &= \mathrm{A}^L(\lambda)\, \langle\Uparrow \cdots \Uparrow| \otimes \langle t + \eta|\,, & (5.10)\\
\langle\Uparrow \cdots \Uparrow| \otimes \langle t| \left(\mathcal{B}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) &= 0, & (5.11)\\
\langle\Uparrow \cdots \Uparrow| \otimes \langle t| \left(\mathcal{D}^L(\lambda) \otimes \mathbb{I}^{\otimes(N-L)}\right) &= \mathrm{e}^{-2\mathrm{i}y\eta L} \frac{\vartheta_1(t - \eta)}{\vartheta_1(t + \eta(2L - 1))} \\
&\quad \times \mathrm{D}^L(\lambda)\, \langle\Uparrow \cdots \Uparrow| \otimes \langle t - \eta|\,. & (5.12)
\end{aligned}$$

Proof. We focus on the proof of (5.7). The reasoning behind the other identities is similar. Graphically, we have

$$\left(\mathcal{A}^L(\lambda_0) \otimes \mathbb{I}^{\otimes(N-L)}\right) |\Downarrow \cdots \Downarrow\rangle \otimes |t\rangle$$

$=$  . (5.13)

The ice rule forces the first L vertices on the left to be of type h. Thus, the only non-zero entry of (5.13) reads



$$= \prod_{j=1}^{L} \mathsf{h}(\lambda_{0j}|t-\eta(2j-1)). \tag{5.14}$$

Using (4.22), the product on the right-hand side simplifies to

$$\prod_{j=1}^{L} \mathsf{h}(\lambda_{0j}|t-\eta(2j-1)) = \mathrm{e}^{2\mathrm{i}y\eta L} \frac{\vartheta_1(t)}{\vartheta_1(t-2\eta L)} \mathrm{D}^L(\lambda), \tag{5.15}$$

which establishes (5.7). $\square$

Action on the qSoV states We study the action of $\mathcal{D}(\lambda)$ on the qSoV states $\|\boldsymbol{h}, r\rangle\rangle$ and $\langle\langle \boldsymbol{h}, r\|$. It acts by shifting the parameter r.

Proposition 5.1.5. *We have*

$$\mathcal{D}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = \mathrm{d}_{\boldsymbol{h},r+1}(\lambda)\|\boldsymbol{h}, r+1\rangle\rangle, \tag{5.16}$$

$$\langle\langle \boldsymbol{h}, r\|\mathcal{D}(\lambda) = \mathrm{d}_{\boldsymbol{h},r-1}(\lambda)\langle\langle \boldsymbol{h}, r-1\|, \tag{5.17}$$

where

$$\mathrm{d}_{\boldsymbol{h},r}(\lambda) = \mathrm{e}^{-\mathrm{i}y\eta(N+s_{\boldsymbol{h}})} \frac{\vartheta_1(t_{\boldsymbol{h},r})}{\vartheta_1(t_{\boldsymbol{2},r})} \prod_{j=1}^{N} \vartheta_1(\lambda-\lambda_j+\eta h_j). \tag{5.18}$$

Proof. The proof is based on a standard technique of the Quantum Inverse Scattering Method [19]. Its idea is to explicitly write out the qSoV state, using Definition 5.1.1, and commute $\mathcal{D}(\lambda)$ to the right of the product of operators $\mathcal{C}$. After this commutation, we use Proposition 5.1.4 to compute the action of $\mathcal{D}(\lambda)$ on the reference state.

By Proposition 4.1.12, the commutation of $\mathcal{D}(\lambda)$ and $\mathcal{C}(\mu)$ is a linear combination of $\mathcal{C}(\lambda)\mathcal{D}(\mu)$ and $\mathcal{C}(\mu)\mathcal{D}(\lambda)$. This property allows us to identify three possible contributions to the action of $\mathcal{D}(\lambda)$ on the qSoV state:

$$\mathcal{D}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = \Bigg(X_{\boldsymbol{h}} \left[\prod_{j=1}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))} \right] \mathcal{D}(\lambda)$$
$$+ \sum_{k=1}^{N} Y_{\boldsymbol{h},k} \left[\prod_{\substack{j=1 \\ j\neq k}}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))} \right] \left(\frac{\mathcal{C}(\lambda_k - \eta)}{\mathrm{D}(\lambda_k - \eta)} \right)^{1-h_k} \mathcal{C}(\lambda) \frac{\mathcal{D}(\lambda_k - 2\eta)}{\mathrm{D}(\lambda_k - 2\eta)}$$
$$+ \sum_{k=1}^{N} \bar{Y}_{\boldsymbol{h},k} \left[\prod_{\substack{j=1 \\ j\neq k}}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))} \right] \mathcal{C}(\lambda) \frac{\mathcal{C}(\lambda_k - 2\eta)}{\mathrm{D}(\lambda_k - 2\eta)} \frac{\mathcal{D}(\lambda_k - \eta)}{\mathrm{D}(\lambda_k - \eta)} \Bigg) \|\boldsymbol{2}, r\rangle\rangle.$$

Here, $X_{\boldsymbol{h}}, Y_{\boldsymbol{h},k}$ and $\bar{Y}_{\boldsymbol{h},k}$ are computable coefficients, well-defined for generic inhomogeneity parameters. We now analyse each contribution individually.

The second contribution in the sum trivially cancels. Indeed, we show that the factor $\mathcal{D}(\lambda_k - 2\eta)\|\boldsymbol{2}, r\rangle\rangle = 0$, by Proposition 5.1.4, and since $\mathrm{A}(\lambda_k - 2\eta) = 0$, for each $k = 1, \dots, N$.

The third contribution never occurs, i.e. $\bar{Y}_{\boldsymbol{h},k} = 0$. For $h_k = 1, 2$, the claim is clear. For $h_k = 0$, we deduce the cancellation from the commutation sequence necessary to obtain such a term. We start from the following rewriting, due to Proposition 4.1.12:

$$\mathcal{D}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = \mathcal{D}(\lambda) \frac{\mathcal{C}(\lambda_k - \eta)}{\mathrm{D}(\lambda_k - \eta)} \frac{\mathcal{C}(\lambda_k - 2\eta)}{\mathrm{D}(\lambda_k - 2\eta)} \times \left[\prod_{\substack{j=1 \\ j\neq k}}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))} \right] \|\boldsymbol{2}, r\rangle\rangle. \quad (5.19)$$

First, we need to commute $\mathcal{D}(\lambda)$ to the right of $\mathcal{C}(\lambda_k - \eta)$ while exchanging their arguments. Second, we need to pass the operator $\mathcal{D}(\lambda_k - \eta)$ to the right of $\mathcal{C}(\lambda_k - 2\eta)$ while keeping their respective arguments. However, we see from (4.66) that this last commutation is impossible, since $\mathsf{a}(-\eta) = 0$.

Therefore, the first contribution is the only one that remains. From (4.50) and Proposition 4.1.12, we compute the coefficient

$$X_{\boldsymbol{h}} = \mathrm{e}^{-\mathrm{i}\mathrm{y}\eta(N+s_{\boldsymbol{h}})} \frac{\vartheta_1(t_{\boldsymbol{h},r} + \eta)}{\vartheta_1(t_{\boldsymbol{2},r} + \eta)} \frac{\prod_{j=1}^{N} \vartheta_1(\lambda - \lambda_j + \eta h_j)}{\mathrm{A}(\lambda)}. \quad (5.20)$$

Using Proposition 5.1.4, we find

$$\mathcal{D}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = X_{\boldsymbol{h}}\mathrm{A}(\lambda)\|\boldsymbol{h}, r+1\rangle\rangle = \mathrm{d}_{\boldsymbol{h},r+1}(\lambda)\|\boldsymbol{h}, r+1\rangle\rangle, \tag{5.21}$$

which establishes (5.16). □

We now consider $\mathcal{C}(\lambda)$ and $\mathcal{B}(\lambda)$. They act on the qSoV states $\|\boldsymbol{h}, r\rangle\rangle$ and $\langle\langle\boldsymbol{h}, r\|$ as weighted sums of shift operators on the $\boldsymbol{h}$. For each $i = 1, \dots, N$, we define $T_i^{\pm} : \mathbb{Z}^N \to \mathbb{Z}^N$ as

$$T_i^{\pm}(h_1, \cdots, h_i, \cdots, h_N) = (h_1, \cdots, h_i \pm 1, \cdots, h_N). \tag{5.22}$$

We note that $T_i^{\pm}$ does not preserve $\{0, 1, 2\}^N$. By convention, we set $\|\boldsymbol{k}, r\rangle\rangle = 0$ and $\langle\langle\boldsymbol{k}, r\| = 0$ for each $\boldsymbol{k} \in \mathbb{Z}^N \backslash \{0, 1, 2\}^N$.

Proposition 5.1.6. *We have*

$$\begin{aligned}\mathcal{C}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{iy}(\lambda-\lambda_j+\eta h_j)} &\frac{\vartheta_1(-\lambda+\lambda_j-\eta h_j+t_{\boldsymbol{h},r})}{\vartheta_1(t_{\boldsymbol{h},r})} \\ &\times \prod_{k\neq j} \frac{\vartheta_1(\lambda-\lambda_k+\eta h_k)}{\vartheta_1(\lambda_{jk}-\eta h_{jk})}\, \mathrm{D}(\lambda_j-\eta h_j)\|T_j^{-}\boldsymbol{h}, r\rangle\rangle,\end{aligned} \tag{5.23}$$

$$\begin{aligned}\langle\langle\boldsymbol{h}, r\|\mathcal{C}(\lambda) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{iy}(\lambda-\lambda_j+\eta h_j)} &\frac{\vartheta_1(-\lambda+\lambda_j-\eta h_j+t_{\boldsymbol{h},r})}{\vartheta_1(t_{\boldsymbol{h},r})} \\ &\times \prod_{k\neq j} \frac{\vartheta_1(\lambda-\lambda_k+\eta h_k)}{\vartheta_1(\lambda_{jk}-\eta h_{jk})}\, \mathrm{D}(\lambda_j-\eta(h_j+1))\langle\langle T_j^{+}\boldsymbol{h}, r\|,\end{aligned} \tag{5.24}$$

where $h_{jk} = h_j - h_k$.

Proposition 5.1.7. *We have*

$$\begin{aligned}\mathcal{B}(\lambda)\|\boldsymbol{h}, r\rangle\rangle = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{iy}(\lambda-\lambda_j+\eta h_j)} &\frac{\vartheta_1(-\lambda+\lambda_j-\eta h_j+t_{\boldsymbol{h},-r})}{\vartheta_1(t_{\boldsymbol{h},-r})} \\ \times \prod_{k\neq j} &\frac{\vartheta_1(\lambda-\lambda_k+\eta h_k)}{\vartheta_1(\lambda_{jk}-\eta h_{jk})}\, \mathrm{A}_{T_j^{+}\boldsymbol{h},r}(\lambda_j-\eta h_j)\|T_j^{+}\boldsymbol{h}, r\rangle\rangle,\end{aligned} \tag{5.25}$$

$$\begin{aligned}\langle\langle\boldsymbol{h}, r\|\mathcal{B}(\lambda) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{iy}(\lambda-\lambda_j+\eta h_j)} &\frac{\vartheta_1(-\lambda+\lambda_j-\eta h_j+t_{\boldsymbol{h},-r})}{\vartheta_1(t_{\boldsymbol{h},-r})} \\ \times \prod_{k\neq j} &\frac{\vartheta_1(\lambda-\lambda_k+\eta h_k)}{\vartheta_1(\lambda_{jk}-\eta h_{jk})}\, \mathrm{A}_{T_j^{-}\boldsymbol{h},r}(\lambda_j-\eta(h_j-1))\langle\langle T_j^{-}\boldsymbol{h}, r\|,\end{aligned} \tag{5.26}$$

where

$$\mathrm{A}_{\boldsymbol{h},r}(\lambda) = -\mathrm{e}^{2\mathrm{i}\eta \mathrm{y} r} \frac{\vartheta_1(t_{\boldsymbol{h},r})}{\vartheta_1(t_{\boldsymbol{h},-r})} \mathrm{A}(\lambda). \tag{5.27}$$

To prove Propositions 5.1.6 and 5.1.7, we characterise the actions of $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$ on the qSoV states as theta functions of the spectral parameter λ (see Appendix A.2).

Lemma 5.1.8. *The vectors*

$$\mathrm{e}^{\mathrm{iy}\lambda}\mathcal{C}(\lambda)\|\boldsymbol{h}, r\rangle\rangle \qquad \text{and} \qquad \mathrm{e}^{\mathrm{iy}\lambda}\langle\langle \boldsymbol{h}, r\|\mathcal{C}(\lambda)$$

are theta functions in λ of degree N, nome p, and norm $\sum_{j=1}^N \lambda_j + t_{\mathbf{0},r}$. The vectors

$$\mathrm{e}^{-\mathrm{iy}\lambda}\mathcal{B}(\lambda)\|\boldsymbol{h}, r\rangle\rangle \qquad \text{and} \qquad \mathrm{e}^{-\mathrm{iy}\lambda}\langle\langle \boldsymbol{h}, r\|\mathcal{B}(\lambda)$$

are theta functions in λ of degree N, nome p, and norm $\sum_{j=1}^N \lambda_j - t_{\mathbf{2},r}$.

Proof. The lemma straightforwardly follows from the definition of theta functions A.2.1, Corollaries 4.1.19 and 5.1.3 and Lemma 4.1.22. □

By Proposition A.2.4, we can interpolate the vectors

$$\mathrm{e}^{\mathrm{iy}\lambda}\mathcal{C}(\lambda)\|\boldsymbol{h}, r\rangle\rangle, \quad \mathrm{e}^{\mathrm{iy}\lambda}\langle\langle \boldsymbol{h}, r\|\mathcal{C}(\lambda), \quad \mathrm{e}^{-\mathrm{iy}\lambda}\mathcal{B}(\lambda)\|\boldsymbol{h}, r\rangle\rangle, \quad \mathrm{e}^{-\mathrm{iy}\lambda}\langle\langle \boldsymbol{h}, r\|\mathcal{B}(\lambda),$$

provided that we know their expressions at N independent values of λ. Here, we select these values in terms of the qSoV state on which the operators act. They are given by

$$\lambda = \lambda_j - \eta h_j, \qquad j = 1, \ldots, N. \tag{5.28}$$

These points are independent in the sense of Definition A.2.3. Indeed, for generic inhomogeneity parameters, we check

$$\lambda_{ij} - \eta h_{ij} \not\equiv 0 \ (\operatorname{mod} \pi, \pi\omega), \qquad \sum_{i=1}^N \lambda_i - \eta h_i \not\equiv t \ (\operatorname{mod} \pi, \pi\omega), \tag{5.29}$$

with $t = \sum_{j=1}^N \lambda_j + t_{\mathbf{0},r}$ or $\sum_{j=1}^N \lambda_j - t_{\mathbf{2},r}$. The non-congruence on the right-hand side crucially follows from (4.3) and (4.105).

The purpose of the two following lemmas is to explicitly compute the actions of $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$ at the specific values (5.28).

Lemma 5.1.9. *For each $j = 1, \ldots, N$, we have*

$$\mathcal{C}(\lambda_j - \eta h_j)\|\boldsymbol{h}, r\rangle\!\rangle = \mathrm{D}(\lambda_j - \eta h_j)\|T_j^- \boldsymbol{h}, r\rangle\!\rangle \tag{5.30}$$

$$\langle\!\langle \boldsymbol{h}, r\|\mathcal{C}(\lambda_j - \eta h_j) = \mathrm{D}(\lambda_j - \eta(h_j + 1))\langle\!\langle T_j^+ \boldsymbol{h}, r\|. \tag{5.31}$$

Proof. For $h_j = 1, 2$, the equality (5.30) straightforwardly follows from Definition 5.1.1 and the commutativity of the operators $\mathcal{C}$, as established in Proposition 4.1.12. For $h_j = 0$, we find

$$\begin{aligned}\mathcal{C}(\lambda_j)\|\boldsymbol{h}, r\rangle\!\rangle = &\frac{\mathcal{C}(\lambda_j)\mathcal{C}(\lambda_j - \eta)\mathcal{C}(\lambda_j - 2\eta)}{\mathrm{D}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)} \\ &\times \left[\prod_{\substack{k=1\\k\neq j}}^{N} \prod_{h=h_k}^{1} \frac{\mathcal{C}(\lambda_k - \eta(h+1))}{\mathrm{D}(\lambda_k - \eta(h+1))}\right]\|\boldsymbol{2}, r\rangle\!\rangle.\end{aligned} \tag{5.32}$$

The triple product of operators $\mathcal{C}$ in the prefactor vanishes by Proposition 4.1.17, which concludes the proof of (5.30). □

Lemma 5.1.10. *For each $j = 1, \ldots, N$, we have*

$$\mathcal{B}(\lambda_j - \eta h_j)\|\boldsymbol{h}, r\rangle\!\rangle = \mathrm{A}_{T_j^+\boldsymbol{h},r}(\lambda_j - \eta h_j)\|T_j^+ \boldsymbol{h}, r\rangle\!\rangle, \tag{5.33}$$

$$\langle\!\langle \boldsymbol{h}, r\|\mathcal{B}(\lambda_j - \eta h_j) = \mathrm{A}_{T_j^-\boldsymbol{h},r}(\lambda_j - \eta(h_j - 1))\langle\!\langle T_j^- \boldsymbol{h}, r\|. \tag{5.34}$$

Proof. We examine the three possible values $h_j = 0, 1, 2$ separately, using QISM arguments.

First, we consider $h_j = 2$. By Proposition 5.1.4, we have

$$\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{2}, r\rangle\!\rangle = 0. \tag{5.35}$$

For $\boldsymbol{h} \neq \boldsymbol{2}$, let $i = 1, \ldots, N$ be the largest integer such that $h_i \neq 2$. By Lemma 5.1.9, we have

$$\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{h}, r\rangle\!\rangle = \mathcal{B}(\lambda_j - 2\eta)\frac{\mathcal{C}(\lambda_i - \eta(h_i + 1))}{\mathrm{D}(\lambda_i - \eta(h_i + 1))}\|T_i^+ \boldsymbol{h}, r\rangle\!\rangle. \tag{5.36}$$

We commute $\mathcal{B}(\lambda_j - 2\eta)$ right of $\mathcal{C}(\lambda_i - \eta(h_i + 1))$ in this expression. By Proposition 4.1.12, we identify three possible contributions,

$$\begin{aligned}\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{h}, r\rangle\!\rangle = &\big(X\mathcal{C}(\lambda_i - \eta(h_i + 1))\mathcal{B}(\lambda_j - 2\eta) \\ &+ Y\mathcal{A}(\lambda_i - \eta(h_i + 1))\mathcal{D}(\lambda_j - 2\eta) \\ &+ \bar{Y}\mathcal{A}(\lambda_j - 2\eta)\mathcal{D}(\lambda_i - \eta(h_i + 1))\big)\|T_i^+ \boldsymbol{h}, r\rangle\!\rangle,\end{aligned} \tag{5.37}$$

where $X, Y, \bar{Y}$ are computable coefficients, well-defined for generic inhomogeneity parameters. By Proposition 5.1.5, the two last terms are proportional to $\mathrm{d}_{T_i^+\boldsymbol{h},r+1}(\lambda_j - 2\eta) = 0$, and $\mathrm{d}_{T_i^+\boldsymbol{h},r+1}(\lambda_i - \eta(h_i+1)) = 0$, respectively, so that

$$\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{h}, r\rangle\rangle = X\mathcal{C}(\lambda_i - \eta(h_i+1))\mathcal{B}(\lambda_j - 2\eta)\|T_i^+\boldsymbol{h}, r\rangle\rangle. \tag{5.38}$$

We iterate this equation and obtain

$$\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{h}, r\rangle\rangle = \bar{X}\left[\prod_{k=1}^{N}\prod_{h=h_k}^{1}\frac{\mathcal{C}(\lambda_k - \eta(h+1))}{\mathrm{D}(\lambda_k - \eta(h+1))}\right]\mathcal{B}(\lambda_j - 2\eta)\|\mathbf{2}, r\rangle\rangle,$$

where $\bar{X}$ is another computable coefficient. Using (5.35), we find $\mathcal{B}(\lambda_j - 2\eta)\|\boldsymbol{h}, r\rangle\rangle = 0$, which establishes (5.33) for $h_j = 2$.

Second,we consider $h_j = 1$. As above, we use Lemma 5.1.9 to rewrite

$$\mathcal{B}(\lambda_j - \eta)\|\boldsymbol{h}, r\rangle\rangle = \mathcal{B}(\lambda_j - \eta)\frac{\mathcal{C}(\lambda_j - 2\eta)}{\mathrm{D}(\lambda_j - 2\eta)}\|T_j^+\boldsymbol{h}, r\rangle\rangle. \tag{5.39}$$

The quantum determinant relation of Corollary 4.1.14 allows us to write

$$\begin{aligned}\mathcal{B}(\lambda_j - \eta)\mathcal{C}(\lambda_j - 2\eta) &= \mathcal{A}(\lambda_j - \eta)\mathcal{D}(\lambda_j - 2\eta)\\ &\quad - \mathrm{A}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)\frac{\mathrm{e}^{-2\mathrm{i}\mathrm{y}\eta S}\vartheta_1(\tau)}{\vartheta_1(\tau + 2\eta S)}.\end{aligned} \tag{5.40}$$

By Proposition 5.1.5, the action of the first term of the right-hand side of this equality vanishes on $\|T_j^+\boldsymbol{h}, r\rangle\rangle$:

$$\mathcal{D}(\lambda_j - 2\eta)\|T_j^+\boldsymbol{h}, r\rangle\rangle = \mathrm{d}_{T_j^+\boldsymbol{h},r+1}(\lambda_j - 2\eta)\|T_j^+\boldsymbol{h}, r\rangle\rangle = 0. \tag{5.41}$$

Only the action of the second term of (5.40) remains. We find

$$\mathcal{B}(\lambda_j - \eta)\|\boldsymbol{h}, r\rangle\rangle = -\mathrm{A}(\lambda_j - \eta)\frac{\mathrm{e}^{-2\mathrm{i}\mathrm{y}\eta s_{T_j^+\boldsymbol{h}}}\vartheta_1\left(t_{T_j^+\boldsymbol{h},r}\right)}{\vartheta_1\left(t_{T_j^+\boldsymbol{h},r} + 2\eta s_{T_j^+\boldsymbol{h}}\right)}\|T_j^+\boldsymbol{h}, r\rangle\rangle, \tag{5.42}$$

where we used Proposition 5.1.2. Further simplifications follow from the pseudo-periodicity of the Jacobi theta functions and the identity $t_{\boldsymbol{h},r} + 2\eta s_{\boldsymbol{h}} = -t_{\boldsymbol{h},-r} + \mathsf{x}\pi + \mathsf{y}\pi\omega$, which is an immediate consequence of (5.2). These simplifications lead to (5.33) with $h_j = 1$.

Finally, the proof for $h_j = 0$, albeit more involved, follows the same lines as for $h_j = 1$. $\square$

Proof of Propositions 5.1.6 and 5.1.7. The propositions follow from the elliptic interpolation of the vectors

$$\mathrm{e}^{\mathrm{i}y\lambda}\mathcal{C}(\lambda)\|\boldsymbol{h},r\rangle\rangle,\quad \mathrm{e}^{\mathrm{i}y\lambda}\langle\langle\boldsymbol{h},r\|\mathcal{C}(\lambda),\quad \mathrm{e}^{-\mathrm{i}y\lambda}\mathcal{B}(\lambda)\|\boldsymbol{h},r\rangle\rangle,\quad \mathrm{e}^{-\mathrm{i}y\lambda}\langle\langle\boldsymbol{h},r\|\mathcal{B}(\lambda),$$

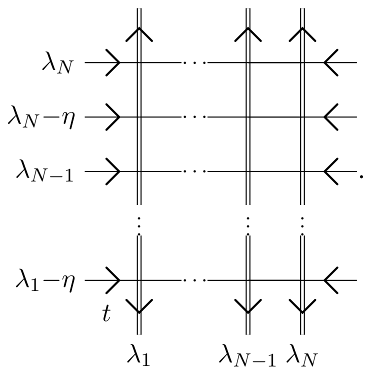
based on their values at the points $\lambda = \lambda_j - \eta h_j$, $j = 1, \ldots, N$, provided by Lemmas 5.1.9 and 5.1.10. $\square$

5.1.3 Dual pairings

We now establish a formula for the dual pairings between left and right qSoV states. We proceed by induction. The first step is computing the dual pairing for a simple choice of qSoV states.

Lemma 5.1.11. *We have* $\langle\langle\mathbf{2},r\|\mathbf{2},r\rangle\rangle = 1$.

Proof. We compute the dual pairing with the help of graphical arguments. To this end, we first evaluate a partition function of the d10V model on a $2N \times N$ rectangle with *domain-wall boundary conditions*:

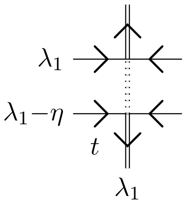
$W_N(t) =$ 

. (5.43)

We show that $W_N(t)$ obeys a simple recurrence relation that we solve explicitly.

To understand the mechanism that leads to this recurrence relation, we first consider $N = 1$. For this base case, the configuration of the only two vertices is uniquely determined by the ice rule and the boundary conditions. The partition function is

$$W_1(t) =$$

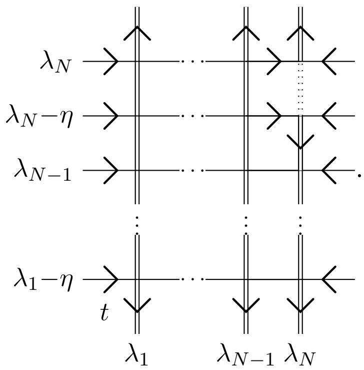


$$= \bar{\mathsf{g}}(0|t-2\eta)\bar{\mathsf{j}}(-\eta|t-\eta) = \vartheta_1(\eta)\vartheta_1(2\eta), \quad (5.44)$$

where the last equality follows from (4.22).

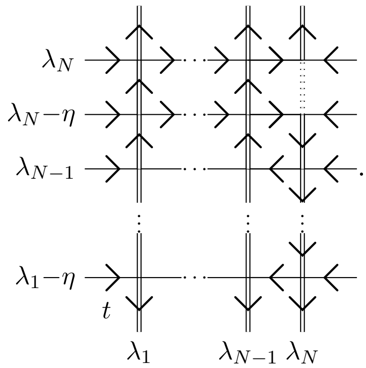
We now examine the case $N \geqslant 2$. By the ice rule, the vertex in the upper-right corner is either of type $\bar{\mathsf{g}}$ or $\bar{\mathsf{h}}$, with a weight $\bar{\mathsf{g}}(0|t-2\eta)$ or $\bar{\mathsf{h}}(0|t-2\eta)$, respectively. By (4.22), we have $\bar{\mathsf{h}}(0|t-2\eta) = 0$, which implies that the upper-right vertex is *frozen* to type $\bar{\mathsf{g}}$. By the ice rule, the vertex just below can be either of type $\bar{\mathsf{j}}$ or $\bar{\mathsf{f}}$, with a weight $\bar{\mathsf{j}}(-\eta|t-\eta)$ or $\bar{\mathsf{f}}(-\eta|t-\eta)$, respectively. By (4.22), we have $\bar{\mathsf{f}}(-\eta|t-\eta) = 0$, which implies that the vertex is frozen to type $\bar{\mathsf{j}}$:

$$W_N(t) = \quad (5.45)$$



Hence, the configuration of the two vertices in the upper-right corner is frozen to the unique configuration contributing to the partition function for $N = 1$. Given this configuration, the remaining vertices along the rightmost vertical line and the two upper horizontal lines are frozen to type e by the ice rule:

$$W_N(t) = $$

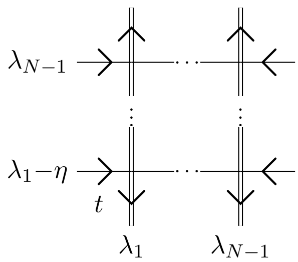


(5.46)

We may, thus, 'peel off' the two upper horizontal lines and the rightmost vertical line and replace them with the product of their vertex weights. Thus, we obtain

$$W_N(t) = \bar{\mathsf{g}}(0|t-2\eta)\bar{\mathsf{j}}(-\eta|t-\eta)\left(\prod_{j=1}^{N-1}\mathsf{e}(\lambda_{Nj})\mathsf{e}(\lambda_{Nj}-\eta)\right)$$

$$\times\left(\prod_{j=1}^{N-1}\mathsf{e}(\lambda_{jN})\mathsf{e}(\lambda_{jN}-\eta)\right)$$



. (5.47)

On the right-hand side of this equality, we recognise the graphical representation of $W_{N-1}(t)$. Hence, upon evaluation of the vertex weights with the help of (4.22), we obtain the recurrence relation

$$W_N(t) = \vartheta_1(\eta)\vartheta_1(2\eta)\left(\prod_{j=1}^{N-1}\vartheta_1(\lambda_{Nj}+2\eta)\vartheta_1(\lambda_{Nj}+\eta)\right)$$
$$\times\left(\prod_{j=1}^{N-1}\vartheta_1(\lambda_{jN}+2\eta)\vartheta_1(\lambda_{jN}+\eta)\right)W_{N-1}(t). \quad (5.48)$$

Using the initial condition $W_1(t) = \vartheta_1(\eta)\vartheta_1(2\eta)$, the unique solution to the recurrence relation reads

$$W_N(t) = \prod_{j=1}^{N}\mathrm{D}(\lambda_j-\eta)\mathrm{D}(\lambda_j-2\eta), \quad (5.49)$$

where $\mathrm{D}(\lambda)$ is defined in (3.49).

Finally, we combine Definition 5.1.1 with the graphical representations of Section 4.1.3 to identify

$$\langle\!\langle \mathbf{2}, r \| \mathbf{2}, r\rangle\!\rangle = \frac{\langle\!\langle \mathbf{0}, r\| \prod_{j=1}^N \mathcal{C}(\lambda_j) C(\lambda_j - \eta)\|\mathbf{2}, r\rangle\!\rangle}{\prod_{j=1}^N \mathrm{D}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)}, \tag{5.50}$$

$$= \frac{W_N(t_{\mathbf{2},r})}{\prod_{j=1}^N \mathrm{D}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)}. \tag{5.51}$$

The lemma follows from this equality and (5.49). □

We now establish a closed-form expression for the dual pairing of qSoV states. To this end, we use the notation

$$\delta_{\boldsymbol{h},\tilde{\boldsymbol{h}}} = \prod_{i=1}^N \delta_{h_i, h'_i}, \tag{5.52}$$

for all $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in \{0,1,2\}^N$.

Proposition 5.1.12. *We have*

$$\langle\!\langle \boldsymbol{h}, r\|\tilde{\boldsymbol{h}}, \tilde{r}\rangle\!\rangle = \delta_{\boldsymbol{h},\tilde{\boldsymbol{h}}}\delta_{r,\tilde{r}}\, \mathrm{e}^{\mathrm{i}\mathsf{y}\eta(N+s_{\boldsymbol{h}})} \frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\boldsymbol{h},r})} \prod_{1\le i<j\le N} \frac{\vartheta_1(\lambda_{ij})}{\vartheta_1(\lambda_{ij} - \eta h_{ij})}. \tag{5.53}$$

Proof. First, we show that $\langle\!\langle \boldsymbol{h}, r\|\tilde{\boldsymbol{h}}, \tilde{r}\rangle\!\rangle = 0$ if $\boldsymbol{h} \neq \tilde{\boldsymbol{h}}$ or $r \neq \tilde{r}$. By Corollary 5.1.3, $\langle\!\langle \boldsymbol{h}, r\|$ and $\|\tilde{\boldsymbol{h}}, \tilde{r}\rangle\!\rangle$ are left and right eigenvectors of S_τ with eigenvalues $r\eta + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi\omega}{2}$ and $\tilde{r}\eta + \mathsf{x}\frac{\pi}{2} + \mathsf{y}\frac{\pi\omega}{2}$, respectively. Hence, their dual pairing vanishes if $r \neq \tilde{r}$. Furthermore, using Proposition 5.1.5, we consider the quantity,

$$\langle\!\langle \boldsymbol{h}, r+1\|\mathcal{D}(\lambda)\|\tilde{\boldsymbol{h}}, r\rangle\!\rangle = \mathrm{d}_{\boldsymbol{h},r}(\lambda)\langle\!\langle \boldsymbol{h}, r+1\|\tilde{\boldsymbol{h}}, r+1\rangle\!\rangle \tag{5.54a}$$

$$= \mathrm{d}_{\tilde{\boldsymbol{h}},r+1}(\lambda)\langle\!\langle \boldsymbol{h}, r\|\tilde{\boldsymbol{h}}, r\rangle\!\rangle. \tag{5.54b}$$

The only dependence in λ of the right-hand sides of (5.54) is featured in the prefactors $\mathrm{d}_{\boldsymbol{h},r}(\lambda)$ and $\mathrm{d}_{\tilde{\boldsymbol{h}},r+1}(\lambda)$. If $\boldsymbol{h} \neq \tilde{\boldsymbol{h}}$, there exists $i = 1, \dots, N$ such that $h_i \neq \tilde{h}_i$. We evaluate (5.54) with $\lambda = \lambda_i - \eta h_i$. Since $\mathrm{d}_{\boldsymbol{h},r}(\lambda_i - \eta h_i) = 0$, we find

$$0 = \mathrm{d}_{\tilde{\boldsymbol{h}},r+1}(\lambda_i - \eta h_i)\langle\!\langle \boldsymbol{h}, r\|\tilde{\boldsymbol{h}}, r\rangle\!\rangle. \tag{5.55}$$

One checks that the prefactor $\mathrm{d}_{\tilde{\boldsymbol{h}},r+1}(\lambda_i - \eta h_i)$ is non-zero for generic inhomogeneity parameters, which implies $\langle\!\langle \boldsymbol{h}, r \| \tilde{\boldsymbol{h}}, r\rangle\!\rangle = 0$, for $\boldsymbol{h} \neq \tilde{\boldsymbol{h}}$.

Second, we compute the dual pairing $\langle\!\langle \boldsymbol{h}, r \| \boldsymbol{h}, r\rangle\!\rangle$. For $\boldsymbol{h} = \mathbf{2}$, we have $\langle\!\langle \mathbf{2}, r \| \mathbf{2}, r\rangle\!\rangle = 1$ by Lemma 5.1.11. Hence, we consider $\boldsymbol{h} \neq \mathbf{2}$. Let $i = 1, \dots, N$ be such that $h_i < 2$, by Lemma 5.1.9, we have

$$\langle\!\langle \boldsymbol{h}, r \| \mathcal{C}(\lambda_i - \eta h_i) \| T_i^+ \boldsymbol{h}, r\rangle\!\rangle = \mathrm{D}(\lambda_i - \eta(h_i + 1)) \langle\!\langle T_i^+ \boldsymbol{h}, r \| T_i^+ \boldsymbol{h}, r\rangle\!\rangle. \tag{5.56}$$

Similarly, by Proposition 5.1.6, we obtain

$$\begin{aligned}\langle\!\langle \boldsymbol{h}, r \| \mathcal{C}(\lambda_i - \eta h_i) \| T_i^+ \boldsymbol{h}, r\rangle\!\rangle = \mathrm{e}^{-\mathrm{i}\mathrm{y}\eta} \frac{\vartheta_1(t_{\boldsymbol{h},r})}{\vartheta_1(t_{T_i^+ \boldsymbol{h},r})} \prod_{\substack{k=1\\k\neq a}}^{N} \frac{\vartheta_1(\lambda_{ik} - \eta h_{ik})}{\vartheta_1(\lambda_{ik} - \eta(h_{ik}+1))} \\ \times \mathrm{D}(\lambda_i - \eta(h_i + 1)) \langle\!\langle \boldsymbol{h}, r \| \boldsymbol{h}, r\rangle\!\rangle. \end{aligned}\tag{5.57}$$

We note that the prefactors of the dual pairings on the right-hand sides of these evaluations never vanish (nor diverge) for generic inhomogeneity parameters. Thus, upon equating the right-hand sides, we obtain the recurrence relation:

$$\begin{aligned}\langle\!\langle \boldsymbol{h}, r \| \boldsymbol{h}, r\rangle\!\rangle = \mathrm{e}^{\mathrm{i}\mathrm{y}\eta} \frac{\vartheta_1(t_{T_i^+ \boldsymbol{h},r})}{\vartheta_1(t_{\boldsymbol{h},r})} \prod_{\substack{k=1\\k\neq i}}^{N} \frac{\vartheta_1(\lambda_{ik} - \eta(h_{ik}+1))}{\vartheta_1(\lambda_{ik} - \eta h_{ik})} \\ \times \langle\!\langle T_i^+ \boldsymbol{h}, r \| T_i^+ \boldsymbol{h}, r\rangle\!\rangle. \end{aligned}\tag{5.58}$$

The iteration of this equality allows us to obtain any dual pairing for $\boldsymbol{h} \neq \mathbf{2}$ from the base case $\langle\!\langle \mathbf{2}, r \| \mathbf{2}, r\rangle\!\rangle = 1$, provided by Lemma 5.1.11. □

Beside the reference states (5.1), the components of the qSoV states with respect to the canonical basis are, in general, rather involved (even for simple choices of $\boldsymbol{h}$, as we shall see below). Nonetheless, the dual partners of the reference states take a simple form:

Proposition 5.1.13. *We have*

$$\| \mathbf{0}, r\rangle\!\rangle = \mathrm{e}^{2\mathrm{i}\mathrm{y}\eta N} \frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\mathbf{0},r})} \left| \Uparrow \cdots \Uparrow \right\rangle \otimes \left| t_{\mathbf{0},r} \right\rangle, \tag{5.59a}$$

$$\langle\!\langle \mathbf{2}, r \| = \left\langle \Downarrow \cdots \Downarrow \right| \otimes \left\langle t_{\mathbf{2},r} \right|. \tag{5.59b}$$

Proof. By Proposition 5.1.2, we have

$$\|\mathbf{0}, r\rangle\rangle = \mu \,|\Uparrow \cdots \Uparrow\rangle \otimes |t_{\mathbf{0},r}\rangle \,, \tag{5.60}$$

$$\langle\langle \mathbf{2}, r\| = \xi \,\langle\Downarrow \cdots \Downarrow| \otimes \langle t_{\mathbf{2},r}| \,, \tag{5.61}$$

where ξ, μ are complex numbers. Using Proposition 5.1.12, and the explicit form (5.1) of the reference states, we evaluate

$$\xi = \langle\langle \mathbf{2}, r\|(|\Downarrow \cdots \Downarrow\rangle \otimes |t_{\mathbf{2},r}\rangle) = \langle\langle \mathbf{2}, r\|\mathbf{2}, r\rangle\rangle = 1, \tag{5.62}$$

$$\mu = (\langle\Uparrow \cdots \Uparrow| \otimes \langle t_{\mathbf{0},r}|)\|\mathbf{0}, r\rangle\rangle = \langle\langle \mathbf{0}, r\|\mathbf{0}, r\rangle\rangle = \mathrm{e}^{2\mathrm{i}y\eta N}\frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\mathbf{0},r})}. \tag{5.63}$$

The proposition follows from these evaluations. □

Bases of the spin-dynamical spaces We show that the qSoV states form bases of the eigenspaces of S_τ.

