

Uniform tail asymptotics for Airy kernel determinant solutions to KdV and for the narrow wedge solution to KPZ

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Abstract

We obtain uniform asymptotics for deformed Airy kernel determinants, which arise in models of finite temperature free fermions and which characterize the narrow wedge solution of the Kardar–Parisi–Zhang equation. The asymptotics for the determinants yield uniform initial data for an associated family of solutions to the Korteweg–de Vries equation, and uniform lower tail asymptotics for the narrow wedge solution of the Kardar–Parisi–Zhang equation.

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1 Introduction

We study Fredholm determinants of deformed Airy kernels of the form

$$Q_\sigma(x, t) := \det(1 - \mathbb{K}_{\sigma, x, t}^{\text{Ai}}) \quad (1.1)$$

where $\mathbb{K}_{\sigma, x, t}^{\text{Ai}}$ is the integral operator acting on $L^2(\mathbb{R})$ with kernel

$$K_{\sigma, x, t}^{\text{Ai}}(u, v) = \sigma(r(u; x, t))K^{\text{Ai}}(u, v), \quad r(u; x, t) = \frac{u}{t^{2/3}} + \frac{x}{t}, \quad (1.2)$$

where K^{Ai} is the Airy kernel

$$K^{\text{Ai}}(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u - v},$$

Ai is the Airy function, and $\sigma : \mathbb{R} \rightarrow [0, 1]$ belongs to a class of functions that we will specify below, and which is in particular such that $\mathbb{K}_{\sigma, x, t}^{\text{Ai}}$ is trace class for all $x \in \mathbb{R}$ and $t > 0$. Expressing the Fredholm determinant as a Fredholm series, we have

$$Q_\sigma(x, t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \det(K^{\text{Ai}}(u_i, u_j))_{i,j=1}^k \prod_{j=1}^k \sigma(r(u_j; x, t)) du_j. \quad (1.3)$$

Fredholm determinants of this form appear in various contexts. If $\sigma = 1_{(0,+\infty)}$ is the Heaviside function, $Q_\sigma(x, t)$ is equal to the Tracy–Widom distribution $F^{\text{TW}}(s)$ evaluated at $s = -xt^{-1/3}$ [39]. In this case, the Tracy–Widom formula [39] states that

$$\partial_x^2 \log Q_{1_{(0,+\infty)}}(x, t) = -t^{-2/3} y^2 \left(-xt^{-1/3} \right), \quad (1.4)$$

where y is the Hastings–McLeod solution of the Painlevé II equation $y'' = \xi y + 2y^3$, uniquely characterized by the asymptotics

$$y(\xi) \sim \text{Ai}(\xi) \quad \text{as } \xi \rightarrow +\infty. \quad (1.5)$$

Moreover, this solution also satisfies the asymptotics

$$y(\xi) = \sqrt{\frac{-\xi}{2}} \left(1 + \frac{1}{8\xi^3} + \mathcal{O}(\xi^{-6}) \right) \quad \text{as } \xi \rightarrow -\infty. \quad (1.6)$$

Upper and lower tail asymptotics for $\log Q_{1_{(0,+\infty)}}(x, t)$ as $s = -xt^{-1/3}$ goes to $\pm\infty$ can be derived from the Tracy–Widom formula (1.4), the fact that $\lim_{x \rightarrow +\infty} \partial_x \log Q_{1_{(0,+\infty)}}(x, t) = 0$, and the asymptotics (1.5) and (1.6) for y : as $s = -xt^{-1/3} \rightarrow -\infty$, we have

$$\log F^{\text{TW}}(-xt^{-1/3}) = \log Q_{1_{(0,+\infty)}}(x, t) = -\frac{x^3}{12t} - \frac{1}{8} \log(xt^{-1/3}) + c + o(1), \quad (1.7)$$

and the value of c was proven to be $c = \frac{\log 2}{24} + \zeta'(-1)$ in [13, 2]. Moreover, it is straightforward to verify that

$$u_{1_{(0,+\infty)}}(x, t) := \partial_x^2 \log Q_{1_{(0,+\infty)}}(x, t) + \frac{x}{2t} \quad (1.8)$$

is a solution to the Korteweg–de Vries (KdV) equation

$$\partial_t u + 2u \partial_x u + \frac{1}{6} \partial_x^3 u = 0. \quad (1.9)$$

We will be interested here in Q_σ for a class of smooth functions σ , and the most important choice of σ for our concerns will be

$$\sigma(r) = \sigma_{\text{KPZ}}(r) := \frac{1}{1 + e^{-r}}.$$

Then $Q_\sigma(x, t)$ characterizes the probability distribution of the narrow wedge solution of the KPZ equation, see e.g. [1, 37, 8, 4], and encodes information about the tail probabilities of the KPZ solution. See also [28] for the $t \rightarrow +\infty$ asymptotics of $Q_\sigma(x, t)$ for $\sigma = \sigma_{\text{KPZ}}$. The lower tail expansion is particularly challenging, and it is connected to the $x \rightarrow +\infty$ asymptotics for $Q_\sigma(x, t)$, as we will explain in more detail in Section 2. Lower tail expansions have been obtained as $x \rightarrow +\infty$ for fixed t [10, 40, 6, 29, 25], but without uniformity for t small. In this work, we will complete the description of the lower tail of the KPZ solution with narrow wedge initial data, by deriving all growing terms in the $x \rightarrow +\infty$ asymptotics, with uniformity for t small.

By algebraic manipulations of the Fredholm determinant (see e.g. [1, Proof of Proposition 5.1]), one can derive an alternative expression for $Q_{\sigma_{\text{KPZ}}}$,

$$Q_{\sigma_{\text{KPZ}}}(x, t) = \det \left(1 - 1_{(-xt^{-1/3}, \infty)} \mathbb{L}_{\sigma_{\text{KPZ}}, x, t}^{\text{Ai}} \right), \quad (1.10)$$

where $1_{(-xt^{-1/3}, \infty)}$ is the projection operator onto $L^2(-xt^{-1/3}, \infty)$, and $\mathbb{L}_{\sigma, x, t}^{\text{Ai}}$ is the integral operator acting on $L^2(\mathbb{R})$ corresponding to the deformed Airy kernel

$$L_{\sigma, x, t}^{\text{Ai}}(u, v) = \int_{\mathbb{R}} \sigma(st^{-2/3}) \text{Ai}(u+s) \text{Ai}(v+s) ds. \quad (1.11)$$

This representation of $Q_{\sigma_{\text{KPZ}}}$ is known as the finite temperature analogue of the Tracy–Widom distribution, arising in the physics of free fermion models at finite temperature [12] and as the largest particle distribution in the MNS model [32, 22, 31]. In this context, $x \rightarrow +\infty$ asymptotics of $Q_{\sigma_{\text{KPZ}}}(x, t)$ describe the large gap probability for the largest free fermion.

For a large class of functions σ , including all σ satisfying Assumptions 1.1 below, we have

$$\partial_x^2 \log Q_{\sigma}(x, t) = -\frac{1}{t} \int_{\mathbb{R}} \phi_{\sigma}^2(r; x, t) d\sigma(r),$$

where ϕ_{σ} satisfies the integro-differential Painlevé II equation

$$\partial_x^2 \phi_{\sigma}(z; x, t) = \left(z - \frac{x}{t} + \frac{2}{t} \int_{\mathbb{R}} \phi_{\sigma}^2(r; x, t) d\sigma(r) \right) \phi_{\sigma}(z; x, t), \quad (1.12)$$

see [1] and [7, Corollary 1.4], and like in the case $\sigma = 1_{(0, +\infty)}$, we have that u_{σ} defined by

$$u_{\sigma}(x, t) := \partial_x^2 \log Q_{\sigma}(x, t) + \frac{x}{2t} \quad (1.13)$$

solves the KdV equation (1.9). This result was proved in [7, Theorem 1.3], but the relation between Fredholm determinants of a general class of operators and the KdV equation was already understood much earlier by Pöppe [33]; for other similar relations between Fredholm determinants and integrable systems (in particular, the KP equation) see also [34] and the recent works [3, 5, 24, 29, 30, 36].

In the case $\sigma = 1_{(0, +\infty)}$, the $xt^{-1/3} \rightarrow \pm\infty$ asymptotics for the KdV solution are obtained easily from (1.4)–(1.8): we have

$$\begin{aligned} u_{1_{(0, +\infty)}}(x, t) &= \frac{x}{2t} + \mathcal{O}\left(\frac{e^{-\frac{4}{3}(-xt^{-1/3})^{3/2}}}{t^{2/3}(xt^{-1/3})}\right), & \text{as } xt^{-1/3} \rightarrow -\infty, \\ u_{1_{(0, +\infty)}}(x, t) &= \frac{1}{8x^2} + \mathcal{O}\left(\frac{1}{t^{2/3}(xt^{-1/3})^5}\right), & \text{as } xt^{-1/3} \rightarrow +\infty. \end{aligned}$$

In [7, Theorem 1.8], it was proved, again for a class of functions σ including those satisfying Assumptions 1.1 below, that the $t \rightarrow 0$ asymptotics for $u_{\sigma}(x, t)$ are as follows, see also Figure 1.

- (i) For any $t_0 > 0$, there exist $M, c > 0$ such that we have uniformly for $x \leq -Mt^{1/3}$ and for $0 < t < t_0$ that

$$u_{\sigma}(x, t) = \frac{x}{2t} + \mathcal{O}\left(e^{-c\frac{|x|}{t^{1/3}}}\right). \quad (1.14)$$

- (ii) There exists $\epsilon > 0$ such that for any $M > 0$, we have uniformly for $|x| \leq Mt^{1/3}$ and for $0 < t < \epsilon$ that

$$u_{\sigma}(x, t) = \frac{x}{2t} - t^{-2/3} y^2(-xt^{-1/3}) + \mathcal{O}(1), \quad (1.15)$$

with y the Hastings–McLeod solution of Painlevé II.

- (iii) There exist $\epsilon, M > 0$ such that for any $K > 0$, we have uniformly for $Mt^{1/3} \leq x \leq K$ and for $0 < t < \epsilon$ that

$$u_\sigma(x, t) = v_\sigma(x) \left(1 + \mathcal{O}\left(x^{-1}t^{1/3}\right)\right), \quad (1.16)$$

where v_σ is a function of $x > 0$, independent of t , with asymptotics

$$v_\sigma(x) = \frac{1}{8x^2} + \frac{1}{2} \int_{-\infty}^{+\infty} (1_{(0,+\infty)}(r) - \sigma(r)) dr + \mathcal{O}(x^2), \quad \text{as } x \rightarrow 0_+. \quad (1.17)$$

Moreover, v_σ satisfies an integro-differential version of the Painlevé V equation [7, Theorem 1.12].

It follows from these asymptotics that the KdV solutions u_σ are unbounded for all $t > 0$ and have ill-defined initial data at $t = 0$; in particular their direct and inverse scattering theory is non-standard (see also [17, 19]).

The analogous result for $Q_\sigma(x, t)$ [7, Theorem 1.14] is as follows.

- (i) For any $t_0 > 0$, there exist $M, c > 0$ such that we have uniformly for $x \leq -Mt^{1/3}$ and for $0 < t < t_0$ that

$$\log Q_\sigma(x, t) = \mathcal{O}\left(e^{-c \frac{|x|}{t^{1/3}}}\right). \quad (1.18)$$

- (ii) There exists $\epsilon > 0$ such that for any $M > 0$, we have uniformly for $|x| \leq Mt^{1/3}$ and for $0 < t < \epsilon$ that

$$\log Q_\sigma(x, t) = \log F_{\text{TW}}(-xt^{-1/3}) + \mathcal{O}(t^{1/3}), \quad (1.19)$$

where F_{TW} is the Tracy–Widom distribution.

- (iii) There exist $\epsilon, M > 0$ such that for any $K > 0$, we have uniformly for $Mt^{1/3} \leq x \leq K$ and for $0 < t < \epsilon$ that

$$\log Q_\sigma(x, t) = -\frac{x^3}{12t} - \frac{1}{8} \log(xt^{-1/3}) + \frac{\log 2}{24} + \zeta'(-1) + \int_0^x (x-\xi) \left(v_\sigma(\xi) - \frac{1}{8\xi^2}\right) d\xi + \mathcal{O}(x^{-1}t^{1/3}), \quad (1.20)$$

where ζ is the Riemann zeta function.

It is important to note that these asymptotics are uniform for x large and negative, but not for x large and positive. The above expansion (1.20) should be compared with the tail expansion (1.7) for the Tracy–Widom distribution.

Besides for $\sigma = 1_{(0,+\infty)}$, precise asymptotics for $Q_\sigma(x, t)$ as $x \rightarrow +\infty$ are only known for $\sigma = \sigma_{\text{KPZ}}$ [6, Theorem 1.1]; they hold for $t > 0$ fixed and also as $t \rightarrow 0$ slow enough such that $t \geq \delta/x$, where $\delta > 0$ is fixed (see also Figure 1).¹

$$\log Q_{\sigma_{\text{KPZ}}}(x, t) = -\frac{1}{\pi^6 t^4} F_1(\pi^2 xt) - \frac{1}{6} \sqrt{1 + \pi^2 xt} + \mathcal{O}(\log^2(xt^{-1/3})) + \mathcal{O}(t^{-2/3}), \quad (1.21)$$

where

$$F_1(y) := \frac{4}{15}(1+y)^{5/2} - \frac{4}{15} - \frac{2}{3}y - \frac{1}{2}y^2. \quad (1.22)$$

Observe that $F_1(y) \sim \frac{y^3}{12}$ as $y \rightarrow 0$, and $F_1(y) \sim \frac{4}{15}y^{5/2}$ as $y \rightarrow \infty$.

