THE INCOMPRESSIBLE NAVIER STOKES FLOW IN TWO DIMENSIONS WITH PRESCRIBED VORTICITY

SAGUN CHANILLO, JEAN VAN SCHAFTINGEN, AND PO-LAM YUNG

To Dick Wheeden in friendship and appreciation

Abstract. We study the incompressible two dimensional Navier–Stokes equation with initial vorticity in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. This complements our earlier work for the case when the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$.

The two-dimensional incompressible Navier–Stokes equation

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1}$$

models an incompressible flow a fluid whose velocity and mechanical pressure at position $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}$ is represented by the vector $\mathbf{v}(x,t) \in \mathbb{R}^2$ and with a kinematic viscosity coefficient ν. Note we have divided the Navier–Stokes equation by the constant density of the fluid ρ and thus ν in our equation is the dynamic viscosity coefficient divided by the density, assumed constant. Throughout this paper, ∇ will refer only to the spatial derivatives (i.e. derivative in the x variables). We also sometimes use the notation ∂_x to denote a derivative in the x variables when we have no need to be specific which space variable we are differentiating in.

The vorticity of the Navier–Stokes flow is a scalar in the two-dimensional case, defined by

$$\omega = \partial_{x_1}v_2 - \partial_{x_2}v_1$$

where we wrote $\mathbf{v} = (v_1, v_2)$. It propagates according to the convection-diffusion equation

$$\omega_t - \nu\Delta\omega = -\nabla\cdot(\mathbf{v}\omega),$$

which one obtains from (1) by taking the curl of both sides. Formally the velocity $\mathbf{v}$ in the Navier–Stokes equation can be expressed in terms of the vorticity through the Biot–Savart relation

$$\mathbf{v} = (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega). \tag{2}$$

This follows formally by differentiating $\omega = \partial_{x_1}v_2 - \partial_{x_2}v_1$, and using that $\nabla\cdot\mathbf{v} = 0$.

Our theorems concern the solution of the vorticity equation when the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. Here $\dot{W}^{1,1}(\mathbb{R}^2)$ is the completion of $C_c^\infty(\mathbb{R}^2)$ under the norm $\|u\|_{\dot{W}^{1,1}(\mathbb{R}^2)} := \|\nabla u\|_{L^1(\mathbb{R}^2)}$. The theorems are as follows:

Theorem 1. *Consider the two-dimensional vorticity equation*

$$\omega_t - \nu\Delta\omega = -\nabla\cdot(\mathbf{v}\omega), \tag{3}$$

where $\mathbf{v}$ is defined through the Biot–Savart relation (2). Suppose we are given an initial vorticity $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$ at time $t = 0$. If

$$\|\nabla\omega_0(x)\|_{L^1(\mathbb{R}^2)} \le A_0,$$

then there exists a unique solution to the integral formulation of this vorticity equation up to time $t_0 = C\nu/A_0^2$*, such that*

$$\sup_{t\le t_0} \|\nabla\omega(x,t)\|_{L^1(\mathbb{R}^2)} \le 2A_0. \tag{4}$$

Moreover, the solution ω *depends continuously on the initial data* ω_0*, in the sense that if* $\omega_0^{(i)}$ *converges to* ω_0 *in* $\dot W^{1,1}(\mathbb{R}^2)$ *as* $i\to\infty$*, then the sequence of solutions* $\omega^{(i)}(x,t)$ *to (3) with initial data* $\omega_0^{(i)}$ *converges to* $\omega(x,t)$ *in* $L^\infty([0,t_0),\dot W^{1,1}(\mathbb{R}^2))$ *as* $i\to\infty$*.*

Theorem 2. *Let* $\omega_0 \in \dot W^{1,1}(\mathbb{R}^2)$*, and* ω *be the solution to the integral formulation of the vorticity equation (3) given by Theorem 1, with initial vorticity* ω_0*. Define a velocity vector* $\mathbf{v}$ *by the Biot–Savart relation (2). Then* $\mathbf{v}$ *is a distributional solution to the 2-dimensional incompressible Navier–Stokes (1) up to time* $t_0 := C\nu\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}^{-2}$*, in the sense that*

$$\begin{cases} \displaystyle\int_0^{t_0}\int_{\mathbb{R}^2} [\mathbf{v}\cdot\partial_t\Phi + \nu\mathbf{v}\cdot\Delta\Phi + \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\Phi]\,dx\,dt = -\int_{\mathbb{R}^2}\mathbf{v}(x,0)\cdot\Phi(x,0)\,dx \\ \displaystyle\int_0^{t_0}\int_{\mathbb{R}^2}\mathbf{v}\cdot\nabla\psi\,dx\,dt = 0 \end{cases}$$