Proposition 5.1.14. *The sets* $\{\|\boldsymbol{h}, r\rangle\rangle\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ *and* $\{\langle\langle \boldsymbol{h}, r\|\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ *form bases of* $\mathbb{D}_N^{(r)}$ *and* $\mathbb{D}_N^{(r)*}$*, respectively.*

Proof. Note that, by Proposition 5.1.2, $\{\|\boldsymbol{h}, r\rangle\rangle\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ is a subset of $\mathbb{D}_N^{(r)}$. First, we show that it is linearly independent. Let us assume the opposite, that there exists a set of complex numbers $\{c_{\boldsymbol{h}}\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ such that

$$\sum_{\boldsymbol{h}\in\{0,1,2\}^N} c_{\boldsymbol{h}}\|\boldsymbol{h}, r\rangle\rangle = 0. \tag{5.64}$$

For each $\boldsymbol{k} \in \{0,1,2\}^N$, we consider the dual pairing of $\langle\langle \boldsymbol{k}, r\|$ with (5.64). By Proposition 5.1.12, we obtain

$$\mathrm{e}^{\mathrm{i}y\eta(N+s_{\boldsymbol{k}})}\frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\boldsymbol{k},r})}\left(\prod_{1\le i<j\le N}\frac{\vartheta_1(\lambda_{ij})}{\vartheta_1(\lambda_{ij}-\eta k_{ij})}\right)c_{\boldsymbol{k}} = 0. \tag{5.65}$$

In this equality, the prefactor of $c_{\boldsymbol{k}}$ never vanishes for generic inhomogeneity parameters. Hence, $c_{\boldsymbol{k}} = 0$, which proves the linear independence.

Second, the cardinality of $\{\|\boldsymbol{h}, r\rangle\rangle\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ matches the dimension of $\mathbb{D}_N^{(r)}$. We deduce that it is a basis of the subspace. □

It follows from Proposition 5.1.12 that the identity operator on $\mathbb{D}_N^{(r)}$ decomposes as

$$\mathbb{I} = \sum_{\boldsymbol{h}\in\{0,1,2,\}^N} \frac{\|\boldsymbol{h},r\rangle\rangle\langle\langle\boldsymbol{h},r\|}{\langle\langle\boldsymbol{h},r\|\boldsymbol{h},r\rangle\rangle}, \tag{5.66}$$

where $\langle\langle\boldsymbol{h},r\|\boldsymbol{h},r\rangle\rangle$ is given by Proposition 5.1.12.

5.1.4 Properties

In this section, we present several properties of the qSoV states. First, we derive their pseudo-periodicity and exchange relations, based on the corresponding relations for $\mathcal{C}(\lambda)$. Second, we study their analyticity and pole structure. Finally, we find that the specialisation of inhomogeneity parameters of the qSoV states leads to reduction relations. In Section 5.2.3, these properties of the qSoV states give rise to analogous ones for special eigenvectors of the anti-periodic d10V transfer matrix.

Pseudo-periodicity Corollary 4.1.19 and Proposition 4.1.20 allows us to establish pseudo-periodicity properties for the qSoV basis states with respect to their inhomogeneity parameters.

Proposition 5.1.15. *For $i = 1,\dots,N$, we have*

$$\begin{aligned}
\|\boldsymbol{h},r\rangle\rangle\big|_{\lambda_i\to\lambda_i+\pi} &= (-1)^{\mathsf{y}(1-h_i)}(\Omega_i^3)^{\mathsf{y}}\|\boldsymbol{h},r\rangle\rangle, &(5.67)\\
\|\boldsymbol{h},r\rangle\rangle\big|_{\lambda_i\to\lambda_i+\pi\omega} &= (-1)^{\mathsf{x}(1-h_i)}\mathrm{e}^{-2\mathrm{i}\eta(N+1-2i)}\\
&\quad\times \mathrm{e}^{2\mathrm{i}\eta r(1-h_i)}\mathrm{e}^{2\mathrm{i}\eta\sum_{j=1}^N h_{ij}}(\Gamma^i)^{-1}\|\boldsymbol{h},r\rangle\rangle, &(5.68)
\end{aligned}$$

and

$$\begin{aligned}
\langle\langle\boldsymbol{h},r\|\big|_{\lambda_i\to\lambda_i+\pi} &= (-1)^{-\mathsf{y}(1-h_i)}\langle\langle\boldsymbol{h},r\|(\Omega_i^3)^{\mathsf{y}}, &(5.69)\\
\langle\langle\boldsymbol{h},r\|\big|_{\lambda_i\to\lambda_i+\pi\omega} &= (-1)^{-\mathsf{x}(1-h_i)}\mathrm{e}^{2\mathrm{i}\eta(N+1-2i)}\\
&\quad\times \mathrm{e}^{-2\mathrm{i}\eta r(1-h_i)}\mathrm{e}^{-4\mathrm{i}\eta\sum_{j=1}^N h_{ij}}\langle\langle\boldsymbol{h},r\|\Gamma^i, &(5.70)
\end{aligned}$$

where Γ^i is defined in (4.89).

Proof. We focus on the shift $\lambda_i \to \lambda_i + \pi\omega$ for the right qSoV states. The reasoning for the other cases is similar. We examine the definition

(5.3). By means of Corollary 4.1.19 and Proposition 4.1.20, we apply the transformation $\lambda_i \to \lambda_i + \pi\omega$ to each factor of the product. We identify two different types of contribution, for $j \neq i$ and $h = 0, 1$, they are

$$\left.\frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))}\right|_{\lambda_i \to \lambda_i + \pi\omega} = \mathrm{e}^{2\mathrm{i}\eta}(\Gamma^i)^{-1}\frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))}\,\Gamma^i, \tag{5.71a}$$

$$\left.\frac{\mathcal{C}(\lambda_i - \eta(h+1))}{\mathrm{D}(\lambda_i - \eta(h+1))}\right|_{\lambda_i \to \lambda_i + \pi\omega} = p^{-\mathsf{y}}\mathrm{e}^{-2\mathrm{i}\eta(N-1)}\mathrm{e}^{2\mathrm{i}S_\tau} \times (\Gamma^i)^{-1}\frac{\mathcal{C}(\lambda_i - \eta(h+1))}{\mathrm{D}(\lambda_i - \eta(h+1))}\,\Gamma^i, \tag{5.71b}$$

where we inferred the behaviour of $\mathrm{D}(\lambda)$ from Proposition A.1.2. Given Lemma 4.1.22, and $[S_\tau, \Gamma^i] = 0$, we have

$$\|\boldsymbol{h}, r\rangle\!\rangle\big|_{\lambda_i \to \lambda_i + \pi\omega} = (p^{-\mathsf{y}}\mathrm{e}^{2\mathrm{i}S_\tau})^{(2-h_i)}\mathrm{e}^{2\mathrm{i}\eta\sum_{j=1}^N h_{ij}}(\Gamma^i)^{-1} \times \left[\prod_{j=1}^{N}\prod_{h=h_j}^{1}\frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))}\right]\Gamma^i\|\mathbf{2}, r\rangle\!\rangle. \tag{5.72}$$

We compute explicitly $\Gamma^i\|\mathbf{2}, r\rangle\!\rangle = (-1)^{\mathsf{x}}\mathrm{e}^{-2\mathrm{i}\eta(r+N+1-2i)}\|\mathbf{2}, r\rangle\!\rangle$. Lastly, Proposition 5.1.2 leads to (5.68). $\square$

The dual pairing $\langle\!\langle \boldsymbol{h}, r\|\boldsymbol{h}, r\rangle\!\rangle$ also possesses pseudo-periodicity relations with respect to the inhomogeneity parameters. We deduce them from the explicit formula of Proposition 5.1.12 and the pseudo-periodicity of Jacobi theta functions of Proposition A.1.2.

Proposition 5.1.16. *For $i = 1, \ldots, N$, we have*

$$\langle\!\langle \boldsymbol{h}, r\|\boldsymbol{h}, r\rangle\!\rangle\big|_{\lambda_i \to \lambda_i + \pi} = \langle\!\langle \boldsymbol{h}, r\|\boldsymbol{h}, r\rangle\!\rangle, \tag{5.73}$$

$$\langle\!\langle \boldsymbol{h}, r\|\boldsymbol{h}, r\rangle\!\rangle\big|_{\lambda_i \to \lambda_i + \pi\omega} = \mathrm{e}^{-2\mathrm{i}\eta\sum_{j=1}^N h_{ij}}\langle\!\langle \boldsymbol{h}, r\|\boldsymbol{h}, r\rangle\!\rangle. \tag{5.74}$$

Exchange relation We use Proposition 4.1.16 to establish an exchange relation for the qSoV states. The multiplication of the qSoV state $\|\boldsymbol{h}, r\rangle\!\rangle$ by an appropriate $\check{\mathsf{R}}$-matrix swaps two of its inhomogeneity parameters and the corresponding entries of $\boldsymbol{h}$.

Proposition 5.1.17. *For $i = 1, \dots, N-1$, we have*

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau + 2\eta S^{i-1})\|\boldsymbol{h}, r\rangle\rangle$$
$$= \vartheta_1(\lambda_{i,i+1} + \eta)\vartheta_1(\lambda_{i,i+1} + 2\eta) \left(\|\boldsymbol{h}, r\rangle\rangle\Big|_{\substack{\lambda_i \leftrightarrow \lambda_{i+1} \\ h_i \leftrightarrow h_{i+1}}} \right), \quad (5.75)$$

and

$$\langle\langle \boldsymbol{h}, r\|\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i+1,i}|\tau + 2\eta S^{i-1})$$
$$= \vartheta_1(\lambda_{i+1,i} + \eta)\vartheta_1(\lambda_{i+1,i} + 2\eta) \left(\langle\langle \boldsymbol{h}, r\|\Big|_{\substack{\lambda_i \leftrightarrow \lambda_{i+1} \\ h_i \leftrightarrow h_{i+1}}} \right). \quad (5.76)$$

Proof. Using the exchange relation of Proposition 4.1.16 and Definition 5.1.1, we obtain

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau + 2\eta S^{i-1})\|\boldsymbol{h}, r\rangle\rangle$$
$$= \left[\prod_{j=1}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1); \cdots \lambda_{i+1}, \lambda_i \cdots)}{\mathrm{D}(\lambda_j - \eta(h+1))} \right]$$
$$\times \check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau + 2\eta S^{i-1})\|\mathbf{2}, r\rangle\rangle, \quad (5.77)$$

where we highlighted the dependence on the inhomogeneity parameters by writing them as arguments. A straightforward calculation yields

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau + 2\eta S^{i-1})\|\mathbf{2}, r\rangle\rangle$$
$$= \vartheta_1(\lambda_{i,i+1} + \eta)\vartheta_1(\lambda_{i,i+1} + 2\eta)\|\mathbf{2}, r\rangle\rangle. \quad (5.78)$$

Next, we inspect the product in (5.77). For $j \neq i, i+1$, we note that the factors do not depend on h_i and h_{i+1}. Moreover, $\mathrm{D}(\lambda_j - \eta(h+1))$ is invariant under the exchange $\lambda_i \leftrightarrow \lambda_{i+1}$. For $j = i, i+1$, the following product equality is straightforward:

$$\prod_{j=i,i+1}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1); \cdots \lambda_{i+1}, \lambda_i \cdots)}{\mathrm{D}(\lambda_j - \eta(h+1))}$$
$$= \prod_{j=i,i+1}^{N} \prod_{h=h_j}^{1} \frac{\mathcal{C}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta(h+1))}\Bigg|_{\substack{\lambda_i \leftrightarrow \lambda_{i+1} \\ h_i \leftrightarrow h_{i+1}}}. \quad (5.79)$$

Combining (5.77), (5.78) and (5.79), we obtain (5.75). □

Analyticity In this paragraph, we examine the analyticity properties of the qSoV states $\|\boldsymbol{h}, r\rangle\rangle$, $\langle\langle\boldsymbol{h}, r\|$, for $\boldsymbol{h} \in \{0,2\}^N$. The reason why we focus on this subset becomes clear in Section 5.2. As a first step, we compute explicitly the elements

$$\|2\cdots 220, r\rangle\rangle, \quad \|2\cdots 202, r\rangle\rangle, \quad \|2\cdots 200, r\rangle\rangle, \tag{5.80}$$

and

$$\langle\langle 0\cdots 002, r\|, \quad \langle\langle 0\cdots 020, r\|, \quad \langle\langle 0\cdots 022, r\|. \tag{5.81}$$

As these calculations prove somewhat involved, we relegate them to the appendix in Section 5.A. From the definition 5.1.1, we know that inhomogeneity parameters always appear in qSoV states through the differences λ_{ij}. The elements (5.80) and (5.81) allow us to probe for poles in $\lambda_{N-1,N}$. As a second step, we extend the characterisation to poles in λ_{ij}, with the help of the exchange relations.

Proposition 5.1.18. *For each $\boldsymbol{h} \in \{0,2\}^N$, and $1 \leq i < j \leq N$, the qSoV state $\|\boldsymbol{h}, r\rangle\rangle$ is an entire function of λ_{ij} if and only if $(h_i, h_j) \neq (0,2)$. For $(h_i, h_j) = (0,2)$, $\|\boldsymbol{h}, r\rangle\rangle$ possesses simple poles at $\lambda_{ij} = \eta, 2\eta$.*

Proof. First, we consider $i = N-1$. By Proposition 5.A.1, the states $\|2\cdots 220, r\rangle\rangle$ and $\|2\cdots 200, r\rangle\rangle$ do not depend on the inhomogeneity parameters. Consequently, they do not possess poles in $\lambda_{N-1,N}$. Moreover, the state $\|2\cdots 202, r\rangle\rangle$ possesses two simple poles at $\lambda_{N-1,N} = \eta, 2\eta$. Furthermore, from Definition 5.1.1, we can build any $\|\boldsymbol{h}, r\rangle\rangle$, with $\boldsymbol{h} \in \{0,2\}^N$, from the application of operators $\mathcal{C}$ on $\|2\cdots 222, r\rangle\rangle$, $\|2\cdots 220, r\rangle\rangle$, $\|2\cdots 202, r\rangle\rangle$ or $\|2\cdots 200, r\rangle\rangle$. This construction process does not add extra dependence on $\lambda_{N-1,N}$.

Second, we consider $1 \leq i < j \leq N$. We use Proposition 5.1.17 to relate

$$\begin{aligned}\|\boldsymbol{h}, r\rangle\rangle = &\prod_{k=j}^{N-1} \frac{\check{\mathsf{R}}^{(2,2)}_{k,k+1}\left(\lambda_{k+1,j} \,\middle|\, \tau + 2\eta S^{k-1}\right)}{\vartheta_1(\lambda_{k+1,j}+\eta)\vartheta_1(\lambda_{k+1,j}+2\eta)} \\ &\times \prod_{\ell=i}^{j-2} \frac{\check{\mathsf{R}}^{(2,2)}_{\ell,\ell+1}\left(\lambda_{\ell+1,i} \,\middle|\, \tau + 2\eta S^{\ell-1}\right)}{\vartheta_1(\lambda_{\ell+1,i}+\eta)\vartheta_1(\lambda_{\ell+1,i}+2\eta)} \\ &\times \prod_{m=j-1}^{N-2} \frac{\check{\mathsf{R}}^{(2,2)}_{m,m+1}\left(\lambda_{m+2,i} \,|\, \tau + 2\eta S^{m-1}\right)}{\vartheta_1(\lambda_{m+2,i}+\eta)\vartheta_1(\lambda_{m+2,i}+2\eta)}\end{aligned} \tag{5.82}$$

$$\times \| \cdots h_{i-1} h_{i+1} \cdots h_{j-1} h_{j+1} \cdots h_N h_i h_j, r\rangle\!\rangle \Big|_{\substack{\lambda_{N-1}\to\lambda_i,\, \lambda_N\to\lambda_j \\ \lambda_k\to\lambda_{k+2},\, k=j-1,\cdots,N-2 \\ \lambda_k\to\lambda_{k+1},\, k=i,\cdots,j-2}} .$$

On the one hand, we note that the products in (5.82) bear no dependence on λ_{ij}. On the other hand, from the first part of this proof, we know the pole structure in λ_{ij} of the state on the right-hand side of (5.82). We thus deduce that $\|\boldsymbol{h}, r\rangle\!\rangle$ is an entire function of λ_{ij} if and only if $(h_i, h_j) \neq (0, 2)$. For $(h_i, h_j) = (0, 2)$, $\|\boldsymbol{h}, r\rangle\!\rangle$ possesses simple poles in $\lambda_{ij} = \eta, 2\eta$. $\square$

Proposition 5.1.19. *For each $\boldsymbol{h} \in \{0, 2\}^N$, and $1 \leq i < j \leq N$, the qSoV state $\langle\!\langle \boldsymbol{h}, r\|$ is an entire function of λ_{ij} if and only if $h_i = h_j$. For $(h_i, h_j) = (0, 2)$, $\langle\!\langle \boldsymbol{h}, r\|$ possesses simple poles in $\lambda_{ij} = -\eta, -2\eta$. For $(h_i, h_j) = (2, 0)$, $\langle\!\langle \boldsymbol{h}, r\|$ possesses simple poles in $\lambda_{ij} = \eta, 2\eta$.*

Proof. The proof works similarly as the previous one. Proposition 5.A.8 provides the poles structure of the qSoV states (5.81) with respect to $\lambda_{N-1,N}$. $\square$

Reduction *Reduction relations* relate a quantity of the size-N model to its lower-size version. They occur through the specialisation of inhomogeneity parameter(s) [1, 69, 111, 128]. The analyticity properties discussed in the previous section give rise to such relations for the qSoV states. To state them, we introduce for each $N \geq 1$ and $i = 1, \ldots, N+1$ the linear operator

$$\Xi^i : (\mathbb{C}^3)^{\otimes N} \to (\mathbb{C}^3)^{\otimes (N+2)}, \tag{5.83}$$

defined for each element $|\alpha_1 \cdots \alpha_N\rangle$ of the canonical basis of $(\mathbb{C}^3)^{\otimes N}$ as

$$\Xi^i \, |\alpha_1 \cdots \alpha_N\rangle = |\alpha_1 \cdots \alpha_{i-1}\rangle \otimes |s\rangle \otimes |\alpha_i \cdots \alpha_N\rangle \,, \tag{5.84}$$

with $|s\rangle = |\Uparrow\Downarrow\rangle - |00\rangle + |\Downarrow\Uparrow\rangle$, the singlet of $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Proposition 5.1.20. *For $N > 2$, we have*

$$
\begin{aligned}
&\lim_{\lambda_N \to \lambda_{N-1}-\eta} \vartheta_1(\lambda_{N-1,N}-\eta)\|h_1 \cdots h_N, r\rangle\rangle \\
&= -\delta_{2,h_N}\delta_{0,h_{N-1}} \mathrm{e}^{\mathrm{i}y\eta(2(N+s_{\boldsymbol{h}})+4r-1)} \frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)} \frac{\vartheta_1(t_{\mathbf{2},r})\vartheta_1(t_{\mathbf{2},r}-\eta)}{\vartheta_1(t_{\boldsymbol{h},-r}+\eta)\vartheta_1(t_{\boldsymbol{h},-r}-\eta)} \\
&\quad \times \prod_{j=1}^{N-2} \frac{\vartheta_1(\lambda_{jN-1}-\eta(h_j-2))\vartheta_1(\lambda_{jN-1}-\eta(h_j-1))}{\vartheta_1(\lambda_{jN-1})\vartheta_1(\lambda_{jN-1}-\eta)} \\
&\qquad \times \mathsf{c}\Big(\eta s_N^3 \,\Big|\, \tau + 2\eta S^{N-2}\Big)\, \Xi^{N-1}\, \|h_1 \cdots h_{N-2}, r\rangle\rangle. \qquad (5.85)
\end{aligned}
$$

For $N = 2$, the relation still applies, provided that one replaces the vector $\Xi^{N-1}\|h_1 \cdots h_{N-2}, r\rangle\rangle$ with $|s\rangle \otimes |t_{\mathbf{1},r}\rangle$ in the last line.

Before proving Proposition 5.1.20, we present several necessary intermediary results. The first element that we need is the computation of a residue-like quantity, which follows from Proposition 5.A.1.

Lemma 5.1.21. *For $N \geq 2$, we have*

$$
\begin{aligned}
&\lim_{\lambda_N \to \lambda_{N-1}-\eta} \vartheta_1(\lambda_{N-1,N}-\eta)\|2 \cdots 2 h_{N-1} h_N, r\rangle\rangle \\
&\quad = -\delta_{2,h_N}\delta_{0,h_{N-1}} \mathrm{e}^{\mathrm{i}y\eta(4N-5)} \frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)} \frac{\vartheta_1(t_{\mathbf{2},r})\vartheta_1(t_{\mathbf{2},r}-\eta)}{\vartheta_1(t_{\mathbf{0},r}+\eta)\vartheta_1(t_{\mathbf{0},r}+3\eta)} \\
&\qquad \times \mathsf{c}(\eta s_N^3|\tau+2\eta S^{N-2})\, |\Downarrow \cdots \Downarrow\rangle \otimes |s\rangle \otimes |t_{\mathbf{2},r}-2\eta\rangle\,. \qquad (5.86)
\end{aligned}
$$

Using Definition 5.1.1, we find

$$
\begin{aligned}
&\lim_{\lambda_N \to \lambda_{N-1}-\eta} \vartheta_1(\lambda_{N-1,N}-\eta)\|h_1 \cdots h_N, r\rangle\rangle \\
&\quad = -\delta_{2,h_N}\delta_{0,h_{N-1}} \mathrm{e}^{\mathrm{i}y\eta(4N-5)} \frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)} \frac{\vartheta_1(t_{\mathbf{2},r})\vartheta_1(t_{\mathbf{2},r}-\eta)}{\vartheta_1(t_{\mathbf{0},r}+\eta)\vartheta_1(t_{\mathbf{0},r}+3\eta)} \\
&\quad \times \prod_{j=1}^{N-2}\prod_{h=h_j}^{1} \frac{\mathcal{C}^N(\lambda_j-\eta(h+1))}{\mathrm{D}^N(\lambda_j-\eta(h+1))}\Bigg|_{\lambda_N\to\lambda_{N-1}-\eta} \mathsf{c}(\eta s_N^3|\tau+2\eta S^{N-2}) \\
&\qquad \times |\Downarrow \cdots \Downarrow\rangle \otimes |s\rangle \otimes |t_{\mathbf{2},r}-2\eta\rangle\,. \qquad (5.87)
\end{aligned}
$$

The second element required for the proof of Proposition 5.1.20 is understanding how

$$\mathcal{C}^N(\lambda_0; \cdots, \lambda_{N-1}, \lambda_{N-1}-\eta)$$

acts on the state (5.86). Specifically, we study its action on the $(N-1)$-th and N-th factors $\mathbb{C}^3$ of the spin space. To this end, we consider the decomposition

$$\begin{aligned}\mathcal{M}_0^N(\lambda_0)\Big|_{\lambda_N\to\lambda_{N-1}-\eta} &= \mathsf{R}_{0N}^{(1,2)}\left(\lambda_{0N-1}+\eta\,\middle|\,\tau+2\eta S^{N-2}+2\eta s_{N-1}^3\right)\\ &\times \mathsf{R}_{0N-1}^{(1,2)}\left(\lambda_{0N-1}\,\middle|\,\tau+2\eta S^{N-2}\right)\left(\mathcal{M}_0^{N-2}(\lambda_0)\otimes\mathbb{I}^{\otimes 2}\right). \quad (5.88)\end{aligned}$$

We find that the product of R-matrices that appears in this expression fulfils the following property:

Lemma 5.1.22. *For all $t\in\mathbb{C}$, we have*

$$\begin{aligned}\mathsf{R}_{0N}^{(1,2)}\left(\lambda_{0N-1}+\eta\,\middle|\,t+2\eta s_{N-1}^3\right)\mathsf{R}_{0N-1}^{(1,2)}(\lambda_{0N-1}\,|\,t)\,\mathsf{c}(\eta s_N^3|t+\eta\sigma_0^3)\,|s\rangle_{N-1,N}&\\ = E_0(\lambda_{0N-1}|t)\mathsf{c}(\eta s_N^3|t)\,|s\rangle_{N-1,N}\,, \quad (5.89)\end{aligned}$$

with

$$E(\lambda|t)=\mathrm{e}^{2\mathrm{i}y\eta\sigma^3}\vartheta_1(\lambda)\vartheta_1(\lambda+3\eta)\frac{\vartheta_1(t)\vartheta_1(t+2\eta\sigma^3)}{\vartheta_1(t+\eta)\vartheta_1(t-\eta)}. \quad (5.90)$$

Proof. Using the fused dYBE (4.29), we have

$$\begin{aligned}&\mathsf{R}_{0N}^{(1,2)}\left(\lambda_{0N-1}+\eta\,\middle|\,t+2\eta s_{N-1}^3\right)\mathsf{R}_{0N-1}^{(1,2)}(\lambda_{0N-1}\,|\,t)\,\mathsf{R}_{NN-1}^{(2,2)}(-\eta|t+\eta\sigma_0^3)\\ &\quad=\mathsf{R}_{NN-1}^{(2,2)}(-\eta|t)\mathsf{R}_{0N-1}^{(1,2)}\left(\lambda_{0N-1}\,\middle|\,t+2\eta s_{N-1}^3\right)\mathsf{R}_{0N}^{(1,2)}(\lambda_{0N-1}+\eta\,|\,t)\,.\end{aligned}$$

From Proposition 4.1.9, we obtain

$$\begin{aligned}&\mathsf{c}(\eta s_N^3|t)^{-1}\mathsf{R}_{0N}^{(1,2)}\left(\lambda_{0N-1}+\eta\,\middle|\,t+2\eta s_{N-1}^3\right) \quad (5.91)\\ &\qquad\times\mathsf{R}_{0N-1}^{(1,2)}(\lambda_{0N-1}\,|\,t)\,\mathsf{c}\left(\eta s_N^3\,\middle|\,t+\eta\sigma_0^3\right)P_{NN-1}^{(2)-}\\ &\quad=P_{NN-1}^{(2)-}\mathsf{d}(\eta s_N^3|t)\mathsf{R}_{0N-1}^{(1,2)}\left(\lambda_{0N-1}\,\middle|\,t+2\eta s_{N-1}^3\right)\\ &\qquad\times\mathsf{R}_{0N}^{(1,2)}(\lambda_{0N-1}+\eta\,|\,t)\,\mathsf{d}(\eta s_N^3|t+\eta\sigma_0^3)^{-1}.\end{aligned}$$

We note that $P_{NN-1}^{(2)-}=P_{N-1,N}^{(2)-}$. From (5.91), we deduce that there exists an operator $E\in\mathrm{End}(\mathbb{C}^2)$ which fulfils (5.89). We obtain the entries of E by computing the dual pairings of (5.89) with $\langle\Uparrow\Downarrow|_{N-1,N}$ and $\langle\Downarrow\Uparrow|_{N-1,N}$. □

Lemma 5.1.22 allows us to compute the action of $\mathcal{C}^N(\lambda_0;\cdots,\lambda_{N-1}-\eta)$ on (5.86).

Lemma 5.1.23. *For $\tilde{\tau} = \tau + 2\eta S^{N-2}$, we have*

$$\begin{aligned}\mathcal{C}^N(\lambda_0)\Big|_{\lambda_N \to \lambda_{N-1}-\eta} &\mathsf{c}(\eta s_N^3|\tilde{\tau})\,|s\rangle_{N-1,N} \\ &= \mathrm{e}^{-2iy\eta}\vartheta_1(\lambda_{0N-1})\vartheta_1(\lambda_{0N-1}+3\eta)\mathsf{c}(\eta s_N^3|\tilde{\tau})\,|s\rangle_{N-1,N} \\ &\qquad\times \frac{\vartheta_1(\tilde{\tau})\vartheta_1(\tilde{\tau}-2\eta)}{\vartheta_1(\tilde{\tau}+\eta)\vartheta_1(\tilde{\tau}-\eta)}\left(\mathcal{C}^{N-2}(\lambda_0)\otimes \mathbb{I}^{\otimes 2}\right). \quad (5.92)\end{aligned}$$

Proof. We recall $\mathcal{C}^N(\lambda_0) = \langle\downarrow|_0\,\mathcal{M}_0^N(\lambda_0)\,|\uparrow\rangle_0$, then use the decomposition (5.88) on the monodromy matrix,

$$\begin{aligned}\mathcal{C}^N(\lambda_0)\Big|_{\lambda_N \to \lambda_{N-1}-\eta} &\mathsf{c}(\eta s_N^3|\tilde{\tau})\,|s\rangle_{N-1,N} \\ &= \langle\downarrow|_0\,\mathsf{R}_{0N}^{(1,2)}\left(\lambda_{0N-1}+\eta\,\middle|\,\tilde{\tau}+2\eta s_{N-1}^3\right)\mathsf{R}_{0N-1}^{(1,2)}\left(\lambda_{0N-1}\,|\,\tilde{\tau}\,\right) \\ &\qquad\times\left(\mathcal{M}_0^{N-2}(\lambda_0)\otimes\mathbb{I}^{\otimes 2}\right)\mathsf{c}(\eta s_N^3|\tilde{\tau})\,|\uparrow\rangle_0\,|s\rangle_{N-1,N}\,. \quad (5.93)\end{aligned}$$

By (4.50) and (4.113), the following commutation holds:

$$\left(\mathcal{M}_0^{N-2}(\lambda_0)\otimes\mathbb{I}^{\otimes 2}\right)\mathsf{c}(\eta s_N^3|\tilde{\tau}) = \mathsf{c}(\eta s_N^3|\tilde{\tau}+\eta\sigma_0^3)\left(\mathcal{M}_0^{N-2}(\lambda_0)\otimes\mathbb{I}^{\otimes 2}\right).$$

We apply it to (5.93). Then, we use Lemma 5.1.22, which leaves us with (5.92). □

Proof of Proposition 5.1.20. We go back to equation (5.87), and examine the quantity

$$\begin{aligned}\prod_{j=1}^{N-2}\prod_{h=h_j}^{1}\frac{\mathcal{C}^N(\lambda_j-\eta(h+1))}{\mathrm{D}^N(\lambda_j-\eta(h+1))}\Bigg|_{\lambda_N\to\lambda_{N-1}-\eta} \\ \times\,\mathsf{c}\left(\eta s_N^3\,\middle|\,\tau+2\eta S^{N-2}\right)|\Downarrow\cdots\Downarrow\rangle\otimes|s\rangle\otimes|t_{\mathbf{2},r}-2\eta\rangle\,. \quad (5.94)\end{aligned}$$

Lemma 5.1.23 allows us to rewrite it as

$$\begin{aligned}\mathsf{c}(\eta s_N^3|\tilde{\tau})&\left[\prod_{j=1}^{N-2}\prod_{h=h_j}^{1}\mathrm{e}^{-2\mathrm{i}y\eta}\frac{\vartheta_1(\lambda_{jN-1}-\eta(h-2))}{\vartheta_1(\lambda_{jN-1}-\eta h)}\frac{\vartheta_1(\tilde{\tau})\vartheta_1(\tilde{\tau}-2\eta)}{\vartheta_1(\tilde{\tau}+\eta)\vartheta_1(\tilde{\tau}-\eta)}\right. \\ &\times\left.\left(\frac{\mathcal{C}^{N-2}(\lambda_j-\eta(h+1))}{\mathrm{D}^{N-2}(\lambda_j-\eta(h+1))}\otimes\mathbb{I}^{\otimes 2}\right)\right]|\Downarrow\cdots\Downarrow\rangle\otimes|s\rangle\otimes|t_{\mathbf{2},r}-2\eta\rangle\,, \quad (5.95)\end{aligned}$$

with $\tilde{\tau} = \tau + 2\eta S^{N-2}$. The scalar factors in (5.95) simplify to

$$\prod_{j=1}^{N-2} \prod_{h=h_j}^{1} \mathrm{e}^{-2\mathrm{i}y\eta} \frac{\vartheta_1(\lambda_{jN-1} - \eta(h-2))}{\vartheta_1(\lambda_{jN-1} - \eta h)}$$
$$= \mathrm{e}^{-2\mathrm{i}y\eta L} \prod_{j=1}^{N-2} \frac{\vartheta_1(\lambda_{jN-1} - \eta(h_j-1))\vartheta_1(\lambda_{jN-1} - \eta(h_j-2))}{\vartheta_1(\lambda_{jN-1})\vartheta_1(\lambda_{jN-1} - \eta)}, \quad (5.96)$$

where $L = \sum_{k=1}^{N-2}(2-h_k)$ is the total number of factors in the product in (5.95). On the other hand, to deal with the operator factors, we need to commute the terms

$$G(\tilde{\tau}) = \frac{\vartheta_1(\tilde{\tau})\vartheta_1(\tilde{\tau} - 2\eta)}{\vartheta_1(\tilde{\tau} + \eta)\vartheta_1(\tilde{\tau} - \eta)} \quad (5.97)$$

to the right of all the operators $\mathcal{C}^{N-2}$ in (5.95). The process requires the relation

$$\left[\mathcal{C}^N(\lambda), \tau + 2\eta S^N\right] = -\eta \mathcal{C}^N(\lambda), \quad (5.98)$$

issued from Lemma 4.1.22. From this commutation, we write

$$\prod_{j=1}^{N-2} \prod_{h=h_j}^{1} G(\tilde{\tau}) \left(\frac{\mathcal{C}^{N-2}(\lambda_j - \eta(h+1))}{\mathrm{D}^{N-2}(\lambda_j - \eta(h+1))} \otimes \mathbb{I}^{\otimes 2} \right)$$
$$= \prod_{j=1}^{N-2} \prod_{h=h_j}^{1} \left(\frac{\mathcal{C}^{N-2}(\lambda_j - \eta(h+1))}{\mathrm{D}^{N-2}(\lambda_j - \eta(h+1))} \otimes \mathbb{I}^{\otimes 2} \right) \prod_{j=1}^{L} G(\tilde{\tau} + \eta j). \quad (5.99)$$

The product of G on the right-hand side is telescoping. We find

$$\prod_{j=1}^{L} G(\tilde{\tau} + \eta j) = \frac{\vartheta_1(\tilde{\tau} + \eta)\vartheta_1(\tilde{\tau} - \eta)}{\vartheta_1(\tilde{\tau} + \eta(L+1))\vartheta_1(\tilde{\tau} + \eta(L-1))}. \quad (5.100)$$

The last step of the proof consists in applying (5.100) to the vector in (5.95). We note

$$(\tau + 2\eta S^{N-2}) \left|\Downarrow \cdots \Downarrow\right\rangle \otimes \mathbb{I}^{\otimes 2} \otimes \left|t_{\mathbf{2},r} - 2\eta\right\rangle$$
$$= (t_{\mathbf{0},r} + 2\eta) \left|\Downarrow \cdots \Downarrow\right\rangle \otimes \mathbb{I}^{\otimes 2} \otimes \left|t_{\mathbf{2},r} - 2\eta\right\rangle. \quad (5.101)$$

Moreover, for $h_{N-1} = 0$ and $h_N = 2$, we have $L = N - 2 + s_{\boldsymbol{h}}$. This remark, along with (A.4) and the definition (5.2) lead to

$$\prod_{j=1}^{N-2+s_{\boldsymbol{h}}} G\left(\tilde{\tau} + \eta j\right) |{\Downarrow} \cdots {\Downarrow}\rangle \otimes \mathbb{I}^{\otimes 2} \otimes |t_{\mathbf{2},r} - 2\eta\rangle$$
$$= \mathrm{e}^{4\mathrm{i}y\eta(r+s_{\boldsymbol{h}})} \frac{\vartheta_1(t_{\mathbf{0},r} + \eta)\vartheta_1(t_{\mathbf{0},r} + 3\eta)}{\vartheta_1(t_{\boldsymbol{h},-r} + \eta)\vartheta_1(t_{\boldsymbol{h},-r} - \eta)}$$
$$\times |{\Downarrow} \cdots {\Downarrow}\rangle \otimes \mathbb{I}^{\otimes 2} \otimes |t_{\mathbf{2},r} - 2\eta\rangle \,. \quad (5.102)$$

To conclude the proof, we combine and simplify the expressions (5.87), (5.96) and (5.102). □

Parallel to Proposition 5.1.20, the left qSoV states also possess a reduction relation. However, in this case, the reduction does not arise from a residue-like quantity at a pole. Indeed, the state $\langle\langle h_1 \dots h_{N-2}20, r\|$ is analytic in $\lambda_N = \lambda_{N+1} + \eta$, but still reduces when specialised to this point.