¹The parameters s and T of [6] correspond respectively to $xt^{-1/3}$ and t^{-2} in this paper.

Our main objective is to derive $x \rightarrow +\infty$ asymptotics for $u_\sigma(x, t)$ and $Q_\sigma(x, t)$ that are uniform for t sufficiently small (also for $t \leq \delta/x$), for a rather broad class of functions σ , in order to complete the description of the initial data for the KdV solutions $u_\sigma(x, t)$, and to describe the cross-over between the small t asymptotics (1.20) and the tail expansion (1.21) of $Q_\sigma(x, t)$. In the particular case $\sigma = \sigma_{\text{KPZ}}$, the asymptotics for $Q_\sigma(x, t)$ will allow us to derive uniform lower tail asymptotics for the KPZ solution with narrow wedge initial data, see Section 2. Our results are more general and also more precise than (1.21): Theorem 1.3 below, even when specialized to $\sigma = \sigma_{\text{KPZ}}$ and to the subsector $t \geq \delta/x$, $t \leq t_0$, contains new results, see Remark 1.4, part 3.

Let us now formulate our assumptions on σ .

Assumptions 1.1. *The function $\sigma : \mathbb{R} \rightarrow [0, 1)$ is such that $F(r) := \frac{1}{1-\sigma(r)}$ satisfies the following conditions.*

1. *F can be extended to an entire function in the complex plane.*
2. *$F'(x) \geq 0$ and $(\log F)''(x) \geq 0$ for all $x \in \mathbb{R}$.*
3. *There exist $c_-, c'_- > 0$ such that $F(r) = 1 + c'_- e^{-c_-|r|}(1 + o(1))$ as $r \rightarrow -\infty$.*
4. *There exist $c_+, c'_+, \epsilon > 0$ such that $F(r) = c'_+ e^{c_+r}(1 + \mathcal{O}(e^{-\epsilon r}))$ as $r \rightarrow +\infty$, and for $z \in \mathbb{C}$, we have*

$$|F(z)| = \mathcal{O}(e^{c_+ \operatorname{Re} z}), \quad \text{as } \operatorname{Re} z \rightarrow +\infty.$$

Remark 1.2. 1. *Functions σ with discontinuities like $\sigma = 1_{(0, +\infty)}$ are not admissible.*

2. *A class of admissible functions F consists of the Laplace transforms of measures of the form $\mu = \delta_0 + a_- \delta_{c_-} + \nu + a_+ \delta_{c_+}$, where $c_\pm, a_\pm > 0$, $c_- \leq c_+$, where δ_x is the Dirac distribution at x , and where ν is a finite Borel measure whose support is a subset of $[c_-, c_+)$ without atom at c_- . Then, it is indeed straightforward to verify that*

$$F(z) = \int_{\mathbb{R}} e^{tz} d\mu(t) = 1 + a_- e^{c_- z} + \int_{\mathbb{R}} e^{tz} d\nu(t) + a_+ e^{c_+ z}$$

satisfies the assumptions, with $c'_\pm = a_\pm$ if $c_- < c_+$, and $c'_+ = c'_- = a_+ + a_-$ if $c_+ = c_-$. Observe that these functions are completely monotonic, i.e. F and all its derivatives are positive. In particular, our prototype example $\sigma = \sigma_{\text{KPZ}}$ is admissible and belongs to this class, with

$$F(z) = 1 + e^z, \quad \mu = \delta_0 + \delta_1, \quad c_\pm = c'_\pm = 1, \quad \epsilon \leq 1.$$

3. *Assumptions 1.1 imply that*

$$(\log F)'(r) \leq \frac{\log F(2r) - \log F(r)}{r} = c_+ + \mathcal{O}(e^{-\epsilon r}) \leq (\log F)'(2r), \quad r \rightarrow +\infty,$$

from which we conclude that

$$(\log F)'(r) = c_+ + \mathcal{O}(e^{-\frac{\epsilon}{2}r}), \quad r \rightarrow +\infty. \quad (1.23)$$

Similarly, since

$$(\log F)'(2r) \leq \frac{\log F(2r) - \log F(r)}{r} = \mathcal{O}(e^{-c_-|r|}) \leq (\log F)'(r), \quad r \rightarrow -\infty,$$

we deduce that

$$(\log F)'(r) = \mathcal{O}(e^{-\frac{c_-}{2}|r|}), \quad r \rightarrow -\infty. \quad (1.24)$$

4. In [7] a broader class of functions σ is considered. Namely, it is assumed in [7, Assumptions 1.1] that σ is non-decreasing and $\mathcal{C}^\infty$ on the real line (except possibly at a finite number of points), that $r \mapsto r^2\sigma(r) \in L^2((-\infty, 0), dr)$, and in [7, equations (1.21) and (1.22)] that σ is exponentially close to $1_{(0, +\infty)}$ and σ' vanishes at least quadratically at $\pm\infty$. It is straightforward to check that if σ satisfies Assumptions 1.1, then it also satisfies all the assumptions made in [7] which we just recalled. In particular, the operator $\mathbb{K}_{\sigma, x, t}$ is trace-class [7, Remark 1.2] and all the above mentioned results from [7] hold. The stronger Assumptions 1.1 are needed here for technical reasons in the asymptotic analysis.

Theorem 1.3. *Let σ satisfy Assumptions 1.1, let $Q_\sigma(x, t)$ be as in (1.1)–(1.2), and let $u_\sigma(x, t)$ be as in (1.13). For any $t_0 > 0$, there exists $K > 0$ such that we have the asymptotics*

$$u_\sigma(x, t) = \frac{x}{2t} a_0 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{1}{2\sqrt{xt}} a_1 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{t^{1/2}}{2x^{3/2}} a_2 \left(\frac{\pi^2}{c_+^2} xt \right) + \mathcal{O}(x^{-2}), \quad (1.25)$$

$$\log Q_\sigma(x, t) = -\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + F_3 \left(\frac{\pi^2}{c_+^2} xt \right) - C \log t + \mathcal{O}(1), \quad (1.26)$$

with error terms that are uniform for $x \geq K$ and $0 < t \leq t_0$. Here

$$a_0(y) = \frac{(\sqrt{y+1}-1)^2}{y}, \quad a_1(y) = -\frac{\log c'_+}{\pi} \sqrt{\frac{y}{1+y}}, \quad (1.27)$$

$$a_2(y) = \frac{y^{3/2}}{\sqrt{y+1}(\sqrt{y+1}-1)^2} \left(\frac{1}{4\pi c_+} \frac{1-2\sqrt{y+1}}{y+1} \log^2 c'_+ - j_\sigma \right), \quad (1.28)$$

$$F_1(y) = \frac{4}{15} (1+y)^{5/2} - \frac{4}{15} - \frac{2}{3}y - \frac{1}{2}y^2, \quad F_2(y) = \frac{2}{3} (1+y)^{3/2} - \frac{2}{3} - y = F'_1(y), \quad (1.29)$$

$$F_3(y) = \frac{2c_+ j_\sigma}{\pi} \sqrt{1+y} - \frac{1}{48} \log(1+y) + \left(\frac{2c_+ j_\sigma}{\pi} + \frac{\log^2 c'_+}{2\pi^2} + \frac{1}{24} \right) \log(\sqrt{1+y}-1), \quad (1.30)$$

and

$$C = \frac{2c_+ j_\sigma}{\pi} + \frac{\log^2 c'_+}{2\pi^2}, \quad j_\sigma = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\log(1-\sigma(r)) + (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr.$$

Remark 1.4. 1. The form of the above expansions was predicted on a non-rigorous basis in [29] by checking consistency with the cylindrical KdV equation, which is a reduction of the KP equation and equivalent to the KdV equation through a change of variables.

2. Using (1.29)–(1.30), we obtain

$$\begin{aligned} & -\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} Kt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} Kt \right) + F_3 \left(\frac{\pi^2}{c_+^2} Kt \right) \\ & = -\frac{K^3}{12t} + \left(\frac{2c_+ j_\sigma}{\pi} + \frac{\log^2 c'_+}{2\pi^2} + \frac{1}{24} \right) \log t + \mathcal{O}(1), \quad \text{as } t \rightarrow 0. \end{aligned}$$

Substituting the above in (1.26) yields $\log Q_\sigma(K, t) = -\frac{K^3}{12t} + \frac{1}{24} \log t + \mathcal{O}(1)$ as $t \rightarrow 0$, which is consistent with (1.20).

3. Recall the relation $\partial_x^2 \log Q_\sigma(x, t) + \frac{x}{2t} = u_\sigma(x, t)$, see (1.13). Even though we have no control on the second derivative of the error term in the asymptotic expansion of $\log Q_\sigma$, it is instructive to formally substitute the asymptotics for $\log Q_\sigma$ and u_σ as $x \rightarrow \infty$ with $t > 0$ fixed into this relation (note that $\partial_x = \frac{y}{x} \partial_y$ with $y = \pi^2 xt / c_+^2$), and to verify order by order whether the result is consistent. We then observe that the leading and sub-leading order in the relation are consistent because of the identities

$$-\left(\frac{y}{x}\right)^2 \frac{c_+^6}{\pi^6 t^4} F_1''(y) + \frac{x}{2t} = \frac{x}{2t} a_0(y), \quad -\left(\frac{y}{x}\right)^2 \frac{c_+^3 \log c_+'}{\pi^4 t^2} F_2''(y) = \frac{1}{2\sqrt{xt}} a_1(y).$$

For the next order, we should have

$$\left(\frac{y}{x}\right)^2 F_3''(y) = \frac{t^{1/2}}{2x^{3/2}} a_2(y) + \mathcal{O}(x^{-2}),$$

and this is indeed the case, by the identity

$$\left(\frac{y}{x}\right)^2 F_3''(y) - \frac{t^{1/2}}{2x^{3/2}} a_2(y) = \frac{y^2 (-3y + 2\sqrt{y+1} - 3)}{96x^2 (y+1)^{5/2} (\sqrt{y+1} - 1)^2},$$

where we note that the right-hand side is a uniformly bounded function of $y \in [0, +\infty)$ times x^{-2} .

4. For xt bounded, the term $\frac{t^{1/2}}{2x^{3/2}} a_2 \left(\frac{\pi^2}{c_+^2} xt \right)$ in the expansion (1.25) of u_σ can simply be absorbed by the error term. However, as $xt \rightarrow \infty$, this term is of order $\mathcal{O}(\frac{(xt)^{1/2}}{x^2})$.
5. For $\sigma = \sigma_{\text{KPZ}}$, we have $c_+ = c_+' = 1$, $j_{\sigma_{\text{KPZ}}} = -\frac{\pi}{12}$, $C = -\frac{1}{6}$, and the asymptotics (1.26) become

$$\begin{aligned} \log Q_{\sigma_{\text{KPZ}}}(x, t) = & -\frac{F_1(\pi^2 xt)}{\pi^6 t^4} - \frac{\sqrt{1 + \pi^2 xt}}{6} - \frac{\log(1 + \pi^2 xt)}{48} \\ & - \frac{\log(\sqrt{1 + \pi^2 xt} - 1)}{8} + \frac{\log t}{6} + \mathcal{O}(1), \end{aligned} \quad (1.31)$$

uniformly for $x \geq K$ and $0 < t \leq t_0$. This result is more precise and valid for a wider sector of the (x, t) -half-plane than (1.21). Up to the coefficient of the logarithmic term in t , this expansion was predicted in [29, equation (50)]. We thus confirm the result of [29] rigorously with uniform error terms, and refine it by computing the logarithmic term.

The starting point of our analysis is a characterization of $\log Q_\sigma(x, t)$ in terms of a Riemann–Hilbert (RH) problem, obtained in [7] by relying on the theory of integrable operators developed by Its, Izergin, Korepin, and Slavnov [18]: from [7, (3.14) and the second equation after (5.3)], we know that

$$\partial_x \log Q_\sigma(x, t) = p(x, t) - \frac{x^2}{4t}, \quad (1.32)$$

and that

$$u_\sigma(x, t) = \partial_x p(x, t) = -2q(x, t) - p(x, t)^2, \quad (1.33)$$

where $p(x, t), q(x, t)$, which depend also on σ , are defined in terms of the unique solution $\Psi(z; x, t)$ to the following RH problem.

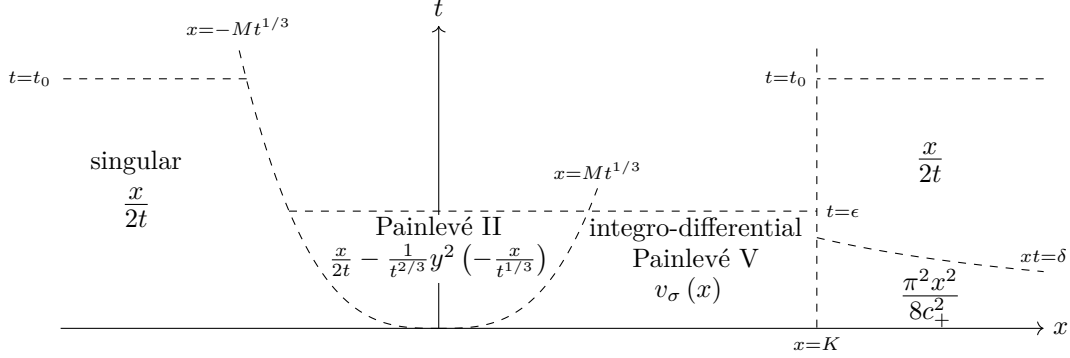


Figure 1: Phase diagram showing the different leading asymptotics for $u_\sigma(x, t)$ when t is small or x is large (or both, uniformly in the indicated regions); sub-leading corrections are omitted and can be found in Theorem 1.3 and [7, Theorem 1.8]. When x is large and positive, the leading asymptotics are given by $\frac{x}{2t} a_0(\pi^2 x t / c_+^2)$. Since $a_0(y) \sim y/4$ as $y \rightarrow 0$ and $a_0(y) \sim 1$ as $y \rightarrow +\infty$, this asymptotic behavior exhibits a phase transition in the critical regime when $x \rightarrow +\infty$ and $t \rightarrow 0_+$ in such a way that xt is bounded away from zero and infinity. Outside of this critical regime we have the two asymptotic behaviors indicated in the diagram. It is worth mentioning that the leading behavior $x/(2t)$, emerging when $x \rightarrow \pm\infty$, is itself a KdV solution.