holds for any $\psi\in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R})$*, and any* $\Phi\in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R}^2)$ *that satisfies* $\nabla\cdot\Phi = 0$ *for all* $t\in[0,t_0)$*. We also have*

$$\sup_{t\le t_0}\|\mathbf{v}(x,t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t\le t_0}\|\nabla\mathbf{v}(x,t)\|_{L^2(\mathbb{R}^2)} \le c\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}, \tag{5}$$

and the pressure $p(x,t) := (-\Delta)^{-1}\nabla\cdot((\mathbf{v}\cdot\nabla)\mathbf{v})$ *satisfies*

$$\sup_{t\le t_0}\|\nabla p(x,t)\|_{L^2(\mathbb{R}^2)} \le c\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}^2. \tag{6}$$

Note that in these theorems, we are only assuming that the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot W^{1,1}(\mathbb{R}^2)$, contrary to [6] where we assumed the stronger assumption that the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2) = \dot W^{1,1}(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$. Giga, Miyakawa, Osada [8] and Kato [9] showed that the vorticity equation is globally well-posed under the hypothesis that the initial vorticity is a measure; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brezis [4]. We point out though that the scaling of their results is different from ours: we are assuming that the *gradient* of the initial vorticity is in L^1. This explains why the solution constructed by Kato satisfies the estimate

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le Ct^{-\frac12},\ t\to 0$$

(see equation (0.5) of [9]), whereas we can obtain bounds on $\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$ and $\|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)}$ that are uniform in t. Indeed it is known that they could not have done better, without further assumptions on the vorticity: the famous example of the *Lamb–Oseen vortex* for $\nu = 1$ [10] consists of an initial vorticity $\omega_0 = \alpha_0\delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ where α_0 is a constant (called the total circulation of the vortex). The corresponding solution ω to the vorticity equation (3) with this initial vorticity, and its corresponding velocity $\mathbf{v}$, are given by

$$\omega(x,t) = \frac{\alpha_0}{4\pi t}e^{-\frac{|x|^2}{4t}}, \qquad \mathbf{v}(x,t) = \frac{\alpha_0}{2\pi}\frac{(-x_2,x_1)}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4t}}\Big).$$

We then have

$$\|\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \sim \|\nabla \mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \sim ct^{-\frac{1}{2}},\ t \to 0.$$

Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorems 1 or 2.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brezis [2, 3].

Proposition 3. *If $\omega(\cdot,t) \in \dot{W}^{1,1}(\mathbb{R}^2)$ at a time t, then one can define a vector-valued function $\mathbf{v}(\cdot,t)$ via the Biot–Savart relation (2) at this time t, in which case we will have*

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\omega(\cdot,t)\|_{L^1(\mathbb{R}^2)}$$

at this time t. Here C is a constant independent of t and ω.

Proof of Proposition 3. For simplicity, let's fix the time t, and drop the dependence of ω and $\mathbf{v}$ on t in the notation. Note that $(-\partial_{x_2}\omega, \partial_{x_1}\omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. The desired conclusion then follows from (2) and the two-dimensional result of Bourgain and Brezis [3].

Since the proof of the two-dimensional result of Bourgain and Brezis [3] is actually quite simple, we adapt it here in our particular setting, for the convenience of the reader.

The main point here is that if $\omega \in C_c^\infty(\mathbb{R}^2)$, then $\mathbf{v} = (v_1, v_2) := (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega)$ satisfies

$$v_1(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \partial_2\omega(x-y)\log\frac{1}{|y|}\,dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} \omega(x-y)\frac{-y_2}{|y|^2}\,dy$$

so

$$|v_1(x)| \le \frac{1}{2\pi}\int_{\mathbb{R}^2} |\omega(x-y)|\frac{1}{|y|}\,dy \le c\|\nabla\omega(x-\cdot)\|_{L^1(\mathbb{R}^2)}$$

the last inequality following from an application of Hardy's inequality (alternatively, one can see that the last inequality holds, by writing $\frac{1}{|y|}$ as $\nabla\cdot\frac{y}{|y|}$, and by integrating by parts). This shows

$$\|v_1\|_{L^\infty(\mathbb{R}^2)} \le c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Similarly one shows $\|v_2\|_{L^\infty(\mathbb{R}^2)} \le c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}$, so

$$\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \le c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Finally,

$$\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)} \le \|\nabla^2(-\Delta)^{-1}\omega\|_{L^2(\mathbb{R}^2)} \le \|\omega\|_{L^2(\mathbb{R}^2)} \le c\|\nabla\omega\|_{L^1(\mathbb{R}^2)},$$

the last inequality following from the Gagliardo–Nirenberg inequality. The above proves the desired conclusion of the proposition under the extra assumption that $\omega \in C_c^\infty(\mathbb{R}^2)$. Since such functions are dense in $\dot{W}^{1,1}(\mathbb{R}^2)$, a standard approximation argument shows that these estimates extend to the general case when $\omega \in \dot{W}^{1,1}(\mathbb{R}^2)$. Hence the full proposition follows. □