Proposition 5.1.24. *For $N > 2$, we have*

$$\langle\langle h_1 \dots h_{N-2}20, r\| \Big|_{\lambda_N = \lambda_{N+1}+\eta} = \mathrm{e}^{\mathrm{i}y\eta(3-2(N+s_{\boldsymbol{h}})-4r)} \frac{\vartheta_1(\eta)}{\vartheta_1(3\eta)}$$
$$\times \frac{\vartheta_1(t_{\boldsymbol{h},-r} + \eta)\vartheta_1(t_{\boldsymbol{h},-r} - \eta)}{\vartheta_1(t_{\mathbf{2},r} - \eta)\vartheta_1(t_{\mathbf{2},r} - 2\eta)} \frac{\mathrm{D}^{N-2}(\lambda_{N-1} + \eta)\mathrm{D}^{N-2}(\lambda_{N-1} + 2\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1} - \eta)}$$
$$\times \prod_{j=1}^{N-2} \frac{\vartheta_1(\lambda_{N-1,j} - \eta(h_j - 1))}{\vartheta_1(\lambda_{N-1,j} + \eta(h_j + 1))\vartheta_1(\lambda_{N-1,j} + 2\eta(h_j - 1))}$$
$$\times \langle\langle h_1 \cdots h_{N-2}, r\|(\Xi^{N-1})^t \mathsf{d}\left(\eta s^3_{N-1} \,\Big|\, \tau + 2\eta S^{N-2}\right) \,, \quad (5.103)$$

where $\mathsf{d}(\lambda|t)$ is defined in (4.41).

For $N = 2$, the relation still applies, provided that one replaces the vector $\langle\langle h_1 \cdots h_{N-2}, r\|(\Xi^{N-1})^t$ with $\langle s| \otimes \langle t_{\mathbf{1},r}|$ in the last line.

The proof of Proposition 5.1.24 is similar to that of Proposition 5.1.20.

To conclude this section, we remark that it is possible to state the right and left reduction relations for the specialisations $\lambda_i \to \lambda_{i-1} \mp \eta$, $i = 2, \dots, N$, respectively. One achieves this generalisation by applying Proposition 5.1.17 to Propositions 5.1.20 and 5.1.24.

5.2 The eigenvalue problem

In this section, we determine the spectrum and the common eigenvectors of the family of anti-periodic d10V transfer matrices, restricted to the subspace $\mathbb{D}_N^{(0)}$. The eigenvalue problem consists in finding $\theta(\lambda)$ and non-zero $|\varphi^R\rangle$ such that

$$\mathcal{T}(\lambda)\,|\varphi^R\rangle = \theta(\lambda)\,|\varphi^R\rangle\,, \qquad \forall\,\lambda \in \mathbb{C}. \tag{5.104}$$

Since the spectra of $\mathcal{T}(\lambda)$ and its transpose coincide, we also seek non-zero $\langle\varphi^L|$ such that

$$\langle\varphi^L|\,\mathcal{T}(\lambda) = \langle\varphi^L|\,\theta(\lambda), \qquad \forall\,\lambda \in \mathbb{C}. \tag{5.105}$$

We refer to $|\varphi^R\rangle$ and $\langle\varphi^L|$ as the right and left eigenvectors, respectively.

This section is structured as follows: first, using qSoV techniques, we characterise the spectrum of $\mathcal{T}(\lambda)$. We express the eigenvalues in terms of theta functions, solutions to a discrete set of equations. Next, we construct the associated left and right eigenvectors. We show them to be unique, up to normalisation.

Second, we find that one eigenvalue possesses a particularly simple expression. We compute explicitly the associated eigenvectors. We proceed by stating some of their properties. These include pseudo-periodicity, exchange and reduction relations.

Finally, we consider a quadratic sum rule involving the explicit eigenvectors. We find an expression for it as a determinant formula.

Below, we assume that any vector belongs to $\mathbb{D}_N^{(0)}$, while any dual vector belongs to its dual. We also adopt the shorthand notations $\|\boldsymbol{h}\rangle\rangle = \|\boldsymbol{h},0\rangle\rangle$, $\langle\langle\boldsymbol{h}\| = \langle\langle\boldsymbol{h},0\|$, and $t_{\boldsymbol{h}} = t_{\boldsymbol{h},0}$.

5.2.1 Solutions

We start the section by addressing two preliminary considerations. First, we show that there exists at least one common eigenvector.

Proposition 5.2.1. *The one-parameter family of anti-periodic d10V transfer matrices possesses at least one common eigenvector.*

Proof. Let $z_1, \ldots, z_N$ be independent complex numbers in the sense of Definition A.2.3. By Proposition 4.1.24, the restriction of $\{\mathcal{T}(z_i)\}_{i=1}^{N}$to $\mathbb{D}_N^{(0)}$ is a finite set of commuting operators. Burnside's theorem on matrix algebras ensures that there exists at least one common eigenvector of this set [156, 157]. By Lemma 4.1.25 and Proposition A.2.4, this common eigenvector must be an eigenvector of $\mathcal{T}(\lambda)$, for all $\lambda \in \mathbb{C}$. □

Second, any eigenvalue $\theta(\lambda)$ must share the same analyticity and pseudo-periodicity properties with respect to λ as $\mathcal{T}(\lambda)$. We infer these properties from Lemma 4.1.25:

Lemma 5.2.2. *Let $\theta(\lambda)$ be an eigenvalue of $\mathcal{T}(\lambda)$, then $\mathrm{e}^{\mathrm{i}\mathsf{y}\lambda}\theta(\lambda)$ is a theta function in λ of degree N, nome p, and norm $\sum_{j=1}^{N}\lambda_j + t_{\mathbf{0}}$.*

As a result of the previous lemma, it is possible to write any eigenvalue as an interpolated form, which yields expressions as in [59, 94]. By Proposition A.2.4, for all $\boldsymbol{h} \in \{0, 1, 2\}^N$, we can interpolate $\mathrm{e}^{\mathrm{i}\mathsf{y}\lambda}\theta(\lambda)$ from its specialisations to $\lambda = \lambda_j - \eta h_j$, $j = 1, \ldots, N$. We obtain

$$\theta(\lambda) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\mathsf{y}(\lambda-\lambda_j+\eta h_j)}\theta(\lambda_j - \eta h_j)\frac{\vartheta_1(-\lambda + \lambda_j - \eta h_j + t_{\boldsymbol{h}})}{\vartheta_1(t_{\boldsymbol{h}})} \times \prod_{k\neq j}\frac{\vartheta_1(\lambda - \lambda_k + \eta h_k)}{\vartheta_1(\lambda_{jk} - \eta h_{jk})}. \quad (5.106)$$

We now begin the characterisation of the spectrum of $\mathcal{T}(\lambda)$.

Theorem 5.2.3. *The spectrum of $\mathcal{T}(\lambda)$ coincides with the set of $\theta(\lambda)$ that fulfil the two following conditions:*

(i) $\mathrm{e}^{\mathrm{i}\mathsf{y}\lambda}\theta(\lambda)$ *is a theta function in λ of degree N, nome p, and norm $\sum_{j=1}^{N}\lambda_j + t_{\mathbf{0}}$,*

(ii) the following discrete set of equations is satisfied:

$$\theta(\lambda_j)\theta(\lambda_j - \eta)\theta(\lambda_j - 2\eta) + \theta(\lambda_j)\mathrm{A}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta) + \theta(\lambda_j - 2\eta)\mathrm{A}(\lambda_j)\mathrm{D}(\lambda_j - \eta) = 0, \qquad j = 1, \ldots, N. \quad (5.107)$$

Moreover, the right and left eigenvectors associated to the eigenvalue $\theta(\lambda)$ are unique, up to normalisation.

Proof. We consider the eigenvalue problem for the left eigenvectors, then point out the modifications needed for the right ones.

Let $\theta(\lambda)$ be an eigenvalue of $\mathcal{T}(\lambda)$. By Lemma 5.2.2, it automatically fulfils condition *(i)*. We now show that it also obeys condition *(ii)*. By hypothesis, there exists $\langle\varphi^L| \neq 0$, solution to the eigenvalue problem (5.105). By Proposition 5.1.14, we know that $\{\|\boldsymbol{h}\rangle\rangle\}_{\boldsymbol{h}\in\{0,1,2\}^N}$ is a basis of the subspace $\mathbb{D}_N^{(0)}$. The equation (5.105) is therefore equivalent to

$$\langle\varphi^L|\,\mathcal{T}(\lambda)\|\boldsymbol{h}\rangle\rangle = \langle\varphi^L|\,(\mathcal{B}(\lambda)+\mathcal{C}(\lambda))\|\boldsymbol{h}\rangle\rangle = \theta(\lambda)\langle\varphi^L\|\boldsymbol{h}\rangle\rangle, \tag{5.108}$$

for all $\boldsymbol{h} \in \{0,1,2\}^N$. By Lemma 4.1.25, and Lemma 5.2.2, we know that both sides of (5.108) are characterised in terms of a theta function of degree N, and of same nome and norm. Theorem A.2.5 states that it is sufficient that (5.108) holds for N independent values of λ in order for it to hold for all $\lambda \in \mathbb{C}$. We thus consider the equation at the specific points $\lambda = \lambda_j - \eta h_j$, $j = 1,\ldots,N$. By Lemmas 5.1.9 and 5.1.10, we get

$$\begin{aligned} -\,\mathrm{A}(\lambda_j-\eta h_j)\langle\varphi^L\|T_j^+\boldsymbol{h}\rangle\rangle + \mathrm{D}(\lambda_j-\eta h_j)\langle\varphi^L\|T_j^-\boldsymbol{h}\rangle\rangle \\ = \theta(\lambda_j-\eta h_j)\langle\varphi^L\|\boldsymbol{h}\rangle\rangle. \end{aligned} \tag{5.109}$$

We evaluate (5.109) for $h_j = 0,1,2$. Thanks to the cancellations $\mathrm{A}(\lambda_j - 2\eta) = \mathrm{D}(\lambda_j) = 0$, we obtain the following system of equations:

$$\begin{pmatrix} \theta(\lambda_j) & \mathrm{A}(\lambda_j) & 0 \\ -\mathrm{D}(\lambda_j-\eta) & \theta(\lambda_j-\eta) & \mathrm{A}(\lambda_j-\eta) \\ 0 & -\mathrm{D}(\lambda_j-2\eta) & \theta(\lambda_j-2\eta) \end{pmatrix} \begin{pmatrix} \langle\varphi^L\|\cdots h_j=0\cdots\rangle\rangle \\ \langle\varphi^L\|\cdots h_j=1\cdots\rangle\rangle \\ \langle\varphi^L\|\cdots h_j=2\cdots\rangle\rangle \end{pmatrix} = 0. \tag{5.110}$$

We denote by Q the matrix on the left of this equation. This linear system possesses a non-zero solution if and only if

$$\begin{aligned} \det\mathsf{Q} = \theta(\lambda_j)\theta(\lambda_j-\eta)\theta(\lambda_j-2\eta) + \theta(\lambda_j)\mathrm{A}(\lambda_j-\eta)\mathrm{D}(\lambda_j-2\eta) \\ + \theta(\lambda_j-2\eta)\mathrm{A}(\lambda_j)\mathrm{D}(\lambda_j-\eta) = 0. \end{aligned} \tag{5.111}$$

This establishes *(ii)*.

Furthermore, the determinant of the $(3,1)$-minor of Q is given by

$$\det_{\substack{k=1,2\\ \ell=2,3}}(\mathsf{Q})_{k\ell} = \mathrm{A}(\lambda_j)\mathrm{A}(\lambda_j-\eta), \tag{5.112}$$

and is non-zero for generic inhomogeneity parameters. This non-cancelling minor indicates that Q is a rank-2 matrix, which makes the solutions to (5.110) unique, up to a normalisation factor.

Let us now assume that $\theta(\lambda)$ fulfils conditions *(i)* and *(ii)*. Since $\theta(\lambda)$ obeys *(i)*, we show through similar arguments as above that the eigenvalue problem is equivalent to (5.110), for $j = 1, \dots, N$. Condition *(ii)* then ensures that each system possesses a non-zero solution. The rank of Q makes the solutions unique, up to a factor. Therefore, there exists a non-zero $\langle\varphi^L|$, solution to the eigenvalue problem, unique up to normalisation. This argument concludes the proof for the left eigenvectors.

The reasoning for the right eigenvectors is similar. In both directions of implication, we find that the eigenvalue problem (5.104) is equivalent to

$$\mathsf{Q}^t \begin{pmatrix} \langle\!\langle \cdots h_j = 0 \cdots \| \varphi^R\rangle \\ \langle\!\langle \cdots h_j = 1 \cdots \| \varphi^R\rangle \\ \langle\!\langle \cdots h_j = 2 \cdots \| \varphi^R\rangle \end{pmatrix} = 0, \qquad j = 1, \dots, N. \tag{5.113}$$

We note that $\det \mathsf{Q}^t = \det \mathsf{Q}$ and $\operatorname{rank} \mathsf{Q}^t = \operatorname{rank} \mathsf{Q}$, which allows us to reach the same conclusions as for the left eigenvectors. $\square$

Theorem 5.2.4. *Let $\theta(\lambda)$ be an eigenvalue of $\mathcal{T}(\lambda)$. The associated right and left eigenspaces are respectively spanned by*

$$|\varphi^R\rangle = \sum_{\boldsymbol{h}\in\{0,1,2\}^N} \left[\prod_{j=1}^{N} (-1)^{h_j} \frac{\mathsf{q}_j^{(h_j)}}{\mathsf{q}_j^{(0)}} \prod_{k=0}^{h_j-1} \frac{\mathrm{A}(\lambda_j - \eta k)}{\mathrm{D}(\lambda_j - \eta(k+1))}\right] \frac{\|\boldsymbol{h}\rangle\!\rangle}{\langle\!\langle \boldsymbol{h}\|\boldsymbol{h}\rangle\!\rangle},$$
$$\langle\varphi^L| = \sum_{\boldsymbol{h}\in\{0,1,2\}^N} \left[\prod_{j=1}^{N} \mathsf{q}_j^{(h_j)}\right] \frac{\langle\!\langle \boldsymbol{h}\|}{\langle\!\langle \boldsymbol{h}\|\boldsymbol{h}\rangle\!\rangle},$$

where $\langle\!\langle \boldsymbol{h}\|\boldsymbol{h}\rangle\!\rangle$ is expressed in (5.53) *and the coefficients $\mathsf{q}_j^{(h)}$ are defined as*

$$\mathsf{q}_j^{(0)} = \left(\frac{\mathrm{A}(\lambda_j - \eta)}{\mathrm{D}(\lambda_j - \eta)} + \frac{\theta(\lambda_j - \eta)\theta(\lambda_j - 2\eta)}{\mathrm{D}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)}\right), \tag{5.114a}$$

$$\mathsf{q}_j^{(1)} = \frac{\theta(\lambda_j - 2\eta)}{\mathrm{D}(\lambda_j - 2\eta)}, \qquad \mathsf{q}_j^{(2)} = 1. \tag{5.114b}$$

Proof. We start the proof with the explicit construction of the left eigenvector $\langle\varphi^L|$. We fix the normalisation by imposing $\langle\varphi^L\|\mathbf{2}\rangle\!\rangle = 1$. The system (5.110) provides recurrence relations for the eigenvector components,

$$\langle \varphi^L \| \cdots h_j = h \cdots \rangle\rangle = \mathsf{q}_j^{(h)} \, \langle \varphi^L \| \cdots h_j = 2 \cdots \rangle\rangle, \qquad h = 0, 1, 2, \tag{5.115}$$

with the proportionality coefficients $\mathsf{q}_j^{(h)}$ defined in (5.114). We note that $\mathsf{q}_j^{(0)} \neq 0$. Indeed, assuming otherwise, (5.110) forces $\mathsf{q}_j^{(1)} = 0$ and $\mathsf{q}_j^{(2)} = 0$, which is a contradiction. Successive applications of (5.115) to the base case $\langle \varphi^L \| \mathbf{2} \rangle\rangle = 1$ lead to the following expression for the components:

$$\langle \varphi^L \| \boldsymbol{h} \rangle\rangle = \prod_{j=1}^{N} \mathsf{q}_j^{(h_j)}. \tag{5.116}$$

Lastly, we obtain the complete expression for the left eigenvector from the decomposition of the identity operator (5.66).

We move on to the construction of the right eigenvector. Here, we decide to fix the normalisation by imposing $\langle\langle \mathbf{0} \| \varphi^R \rangle = 1$. The system (5.113) provides us with recurrence relations for the eigenvector components,

$$\langle\langle \cdots h_j = h \cdots \| \varphi^R \rangle = \tilde{\mathsf{q}}_j^{(h)} \, \langle\langle \cdots h_j = 0 \cdots \| \varphi^R \rangle, \qquad h = 0, 1, 2, \tag{5.117}$$

where the proportionality coefficients $\tilde{\mathsf{q}}_j^{(h)}$ fulfil

$$\mathsf{Q}^t . \left(\tilde{\mathsf{q}}_j^{(0)} \;\; \tilde{\mathsf{q}}_j^{(1)} \;\; \tilde{\mathsf{q}}_j^{(2)} \right)^t = 0,$$

with $\tilde{\mathsf{q}}_j^{(0)} = 1$, to match the normalisation. We now express the $\tilde{\mathsf{q}}_j^{(h)}$ in terms of their counterparts $\mathsf{q}_j^{(h)}$. To this end, we note that Q is related to its transpose by a conjugation with a diagonal matrix,

$$\mathsf{Q}^t = W^{-1} \mathsf{Q} W, \tag{5.118}$$

where

$$W = \operatorname{diag}\left(1, -\frac{\mathrm{D}(\lambda_j - \eta)}{\mathrm{A}(\lambda_j)}, \frac{\mathrm{D}(\lambda_j - \eta)\mathrm{D}(\lambda_j - 2\eta)}{\mathrm{A}(\lambda_j)\mathrm{A}(\lambda_j - \eta)} \right). \tag{5.119}$$

Since $\mathsf{Q} . \left(\mathsf{q}_j^{(0)} \;\; \mathsf{q}_j^{(1)} \;\; \mathsf{q}_j^{(2)} \right)^t = 0$, we deduce

$$\tilde{\mathsf{q}}_j^{(h)} = (-1)^h \frac{\mathsf{q}_j^{(h)}}{\mathsf{q}_j^{(0)}} \prod_{k=0}^{h-1} \frac{\mathrm{A}(\lambda_j - \eta k)}{\mathrm{D}(\lambda_j - \eta(k+1))}. \tag{5.120}$$

Similarly to the previous case, the eigenvector components are given by

$$\langle\langle \boldsymbol{h} \| \varphi^R \rangle = \prod_{j=1}^{N} \tilde{\mathsf{q}}_j^{(h_j)}. \tag{5.121}$$

Finally, the decomposition (5.66) leads to the expression for the right eigenvector, which concludes the proof. □

5.2.2 An explicit solution

Proposition 5.2.3 provides us with a complete, but implicit, characterisation of the spectrum of $\mathcal{T}(\lambda)$. We now focus on an explicit example of an eigenvalue with a simple expression. Aside from its simplicity, this eigenvalue and its associated eigenvectors are related to the remarkable eigenvectors of the higher-spin elliptic models of Chapter 3, as we demonstrate in Chapter 6. The following proposition presents the eigenvalue and its associated left and right eigenvectors:

Theorem 5.2.5. *The transfer matrix $\mathcal{T}(\lambda)$ possesses the eigenvalue $\theta(\lambda) = 0$. Moreover, the associated right and left eigenvectors, unique up to normalisation, are given by*

$$|\varphi^R\rangle = \sum_{\boldsymbol{h} \in \{0,2\}^N} \left[\prod_{j,k=1}^{N} \frac{\vartheta_1(\lambda_{jk} - 2\eta(1-h_j))}{\vartheta_1(\lambda_{jk} - 2\eta)} \right] \frac{\|\boldsymbol{h}\rangle\rangle}{\langle\langle \boldsymbol{h} \| \boldsymbol{h} \rangle\rangle}, \tag{5.122a}$$

$$\langle \varphi^L | = \sum_{\boldsymbol{h} \in \{0,2\}^N} \left[\prod_{j,k=1}^{N} \frac{\vartheta_1(\lambda_{jk} - \eta(h_j - 1))}{\vartheta_1(\lambda_{jk} - \eta)} \right] \frac{\langle\langle \boldsymbol{h} \|}{\langle\langle \boldsymbol{h} \| \boldsymbol{h} \rangle\rangle}. \tag{5.122b}$$

Proof. We check that $\theta(\lambda) = 0$ fulfils the two conditions of Theorem 5.2.3. It is therefore an eigenvalue of $\mathcal{T}(\lambda)$. We consider Theorem 5.2.4 for $\theta(\lambda) = 0$. The left and right proportionality coefficients respectively simplify to

$$\mathsf{q}_j^{(h)} = \frac{\mathrm{A}(\lambda_j - \eta(h+1))}{\mathrm{D}(\lambda_j - \eta)}, \tag{5.123}$$

$$(-1)^h \frac{\mathsf{q}_j^{(h)}}{\mathsf{q}_j^{(0)}} \prod_{k=0}^{h-1} \frac{\mathrm{A}(\lambda_j - \eta k)}{\mathrm{D}(\lambda_j - \eta(k+1))} = \frac{\mathrm{D}(\lambda_j - 2\eta(1-h))}{\mathrm{D}(\lambda_j - 2\eta)}. \tag{5.124}$$

In particular, we have $\mathsf{q}_j^{(1)} = 0$. From (5.116) and (5.121), we deduce the following systematic cancellations:

$$\langle\!\langle \boldsymbol{h} \| \varphi^R \rangle = 0 = \langle \varphi^L \| \boldsymbol{h} \rangle\!\rangle, \qquad \forall\, \boldsymbol{h} \in \{0,1,2\}^N \backslash \{0,2\}^N. \tag{5.125}$$

Lastly, we reach (5.122) by replacing $\mathrm{A}(\lambda)$ and $\mathrm{D}(\lambda)$ by their definitions. □

As an immediate consequence of Proposition 4.1.28, we find that the special eigenvectors are eigenvectors of the transfer matrix of the d19V model, as well.

Corollary 5.2.6. *We have*

$$\mathcal{T}^{(2)}(\lambda)|\varphi^R\rangle = \mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta)|\varphi^R\rangle, \tag{5.126}$$

$$\langle\varphi^L|\mathcal{T}^{(2)}(\lambda) = \mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta)\langle\varphi^L|. \tag{5.127}$$

Small sizes We explicitly compute the special eigenvectors for $N = 1, 2$, using MATHEMATICA. In the canonical basis of the spin-dynamical space, we have the decomposition

$$|\varphi^R\rangle = \sum_{\boldsymbol{\alpha}\in\{\Uparrow,0,\Downarrow\}^N} (\varphi^R)_{\boldsymbol{\alpha}} \,|\boldsymbol{\alpha}\rangle \otimes |t_{\boldsymbol{\alpha}}\rangle\,, \tag{5.128}$$

$$\langle\varphi^L| = \sum_{\boldsymbol{\alpha}\in\{\Uparrow,0,\Downarrow\}^N} (\varphi^L)_{\boldsymbol{\alpha}} \,\langle\boldsymbol{\alpha}| \otimes \langle t_{\boldsymbol{\alpha}}|\,. \tag{5.129}$$

We refer to $(\varphi^R)_{\boldsymbol{\alpha}}$ and $(\varphi^L)_{\boldsymbol{\alpha}}$ as the components of the special eigenvectors in the canonical basis. For $N = 1$, the non-zero components of the eigenvectors are

$$(\varphi^R)_{\Uparrow} = 1, \qquad\qquad (\varphi^L)_{\Uparrow} = -1, \tag{5.130a}$$

$$(\varphi^R)_{\Downarrow} = -1, \qquad\qquad (\varphi^L)_{\Downarrow} = 1. \tag{5.130b}$$

For $N = 2$, the non-zero components of the eigenvectors are

$$
\begin{aligned}
(\varphi^R)_{\Uparrow\Uparrow} &= 1, & (\varphi^L)_{\Uparrow\Uparrow} &= 1,\\
(\varphi^R)_{\Uparrow\Downarrow} &= -\frac{\vartheta_1(\lambda_{12}+\eta)}{\vartheta_1(\lambda_{12}-\eta)}, & (\varphi^L)_{\Uparrow\Downarrow} &= -\frac{\vartheta_1(\lambda_{12}-\eta)}{\vartheta_1(\lambda_{12}+\eta)},\\
(\varphi^R)_{00} &= -\frac{\vartheta_1(2\eta)\vartheta_a(0)}{\vartheta_a(\eta)^2}\frac{\vartheta_a(\lambda_{12})}{\vartheta_1(\lambda_{12}-\eta)}, & (\varphi^L)_{00} &= -\frac{\vartheta_1(2\eta)}{\vartheta_a(2\eta)}\frac{\vartheta_a(\lambda_{12})}{\vartheta_1(\lambda_{12}+\eta)},\\
(\varphi^R)_{\Downarrow\Uparrow} &= -\frac{\vartheta_1(\lambda_{12}+\eta)}{\vartheta_1(\lambda_{12}-\eta)}, & (\varphi^L)_{\Downarrow\Uparrow} &= -\frac{\vartheta_1(\lambda_{12}-\eta)}{\vartheta_1(\lambda_{12}+\eta)},\\
(\varphi^R)_{\Downarrow\Downarrow} &= 1, & (\varphi^L)_{\Downarrow\Downarrow} &= 1,
\end{aligned}
$$

where $a = 3 + \mathsf{x} - \mathsf{y}$.

5.2.3 Properties of the special eigenvectors

In principle, Theorem 5.2.5 allows us to compute the special eigenvectors explicitly. Unfortunately, this computation becomes rather involved for $N \geqslant 3$. The purpose of this section is to present several properties of the special eigenvectors that characterise them implicitly. These include their behaviour under spin-parity and spin-reversal transformations, their pseudo-periodicity properties with respect to the inhomogeneity parameters, a set of exchange and reduction relations, and a conjecture regarding their analyticity properties with respect to the inhomogeneity parameters.

The implicit characterisation presented in this section mirrors that of the special eigenvectors of the supersymmetric 8V model of Section 2.1. As such, it opens the door to new types of analysis for the d10V model. First, we expect it to help construct the special eigenvectors by recursion in N. Second, the characterisation makes possible the use of induction techniques *à la* Izergin and Korepin. These could allow the computation of special eigenvectors components and sum rules [1, 111], emptiness formation probabilities [128, 129],. . .

Spin-parity and spin-reversal symmetry

Proposition 5.2.7. *We have*

$$(-1)^S|\varphi^R\rangle = (-1)^N|\varphi^R\rangle, \qquad \langle\varphi^L|(-1)^S = (-1)^N\langle\varphi^L|. \qquad (5.131)$$

Proof. By Definition 5.1.1 and Proposition 5.1.2, we have

$$(-1)^S \|\boldsymbol{h}\rangle\rangle = (-1)^N \|\boldsymbol{h}\rangle\rangle, \qquad \langle\langle \boldsymbol{h}\|(-1)^S = (-1)^N \langle\langle \boldsymbol{h}\|, \tag{5.132}$$

for each $\boldsymbol{h} \in \{0,2\}^N$. The proposition straightforwardly follows from these equalities and the definitions (5.122). $\square$

For the symmetry under spin reversal, we need the following lemma:

Lemma 5.2.8. *We have*

$$(\varphi^R)_{\Uparrow\cdots\Uparrow} = 1, \qquad (\varphi^R)_{\Downarrow\cdots\Downarrow} = (-1)^N, \tag{5.133}$$

and

$$(\varphi^L)_{\Uparrow\cdots\Uparrow} = (-1)^N, \qquad (\varphi^L)_{\Downarrow\cdots\Downarrow} = 1. \tag{5.134}$$

Proof. Using the simple form of the reference states (5.1) and their duals (5.59) (with $r = 0$), we obtain

$$(\varphi^R)_{\Uparrow\cdots\Uparrow} = (\langle \Uparrow \cdots \Uparrow| \otimes \langle t_{\mathbf{0}}|)\, |\varphi^R\rangle = \langle\langle \mathbf{0}\|\varphi^R\rangle = 1, \tag{5.135}$$

$$(\varphi^R)_{\Downarrow\cdots\Downarrow} = (\langle \Downarrow \cdots \Downarrow| \otimes \langle t_{\mathbf{2}}|)\, |\varphi^R\rangle = \langle\langle \mathbf{2}\|\varphi^R\rangle = (-1)^N, \tag{5.136}$$

which proves (5.133). The components (5.134) follow along the same lines. $\square$

Proposition 5.2.9. *We have*

$$F|\varphi^R\rangle = (-1)^N T_\tau^{-2S}|\varphi^R\rangle, \qquad \langle\varphi^L|F = (-1)^N \langle\varphi^L|T_\tau^{-2S}. \tag{5.137}$$

Proof. By Proposition 4.1.27, $\mathcal{T}(\lambda)$ commutes with the operator $F_\tau = FT_\tau^{-2S}$ on $\mathbb{D}_N^{(0)}$. Hence, since the eigenspace spanned by $|\varphi^R\rangle$ is one-dimensional, there must exists $\mu \in \mathbb{C}$ such that

$$FT_\tau^{-2S}|\varphi^R\rangle = \mu|\varphi^R\rangle. \tag{5.138}$$

Using $\langle\langle \mathbf{0}\| = \langle\langle \mathbf{2}\|FT_\tau^{-2S}$, we infer

$$(\varphi^R)_{\Uparrow\cdots\Uparrow} = \langle\langle \mathbf{0}\|\varphi^R\rangle = \langle\langle \mathbf{2}\|FT_\tau^{-2S}|\varphi^R\rangle = \mu\langle\langle \mathbf{2}\|\varphi^R\rangle = \mu(\varphi^R)_{\Downarrow\cdots\Downarrow},$$

which yields $\mu = (-1)^N$ by Lemma 5.2.8. $\square$

Pseudo-periodicity We now establish that $\langle\varphi^R|$ and $|\varphi^L\rangle$ possess pseudo-periodicity relations with respect to the inhomogeneity parameters.

Proposition 5.2.10. *For each $i = 1,\ldots,N$, we have*

$$|\varphi^R\rangle\Big|_{\lambda_i\to\lambda_i+\pi} = (-\Omega_i^3)^{\mathsf{y}}\,|\varphi^R\rangle\,, \tag{5.139a}$$

$$|\varphi^R\rangle\Big|_{\lambda_i\to\lambda_i+\pi\omega} = (-1)^{\mathsf{x}}\mathrm{e}^{-2\mathrm{i}\eta(N+1-2i)}(\Gamma^i)^{-1}\,|\varphi^R\rangle\,, \tag{5.139b}$$

$$\langle\varphi^L|\Big|_{\lambda_i\to\lambda_i+\pi} = \langle\varphi^L|\,(-\Omega_i^3)^{\mathsf{y}}, \tag{5.139c}$$

$$\langle\varphi^L|\Big|_{\lambda_i\to\lambda_i+\pi\omega} = (-1)^{\mathsf{x}}\mathrm{e}^{2\mathrm{i}\eta(N+1-2i)}\,\langle\varphi^L|\,\Gamma^i, \tag{5.139d}$$

with Γ^i as defined in (4.89)*.*

Proof. We focus on the shift $\lambda_i \to \lambda_i + \pi\omega$ for the right eigenvector. The reasoning behind the other cases is similar. We use the decomposition of the special eigenvector provided in Theorem 5.2.5. We consider each summand separately and inspect the behaviour of their factors under $\lambda_i \to \lambda_i + \pi\omega$. From Proposition A.1.2, we compute the effect on the components of the eigenvector in the qSOV basis. It reads

$$\langle\langle\boldsymbol{h},0\|\varphi^R\rangle\Big|_{\lambda_i\to\lambda_i+\pi\omega} = \mathrm{e}^{-4\mathrm{i}\eta\sum_{j=1}^N h_{ij}}\,\langle\langle\boldsymbol{h},0\|\varphi^R\rangle. \tag{5.140}$$

We combine this expression with Propositions 5.1.15 and 5.1.16 and deduce that the shift acts on each summand of (5.122a) identically. We express the final result in (5.139d). ☐

Exchange The special eigenvectors obey exchange relations directly inferred from those obeyed by the qSoV states. The multiplication of the eigenvectors by an appropriate $\check{\mathsf{R}}$-matrix allows to swap two of their inhomogeneity parameters.

Proposition 5.2.11. *For each $i = 1,\ldots,N-1$, we have*

$$\begin{aligned}&\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau+2\eta S^{i-1})\,|\varphi^R\rangle\\ &\qquad= \vartheta_1(\lambda_{i,i+1}+\eta)\vartheta_1(\lambda_{i,i+1}+2\eta)\;|\varphi^R\rangle\Big|_{\lambda_i\leftrightarrow\lambda_{i+1}},\end{aligned} \tag{5.141a}$$

and

$$\begin{aligned}\langle\varphi^L|\,\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i+1,i}|\tau+2\eta S^{i-1})\\ =\vartheta_1(\lambda_{i+1,i}+\eta)\vartheta_1(\lambda_{i+1,i}+2\eta)\ \langle\varphi^L|\,\Big|_{\lambda_i\leftrightarrow\lambda_{i+1}}.\end{aligned}\tag{5.141b}$$

Proof. We use the decomposition of the special eigenvectors provided in Theorem 5.2.5. From the explicit formulas for the components, and Proposition 5.1.12, we note that the quantities $\langle\!\langle \boldsymbol{h}\|\varphi^R\rangle$, $\langle\varphi^L\|\boldsymbol{h}\rangle\!\rangle$ and $\langle\!\langle \boldsymbol{h}\|\boldsymbol{h}\rangle\!\rangle$ are invariant under the simultaneous transformations $\lambda_i\leftrightarrow\lambda_{i+1}$, $h_i\leftrightarrow h_{i+1}$. Therefore, Proposition 5.1.17 leads to (5.141). □

Reduction The special eigenvectors obey reduction relations. To state them, we denote by $|\varphi^R_N\rangle=|\varphi^R\rangle$ and $\langle\varphi^L_N|=\langle\varphi^L|$, respectively, the right and left eigenvectors of $\mathcal{T}(\lambda)$ in $\mathbb{D}^{(0)}_N$. By convention, we set $|\varphi^R_0\rangle=1$ and $\langle\varphi^L_0|=1$.

Proposition 5.2.12. *For $N\geq 2$, we have*

$$\begin{aligned}\lim_{\lambda_N\to\lambda_{N-1}-\eta}\vartheta_1(\lambda_{N-1,N}-\eta)\,|\varphi^R_N\rangle&=-\vartheta_1(t_{\mathbf{2}}-\eta)\vartheta_1(t_{\mathbf{2}}-2\eta)\frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)}\\&\times\prod_{j=1}^{N-2}\frac{\vartheta_1(\lambda_{N-1,j}-2\eta)}{\vartheta_1(\lambda_{N-1,j})}\frac{\mathrm{e}^{-\mathrm{i}y\eta\left(3-2\left(N+S^{N-2}\right)\right)}}{\vartheta_1(\tau+\eta)\vartheta_1(\tau-\eta)}\\&\times\mathsf{c}\Big(\eta s^3_N\,\Big|\tau+2\eta S^{N-2}\Big)\,\Xi^{N-1}\,|\varphi^R_{N-2}\rangle\,,\end{aligned}\tag{5.142}$$

and

$$\begin{aligned}\lim_{\lambda_N\to\lambda_{N-1}+\eta}\vartheta_1(\lambda_{NN-1}-\eta)\,\langle\varphi^L_N|&=-\frac{\mathrm{e}^{-\mathrm{i}y\eta(2N-1)}\vartheta_1(2\eta)}{\vartheta_1(t_{\mathbf{2}})\vartheta_1(t_{\mathbf{2}}-\eta)}\\&\times\prod_{j=1}^{N-2}\frac{\vartheta_1(\lambda_{N-1,j}+2\eta)}{\vartheta_1(\lambda_{N-1,j})}\,\langle\varphi^L_{N-2}|\,(\Xi^{N-1})^t\mathsf{d}\Big(\eta s^3_{N-1}\,\Big|\tau+2\eta S^{N-2}\Big)\\&\times\vartheta_1(\tau+\eta)\vartheta_1(\tau-\eta)\mathrm{e}^{-2\mathrm{i}y\eta S^{N-2}},\end{aligned}\tag{5.143}$$

with $\mathsf{c}(\lambda|t)$ *and* $\mathsf{d}(\lambda|t)$ *defined in* (4.4) *and* (4.41), *respectively.*

Proof. We recall the decomposition of the special eigenvector provided in Theorem 5.2.5. Each summand is composed of three factors: a qSoV state

$\|\boldsymbol{h}\rangle\rangle$, an eigenvector component $\langle\langle\boldsymbol{h}\|\varphi_N^R\rangle$, and a dual pairing $\langle\langle\boldsymbol{h}\|\boldsymbol{h}\rangle\rangle$. We individually examine how these behave in the limit $\lambda_N \to \lambda_{N-1} - \eta$.