RH problem for Ψ

- (a) $\Psi : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- (b) The boundary values of Ψ are continuous on $\mathbb{R}$ and are related by

$$\Psi_+(z; x, t) = \Psi_-(z; x, t) \begin{pmatrix} 1 & 1 - \sigma(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}.$$

The subscript $+$ (resp. $-$) indicates the boundary value from above (resp. below) the real axis.

- (c) As $z \rightarrow \infty$, we have

$$\Psi(z; x, t) = \left(I + \frac{\Psi_1(x, t)}{z} + \mathcal{O}(z^{-2}) \right) z^{\frac{1}{4}\sigma_3} A^{-1} e^{(-\frac{2}{3}tz^{\frac{3}{2}} + xz^{\frac{1}{2}})\sigma_3} \begin{cases} I, & |\arg z| < \pi - \delta, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm \arg z < \pi, \end{cases}$$

for any $\delta \in (0, \frac{\pi}{2})$, where we take the principal branches of $z^{\frac{1}{4}\sigma_3}$, $z^{\frac{3}{2}}$ and $z^{\frac{1}{2}}$, analytic in $\mathbb{C} \setminus [0, +\infty)$ and positive for $z > 0$, and

$$\Psi_1(x, t) = \begin{pmatrix} q(x, t) & -ir(x, t) \\ ip(x, t) & -q(x, t) \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad (1.34)$$

Our main task will be to derive asymptotics for the RH solution $\Psi(z; x, t)$ as $x \rightarrow +\infty$, uniform in $t \leq t_0$ for any $t_0 > 0$ and uniform for large z ; this is achieved via the Deift–Zhou nonlinear steepest descent method [15]. The large z asymptotics in condition (c) of the RH problem for Ψ will allow us to derive asymptotics for $\log Q_\sigma(x, t)$ and $u_\sigma(x, t)$ using the identities (1.32) and (1.33).

In order to obtain asymptotics for Ψ , we will need to consider two asymptotic regimes separately. First, we will consider $K \leq x \leq \delta/t$ for a sufficiently large $K > 0$ and a sufficiently small $\delta > 0$. The

asymptotic RH analysis in this regime requires novel ideas, in particular the manner in which the jump is decomposed into an explicit part taken care of by the g -function and a σ -dependent part taken care of by the global parametrix, and is the most challenging part of the asymptotic analysis; this is the subject of Section 3. Secondly, we will consider the regime $x \geq \delta/t$, and in this part of the analysis $\delta > 0$ will be arbitrary. Here, the asymptotic analysis is a generalization of the analysis done in [6] for $\sigma = \sigma_{\text{KPZ}}$, which we also refine in order to cover general σ and in order to compute all growing terms in the asymptotic expansions; this is done in Section 4. Finally, the results for the two regimes are combined in Section 4.9, where the proof of Theorem 1.3 is completed.

2 Lower tail of the KPZ solution with narrow wedge initial data

The KPZ equation was introduced by Kardar, Parisi, and Zhang in 1986 [23] as a universal model for interface growth, and is believed to model a variety of real-life phenomena such as bacterial colony growth, see e.g. the review article [21]. It is a stochastic PDE, given by

$$\partial_T \mathcal{H}(T, X) = \frac{1}{2} \partial_X^2 \mathcal{H}(T, X) + \frac{1}{2} (\partial_X \mathcal{H}(T, X))^2 + \xi(T, X), \quad (2.1)$$

where the first term in the right-hand side stands for relaxation, the second term for non-linear slope-dependent growth, and the random forcing term $\xi(T, X)$ is space-time white noise. Mathematical sense can be given to this equation through the Hopf–Cole transformation $\mathcal{H}(T, X) = \log Z(T, X)$, which transforms the KPZ equation to the stochastic heat equation

$$\partial_T Z(T, X) = \frac{1}{2} \partial_X^2 Z(T, X) + Z(T, X) \xi(T, X),$$

a stochastic PDE for which the solution theory is classical. A more general notion of solution was constructed by Hairer [20]. The narrow wedge solution of KPZ is one of the physically relevant solutions, characterized by Dirac initial data of the Hopf–Cole transform $Z(0, X) = \delta_{X=0}$. With this initial condition, typical KPZ solutions behave roughly like a narrowing parabola $\frac{X^2}{2T}$ as $T \rightarrow 0$. We refer to [9] for more details about this solution and to [35] for a general discussion on Hopf–Cole solutions of KPZ.

In what follows, $\mathcal{H}(T, X)$ denotes the Hopf–Cole solution of KPZ equation with narrow wedge initial condition. Amir, Corwin, and Quastel [1] proved in 2010 that the probability distribution of $\mathcal{H}(T, X)$ can be characterized by the Fredholm determinant $Q_{\sigma_{\text{KPZ}}}$. Such a characterization was obtained independently, but without rigorous justification of certain steps, by Sasamoto and Spohn [37], and predicted also by Dotsenko [16] and Calabrese, Le Doussal, and Rosso [8]. Let us now describe this characterization.

The probability distribution of $\mathcal{H}(T, X) - \frac{X^2}{2T}$ is independent of X [9, Theorem 1.1]), hence we can restrict ourselves to studying the probability distribution of $\mathcal{H}(T, 0)$, or to the distribution of the suitably re-scaled random variable

$$\Upsilon_T = \frac{\mathcal{H}(2T, 0) + \frac{T}{12}}{T^{1/3}}. \quad (2.2)$$

Borodin and Gorin [4, Theorem 2.1] proved that the results from [1] imply the identity

$$\mathbb{E} \left[e^{-e^{T^{1/3}(\Upsilon_T + s)}} \right] = Q_{\sigma_{\text{KPZ}}} \left(x = sT^{-1/6}, t = T^{-1/2} \right), \quad (2.3)$$

or, in other words, $Q_{\sigma_{\text{KPZ}}}(x, t)$ is a Laplace-type transform of the solution to the stochastic heat equation.

Corwin and Ghosal [10, Theorem 1.1] obtained robust upper and lower bounds for the lower tail of the probability distribution of Υ_T : given $\epsilon, \delta \in (0, \frac{1}{3})$ and $T_0 > 0$, there exist $S = S(\epsilon, \delta, T_0)$, $C = C(T_0) > 0$, $K_1 = K_1(\epsilon, \delta, T_0) > 0$, and $K_2 = K_2(T_0) > 0$ such that for all $s \geq S$ and $T \geq T_0$,

$$\mathbb{P}(\Upsilon_T \leq -s) \leq e^{-\frac{4(1-C\epsilon)}{15\pi}T^{\frac{1}{3}}s^{5/2}} + e^{-K_1s^{3-\delta}-\epsilon T^{1/3}s} + e^{-\frac{(1-C\epsilon)}{12}s^3}, \quad (2.4)$$

and

$$\mathbb{P}(\Upsilon_T \leq -s) \geq e^{-\frac{4(1+C\epsilon)}{15\pi}T^{\frac{1}{3}}s^{5/2}} + e^{-K_2s^3}. \quad (2.5)$$

For different values of s and T , different terms at the right can dominate the others. In [6], these bounds were refined in the region where $M^{-1} \leq T \leq Ms^{3/2}$, for any $M > 0$: then for any $\epsilon > 0$,

$$B(s + (3 + \epsilon)T^{-1/3} \log s, T) \leq \log \mathbb{P}(\Upsilon_T < -s) \leq B(s, T),$$

with

$$B(s, T) = -\frac{T^2}{\pi^6} F_1\left(\frac{\pi^2 s}{T^{2/3}}\right) - \frac{1}{6} \sqrt{1 + \frac{\pi^2 s}{T^{2/3}}} + \mathcal{O}(\log^2 s) + \mathcal{O}(T^{1/3}), \quad \text{as } s \rightarrow +\infty, \quad (2.6)$$

where F_1 is defined in (1.29).

We extend these estimates uniformly to $s \geq s_0, T \geq T_0$ for any $s_0, T_0 > 0$. To this end, we first obtain the asymptotic properties of $\log Q_{\sigma_{\text{KPZ}}}$ in this regime, see also Figure 2.

Theorem 2.1. *We have, uniformly for $s \geq s_0, T \geq T_0$,*

$$\log Q_{\sigma_{\text{KPZ}}}(x = sT^{-1/6}, t = T^{-1/2}) = -G(s, T) + \mathcal{O}(1), \quad (2.7)$$

where

$$G(s, T) := \frac{T^2}{\pi^6} F_1\left(\frac{\pi^2 s}{T^{2/3}}\right) + \frac{1}{6} \sqrt{1 + \frac{\pi^2 s}{T^{2/3}}} + \frac{1}{48} \log\left(1 + \frac{\pi^2 s}{T^{2/3}}\right) + \frac{1}{8} \log\left(\sqrt{1 + \frac{\pi^2 s}{T^{2/3}}} - 1\right) + \frac{1}{12} \log T, \quad (2.8)$$

with F_1 defined in (1.29).

Proof. The asymptotics (1.31) imply

$$\log Q_{\sigma_{\text{KPZ}}}(x = sT^{-1/6}, t = T^{-1/2}) = -G(s, T) + \mathcal{O}(1), \quad (2.9)$$

uniformly for $s \geq KT^{1/6}, T \geq T_0$. The remaining region $s_0 \leq s \leq KT^{1/6}, T \geq T_0$ is covered precisely by [7, Theorem 1.14 (iii)], see also (1.20); in terms of the variables s, T we obtain

$$\log Q_{\sigma_{\text{KPZ}}}(x = sT^{-1/6}, t = T^{-1/2}) = -\frac{s^3}{12} - \frac{1}{8} \log s + \mathcal{O}(1), \quad (2.10)$$

uniformly in $s_0 \leq s \leq KT^{1/6}, T \geq T_0$. We complete the proof by observing that

$$G(s, T) = \frac{s^3}{12} + \frac{1}{8} \log s + \mathcal{O}(1) \quad (2.11)$$

uniformly in the same regime. $\square$

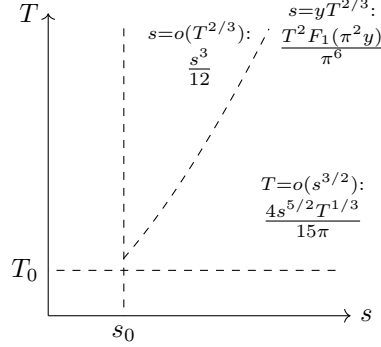


Figure 2: Phase diagram showing the different leading asymptotics for $-\log Q_{\sigma_{\text{KPZ}}}(sT^{-1/6}, T^{-1/2})$ when $s, T \rightarrow +\infty$, uniform for $s \geq s_0, T \geq T_0$. This is given by the leading behavior for large s, T of the function $G(s, T)$ in (2.8). A phase transition occurs in the critical regime when $s, T \rightarrow +\infty$ such that $sT^{-2/3}$ remains bounded away from zero and infinity.

The following result is a direct consequence of Theorem 2.1, whose proof is almost identical to that in [1, Section 3.1] or [6, Corollary 1.2].

Corollary 2.2. *Let Υ_T be as defined in (2.2) and $G(s, T)$ be as defined in (2.8).*

1. *For any $s_0, T_0 > 0$, there exists a real constant $D_+ = D_+(s_0, T_0)$ such that the inequality*

$$\log \mathbb{P}(\Upsilon_T < -s) \leq p - G(s + T^{-1/3} \log p, T) + D_+ \quad (2.12)$$

holds for all $s \geq s_0, T \geq T_0$, and $p \geq 1$.

2. *For any $\epsilon > 0$ and for any $s_0, T_0 > 0$ sufficiently large, there exists a real constant $D_- = D_-(s_0, T_0)$ such that the inequality*

$$\log \mathbb{P}(\Upsilon_T < -s) \geq -G(s + T^{-1/3} \log(s^{3+\epsilon} + T^\epsilon), T) + D_- \quad (2.13)$$

holds for all $s \geq s_0, T \geq T_0$.

Proof. Let $\Omega(s, T) := e^{-e^{T^{1/3}(\Upsilon_T + s)}}$ so that $\mathbb{E}[\Omega(s, T)] = Q_{\sigma_{\text{KPZ}}}(sT^{-1/6}, T^{-1/2})$ by (2.3). Moreover, for all $p > 0$ we have

$$\Omega(s, T)^p = e^{-pe^{T^{1/3}(\Upsilon_T + s)}} = e^{-e^{T^{1/3}(\Upsilon_T + s + T^{-1/3} \log p)}} = \Omega(s + T^{-1/3} \log p, T).$$

1. Markov's inequality implies (for any $p > 0$)

$$\mathbb{P}(\Upsilon_T \leq -s) = \mathbb{P}(\Omega(s, T)^p \geq e^{-p}) \leq e^p \mathbb{E}[\Omega(s, T)^p] = e^p Q_{\sigma_{\text{KPZ}}}((s + T^{-1/3} \log p)T^{-1/6}, T^{-1/2})$$

and taking logarithms it follows that

$$\log \mathbb{P}(\Upsilon_T \leq -s) \leq p + \log Q_{\sigma_{\text{KPZ}}}((s + T^{-1/3} \log p)T^{-1/6}, T^{-1/2}).$$

Then it suffices to use (2.7).