Proof of Theorem 1. In the sequel by a scaling we may assume without loss of generality that the viscosity coefficient $\nu = 1$.

Let K_t be the heat kernel on $\mathbb{R}^2$, i.e.

$$K_t(x) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}.$$

Rewriting (3) as an integral equation for ω using Duhamel's theorem, where ω_0 is the initial vorticity, we have,

$$\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]\, ds \tag{7}$$

where $\mathbf{v}$ is given by (2).

We apply a Banach fixed point argument to the operator T given by

$$T\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]\, ds, \tag{8}$$

where again $\mathbf{v}$ is given by (2). Let us set

$$E = \Big\{ g : \mathbb{R}^2 \times [0,t_0] \to \mathbb{R} \mid \sup_{0<t<t_0} \|\nabla g(x,t)\|_{L^1(\mathbb{R}^2)} \leq A \Big\}.$$

We will first show that T maps E into itself, for t_0 chosen as in the theorem.

Differentiating (8) in the space variable once, we get

$$(T\omega(x,t))_x = K_t \star (\omega_0)_x + \int_0^t \partial_x K_{t-s} \star \Big(\mathbf{v}_x \omega\Big)\, ds + \int_0^t \partial_x K_{t-s} \star \Big(\mathbf{v}\omega_x\Big)\, ds.$$

By Young's convolution inequality, we have

$$\|(T\omega(\cdot,t))_x\|_{L^1(\mathbb{R}^2)} \leq \|(\omega_0)_x\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2} (\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)})\, ds.$$

Now we apply Proposition 3 to each of the terms in the integral on the right hand side. For the first term we have, by Cauchy-Schwarz followed by Gagliardo–Nirenberg, that

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \leq C\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \|\omega\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} \|\omega_x\|_{L^1(\mathbb{R}^2)}.$$

We control $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}$ with Proposition 3: this gives

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \leq C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Similarly, by Proposition 3, for the second term, we have

$$\|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)} \leq \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \|\omega_x\|_{L^1(\mathbb{R}^2)} \leq C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Altogether, we have,

$$\|(T\omega)_x\|_{L^1(\mathbb{R}^2)} \leq \|\nabla\omega_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2} \|\nabla\omega\|^2_{L^1(\mathbb{R}^2)}\, ds.$$

Thus if $\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)} \leq A_0$, then since $\omega \in E$, we have

$$\sup_{0\leq t\leq t_0} \|\nabla(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \leq A_0 + Ct_0^{1/2}A^2.$$

By choosing A so that $A_0 = A/2$ and $t_0 = 1/(2CA)^2$, we see that if $\omega \in E$, then

$$\sup_{0\leq t\leq t_0} \|\nabla_x(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \leq A,$$

i.e. $T\omega \in E$. It remains to show that T is a contraction on E.

For this let $\omega_1(x,t)$, $\omega_2(x,t) \in E$. We just need to observe that from Proposition 3, we get

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_\infty + \|\nabla\mathbf{v}_1 - \nabla\mathbf{v}_2\|_2 \leq C\|\nabla(\omega_1 - \omega_2)\|_{L^1(\mathbb{R}^2)}.$$

Thus repeating our earlier computations, we see that

$$\sup_{0\le t\le t_0} \|\nabla(T\omega_1 - T\omega_2)\|_{L^1(\mathbb{R}^2)} \le Ct_0^{1/2} A \sup_{0\le t\le t_0} \|\nabla(\omega_1-\omega_2)\|_{L^1(\mathbb{R}^2)}.$$

By the choice of t_0, it is seen that T is a contraction. Thus using the Banach fixed point theorem on E, we obtain our operator T has a fixed point and so the integral equation (7) has a unique solution in E. The continuous dependence on initial data can be proved in an identical way, and we will not repeat the details here. □