By Proposition 5.1.20, $\|\boldsymbol{h}\rangle\rangle$ possesses a reduction relation in the limit. Moreover, this limit cancels unless $h_{N-1} = 0$ and $h_N = 2$.

By Theorem 5.2.5, $\langle\langle\boldsymbol{h}\|\varphi_N^R\rangle$ is analytic and non-zero in $\lambda_N = \lambda_{N-1} - \eta$, for all $\boldsymbol{h} \in \{0,2\}^N$. We compute

$$\begin{aligned}
&\langle\langle h_1 \cdots h_{N-2}02\|\varphi_N^R\rangle\big|_{\lambda_N=\lambda_{N-1}-\eta} \\
&\quad= \frac{\vartheta_1(\eta)}{\vartheta_1(3\eta)}\left(\prod_{j=1}^{N-2}\frac{\vartheta_1(\lambda_{N-1,j}-\eta(2h_j-1))}{\vartheta_1(\lambda_{N-1,j}+2\eta)}\frac{\vartheta_1(\lambda_{N-1,j}-2\eta(h_j-1))}{\vartheta_1(\lambda_{N-1,j}-3\eta)}\right) \\
&\qquad\qquad \times \langle\langle h_1\cdots h_{N-2}\|\varphi_{N-2}^R\rangle. \qquad (5.144)
\end{aligned}$$

By Proposition 5.1.12, $\langle\langle\boldsymbol{h}\|\boldsymbol{h}\rangle\rangle$ is analytic and non-zero in $\lambda_N = \lambda_{N-1}-\eta$, for all $\boldsymbol{h} \in \{0,2\}^N$. In particular, we find

$$\begin{aligned}
\langle\langle h_1\cdots h_{N-2}02\|h_1\cdots h_{N-2}02\rangle\rangle\big|_{\lambda_N=\lambda_{N-1}-\eta} &= \mathrm{e}^{2\mathrm{i}y\eta}\frac{\vartheta_1(\eta)\vartheta_1(t_{\boldsymbol{2}})}{\vartheta_1(3\eta)\vartheta_1(t_{\boldsymbol{2}}-2\eta)} \\
\times \prod_{j=1}^{N-2}&\frac{\vartheta_1(\lambda_{N-1,j})\vartheta_1(\lambda_{N-1,j}-\eta)}{\vartheta_1(\lambda_{N-1,j}+\eta h_j)\vartheta_1(\lambda_{N-1,j}+\eta(h_j-3))} \\
&\times \langle\langle h_1\cdots h_{N-2}\|h_1\cdots h_{N-2}\rangle\rangle. \qquad (5.145)
\end{aligned}$$

Combining Proposition 5.1.20 with (5.144) and (5.145), we obtain

$$\begin{aligned}
&\lim_{\lambda_N\to\lambda_{N-1}-\eta}\vartheta_1(\lambda_{N-1,N}-\eta)\|\boldsymbol{h}\rangle\rangle\frac{\langle\langle\boldsymbol{h}\|\varphi_N^R\rangle}{\langle\langle\boldsymbol{h}\|\boldsymbol{h}\rangle\rangle} \\
&\quad= -\delta_{2,h_N}\delta_{0,h_{N-1}}\mathrm{e}^{\mathrm{i}y\eta(2(N+s_{\boldsymbol{h}})-3)}\frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)}\frac{\vartheta_1(t_{\boldsymbol{2}}-\eta)\vartheta_1(t_{\boldsymbol{2}}-2\eta)}{\vartheta_1(t_{\boldsymbol{h}}+\eta)\vartheta_1(t_{\boldsymbol{h}}-\eta)} \\
&\qquad\times\prod_{j=1}^{N-2}\frac{\vartheta_1(\lambda_{N-1,j}-2\eta)}{\vartheta_1(\lambda_{N-1,j})}\mathsf{c}\Big(\eta s_N^3\,\Big|\,\tau+2\eta S^{N-2}\Big)\,\Xi^{N-1} \\
&\qquad\qquad\times\|h_1\cdots h_{N-2}\rangle\rangle\frac{\langle\langle h_1\cdots h_{N-2}\|\varphi_{N-2}^R\rangle}{\langle\langle h_1\cdots h_{N-2}\|h_1\cdots h_{N-2}\rangle\rangle}. \qquad (5.146)
\end{aligned}$$

We swap $s_{\boldsymbol{h}}$ for S^{N-2} and $t_{\boldsymbol{h}}$ for τ in (5.146). As a result, the same operator multiplies each summand of (5.122a) in the limit. We factor this operator out of the sum, and obtain (5.142). □

Analyticity One can deduce analyticity properties for each element in the decomposition of the special eigenvectors from Propositions 5.1.12, 5.1.18 and 5.1.19. Based on this, and on our investigation of the eigenvectors for small sizes, we formulate a conjecture regarding their analyticity.

Conjecture 5.2.13. *The vectors*

$$\prod_{1\le i<j\le N} \vartheta_1(\lambda_{ij}-\eta)\,|\varphi^R\rangle\,, \qquad \prod_{1\le i<j\le N} \vartheta_1(\lambda_{ji}-\eta)\,\langle\varphi^L|\,, \tag{5.147}$$

are entire functions of the inhomogeneity parameters λ_i, $i=1,\dots,N$.

This conjecture is essential to complete the implicit characterisation of the special eigenvectors of the d10V model.

5.2.4 Quadratic sum rule

We discuss a sum rule involving the special eigenvectors of the antiperiodic d10V transfer matrix,

$$Z(\xi;\lambda_1,\dots,\lambda_N) = \langle\varphi^L|\xi^S|\varphi^R\rangle\,. \tag{5.148}$$

We express the quantity as a determinant formula. The proof generalises a corresponding result obtained for the ten- and nineteen-vertex models [92]. In Chapter 6, we relate this sum rule to the square norm of the conjectured ground state of the spin-1 XYZ chain.

Theorem 5.2.14. *For each $\xi\in\mathbb{C}$, we have*

$$\begin{aligned} Z(\xi;\lambda_1,\dots,\lambda_N) &= \frac{\mathrm{e}^{-\mathrm{i}y\eta N}}{\vartheta_1(t_{\mathbf{2}})\vartheta_1(t_{\mathbf{1}})^{N-1}}\frac{\prod_{i,j=1}^N\vartheta_1(\lambda_{ij}+\eta)}{\prod_{1\le i<j\le N}\vartheta_1(\lambda_{ij})\vartheta_1(\lambda_{ji})}\\ &\times \det_{i,j=1}^N\left[\frac{\vartheta_1(\lambda_{ij}-\eta+t_{\mathbf{1}})}{\vartheta_1(\lambda_{ij}-\eta)}\mathrm{e}^{-\mathrm{i}y\eta}\xi-\frac{\vartheta_1(\lambda_{ij}+\eta+t_{\mathbf{1}})}{\vartheta_1(\lambda_{ij}+\eta)}\mathrm{e}^{\mathrm{i}y\eta}\xi^{-1}\right]. \end{aligned} \tag{5.149}$$

Proof. Our proof follows the strategy of [92, Proposition 4.1], while adapting it to the specificities of Jacobi theta functions. We use (5.5) as well as (5.53), and obtain

$$Z(\xi;\lambda_1,\dots,\lambda_N) = \sum_{\boldsymbol{h}\in\{0,1,2\}^N}\xi^{s_{\boldsymbol{h}}}\frac{\langle\varphi^L\|\boldsymbol{h}\rangle\langle\boldsymbol{h}\|\varphi^R\rangle}{\langle\boldsymbol{h}\|\boldsymbol{h}\rangle}. \tag{5.150}$$

From Theorem 5.2.5 and Proposition 5.1.12, we write

$$Z(\xi;\lambda_1,\dots,\lambda_N)=\sum_{\boldsymbol{h}\in\{0,2\}^N}\xi^{s_{\boldsymbol{h}}}\mathrm{e}^{-\mathrm{i}y\eta(N+s_{\boldsymbol{h}})}\frac{\vartheta_1(t_{\boldsymbol{h}})}{\vartheta_1(t_{\boldsymbol{2}})}\prod_{1\le i<j\le N}\frac{\vartheta_1(\lambda_{ij}-\eta h_{ij})}{\vartheta_1(\lambda_{ij})}$$
$$\times\prod_{i,j=1}^{N}\frac{\vartheta_1(\lambda_{ij}-\eta(h_i-1))}{\vartheta_1(\lambda_{ij}-\eta)}\frac{\vartheta_1(\lambda_{ij}-2\eta(1-h_i))}{\vartheta_1(\lambda_{ij}-2\eta)}. \quad (5.151)$$

For $i\neq j$, $h_i,h_j=0,2$, the following equality holds:

$$\vartheta_1(\lambda_{ij}-\eta h_{ij})\vartheta_1(\lambda_{ij}-2\eta(1-h_i))\vartheta_1(\lambda_{ji}-2\eta(1-h_j))$$
$$=\vartheta_1(\lambda_{ij}+\eta h_{ij})\vartheta_1(\lambda_{ij}+2\eta)\vartheta_1(\lambda_{ji}+2\eta). \quad (5.152)$$

It allows us to simplify the terms of the sum (5.151),

$$Z(\xi;\lambda_1,\dots,\lambda_N)=\sum_{\boldsymbol{h}\in\{0,2\}^N}\xi^{s_{\boldsymbol{h}}}(-1)^{(N-s_{\boldsymbol{h}})/2}\mathrm{e}^{-\mathrm{i}y\eta(N+s_{\boldsymbol{h}})}\frac{\vartheta_1(t_{\boldsymbol{h}})}{\vartheta_1(t_{\boldsymbol{2}})}$$
$$\times\prod_{1\le i<j\le N}\frac{\vartheta_1(\lambda_{ij}+\eta h_{ij})}{\vartheta_1(\lambda_{ij})}\prod_{i,j=1}^{N}\frac{\vartheta_1(\lambda_{ij}-\eta(h_i-1))}{\vartheta_1(\lambda_{ij}-\eta)}. \quad (5.153)$$

We use the Frobenius determinant formula of Proposition A.1.4, specialised to $x_i=\lambda_i+\eta h_i, y_i=\lambda_i+\eta$, for $i=1,\dots,N$ and $\rho=t_{\boldsymbol{1}}$,

$$\det_{i,j=1}^{N}\left[\frac{\vartheta_1(\lambda_{ij}+\eta(h_i-1)+t_{\boldsymbol{1}})}{\vartheta_1(\lambda_{ij}+\eta(h_i-1))\vartheta_1(t_{\boldsymbol{1}})}\right]$$
$$=\frac{\vartheta_1(t_{\boldsymbol{1}}-\eta s_{\boldsymbol{h}})}{\vartheta_1(t_{\boldsymbol{1}})}\frac{\prod_{1\le i<j\le N}\vartheta_1(\lambda_{ij}+\eta h_{ij})\vartheta_1(\lambda_{ji})}{\prod_{i,j}\vartheta_1(\lambda_{ij}+\eta(h_i-1))}. \quad (5.154)$$

We also note that $t_{\boldsymbol{h}}=t_{\boldsymbol{1}}-\eta s_{\boldsymbol{h}}$. These observations help us rewrite (5.153) as

$$Z(\xi;\lambda_1,\dots,\lambda_N)=\sum_{\boldsymbol{h}\in\{0,2\}^N}\frac{\mathrm{e}^{-2\mathrm{i}y\eta N}}{\vartheta_1(t_{\boldsymbol{2}})\vartheta_1(t_{\boldsymbol{1}})^{N-1}}\prod_{1\le i<j\le N}\frac{1}{\vartheta_1(\lambda_{ij})\vartheta_1(\lambda_{ji})}$$
$$\times\prod_{i,j=1}^{N}\frac{\vartheta_1(\lambda_{ij}-\eta(h_i-1))\vartheta_1(\lambda_{ij}+\eta(h_i-1))}{\vartheta_1(\lambda_{ij}-\eta)}$$
$$\times\det_{i,j=1}^{N}\left[(-1)^{h_i/2}\mathrm{e}^{\mathrm{i}y\eta h_i}\xi^{1-h_i}\frac{\vartheta_1(\lambda_{ij}+\eta(h_i-1)+t_{\boldsymbol{1}})}{\vartheta_1(\lambda_{ij}+\eta(h_i-1))}\right]. \quad (5.155)$$

We notice that for $\boldsymbol{h} \in \{0,2\}^N$,

$$\prod_{i,j=1}^{N} \frac{\vartheta_1(\lambda_{ij} - \eta(h_i - 1))\vartheta_1(\lambda_{ij} + \eta(h_i - 1))}{\vartheta_1(\lambda_{ij} - \eta)} = \prod_{i,j=1}^{N} \vartheta_1(\lambda_{ij} + \eta). \quad (5.156)$$

In each summand of (5.155), the factors multiplying the determinant are thus independent of $\boldsymbol{h}$. We factor them out of the sum.

To conclude the proof, we sum the determinants. This is equivalent to performing elementary row operations and leads to (5.149). □

5.A Explicit qSoV states

In this appendix, we present the explicit computation of the qSoV states (5.80) and (5.80). These are essential elements to establish the analyticity properties of Propositions 5.1.18 and 5.1.19.

Right qSoV states We begin our analysis with the explicit computation of the right qSoV states

$$\|2\cdots 220, r\rangle\rangle, \qquad \|2\cdots 202, r\rangle\rangle, \qquad \|2\cdots 200, r\rangle\rangle. \quad (5.157)$$

Proposition 5.A.1. *We have*

$$\begin{aligned} \|2\cdots 220, r\rangle\rangle &= \mathrm{e}^{2\mathrm{i}y\eta(2N-1)} \frac{\vartheta_1(t_{\mathbf{2},r})\vartheta_1(t_{\mathbf{2},r} - \eta)}{\vartheta_1(t_{\mathbf{0},r})\vartheta_1(t_{\mathbf{0},r} + \eta)} \\ &\qquad \times |\Downarrow \cdots \Downarrow\Uparrow\rangle \otimes |t_{\mathbf{2},r} - 2\eta\rangle\,, && (5.158) \\ \|2\cdots 200, r\rangle\rangle &= \mathrm{e}^{4\mathrm{i}y\eta(2N-3)} \prod_{j=0}^{3} \frac{\vartheta_1(t_{\mathbf{2},r} - j\eta)}{\vartheta_1(t_{\mathbf{0},r} + j\eta)} \\ &\qquad \times |\Downarrow \cdots \Downarrow\Uparrow\Uparrow\rangle \otimes |t_{\mathbf{2},r} - 4\eta\rangle\,, && (5.159) \\ \|2\cdots 202, r\rangle\rangle &= \frac{\mathrm{e}^{-\mathrm{i}y(\lambda_{N-1,N} - 4\eta(N-1))}\vartheta_1(t_{\mathbf{2},r})\vartheta_1(t_{\mathbf{2},r} - \eta)}{\vartheta_1(\lambda_{N-1,N} - \eta)\vartheta_1(\lambda_{N-1,N} - 2\eta)} \\ &\qquad \times \big(X_{\Uparrow\Downarrow}\,|\Downarrow \cdots \Downarrow\Uparrow\Downarrow\rangle + X_{00}\,|\Downarrow \cdots \Downarrow 00\rangle \\ &\qquad + X_{\Downarrow\Uparrow}\,|\Downarrow \cdots \Downarrow\Downarrow\Uparrow\rangle\big) \otimes |t_{\mathbf{2},r} - 2\eta\rangle\,, && (5.160) \end{aligned}$$

where

$$X_{\Uparrow\Downarrow} = \mathrm{e}^{\mathrm{iy}(\lambda_{N-1,N}-2\eta)} \frac{\vartheta_1(\lambda_{N-1,N})\vartheta_1(\lambda_{N-1,N}+\eta)}{\vartheta_1(t_{\mathbf{0},r}+2\eta)\vartheta_1(t_{\mathbf{0},r}+3\eta)}, \tag{5.161a}$$

$$X_{00} = \alpha^2 \frac{\vartheta_1(2\eta)^2}{\vartheta_1(\eta)} \frac{\vartheta_1(\lambda_{N-1,N})\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-2\eta)}{\vartheta_1(t_{\mathbf{0},r}+\eta)^2\vartheta_1(t_{\mathbf{0},r}+3\eta)}, \tag{5.161b}$$

$$\begin{aligned} X_{\Downarrow\Uparrow} = \mathrm{e}^{-\mathrm{iy}(\lambda_{N-1,N}-2\eta)} & \vartheta_1(\eta)\vartheta_1(2\eta) \\ & \times \frac{\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-\eta)\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-2\eta)}{\vartheta_1(t_{\mathbf{0},r})\vartheta_1(t_{\mathbf{0},r}+\eta)^2\vartheta_1(t_{\mathbf{0},r}+2\eta)}. \end{aligned} \tag{5.161c}$$

We proceed by successively applying the operators $\mathcal{C}$ as prescribed by (5.3). The following lemma is the stepping stone which leads to the intermediary results $\|2\cdots 221, r\rangle\!\rangle$ and $\|2\cdots 212, r\rangle\!\rangle$.

Lemma 5.A.2. *We have*

$$\begin{aligned} \frac{\mathcal{C}^N(\lambda_0)}{\mathrm{D}^N(\lambda_0)}\|\mathbf{2}, r\rangle\!\rangle = & -\alpha \mathrm{e}^{-\mathrm{iy}(\lambda_{0N}-2\eta(N-1))} \\ & \times \frac{\vartheta_1(2\eta)\vartheta_1(t_{\mathbf{2},r})\vartheta_1(\lambda_{0N}-t_{\mathbf{0},r})}{\vartheta_1(t_{\mathbf{0},r})\vartheta_1(t_{\mathbf{0},r}+2\eta)\vartheta_1(\lambda_{0N})} |\Downarrow\cdots\Downarrow 0\rangle \otimes |t_{\mathbf{2},r}-\eta\rangle \\ & + \frac{\vartheta_1(\lambda_{0N}+2\eta)}{\vartheta_1(\lambda_{0N})} \left(\frac{\mathcal{C}^{N-1}(\lambda_0)}{\mathrm{D}^{N-1}(\lambda_0)} \otimes \mathbb{I}\right) |\Downarrow\cdots\Downarrow\Downarrow\rangle \otimes |t_{\mathbf{2},r}\rangle\,. \end{aligned} \tag{5.162}$$

Proof. The lemma follows from Lemma 4.1.21 and Proposition 5.1.4. □

For $\lambda_0 = \lambda_N - 2\eta$, the second term in (5.162) trivially cancels. This leads to the following expression for $\|2\cdots 221, r\rangle\!\rangle$:

Lemma 5.A.3. *We have*

$$\|2\cdots 221, r\rangle\!\rangle = -\alpha \mathrm{e}^{2\mathrm{iy}\eta N} \frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\mathbf{0},r})} |\Downarrow\cdots\Downarrow 0\rangle \otimes |t_{\mathbf{2},r}-\eta\rangle\,. \tag{5.163}$$

To obtain $\|2\cdots 220, r\rangle\!\rangle$ and $\|2\cdots 200, r\rangle\!\rangle$, we successively apply the appropriate monodromy-matrix entries $\mathcal{C}(\lambda)$ on (5.163). Then, we use Lemma 4.1.21. Each time, the choice of the spectral parameter of $\mathcal{C}(\lambda)$ leads to only one remaining term. The results of these computations are, in sequence, (5.158), Lemma 5.A.4 and (5.159).

Lemma 5.A.4. *We have*

$$\|2\cdots 210, r\rangle\rangle = -\alpha \mathrm{e}^{6\mathrm{i}y\eta(N-1)} \prod_{j=0}^{2} \frac{\vartheta_1(t_{\mathbf{2},r} - j\eta)}{\vartheta_1(t_{\mathbf{0},r} + j\eta)} \times |\Downarrow \cdots \Downarrow 0 \Uparrow\rangle \otimes |t_{\mathbf{2},r} - 3\eta\rangle \,. \quad (5.164)$$

Proof. We apply $\frac{\mathcal{C}^N(\lambda_{N-1}-2\eta)}{\mathrm{D}^N(\lambda_{N-1}-2\eta)}$ to the state in (5.158). By Lemma 4.1.21, it becomes

$$\begin{aligned}
&\frac{\mathcal{C}^N(\lambda_{N-1} - 2\eta)}{\mathrm{D}^N(\lambda_{N-1} - 2\eta)} |\Downarrow \cdots \Downarrow \Uparrow\rangle \otimes |t_{\mathbf{2},r} - 2\eta\rangle \\
&\quad = \frac{\bar{\mathsf{h}}\left(\lambda_{N-1,N} - 2\eta \,\middle|\, \tau + 2\eta S^{N-1}\right)}{\vartheta_1(\lambda_{N-1,N} - 2\eta)} \left(\frac{\mathcal{C}^{N-1}(\lambda_{N-1} - 2\eta)}{\mathrm{D}^{N-1}(\lambda_{N-1} - 2\eta)} \otimes \mathbb{I}\right) \\
&\qquad \times |\Downarrow \cdots \Downarrow\Downarrow\Uparrow\rangle \otimes |t_{\mathbf{2},r} - 2\eta\rangle \,. \quad (5.165)
\end{aligned}$$

A second application of Lemma 4.1.21 yields

$$\begin{aligned}
&\left(\frac{\mathcal{C}^{N-1}(\lambda_{N-1} - 2\eta)}{\mathrm{D}^{N-1}(\lambda_{N-1} - 2\eta)} \otimes \mathbb{I}\right) |\Downarrow \cdots \Downarrow\Downarrow\Uparrow\rangle \otimes |t_{\mathbf{2},r} - 2\eta\rangle \\
&\quad = -\frac{\bar{\mathsf{j}}\left(-2\eta \,\middle|\, \tau + 2\eta S^{N-2}\right)}{\vartheta_1(2\eta)} \left(\frac{\mathcal{A}^{N-2}(\lambda_{N-1} - 2\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1} - 2\eta)} \otimes \mathbb{I}^{\otimes 2}\right) \\
&\qquad \times |\Downarrow \cdots \Downarrow 0 \Uparrow\rangle \otimes |t_{\mathbf{2},r} - 2\eta\rangle \,. \quad (5.166)
\end{aligned}$$

Proposition 5.1.4 informs us on the effect of $\mathcal{A}^{N-2}(\lambda_{N-1} - 2\eta)$ in (5.166). We gather the coefficients from (5.158), (5.165) and (5.166). Simplifying them with the help of definition (5.2) leads to (5.164). □

The same reasoning as in Lemma 5.A.4 leads to (5.159).

We now turn to the computation of $\|2\cdots 202, r\rangle\rangle$. Looking back to Lemma 5.A.2, we realise that the choice $\lambda_0 = \lambda_{N-1} - 2\eta$ produces two non-zero terms.

Lemma 5.A.5. *We have*

$$\begin{aligned}\|2\cdots 212,r\rangle\!\rangle = -\alpha \mathrm{e}^{2\mathrm{i}\mathrm{y}\eta N}\frac{\vartheta_1(t_{\mathbf{2},r})}{\vartheta_1(t_{\mathbf{0},r}+2\eta)}\frac{1}{\vartheta_1(\lambda_{N-1,N}-2\eta)}\\ \times\Bigg(\mathrm{e}^{-i\mathrm{y}\lambda_{N-1,N}}\frac{\vartheta_1(2\eta)}{\vartheta_1(t_{\mathbf{0},r})}\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-2\eta)\,|\Downarrow\cdots\Downarrow\Downarrow 0\rangle\\ +\mathrm{e}^{-2\mathrm{i}\mathrm{y}\eta}\vartheta_1(\lambda_{N-1,N})\,|\Downarrow\cdots\Downarrow 0\Downarrow\rangle\Bigg)\otimes|t_{\mathbf{2},r}-\eta\rangle\,. \quad (5.167)\end{aligned}$$

To obtain $\|2\cdots 202,r\rangle\!\rangle$, we apply the appropriate $\mathcal{C}(\lambda)$ to both terms in (5.167). The computation requires, once more, the use of Lemma 4.1.21 and Proposition 5.1.4.

Lemma 5.A.6. *We have*

$$\begin{aligned}&\frac{\mathcal{C}^N(\lambda_{N-1}-\eta)}{\mathrm{D}^N(\lambda_{N-1}-\eta)}\,|\Downarrow\cdots\Downarrow\Downarrow 0\rangle\otimes|t_{\mathbf{2},r}-\eta\rangle\\ &\quad=(W_{\Downarrow\Uparrow}\,|\Downarrow\cdots\Downarrow\Downarrow\Uparrow\rangle+W_{00}\,|\Downarrow\cdots\Downarrow 00\rangle)\otimes|t_{\mathbf{2},r}-2\eta\rangle\\ &\quad+W_{\Downarrow 0}\left(\frac{\mathcal{C}^{N-2}(\lambda_{N-1}-\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-\eta)}\otimes\mathbb{1}^{\otimes 2}\right)|\Downarrow\cdots\Downarrow\Downarrow 0\rangle\otimes|t_{\mathbf{2},r}-\eta\rangle\,, \quad (5.168)\end{aligned}$$

with

$$W_{\Downarrow\Uparrow}=-\mathrm{e}^{-\mathrm{i}\mathrm{y}\lambda_{N-1,N}}\mathrm{e}^{2\mathrm{i}\mathrm{y}\eta(N-1)}\frac{\vartheta_1(\eta)\vartheta_1(t_{\mathbf{2},r}-\eta)}{\alpha\vartheta_1(t_{\mathbf{0},r}+\eta)^2}\times\frac{\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-\eta)}{\vartheta_1(\lambda_{N-1,N}-\eta)}, \quad (5.169a)$$

$$W_{00}=-\alpha\mathrm{e}^{2\mathrm{i}\mathrm{y}\eta(N-2)}\frac{\vartheta_1(2\eta)}{\vartheta_1(\eta)}\frac{\vartheta_1(t_{\mathbf{2},r}-\eta)\vartheta_1(t_{\mathbf{0},r})\vartheta_1(t_{\mathbf{0},r}+2\eta)}{\vartheta_1(t_{\mathbf{0},r}+\eta)^2\vartheta_1(t_{\mathbf{0},r}+3\eta)}\times\frac{\vartheta_1(\lambda_{N-1,N})}{\vartheta_1(\lambda_{N-1,N}-\eta)}, \quad (5.169b)$$

$$W_{\Downarrow 0}=-\mathrm{e}^{-\mathrm{i}\mathrm{y}\eta}\frac{\vartheta_1(t_{\mathbf{0},r})}{\vartheta_1(t_{\mathbf{0},r}+\eta)}\frac{\vartheta_1(\lambda_{N-1,N})}{\vartheta_1(\lambda_{N-1,N}-\eta)}. \quad (5.169c)$$

Lemma 5.A.7. *We have*

$$\begin{aligned}
&\frac{\mathcal{C}^N(\lambda_{N-1}-\eta)}{\mathrm{D}^N(\lambda_{N-1}-\eta)}\left|\Downarrow\cdots\Downarrow 0\Downarrow\right\rangle\otimes\left|t_{\mathbf{2},r}-\eta\right\rangle \\
&=-\frac{\mathrm{e}^{2\mathrm{i}y\eta(N-2)}}{\alpha}\frac{\vartheta_1(t_{\mathbf{2},r}-\eta)}{\vartheta_1(t_{\mathbf{0},r}+3\eta)}\frac{\vartheta_1(\lambda_{N-1,N}+\eta)}{\vartheta_1(\lambda_{N-1,N}-\eta)}\left|\Downarrow\cdots\Downarrow\Uparrow\Downarrow\right\rangle\otimes\left|t_{\mathbf{2},r}-2\eta\right\rangle \\
&\qquad+\mathrm{e}^{-\mathrm{i}y(\lambda_{N-1,N}-\eta)}\frac{\vartheta_1(2\eta)}{\vartheta_1(t_{\mathbf{0},r}+\eta)}\frac{\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{0},r}-2\eta)}{\vartheta_1(\lambda_{N-1,N}-\eta)} \\
&\qquad\times\left(\frac{\mathcal{C}^{N-2}(\lambda_{N-1}-\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-\eta)}\otimes\mathbb{I}^{\otimes 2}\right)\left|\Downarrow\cdots\Downarrow\Downarrow 0\right\rangle\otimes\left|t_{\mathbf{2},r}-\eta\right\rangle. \qquad (5.170)
\end{aligned}$$

Combining Lemmas 5.A.5 to 5.A.7 establishes (5.160).

Left qSoV states We now turn to the computation of the left qSoV states

$$\langle\!\langle 0\cdots 002,r\|,\qquad \langle\!\langle 0\cdots 020,r\|,\qquad \langle\!\langle 0\cdots 022,r\|. \qquad (5.171)$$

We proceed similarly to the previous case, by successive application of operators $\mathcal{C}$ on the reference state $\langle\!\langle \mathbf{0},r\|$.

Proposition 5.A.8. *We have*

$$\langle\!\langle 0\cdots 002,r\| = \prod_{a=1}^{2}\frac{\mathrm{D}^{N-1}(\lambda_N+a\eta)}{\mathrm{D}^{N-1}(\lambda_N-a\eta)}\left\langle\Uparrow\cdots\Uparrow\Uparrow\Downarrow\right|\otimes\left\langle t_{\mathbf{0},r}+2\eta\right|, \qquad (5.172)$$

$$\begin{aligned}
\langle\!\langle 0\cdots 022,r\| = &\prod_{k=N-1}^{N}\prod_{a=1}^{2}\frac{\mathrm{D}^{N-2}(\lambda_k+a\eta)}{\mathrm{D}^{N-2}(\lambda_k-a\eta)} \qquad (5.173)\\
&\times\left\langle\Uparrow\cdots\Uparrow\Downarrow\Downarrow\right|\otimes\left\langle t_{\mathbf{0},r}+4\eta\right|,
\end{aligned}$$

and

$$\begin{aligned}
\langle\!\langle 0\cdots 020,r\| = &\frac{\mathrm{e}^{-\mathrm{i}y(\lambda_{N-1,N}+2\eta)}}{\vartheta_1(t_{\mathbf{2},r}-\eta)\vartheta_1(t_{\mathbf{2},r}-2\eta)} \qquad (5.174)\\
&\times\frac{1}{\vartheta_1(\lambda_{N-1,N}-\eta)\vartheta_1(\lambda_{N-1,N}-2\eta)}\prod_{a=1}^{2}\frac{\mathrm{D}^{N-2}(\lambda_{N-1}+a\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-a\eta)} \\
&\times\left(Y_{\Uparrow\Downarrow}\left\langle\Uparrow\cdots\Uparrow\Uparrow\Downarrow\right|+Y_{00}\left\langle\Uparrow\cdots\Uparrow 00\right|+Y_{\Downarrow\Uparrow}\left\langle\Uparrow\cdots\Uparrow\Downarrow\Uparrow\right|\right)\otimes\left\langle t_{\mathbf{0},r}+2\eta\right|,
\end{aligned}$$

where

$$Y_{\Uparrow\Downarrow} = \mathrm{e}^{-\mathrm{iy}(\lambda_{N-1,N}-2\eta)}\vartheta_1(\eta)\vartheta_1(2\eta)\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{2},r}+\eta) \times \vartheta_1(\lambda_{N-1,N}-t_{\mathbf{2},r}+2\eta), \tag{5.175a}$$

$$Y_{00} = -\frac{\vartheta_1(\eta)}{\alpha^2}\vartheta_1(t_{\mathbf{2},r}-3\eta)\vartheta_1(\lambda_{N-1,N})\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{2},r}+2\eta), \tag{5.175b}$$

$$Y_{\Downarrow\Uparrow} = \mathrm{e}^{\mathrm{iy}(\lambda_{N-1,N}-2\eta)}\vartheta_1(t_{\mathbf{2},r}-3\eta)\vartheta_1(t_{\mathbf{2},r}-4\eta) \times \vartheta_1(\lambda_{N-1,N})\vartheta_1(\lambda_{N-1,N}-\eta). \tag{5.175c}$$

The following lemma allows us to obtain the intermediary elements $\langle\langle 0\cdots 001, r\|$ and $\langle\langle 0\cdots 010, r\|$:

Lemma 5.A.9. *We have*

$$\begin{aligned}\langle\langle \mathbf{0}, r\| \frac{\mathcal{C}^N(\lambda_0)}{\mathrm{D}^N(\lambda_0-\eta)} &= -\frac{\mathrm{e}^{-\mathrm{iy}(\lambda_{0N}+\eta)}}{\alpha}\frac{\vartheta_1(\eta)\vartheta_1(\lambda_{0N}-t_{\mathbf{2},r}+2\eta)}{\vartheta_1(t_{\mathbf{2},r}-\eta)\vartheta_1(\lambda_{0N}-\eta)} \\ &\times \frac{\mathrm{D}^{N-1}(\lambda_0+2\eta)}{\mathrm{D}^{N-1}(\lambda_0-\eta)}\langle\Uparrow\cdots\Uparrow 0|\otimes\langle t_{\mathbf{0},r}+\eta| \\ &+ \mathrm{e}^{-2\mathrm{iy}\eta}\frac{\vartheta_1(t_{\mathbf{2},r}-3\eta)\vartheta_1(\lambda_{0N})}{\vartheta_1(t_{\mathbf{2},r}-\eta)\vartheta_1(\lambda_{0N}-\eta)} \\ &\times \langle\Uparrow\cdots\Uparrow\Uparrow|\otimes\langle t_{\mathbf{0},r}|\left(\frac{\mathcal{C}^{N-1}(\lambda_0)}{\mathrm{D}^{N-1}(\lambda_0-\eta)}\otimes\mathbb{I}\right).\end{aligned} \tag{5.176}$$

For $\lambda_0 = \lambda_N$, the second term in (5.176) trivially cancels. This leads to the following expression (5.177) for $\langle\langle 0\cdots 001, r\|$:

$$\langle\langle 0\cdots 001, r\| = -\frac{\mathrm{e}^{-\mathrm{iy}\eta}}{\alpha}\frac{\vartheta_1(t_{\mathbf{2},r}-2\eta)}{\vartheta_1(t_{\mathbf{2},r}-\eta)}\frac{\mathrm{D}^{N-1}(\lambda_N+2\eta)}{\mathrm{D}^{N-1}(\lambda_N-\eta)} \times\langle\Uparrow\cdots\Uparrow 0|\otimes\langle t_{\mathbf{0},r}+\eta|. \tag{5.177}$$

Next, we apply the appropriate operators $\mathcal{C}$ on $\langle\langle 0\cdots 001, r\|$ to obtain $\langle\langle 0\cdots 002, r\|$, $\langle\langle 0\cdots 012, r\|$ and $\langle\langle 0\cdots 022, r\|$. For each of these states, the specialisation of the spectral parameter in $\mathcal{C}(\lambda)$ gives rise to a single non-zero term. The results of these computations are (5.172), (5.173),

and

$$\langle\!\langle 0\cdots 012, r\| = -\frac{\mathrm{e}^{-\mathrm{i}y\eta}}{\alpha}\frac{\vartheta_1(t_{\mathbf{2},r}-2\eta)}{\vartheta_1(t_{\mathbf{2},r}-\eta)}\frac{\mathrm{D}^{N-1}(\lambda_N+2\eta)}{\mathrm{D}^{N-1}(\lambda_N-\eta)}\frac{\mathrm{D}^{N-2}(\lambda_N+\eta)}{\mathrm{D}^{N-2}(\lambda_N-2\eta)}$$
$$\times\frac{\mathrm{D}^{N-2}(\lambda_{N-1}+2\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-\eta)}\langle\Uparrow\cdots\Uparrow 0\Downarrow|\otimes\langle t_{\mathbf{0},r}+3\eta|\,. \quad (5.178)$$

We now consider (5.176) with $\lambda_0=\lambda_{N-1}$, and obtain

$$\langle\!\langle 0\cdots 010, r\| = -\frac{1}{\alpha\vartheta_1(t_{\mathbf{2},r}-\eta)\vartheta_1(\lambda_{N-1,N}-\eta)}\frac{\mathrm{D}^{N-2}(\lambda_{N-1}+2\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-\eta)}$$
$$\times\big(-\mathrm{e}^{-\mathrm{i}y(\lambda_{N-1,N}+\eta)}\vartheta_1(2\eta)\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{2},r}+2\eta)\langle\Uparrow\cdots\Uparrow\Uparrow 0|$$
$$+\mathrm{e}^{-3\mathrm{i}y\eta}\vartheta_1(t_{\mathbf{2},r}-4\eta)\vartheta_1(\lambda_{N-1,N})\langle\Uparrow\cdots\Uparrow 0\Uparrow|\big)\otimes\langle t_{\mathbf{0},r}+\eta|\,. \quad (5.179)$$

To reach $\langle\!\langle 0\cdots 020, r\|$, we apply one last operator $\mathcal{C}$ to the two vectors in (5.179). We present intermediary results in the following lemma:

Lemma 5.A.10. *We have*

$$\langle\Uparrow\cdots\Uparrow 0\Uparrow|\otimes\langle t_{\mathbf{0},r}+\eta|\frac{\mathcal{C}^N(\lambda_{N-1}-\eta)}{\mathrm{D}^N(\lambda_{N-1}-2\eta)}$$
$$= -\alpha\mathrm{e}^{-\mathrm{i}y\eta}\frac{\vartheta_1(t_{\mathbf{2},r}-3\eta)}{\vartheta_1(t_{\mathbf{2},r}-2\eta)}\frac{\vartheta_1(\lambda_{N-1,N}-\eta)}{\vartheta_1(\lambda_{N-1,N}-2\eta)}$$
$$\times\frac{\mathrm{D}^{N-2}(\lambda_{N-1}+\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-2\eta)}\langle\Uparrow\cdots\Uparrow\Downarrow\Uparrow|\otimes\langle t_{\mathbf{0},r}+2\eta|$$
$$+\mathrm{e}^{-\mathrm{i}y(\lambda_{N-1,N}+\eta)}\frac{\vartheta_1(\eta)\vartheta_1(t_{\mathbf{2},r}-3\eta)}{\vartheta_1(t_{\mathbf{2},r}-2\eta)^2}\frac{\vartheta_1(\lambda_{N-1,N}-t_{\mathbf{2},r}+2\eta)}{\vartheta_1(\lambda_{N-1,N}-2\eta)}$$
$$\times\langle\Uparrow\cdots\Uparrow\Uparrow 0|\otimes\langle t_{\mathbf{0},r}+\eta|\left(\frac{\mathcal{C}^{N-2}(\lambda_{N-1}-\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-2\eta)}\otimes\mathbb{I}^{\otimes 2}\right), \quad (5.180)$$

and

$$\langle\Uparrow\cdots\Uparrow\Uparrow 0|\otimes\langle t_{\mathbf{0},r}+\eta|\frac{\mathcal{C}^N(\lambda_{N-1}-\eta)}{\mathrm{D}^N(\lambda_{N-1}-2\eta)}$$
$$=\frac{\vartheta_1(\eta)}{\vartheta_1(t_{\mathbf{2},r}-2\eta)\vartheta_1(\lambda_{N-1,N}-2\eta)}\frac{\mathrm{D}^{N-2}(\lambda_{N-1}+\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-2\eta)}$$
$$\times\Big(\tilde{W}_{\Uparrow\Downarrow}\langle\Uparrow\cdots\Uparrow\Uparrow\Downarrow|+\tilde{W}_{00}\langle\Uparrow\cdots\Uparrow 00|\Big)\otimes\langle t_{\mathbf{0},r}+2\eta|$$
$$+\tilde{W}_{\Uparrow 0}\langle\Uparrow\cdots\Uparrow\Uparrow 0|\otimes\langle t_{\mathbf{0},r}+\eta|\left(\frac{\mathcal{C}^{N-2}(\lambda_{N-1}-\eta)}{\mathrm{D}^{N-2}(\lambda_{N-1}-2\eta)}\otimes\mathbb{I}^{\otimes 2}\right), \quad (5.181)$$

with

$$\tilde{W}_{\Uparrow\Downarrow} = \alpha \mathrm{e}^{-\mathrm{iy}(\lambda_{N-1,N}-\eta)} \vartheta_1(\lambda_{N-1,N} - t_{\mathbf{2},r} + \eta), \tag{5.182a}$$

$$\tilde{W}_{00} = -\mathrm{e}^{-\mathrm{iy}\eta} \frac{\vartheta_1(t_{\mathbf{2},r} - 3\eta)\vartheta_1(\lambda_{N-1,N})}{\alpha\vartheta_1(2\eta)}, \tag{5.182b}$$

$$\begin{aligned} \tilde{W}_{\Uparrow 0} = \mathrm{e}^{-3\mathrm{iy}\eta} \frac{\vartheta_1(\eta)}{\vartheta_1(2\eta)} \frac{\vartheta_1(t_{\mathbf{2},r} - 3\eta)\vartheta_1(t_{\mathbf{2},r} - 4\eta)}{\vartheta_1(t_{\mathbf{2},r} - 2\eta)^2} \\ \times \frac{\vartheta_1(\lambda_{N-1,N})}{\vartheta_1(\lambda_{N-1,N} - 2\eta)}. \end{aligned} \tag{5.182c}$$

Finally, (5.179) and Lemma 5.A.10 lead to (5.174).