2. For any $q > 0$ we have

$$\mathbb{E} [\Omega(s, T)^q] = \mathbb{E} [1_{\{\Upsilon_T + s < 0\}} \Omega(s, T)^q] + \mathbb{E} [1_{\{\Upsilon_T + s \geq 0\}} \Omega(s, T)^q] \leq \mathbb{P}(\Upsilon_T + s < 0) + e^{-q}$$

where in the last step we used that $\Omega(s, T) \leq 1$ and $1_{\{\Upsilon_T + s \geq 0\}} \Omega(s, T)^q \leq e^{-q}$. Therefore

$$\mathbb{P}(\Upsilon_T + s < 0) \geq Q_{\sigma_{\text{KPZ}}} \left((s + T^{-1/3} \log q) T^{-1/6}, T^{-1/2} \right) - e^{-q}.$$

On the other hand, the uniform asymptotics (2.7) imply that if $q = s^{3+\epsilon} + T^\epsilon$ (for any $\epsilon > 0$) then e^{-q} is negligible with respect to the other term. Therefore, for this choice of q and s_0, T_0 sufficiently large, we have

$$Q_{\sigma_{\text{KPZ}}} \left((s + T^{-1/3} \log q) T^{-1/6}, T^{-1/2} \right) - e^{-q} \geq \frac{1}{2} Q_{\sigma_{\text{KPZ}}} \left((s + T^{-1/3} \log q) T^{-1/6}, T^{-1/2} \right)$$

for all $s \geq s_0, T \geq T_0$. The proof is completed by combining the last two inequalities, taking logarithms, and using (2.7). $\square$

Remark 2.3. The bounds of Corollary 2.2 imply the following estimates in various regimes.

1. When $s = yT^{2/3}$ and T sufficiently large with y in any compact subset of $(0, +\infty)$, we set $p = 1$ in the above result, and use the expressions (1.29) for F_1 and (2.8) for G . After a straightforward computation, we then obtain that for any compact $K \subset (0, +\infty)$, there exist $s_0, T_0, C > 0$ such that for $s \geq s_0, T \geq T_0$ and $y = sT^{-2/3} \in K$, we have

$$-CT \log T \leq \log \mathbb{P}(\Upsilon_T < -yT^{2/3}) + \frac{T^2 F_1(\pi^2 y)}{\pi^6} \leq \frac{1}{12} \log T + C.$$

This confirms the large deviation function which was predicted using different approaches in [38, 27, 11, 26] and rigorously proved in [40]; our approach provides an alternative proof via the RH method and allows to quantify the error term.

2. When $s \rightarrow +\infty$ and $sT^{-2/3} \rightarrow +\infty$ (in particular when $s \rightarrow +\infty$ with $T > 0$ fixed), we obtain a sharper bound by choosing $p = s^{3/2}$. We then get

$$\begin{aligned} \log \mathbb{P}(\Upsilon_T < -s) &\leq -\frac{4s^{\frac{5}{2}} T^{\frac{1}{3}}}{15\pi} + \frac{s^2 T^{\frac{2}{3}}}{2\pi^2} - \frac{2s^{\frac{3}{2}} T}{3\pi^3} + \frac{2sT^{\frac{4}{3}}}{3\pi^4} + \frac{s^{\frac{1}{2}} T^{\frac{5}{3}}}{2\pi^5} \\ &\quad + \frac{4T^2}{15\pi^6} - \frac{s^{\frac{3}{2}} \log s}{\pi} + \mathcal{O}(T^{7/3} s^{-1/2} + s^{3/2} + T^{1/3} s \log s). \end{aligned}$$

Combining this with the lower bound (2.13), we obtain that for any $T_0 > 0$, there exist $C, s_0, y_0 > 0$ such that for $s \geq s_0, T \geq T_0, sT^{-2/3} \geq y_0$, we have the upper and lower bounds

$$\begin{aligned} -C(T^{7/3} s^{-1/2} + T^{1/3} s \log s) &\leq \left(\log \mathbb{P}(\Upsilon_T < -s) + \frac{4s^{\frac{5}{2}} T^{\frac{1}{3}}}{15\pi} - \frac{s^2 T^{\frac{2}{3}}}{2\pi^2} + \frac{2s^{\frac{3}{2}} T}{3\pi^3} \right. \\ &\quad \left. - \frac{2sT^{\frac{4}{3}}}{3\pi^4} + \frac{s^{\frac{1}{2}} T^{\frac{5}{3}}}{2\pi^5} - \frac{4T^2}{15\pi^6} + \frac{s^{\frac{3}{2}} \log s}{\pi} \right) \leq C(T^{7/3} s^{-1/2} + s^{3/2} + T^{1/3} s \log s). \end{aligned}$$

3. When T is sufficiently large and $sT^{-2/3}$ is sufficiently small, we take $p = 1$, and obtain the following bounds. For any $\epsilon > 0$, there exist s_0, T_0, y_0, C such that for $s \geq s_0, T \geq T_0, sT^{-2/3} \leq y_0$, we have the bounds

$$\begin{aligned} -\frac{s^2 \log(s^{3+\epsilon} + T^\epsilon)}{4T^{\frac{1}{3}}} - Cs^4 T^{-\frac{2}{3}} - CsT^{-\frac{2}{3}} \log^2 T - C &\leq \log \mathbb{P}(\Upsilon_T < -s) + \frac{s^3}{12} + \frac{1}{8} \log s \\ &\leq C + Cs^4 T^{-\frac{2}{3}}. \end{aligned}$$

3 Asymptotic analysis for Ψ for $K \leq x \leq \delta/t$

3.1 Rescaling

In this section, x is large while xt remains bounded. Since the exponential functions in the large z asymptotics of $\Psi(z; x, t)$ are $e^{\pm(-\frac{2}{3}tz^{\frac{3}{2}}+xz^{\frac{1}{2}})}$, it is not convenient to directly work with Ψ . The change of variables $z = x^2\zeta$ is suited for this section as it transforms these exponential functions into $e^{\pm x^2(-\frac{2}{3}xt\zeta^{\frac{3}{2}}+\zeta^{\frac{1}{2}})}$. Therefore, we define

$$\widehat{\Psi}(\zeta; x, t) = x^{-\frac{\sigma_3}{2}} \Psi(x^2\zeta; x, t).$$

The RH problem for Ψ then transforms into the follow RH problem for $\widehat{\Psi}$.

RH problem for $\widehat{\Psi}$

(a) $\widehat{\Psi} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $\widehat{\Psi}$ has the jumps

$$\widehat{\Psi}_+(\zeta; x, t) = \widehat{\Psi}_-(\zeta; x, t) \begin{pmatrix} 1 & \frac{1}{F(x^2\zeta)} \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R},$$

where we recall that $F = \frac{1}{1-\sigma}$.

(c) As $\zeta \rightarrow \infty$, we have

$$\begin{aligned} \widehat{\Psi}(\zeta; x, t) &= \left(I + \frac{1}{\zeta} \widehat{\Psi}_1(x, t) + \mathcal{O}(\zeta^{-2}) \right) \zeta^{\frac{1}{4}\sigma_3} A^{-1} e^{x^2(-\frac{2}{3}xt\zeta^{\frac{3}{2}}+\zeta^{\frac{1}{2}})\sigma_3} \\ &\quad \times \begin{cases} I, & |\arg \zeta| < \pi - \varepsilon, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \varepsilon < \pm \arg \zeta < \pi, \end{cases} \end{aligned}$$

for any $\varepsilon \in (0, \frac{\pi}{2})$, where

$$\widehat{\Psi}_1(x, t) = \begin{pmatrix} \frac{q(x, t)}{x^2} & -\frac{ir(x, t)}{x^3} \\ \frac{ip(x, t)}{x} & -\frac{q(x, t)}{x^2} \end{pmatrix}. \quad (3.1)$$

3.2 Construction of the g -function

By Assumptions 1.1, we have, for $M > 0$ sufficiently large, the estimates

$$\log F(x^2\zeta) = \begin{cases} \mathcal{O}\left(e^{-c-x^2|\zeta|}\right), & \text{for } \zeta < -M/x^2, \\ \mathcal{O}(1), & \text{for } -M/x^2 \leq \zeta \leq M/x^2, \\ \log c'_+ + c_+x^2\zeta + \mathcal{O}\left(e^{-\epsilon x^2\zeta}\right), & \text{for } M/x^2 < \zeta, \end{cases} \quad (3.2)$$

which are valid as $x \rightarrow +\infty$, uniformly in $\zeta \in \mathbb{R}$. Let us decompose $\frac{1}{F(x^2\zeta)}$ as follows:

$$\frac{1}{F(x^2\zeta)} = e^{-x^2V(\zeta)} e^{W(\zeta; x)},$$

where

$$V(\zeta) := \begin{cases} 0, & \zeta < 0, \\ c_+\zeta, & \zeta > 0, \end{cases} \quad W(\zeta; x) := -\log F(x^2\zeta) + x^2V(\zeta). \quad (3.3)$$

Whereas $e^{-x^2V(\zeta)}$ decays rapidly as soon as $\zeta > 0$ moves away from 0, $e^{W(\zeta;x)}$ is uniformly bounded and bounded away from 0 for $\zeta \in \mathbb{R}$ and x sufficiently large. More precisely, for sufficiently large $M > 0$, we have the uniform in ζ large x asymptotics

$$W(\zeta; x) = \begin{cases} \mathcal{O}\left(e^{-c-x^2|\zeta|}\right), & \text{for } \zeta < -M/x^2, \\ \mathcal{O}(1), & \text{for } -M/x^2 \leq \zeta \leq M/x^2, \\ -\log c'_+ + \mathcal{O}\left(e^{-\epsilon x^2\zeta}\right), & \text{for } M/x^2 < \zeta. \end{cases} \quad (3.4)$$

Note that W and V are both (Lipschitz) continuous at $\zeta = 0$, but not differentiable. In view of our upcoming RH analysis, we need to construct a g -function, which will be the key towards a successful application of the Deift–Zhou nonlinear steepest descent method [15], and which will later turn out to determine the leading order asymptotic behavior of $Q_\sigma(x, t)$ as $t \rightarrow 0$ and $K \leq x \leq \delta/t$. We aim to construct g such that

Conditions for g

- (a) $g(\zeta; xt)$ is analytic in $\zeta \in \mathbb{C} \setminus (-\infty, \alpha(xt)]$,
- (b) $g_+(\zeta; xt) + g_-(\zeta; xt) + \ell(xt) = V(\zeta)$ for $\zeta \in (-\infty, \alpha(xt))$,
- (c) $g(\zeta; xt) = -\frac{2}{3}xt\zeta^{3/2} + \zeta^{1/2} + g_0(xt) + g_1(xt)\zeta^{-1/2} + \mathcal{O}(\zeta^{-3/2})$ as $\zeta \rightarrow \infty$,
- (d) $g_+(\zeta; xt) - g_-(\zeta; xt) = \mathcal{O}((\zeta - \alpha(xt))^{3/2})$ as $\zeta \rightarrow \alpha(xt)$, $\zeta < \alpha(xt)$,

Here, $\alpha(xt) > 0$, $g_0(xt)$, $g_1(xt)$, and $\ell(xt)$ are to be determined. The derivative g' then has to satisfy the following.

Conditions for g'

- (a) $g'(\zeta; xt)$ is analytic in $\zeta \in \mathbb{C} \setminus (-\infty, \alpha(xt)]$,
- (b) $g'_+(\zeta; xt) + g'_-(\zeta; xt) = V'(\zeta)$ for $\zeta \in (-\infty, \alpha(xt)) \setminus \{0\}$,
- (c) $g'(\zeta; xt) = -xt\zeta^{1/2} + \frac{1}{2}\zeta^{-\frac{1}{2}} - \frac{1}{2}g_1(xt)\zeta^{-\frac{3}{2}} + \mathcal{O}(\zeta^{-5/2})$ as $\zeta \rightarrow \infty$,
- (d) $g'_+(\zeta; xt) - g'_-(\zeta; xt) = \mathcal{O}((\zeta - \alpha(xt))^{1/2})$ as $\zeta \rightarrow \alpha(xt)$, $\zeta < \alpha(xt)$.