Proof of Theorem 2. Let $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$, and $\omega(x,t)$ be the unique solution to (7) given above. Let $\mathbf{v}(x,t)$ be defined by the Biot–Savart relation (2) as in Proposition 3. If $\omega_0^{(i)}$ is a sequence of functions in $C_c^\infty(\mathbb{R}^2)$ converging to ω_0 in $\dot{W}^{1,1}(\mathbb{R}^2)$, then the corresponding solution $\omega^{(i)}(x,t)$ to the vorticity equation (3) converges to $\omega(x,t)$ in $L^\infty([0,t_0),\dot{W}^{1,1}(\mathbb{R}^2))$. Thus the velocities $\mathbf{v}^{(i)} := (-\Delta)^{-1}(-\partial_{x_2}\omega^{(i)}, \partial_{x_1}\omega^{(i)})$ converges in $L^\infty([0,t_0);L^\infty(\mathbb{R}^2))$ to $\mathbf{v}$. But since $\omega_0^{(i)} \in C_c^\infty(\mathbb{R}^2)$, which are in particular in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$, so we may apply Theorem II of Kato [9] as in [6], and conclude that the $\mathbf{v}^{(i)}$ defined above solves the Navier–Stokes equation (1), at least in the distributional sense. We can now pass to limit as $i\to\infty$, using the convergence of $\mathbf{v}^{(i)}$ to $\mathbf{v}$ in $L^\infty([0,t_0),L^\infty(\mathbb{R}^2))$ we obtained above, and appealing to the dominated convergence theorem: this shows that $\mathbf{v}(x,t)$ is also a distributional solution to (1) up to time t_0, in the sense that

$$\begin{cases} \displaystyle\int_0^{t_0}\int_{\mathbb{R}^2} [\mathbf{v}\cdot\partial_t\Phi + \mathbf{v}\cdot\Delta\Phi + \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\Phi]\,dx\,dt = -\int_{\mathbb{R}^2} \mathbf{v}(x,0)\cdot\Phi(x,0)\,dx \\ \displaystyle\int_0^{t_0}\int_{\mathbb{R}^2} \mathbf{v}\cdot\nabla\psi\,dx\,dt = 0 \end{cases}$$

holds for any $\psi \in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R})$, and any $\Phi\in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R}^2)$ that satisfies $\nabla\cdot\Phi = 0$ for all $t\in[0,t_0)$. The estimate (5) then follows from Proposition 3 and (4). Lastly we observe that the estimate (6) follows, from the fact that the pressure $p(x,t)$ satisfies the equation

$$-\Delta p = \nabla\cdot((\mathbf{v}\cdot\nabla)\mathbf{v}),$$

which is a consequence of taking the divergence of the Navier–Stokes equation. □

References

[1] M. Ben-Artzi, *Global solutions of two-dimensional Navier–Stokes and Euler equations*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 329–358.

[2] J. Bourgain and H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 7, 539–543.

[3] ______, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 2, 277–315.

[4] H. Brezis, *Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier–Stokes and Euler equations"*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 359–360.

[5] S. Chanillo, J. Van Schaftingen, and P.-L. Yung, *Variations on a proof of a borderline Bourgain–Brezis Sobolev embedding theorem*, Submitted July 2015, available at `arXiv:1612.02888`.

[6] ______, *Applications of Bourgain-Brézis inequalities to fluid mechanics and magnetism*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 1, 51–55.

[7] S. Chanillo and P.-L. Yung, *An improved Strichartz estimate for systems with divergence free data*, Comm. Partial Differential Equations **37** (2012), no. 2, 225–233.

[8] Y. Giga, T. Miyakawa, and H. Osada, *Two-dimensional Navier–Stokes flow with measures as initial vorticity*, Arch. Rational Mech. Anal. **104** (1988), no. 3, 223–250.

[9] T. Kato, *The Navier–Stokes equation for an incompressible fluid in $\mathbf{R}^2$ with a measure as the initial vorticity*, Differential Integral Equations **7** (1994), no. 3-4, 949–966.
[10] C. W. Oseen, *Über Wirbelbewegung in einer reibenden Flüssigheit*, Ark. Mat. Astr. Fys. **7** (1912), 1–13.
[11] J. Van Schaftingen, *Estimates for L^1-vector fields*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 181–186.

Department of Mathematics, Rutgers, the State University of New Jersey, NJ 08854
E-mail address: chanillo@math.rutgers.edu

Institut de Recherche en Mathématique et en Physique, Université catholique de Louvain, Chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
E-mail address: Jean.VanSchaftingen@uclouvain.be

Department of Mathematics, the Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: plyung@math.cuhk.edu.hk