Chapter 6

Special eigenvectors of higher-spin models

In this chapter, we pursue the investigation of special eigenvectors of the 18V and 41V models, and the related spin-1 XYZ chain. We recall that, in Chapter 3, simple algebraic arguments established the existence of eigenvectors of the 18V and 41V transfer matrices with remarkably simple eigenvalues. To examine them further, in Chapter 4, we related the transfer matrix of the 18V model to that of the d10V model. Then, in Chapter 5, we solved the eigenvalue problem of the latter via qSoV techniques. We now exploit these relations and results to define the special eigenvectors of the 18V and 41V models. Through these definitions, we derive a number of properties that characterise the special eigenvectors implicitly. These include symmetry properties, exchange and reduction relations, and pseudo-periodicity properties with respect to the inhomogeneity parameters, as well as a conjecture regarding their analyticity with respect to the inhomogeneity parameters. Additionally, we compute a quadratic sum rule involving them in terms of a determinant formula. Taking the homogeneous limit of the special eigenvectors allows us to define a zero-energy eigenstate of the spin-1 XYZ chain. We obtain a symmetry property of this last vector, as well as a closed-form expression for its square norm.

We start our investigation with preliminary work on the sum rule of Theorem 5.2.14. Indeed, its elliptic determinant expression tends to not be the most convenient to manipulate, especially when taking the homogeneous limit. We thus proceed to its *uniformisation*. Through this procedure, we find an elliptic parameterisation that expresses the determinant formula in terms of polynomials. We study several properties of these polynomials, and find that they are generalisations of those introduced by Zinn-Justin and Rosengren [78–81, 111].

This chapter is structured as follows: in Section 6.1, we proceed to the uniformisation of the determinant formula of Theorem 5.2.14. We analyse the properties of the resulting polynomials. In Section 6.2, we present results for the special eigenvectors of the 18V, 41V models and the spin-1 XYZ chain. We conclude the chapter with some research perspectives related to the present work.

6.1 Uniformisation of an elliptic determinant

The goal of this section is to study the quantity

$$Z(\xi = 1; \boldsymbol{\lambda}) = \langle \varphi^L | \varphi^R \rangle \,, \tag{6.1}$$

where $\langle \varphi^L|$ and $|\varphi^R\rangle$ are the left and right special eigenvectors of the d10V transfer matrix, as defined in Theorem 5.2.5. In Chapter 5, we used qSoV techniques to find an explicit expression for $Z(\xi; \boldsymbol{\lambda})$. The expression depends on the inhomogeneity parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ and

$$(\mathsf{x}, \mathsf{y}) \in \{(1, 0),\ (1, 1),\ (0, 1)\}. \tag{6.2}$$

Here, we prefer to label these three choices by a single integer

$$a \in \{2, 3, 4\}. \tag{6.3}$$

The correspondence between the labels is

$$\begin{array}{c||c|c|c} (\mathsf{x}, \mathsf{y}) & (1,0) & (1,1) & (0,1) \\ \hline a & 2 & 3 & 4 \end{array}. \tag{6.4}$$

Using this correspondence and Proposition A.1.3, we infer from Theorem 5.2.14 the explicit expression

$$Z(\xi;\boldsymbol{\lambda}) = Z_N^{(a)}(\xi;\boldsymbol{\lambda}) = \frac{1}{\vartheta_a^{N-1}\vartheta_a(N\eta)} \frac{\prod_{i,j=1}^N \vartheta_1(\lambda_{ij}+\eta)}{\prod_{1\leqslant i<j\leqslant N}\vartheta_1(\lambda_{ij})\vartheta_1(\lambda_{ji})} \\ \times \det_{i,j=1}^N \left(\frac{\vartheta_a(\lambda_{ij}-\eta)}{\vartheta_1(\lambda_{ij}-\eta)}\xi - \frac{\vartheta_a(\lambda_{ij}+\eta)}{\vartheta_1(\lambda_{ij}+\eta)}\xi^{-1} \right), \quad (6.5)$$

where $\lambda_{ij} = \lambda_i - \lambda_j$. In Section 6.2, we find an interpretation for $Z(\xi = 1;\boldsymbol{\lambda})$ as a quadratic sum rule for the special eigenvectors of the 41V and 18V models. Moreover, we relate the homogeneous limit $Z(\xi = 1;\mathbf{0})$ to the square norm of a zero-energy eigenstate of the spin-1 XYZ chain. To make this last process easier, we now proceed to the uniformisation of the determinant formula of $Z(\xi = 1;\boldsymbol{\lambda})$.

Generalisation We consider the generalisation

$$\Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu}) = \frac{\prod_{i,j=1}^N \vartheta_1(\lambda_i-\mu_j+\eta)\vartheta_1(\lambda_i-\mu_j-\eta)}{\prod_{1\leqslant i<j\leqslant N}\vartheta_1(\lambda_{ij})\vartheta_1(\mu_{ji})} \\ \times \det_{i,j=1}^N f^{(a)}(\lambda_i-\mu_j), \quad (6.6)$$

where

$$f^{(a)}(\lambda) = \frac{\vartheta_a(\lambda-\eta)}{\vartheta_1(\lambda-\eta)} - \frac{\vartheta_a(\lambda+\eta)}{\vartheta_1(\lambda+\eta)}. \quad (6.7)$$

Clearly, we have

$$Z_N^{(a)}(\xi = 1;\boldsymbol{\lambda}) = \frac{\Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\lambda})}{\vartheta_a^{N-1}\vartheta_a(N\eta)\prod_{i,j=1}^N \vartheta_1(\lambda_{ij}-\eta)}. \quad (6.8)$$

Moreover, by convention, we set $Z_0^{(a)} = \Upsilon_0^{(a)} = 1$.

6.1.1 Modular and factorisation properties

Modular properties We begin our analysis by showing that the three functions $\Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu})$ transform into one another under the action of the *modular group*. The *modular group* is the group of fractional linear transformations of the upper complex plane of the form

$$\omega \mapsto \frac{a\omega+b}{c\omega+d}, \quad (6.9)$$

where a, b, c, d are integers obeying $ad - bc = 1$. Each of them is a composition of the following shift and inversion [158]:

$$T : \omega \mapsto \omega + 1, \qquad\qquad S : \omega \mapsto -\frac{1}{\omega}. \tag{6.10}$$

Thus, T and S are generators of the modular group. We often refer to them and their compositions as the *modular transformations*. We specify their actions on the Jacobi theta functions in Appendix A.1.3.

We consider the actions of T and S on $\Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu}) = \Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu};\eta|\omega)$:

$$(T\Upsilon_N^{(a)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = \Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu};\eta|\omega+1), \tag{6.11}$$

$$(S\Upsilon_N^{(a)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = \Upsilon_N^{(a)}(\boldsymbol{\lambda}/\omega;\boldsymbol{\mu}/\omega;\eta/\omega|-1/\omega). \tag{6.12}$$

Lemma 6.1.1. *We have*

$$(T\Upsilon_N^{(2)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = \mathrm{e}^{\mathrm{i}\pi N(N+1)/4}\Upsilon_N^{(2)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.13}$$

$$(T\Upsilon_N^{(3)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = \mathrm{e}^{\mathrm{i}\pi N^2/4}\Upsilon_N^{(4)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.14}$$

$$(T\Upsilon_N^{(4)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = \mathrm{e}^{\mathrm{i}\pi N^2/4}\Upsilon_N^{(3)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.15}$$

and

$$(S\Upsilon_N^{(2)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = c_N(\boldsymbol{\lambda};\boldsymbol{\mu})\Upsilon_N^{(4)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.16}$$

$$(S\Upsilon_N^{(3)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = c_N(\boldsymbol{\lambda};\boldsymbol{\mu})\Upsilon_N^{(3)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.17}$$

$$(S\Upsilon_N^{(4)})(\boldsymbol{\lambda};\boldsymbol{\mu}) = c_N(\boldsymbol{\lambda};\boldsymbol{\mu})\Upsilon_N^{(2)}(\boldsymbol{\lambda};\boldsymbol{\mu}), \tag{6.18}$$

where

$$\begin{aligned} c_N(\boldsymbol{\lambda};\boldsymbol{\mu}) &= \mathrm{i}^{N^2}A^{N(N+1)}\mathrm{e}^{2\mathrm{i}(N\eta)^2/\pi\omega} \\ &\quad\times \exp\left(\frac{\mathrm{i}}{\pi\omega}\left(2\sum_{i,j=1}^{N}(\lambda_i-\mu_j)^2 - \sum_{1\leqslant i<j\leqslant N}\left(\lambda_{ij}^2+\mu_{ij}^2\right)\right)\right), \end{aligned} \tag{6.19}$$

and $A = (-\mathrm{i}\omega)^{1/2}$.

Proof. The proof is a straightforward calculation, based on the transformation behaviour (A.30) and (A.31) of the Jacobi theta functions under modular transformations. □

Half-specialisation and factorisation By Lemma 6.1.1, we may restrict our analysis of $\Upsilon_N^{(a)}(\boldsymbol{\lambda};\boldsymbol{\mu})$ to $a=2$, and recover the cases $a=3,4$ through modular transformations. In this and the following sections, we use the simplified notations

$$f(\lambda)=f^{(2)}(\lambda), \qquad \Upsilon_N(\boldsymbol{\lambda};\boldsymbol{\mu})=\Upsilon_N^{(2)}(\boldsymbol{\lambda};\boldsymbol{\mu}). \tag{6.20}$$

A first step towards the uniformisation of $\Upsilon_N(\boldsymbol{\lambda};\boldsymbol{\mu})$ is to *half-specialise* the arguments $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Let $n\geqslant 0$ be an integer and $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)$ be n-tuples of complex numbers. (If $n=0$, $\boldsymbol{x}=()$ is just the empty list by convention.) We denote by

$$(\boldsymbol{x},\boldsymbol{y})=(x_1,\dots,x_n,y_1,\dots,y_n) \tag{6.21}$$

their concatenation. We use the same notation for the concatenation of lists with an unequal number of elements. Moreover, we omit the parentheses of the concatenation when it appears as an argument of a function.

For $N=2n$, the half-specialisation amounts to setting the arguments of $\Upsilon_N(\boldsymbol{\lambda};\boldsymbol{\mu})$ to

$$\boldsymbol{\lambda}=(\boldsymbol{x},-\boldsymbol{x}), \quad \boldsymbol{\mu}=(\boldsymbol{y},-\boldsymbol{y}). \tag{6.22}$$

For $N=2n+1$, the half-specialisation is

$$\boldsymbol{\lambda}=(\boldsymbol{x},-\boldsymbol{x},0), \quad \boldsymbol{\mu}=(\boldsymbol{y},-\boldsymbol{y},0). \tag{6.23}$$

Following Kuperberg [114], we show that $\Upsilon_N(\boldsymbol{\lambda};\boldsymbol{\mu})$ factorises when its arguments are half-specialised. We write the factors in terms of determinant formulas that involve three functions. To define them, we use the shorthand notation:

$$\vartheta_i(x\pm y)=\vartheta_i(x+y)\vartheta_i(x-y), \qquad i=1,\dots,4. \tag{6.24}$$

The three functions are

$$\mathbb{h}(x,y)=\vartheta_1(x-y\pm\eta)\vartheta_1(x+y\pm\eta), \tag{6.25}$$

and

$$\begin{aligned}\mathbb{g}(x,y)&=\vartheta_3(x-y)\vartheta_4(x-y)\vartheta_1(x+y\pm\eta)\\&\qquad+\vartheta_3(x+y)\vartheta_4(x+y)\vartheta_1(x-y\pm\eta),\end{aligned} \tag{6.26}$$

$$\begin{aligned}\bar{\mathbb{g}}(x,y)&=\vartheta_3(x-y)\vartheta_4(x-y)\vartheta_1(x+y\pm\eta)\\&\qquad-\vartheta_3(x+y)\vartheta_4(x+y)\vartheta_1(x-y\pm\eta).\end{aligned} \tag{6.27}$$

We define $\Xi_0 = \bar{\Xi}_0 = 1$, and for $n \geq 1$,

$$\Xi_{2n}(\boldsymbol{x};\boldsymbol{y}) = \frac{\prod_{i,j=1}^{n} \mathbb{h}(x_i, y_j)}{\mathbb{A}(\boldsymbol{x})\mathbb{A}(\boldsymbol{y})} \det_{i,j=1}^{n} \left(\frac{\mathbb{g}(x_i, y_j)}{\mathbb{h}(x_i, y_j)} \right), \tag{6.28}$$

$$\bar{\Xi}_{2n}(\boldsymbol{x};\boldsymbol{y}) = \frac{\prod_{i,j=1}^{n} \mathbb{h}(x_i, y_j)}{\bar{\mathbb{A}}(\boldsymbol{x})\bar{\mathbb{A}}(\boldsymbol{y})} \det_{i,j=1}^{n} \left(\frac{\bar{\mathbb{g}}(x_i, y_j)}{\mathbb{h}(x_i, y_j)} \right). \tag{6.29}$$

Here,

$$\mathbb{A}(\boldsymbol{x}) = \prod_{1 \leqslant i < j \leqslant n} \vartheta_1(x_j \pm x_i), \qquad \bar{\mathbb{A}}(\boldsymbol{x}) = \mathbb{A}(\boldsymbol{x}) \prod_{i=1}^{n} \vartheta_1(2x_i) \tag{6.30}$$

are theta-function generalisations of the classical Weyl denominators of type D_n and C_n, respectively [63].

Proposition 6.1.2. *For each $n \geq 0$, we have*

$$\Upsilon_{2n}(\boldsymbol{x}, -\boldsymbol{x}; \boldsymbol{y}, -\boldsymbol{y}) = (-1)^n \Lambda^{2n} \Xi_{2n}(\boldsymbol{x};\boldsymbol{y}) \bar{\Xi}_{2n}(\boldsymbol{x};\boldsymbol{y}), \tag{6.31}$$

$$\begin{aligned} \Upsilon_{2n+1}(\boldsymbol{x}, -\boldsymbol{x}, 0; \boldsymbol{y}, -\boldsymbol{y}, 0) = {} & \frac{(-1)^{n+1}\Lambda^{2n+1}}{2\vartheta_1(\eta)^2} \\ & \times \Xi_{2(n+1)}(\boldsymbol{x}, 0; \boldsymbol{y}, 0) \bar{\Xi}_{2n}(\boldsymbol{x}, \boldsymbol{y}), \end{aligned} \tag{6.32}$$

where

$$\Lambda = \frac{2\vartheta_1(\eta)\vartheta_2(\eta)}{\vartheta_3\vartheta_4}. \tag{6.33}$$

Proof. First, we consider $\Upsilon_{2n}(\boldsymbol{\lambda}; \boldsymbol{\mu})$ with the half-specialisation (6.22). We evaluate

$$\prod_{1 \leqslant i < j \leqslant 2n} \vartheta_1(\lambda_i - \lambda_j) = (-1)^{n(n-1)/2} \mathbb{A}(\boldsymbol{x}) \bar{\mathbb{A}}(\boldsymbol{x}), \tag{6.34}$$

$$\prod_{1 \leqslant i < j \leqslant 2n} \vartheta_1(\mu_j - \mu_i) = (-1)^{n(n+1)/2} \mathbb{A}(\boldsymbol{y}) \bar{\mathbb{A}}(\boldsymbol{y}), \tag{6.35}$$

and

$$\prod_{i,j=1}^{2n} \vartheta_1(\lambda_i - \mu_j \pm \eta) = \left(\prod_{i,j=1}^{n} \mathbb{h}(x_i, y_j) \right)^2. \tag{6.36}$$

These evaluations suggest that we consider the ratio

$$\hat{\Upsilon}_{2n}(\boldsymbol{x};\boldsymbol{y}) = (-1)^n \frac{\mathbb{A}(\boldsymbol{x})\mathbb{A}(\boldsymbol{y})\bar{\mathbb{A}}(\boldsymbol{x})\bar{\mathbb{A}}(\boldsymbol{y}) \Upsilon_{2n}(\boldsymbol{x}, -\boldsymbol{x}; \boldsymbol{y}, -\boldsymbol{y})}{\left(\prod_{i,j=1}^{n} \mathbb{h}(x_i, y_j) \right)^2}. \tag{6.37}$$

Since $f(\lambda)$ is an even function of λ, $\hat{\Upsilon}_{2n}(\boldsymbol{x};\boldsymbol{y})$ is the determinant of a block matrix:

$$\hat{\Upsilon}_{2n}(\boldsymbol{x};\boldsymbol{y}) = \det\left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y}) & B_n(\boldsymbol{x};\boldsymbol{y}) \\ \hline B_n(\boldsymbol{x};\boldsymbol{y}) & A_n(\boldsymbol{x};\boldsymbol{y}) \end{array}\right). \tag{6.38}$$

The blocks are given by

$$A_n(\boldsymbol{x};\boldsymbol{y}) = (f(x_i - y_i))_{i,j=1}^n, \qquad B_n(\boldsymbol{x};\boldsymbol{y}) = (f(x_i + y_i))_{i,j=1}^n. \tag{6.39}$$

We exploit the symmetry of the block matrix in (6.38) to block-diagonalise it with the help of the $2n \times 2n$ orthogonal matrix

$$U = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} \mathbb{1}_n & \mathbb{1}_n \\ \hline -\mathbb{1}_n & \mathbb{1}_n \end{array}\right), \tag{6.40}$$

where $\mathbb{1}_n$ denotes the $n \times n$ identity matrix. Indeed, we have

$$\begin{aligned} U^t &\left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y}) & B_n(\boldsymbol{x};\boldsymbol{y}) \\ \hline B_n(\boldsymbol{x};\boldsymbol{y}) & A_n(\boldsymbol{x};\boldsymbol{y}) \end{array}\right) U \\ &= \left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y}) - B_n(\boldsymbol{x};\boldsymbol{y}) & \mathbb{0} \\ \hline \mathbb{0} & A_n(\boldsymbol{x};\boldsymbol{y}) + B_n(\boldsymbol{x};\boldsymbol{y}) \end{array}\right). \end{aligned} \tag{6.41}$$

Taking the determinant on both sides of this equality, we obtain

$$\begin{aligned} \hat{\Upsilon}_{2n}(\boldsymbol{x};\boldsymbol{y}) &= \det(A_n(\boldsymbol{x};\boldsymbol{y}) - B_n(\boldsymbol{x};\boldsymbol{y})) \\ &\qquad \times \det(A_n(\boldsymbol{x};\boldsymbol{y}) + B_n(\boldsymbol{x};\boldsymbol{y})). \end{aligned} \tag{6.42}$$

The substitution of this expression into (6.37) and the identities

$$A_n(\boldsymbol{x};\boldsymbol{y}) + B_n(\boldsymbol{x};\boldsymbol{y}) = \Lambda\left(\frac{\mathbb{g}(x_i,y_j)}{\mathbb{h}(x_i,y_j)}\right)_{i,j=1}^n, \tag{6.43a}$$

$$A_n(\boldsymbol{x};\boldsymbol{y}) - B_n(\boldsymbol{x};\boldsymbol{y}) = \Lambda\left(\frac{\bar{\mathbb{g}}(x_i,y_j)}{\mathbb{h}(x_i,y_j)}\right)_{i,j=1}^n, \tag{6.43b}$$

lead to the factorisation (6.31).

Second, we consider $\Upsilon_{2n+1}(\boldsymbol{\lambda};\boldsymbol{\mu})$ with the half-specialisation (6.23). We have

$$\prod_{1\leqslant i<j\leqslant 2n+1} \vartheta_1(\lambda_i - \lambda_j) = (-1)^{n(n+1)/2}\mathbb{\Delta}(\boldsymbol{x},0)\bar{\mathbb{\Delta}}(\boldsymbol{x}), \tag{6.44}$$

$$\prod_{1\leqslant i<j\leqslant 2n+1} \vartheta_1(\mu_j - \mu_i) = (-1)^{n(n-1)/2}\mathbb{\Delta}(\boldsymbol{y},0)\bar{\mathbb{\Delta}}(\boldsymbol{y}), \tag{6.45}$$

and

$$\prod_{1\leqslant i<j\leqslant 2n+1} \vartheta_1(\lambda_i-\mu_j\pm\eta) \\ = -\frac{1}{\vartheta_1(\eta)^2}\left(\prod_{i,j=1}^{n}\mathbb{h}(x_i,y_j)\right)\left(\prod_{i=1}^{n}\mathbb{h}(x_i,0)\mathbb{h}(0,y_i)\right)\mathbb{h}(0,0). \quad (6.46)$$

Hence, it is natural to analyse the ratio

$$\hat{\Upsilon}_{2n+1}(\boldsymbol{x};\boldsymbol{y}) = \frac{(-1)^{n+1}\vartheta_1(\eta)^2\mathbb{A}(\boldsymbol{x},0)\mathbb{A}(\boldsymbol{y},0)\bar{\mathbb{A}}(\boldsymbol{x})\bar{\mathbb{A}}(\boldsymbol{y})}{\left(\prod_{i,j=1}^{n}\mathbb{h}(x_i,y_j)\right)^2\left(\prod_{i=1}^{n}\mathbb{h}(x_i,0)\mathbb{h}(0,y_i)\right)\mathbb{h}(0,0)} \\ \times\Upsilon_{2n+1}(\boldsymbol{x},-\boldsymbol{x},0;\boldsymbol{y},-\boldsymbol{y},0). \quad (6.47)$$

It is given by the determinant of a block matrix, too:

$$\hat{\Upsilon}_{2n+1}(\boldsymbol{x};\boldsymbol{y}) = \det\left(\begin{array}{c|c|c} A_n(\boldsymbol{x};\boldsymbol{y}) & B_n(\boldsymbol{x};\boldsymbol{y}) & f(\boldsymbol{x})^t \\ \hline B_n(\boldsymbol{x};\boldsymbol{y}) & A_n(\boldsymbol{x};\boldsymbol{y}) & f(\boldsymbol{x})^t \\ \hline f(\boldsymbol{y}) & f(\boldsymbol{y}) & f(0)\end{array}\right). \quad (6.48)$$

Here, $f(\boldsymbol{x}) = (f(x_1),\ldots,f(x_n))$ and $f(\boldsymbol{y}) = (f(y_1),\ldots,f(y_n))$. Next, we define the orthogonal block matrix

$$\hat{U} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c|c} \mathbb{1}_n & \mathbb{1}_n & \mathbb{0} \\ \hline -\mathbb{1}_n & \mathbb{1}_n & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \sqrt{2}\end{array}\right). \quad (6.49)$$

We have the identity

$$\hat{U}^t\left(\begin{array}{c|c|c} A_n(\boldsymbol{x};\boldsymbol{y}) & B_n(\boldsymbol{x};\boldsymbol{y}) & f(\boldsymbol{x})^t \\ \hline B_n(\boldsymbol{x};\boldsymbol{y}) & A_n(\boldsymbol{x};\boldsymbol{y}) & f(\boldsymbol{x})^t \\ \hline f(\boldsymbol{y}) & f(\boldsymbol{y}) & f(0)\end{array}\right)\hat{U} \\ = \left(\begin{array}{c|c|c} A_n(\boldsymbol{x};\boldsymbol{y})-B_n(\boldsymbol{x};\boldsymbol{y}) & \mathbb{0} & \boldsymbol{0}^t \\ \hline \mathbb{0} & A_n(\boldsymbol{x};\boldsymbol{y})+B_n(\boldsymbol{x};\boldsymbol{y}) & \sqrt{2}f(\boldsymbol{x})^t \\ \hline \boldsymbol{0} & \sqrt{2}f(\boldsymbol{y}) & f(0)\end{array}\right). \quad (6.50)$$

Taking the determinant on both sides of this equality leads to

$$\hat{\Upsilon}_{2n+1}(\boldsymbol{x},\boldsymbol{y}) = \det(A_n(\boldsymbol{x};\boldsymbol{y})-B_n(\boldsymbol{x};\boldsymbol{y})) \\ \times\det\left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y})+B_n(\boldsymbol{x};\boldsymbol{y}) & \sqrt{2}f(\boldsymbol{x})^t \\ \hline \sqrt{2}f(\boldsymbol{y}) & f(0)\end{array}\right). \quad (6.51)$$

Furthermore, we note that

$$\det\left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y})+B_n(\boldsymbol{x};\boldsymbol{y}) & \sqrt{2}f(\boldsymbol{x})^t \\ \hline \sqrt{2}f(\boldsymbol{y}) & f(0)\end{array}\right) = \frac{1}{2}\det\left(\begin{array}{c|c} A_n(\boldsymbol{x};\boldsymbol{y})+B_n(\boldsymbol{x};\boldsymbol{y}) & 2f(\boldsymbol{x})^t \\ \hline 2f(\boldsymbol{y}) & 2f(0)\end{array}\right). \quad (6.52)$$

On the right-hand side, we recognise the determinant of the $(n+1)\times(n+1)$ matrix $A_{n+1}(\boldsymbol{x},0;\boldsymbol{y},0)+B_{n+1}(\boldsymbol{x},0;\boldsymbol{y},0)$:

$$\hat{\Upsilon}_{2n+1}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\det(A_n(\boldsymbol{x};\boldsymbol{y})-B_n(\boldsymbol{x};\boldsymbol{y})) \times \det(A_{n+1}(\boldsymbol{x},0;\boldsymbol{y},0)+B_{n+1}(\boldsymbol{x},0;\boldsymbol{y},0)). \quad (6.53)$$

The factorisation (6.32) follows from the substitution of this expression into (6.47) and the identities (6.43). □

6.1.2 Uniformisation variables

Let

$$w(x) = \frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\frac{\vartheta_1(x)^2}{\vartheta_1(x-\eta)\vartheta_1(x+\eta)}. \quad (6.54)$$

We shall express the functions $\Xi_{2n}(\boldsymbol{x};\boldsymbol{y}), \bar{\Xi}_{2n}(\boldsymbol{x};\boldsymbol{y})$ in terms of two multivariate polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$, $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$, where

$$\boldsymbol{w} = (w(x_1),\dots,w(x_n)), \qquad \boldsymbol{w}' = (w(y_1),\dots,w(y_n)). \quad (6.55)$$

The polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$, $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ also depend polynomially on the X,Y parameters defined in (3.17). We refer to the change of variables from $\boldsymbol{x},\boldsymbol{y},\eta,\omega$ to $\boldsymbol{w},\boldsymbol{w}',X,Y$ as *uniformisation*, and to $\boldsymbol{w},\boldsymbol{w}',X,Y$ as the *uniformisation variables.*

As preliminary steps toward uniformisation, we discuss several properties of $w(x)$ and X,Y. These properties include the evaluations of $w(x)$ at special points and the behaviour of $w(x)$ and X,Y under modular transformations.

Evaluation at special points

Lemma 6.1.3. *We have*

$$w(x) - w(y) = -\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\frac{\vartheta_1(\eta)^2\vartheta_1(x \pm y)}{\vartheta_1(x \pm \eta)\vartheta_1(y \pm \eta)}. \tag{6.56}$$

Proof. The proof is a straightforward application of Weierstrass' identity for the Jacobi theta functions (A.9). □

In the next two lemmas, we evaluate $w(x)$ at special points. First, we focus on the half-periods

$$\gamma_1 = 0, \qquad \gamma_2 = \frac{\pi}{2}, \qquad \gamma_3 = \frac{\pi}{2} + \frac{\pi\omega}{2}, \qquad \gamma_4 = \frac{\pi\omega}{2}. \tag{6.57}$$

Lemma 6.1.4. *We have*

$$w(\gamma_1) = 0, \qquad w(\gamma_2) = \frac{4}{X^2 - Y^2}, \tag{6.58}$$

$$w(\gamma_3) = \frac{X+Y}{X-Y}, \qquad w(\gamma_4) = \frac{X-Y}{X+Y}. \tag{6.59}$$

Proof. The equality $w(\gamma_1) = 0$ is immediate. For $i = 2, 3, 4$, we use the transformations of the Jacobi theta functions under half-period shifts of Proposition A.1.3 to obtain

$$w(\gamma_i) = \frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\left(\frac{\vartheta_i}{\vartheta_i(\eta)}\right)^2. \tag{6.60}$$

Using Lemma A.1.5, it is straightforward to write these expressions in terms of $X + Y$ and $X - Y$. □

Next, we consider the shifted half-periods

$$\beta_1 = \eta, \quad \beta_2 = \frac{\pi}{2} + \eta, \quad \gamma_3 = \frac{\pi}{2} + \frac{\pi\omega}{2} + \eta, \quad \gamma_4 = \frac{\pi\omega}{2} + \eta. \tag{6.61}$$

Lemma 6.1.5. *We have*

$$w(\beta_1) = \infty, \qquad w(\beta_2) = \frac{X^2 - Y^2}{2(X^2 + Y^2 - 2)}, \tag{6.62}$$

$$w(\beta_3) = \frac{X-Y}{(X+Y)(1-XY)}, \quad w(\beta_4) = \frac{X+Y}{(X-Y)(1+XY)}. \tag{6.63}$$

Proof. The equality $w(\beta_1) = \infty$ is immediate. To evaluate $w(\beta_i)$, $i = 2, 3, 4$, we note that

$$w(x)w(x+\eta) = \left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^2 \frac{\vartheta_1(x)\vartheta_1(x+\eta)}{\vartheta_1(x-\eta)\vartheta_1(x+2\eta)}. \tag{6.64}$$

Setting $x = \gamma_i$, we find

$$w(\gamma_i)w(\beta_i) = \left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^2 \frac{\vartheta_i}{\vartheta_i(2\eta)}. \tag{6.65}$$

We use Lemma 6.1.4 and the identities given in Lemma A.1.6 to express the left-hand and the right-hand sides, respectively, in terms of X and Y. □

Modular properties The uniformisation variables $w(x)$, X, Y have a simple transformation behaviour under the action of the modular group:

Lemma 6.1.6. *We have*

$$TX = X, \qquad TY = -Y, \tag{6.66}$$

$$SX = \frac{X+Y+2}{X-Y}, \qquad SY = \frac{X+Y-2}{X-Y}, \tag{6.67}$$

and

$$(Tw)(x) = w(x), \qquad (Sw)(x) = \frac{X-Y}{2}w(x). \tag{6.68}$$

Proof. Combining Lemma A.1.5 and the transformation properties of the Jacobi theta functions (A.30) and (A.31), we find

$$T(X+Y) = T\left(\frac{2\vartheta_3\vartheta_2(\eta)}{\vartheta_2\vartheta_3(\eta)}\right) = \frac{2\vartheta_4\vartheta_2(\eta)}{\vartheta_2\vartheta_4(\eta)} = X - Y, \tag{6.69}$$

$$T(X-Y) = T\left(\frac{2\vartheta_4\vartheta_2(\eta)}{\vartheta_2\vartheta_4(\eta)}\right) = \frac{2\vartheta_3\vartheta_2(\eta)}{\vartheta_2\vartheta_3(\eta)} = X + Y. \tag{6.70}$$

Likewise, we have

$$S(X+Y) = S\left(\frac{2\vartheta_3\vartheta_2(\eta)}{\vartheta_2\vartheta_3(\eta)}\right) = \frac{2\vartheta_3\vartheta_4(\eta)}{\vartheta_4\vartheta_3(\eta)} = \frac{2(X+Y)}{X-Y}, \tag{6.71}$$

$$S(X-Y) = S\left(\frac{2\vartheta_4\vartheta_2(\eta)}{\vartheta_2\vartheta_4(\eta)}\right) = \frac{2\vartheta_2\vartheta_4(\eta)}{\vartheta_4\vartheta_2(\eta)} = \frac{4}{X-Y}. \tag{6.72}$$

Taking linear combinations of these equalities leads to (6.66). The proof of (6.68) is similar. □

6.1.3 Uniformisation of $\Xi_n, \bar{\Xi}_n$

We now address the problem of the uniformisation of $\Xi_n, \bar{\Xi}_n$. To this end, we note that $w(x)$ is an even elliptic function with periods $\pi, \pi\omega$ and simple poles at $x \equiv \pm\eta \ (\mathrm{mod}\,\pi, \pi\omega)$. By classical theory, it generates the field of even elliptic functions with these periods [158]. In particular, each elliptic function of x with poles at $x \equiv \pm\eta \ (\mathrm{mod}\,\pi, \pi\omega)$ is a polynomial in $w(x)$.