We can construct g' as

$$g'(\zeta; xt) = \frac{1}{(\zeta - \alpha(xt))^{1/2}} \left(-xt\zeta + \frac{1 + \alpha(xt)xt}{2} + \frac{c_+}{2\pi} \int_0^{\alpha(xt)} \sqrt{\alpha(xt) - s} \frac{ds}{s - \zeta} \right), \quad (3.5)$$

with $(\zeta - \alpha(xt))^{1/2}$ analytic except for $\zeta \leq \alpha(xt)$ and positive for $\zeta > \alpha(xt)$. Then, conditions (a), (b), and (c) are straightforward to verify. In order to have also (d), we need to impose that $\alpha(xt)$ is such that $g'_+(\zeta; xt) - g'_-(\zeta; xt)$ vanishes as $\zeta \rightarrow \alpha(xt)$. This is the case if

$$\frac{c_+}{2\pi} \int_0^{\alpha(xt)} \frac{ds}{\sqrt{\alpha(xt) - s}} = \frac{1 - \alpha(xt)xt}{2}. \quad (3.6)$$

This equation admits two solutions, and we take the solution for $\alpha(xt)$ which remains bounded as $xt \rightarrow 0$, namely we take

$$\alpha(xt) := \frac{2c_+^2 + \pi^2 xt - 2c_+ \sqrt{c_+^2 + \pi^2 xt}}{\pi^2 (xt)^2}. \quad (3.7)$$

Remark 3.1. The function $\alpha(xt)$ is a smooth monotonically decreasing function of $xt > 0$, with $\alpha(xt) \rightarrow \frac{\pi^2}{4c_+^2}$ as $xt \rightarrow 0_+$ and $\alpha(xt) \rightarrow 0_+$ as $xt \rightarrow +\infty$. For later convenience we point out that for $xt < \delta$, $\alpha(xt)$ is positive and bounded away from 0.

Remark 3.2. Observe that g' can equivalently be constructed as

$$g'(\zeta; xt) = \sqrt{\zeta - \alpha(xt)} \left(-xt - \frac{c_+}{2\pi} \int_0^{\alpha(xt)} \frac{1}{\sqrt{\alpha(xt) - s}} \frac{ds}{s - \zeta} \right), \quad (3.8)$$

where $\alpha(xt)$ is found by requiring that $g'(\zeta; xt) = -xt\zeta^{1/2} + \frac{1}{2}\zeta^{-\frac{1}{2}} + \mathcal{O}(\zeta^{-3/2})$ as $\zeta \rightarrow \infty$. Here too, we see that $\alpha(xt)$ must satisfy (3.6). Using (3.6), we easily show that the right-hand sides of (3.5) and (3.8) are indeed equal. With this expression, we immediately obtain that $g'(\zeta; xt)$ remains bounded as $\zeta \rightarrow \alpha(xt)$, and that condition (d) holds.

Using a direct primitive, we note that g' can also be expressed more explicitly as

$$g'(\zeta; xt) = -xt(\zeta - \alpha(xt))^{1/2} + \frac{c_+}{2\pi i} \log \left(\frac{(\zeta - \alpha(xt))^{1/2} + i\sqrt{\alpha(xt)}}{(\zeta - \alpha(xt))^{1/2} - i\sqrt{\alpha(xt)}} \right), \quad (3.9)$$

where the principal branches for the log and the square roots are taken. Using this and (3.6), we also easily see that

$$g_1(xt) = -\left(\frac{\alpha(xt)^{3/2} c_+}{3\pi} + \frac{\alpha(xt)^2 xt}{4} \right) = \alpha(xt) \left(\frac{\sqrt{\alpha(xt)} c_+}{6\pi} - \frac{1}{4} \right). \quad (3.10)$$

We can now construct g from g' by setting

$$g(\zeta; xt) := \int_{\alpha(xt)}^{\zeta} g'(s; xt) ds.$$

The values of the remaining constants $\ell(xt), g_0(xt)$ now follow easily. Since $g(\alpha(xt)) = 0$, we must choose

$$\ell(xt) = V(\alpha(xt)) = c_+ \alpha(xt). \quad (3.11)$$

By construction, g satisfies the required jumps. Because $g_+(\zeta; xt) + g_-(\zeta; xt) + \ell(xt) = 0$ for all $\zeta \in (-\infty, 0]$, we have

$$g_0(xt) = -\frac{\ell(xt)}{2} = -\frac{c_+ \alpha(xt)}{2}.$$

3.3 Normalization of the RH problem

Let us define

$$\begin{aligned} T(\zeta; x, t) &:= \begin{pmatrix} 1 & ig_1(xt)x^2 \\ 0 & 1 \end{pmatrix} \widehat{\Psi}(\zeta; x, t) e^{-x^2(g(\zeta; xt) - g_0(xt))\sigma_3}, \\ \phi(\zeta; xt) &:= 2g(\zeta; xt) + \ell(xt) - V(\zeta). \end{aligned} \quad (3.12)$$

We define an analytic continuation of $V(\zeta)$ on $\mathbb{C} \setminus i\mathbb{R}$ by setting it equal to 0 for $\operatorname{Re} \zeta < 0$, and to $c_+\zeta$ for $\operatorname{Re} \zeta > 0$, so that we can also extend ϕ to an analytic function in $\mathbb{C} \setminus (i\mathbb{R} \cup (-\infty, \alpha(xt)])$. By the jump condition for g , we have the relation

$$\phi_{\pm}(\zeta; xt) = \pm (g_+(\zeta; xt) - g_-(\zeta; xt)), \quad \zeta \in (-\infty, \alpha(xt)), \quad (3.13)$$

from which RH conditions for T follow in a straightforward way.

RH problem for T

(a) $T : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) T has the jumps

$$\begin{aligned} T_+(\zeta; x, t) &= T_-(\zeta; x, t) \begin{pmatrix} e^{-x^2\phi_+(\zeta; xt)} & e^{W(\zeta; x)} \\ 0 & e^{-x^2\phi_-(\zeta; xt)} \end{pmatrix}, & \zeta \in (-\infty, \alpha(xt)), \\ T_+(\zeta; x, t) &= T_-(\zeta; x, t) \begin{pmatrix} 1 & e^{x^2\phi(\zeta; xt)}e^{W(\zeta; x)} \\ 0 & 1 \end{pmatrix}, & \zeta \in (\alpha(xt), +\infty). \end{aligned}$$

(c) We have

$$T(\zeta; x, t) = \left(I + \frac{1}{\zeta} T_1(x, t) + \mathcal{O}(\zeta^{-2}) \right) \zeta^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } \zeta \rightarrow \infty, \quad (3.14)$$

where

$$T_1(x, t) = \begin{pmatrix} -\frac{1}{2}g_1^2x^4 - xg_1p + \frac{q}{x^2} & \frac{ix^6g_1^3}{3} + ix^3g_1^2p - ix^2g_2 - 2ig_1q - \frac{ir}{x^3} \\ ix^2g_1 + \frac{ip}{x} & \frac{1}{2}g_1^2x^4 + xg_1p - \frac{q}{x^2} \end{pmatrix} \quad (3.15)$$

where $g_i = g_i(xt)$, $p = p(x, t)$, $q = q(x, t)$, and $r = r(x, t)$.

(d) As $\zeta \rightarrow 0$ and as $\zeta \rightarrow \alpha(xt)$, we have $T(\zeta; x, t) = \mathcal{O}(1)$.

3.4 Opening of the lenses

Note that the jump of T for $\zeta \in (-\infty, \alpha(xt))$ can be written as

$$\begin{aligned} & \begin{pmatrix} e^{-x^2\phi_+(\zeta; xt)} & e^{W(\zeta; x)} \\ 0 & e^{-x^2\phi_-(\zeta; xt)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e^{-W(\zeta; x) - x^2\phi_-(\zeta; xt)} & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{W(\zeta; x)} \\ -e^{-W(\zeta; x)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-W(\zeta; x) - x^2\phi_+(\zeta; xt)} & 1 \end{pmatrix}. \end{aligned} \quad (3.16)$$

Now we will use this factorization to open lenses around $(-\infty, \alpha(xt))$: we will split it into two parts, and deform the upper part into the upper half plane, and the lower part into the lower half plane. Although the general principles of RH asymptotic analysis [14] suggest to choose these curves close to the real line, we prefer to deform the upper and lower parts of $(-\infty, \alpha(xt))$ all the way to the vertical half-lines $\alpha(xt) \pm i\mathbb{R}^+$. This is not essential, but it will be convenient later to prove that the jump matrices are small on these parts of the jump contour. The function ϕ is analytic in $\mathbb{C} \setminus (i\mathbb{R} \cup (-\infty, \alpha(xt)])$ only, but the identity

$$x^2\phi(\zeta; xt) + W(\zeta; x) = 2x^2(g(\zeta; xt) - g_0) - \log F(x^2\zeta)$$

shows that the expressions

$$\exp(-x^2\phi_{\pm}(\zeta; xt) - W(\zeta; x)) = F(x^2\zeta)\exp(-2x^2(g_{\pm}(\zeta; xt) - g_0)), \quad \zeta \in \mathbb{R}, \quad (3.17)$$

appearing in the factorization (3.16), are boundary values of a function analytic in $\mathbb{C} \setminus (-\infty, \alpha(xt)]$, due to the conditions defining g and Assumptions 1.1. The next transformation can then be defined by

$$S(\zeta; x, t) := \begin{cases} T(\zeta; x, t) \begin{pmatrix} 1 & 0 \\ \mp e^{-x^2\phi(\zeta; xt)} e^{-W(\zeta; x)} & 1 \end{pmatrix}, & \text{if } \operatorname{Re} \zeta < \alpha(xt) \text{ and } \pm \operatorname{Im} \zeta > 0, \\ T(\zeta; x, t), & \text{if } \operatorname{Re} \zeta > \alpha(xt). \end{cases} \quad (3.18)$$

Indeed, by the above discussion, $e^{-x^2\phi(\zeta; xt)} e^{-W(\zeta; x)} = F(x^2\zeta)\exp(-2x^2(g(\zeta; xt) - g_0))$ is an analytic function of ζ in the relevant regions. RH conditions for S follow from those for T along with the identity (3.16).

RH problem for S

- (a) $S : \mathbb{C} \setminus (\mathbb{R} \cup (i\mathbb{R} + \alpha(xt))) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- (b) S has the jumps

$$\begin{aligned} S_+(\zeta; x, t) &= S_-(\zeta; x, t) \begin{pmatrix} 1 & 0 \\ e^{-x^2\phi(\zeta; xt)} e^{-W(\zeta; x)} & 1 \end{pmatrix}, & \zeta \in i\mathbb{R} + \alpha(xt), \\ S_+(\zeta; x, t) &= S_-(\zeta; x, t) \begin{pmatrix} 0 & e^{W(\zeta; x)} \\ -e^{-W(\zeta; x)} & 0 \end{pmatrix}, & \zeta \in (-\infty, \alpha(xt)), \\ S_+(\zeta; x, t) &= S_-(\zeta; x, t) \begin{pmatrix} 1 & e^{x^2\phi(\zeta; xt)} e^{W(\zeta; x)} \\ 0 & 1 \end{pmatrix}, & \zeta \in (\alpha(xt), +\infty), \end{aligned}$$

where $\mathbb{R}$ is oriented from left to right, and the vertical half-lines $\alpha(xt) \pm i\mathbb{R}^+$ are oriented towards $\alpha(xt)$. According to this orientation, the subscript $+$ (resp. $-$) indicates the boundary value from the left (resp. right).

- (c) We have

$$S(\zeta; x, t) = \left(I + \frac{1}{\zeta} T_1(x, t) + \mathcal{O}(\zeta^{-2}) \right) \zeta^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } \zeta \rightarrow \infty,$$

where $T_1(x, t)$ is the same as in (3.14).

- (d) As $\zeta \rightarrow 0$ and as $\zeta \rightarrow \alpha(xt)$, we have $S(\zeta; x, t) = \mathcal{O}(1)$.

We now show that the jump matrix for S is close to the identity except on $(-\infty, \alpha(xt))$ and near $\alpha(xt)$.

Proposition 3.3. *1. For any $\rho > 0$, there exists $\eta > 0$ such that the inequality*

$$\phi(\alpha(xt) + v; xt) < -\eta v$$

holds for $v > \rho$ and $xt \leq \delta$.

2. For any $\rho > 0$, there exists $\eta > 0$ such that the inequalities

$$\operatorname{Re} \phi(\alpha(xt) \pm iv; xt) > \eta \sqrt{v}$$

hold for $v > \rho$ and $xt \leq \delta$.

Proof. 1. Let us rewrite (3.9) for $\zeta = \alpha(xt) + v$, $v > 0$, as

$$g'(\alpha(xt) + v) = -xt\sqrt{v} + \frac{c_+}{\pi} \arctan \sqrt{\frac{\alpha(xt)}{v}}. \quad (3.19)$$

Integration of this expression, using $g(\alpha(xt)) = 0$, yields for $v > 0$,

$$g(\alpha(xt) + v) = -\frac{2}{3}xtv^{3/2} + \frac{c_+}{\pi} \left(\sqrt{\alpha(xt)v} - \frac{\alpha(xt)\pi}{2} + (\alpha(xt) + v) \arctan \sqrt{\frac{\alpha(xt)}{v}} \right).$$

By omitting the negative term $-\frac{2}{3}xtv^{3/2}$ we obtain, using (3.11) and (3.12),

$$\phi(\alpha(xt) + v) = 2g(\alpha(xt) + v) - c_+v < \frac{2c_+}{\pi} \left(\sqrt{\alpha(xt)v} - (\alpha(xt) + v) \arctan \sqrt{\frac{v}{\alpha(xt)}} \right).$$

The function $\mathcal{F}(\alpha, v) := \frac{2c_+}{\pi} (\sqrt{\alpha v} - (\alpha + v) \arctan \sqrt{\frac{v}{\alpha}})$ (for $\alpha, v > 0$) is concave in v , $\mathcal{F}(\alpha, 0) = 0$, and $\mathcal{F}(\alpha, v) < 0$ for $v > 0$; therefore, for any $\rho > 0$ we have $\mathcal{F}(\alpha, v) < -\eta v$ for $v > \rho$, where $\eta(\alpha) := -\mathcal{F}(\alpha, \rho)/\rho > 0$. To see that we can take η in the statement of the proposition independent of $x, t > 0$, note that $\eta(\alpha)$ is decreasing in α and $\alpha(xt) \leq \pi^2/(4c_+^2)$, see Remark 3.1, so that the proof is complete by taking $\eta = \eta(\pi^2/(4c_+^2))$.