Auxiliary functions The first step of the uniformisation is to express $\mathbb{h}(x,y)$, $\mathbb{g}(x,y)$ and $\mathbb{g}(x,y)$ in terms of the polynomials

$$\begin{aligned} h(w,w') = 1 &- (X^2+Y^2+2)ww' \\ &+ (X^2-Y^2)ww'(w+w') + (1-X^2)(1-Y^2)w^2w'^2, \end{aligned} \tag{6.73}$$

and

$$g(w,w') = 1 - \frac{1}{2}(X^2+Y^2-2)ww', \tag{6.74}$$

$$\begin{aligned} \bar{g}(w,w') = X^2+Y^2 &- (X^2-Y^2)(w+w') \\ &+ (X^2+Y^2-2X^2Y^2)ww'. \end{aligned} \tag{6.75}$$

Lemma 6.1.7. *We have*

$$\mathbb{h}(x,y) = \frac{((\vartheta_1(x\pm\eta)\vartheta_1(y\pm\eta))^2}{\vartheta_1(\eta)^4} h(w(x),w(y)). \tag{6.76}$$

Proof. We consider the function

$$\hat{\mathbb{h}}(x,y) = \frac{\vartheta_1(\eta)^4 \mathbb{h}(x,y)}{(\vartheta_1(x\pm\eta)\vartheta_1(y\pm\eta))^2}. \tag{6.77}$$

With respect to both x and y, it is an even elliptic function with poles of order 2 at $x, y \equiv \pm\eta \ (\mathrm{mod}\,\pi, \pi\omega)$. Hence, it can be written as a quadratic polynomial in both $w(x)$ and $w(y)$. Using the symmetry under the exchange of x, y, we find

$$\begin{aligned} \hat{\mathbb{h}}(x,y) = a_0 + a_1 w(x)w(y) + a_2(w(x)+w(y)) + a_3 w(x)w(y)(w(x)+w(y)) \\ + a_4(w(x)+w(y))^2 + a_5 w(x)^2 w(y)^2. \end{aligned} \tag{6.78}$$

The coefficients a_i, $i = 1, \ldots, 5$, are independent of x, y. We determine them through specialisations of x, y. First, setting $y = 0$, we find

$$1 = a_0 + a_2 w(x) + a_4 w(x)^2. \tag{6.79}$$

A series expansion of the right-hand side in x leads to

$$a_0 = 1, \qquad a_2 = 0, \qquad a_4 = 0. \tag{6.80}$$

Second, we observe that $\mathbb{h}(x, x+\eta) = 0$. Hence, we obtain

$$0 = 1 + a_1 w(x) w(x+\eta) + a_3 w(x) w(x+\eta)(w(x) + w(x+\eta)) \\ + a_5 w(x)^2 w(x+\eta)^2. \tag{6.81}$$

The specialisations $x = \gamma_i$, $i = 2, \ldots, 4$, lead to a system of linear equations for a_1, a_3, a_5 whose coefficients are rational functions in X, Y by Lemmas 6.1.4 and 6.1.5. Solving this system is straightforward and yields

$$a_1 = -(2 + X^2 + Y^2), \quad a_3 = X^2 - Y^2, \quad a_5 = (1 - X^2)(1 - Y^2). \tag{6.82}$$

The substitution of (6.80) and (6.82) into (6.78) leads to $\hat{\mathbb{h}}(x, y) = h(w(x), w(y))$. The lemma follows from this equality and (6.77). $\square$

Lemma 6.1.8. *We have*

$$\mathbb{g}(x, y) = -\frac{2\vartheta_3(x)\vartheta_4(x)\vartheta_3(y)\vartheta_4(y)}{\vartheta_1(\eta)^2\vartheta_3\vartheta_4} \\ \times \vartheta_1(x \pm \eta)\vartheta_1(y \pm \eta) g(w(x), w(y)) \tag{6.83}$$

and

$$\bar{\mathbb{g}}(x, y) = \left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_1(\eta)\vartheta_2(\eta)}\right)^2 \frac{\vartheta_1(x)\vartheta_2(x)\vartheta_1(y)\vartheta_2(y)}{\vartheta_1(\eta)^2\vartheta_3\vartheta_4} \\ \times \vartheta_1(x \pm \eta)\vartheta_1(y \pm \eta)\bar{g}(w(x), w(y)). \tag{6.84}$$

Proof. We focus on the proof of (6.84), as the other one is similar.

First, one checks that $\bar{\mathbb{g}}(x, \gamma_1) = \bar{\mathbb{g}}(x, \gamma_2) = 0$. Using the factorisation properties of theta functions, we may write

$$\bar{\mathbb{g}}(x, y) = \vartheta_1(x)\vartheta_2(x)\vartheta_1(y)\vartheta_2(y)\bar{\mathbb{G}}(x, y), \tag{6.85}$$

where $\bar{\mathbb{G}}(x,y)$ is an even theta functions of order 2 with respect to x and y, obeying $\bar{\mathbb{G}}(x,y)=\bar{\mathbb{G}}(y,x)$. Using (6.27) and the duplication formulas (A.8), we obtain the specialisation

$$\bar{\mathbb{G}}(x,x+\eta)=\frac{4\vartheta_3(\eta)\vartheta_4(\eta)}{(\vartheta_2\vartheta_3\vartheta_4)^2}\vartheta_3(x)\vartheta_4(x)\vartheta_3(x+\eta)\vartheta_4(x+\eta). \tag{6.86}$$

In particular, we find

$$\bar{\mathbb{G}}(\gamma_3,\beta_3)=\bar{\mathbb{G}}(\gamma_4,\beta_4)=0, \qquad \bar{\mathbb{G}}(\gamma_1,\beta_1)=\frac{4(\vartheta_3(\eta)\vartheta_4(\eta))^2}{\vartheta_2^2\vartheta_3\vartheta_4}, \tag{6.87}$$

which we use below.

Second, one checks that the ratio

$$\frac{\vartheta_1(\eta)^4\bar{\mathbb{G}}(x,y)}{\vartheta_1(x\pm\eta)\vartheta_1(y\pm\eta)} \tag{6.88}$$

is an even elliptic function in both x and y, with simple poles at $x,y\equiv\pm\eta \pmod{\pi,\pi\omega}$. Using its symmetry under the exchange of x and y, we write

$$\frac{\vartheta_1(\eta)^4\bar{\mathbb{G}}(x,y)}{\vartheta_1(x\pm\eta)\vartheta_1(y\pm\eta)}=\bar{a}_0+\bar{a}_1w(x)w(y)+\bar{a}_2(w(x)+w(y)). \tag{6.89}$$

The coefficients $\bar{a}_0,\bar{a}_1,\bar{a}_2$ are independent of x and y. To find these coefficients, we set $y=x+\eta$ and obtain

$$\begin{aligned}\frac{\vartheta_1(\eta)^4\bar{\mathbb{G}}(x,x+\eta)}{\vartheta_1(x\pm\eta)\vartheta_1(x)\vartheta_1(x+2\eta)}&\\ =\bar{a}_0+\bar{a}_1w(x)w(x+\eta)+\bar{a}_2(w(x)+w(x+\eta)).\end{aligned} \tag{6.90}$$

Setting $x=\gamma_3,\gamma_4$, the left-hand side of this equality vanishes by (6.87). We thus obtain two linear equations for $\bar{a}_0,\bar{a}_1,\bar{a}_2$. We express their coefficients in terms of X,Y, using Lemmas 6.1.4 and 6.1.5. Solving for $\bar{a}_0$ and $\bar{a}_2$ yields

$$\bar{a}_0=-\frac{X^2+Y^2}{X^2-Y^2}\bar{a}_1, \qquad \bar{a}_1=\frac{X^2+Y^2-2X^2Y^2}{X^2-Y^2}\bar{a}_2. \tag{6.91}$$

To find $\bar{a}_2$, we note that both sides of (6.90) have a simple pole at $x=0$. Hence, the corresponding residues must be equal. Using (6.87), we find

$$\bar{a}_2=-\frac{4\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_2^2}=-\frac{1}{\vartheta_3\vartheta_4}\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_2(\eta)}\right)^2(X^2-Y^2). \tag{6.92}$$

The coefficients (6.91) and (6.92) determine $\bar{\mathbb{G}}(x,y)$ in terms of $g(w(x), w(y))$. The expression (6.84) follows from (6.85). □

Uniformisation Let us define $G_0 = \bar{G}_0 = 1$, and for $n \geq 1$,

$$G_{2n}(\boldsymbol{w};\boldsymbol{w}') = \frac{\prod_{i,j=1}^n h(w_i,w_j')}{\prod_{1\leqslant i<j\leqslant n}(w_i-w_j)(w_i'-w_j')} \det_{i,j=1}^n \left(\frac{g(w_i,w_j')}{h(w_i,w_j')}\right), \quad (6.93)$$

$$\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}') = \frac{\prod_{i,j=1}^n h(w_i,w_j')}{\prod_{1\leqslant i<j\leqslant n}(w_i-w_j)(w_i'-w_j')} \det_{i,j=1}^n \left(\frac{\bar{g}(w_i,w_j')}{h(w_i,w_j')}\right). \quad (6.94)$$

Both $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ are polynomials in $\boldsymbol{w},\boldsymbol{w}'$ and X,Y. Clearly, they are separately symmetric in $\boldsymbol{w}$ and $\boldsymbol{w}'$.

We now use Lemmas 6.1.7 and 6.1.8 to express the functions $\Xi_n(\boldsymbol{x};\boldsymbol{y})$, $\bar{\Xi}_n(\boldsymbol{x};\boldsymbol{y})$ in terms of $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$.

Proposition 6.1.9. *Let $n \geqslant 0$ be an integer, and $w_i = w(x_i)$, $w_i' = w(y_i)$, $i = 1,\dots,n$, then*

$$\Xi_{2n}(\boldsymbol{x};\boldsymbol{y}) = \frac{1}{\vartheta_1(\eta)^{2n^2}}\left(-\frac{2}{\vartheta_3\vartheta_4}\right)^n\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^{n(n-1)} \quad (6.95)$$
$$\times\left(\prod_{i=1}^n\left(\vartheta_1(x_i\pm\eta)\vartheta_1(y_i\pm\eta)\right)^n\vartheta_3(x_i)\vartheta_4(x_i)\vartheta_3(y_i)\vartheta_4(y_i)\right)G_{2n}(\boldsymbol{w};\boldsymbol{w}'),$$

and

$$\bar{\Xi}_{2n}(\boldsymbol{x};\boldsymbol{y}) = \frac{1}{\vartheta_1(\eta)^{2n^2}}\left(\frac{\vartheta_2^2\vartheta_3\vartheta_4}{4}\right)^n\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^{n(n-1)} \quad (6.96)$$
$$\times\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_1(\eta)\vartheta_2(\eta)}\right)^{2n}\left(\prod_{i=1}^n\frac{(\vartheta_1(x_i\pm\eta)\vartheta_1(y_i\pm\eta))^n}{\vartheta_3(x_i)\vartheta_4(x_i)\vartheta_3(y_i)\vartheta_4(y_i)}\right)\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}').$$

Proof. The proof is a calculation, which we present for (6.96). First, we note that the difference property of Lemma 6.1.3 leads to

$$\bar{\mathbb{A}}(\boldsymbol{x}) = \frac{1}{\vartheta_1(\eta)^{n(n-1)}}\left(-\frac{\vartheta_3\vartheta_4}{\vartheta_3(\eta)\vartheta_4(\eta)}\right)^{n(n-1)/2}$$
$$\times\prod_{i=1}^n\left(\vartheta_1(x_i\pm\eta)^{n-1}\vartheta_1(2x_i)\right)\prod_{1\leqslant i<j\leqslant n}(w_j-w_i). \quad (6.97)$$

We combine this equality with Lemma 6.1.7 and obtain

$$\frac{\prod_{i,j=1}^n \mathbb{h}(x_i,y_j)}{\bar{\mathbb{A}}(\boldsymbol{x})\bar{\mathbb{A}}(\boldsymbol{y})} = \frac{1}{\vartheta_1(\eta)^{2n(n+1)}}\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^{n(n-1)} \quad (6.98)$$
$$\times\prod_{i=1}^n\frac{\vartheta_1(x_i\pm\eta)^{n+1}\vartheta_1(y_i\pm\eta)^{n+1}}{\vartheta_1(2x_i)\vartheta_1(2y_i)}\frac{\prod_{i,j=1}^n h(w_i,w_j')}{\prod_{1\leqslant i<j\leqslant n}(w_j-w_i)(w_j'-w_i')}.$$

Second, by Lemmas 6.1.7 and 6.1.8, we find

$$\det_{i,j=1}^{n}\left(\frac{\mathbb{g}(x_i,y_j)}{\mathbb{h}(x_i,y_j)}\right) = \left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_2(\eta)}\right)^{2n}\left(\frac{1}{\vartheta_3\vartheta_4}\right)^{n}$$
$$\times \prod_{i=1}^{n}\frac{\vartheta_1(x_i)\vartheta_2(x_i)\vartheta_1(y_i)\vartheta_2(y_i)}{\vartheta_1(x_i\pm\eta)\vartheta_1(y_i\pm\eta)}\det_{i,j=1}^{n}\left(\frac{g(w_i,w_j')}{h(w_i,w_j')}\right). \quad (6.99)$$

The relation (6.96) follows from (6.98), (6.99) and the duplication identity (A.8). □

6.1.4 Uniformisation of $\Upsilon_N^{(a)}$

Propositions 6.1.2 and 6.1.9 complete the uniformisation of

$$\Upsilon_{2n}^{(a)}(\boldsymbol{x},-\boldsymbol{x};\boldsymbol{y},-\boldsymbol{y}) \qquad \text{and} \qquad \Upsilon_{2n+1}^{(a)}(\boldsymbol{x},-\boldsymbol{x},0;\boldsymbol{y},-\boldsymbol{y},0), \quad (6.100)$$

for $a=2$. We now turn to the uniformisation of these functions for the cases $a=3,4$. To this end, we apply modular transformations to $\Upsilon_N^{(2)}$, as prescribed by Lemma 6.1.1. We introduce the notations

$$(X^{(2)},Y^{(2)}) = (X,Y), \quad (6.101a)$$
$$(X^{(3)},Y^{(3)}) = TS(X,Y), \quad (6.101b)$$
$$(X^{(4)},Y^{(4)}) = S(X,Y), \quad (6.101c)$$

where T,S are the generators of the modular group (6.10). The explicit expressions of (6.101) follow from Lemma 6.1.6. We apply the same modular transformations to

$$G_{2n}(\boldsymbol{w};\boldsymbol{w}') = G_{2n}\left(\boldsymbol{w};\boldsymbol{w}';X^{(2)},Y^{(2)}\right), \quad (6.102)$$
$$\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}') = \bar{G}_{2n}\left(\boldsymbol{w};\boldsymbol{w}';X^{(2)},Y^{(2)}\right). \quad (6.103)$$

Lemma 6.1.10. *We have*

$$(TSG_{2n})(\boldsymbol{w};\boldsymbol{w}') = G_{2n}\left(\frac{X+Y}{2}\boldsymbol{w};\frac{X+Y}{2}\boldsymbol{w}';X^{(3)},Y^{(3)}\right), \quad (6.104a)$$
$$(TS\bar{G}_{2n})(\boldsymbol{w};\boldsymbol{w}') = \bar{G}_{2n}\left(\frac{X+Y}{2}\boldsymbol{w};\frac{X+Y}{2}\boldsymbol{w}';X^{(3)},Y^{(3)}\right), \quad (6.104b)$$
$$(SG_{2n})(\boldsymbol{w};\boldsymbol{w}') = G_{2n}\left(\frac{X-Y}{2}\boldsymbol{w};\frac{X-Y}{2}\boldsymbol{w}';X^{(4)},Y^{(4)}\right), \quad (6.104c)$$
$$(S\bar{G}_{2n})(\boldsymbol{w};\boldsymbol{w}') = \bar{G}_{2n}\left(\frac{X-Y}{2}\boldsymbol{w};\frac{X-Y}{2}\boldsymbol{w}';X^{(4)},Y^{(4)}\right). \quad (6.104d)$$

From their expressions, it is clear that the functions (6.104) are polynomials in $\boldsymbol{w}$ and $\boldsymbol{w}'$. Additionally, direct examination reveals that, for $n \geq 2$, they are rational functions of X, Y.

We define

$$G_{2n}^{(2)}(\boldsymbol{w};\boldsymbol{w}') = G_{2n}\left(\boldsymbol{w};\boldsymbol{w}'\right), \qquad \bar{G}_{2n}^{(2)}(\boldsymbol{w};\boldsymbol{w}') = \bar{G}_{2n}\left(\boldsymbol{w};\boldsymbol{w}'\right), \tag{6.105}$$

and

$$G_{2n}^{(3)}(\boldsymbol{w};\boldsymbol{w}') = \left(\frac{X+Y}{2}\right)^{n(n-1)}(TSG_{2n})(\boldsymbol{w};\boldsymbol{w}'), \tag{6.106a}$$

$$\bar{G}_{2n}^{(3)}(\boldsymbol{w};\boldsymbol{w}') = \left(\frac{X+Y}{2}\right)^{n(n+1)}(TS\bar{G}_{2n})(\boldsymbol{w};\boldsymbol{w}'), \tag{6.106b}$$

$$G_{2n}^{(4)}(\boldsymbol{w};\boldsymbol{w}') = \left(\frac{X-Y}{2}\right)^{n(n-1)}(SG_{2n})(\boldsymbol{w};\boldsymbol{w}'), \tag{6.106c}$$

$$\bar{G}_{2n}^{(4)}(\boldsymbol{w};\boldsymbol{w}') = \left(\frac{X-Y}{2}\right)^{n(n+1)}(S\bar{G}_{2n})(\boldsymbol{w};\boldsymbol{w}'). \tag{6.106d}$$

The prefactors in (6.106) ensure that $G_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}')$ are polynomials in X, Y of minimal degree, for $a = 3, 4$.

Proposition 6.1.11. *Let $n \geqslant 0$ be an integer, and $w_i = w(x_i)$, $w_i' = w(y_i)$, $i = 1, \ldots, n$, then*

$$\begin{aligned}\Upsilon_{2n}^{(a)}(\boldsymbol{x},-\boldsymbol{x};\boldsymbol{y},-\boldsymbol{y}) &= \frac{2^n\vartheta_a^{2n}}{\vartheta_1(\eta)^{4n^2}}\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^{2n^2} \\ &\times\left(\prod_{i=1}^{n}\vartheta_1(x_i\pm\eta)\vartheta_1(y_i\pm\eta)\right)^{2n} G_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}')\bar{G}_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}'),\end{aligned} \tag{6.107}$$

and

$$\begin{aligned}\Upsilon_{2n+1}^{(a)}(\boldsymbol{x},-\boldsymbol{x},0;\boldsymbol{y},-\boldsymbol{y},0) &= \frac{2^{n+1}\vartheta_a(\eta)\vartheta_a^{2n}}{\vartheta_1(\eta)^{4n^2-1}}\left(\frac{\vartheta_3(\eta)\vartheta_4(\eta)}{\vartheta_3\vartheta_4}\right)^{2n(n+1)} \\ \times\left(\prod_{i=1}^{n}\vartheta_1(x_i\pm\eta)\vartheta_1(y_i\pm\eta)\right)^{2n+1} &G_{2(n+1)}^{(a)}(\boldsymbol{w},0;\boldsymbol{w}',0)\bar{G}_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}').\end{aligned}$$

Proof. For $a = 2$, the proposition follows from Propositions 6.1.2 and 6.1.9. The cases $a = 3, 4$ follow from $a = 2$ and the transformation properties under the action of the generators of the modular group, given in Lemmas 6.1.1, 6.1.6 and 6.1.10. ☐

6.1.5 Properties of G_{2n}, $\bar{G}_{2n}$

In this section, we discuss several properties of the polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$. First, we present a condition under which they are symmetric in all of their variables. Second, by examining the limit $X = 1$, $Y = \zeta$, we find that the polynomials are generalisations of those of Rosengren and Zinn-Justin, introduced in Section 2.2.4. Third, we investigate the homogeneous limit of the polynomials. Finally, we relate their trigonometric limit $Y = 0$ to weighted enumerations of symmetry classes of ASMs.

Symmetry As previously mentioned, it is clear from their definition that both $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ are separately symmetric in $\boldsymbol{w}$ and $\boldsymbol{w}'$. In this paragraph, we study the conditions under which the polynomials are symmetric in all of their variables. The techniques that we implement mirror those of Section 2.2.4.

First, we present an alternate definition for $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ based on determinant condensation [63].

Lemma 6.1.12. *For each $n \geqslant 2$, and $P_{2n} = G_{2n}, \bar{G}_{2n}$, we have*

$$P_{2n}(\boldsymbol{w};\boldsymbol{w}') = \frac{1}{p(w_n,w_n')^{n-2}} \frac{\prod_{i,j=1}^{n-1} h(w_i,w_j')}{\prod_{1\leqslant i<j\leqslant n-1}(w_j-w_i)(w_j'-w_i')} \times \det_{i,j=1}^{n-1}\left(\frac{P_4(w_i,w_n;w_j',w_n')}{h(w_i,w_j')}\right). \quad (6.108)$$

Proof. We focus on the proof for $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$. Let us consider the condensation identity (2.77) for $A = (g(w_i,w_j')/h(w_i,w_j'))_{i,j=1}^n$. An explicit calculation yields

$$\det\begin{pmatrix} a_{ij} & a_{in} \\ a_{nj} & a_{nn}\end{pmatrix} = \frac{(w_n-w_i)(w_n'-w_j')}{h(w_i,w_n')h(w_n,w_j')h(w_n,w_n')} \frac{G_4(w_i,w_n;w_j',w_n')}{h(w_i,w_j')},$$

and, therefore,

$$\det_{i,j=1}^{n}\left(\frac{g(w_i,w_j')}{h(w_i,w_j')}\right)=\frac{1}{g(w_n,w_n')^{n-2}} \times \frac{\prod_{i=1}^{n-1}(w_n-w_i)(w_n'-w_i')}{h(w_n,w_n')\prod_{i=1}^{n-1}h(w_i,w_n')h(w_n,w_i')}\det_{i,j=1}^{n-1}\left(\frac{G_4(w_i,w_n;w_j',w_n')}{h(w_i,w_j')}\right). \tag{6.109}$$

We insert this expression into (6.93). Simplifying the prefactors leads to (6.108). □

Second, Lemma 6.1.12 allows us to infer a symmetry property of $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ from their expressions for $n=2$.

Proposition 6.1.13. *For each $n\geqslant 1$, and $X^2=1$ or $Y^2=1$, the polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ are symmetric in $\boldsymbol{w},\boldsymbol{w}'$.*

Proof. We focus on the proof of the symmetry of $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$. For $n=1$, the proposition trivially holds since $G_2(w;w')=g(w,w')=g(w',w)$. For $n\geqslant 2$, we use that $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ is separately symmetric in $\boldsymbol{w}$ and $\boldsymbol{w}'$. To establish the full symmetry, it is sufficient to show that it is invariant under the exchange of w_n and w_n'. This invariance holds for $n=2$, since

$$\begin{aligned}&G_4(w_1,w_2;w_1',w_2')-G_4(w_1,w_2';w_2,w_1')\\&=(1-X^2)(1-Y^2)(w_1-w_1')(w_2-w_2')g(w_1,w_2')g(w_1',w_2)=0.\end{aligned} \tag{6.110}$$

For $n>2$, the invariance follows from Lemma 6.1.12. □

Relation to the polynomials of Rosengren and Zinn-Justin We now consider the case

$$X=1,\qquad Y=\zeta, \tag{6.111}$$

and show that $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ are specialisations of the symmetric polynomials $H_{2n}(\boldsymbol{w},\boldsymbol{w}')$ introduced in Section 2.2.4.

By Proposition 6.1.13, for the choice (6.111), $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ are symmetric polynomials in $\boldsymbol{w},\boldsymbol{w}'$. Their building blocks are

$$h(w,w')=1-(\zeta^2+3)ww'-(\zeta^2-1)ww'(w+w') \tag{6.112}$$

and

$$g(w, w') = 1 - \frac{1}{2}(\zeta^2 - 1)ww', \tag{6.113}$$
$$\bar{g}(w, w') = \zeta^2 + 1 + (\zeta^2 - 1)(w + w' + ww'). \tag{6.114}$$

To express the relation between $G_{2n}(\boldsymbol{w}; \boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w}; \boldsymbol{w}')$ and $H_{2n}(\boldsymbol{w}, \boldsymbol{w}')$, we recall the notations

$$J_2 = -\frac{1}{2}, \qquad J_3 = \frac{1}{1+\zeta}, \qquad J_4 = \frac{1}{1-\zeta}. \tag{6.115}$$

Proposition 6.1.14. *For $X = 1$, $Y = \zeta$, we have*

$$G_{2n}(w_1, \ldots, w_{n-1}, 0; \boldsymbol{w}') = H_{2n}(w_1, \ldots, w_{n-1}, J_2, \boldsymbol{w}'), \tag{6.116}$$
$$\bar{G}_{2n}(\boldsymbol{w}; \boldsymbol{w}') = H_{2(n+1)}(\boldsymbol{w}, J_3, \boldsymbol{w}', J_4). \tag{6.117}$$

Proof. It is straightforward to check (6.116) and (6.117) for $n = 0, 1$ and $n = 0$, respectively. Then, by explicit calculation, we find

$$G_4(w_1, 0; w_1', w_2') = H_4(w_1, J_2, w_1', w_2), \tag{6.118}$$
$$\bar{g}(w_1, w_1') = H_4(w_1, J_3, w_1', J_4). \tag{6.119}$$

The proposition follows from these equalities and Lemma 6.1.12. □

Homogeneous limit The homogeneous limit is achieved by setting all inhomogeneity parameters to zero. Here, given the parameterisation (6.54), this limit corresponds to

$$\boldsymbol{w} = \boldsymbol{w}' = \boldsymbol{0} = (0, \ldots, 0). \tag{6.120}$$

For this choice, $G_{2n}(\boldsymbol{w}; \boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w}; \boldsymbol{w}')$ become polynomials in X, Y. Table 6.1 provides these polynomials for $n = 1, 2, 3$.

The examples of Table 6.1 suggest that $2^{n-1}G_{2n}(\boldsymbol{0}; \boldsymbol{0})$ and $\bar{G}_{2n}(\boldsymbol{0}; \boldsymbol{0})$ have positive integer coefficients. The positivity hints at a possible relation to combinatorics, similar to what Hietala found for the $H_{2n}(\boldsymbol{w}, \boldsymbol{w}')$ [82, 83]. For our case, however, such a relation remains to be discovered. Note also that establishing the positivity in itself is not a trivial task. Indeed, this question is still open, even for $X = 1, Y = \zeta$ [82, 83].

n	$2^{n-1}G_{2n}(\mathbf{0};\mathbf{0})$
1	1
2	X^2+Y^2+6
3	$X^6+12X^4+70X^2+X^4Y^2+28X^2Y^2+X^2Y^4+70Y^2+12Y^4+Y^6+60$

n	$\bar{G}_{2n}(\mathbf{0};\mathbf{0})$
1	X^2+Y^2
2	$X^6+2X^4+X^4Y^2+8X^2Y^2+X^2Y^4+2Y^4+Y^6$
3	$X^{12}+6X^{10}+X^{10}Y^2+13X^8+16X^8Y^2+X^8Y^4+6X^6+82X^6Y^2+30X^6Y^4+54X^4Y^2+2X^6Y^6+90X^4Y^4+54X^2Y^4+30X^4Y^6+82X^2Y^6+6Y^6+X^4Y^8+16X^2Y^8+13Y^8+X^2Y^{10}+6Y^{10}+Y^{12}$

Table 6.1: The homogeneous limits of the polynomials $2^{n-1}G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ for $n=1,2,3$.

The trigonometric limit In the trigonometric case $Y=0$, exact computations for sizes up to $n=4$, and preliminary analytical work seem to indicate that $G_{2n}^{(a)}(\mathbf{0};\mathbf{0})$ and $\bar{G}_{2n}^{(a)}(\mathbf{0};\mathbf{0})$ reduce to polynomials that appear in the weighted enumeration of ASMs (see Section 1.4). These polynomials possess explicit determinant formulas [114, 116].

The case $a=2$. We begin our discussion with the polynomials

$$G_{2n}(\mathbf{0};\mathbf{0};X,Y=0), \qquad \bar{G}_{2n}(\mathbf{0};\mathbf{0};X,Y=0). \tag{6.121}$$

To this end, we recall the t-enumerations of $N\times N$ ASMs $A(N;t)$, and of $(2n{+}1)\times(2n{+}1)$ vertically-symmetric ASMs $A_{\rm V}(2n+1;t)$, introduced in Section 1.4. We also need the polynomials $\tilde{A}_{\rm V}(2n;t)$, defined through the following factorisations [114]:

$$A(2n;t)=2A_{\rm V}(2n+1;t)\tilde{A}_{\rm V}(2n;t), \tag{6.122a}$$
$$A(2n+1;t)=A_{\rm V}(2n+1;t)\tilde{A}_{\rm V}(2n+2;t). \tag{6.122b}$$

Conjecture 6.1.15. *We have*

$$2^{n-1}G_{2n}(\mathbf{0};\mathbf{0};X,Y=0)=\tilde{A}_{\mathrm{V}}(2n;X^2), \tag{6.123}$$
$$\bar{G}_{2n}(\mathbf{0};\mathbf{0};X,Y=0)=X^{2n}A_{\mathrm{V}}(2n+1;X^2). \tag{6.124}$$

Note that the trigonometric case $Y=0$ can be realised by taking the limit $\omega\to\mathrm{i}\infty$ or, alternatively, $p\to 0$. Incidentally, the Jacobi theta functions have known series expansions in this limit (see Definition A.1.1). One could therefore prove Conjecture 6.1.15 by considering the leading terms in the serie expansions of $\Xi_{2n}(\mathbf{0};\mathbf{0})$, $\bar{\Xi}_{2n}(\mathbf{0};\mathbf{0})$ for $p\to 0$, in relation to Proposition 6.1.9. We expect this computation to require the use of the method of divided differences, as performed in [92]. The method should lead to determinant formulas for $G_{2n}(\mathbf{0};\mathbf{0};X,Y=0)$ and $\bar{G}_{2n}(\mathbf{0};\mathbf{0};X,Y=0)$, to be compared to those of $\tilde{A}_{\mathrm{V}}(2n;t)$ and $A_{\mathrm{V}}(2n+1;t)$ found in litterature [114, 116].

Assuming that Conjecture 6.1.15 holds, and combining it with (6.122) leads to the following result, useful for analysing the normalisation of the special eigenvector of the spin-1 XYZ Hamiltonian with the diagonal twist:

Proposition 6.1.16. *Assuming that Conjecture 6.1.15 holds, we have*

$$2^{n}G_{2n}(\mathbf{0};\mathbf{0};X,0)\bar{G}_{2n}(\mathbf{0};\mathbf{0};X,0)=X^{2n}A(2n;X^2), \tag{6.125}$$
$$2^{n}G_{2(n+1)}(\mathbf{0};\mathbf{0};X,0)\bar{G}_{2n}(\mathbf{0};\mathbf{0};X,0)=X^{2n}A(2n+1;X^2). \tag{6.126}$$

The cases $a=3,4$. We continue our discussion with the polynomials

$$G^{(a)}_{2n}(\mathbf{0},\mathbf{0};X,Y=0),\qquad \bar{G}^{(a)}_{2n}(\mathbf{0},\mathbf{0};X,Y=0),\qquad a=3,4. \tag{6.127}$$

We formulate a conjecture regarding their expression. We write it in terms of the polynomials $A^{(2)}_{\mathrm{UU}}(4n;t,y,z)$ and $\tilde{A}^{(2)}_{\mathrm{UU}}(4n;t)$ defined in (1.42) and (1.44), respectively. Recall that both of them appear in the weighted enumeration of double U-turn ASMs.

Conjecture 6.1.17. *For $a=3,4$, we have*

$$G^{(a)}_{2n}(\mathbf{0};\mathbf{0})=\tilde{A}^{(2)}_{\mathrm{UU}}(4n;X^2),\quad 2^n\bar{G}^{(a)}_{2n}(\mathbf{0};\mathbf{0})=A^{(2)}_{\mathrm{UU}}(4n;X^2,1,1). \tag{6.128}$$

The $G_{2n}^{(a)}(\boldsymbol{w},\boldsymbol{w}';X,Y)$, $\bar{G}_{2n}^{(a)}(\boldsymbol{w},\boldsymbol{w}';X,Y)$ are defined in terms of modular transformations of $G_{2n}(\boldsymbol{w},\boldsymbol{w}';X,Y)$ and $\bar{G}_{2n}(\boldsymbol{w},\boldsymbol{w}';X,Y)$. The proof of Conjecture 6.1.17 is expected to follow the same lines as for Conjecture 6.1.15. The first step is to apply the modular transformations to $\Xi_{2n}(\boldsymbol{x};\boldsymbol{y})$ and $\bar{\Xi}_{2n}(\boldsymbol{x};\boldsymbol{y})$. The second step consists in extracting the leading terms in their series expansion with respect to p, for $p\to 0$.

Let $A_{\mathrm{HT}}(2n;t)$ be the t-enumeration of $2n\times 2n$ half-turn symmetric ASMs, as defined in Section 1.4. Kuperberg has established the following factorisations [114]:

$$\frac{A_{\mathrm{HT}}(4n;t)}{A(2n;t)} = A_{\mathrm{UU}}^{(2)}(4n;t,1,1)\tilde{A}_{\mathrm{UU}}^{(2)}(4n;t), \tag{6.129a}$$

$$\frac{A_{\mathrm{HT}}(4n+2;t)}{A(2n+1;t)} = 2A_{\mathrm{UU}}^{(2)}(4n;t,1,1)\tilde{A}_{\mathrm{UU}}^{(2)}(4n+4;t). \tag{6.129b}$$

Assuming that Conjecture 6.1.17 holds, and combining it with these factorisations leads to the following result, useful for analysing the normalisation of the special eigenvector of the spin-1 XYZ Hamiltonian with the anti-diagonal and the mixed twist:

Proposition 6.1.18. *For $a=3,4$, assuming that Conjecture 6.1.17 holds, we have*

$$2^n G_{2n}^{(a)}(\mathbf{0};\mathbf{0})\bar{G}_{2n}^{(a)}(\mathbf{0};\mathbf{0}) = \frac{A_{\mathrm{HT}}(4n;X^2)}{A(2n;X^2)}, \tag{6.130}$$

$$2^{n+1} G_{2(n+1)}^{(a)}(\mathbf{0};\mathbf{0})\bar{G}_{2n}^{(a)}(\mathbf{0};\mathbf{0}) = \frac{A_{\mathrm{HT}}(4n+2;X^2)}{A(2n+1;X^2)}. \tag{6.131}$$

6.2 Special eigenvectors

In Chapter 3, we proved the existence of eigenvectors of the 18V and 41V transfer matrices with remarkably simple eigenvalues. In the homogeneous limit, these are related to zero-energy eigenstates of the spin-1 XYZ chain.

In this section, we exploit the relations between higher-spin elliptic models uncovered in Chapters 3 and 4 (see Figure 6.1 for a graphical summary). Doing so allows us to transpose the results obtained in Chapter 5 for the d10V model, to define the special eigenvectors of the 18V and 41V models. Through these definitions, we derive a number of properties that characterise the special eigenvectors implicitly. These include symmetry properties, exchange and reduction relations, and pseudo-periodicity properties with respect to the inhomogeneity parameters, as well as a conjecture regarding their analyticity with respect to the inhomogeneity parameters. These properties are of particular interest, as they pave the way for the analysis of the special eigenvectors via induction-style methods, such as that of Izergin and Korepin (see Chapter 2). Additionally, we compute a quadratic sum rule involving the special eigenvectors in terms of a determinant formula. Then, based on Proposition 3.2.3, we define a zero-energy eigenstate of the spin-1 XYZ chain. Notably, we express its square norm in terms of the polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$ and $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$.

$$
\begin{tikzcd}
\text{Spin-1 XYZ} & \text{41V} \arrow[l, "\text{Prop. 3.2.3}", "{[91]}"'] & & \text{d19V} \\
 & \text{18V} \arrow[u, "\text{Prop. 3.1.17}", "\text{Fusion [89]}"'] \arrow[rr, leftrightarrow, "\text{Prop. 4.2.13}", "\text{VF corresp.}"'] & & \text{d10V} \arrow[u, "\text{Prop. 4.1.28}", "\text{Fusion [95]}"'] \\
\text{Spin-1/2 XYZ} & \text{8V} \arrow[l, "\text{Prop. 1.3.1}", "{[13,28,29]}"'] \arrow[u, "\text{Fusion [89]}"'] \arrow[rr, leftrightarrow, "\text{VF corresp. [45,94]}"'] & & \text{d6V} \arrow[u, "\text{Fusion [95]}"']
\end{tikzcd}
$$

Figure 6.1: Summary of the relations between the elliptic integrable models studied in this thesis.

6.2.1 Eighteen- and forty-one-vertex models

We define the vectors

$$|\psi^R\rangle = \prod_{i=1}^{N} \vartheta_1(\lambda_i)^{-1} \prod_{1\le i<j\le N} \vartheta_1(\lambda_{ij}-\eta)\mathbf{V}_q\mathbf{P}\,|\varphi^R\rangle\,, \tag{6.132a}$$

$$\langle\psi^L| = \prod_{i=1}^{N} \vartheta_1(\lambda_i) \prod_{1\le i<j\le N} \vartheta_1(\lambda_{ji}-\eta)\,\langle\varphi^L|\,[\mathbf{V}_q\mathbf{P}]^{-1}, \tag{6.132b}$$

where $\langle\varphi^L|$ and $|\varphi^R\rangle$ are the eigenvectors of the d10V transfer matrix defined in Theorem 5.2.5, and $\mathbf{V}_q\mathbf{P}$ is the vertex-face transformation (4.177).

Proposition 6.2.1. *The vectors $|\psi^R\rangle$, $\langle\psi^L|$ are eigenvectors of $T^{(1)}(\lambda)$ and $T^{(2)}(\lambda)$, with eigenvalues*

$$\theta^{(1)}(\lambda)=0, \qquad \textit{and} \qquad \theta^{(2)}(\lambda) = -\mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta), \tag{6.133}$$

respectively. Moreover, these eigenvectors are unique, up to normalisation.

Proof. First, by Proposition 4.2.13, $|\psi^R\rangle$ and $\langle\psi^L|$ are eigenvectors of $T^{(1)}(\lambda)$ with eigenvalue $\theta^{(1)}(\lambda)=0$. By Proposition 3.1.17, $|\psi^R\rangle$ and $\langle\psi^L|$ are eigenvectors of $T^{(2)}(\lambda)$ with eigenvalue $\theta^{(2)}(\lambda) = -\mathrm{A}(\lambda)\mathrm{D}(\lambda-\eta)$.

Second, from Theorem 5.2.5 and Assumption 4.2.11, the eigenspace associated to $\theta^{(1)}(\lambda)$ must be one-dimensional. To prove that it is also the case for $\theta^{(2)}(\lambda)$, let assume that there exists $|\tilde{\psi}^R\rangle$ such that

$$T^{(2)}(\lambda)\,|\tilde{\psi}^R\rangle = \theta^{(2)}(\lambda)\,|\tilde{\psi}^R\rangle\,, \quad \text{and} \quad T^{(1)}(\lambda-\eta)\,|\tilde{\psi}^R\rangle \neq 0. \tag{6.134}$$

By Proposition 3.1.17, we have $T^{(1)}(\lambda)T^{(1)}(\lambda-\eta)\,|\tilde{\psi}^R\rangle = 0$. By uniqueness of the special eigenvectors of $T^{(1)}(\lambda)$, we deduce that there exists a non-zero ξ such that $T^{(1)}(\lambda-\eta)\,|\tilde{\psi}^R\rangle = \xi\,|\psi^R\rangle$. We consider the following dual pairing:

$$\langle\psi^L|T^{(1)}(\lambda-\eta)|\tilde{\psi}^R\rangle = 0 = \xi\,\langle\psi^L|\psi^R\rangle\,. \tag{6.135}$$

From (6.132) and Theorem 5.2.14, one deduces that $\langle\psi^L|\psi^R\rangle \neq 0$ for generic inhomogeneity parameters. Therefore, $\xi=0$, which is a contradiction. $\square$

We make the following conjecture regarding the analyticity of the special eigenvectors:

Conjecture 6.2.2. *The vectors $|\psi^R\rangle$ and $\langle\psi^L|$ are entire functions of the inhomogeneity parameters λ_i, $i = 1, \ldots, N$.*

This conjecture ensures the regularity of the homogeneous limit of the special eigenvectors.

Small size We now provide explicit expressions for $|\psi^R\rangle$ and $\langle\psi^L|$ for $N = 1$. We base our computations on (5.130), and the explicit expression of the vertex-face correspondence.

For the diagonal twist, $(\mathsf{x}, \mathsf{y}) = (1, 0)$, the eigenvectors are

$$|\psi^R\rangle = 2\alpha\vartheta_2(\eta)(0 \quad 1 \quad 0)^t, \tag{6.136a}$$

$$\langle\psi^L| = -(\alpha\vartheta_2(\eta))^{-1}(0 \quad 1 \quad 0). \tag{6.136b}$$

For the mixed twist, $(\mathsf{x}, \mathsf{y}) = (1, 1)$, the eigenvectors are

$$|\psi^R\rangle = p^{1/4}\vartheta_3(\eta)(1 \quad 0 \quad 1)^t, \tag{6.136c}$$

$$\langle\psi^L| = -(p^{-1/4}\vartheta_3(\eta))^{-1}(1 \quad 0 \quad 1). \tag{6.136d}$$

For the anti-diagonal twist, $(\mathsf{x}, \mathsf{y}) = (0, 1)$, the eigenvectors are

$$|\psi^R\rangle = p^{1/4}\vartheta_4(\eta)(1 \quad 0 \quad -1)^t, \tag{6.136e}$$

$$\langle\psi^L| = -(p^{-1/4}\vartheta_4(\eta))^{-1}(1 \quad 0 \quad -1). \tag{6.136f}$$

Properties The eigenvectors obey the following pseudo-periodicity properties:

Proposition 6.2.3. *For each $i = 1, \ldots, N$, we have*

$$|\psi^R\rangle\Big|_{\lambda_i \to \lambda_i + \pi} = (-1)^{N-1+\mathsf{y}}\Omega_i^3\,|\psi^R\rangle\,, \tag{6.137a}$$

$$|\psi^R\rangle\Big|_{\lambda_i \to \lambda_i + \pi\omega} = (-1)^{\mathsf{x}}(-p^{-1}\mathrm{e}^{-2\mathrm{i}\lambda_i})^{N-1}\prod_{\substack{j=1\\j\neq i}}^{N}\mathrm{e}^{2\mathrm{i}\lambda_j}\,\Omega_i^1\,|\psi^R\rangle\,, \tag{6.137b}$$

and

$$\langle\psi^L|\Big|_{\lambda_i\to\lambda_i+\pi} = (-1)^{N-1+\mathsf{y}}\,\langle\psi^L|\,\Omega_i^3, \tag{6.138a}$$

$$\langle\psi^L|\Big|_{\lambda_i\to\lambda_i+\pi\omega} = (-1)^{\mathsf{x}}(-p^{-1}\mathrm{e}^{-2\mathrm{i}\lambda_i})^{N-1}\prod_{\substack{j=1\\j\neq i}}^{N}\mathrm{e}^{2\mathrm{i}\lambda_j}\,\langle\psi^L|\,\Omega_i^1. \tag{6.138b}$$

Proof. The proposition follows from Propositions 4.2.9 and 5.2.10, and (4.178). □

Next, the eigenvectors fulfil exchange relations.

Proposition 6.2.4. *For each $i=1,\dots,N-1$, we have*

$$\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1})\,|\psi^R\rangle = -\vartheta_1(\lambda_{i,i+1}-\eta)\vartheta_1(\lambda_{i,i+1}+2\eta)\;\;|\psi^R\rangle\Big|_{\lambda_i\leftrightarrow\lambda_{i+1}}\,,$$

$$\langle\psi^L|\,\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i+1,i}) = -\vartheta_1(\lambda_{i+1,i}-\eta)\vartheta_1(\lambda_{i+1,i}+2\eta)\;\;\langle\psi^L|\Big|_{\lambda_i\leftrightarrow\lambda_{i+1}}\,.$$

Proof. From Proposition 4.2.10 and (4.177), we find

$$\check{R}^{(2,2)}_{i,i+1}(\lambda_{i,i+1})\mathbf{V}_q\mathbf{P} = \mathbf{V}_q|_{\lambda_i\leftrightarrow\lambda_{i+1}}\,\mathbf{P}\check{\mathsf{R}}^{(2,2)}_{i,i+1}(\lambda_{i,i+1}|\tau+2\eta S^{i-1}). \tag{6.139}$$

The proposition follows from Proposition 5.2.11, along with (6.139) and (4.177). □

We state a reduction relation obeyed by $|\psi^R\rangle = |\psi^R_N\rangle$ and $\langle\psi^L| = \langle\psi^L_N|$. By convention, we set $|\psi^R_0\rangle = 1$ and $\langle\psi^L_0| = 1$.

Proposition 6.2.5. *For $N\geq 2$, we have*

$$|\psi^R_N\rangle\Big|_{\lambda_N\to\lambda_{N-1}-\eta} = -\mathrm{e}^{2\mathrm{i}\mathsf{y}t_{\mathbf{2}}}\mathrm{e}^{-3\mathrm{i}\mathsf{y}\eta}\vartheta_1(t_{\mathbf{2}}-\eta)\vartheta_1(t_{\mathbf{2}}-2\eta)\frac{\hat\vartheta_4\vartheta_1(2\eta)}{\hat\vartheta_4(2\eta)}$$
$$\times\prod_{j=1}^{N-2}\vartheta_1(\lambda_{jN-1}-\eta)\vartheta_1(\lambda_{jN-1}+2\eta)W_{N-1}^{-1}W_N^{-1}\Xi^{N-1}\,|\psi^R_{N-2}\rangle\,, \tag{6.140}$$

and

$$\langle\psi^L_N|\Big|_{\lambda_N\to\lambda_{N-1}+\eta} = -\frac{\mathrm{e}^{-2\mathrm{i}\mathsf{y}t_{\mathbf{2}}}\mathrm{e}^{\mathrm{i}\mathsf{y}\eta}}{\vartheta_1(t_{\mathbf{2}})\vartheta_1(t_{\mathbf{2}}-\eta)}\frac{\hat\vartheta_4(2\eta)\vartheta_1(2\eta)}{\hat\vartheta_4} \tag{6.141}$$
$$\times\prod_{j=1}^{N-2}\vartheta_1(\lambda_{N-1,j}-\eta)\vartheta_1(\lambda_{N-1,j}+2\eta)\,\langle\psi^L_{N-2}|\,(\Xi^{N-1})^tW_{N-1}W_N,$$

with W defined in (3.34)*, and Ξ^{N-1} in* (5.84)*.*

The proof of Proposition 6.2.5 requires the intermediary result:

Lemma 6.2.6. *We have*

$$\begin{aligned} &V_1^{(2)}(\lambda|t)V_2^{(2)}(\lambda-\eta|t+2\eta s_1^3)\mathsf{c}(\eta s_2^3|t)\,|s\rangle \\ &\quad = \mathrm{e}^{2\mathrm{ity}}\frac{\hat{\vartheta}_4\vartheta_1(\eta)}{\hat{\vartheta}_4(2\eta)}\vartheta_1(\lambda)\vartheta_1(\lambda-\eta)\vartheta(t+\eta)\vartheta_1(t-\eta)W_1^{-1}W_2^{-1}\,|s\rangle\,. \end{aligned} \tag{6.142}$$

Proof. We consider the following specialised version of Proposition 4.2.7:

$$\begin{aligned} &V_1^{(2)}(\lambda|t)V_2^{(2)}(\lambda-\eta|t+2\eta s_1^3)\mathsf{R}_{21}^{(2,2)}(-\eta|t) \\ &= R_{21}^{(2,2)}(-\eta)V_2^{(2)}(\lambda-\eta|t)V_1^{(2)}(\lambda|t+2\eta s_2^3). \end{aligned} \tag{6.143}$$

We use Propositions 3.1.10 and 4.1.9 on this equation, and obtain

$$\begin{aligned} &V_1^{(2)}(\lambda|t)V_2^{(2)}(\lambda-\eta|t+2\eta s_1^3)\mathsf{c}(\eta s_2^3|t)P_{12}^{(2)-} \\ &= \vartheta_1(\eta)W_1^{-1}W_2^{-1}P_{12}^{(2)-}W_2W_1V_2^{(2)}(\lambda-\eta|t)V_1^{(2)}(\lambda|t+2\eta s_2^3)\mathsf{d}(\eta s_2^3|t)^{-1}. \end{aligned} \tag{6.144}$$

From (6.144), we deduce that the vector on the left-hand side of (6.142) belongs to the subspace spanned by $W_1^{-1}W_2^{-1}\,|s\rangle$. To find the proportionality factor between the two vectors, we compute the following component explicitly:

$$\begin{aligned} &\langle 00|\,V_1^{(2)}(\lambda|t)V_2^{(2)}(\lambda-\eta|t+2\eta s_1^3)\mathsf{c}(\eta s_2^3|t)\,|s\rangle \\ &\qquad = -\mathrm{e}^{2\mathrm{ity}}\frac{\hat{\vartheta}_4\vartheta_1(\eta)}{\hat{\vartheta}_4(2\eta)}\vartheta_1(\lambda)\vartheta_1(\lambda-\eta)\vartheta(t+\eta)\vartheta_1(t-\eta). \end{aligned} \tag{6.145}$$

This expression and $\langle 00|\,W_1^{-1}W_2^{-1}\,|s\rangle = -1$ lead to (6.142). $\square$

Proof of Proposition 6.2.5. We focus on the proof of (6.140), as the other case is similar. The idea is to apply the operator

$$\left(\prod_{i=1}^{N}\vartheta_1(\lambda_i)^{-1}\prod_{\substack{1\le i<j\le N\\(i,j)\neq(N-1,N)}}\vartheta_1(\lambda_{ij}-\eta)\mathbf{V}_q\mathbf{P}\right)\Bigg|_{\lambda_N\to\lambda_{N-1}+\eta} \tag{6.146}$$

to the left of (5.142). We explicitly compute the action of this operator on the $(N-1)$-th and N-th factors $\mathbb{C}^3$ of the spin space. Using (4.177) and (4.169), we rewrite

$$\mathbf{V}_q\mathbf{P}|_{\lambda_N\to\lambda_{N-1}+\eta} = \mathbf{P}\prod_{j=1}^{N-2} V_j^{(2)}(\lambda_j|\tau+2\eta S^{j-1}) \\ \times V_{N-1}^{(2)}(\lambda_{N-1}|\tilde{\tau})V_N^{(2)}(\lambda_{N-1}-\eta|\tilde{\tau}+2\eta s_{N-1}^3), \quad (6.147)$$

where $\tilde{\tau} = \tau + 2\eta S^{N-2}$. Then, we use Lemma 6.2.6 to compute the quantity:

$$V_{N-1}^{(2)}(\lambda_{N-1}|\tilde{\tau})V_N^{(2)}(\lambda_{N-1}-\eta|\tilde{\tau}+2\eta s_{N-1}^3)\mathsf{c}\Big(\eta s_N^3\,|\tilde{\tau}\Big)\,\Xi^{N-1}\,|\varphi_{N-2}^R\rangle \\ = \frac{\hat{\vartheta}_4\vartheta_1(\eta)}{\hat{\vartheta}_4(2\eta)}\vartheta_1(\lambda_{N-1})\vartheta_1(\lambda_{N-1}-\eta)\mathrm{e}^{2\mathrm{i}\tilde{\tau}\mathsf{y}}\vartheta(\tilde{\tau}+\eta)\vartheta_1(\tilde{\tau}-\eta) \\ \times W_{N-1}^{-1}W_N^{-1}\Xi^{N-1}\,|\varphi_{N-2}^R\rangle\,. \quad (6.148)$$

We recall that, in $\mathbb{D}_{N-2}^{(0)}$, $\tilde{\tau} = \tau + 2\eta S^{N-2} = -\tau + \mathsf{x}\pi + \mathsf{y}\pi\omega$. We use this relation and (A.4) to simplify the remaining $\tilde{\tau}$-dependent factors in (6.148). Finally, we consider the complete expression of (6.146) acting on the left of (5.142), simplify, and recognise the definition of $|\psi_{N-2}^R\rangle$. □

The eigenvectors fulfil a $\mathbb{Z}_2$ symmetry.

Proposition 6.2.7. *We have*

$$(\Omega)^{\otimes N}\,|\psi^R\rangle = |\psi^R\rangle\,, \qquad \langle\psi^L|\,(\Omega)^{\otimes N} = \langle\psi^L|\,, \quad (6.149a)$$
$$(\Omega^b)^{\otimes N}\,|\psi^R\rangle = (-1)^N\,|\psi^R\rangle\,, \quad \langle\psi^L|\,(\Omega^b)^{\otimes N} = (-1)^N\,\langle\psi^L|\,, \quad (6.149b)$$

with Ω *the twist* (4.179)*, and* $b\in\{1,2,3\}$*, such that* $\Omega^b\neq\Omega$.

Proof. By Propositions 3.1.12 and 6.2.1, for each $c=1,2,3$, there exists $\xi\in\mathbb{C}$ such that

$$(\Omega^c)^{\otimes N}\,|\psi^R\rangle = \xi\,|\psi^R\rangle\,. \quad (6.150)$$

Given $((\Omega^c)^{\otimes N})^2 = \mathbb{I}$, we must have $\xi = \pm1$. Moreover, since ξ takes discrete values, by continuity of the eigenvector with respect to the spectral parameters, ξ must be independent of the spectral parameters.

To prove (6.149), we proceed by induction on the system size. We establish the base case $N=1$ from the explicit expressions of the eigenvectors provided in (6.136). For $N = 2$, we consider Proposition 6.2.5, and the relation

$$(\Omega^c W^{-1})^{\otimes 2} |s\rangle = (W^{-1})^{\otimes 2} |s\rangle \,. \tag{6.151}$$

Let us assume that (6.149) holds for size $N - 2$. Then, we apply $(\Omega^c)^{\otimes N}$ to Proposition 6.2.5. The relation (6.151) establishes (6.149) for size N. □

Sum rule Based on (6.132) and Theorem 5.2.14, we find the following sum rule for the dual pairing of the special eigenvectors:

Proposition 6.2.8. *We have*

$$\langle\psi^L|\psi^R\rangle = \prod_{1\le i<j\le N} \vartheta_1(\lambda_{ij} - \eta)\vartheta_1(\lambda_{ji} - \eta) Z(\xi = 1; \boldsymbol{\lambda}), \tag{6.152}$$

where $Z(\xi; \boldsymbol{\lambda})$ is the elliptic determinant of Theorem 5.2.14.

Relation between right and left eigenvector In Proposition 3.1.15, we express the 18V and 41V transfer matrices in terms of their transposes. This result allows us to find an explicit relation between the eigenvectors $|\psi^R\rangle$ and $\langle\psi^L|$.

Proposition 6.2.9. *There exists $F_N \in \mathbb{C}$ such that*

$$\langle\psi^L| = F_N \, |\psi^R\rangle^t\Big|_{\substack{\lambda_i \to -\lambda_i,\\ i=1,\dots,N}} M^{\otimes N}, \tag{6.153}$$

with M the matrix defined in (3.31).

Proof. The proposition follows from Proposition 3.1.15 and the uniqueness of the eigenvectors. □

We make the following assumption:

Assumption 6.2.10. *F_N is independent of the inhomogeneity parameters.*

Note that the computation of appropriate components of $\langle\psi^L|$ and $|\psi^R\rangle$ would establish Assumption 6.2.10. This, however, goes beyond the scope of this work. In Section 6.3, we indicate how one could perform this calculation.

We find an explicit expression for F_N.

Proposition 6.2.11. *We have*

$$F_N = (-1)^{(\mathsf{x}+\mathsf{y})N} p^{-\mathsf{y}N/2} \left(\frac{\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_4}\right)^N \vartheta_a \vartheta_a(N\eta) \prod_{j=0}^{N} \vartheta_a(j\eta)^{-2}, \quad (6.154)$$

with the label a as in (6.4)*.*

Proof. To prove the proposition, we proceed by induction on the system size. We deduce a recurrence relation for F_N from Propositions 6.2.5 and 6.2.9, and the simplification

$$\begin{aligned} &\mathrm{e}^{4\mathrm{i}\mathsf{y}(t_{\mathbf{2}}-\eta)}\vartheta_1(t_{\mathbf{2}})\vartheta_1(t_{\mathbf{2}}-\eta)^2\vartheta_1(t_{\mathbf{2}}-2\eta) \\ &= p^{\mathsf{y}}\vartheta_a(N\eta)\vartheta_a((N-1)\eta)^2\vartheta_a((N-2)\eta). \end{aligned} \quad (6.155)$$

For $N \geq 3$, it reads

$$F_N = \left(\frac{\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_4}\right)^2 p^{-\mathsf{y}} \left[\vartheta_a(N\eta)\vartheta_a((N-1)\eta)^2\vartheta_a((N-2)\eta)\right]^{-1} F_{N-2}. \quad (6.156)$$

We establish the base case for the odd system sizes from the explicit eigenvectors (6.136). We find

$$F_1 = \frac{\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_4}(-1)^{\mathsf{x}+\mathsf{y}} p^{-\mathsf{y}/2} \left[\vartheta_a\vartheta_a(\eta)\right]^{-1}. \quad (6.157)$$

For the even system sizes, we use Proposition 6.2.5 for $N = 2$ and (6.155). We obtain

$$F_2 = \left(\frac{\hat{\vartheta}_4(2\eta)}{\hat{\vartheta}_4}\right)^2 p^{-\mathsf{y}} \left[\vartheta_a\vartheta_a(\eta)^2\vartheta_a(2\eta)\right]^{-1}. \quad (6.158)$$

The recurrence (6.156) with the base cases (6.157) and (6.158) lead to (6.154). □

6.2.2 Spin-1 XYZ chain

We define the vector

$$|\phi\rangle = \mathcal{N}_N (W)^{\otimes N} \, |\psi^R\rangle\Big|_{\lambda_1,\dots,\lambda_N\to 0}\,, \tag{6.159}$$

where W is the matrix defined in (3.34), and $\mathcal{N}_N$ is a normalisation factor given by

$$\mathcal{N}_{2n} = \vartheta_1(\eta)^{-n(2n-1)} p^{-\mathsf{y}n/2} \left(\frac{\hat{\vartheta}_4}{\hat{\vartheta}_4(2\eta)}\right)^{n(n-1)} \prod_{j=0}^{2n-1} \vartheta_a(j\eta)^{-1}, \tag{6.160a}$$

$$\begin{aligned}\mathcal{N}_{2n+1} &= \epsilon_a \vartheta_1(\eta)^{-n(2n+1)} p^{-\mathsf{y}(2n+1)/4} \\ &\quad\times \sqrt{\frac{\vartheta_a}{\vartheta_a(\eta)}} \left(\frac{\hat{\vartheta}_4}{\hat{\vartheta}_4(2\eta)}\right)^{n^2-1/2} \prod_{j=0}^{2n} \vartheta_a(j\eta)^{-1},\end{aligned} \tag{6.160b}$$

with a as in (6.4) and $\epsilon_2 = 2^{-1/2}$, $\epsilon_{3,4} = 1$. Conjecture 6.2.2 ensures that the vector $|\phi\rangle$ is well-defined.

We use Proposition 3.2.3, to show that $|\phi\rangle$ is a zero-energy eigenvector of the spin-1 XYZ Hamiltonian.

Proposition 6.2.12. *The vector $|\phi\rangle$ is an eigenvector of H with eigenvalue $E = 0$.*

Properties The eigenvector obeys a $\mathbb{Z}_2$ symmetry.

Proposition 6.2.13. *We have*

$$(\Omega)^{\otimes N} |\phi\rangle = |\phi\rangle\,, \qquad (\Omega^b)^{\otimes N} |\phi\rangle = (-1)^N |\phi\rangle\,. \tag{6.161}$$

with Ω the twist (4.179)*, and $b \in \{1,2,3\}$, such that $\Omega^b \neq \Omega$.*

Proof. The statement follows from Proposition 6.2.7, the definition (6.159), and the commutation relation $[\Omega^c, W] = 0$, for $c = 1,2,3$. □

Sum rule We express the square norm of the zero-energy eigenvector $|\phi\rangle$ in terms of the polynomials $G_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}')$, $\bar{G}_{2n}^{(a)}(\boldsymbol{w};\boldsymbol{w}')$ defined in Section 6.1.4.

Theorem 6.2.14. *We have*

$$\langle \phi_{2n}|\phi_{2n}\rangle = 2^n G^{(a)}_{2n}(\mathbf{0};\mathbf{0})\bar{G}^{(a)}_{2n}(\mathbf{0};\mathbf{0}), \tag{6.162}$$

$$\langle \phi_{2n+1}|\phi_{2n+1}\rangle = 2^n \varepsilon_a G^{(a)}_{2(n+1)}(\mathbf{0};\mathbf{0})\bar{G}^{(a)}_{2n}(\mathbf{0};\mathbf{0}), \tag{6.163}$$

with $\varepsilon_2 = 1$, $\varepsilon_{3,4} = 2$.

The proof of Theorem 6.2.14 requires the following lemma:

Lemma 6.2.15. *We have*

$$\langle \phi|\phi\rangle = (-1)^{N(1+\mathsf{x}+\mathsf{y})} F_N^{-1}\mathcal{N}_N^2 \vartheta_1(\eta)^{N(N-1)} Z(\xi=1;\mathbf{0}), \tag{6.164}$$

with $Z(\xi;\boldsymbol{\lambda})$ *the elliptic determinant of Theorem 5.2.14, and* F_N *and* $\mathcal{N}_N$ *as defined in* (6.154) *and* (6.160)*, respectively.*

Proof. We take the homogeneous limit $\lambda_1 = \ldots = \lambda_N = 0$ of Proposition 6.2.8, and obtain

$$\langle \psi^L|\psi^R\rangle\Big|_{\lambda_1,\ldots,\lambda_N\to 0} = \vartheta_1(\eta)^{N(N-1)} Z(\xi=1;\mathbf{0}). \tag{6.165}$$

By Conjecture 6.2.2, the homogeneous limits of $|\psi^R\rangle$ and $\langle\psi^L|$ exist. We relate these vectors to $|\phi\rangle$ and $\langle\phi|$, respectively. For the right eigenvector, we use the definition (6.159). For the left eigenvector, we successively consider Proposition 6.2.9, (6.159) and (3.36). We find:

$$\langle \psi^L|\Big|_{\lambda_1,\ldots,\lambda_N\to 0} = F_N \, |\psi^R\rangle^t\Big|_{\lambda_1,\ldots,\lambda_N\to 0} M^{\otimes N}, \tag{6.166}$$

$$= F_N\mathcal{N}_N^{-1}\, |\phi\rangle^t\, (W^{-1}M)^{\otimes N}, \tag{6.167}$$

$$= (-1)^N F_N\mathcal{N}_N^{-1}\, \langle\phi|\, (W\Omega^2)^{\otimes N}. \tag{6.168}$$

Then, the commutation relation $[W,\Omega^2] = 0$ and Proposition 6.2.13 lead to (6.164). □

Proof of Theorem 6.2.14. The theorem follows from Lemma 6.2.15, Proposition 6.1.11, and (6.8). □

In the *trigonometric limit* $Y = 0$, Hagendorf and Morin-Duchesne showed that the square norms of the zero-energy eigenvectors become polynomials that appear in weighted enumerations of symmetry classes of ASMs [92]. Here, we obtain a similar relation, issued from Propositions 6.1.16 and 6.1.18.

Proposition 6.2.16. *For the diagonal twist, we have*

$$\|\phi_N\|^2\Big|_{Y=0} = X^{2\lfloor N/2\rfloor}A(N;X^2), \tag{6.169}$$

with $A(N;t)$ as defined in (6.122)*.*

For the anti-diagonal and mixed twist, we have

$$\|\phi_N\|^2\Big|_{Y=0} = \frac{A_{\mathrm{HT}}(2N;X^2)}{A(N;X^2)}, \tag{6.170}$$

with $A_{\mathrm{HT}}(2N;t)$ as defined in (6.129)*.*

Additional conjectures We state our results and observations for the Hamiltonian with diagonal twist. We obtain their equivalents for the other twists from Proposition 3.2.6. First, the simple symmetries of Proposition 6.2.13 lead to some cancelling components and sum rules.

Proposition 6.2.17. *For the diagonal twist, we have*

$$\langle \Uparrow \cdots \Uparrow |\phi_{2n+1}\rangle = \langle \Downarrow \cdots \Downarrow |\phi_{2n+1}\rangle = 0, \tag{6.171}$$

$$(\langle \Uparrow| \pm \langle \Downarrow|)^{\otimes(2n+1)}\,|\phi_{2n+1}\rangle = 0. \tag{6.172}$$

Second, we used Mathematica to diagonalise symbolically the Hamiltonian with diagonal twist for small system sizes. To this end, we considered the eigenvalue problem on subspaces invariant under twisted translation [159,160]. Taking into account the $\mathbb{Z}_2$ symmetries of Proposition 3.2.4, we managed to compute the zero-energy eigenvector for sizes up to $N=5$. Our observations support the following conjectures:

Conjecture 6.2.18. *The components of $|\phi\rangle$ are co-prime polynomials in X, Y with rational coefficients.*

Conjecture 6.2.19. *For the diagonal twist, up to multiplication by a sign, we observe the components:*

$$\langle 0\cdots 0|\phi_{2n+1}\rangle = \bar{G}_{2n}(\mathbf{0};\mathbf{0}), \tag{6.173}$$

$$\langle 0\cdots 0|\phi_{2n}\rangle = 0, \tag{6.174}$$

$$\langle \Uparrow\cdots\Uparrow|\phi_{2n}\rangle = \langle \Downarrow\cdots\Downarrow|\phi_{2n}\rangle = Y^n G_{2n}(\mathbf{0};\mathbf{0}), \tag{6.175}$$

and the sum rule:

$$(\langle \Uparrow| \pm \langle \Downarrow|)^{\otimes 2n}\,|\phi_{2n}\rangle = 2^n(Y\mp X)^n G_{2n}(\mathbf{0};\mathbf{0}). \tag{6.176}$$

6.3 Conclusion and outlook

In this chapter, we investigated the eigenvalue problem of the transfer matrices of the inhomogeneous 41V and 18V models with quasi-periodic boundary conditions along the horizontal direction. In particular, we examined the eigenvectors associated to the remarkably simple eigenvalues found in Chapter 3. We provided a formula for them, and established that they are unique, up to normalisation. We also presented several properties of the special eigenvectors that characterise them implicitly. These include symmetry properties, exchange and reduction relations, their pseudo-periodicity properties with respect to the inhomogeneity parameters, as well as a conjecture regarding their analyticity with respect to the inhomogeneity parameters. Furthermore, we computed a quadratic sum rule involving them in terms of a determinant formula. In the homogeneous limit, we used the eigenvectors to define a zero-energy eigenstate of the spin-1 XYZ chain. We had conjectured the latter to be a ground state for most values of the anisotropy parameters, in Chapter 3. We examined its square norm, and found a closed-form expression for it in terms of new special polynomials, generalisations of those of Rosengren and Zinn-Justin. Additionally, we used these polynomials to formulate several conjectures regarding components and sum rules of the zero-energy eigenstates.

We now discuss some related research perspectives. First, we reflect on the implicit characterisation that we found for the special eigenvectors of the inhomogeneous 18V and 41V models. This characterisation is akin to that of the special eigenvector of the supersymmetric 8V model in Section 2.1. As such, it paves the way for a number of future analyses. We expect these properties to help construct the eigenvectors by recursion on the system size, similar to what Hagendorf did for the supersymmetric 8V model [118]. More importantly, one could reproduce the Izergin-Korepin technique of Chapter 2 in order to compute sum rules or components of the special eigenvectors. This would require the development of the theory of the boundary Yang-Baxter equation for spin-1 models. If successful, the computation of appropriate components would establish the relation between left and right special eigenvectors, and thus prove Assumption 6.2.10. Second, the results of this chapter rely on several conjectures or assumptions, notably Assumption 4.2.11 and Conjecture 6.2.2. The proof of these remains a challenge but would undoubtedly bring about a better understanding of the special eigenvectors. Finally, the introduction of the polynomials $G_{2n}(\boldsymbol{w};\boldsymbol{w}')$, $\bar{G}_{2n}(\boldsymbol{w};\boldsymbol{w}')$ brings forward a number of interesting questions. For example, we observed that $2^{n-1}G_{2n}(\mathbf{0};\mathbf{0})$ and $\bar{G}_{2n}(\mathbf{0};\mathbf{0})$ are polynomials in X, Y with positive integer coefficients. This suggests a possible combinatorial interpretation, similar to what Hietala found in the limit $X = 0$, $Y = \zeta$ [82, 83]. Moreover, several properties established in this limit have yet to be generalised. Indeed, differential recurrence relations such as in [111], or a determinant formula equivalent to Lemma 2.2.18 remain to be found. These would provide a more direct way of computing the new polynomials explicitly.

Appendix A

Theta functions

In this appendix, we recall some elements of the theory of (Jacobi) theta functions essential to our work above. Throughout, we consider an integer $n \geq 0$, the *elliptic nome* $p = \mathrm{e}^{\mathrm{i}\pi\omega}$, with $\omega \in \mathbb{C}$ such that $\operatorname{Im}\omega > 0$, and complex numbers z, t. We write p^z for the expression $\mathrm{e}^{\mathrm{i}z\pi\omega}$.

A.1 Jacobi theta functions

In this section, we mostly follow [161,162], and expose some key properties of the Jacobi theta functions.

A.1.1 Definition and properties

Definition A.1.1. *The* Jacobi theta functions *are*

$$\vartheta_1(z,p) = \vartheta_1(z|\omega) = 2\sum_{n=0}^{\infty}(-1)^n p^{(n+1/2)^2}\sin(2n+1)z, \tag{A.1a}$$

$$\vartheta_2(z,p) = \vartheta_2(z|\omega) = 2\sum_{n=0}^{\infty} p^{(n+1/2)^2}\cos(2n+1)z, \tag{A.1b}$$

$$\vartheta_3(z,p) = \vartheta_3(z|\omega) = 1 + 2\sum_{n=1}^{\infty} p^{n^2}\cos 2nz, \tag{A.1c}$$

$$\vartheta_4(z,p) = \vartheta_4(z|\omega) = 1 + 2\sum_{n=1}^{\infty}(-1)^n p^{n^2}\cos 2nz. \tag{A.1d}$$

The Jacobi theta functions are entire functions. Here, and in the main text, we often omit the dependence on the parameters p or ω, and write $\vartheta_i(z) = \vartheta_i(z,p) = \vartheta_i(z|\omega)$. Moreover, to keep expressions from becoming overly cumbersome, we sometimes resort to the shorthand notations

$$\vartheta_i(0) = \vartheta_i, \qquad \vartheta_i(z,p^2) = \vartheta_i(z|2\omega) = \hat{\vartheta}_i(z), \qquad \hat{\vartheta}_i(0) = \hat{\vartheta}_i, \tag{A.2}$$

for $i = 1, \ldots, 4$.

We deduce several key features of the Jacobi theta functions from Definition A.1.1. First, $\vartheta_1(z)$ is odd and $\vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ are even functions of z. Second, the Jacobi theta functions are *pseudo-periodic* in π and $\pi\omega$:

Proposition A.1.2. *We have*

i	1	2	3	4
$\vartheta_i(z+\pi)/\vartheta_i(z)$	-1	-1	1	1
$\vartheta_i(z+\pi\omega)/\vartheta_i(z)$	$-N$	N	N	$-N$

(A.3)

with $N = p^{-1}\mathrm{e}^{-2\mathrm{i}z}$.

In particular, for all $(\mathsf{x},\mathsf{y}) \in \{0,1\}^2$, we have the transformation

$$\vartheta_1(z + \mathsf{x}\pi + \mathsf{y}\pi\omega) = (-1)^{\mathsf{x}+\mathsf{y}} p^{-\mathsf{y}} \mathrm{e}^{-2\mathrm{i}\mathsf{y}z} \vartheta_1(z). \tag{A.4}$$

In the following, we refer to $\pi, \pi\omega$ as *periods* and to $\pi/2, \pi\omega/2$ as *half-periods*. Shifting arguments by half-periods allows us to relate the four Jacobi theta functions.

Proposition A.1.3. *We have*

$$\begin{aligned}
\vartheta_1(z) &= -\vartheta_2(z+\tfrac{1}{2}\pi) = -\mathrm{i}M\vartheta_3(z+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) = -\mathrm{i}M\vartheta_4(z+\tfrac{1}{2}\pi\omega),\\
\vartheta_2(z) &= M\vartheta_3(z+\tfrac{1}{2}\pi\omega) = M\vartheta_4(z+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) = \vartheta_1(z+\tfrac{1}{2}\pi),\\
\vartheta_3(z) &= \vartheta_4(z+\tfrac{1}{2}\pi) = M\vartheta_1(z+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) = M\vartheta_2(z+\tfrac{1}{2}\pi\omega),\\
\vartheta_4(z) &= -\mathrm{i}M\vartheta_1(z+\tfrac{1}{2}\pi\omega) = \mathrm{i}M\vartheta_2(z+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\omega) = \vartheta_3(z+\tfrac{1}{2}\pi),
\end{aligned}$$

with $M = p^{1/4}\mathrm{e}^{\mathrm{i}z}$.

The Jacobi theta functions also have the following infinite-product form [161, 162]:

$$\vartheta_1(z) = 2p^{1/4}\sin z \prod_{n=1}^{\infty}\left(1-p^{2n}\right)\left(1-\mathrm{e}^{2\mathrm{i}z}p^{2n}\right)\left(1-\mathrm{e}^{-2\mathrm{i}z}p^{2n}\right), \quad \text{(A.5a)}$$

$$\vartheta_2(z) = 2p^{1/4}\cos z \prod_{n=1}^{\infty}\left(1-p^{2n}\right)\left(1+\mathrm{e}^{2\mathrm{i}z}p^{2n}\right)\left(1+\mathrm{e}^{-2\mathrm{i}z}p^{2n}\right), \quad \text{(A.5b)}$$

$$\vartheta_3(z) = \prod_{n=1}^{\infty}\left(1-p^{2n}\right)\left(1+\mathrm{e}^{2\mathrm{i}z}p^{2n-1}\right)\left(1+\mathrm{e}^{-2\mathrm{i}z}p^{2n-1}\right), \quad \text{(A.5c)}$$

$$\vartheta_4(z) = \prod_{n=1}^{\infty}\left(1-p^{2n}\right)\left(1-\mathrm{e}^{2\mathrm{i}z}p^{2n-1}\right)\left(1-\mathrm{e}^{-2\mathrm{i}z}p^{2n-1}\right). \quad \text{(A.5d)}$$

A.1.2 Identities

The Jacobi theta functions obey a large number of identities [161–163]. Throughout our work, we often omit the intermediate steps of our computations that involve these identities. We nevertheless consider a few of them here, which we use in our text explicitly. First, several relations between products of Jacobi theta functions follow from the infinite-product formulas. For instance, we have

$$\frac{\vartheta_1(z|2\omega)\vartheta_4(z|2\omega)}{\vartheta_1(z)} = \frac{\vartheta_2(z|2\omega)\vartheta_3(z|2\omega)}{\vartheta_2(z)} = \frac{1}{2}\vartheta_2(0), \quad \text{(A.6)}$$

and

$$\frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_1(2z|2\omega)} = \frac{\vartheta_3(z)\vartheta_4(z)}{\vartheta_4(2z|2\omega)} = \vartheta_4(0|2\omega). \quad \text{(A.7)}$$

Combining (A.6) and (A.7), we obtain the duplication formula for $\vartheta_1(z)$:

$$\vartheta_1(2z) = \frac{2\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)}. \quad \text{(A.8)}$$

Second, another source of relations between the Jacobi theta functions is Weierstrass' identity:

$$\begin{aligned} &\vartheta_1(z_1+z_2)\vartheta_1(z_1-z_2)\vartheta_1(z_3+z_4)\vartheta_1(z_3-z_4) \\ &-\vartheta_1(z_1+z_3)\vartheta_1(z_1-z_3)\vartheta_1(z_2+z_4)\vartheta_1(z_2-z_4) \\ &+\vartheta_1(z_1+z_4)\vartheta_1(z_1-z_4)\vartheta_1(z_2+z_3)\vartheta_1(z_2-z_3) = 0. \end{aligned} \quad \text{(A.9)}$$

The specialisation of z_3, z_4 to half-periods leads to the addition formulas

$$\vartheta_1(x+y)\vartheta_1(x-y)\vartheta_4(0)^2 = \vartheta_1(x)^2\vartheta_4(y)^2 - \vartheta_4(x)^2\vartheta_1(y)^2, \tag{A.10}$$
$$\vartheta_4(x+y)\vartheta_4(x-y)\vartheta_4(0)^2 = \vartheta_4(x)^2\vartheta_4(y)^2 - \vartheta_1(x)^2\vartheta_1(y)^2. \tag{A.11}$$

Moreover, combining similar specialisations with the product identities (A.6) or (A.7) leads to

$$\begin{aligned}\vartheta_1(x)\vartheta_1(y) &= \vartheta_3(x+y|2\omega)\vartheta_2(x-y|2\omega) \\ &\qquad - \vartheta_2(x+y|2\omega)\vartheta_3(x-y|2\omega), \qquad (A.12)\\ \vartheta_1(x)\vartheta_2(y) &= \vartheta_1(x+y|2\omega)\vartheta_4(x-y|2\omega) \\ &\qquad + \vartheta_4(x+y|2\omega)\vartheta_1(x-y|2\omega), \qquad (A.13)\\ \vartheta_4(x)\vartheta_3(y) &= \vartheta_4(x+y|2\omega)\vartheta_4(x-y|2\omega) \\ &\qquad + \vartheta_1(x+y|2\omega)\vartheta_1(x-y|2\omega). \qquad (A.14)\end{aligned}$$

Finally, we recall the following Frobenius determinant identity [164, 165]:

Proposition A.1.4. *Let $x_1, \dots, x_N, y_1, \dots, y_N, \rho \in \mathbb{C}$ such that*

$$\rho \not\equiv 0 \ (mod\, \pi, \pi\omega), \quad x_i - y_j \not\equiv 0 \ (mod\, \pi, \pi\omega), \quad i, j = 1, \dots N, \tag{A.15}$$

we have

$$\det_{i,j=1}^{n}\left[\frac{\vartheta_1(x_i - y_j + \rho)}{\vartheta_1(x_i - y_j)\vartheta_1(\rho)}\right] = \frac{\vartheta_1(\sum_{j=1}^{n}(x_j - y_j) + \rho)}{\vartheta_1(\rho)} \frac{\prod_{1\leqslant i<j\leqslant n}\vartheta_1(x_{ij})\vartheta_1(y_{ji})}{\prod_{i,j=1}^{n}\vartheta_1(x_i - y_j)}, \tag{A.16}$$

where $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$.