2. Let us rewrite (3.9) for $\zeta = \alpha(xt) \pm iv$, $v > 0$, as

$$g'(\alpha(xt) \pm iv) = -xte^{\pm i\pi/4}\sqrt{v} + \frac{c_+}{2\pi i} \log \left(\frac{e^{\pm i\pi/4}\sqrt{v} + i\sqrt{\alpha(xt)}}{e^{\pm i\pi/4}\sqrt{v} - i\sqrt{\alpha(xt)}} \right).$$

For $v > 0$ we have $\operatorname{Re} \phi(\alpha(xt) \pm iv) = 2\operatorname{Re} g(\alpha(xt) \pm iv)$, see (3.11) and (3.12), so that

$$\begin{aligned} \operatorname{Re} \phi(\alpha(xt) \pm iv) &= \pm 2 \operatorname{Re} i \int_0^v g'(\alpha(xt) \pm is) ds \\ &= \int_0^v \left(xt\sqrt{2s} \pm \frac{c_+}{\pi} \log \left| \frac{e^{\pm i\pi/4}\sqrt{s} + i\sqrt{\alpha(xt)}}{e^{\pm i\pi/4}\sqrt{s} - i\sqrt{\alpha(xt)}} \right| \right) ds \\ &= \frac{2\sqrt{2}}{3}xtv^{3/2} + \frac{c_+}{2\pi} \int_0^v \log \left(1 + \frac{4}{\sqrt{\frac{2\alpha(xt)}{s}} + \sqrt{\frac{2s}{\alpha(xt)}} - 2} \right) ds \\ &\geq \frac{\sqrt{2}c_+}{\pi} \int_0^v \frac{ds}{\sqrt{\frac{s}{\alpha(xt)}} + \sqrt{\frac{\alpha(xt)}{s}}} = \frac{2\sqrt{2}c_+}{\pi} \left(\sqrt{\alpha(xt)v} - \alpha(xt) \arctan \sqrt{\frac{v}{\alpha(xt)}} \right), \end{aligned}$$

where we use the inequality $\log(1 + 4(\sqrt{2}(\xi + \xi^{-1}) - 2)^{-1}) \geq 2\sqrt{2}(\xi + \xi^{-1})^{-1}$, valid for all $\xi > 0$.

The function $\mathcal{H}(\alpha, v) := \frac{2\sqrt{2}c_+}{\pi} (\sqrt{\alpha v} - \alpha \arctan \sqrt{\frac{v}{\alpha}})$ (for $\alpha, v > 0$) is convex with respect to the variable $\sqrt{v}$, $\mathcal{H}(\alpha, 0) = 0$, and $\mathcal{H}(\alpha, v) > 0$ for $v > 0$; therefore, for any $\rho > 0$ we have $\mathcal{H}(\alpha, v) > \eta\sqrt{v}$ for $v > \rho$, where $\eta(\alpha) := \mathcal{H}(\alpha, \rho)/\sqrt{\rho} > 0$. To see that we can take η in the statement of the proposition independent of $x, t > 0$, note that $\eta(\alpha)$ is bounded away from 0 provided α does not tend to 0, and that $\alpha(xt)$ does not tend to 0 for $xt \leq \delta$ by Remark 3.1. $\square$

Corollary 3.4. *For any sufficiently large K and any $\rho > 0$, there exists $\eta > 0$ such that for $\zeta \in (\alpha(xt) + i\mathbb{R}) \cup (\alpha(xt), +\infty)$ and satisfying $|\zeta - \alpha(xt)| > \rho$, the jump matrix for S is $I + \mathcal{O}(\exp(-\eta x^2 \sqrt{|\zeta - \alpha(xt)|}))$, uniformly in $x, t > 0$ such that $xt \leq \delta$, $x \geq K$.*

Proof. Equation (3.4) implies that $|e^{W(\zeta;x)}|$ remains bounded for $\zeta \in (\alpha(xt), +\infty)$. Also, for $\zeta \in \alpha(xt) + i\mathbb{R}$, by part 4 of Assumptions 1.1 and the definition (3.3) of W , we have $|e^{-W(\zeta;x)}| = \left| \frac{F(x^2\zeta)}{e^{x^2 c_+ \zeta}} \right| = \mathcal{O}(1)$. The claim now directly follows from Proposition 3.3 and the definition of the jump matrix of S . $\square$

3.5 Global parametrix

Let us for a moment ignore the jumps on $(\alpha(xt), +\infty) \cup (i\mathbb{R} + \alpha(xt))$ in the RH problem for S . We then obtain the following RH problem for $P^{(\infty)}$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, \alpha(xt)] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $P^{(\infty)}$ has the jumps

$$P_+^{(\infty)}(\zeta; x, t) = P_-^{(\infty)}(\zeta; x, t) \begin{pmatrix} 0 & e^{W(\zeta;x)} \\ -e^{-W(\zeta;x)} & 0 \end{pmatrix}, \quad \zeta \in (-\infty, \alpha(xt)).$$

(c) We have

$$P^{(\infty)}(\zeta; x, t) = \left(I + \frac{1}{\zeta} P_1^{(\infty)}(x, t) + \mathcal{O}(\zeta^{-2}) \right) \zeta^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } \zeta \rightarrow \infty. \quad (3.20)$$

If we impose in addition that we have the condition

(d) As $\zeta \rightarrow \alpha(xt)$, $P^{(\infty)}(\zeta; x, t) = \mathcal{O}(|\zeta - \alpha(xt)|^{-\frac{1}{4}})$,

there is a unique solution to this RH problem, and it will turn out to be a good approximation to S for large x , for ζ not too close to 0 and $\alpha(xt)$.

The unique solution to this RH problem is given by

$$P^{(\infty)}(\zeta; x, t) = \begin{pmatrix} 1 & id_1(x, t) \\ 0 & 1 \end{pmatrix} (\zeta - \alpha(xt))^{\frac{1}{4}\sigma_3} A^{-1} e^{-D(\zeta; x, t)\sigma_3},$$

with

$$D(\zeta; x, t) = \frac{\sqrt{\zeta - \alpha(xt)}}{2\pi} \int_{-\infty}^{\alpha(xt)} \frac{W(s; x)}{\sqrt{\alpha(xt) - s}} \frac{ds}{\zeta - s}.$$

Indeed, the function D satisfies $D_+(\zeta; x, t) + D_-(\zeta; x, t) = W(\zeta; x)$ for $\zeta \in (-\infty, \alpha(xt))$, from which the jump condition for $P^{(\infty)}$ follows. Moreover, we have the asymptotics

$$D(\zeta; x, t) = \zeta^{-\frac{1}{2}} d_1(x, t) + \mathcal{O}(\zeta^{-3/2}) \text{ as } \zeta \rightarrow \infty, \quad d_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\alpha(xt)} \frac{W(s; x)}{\sqrt{\alpha(xt) - s}} ds, \quad (3.21)$$

from which (3.20) follows. In particular, let us record the asymptotic behavior of $d_1(x, t)$ for later use.

Proposition 3.5. *As $x \rightarrow +\infty$, we have uniformly in $xt \leq \delta$ for any $\delta > 0$, that*

$$d_1(x, t) = -\frac{\sqrt{\alpha(xt)}}{\pi} \log c'_+ + \frac{j_\sigma}{x^2 \sqrt{\alpha(xt)}} + \mathcal{O}(x^{-4}),$$

$$j_\sigma := -\frac{1}{2\pi} \int_{-\infty}^{+\infty} [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr. \quad (3.22)$$

Proof. Let us write, using (3.3), $d_1(x, t) = \frac{1}{2\pi}(I_1 + I_2 + I_3 + I_4)$, where

$$\begin{aligned} I_1 &:= - \int_{-\infty}^{-\frac{\alpha(xt)}{2}} \frac{\log F(x^2 s)}{\sqrt{\alpha(xt) - s}} ds, \quad I_2 := - \int_{-\frac{\alpha(xt)}{2}}^{\frac{\alpha(xt)}{2}} \frac{\log F(x^2 s) - (c_+ x^2 s + \log c'_+) 1_{(0, +\infty)}(s)}{\sqrt{\alpha(xt) - s}} ds, \\ I_3 &:= - \int_{\frac{\alpha(xt)}{2}}^{\alpha(xt)} \frac{\log F(x^2 s) - (c_+ x^2 s + \log c'_+)}{\sqrt{\alpha(xt) - s}} ds, \quad I_4 := - \int_0^{\alpha(xt)} \frac{\log c'_+}{\sqrt{\alpha(xt) - s}} ds = -2\sqrt{\alpha(xt)} \log c'_+. \end{aligned}$$

The terms I_1, I_3 are exponentially small as $x \rightarrow +\infty$ by (3.4);

$$I_1 = \mathcal{O}\left(\frac{e^{-c_- x^2 \alpha(xt)/2}}{\sqrt{\alpha(xt)}}\right), \quad I_3 = \mathcal{O}\left(e^{-\epsilon x^2 \alpha(xt)/2} \sqrt{\alpha(xt)}\right). \quad (3.23)$$

The estimate is uniform for $xt \leq \delta$, since $\alpha(xt)$ is positive and bounded away from zero (see Remark 3.1). To estimate I_2 we perform the change variable $x^2 s = r$ and obtain

$$\begin{aligned} I_2 &= -\frac{1}{x^2} \int_{-x^2 \frac{\alpha(xt)}{2}}^{x^2 \frac{\alpha(xt)}{2}} \frac{\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)}{\sqrt{\alpha(xt) - r/x^2}} dr \\ &= \frac{-1}{x^2 \sqrt{\alpha(xt)}} \sum_{k \geq 0} \frac{(2k-1)!!}{(2k)!! (\alpha(xt)x^2)^k} \int_{-x^2 \frac{\alpha(xt)}{2}}^{x^2 \frac{\alpha(xt)}{2}} r^k [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr \\ &= \frac{-1}{x^2 \sqrt{\alpha(xt)}} \int_{\mathbb{R}} [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr + \mathcal{O}(x^{-4}). \end{aligned}$$

In the last step we use again (3.2) to show that

$$\begin{aligned} &\int_{-x^2 \frac{\alpha(xt)}{2}}^{x^2 \frac{\alpha(xt)}{2}} [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr \\ &= \int_{\mathbb{R}} [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr + \mathcal{O}\left(e^{-\min\{\epsilon, c_-\} x^2 \alpha(xt)/2}\right). \end{aligned}$$

□

Recall that W is Lipschitz continuous (but not differentiable) at $\zeta = 0$. Hence D admits well-defined boundary values at $\zeta = 0$. Moreover, expanding $P^{(\infty)}(\zeta)$ as $\zeta \rightarrow \infty$, we obtain

$$P_1^{(\infty)}(x, t) = \begin{pmatrix} -\frac{\alpha(xt)}{4} - \frac{d_1(x, t)^2}{2} & \frac{i}{6}(3\alpha(xt)d_1(x, t) + 2d_1(x, t)^3 - 6d_2(x, t)) \\ id_1(x, t) & \frac{\alpha(xt)}{4} + \frac{d_1(x, t)^2}{2} \end{pmatrix} \quad (3.24)$$

in (3.20). Finally, let us discuss the behavior of $P^{(\infty)}$ as $\zeta \rightarrow \alpha(xt)$; to this end write

$$D(\zeta; x, t) = \frac{W(\zeta; x)}{2} - \frac{(\zeta - \alpha(xt))^{1/2}}{2\pi} \int_{-\infty}^{\alpha(xt)} \frac{W(\zeta; x) - W(s; x)}{(\zeta - s)\sqrt{\alpha(xt) - s}} ds. \quad (3.25)$$

Since $W(\zeta; x)$ is analytic near $\zeta = \alpha(xt)$, we have the Taylor series

$$\frac{W(\zeta; x) - W(s; x)}{\zeta - s} = \sum_{\ell=0}^{+\infty} \frac{(\zeta - \alpha(xt))^\ell}{(s - \alpha(xt))^{\ell+1}} \left[W(s; x) - \sum_{j=0}^{\ell} W^{(j)}(\alpha(xt); x) \frac{(s - \alpha(xt))^j}{j!} \right] \quad (3.26)$$

and so we have the following Poincaré asymptotic series, as $\zeta \rightarrow \alpha(xt)$ away from $(-\infty, \alpha(xt))$,

$$\int_{-\infty}^{\alpha(xt)} \frac{W(\zeta; x) - W(s; x)}{(\zeta - s)\sqrt{\alpha(xt) - s}} ds \sim \sum_{\ell=0}^{+\infty} (\zeta - \alpha(xt))^\ell \int_{-\infty}^{\alpha(xt)} \frac{W(s; x) - \sum_{j=0}^{\ell} W^{(j)}(\alpha(xt); x) \frac{(s - \alpha(xt))^j}{j!}}{(s - \alpha(xt))^{\ell+1} \sqrt{\alpha(xt) - s}} ds. \quad (3.27)$$

In particular, as $\zeta \rightarrow \alpha(xt)$,

$$D(\zeta; x, t) = \frac{W(\zeta; x)}{2} - \chi(x, t)(\zeta - \alpha(xt))^{1/2} + \mathcal{O}((\zeta - \alpha(xt))^{3/2}), \quad (3.28)$$

$$\chi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\alpha(xt)} \frac{W(\alpha(xt); x) - W(s; x)}{(\alpha(xt) - s)^{3/2}} ds. \quad (3.29)$$