Identities involving $\boldsymbol{X}, \boldsymbol{Y}$ In the next two lemmas, we use the identities above to simplify certain expressions involving the parameters X and Y, defined in (3.17).

Lemma A.1.5. *We have*

$$X + Y = \frac{2\vartheta_2(\eta)\vartheta_3(0)}{\vartheta_3(\eta)\vartheta_2(0)}, \qquad X - Y = \frac{2\vartheta_2(\eta)\vartheta_4(0)}{\vartheta_4(\eta)\vartheta_2(0)}, \tag{A.17}$$

and

$$X^2 - Y^2 = \left(\frac{\vartheta_3(0)\vartheta_4(0)}{\vartheta_3(\eta)\vartheta_4(\eta)}\right)\left(\frac{2\vartheta_2(\eta)}{\vartheta_2(0)}\right)^2. \tag{A.18}$$

Proof. Using (3.17), we find

$$X \pm Y = \frac{\vartheta_1(2\eta|2\omega)}{\vartheta_4(2\eta|2\omega)\vartheta_1(\eta|2\omega)\vartheta_4(\eta|2\omega)} \left(\vartheta_4(\eta|2\omega)^2 \pm \vartheta_1(\eta|2\omega)^2\right). \tag{A.19}$$

Using (A.6) and (A.7) with $z = \eta$, we obtain

$$\frac{\vartheta_1(2\eta|2\omega)}{\vartheta_4(2\eta|2\omega)\vartheta_1(\eta|2\omega)\vartheta_4(\eta|2\omega)} = \frac{2\vartheta_2(\eta)}{\vartheta_2(0)\vartheta_3(\eta)\vartheta_4(\eta)}. \tag{A.20}$$

Finally, the specialisations $x = \eta, y = 0$ and $x = 0, y = \eta$ in (A.14) lead to

$$\vartheta_4(\eta|2\omega)^2 + \vartheta_1(\eta|2\omega)^2 = \vartheta_4(\eta)\vartheta_3(0), \tag{A.21a}$$
$$\vartheta_4(\eta|2\omega)^2 - \vartheta_1(\eta|2\omega)^2 = \vartheta_3(\eta)\vartheta_4(0). \tag{A.21b}$$

The expressions in (A.17) follow from the substitution of (A.20) and (A.21) into (A.19). The relation (A.18) is an easy consequence of these expressions. □

Lemma A.1.6. *We have*

$$1 - XY = \left(\frac{\vartheta_3(0)\vartheta_4(0)}{\vartheta_3(\eta)\vartheta_4(\eta)}\right)^2 \frac{\vartheta_3(2\eta)}{\vartheta_3(0)}, \tag{A.22a}$$
$$1 + XY = \left(\frac{\vartheta_3(0)\vartheta_4(0)}{\vartheta_3(\eta)\vartheta_4(\eta)}\right)^2 \frac{\vartheta_4(2\eta)}{\vartheta_4(0)}, \tag{A.22b}$$

and

$$X^2 + Y^2 - 2 = \left(\frac{\vartheta_3(0)\vartheta_4(0)}{\vartheta_3(\eta)\vartheta_4(\eta)}\right)^2 \frac{2\vartheta_2(2\eta)}{\vartheta_2(0)}. \tag{A.23}$$

Proof. The definition (3.17) leads to

$$1 - XY = \left(\frac{\vartheta_4(0|2\omega)}{\vartheta_4(2\eta|2\omega)}\right)^2 \frac{\vartheta_4(2\eta|2\omega)^2 - \vartheta_1(2\eta|2\omega)^2}{\vartheta_4(0|2\omega)^2}, \tag{A.24}$$
$$1 + XY = \left(\frac{\vartheta_4(0|2\omega)}{\vartheta_4(2\eta|2\omega)}\right)^2 \frac{\vartheta_4(2\eta|2\omega)^2 + \vartheta_1(2\eta|2\omega)^2}{\vartheta_4(0|2\omega)^2} \tag{A.25}$$

We simplify the right-hand sides with the help of (A.21), with η replaced by 2η, and (A.7), with $z = 0$. This simplification yields (A.22).

Moreover, using specialising (A.10) and (A.11) to $x = 2\eta, y = \eta$, we obtain

$$X^2 - 1 = \left(\frac{\vartheta_4(0|2\omega)}{\vartheta_4(2\eta|2\omega)}\right)^2 \frac{\vartheta_1(3\eta|2\omega)}{\vartheta_1(\eta|2\omega)}, \tag{A.26}$$

$$Y^2 - 1 = -\left(\frac{\vartheta_4(0|2\omega)}{\vartheta_4(2\eta|2\omega)}\right)^2 \frac{\vartheta_4(3\eta|2\omega)}{\vartheta_4(\eta|2\omega)}. \tag{A.27}$$

Hence, we obtain

$$X^2 + Y^2 - 2 = \left(\frac{\vartheta_4(0|2\omega)}{\vartheta_4(2\eta|2\omega)}\right)^2 \frac{\vartheta_1(3\eta|2\omega)\vartheta_4(\eta|2\omega) - \vartheta_4(3\eta|2\omega)\vartheta_1(\eta|2\omega)}{\vartheta_1(\eta|2\omega)\vartheta_4(\eta|2\omega)}.$$

By (A.13) with $x = \eta$ and $y = 2\eta$, we have

$$\vartheta_1(3\eta|2\omega)\vartheta_4(\eta|2\omega) - \vartheta_4(3\eta|2\omega)\vartheta_1(\eta|2\omega) = \vartheta_1(\eta)\vartheta_2(2\eta). \tag{A.28}$$

We combine this result with the duplication formula (A.7), with $z = \eta$, and reach (A.23). □

A.1.3 Modular properties

Let T, S denote the generators of the *modular group*. Their action on the Jacobi theta functions is defined as

$$(T\vartheta_i)(z) = \vartheta_i(z|\omega + 1), \qquad (S\vartheta_i)(z) = \vartheta_i(z/\omega| - 1/\omega), \tag{A.29}$$

for each $i = 1, \ldots, 4$. We infer from Definition A.1.1 the transformations

$$(T\vartheta_1)(z) = \mathrm{e}^{\mathrm{i}\pi/4}\vartheta_1(z), \qquad (T\vartheta_2)(z) = \mathrm{e}^{\mathrm{i}\pi/4}\vartheta_2(z), \tag{A.30a}$$

$$(T\vartheta_3)(z) = \vartheta_4(z), \qquad (T\vartheta_4)(z) = \vartheta_3(z). \tag{A.30b}$$

Moreover, by [161, Section 21.51], we have

$$(S\vartheta_1)(z) = \mathrm{i}A\mathrm{e}^{\mathrm{i}z^2/(\pi\omega)}\vartheta_1(z), \quad (S\vartheta_2)(z) = A\mathrm{e}^{\mathrm{i}z^2/(\pi\omega)}\vartheta_4(z) \tag{A.31a}$$

$$(S\vartheta_3)(z) = A\mathrm{e}^{\mathrm{i}z^2/(\pi\omega)}\vartheta_3(z), \quad (S\vartheta_4)(z) = A\mathrm{e}^{\mathrm{i}z^2/(\pi\omega)}\vartheta_2(z). \tag{A.31b}$$

where $A = (-\mathrm{i}\omega)^{1/2}$.

A.2 Theta functions

Parallel to elliptic functions, theta functions generalise the concept of periodic functions to the complex plane. We use the following definition [166, 167]:

Definition A.2.1. *A* theta function *of degree n, nome p and norm t is an entire function f with the pseudo-periodicity properties*

$$f(\lambda+\pi)=(-1)^n f(\lambda), \qquad f(\lambda+\pi\omega)=(-p)^{-n}\mathrm{e}^{-2\mathrm{i}(n\lambda-t)}f(\lambda). \quad \text{(A.32)}$$

The norm is defined up to integer multiples of π. The resulting ambiguity is, however, not important for our considerations. A simple (non-trivial) example of a theta function of degree $n=1$, nome p and norm t is $f(\lambda)=\vartheta_1(\lambda-t)$.

Theta functions enjoy factorisation properties similar to those of polynomials. For example, the following lemma stems from standard complex analysis [121, 158]:

Lemma A.2.2. *Let f be a theta function of degree $n\geqslant 1$, nome p and norm t. Let ξ be a complex number such that $f(\xi)=0$, then there exists a theta function g of degree $n-1$, nome p and norm $t-\xi$ such that*

$$f(\lambda)=\vartheta_1(\lambda-\xi)g(\lambda). \quad \text{(A.33)}$$

Definition A.2.3. *We say that the points $z_1,\dots,z_n\in\mathbb{C}$ are* independent *if they obey*

$$z_i-z_j\not\equiv 0 \ (mod\,\pi,\pi\omega), \qquad \forall\, i,j=1,\dots,N, \quad \text{(A.34)}$$

$$\textit{and} \quad z_1+\ldots+z_n-t\not\equiv 0\ (mod\,\pi,\pi\omega). \quad \text{(A.35)}$$

Another key feature of theta functions is that any theta function of degree $n\geqslant 1$ can be interpolated from its value at n independent points [56].

Proposition A.2.4. *Let f be a theta function of degree $n\geqslant 1$, nome p and norm t, and $z_1,\dots,z_n\in\mathbb{C}$ be independent points, we have*

$$f(\lambda)=\sum_{j=1}^{n}\frac{\vartheta_1(t-\sum_{k=1}^n z_k+z_j-\lambda)}{\vartheta_1(t-\sum_{k=1}^n z_k)}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{\vartheta_1(\lambda-z_k)}{\vartheta_1(z_j-z_k)}f(z_j). \quad \text{(A.36)}$$

As a consequence of Proposition A.2.4, any two theta functions of same degree $n \geq 1$, nome and norm, whose values match in n independent points must be identical [166].

Theorem A.2.5. *Let f, g be theta functions of degree $n \geqslant 1$, nome p and norm t. If there are n independent complex numbers $z_1, \ldots, z_n$ such that $f(z_i) = g(z_i)$ for each $i = 1, \ldots, n$, then $f = g$.*

Bibliography

[1] S. Brasseur and C. Hagendorf, *Sum rules for the supersymmetric eight-vertex model*, J. Stat. Mech. **2021** (2021) 023102.

[2] S. Brasseur and C. Hagendorf. On the dynamical ten-vertex and nineteen-vertex models with anti-periodic boundary conditions. In preparation.

[3] S. Brasseur and C. Hagendorf. The spin-one XYZ chain with quasi-periodic boundary conditions. In preparation.

[4] B. Diu, C. Guthmann, D. Lederer and B. Roulet, *Physique statistique*, Hermann, 2001.

[5] M. Mézard, G. Parisi and M. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, 1986.

[6] M. Mézard and A. Montanari, *Information, Physics, and Computation*, Oxford University Press, 2009.

[7] C. Texier and G. Roux, *Physique statistique*, Dunod, 2017.

[8] L. Pauling, *The Structure and Entropy of Ice and of Other Crystals with Some Randomness of Atomic Arrangement*, J. Am. Chem. Soc. **57** (1935) 2680–2684.

[9] E. H. Lieb, *Exact Solution of the Two-Dimensional Slater KDP Model of a Ferroelectric*, Phys. Rev. Lett. **19** (1967) 108–110.

[10] E. H. Lieb, *Residual Entropy of Square Ice*, Phys. Rev. **162** (1967) 162–172.

[11] B. Sutherland, *Exact Solution of a Two-Dimensional Model for Hydrogen-Bonded Crystals*, Phys. Rev. Lett. **19** (1967) 103–104.

[12] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Dover Publications, 2013.

[13] B. Sutherland, *Two-Dimensional Hydrogen Bonded Crystals without the Ice Rule*, J. Math. Phys. **11** (1970) 3183–3186.

[14] C. Fan and F. Y. Wu, *General Lattice Model of Phase Transitions*, Phys. Rev. B **2** (1970) 723–733.

[15] R. J. Baxter, *Solving models in statistical mechanics*, Adv. Stud. Pure Math. **19** (1989) 95–116.

[16] H. A. Kramers and G. H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part I*, Phys. Rev. **60** (1941) 252–262.

[17] H. A. Kramers and G. H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part II*, Phys. Rev. **60** (1941) 263–276.

[18] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.

[19] N. Slavnov, *Algebraic Bethe Ansatz and Correlation Functions*, World Scientific, 2022.

[20] C. N. Yang, *Some Exact Results for the Many-Body Problem in one Dimension with Repulsive Delta-Function Interaction*, Phys. Rev. Lett. **19** (1967) 1312.

[21] R. J. Baxter, *Solvable Eight-Vertex Model on an Arbitrary Planar Lattice*, Philos. Trans. R. Soc. London, Ser. A **289** (1978) 315–346.

[22] L. D. Faddeev, *How Algebraic Bethe Ansatz works for integrable model*, arXiv:hep-th/9605187 (1996).

[23] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) 63–69.

[24] V. G. Drinfel'd, *Quantum groups*, J. Math. Sci. **41** (1988) 898–915.

[25] C. Gómez, M. Ruiz-Altaba and G. Sierra, *Quantum Groups in Two-Dimensional Physics*, Cambridge University Press, 1996.

[26] J.-S. Caux and J. Mossel, *Remarks on the notion of quantum integrability*, J. Stat. Mech. **2011** (2011) P02023.

[27] E. H. Lieb, *Exact Solution of the F Model of An Antiferroelectric*, Phys. Rev. Lett. **18** (1967) 1046–1048.

[28] R. J. Baxter, *One-Dimensional Anisotropic Heisenberg Chain*, Phys. Rev. Lett. **26** (1971) 834.

[29] R. J. Baxter, *One-Dimensional anisotropic Heisenberg Chain*, Ann. Phys. **70** (1972) 323–337.

[30] H. Bethe, *Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen der linearen Atomkette*, Z. Physik **71** (1931) 205–226.

[31] E. K. Sklyanin, L. A. Takhtadzhyan and L. D. Faddeev, *Quantum inverse problem method. I*, Theor. Math. Phys. **40** (1979) 688–706.

[32] L. A. Takhtadzhan and L. D. Faddeev, *The quantum method of the inverse problem and the Heisenberg XYZ model*, Russ. Math. Surv. **34** (1979) 11.

[33] J. L. Jacobsen and Y. Ikhlef, Algèbres, Intégrabilité et Modèles Exactement Solubles March 2013.

[34] N. Kitanine, J. M. Maillet and V. Terras, *Form factors of the XXZ Heisenberg spin-$\frac{1}{2}$ finite chain*, Nucl. Phys. B **554** (1999) 647–678.

[35] N. Kitanine, J. M. Maillet and V. Terras, *Correlation functions of the XXZ Heisenberg spin-$\frac{1}{2}$ chain in a magnetic field*, Nucl. Phys. B **567** (2000) 554–582.

[36] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions*, J. Stat. Mech. **2009** (2009) P04003.

[37] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *A form factor approach to the asymptotic behavior of*

correlation functions in critical models, J. Stat. Mech. **2011** (2011) P12010.

[38] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Form factor approach to dynamical correlation functions in critical models*, J. Stat. Mech. **2012** (2012) P09001.

[39] J.-S. Caux and J. M. Maillet, *Computation of Dynamical Correlation Functions of Heisenberg Chains in a Magnetic Field*, Phys. Rev. Lett. **95** (2005) 077201.

[40] F. Göhmann, A. Klümper and A. Seel, *Integral representations for correlation functions of the XXZ chain at finite temperature*, J. Phys. A: Math. Gen. **37** (2004) 7625.

[41] M. Dugave, F. Göhmann and K. K. Kozlowski, *Thermal form factors of the XXZ chain and the large-distance asymptotics of its temperature dependent correlation functions*, J. Stat. Mech. **2013** (2013) P07010.

[42] F. Göhmann, M. Karbach, A. Klümper, K. Kozlowski and J. Suzuki, *Thermal form-factor approach to dynamical correlation functions of integrable lattice models*, J. Stat. Mech. **2017** (2017) 113106.

[43] T. C. Dorlas, *Orthogonality and Completeness of the Bethe Ansatz Eigenstates of the Nonlinear Schroedinger Model*, Commun. Math. Phys. **154** (1993) 347–376.

[44] V. Tarasov and A. Varchenko, *Completeness of Bethe vectors and difference equations with regular singular points*, Int. Math. Res. Not. **1995** (1995) 637–669.

[45] R. J. Baxter, *Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. II. Equivalence to a Generalized Ice-type Lattice Model*, Ann. Phys. **76** (1973) 25–47.

[46] G. Felder, *Conformal Field Theory and Integrable Systems Associated to Elliptic Curves*, in S. D. Chatterji, editor, *Proceedings of the International Congress of Mathematicians*, pages 1247–1255, Birkhäuser, Basel, 1995.

[47] G. Felder and A. Varchenko, *On representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$*, Commun. Math. Phys. **181** (1996) 741–761.

[48] R. J. Baxter, *Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. III. Eigenvectors of the Transfer Matrix and Hamiltonian*, Ann. of Phys. **76** (1973) 48–71.

[49] E. K. Sklyanin, *FUNCTIONAL BETHE ANSATZ*, in *Integrable and Superintegrable Systems*, pages 8–33, World Scientific, 1990.

[50] E. K. Sklyanin, *Quantum Inverse Scattering Method. Selected Topics*, arXiv:hep-th/9211111 (1992).

[51] G. Niccoli, *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*, Nucl. Phys. B **870** (2013) 397–420.

[52] H. Pei and V. Terras, *On scalar products and form factors by separation of variables: the antiperiodic XXZ model*, J. Phys. A: Math. Theor. **55** (2021) 015205.

[53] G. Niccoli, *Form factors and complete spectrum of XXX antiperiodic higher spin chains by quantum separation of variables*, J. Math. Phys. **54** (2013) 053516.

[54] G. Niccoli and V. Terras, *Antiperiodic XXZ Chains with Arbitrary Spins: Complete Eigenstate Construction by Functional Equations in Separation of Variables*, Lett. Math. Phys. **105** (2015) 989–1031.

[55] J. M. Maillet and G. Niccoli, *On quantum separation of variables*, J. Math. Phys. **59** (2018) 091417.

[56] G. Felder and A. Schorr, *Separation of variables for quantum integrable systems on elliptic curves*, J. Phys. A: Math. Gen. **32** (1999) 8001.

[57] A. Schorr, *Separation of variables for the eight-vertex SOS model with antiperiodic boundary conditions*, PhD thesis, ETH Zürich, 2000.

[58] G. Niccoli, *An antiperiodic dynamical six-vertex model: I. Complete spectrum by SOV, matrix elements of the identity on separate states and connections to the periodic eight-vertex model*, J. Phys. A: Math. Theor. **46** (2013) 075003.

[59] D. Levy-Bencheton, G. Niccoli and V. Terras, *Antiperiodic dynamical 6-vertex model by separation of variables II: functional equations and form factors*, J. Stat. Mech. **2016** (2016) 33110.

[60] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, *New construction of eigenstates and separation of variables for $SU(N)$ quantum spin chains*, J. High Energ. Phys. **2017** (2017).

[61] A. Cavaglià, N. Gromov and F. Levkovich-Maslyuk, *Separation of variables and scalar products at any rank*, J. High Energ. Phys. **2019** (2019).

[62] N. Gromov, F. Levkovich-Maslyuk, P. Ryan and D. Volin, *Dual separated variables and scalar products*, Phys. Let. B **806** (2020) 135494.

[63] D. M. Bressoud, *Proofs and Confirmations*, Cambridge University Press, 1999.

[64] W. H. Mills, D. P. Robbins and H. Jr. Rumsey, *Proof of the Macdonald conjecture*, Invent. Math. **66** (1982) 73–87.

[65] W. H. Mills, D. P. Robbins and H. Jr. Rumsey, *Alternating sign matrices and descending plane partitions*, J. Comb. Theory Ser. A **34** (1983) 340–359.

[66] D. Zeilberger, *Proof of the Alternating Sign Matrix Conjecture*, Electron. J. Comb. **3** (1996) R13.

[67] G. Kuperberg, *Another proof of the alternative-sign matrix conjecture*, Int. Math. Res. Not. **1996** (1996) 139–150.

[68] N. Elkies, G. Kuperberg, M. Larsen and J. Propp, *Alternating-Sign Matrices and Domino Tilings (Part II)*, J. Algebraic Comb. **1** (1992) 219–234.

[69] A. G. Izergin, D. A. Coker and V. E. Korepin, *Determinant formula for the six-vertex model*, J. Phys. A: Math. Gen. **25** (1992) 4315.

[70] A. V. Razumov and Y. G. Stroganov, *Spin chains and combinatorics*, J. Phys. A: Math. Gen. **34** (2001) 3185–3190.

[71] A. V. Razumov and Y. G. Stroganov, *A possible combinatorial point for the XYZ spin chain*, Theor. Math. Phys. **164** (2010) 977–991.

[72] V. V. Bazhanov and V. V. Mangazeev, *The eight-vertex model and Painlevé VI*, J. Phys. A: Math. Gen. **39** (2006) 12235.

[73] Bazhanov, V. V. and Mangazeev, V. V., *The eight-vertex model and Painlevé VI equation II: eigenvector results*, J. Phys. A: Math. Theor. **43** (2010) 085206.

[74] K. Fabricius and B. M. McCoy, *New Developments in the Eight Vertex Model II. Chains of Odd Length*, J. Stat. Phys. **120** (2005) 37–70.

[75] H. Rosengren, *Elliptic pfaffians and solvable lattice models*, J. Stat. Mech. **2016** (2016) 083106.

[76] V. V. Bazhanov and V. V. Mangazeev, *Eight-vertex model and non-stationary Lamé equation*, J. Phys. A: Math. Gen. **38** (2005) L145.

[77] C. Hagendorf and H. Rosengren, *Nearest-neighbour correlation functions for the supersymmetric XYZ spin chain and Painlevé VI*, arXiv:2208.13533 (2022).

[78] H. Rosengren, *Special polynomials related to the supersymmetric eight-vertex model. I. Behaviour at cusps*, arXiv:1305.0666 (2013).

[79] H. Rosengren, *Special polynomials related to the supersymmetric eight-vertex model. II. Schrödinger equation*, arXiv:1312.5879 (2013).

[80] H. Rosengren, *Special polynomials related to the supersymmetric eight-vertex model. III. Painlevé VI equation*, arXiv:1405.5318 (2014).

[81] H. Rosengren, *Special Polynomials Related to the Supersymmetric Eight-Vertex Model: A Summary*, Commun. Math. Phys. **340** (2015) 1143–1170.

[82] L. Hietala, *A Combinatorial Description of Certain Polynomials Related to the XYZ Spin Chain*, SIGMA **16** (2020).

[83] L. Hietala, *A Combinatorial Description of Certain Polynomials Related to the XYZ Spin Chain. II. The Polynomials p_n*, SIGMA **18** (2022).

[84] C. Hagendorf and P. Fendley, *The Eight-Vertex Model and Lattice Supersymmetry*, J. Stat. Phys. **146** (2012) 1122–1155.

[85] C. Hagendorf and J. Liénardy, *On the transfer matrix of the supersymmetric eight-vertex model. I. Periodic boundary conditions*, J. Stat. Mech. **2018** (2018) 033106.

[86] P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, *Yang-Baxter Equation and Representation Theory: I*, Lett. Math. Phys. **5** (1981) 393–403.

[87] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method. Recent developments.*, in J. Hietarinta and C. Montonen, editors, *Integrable Quantum Field Theories*, pages 61–119, Springer, Berlin, Heidelberg, 1982.

[88] P. P. Kulish and E. K. Sklyanin, *Solutions of the Yang-Baxter equation*, J. Sov. Math. **19** (1982) 1596–1620.

[89] H. Konno, *Fusion of Baxter's Elliptic R-Matrix and the Vertex-Face Correspondence*, Ann. Henri Poincaré **7** (2006) 1555–1567.

[90] A. B. Zamolodchikov and V. A. Fateev, *Model factorized S-matrix and an integrable spin-1 Heisenberg chain*, Sov. J. Nucl. Phys. **32** (1980).

[91] V. A. Fateev, *A factorized S matrix for particles of opposite parities and an integrable 21-vertex statistical model*, Sov. J. Nucl. Phys. **33** (1981) 761–766.

[92] C. Hagendorf and A. Morin-Duchesne, *Symmetry classes of alternating sign matrices in a nineteen-vertex model*, J. Stat. Mech. **2016** (2016) 053111.

[93] C. Hagendorf, *The nineteen-vertex model and alternating sign matrices*, J. Stat. Mech. **2015** (2015) P01017.

[94] G. Niccoli and V. Terras, *The eight-vertex model with quasi-periodic boundary conditions*, J. Phys. A: Math. Theor. **49** (2016) 044001.

[95] E. Date, M. Jimbo, T. Miwa and M. Okado, *Fusion of the Eight Vertex SOS Model*, Lett. Math. Phys. **12** (1986) 209–215.

[96] J. C. Slater, *Theory of the Transition in* KH_2PO_4, J. Chem. Phys. **9** (1941) 16–33.

[97] G. Algara-Siller et al., *Square ice in graphene nanocapillaries*, Nature **519** (2015) 443–445.

[98] W. Zhou et al., *The observation of square ice in graphene questioned*, Nature **528** (2015) E1–E2.

[99] N. Reshetikhin, *Lectures on the integrability of the 6-vertex model*, in J. Jacobsen et al., editors, *Exact methods in low-dimensional statistical physics and quantum computing*, pages 197–266, Oxford University Press, 2010.

[100] R. J. Baxter, *Partition function of the Eight-Vertex lattice model*, Ann. Phys. **70** (1972) 193–228.

[101] J. Lamers, *On elliptic quantum integrability: vertex models, solid-on-solid models and spin chains*, PhD thesis, Utrecht University, 2016.

[102] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 2013.

[103] A. Auerbach, *Interacting Electrons and Quantum Magnetism*, Springer, 2012.

[104] C. Cohen-Tannoudji, F. Laloë and B. Diu, *Mécanique Quantique – Tomes I et II*, EDP Sciences, 2018.

[105] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *Surface exponents of the quantum XXZ, Ashkin-Teller and Potts models*, J. Phys. A: Math. Gen. **20** (1987) 6397.

[106] F. C. Alcaraz, M. N. Barber and M. T. Batchelor, *Conformal invariance, the XXZ chain and the operator content of two-dimensional critical systems*, Ann. Phys. **182** (1988) 280–343.

[107] Y. Stroganov, *The importance of being odd*, J. Phys. A: Math. Gen. **34** (2001) L179–85.

[108] A. V. Razumov and Y. G. Stroganov, *Spin chains and combinatorics: twisted boundary conditions*, J. Phys. A: Math. Gen. **34** (2001) 5335–5340.

[109] M. T. Batchelor, J. de Gier and B. Nienhuis, *The quantum symmetric XXZ chain at $\Delta = -1/2$, alternating-sign matrices and plane partitions*, J. Phys. A: Math. Gen. **34** (2001) L265.

[110] J. de Gier, M. T. Batchelor, B. Nienhuis and S. Mitra, *The XXZ spin chain at $\Delta = -1/2$: Bethe roots, symmetric functions, and determinants*, J. Math. Phys. **43** (2002) 4135–4146.

[111] P. Zinn-Justin, *Sum rule for the eight-vertex model on its combinatorial line*, in K. Iohara, S. Morier-Genoud and B. Rémy, editors, *Symmetries, Integrable Systems and Representations*, Volume *40*, pages 599–637, Springer, London, 2013.

[112] J. Liénardy, *Integrable lattice models and supersymmetry*, PhD thesis, Université catholique de Louvain, 2020.

[113] D. Zeilberger, *Proof of the Refined Alternating Sign Matrix Conjecture*, New York J. Math **2** (1996) 59–68.

[114] G. Kuperberg, *Symmetry Classes of Alternating-Sign Matrices under One Roof*, Ann. Math. **156** (2002) 835–866.

[115] S. Okada, *Enumeration of symmetry classes of alternating sign matrices and characters of classical groups*, J. Algebr. Comb. **23** (2006) 43–69.

[116] D. P. Robbins, *Symmetry Classes of Alternating Sign Matrices*, arXiv:math/0008045 (2000).

[117] R. E. Behrend, I. Fischer and M. Konvalinka, *Diagonally and antidiagonally symmetric alternating sign matrices of odd order*, Adv. Math. **315** (2017) 324–365.

[118] C. Hagendorf. Unpublished notes. 2022.

[119] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988) 2375.

[120] T. Inami and H. Konno, *Integrable XYZ spin chain with boundaries*, J. Phys. A: Math. Gen. **27** (1994) L913.

[121] G. Filali, *Elliptic dynamical reflection algebra and partition function of SOS model with reflecting end*, J. Geom. Phys. **61** (2011) 1789–1796.

[122] R. E. Behrend, P. Di Francesco and P. Zinn-Justin, *On the weighted enumeration of alternating sign matrices and descending plane partitions*, J. Comb. Theory Ser. A **119** (2012) 331–363.

[123] M. Noumi, *Painlevé Equations through Symmetry*, American Mathematical Society, 2004.

[124] A. V. Razumov and Stroganov Y. G., *Refined Enumerations of Some Symmetry Classes of Alternating-Sign Matrices*, Theor. Math. Phys. **141** (2004) 1609–1630.

[125] A. V. Razumov, Y. G. Stroganov and P. Zinn-Justin, *Polynomial solutions of qKZ equation and ground state of XXZ spin chain at* $\Delta = -1/2$, J. Phys. A: Math. Theor. **40** (2007) 11827.

[126] A. Ayyer and R. E. Behrend, *Factorization theorems for classical group characters, with applications to alternating sign matrices and plane partitions*, J. Comb. Theory Ser. A **165** (2019) 78–105.

[127] S. Brasseur, *The eight-vertex model on its combinatorial line*, Master's thesis, Université catholique de Louvain, 2019.

[128] A. Morin-Duchesne, C. Hagendorf and L. Cantini, *Boundary emptiness formation probabilities in the six-vertex model at* $\Delta = -1/2$, J. Phys. A: Math. Theor. **53** (2020) 255202.

[129] L. Cantini, *Finite size emptiness formation probability of the XXZ spin chain at* $\Delta = -1/2$, J. Phys. A: Math. Theor. **45** (2012) 135207.

[130] C. Hagendorf and J. Liénardy, *On the transfer matrix of the supersymmetric eight-vertex model. II. Open boundary conditions*, J. Stat. Mech. **2020** (2020) 033104.

[131] E. Date, M. Jimbo, Miwa. T. and M. Okado, *Exactly solvable SOS models: Local height probabilities and theta function identities*, Nucl. Phys. B **290** (1987) 231–273.

[132] E. Date, M. Jimbo, A. Kuniba, Miwa. T. and M. Okado, *Exactly Solvable SOS Models II: Proof of the star-triangle relation and combinatorial identities*, Adv. Stud. Pure Math. **16** (1988) 17–122.

[133] T. Takebe, *Generalized Bethe ansatz with the general spin representations of the Sklyanin algebra*, J. Phys. A: Math. Gen. **25** (1992) 1071.

[134] T. Takebe, *Bethe ansatz for higher spin eight-vertex models*, J. Phys. A: Math. Gen. **28** (1995) 6675.

[135] T. Kojima, H. Konno and R. Weston, *The vertex-face correspondence and correlation functions of the fusion eight-vertex model: I: The general formalism*, Nucl. Phys. B **720** (2005) 348–398.

[136] C. Hagendorf, *Spin Chains with Dynamical Lattice Supersymmetry*, J. Stat. Phys. **150** (2013) 609–657.

[137] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer, 2004.

[138] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[139] M. Gaudin, *The Bethe Wavefunction*, Cambridge University Press, 2014.

[140] G. Felder and A. Varchenko, *Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$*, Nucl. Phys. B **480** (1996) 485–503.

[141] G. E. Andrews, R. J. Baxer and P. J. Forrester, *Eight-Vertex SOS Model and Generalized Rogers-Ramanujan-Type Identities*, J. Stat. Phys. **35** (1984) 193–266.

[142] P. J. Forrester and R. J. Baxer, *Further Exact Solutions of the Eight-Vertex SOS Model and Generalizations of the Rogers-Ramanujan Identities*, J. Stat. Phys. **38** (1985) 435–472.

[143] D. Huse, *Exact exponents for infinitely many new multicritical points*, Phys. Rev. B **30** (1984) 3908–3915.

[144] A. Klümper and P. Pearce, *Conformal weights of RSOS lattice models and their fusion hierarchies*, Physica A **183** (1992) 304–350.

[145] P. Pearce and K. Seaton, *Exact Solution of Cyclic Solid-on-Solid Lattice Models*, Ann. Phys. **193** (1989) 326–366.

[146] D. Levy-Bencheton and V. Terras, *An algebraic bethe ansatz approach to form factors and correlation functions of the cyclic eight-vertex solid-on-solid model*, J. Stat. Mech. **2013** (2013) P04015.

[147] M. Jimbo and T. Miwa, *Algebraic Analysis of Solvable Lattice Models*, Volume *85* of *Regional Conference Series in Mathematics*, American Mathematical Society, 1995.

[148] J.-L. Gervais and A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory*, Nucl. Phys. B **238** (1984) 125–141.

[149] B. Enriquez and G. Felder, *Elliptic Quantum Groups $E_{\tau,\eta}(sl_2)$ and Quasi-Hopf Algebras*, Commun. Math. Phys. **195** (1998) 651–689.

[150] W. Miller, *Symmetry and Separation of Variables*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1984.

[151] E. G. Kalnins, J. M. Kress and W. Miller, *Separation of Variables and Superintegrability*, IOP Publishing, 2018.

[152] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, 1989.

[153] M. C. Gutzwiller, *The quantum mechanical Toda lattice, II*, Ann. Phys. **133** (1981) 304–331.

[154] L. Vignoli, *Quantum Separation of Variables in Higher Rank and Supersymmetric Integrable Models*, PhD thesis, Université de Lyon, 2021.

[155] G. Niccoli, H. Pei and V. Terras, *Correlation functions by separation of variables: the XXX spin chain*, SciPost Phys. **10** (2021) 006.

[156] W. Burnside, *On the Condition of Reducibility of any Group of Linear Substitutions*, Proc. London Math. Soc. **s2-3** (1905) 430–434.

[157] V. Lomonosov and P. Rosenthal, *The simplest proof of Burnside's theorem on matrix algebras*, Linear Algebra Appl. **383** (2004) 45–47.

[158] L.V. Ahlfors, *Complex analysis*, McGraw-Hill, 1979.

[159] N. Beisert, *The dilatation operator of N=4 super Yang–Mills theory and integrability*, Phys. Rep. **405** (2004) 1–202.

[160] A. Sandvik, Course on 'Quantum Spin Simulations' April 2010.

[161] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1996.

[162] *NIST Digital Library of Mathematical Functions*, `https://dlmf.nist.gov/`, Release 1.1.11 of 2023-09-15.

[163] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser, 2007.

[164] G. Frobenius, *Ueber die elliptischen Functionen zweiter Art.*, J. für Reine Angew. Math. **1882** (1882) 53–68.

[165] H. Rosengren and M. Schlosser, *Elliptic determinant evaluations and the Macdonald identities for affine root systems*, Compos. Math. **142** (2006) 937–961.

[166] H. Weber, *Elliptische Funktionen und algebraische Zahlen*, Verlag Vieweg und Sohn, 1908.

[167] H. Rosengren, *An Izergin–Korepin-type identity for the 8VSOS model, with applications to alternating sign matrices*, Adv. Appl. Math. **43** (2009) 137–155.