Proposition 3.6. *As $x \rightarrow +\infty$, we have uniformly for $xt \leq \delta$ for any $\delta > 0$, that*

$$\chi(x, t) = -\frac{\log c'_+}{\pi\sqrt{\alpha(xt)}} + \mathcal{O}(x^{-2}). \quad (3.30)$$

Proof. Write $\chi = \chi_1 + \chi_2 - \frac{\log c'_+}{2\pi} \int_{-\infty}^0 (\alpha(xt) - s)^{-3/2} ds = \chi_1 + \chi_2 - \log c'_+ / (\pi\sqrt{\alpha(xt)})$, where

$$\chi_1 := \int_{-\infty}^0 \frac{W(\alpha(xt); x) + \log c'_+ - W(s; x)}{(\alpha(xt) - s)^{3/2}} \frac{ds}{2\pi}, \quad \chi_2 := \int_0^{\alpha(xt)} \frac{W(\alpha(xt); x) - W(s; x)}{(\alpha(xt) - s)^{3/2}} \frac{ds}{2\pi}.$$

As $x \rightarrow +\infty$, by using (3.4) and the change of variables $r = x^2 s$, we have

$$\begin{aligned} \chi_1 &= (W(\alpha(xt); x) + \log c'_+) \int_{-\infty}^0 \frac{ds}{2\pi(\alpha(xt) - s)^{3/2}} + \int_{-\infty}^0 \frac{\log F(x^2 s)}{(\alpha(xt) - s)^{3/2}} \frac{ds}{2\pi} \\ &= \mathcal{O}(e^{-\epsilon x^2 \alpha(xt)}) + \int_{-\infty}^0 \frac{\log F(r)}{(\alpha(xt) - x^{-2}r)^{3/2}} \frac{dr}{2\pi x^2} \\ &= \frac{1}{2\pi x^2 \alpha(xt)^{3/2}} \int_{-\infty}^0 \log F(r) dr + \mathcal{O}(x^{-4}) = \mathcal{O}(x^{-2}), \end{aligned}$$

and these estimates are uniform in $xt \leq \delta$ for any $\delta > 0$, see Remark 3.1. Similarly, let $x \rightarrow +\infty$ and $\beta \rightarrow 0_+$ (in a manner to be specified below) so that

$$\begin{aligned} \chi_2 &= (W(\alpha(xt); x) + \log c'_+) \int_0^{\alpha(xt)-\beta} \frac{ds}{2\pi(\alpha(xt) - s)^{3/2}} \\ &\quad + \int_0^{\alpha(xt)-\beta} \frac{\log F(x^2 s) - c_+ x^2 s - \log c'_+}{(\alpha(xt) - s)^{3/2}} \frac{ds}{2\pi} + \int_{\alpha(xt)-\beta}^{\alpha(xt)} \frac{W(\alpha(xt); x) - W(s; x)}{(\alpha(xt) - s)^{3/2}} \frac{ds}{2\pi} \\ &= \mathcal{O}\left(\frac{e^{-\epsilon x^2 \alpha(xt)}}{\sqrt{\beta}}\right) + \int_0^{x^2(\alpha(xt)-\beta)} \frac{\log F(r) - c_+ r - \log c'_+}{(\alpha(xt) - x^{-2}r)^{3/2}} \frac{dr}{2\pi x^2} + \mathcal{O}\left(\sqrt{\beta} W'(\alpha(xt); x)\right). \end{aligned}$$

By Assumptions 1.1, we obtain

$$W'(\alpha(xt); x) = x^2 (c_+ - (\log F)'(x^2 \alpha(xt))) = \mathcal{O}(x^2 e^{-\frac{\epsilon}{2} x^2 \alpha(xt)}),$$

where we use (1.23), and so, by choosing for instance $\beta = \mathcal{O}(e^{-\epsilon x^2 \alpha(xt)})$, we get

$$\chi_2 = \frac{1}{2\pi x^2 \alpha(xt)^{3/2}} \int_0^{+\infty} (\log F(r) - c_+ r - \log c'_+) dr + \mathcal{O}(x^{-4}) = \mathcal{O}(x^{-2}).$$

These estimates are uniform in $xt \leq \delta$, see again Remark 3.1. □

3.6 Local Airy parametrix near $\alpha(xt)$

We now construct a local parametrix in a neighborhood of $\alpha(xt)$, with exactly the same jumps as S . Let us introduce

$$P^{(\alpha)}(\zeta; x, t) := E(\zeta; x, t) \Phi_{\text{Ai}}^{[j]}(x^{\frac{4}{3}} f(\zeta; xt)) e^{-\frac{x^2}{2} \phi(\zeta; xt) \sigma_3} e^{-\frac{W(\zeta; x)}{2} \sigma_3},$$

where $j = 1$ for $0 < \arg(\zeta - \alpha(xt)) < \pi/2$, $j = 2$ for $\pi/2 < \arg(\zeta - \alpha(xt)) < \pi$, $j = 3$ for $-\pi < \arg(\zeta - \alpha(xt)) < -\pi/2$, and $j = 4$ for $-\pi/2 < \arg(\zeta - \alpha(xt)) < 0$, with $\Phi_{\text{Ai}}^{[1]}, \dots, \Phi_{\text{Ai}}^{[4]}$ the entire functions defined in Appendix A which together form the solution to the Airy model RH problem. The functions f and E are given by

$$f(\zeta; xt) := \left(-\frac{3}{4} \phi(\zeta; xt) \right)^{2/3}, \quad E(\zeta; x, t) := P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3} A^{-1} \left(x^{\frac{4}{3}} f(\zeta; xt) \right)^{\frac{\sigma_3}{4}}. \quad (3.31)$$

We now contend that there exists a sufficiently small $\rho > 0$ such that $f(\zeta; xt)$ is a conformal transformation of $\zeta \in \mathcal{D} := \{\zeta \in \mathbb{C} : |\zeta - \alpha(xt)| < \rho\}$ onto a neighborhood of 0, satisfying $f'(\alpha(xt); xt) > 0$, and that $E(\zeta; x, t)$ is a holomorphic function of $\zeta \in \mathcal{D}$.

For the first statement, we note from (3.9) that, as $\zeta \rightarrow \alpha(xt)$, we have

$$g'(\zeta; xt) = \frac{c_+}{2} - \left(\frac{c_+}{\pi \alpha(xt)^{1/2}} + xt \right) (\zeta - \alpha(xt))^{1/2} + \frac{c_+}{\pi} \sum_{\ell=1}^{+\infty} \frac{(-1)^{\ell+1}}{2\ell+1} \frac{(\zeta - \alpha(xt))^{\ell+1/2}}{\alpha(xt)^{\ell+1/2}}. \quad (3.32)$$

Since $\phi(\zeta; xt) = 2g(\zeta; xt) - c_+(\zeta - \alpha(xt))$ for $\text{Re } \zeta > 0$, see (3.3), (3.11) and (3.12), integrating this relation (using $g(\alpha(xt)) = 0$) we conclude that, as $\zeta \rightarrow \alpha(xt)$, we have

$$\phi(\zeta; xt) = (\zeta - \alpha(xt))^{3/2} \left[-\frac{4}{3} \left(\frac{c_+}{\pi \alpha(xt)^{1/2}} + xt \right) + \frac{4c_+}{\pi} \sum_{\ell=1}^{+\infty} \frac{(-1)^{\ell+1}}{(2\ell+3)(2\ell+1)} \frac{(\zeta - \alpha(xt))^\ell}{\alpha(xt)^{\ell+1/2}} \right]. \quad (3.33)$$

Hence $f(\zeta; xt)$ defined in (3.31) is holomorphic in a neighborhood of $\alpha(xt)$ and $f'(\alpha(xt); xt) > 0$, proving the claim about f . For later reference we compute the first terms in the Taylor series at $\zeta = \alpha(xt)$;

$$\begin{aligned} f(\zeta; xt) &= f_1(xt)(\zeta - \alpha(xt)) + f_2(xt)(\zeta - \alpha(xt))^2 + \mathcal{O}((\zeta - \alpha(xt))^3), \quad \zeta \rightarrow \alpha(xt), \\ f_1(xt) &= \left(\frac{c_+}{\pi \alpha(xt)^{1/2}} + xt \right)^{2/3}, \quad f_2(xt) = -\frac{2c_+}{15\pi \alpha(xt)^{3/2}} \left(\frac{c_+}{\pi \alpha(xt)^{1/2}} + xt \right)^{-1/3}. \end{aligned} \quad (3.34)$$

We now prove that $E(\zeta; x, t)$ is holomorphic for $\zeta \in \mathcal{D}$. To this end, first we note that $f(\zeta; xt)$ is analytic in $\mathcal{D}$ vanishing linearly at $\zeta = \alpha(xt)$ so that

$$\left(f(\zeta; xt)^{\sigma_3/4} \right)_+ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(f(\zeta; xt)^{\sigma_3/4} \right)_-. \quad (3.35)$$

On the other hand, directly from the jump condition for $P^{(\infty)}$ we get

$$\left(P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3} A^{-1} \right)_+ = \left(P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3} A^{-1} \right)_- \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \quad (3.36)$$

Therefore, $E(\zeta; x, t)$ satisfies $E_+(\zeta; x, t) = E_-(\zeta; x, t)$ for $\alpha(xt) - \rho < \zeta < \alpha(xt)$ and so it has at worst an isolated singularity at $\zeta = \alpha(xt)$; however $f(\zeta; xt)^{-1/4}, P^{(\infty)}(\zeta; x, t)$ are both $\mathcal{O}((\zeta - \alpha(xt))^{-1/4})$ as $\zeta \rightarrow \alpha(xt)$ so that $E(\zeta; x, t) = \mathcal{O}((\zeta - \alpha(xt))^{-1/2})$ as $\zeta \rightarrow \alpha(xt)$. Hence $\alpha(xt)$ is a removable singularity and $E(\zeta; x, t)$ is analytic in the whole disk $\mathcal{D}$, as claimed.

3.7 Small norm RH problem

Define

$$R(\zeta; x, t) := \begin{cases} S(\zeta; x, t)P^{(\infty)}(\zeta; x, t)^{-1}, & \zeta \in \mathbb{C} \setminus \overline{\mathcal{D}}, \\ S(\zeta; x, t)P^{(\alpha)}(\zeta; x, t)^{-1}, & \zeta \in \mathcal{D}. \end{cases} \quad (3.37)$$

The RH conditions for R are detailed below and follow directly from those of S , $P^{(\infty)}$, and $P^{(\alpha)}$. Let us denote $\Gamma_R := \partial\mathcal{D} \cup \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ where $\Gamma_0 := (\alpha(xt) + \rho, +\infty)$, $\Gamma_{\pm} := (\alpha(xt) \pm i\rho, \alpha(xt) \pm i\infty)$; we consider Γ_0 oriented from left to right, $\Gamma_{\pm}$ oriented towards $\alpha(xt)$, and $\partial\mathcal{D}$ oriented clockwise (see Figure 3). As usual, boundary values are labeled by $+$ (resp. $-$) when we approach the oriented contour from the left (resp. right).

RH problem for R

(a) $R : \mathbb{C} \setminus \Gamma_R \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) R has the jumps

$$R_+(\zeta; x, t) = R_-(\zeta; x, t)P^{(\infty)}(\zeta; x, t) \begin{pmatrix} 1 & 0 \\ e^{-x^2\phi(\zeta; xt)}e^{-W(\zeta; x)} & 1 \end{pmatrix} P^{(\infty)}(\zeta; x, t)^{-1}, \quad \zeta \in \Gamma_{\pm}, \quad (3.38)$$

$$R_+(\zeta; x, t) = R_-(\zeta; x, t)P^{(\infty)}(\zeta; x, t) \begin{pmatrix} 1 & e^{x^2\phi(\zeta; xt)}e^{W(\zeta; x)} \\ 0 & 1 \end{pmatrix} P^{(\infty)}(\zeta; x, t)^{-1}, \quad \zeta \in \Gamma_0, \quad (3.39)$$

$$R_+(\zeta; x, t) = R_-(\zeta; x, t)P^{(\alpha)}(\zeta; x, t)P^{(\infty)}(\zeta; x, t)^{-1}, \quad \zeta \in \partial\mathcal{D}. \quad (3.40)$$

(c) We have

$$R(\zeta; x, t) = I + \frac{1}{\zeta}R_1(x, t) + \mathcal{O}(\zeta^{-2}), \quad \text{as } \zeta \rightarrow \infty. \quad (3.41)$$

(d) As $\zeta \rightarrow \alpha(xt) + \rho$ and as $\zeta \rightarrow \alpha(xt) \pm i\rho$, $R(\zeta; x, t) = \mathcal{O}(1)$.

In particular, by construction of the global parametrix $P^{(\infty)}$ there is no jump on $(-\infty, \alpha(xt))$, and by construction of the local parametrix $P^{(\alpha)}$ there is no jump inside $\mathcal{D}$. Furthermore, since both S and $P^{(\alpha)}$ remain bounded near $\alpha(xt)$, R has no pole at $\alpha(xt)$.

We now show that the jump for R is close to the identity in the appropriate norms, namely that R satisfies a *small norm* RH problem. To this end we introduce the matrix function $J_R : \Gamma_R \rightarrow \mathbb{C}^{2 \times 2}$, piece-wise defined according to (3.38)–(3.40) so that $R_+ = R_- J_R$ on Γ_R .

Lemma 3.7. *We have $\|J_R - I\|_p = \mathcal{O}(x^{-2})$ for $p = 1, 2, \infty$, uniformly in $K \leq x \leq \delta/t$; here $\|\cdot\|_p$ denotes the norm in $L^p(\Gamma_R, \mathbb{C}^{2 \times 2})$ with respect to any matrix norm on $\mathbb{C}^{2 \times 2}$.*

Proof. By construction of the global parametrix $P^{(\infty)}$ we see that $P^{(\infty)}(\zeta; x, t)$ and $P^{(\infty)}(\zeta; x, t)^{-1}$ are $\mathcal{O}(\sqrt[4]{|\zeta|} + 1)$ for $|\zeta - \alpha(xt)| > \rho$, uniformly in $0 < xt \leq \delta$. It follows from this fact and from Corollary 3.4 that there exists $\eta > 0$ such that $J_R(\zeta) = I + \mathcal{O}(\sqrt{|\zeta|} + 1)e^{-\eta x^2 \sqrt{|\zeta - \alpha(xt)|}}$ for $|\zeta - \alpha(xt)| > \rho$, uniformly in $0 < xt \leq \delta$. Up to possibly choosing a smaller $\eta > 0$, it follows that

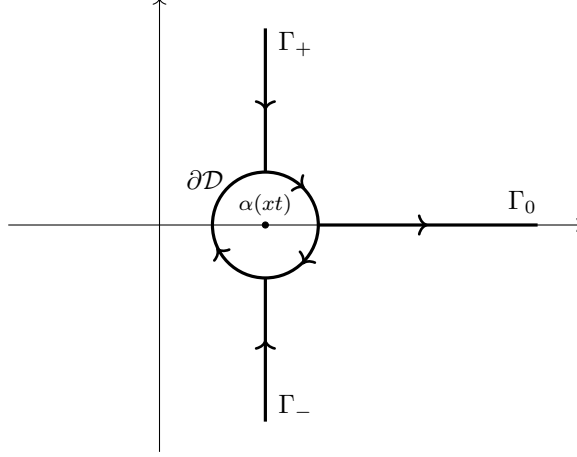


Figure 3: Jump contour $\Gamma_R = \partial\mathcal{D} \cup \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ for R .

the $L^p(\Gamma_R, \mathbb{C}^{2 \times 2})$ -norms of $J_R - I$ for $p = 1, 2, \infty$ are $\mathcal{O}(e^{-\eta x^2 \sqrt{\rho}})$ uniformly in $0 < xt \leq \delta$. On the remaining part of the contour we have, as $x \rightarrow +\infty$,

$$\begin{aligned} J_R(\zeta; x, t) - I &= P^{(\alpha)}(\zeta; x, t) P^{(\infty)}(\zeta; x, t)^{-1} - I \\ &= \frac{P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3} \Phi_{\text{Ai}, 1} e^{-\frac{W(\zeta; x)}{2} \sigma_3} P^{(\infty)}(\zeta; x, t)^{-1}}{f(\zeta; xt)^{3/2}} x^{-2} + \mathcal{O}(x^{-4}), \end{aligned} \quad (3.42)$$

uniformly for $\zeta \in \partial\mathcal{D}$. Here we use (A.2); for instance $\Phi_{\text{Ai}, 1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$, see Appendix A. It remains to note that (3.42) is uniform for x large and $xt \leq \delta$. First, f is bounded away from zero, uniformly for the relevant values of x, t , and ζ ; this is seen by inspection of the series (3.33) and by the definition of f (3.31). Similarly, by Propositions 3.5 and 3.6 we can show that $P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3}$ is bounded for $\zeta \in \partial\mathcal{D}$, uniformly for large x and $xt \leq \delta$. $\square$

It should be noted that the contour Γ_R depends on xt through $\alpha(xt)$. This is not so convenient to apply the standard theory of small-norm RH problems; however one observes easily that the jump contour is independent of x and t after the uniformly bounded shift $\tilde{\zeta} = \zeta - \alpha(xt)$, so that the standard theory applies to the RH problem in that variable. Back in our variable ζ , we obtain that, as $x \rightarrow +\infty$,

$$R(\zeta; x, t) = I + \frac{1}{x^2} R^{(1)}(\zeta; x, t) + \mathcal{O}(x^{-4}), \quad R^{(1)}(\zeta; x, t) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{J_R^{(1)}(s; x, t)}{s - \zeta} ds \quad (3.43)$$

uniformly in ζ , where (see (3.42))

$$J_R^{(1)}(\zeta; x, t) := \frac{1}{f(\zeta; xt)^{3/2}} P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3} \Phi_{\text{Ai}, 1} e^{-\frac{W(\zeta; x)}{2} \sigma_3} P^{(\infty)}(\zeta; x, t)^{-1}. \quad (3.44)$$

By the relations

$$\begin{aligned} (f^{3/2}(\zeta; xt))_+ &= -(f^{3/2}(\zeta; xt))_-, & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_{\text{Ai}, 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= -\Phi_{\text{Ai}, 1}, \\ (P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3})_+ &= (P^{(\infty)}(\zeta; x, t) e^{\frac{W(\zeta; x)}{2} \sigma_3})_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

we infer that $J_R^{(1)}(\zeta; x, t)$ has no jump for $\alpha(xt) - \rho < \zeta < \alpha(xt)$, hence $J_R^{(1)}(\zeta; x, t)$ is meromorphic for $\zeta \in \mathcal{D}$ with a pole at $\zeta = \alpha(xt)$ only, which is readily seen to be of second order. Thus we can compute $R^{(1)}$ in (3.43) for $\zeta \in \mathbb{C} \setminus \overline{\mathcal{D}}$ by extracting the polar part of $J_R^{(1)}$ at $\zeta = \alpha(xt)$ (note that $\partial\mathcal{D}$ is oriented clockwise); namely, if $|\zeta - \alpha(xt)| > \rho$,

$$R^{(1)}(\zeta; x, t) = \frac{5}{48f_1^{3/2}} \begin{pmatrix} -d_1 & id_1^2 \\ i & d_1 \end{pmatrix} \frac{1}{(\zeta - \alpha(xt))^2} + \frac{1}{32f_1^{5/2}} \begin{pmatrix} -8\chi f_1(1 + d_1\chi) + 5d_1f_2 & \frac{i}{3}(14f_1 + 48d_1\chi f_1 + 24d_1^2\chi^2 f_1 - 15d_1^2 f_2) \\ i(8\chi^2 f_1 - 5f_2) & 8\chi f_1(1 + d_1\chi) - 5d_1f_2 \end{pmatrix} \frac{1}{\zeta - \alpha(xt)}, \quad (3.45)$$

where $f_1 = f_1(xt)$, $f_2 = f_2(xt)$ are given in (3.34), $d_1 = d_1(x, t)$ in (3.21), and $\chi = \chi(x, t)$ in (3.29).

Since (3.43) is uniform in ζ we also obtain

$$R_1(x, t) = \frac{1}{32x^2 f_1^{3/2}} \begin{pmatrix} -8\chi f_1(1 + d_1\chi) + 5d_1f_2 & \frac{i}{3}(14f_1 + 48d_1\chi f_1 + 24d_1^2\chi^2 f_1 - 15d_1^2 f_2) \\ i(8\chi^2 f_1 - 5f_2) & 8\chi f_1(1 + d_1\chi) - 5d_1f_2 \end{pmatrix} + \mathcal{O}(x^{-4}). \quad (3.46)$$

3.8 Asymptotics for $u_\sigma(x, t)$ and $\partial_x \log Q_\sigma(x, t)$

Proposition 3.8. *As $x \rightarrow +\infty$ we have, uniformly in $0 < xt \leq \delta$ for any $\delta > 0$,*

$$u_\sigma(x, t) = \frac{x^2 \alpha(xt)}{2} - \frac{\log c'_+}{2\pi x t \sqrt{\alpha(xt)} + 2c_+} + \mathcal{O}(x^{-2}),$$

or equivalently

$$u_\sigma(x, t) = \frac{x}{2t} a_0 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{1}{2\sqrt{xt}} a_1 \left(\frac{\pi^2}{c_+^2} xt \right) + \mathcal{O}(x^{-2}),$$

with a_0 and a_1 as in Theorem 1.3.

Proof. Combining (1.33) and (3.15), we obtain the identity

$$u_\sigma(x, t) = x^2 (T_{1,21}(x, t)^2 - 2T_{1,11}(x, t)). \quad (3.47)$$

For ζ satisfying $\operatorname{Re} \zeta > \alpha(xt) + \rho$, using (3.18) and (3.37), we have

$$T(\zeta; x, t) = S(\zeta; x, t) = R(\zeta; x, t) P^{(\infty)}(\zeta; x, t). \quad (3.48)$$

Substituting the large ζ asymptotics (3.14) for T , (3.41) for R , and (3.20) for $P^{(\infty)}$, we get

$$T_1(x, t) = P_1^{(\infty)}(x, t) + R_1(x, t). \quad (3.49)$$

We then complete the proof using (3.24), (3.46), and Proposition 3.6. $\square$

Proposition 3.9. *We have the identity*

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{4t} - x^3 g_1(xt) + x d_1(x, t) - i x R_{1,21}(x, t), \quad (3.50)$$

where $g_1(xt)$ is given in (3.10), $d_1(x, t)$ in (3.21), and $R_1(x, t)$ in (3.41).

Proof. Combining (1.32) and (3.15), we obtain the identity

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{4t} - x^3 g_1(xt) - ixT_{1,21}(x, t). \quad (3.51)$$

By (3.49) we get

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{4t} - x^3 g_1(xt) - ixP_{1,21}^{(\infty)}(x, t) - ixR_{1,21}(x, t), \quad (3.52)$$

and it suffices to use (3.24) to obtain the result. $\square$

We now compute the large x asymptotics for the above differential identity, and we write the asymptotics as a derivative, so that we will be able to easily integrate it later.

Corollary 3.10. *As $x \rightarrow +\infty$ we have, uniformly in $0 < xt \leq \delta$ for any $\delta > 0$,*

$$\partial_x \log Q_\sigma(x, t) = \partial_x \left[-\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + F_3 \left(\frac{\pi^2}{c_+^2} xt \right) \right] + \mathcal{O}(x^{-3}) \quad (3.53)$$

with F_1 , F_2 , and F_3 as in Theorem 1.3.

Proof. By (3.10) and (3.7) it is straightforward to check that

$$\frac{x^2}{4t} + x^3 g_1(xt) = \partial_x \left[\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) \right]. \quad (3.54)$$

Moreover, by Proposition 3.5 and an explicit integration, as $x \rightarrow +\infty$ uniformly in $0 < xt \leq \delta$, we have

$$xd_1(x, t) = \partial_x \left[-\frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{2c_+ j_\sigma}{\pi} \left(\log \left(\sqrt{1 + \frac{\pi^2}{c_+^2} xt} - 1 \right) + \sqrt{1 + \frac{\pi^2}{c_+^2} xt} \right) \right] + \mathcal{O}(x^{-3}). \quad (3.55)$$

Next, using (3.46) we get

$$-ixR_{1,21}(x, t) = \frac{\chi^2(x, t)}{4xf_1(xt)^{3/2}} - \frac{5f_2(xt)}{32xf_1(xt)^{5/2}} + \mathcal{O}(x^{-3}). \quad (3.56)$$

It follows from Proposition 3.6 that, as $x \rightarrow +\infty$, we have

$$-ixR_{1,21}(x, t) = \frac{\log^2 c'_+}{4\pi^2 x \alpha(xt) f_1(xt)^{3/2}} - \frac{5f_2(xt)}{32xf_1(xt)^{5/2}} + \mathcal{O}(x^{-3}), \quad (3.57)$$

which can be explicitly integrated as

$$-ixR_{1,21}(x, t) = \partial_x \left[\left(\frac{\log^2 c'_+}{2\pi^2} + \frac{1}{24} \right) \log \left(\sqrt{1 + \frac{\pi^2}{c_+^2} xt} - 1 \right) - \frac{1}{48} \log \left(1 + \frac{\pi^2}{c_+^2} xt \right) \right] + \mathcal{O}(x^{-3}). \quad (3.58)$$

The proof is complete by (3.50). $\square$

4 Asymptotic analysis for Ψ if $x \geq K$, $x \geq \delta/t$, $t \leq t_0$

In this section, we analyze the solution to the RH problem for Ψ asymptotically as $x \rightarrow \infty$ and xt bounded from below. This is the same regime as the one studied in [6] for $\sigma(r) = \frac{1}{1+e^{-r}}$. Here, we extend the analysis of [6] to any σ satisfying Assumptions 1.1, and we compute further sub-leading terms in the asymptotic expansion.

Throughout this section, we will construct a number of auxiliary functions which will carry the same names g, ϕ, T, S, R as their natural counterparts in Section 3. We emphasize however that these functions are not identical to the ones constructed in Section 3.

4.1 Rescaling

In this section, both x and xt are large while t remains bounded, and we therefore need another change of variables than in Subsection 3.1. Instead of working in the z -plane, it will be more convenient to make the change of variable $w = \frac{t}{x}z$ and to work in the w -plane. Indeed, this will transform the functions $e^{\pm(-\frac{2}{3}tz^{\frac{3}{2}}+xz^{\frac{1}{2}})}$, which appear in the large z asymptotics of $\Psi(z; x, t)$, into $e^{\pm x^{3/2}t^{-1/2}(-\frac{2}{3}w^{\frac{3}{2}}+w^{\frac{1}{2}})}$. We define

$$\tilde{\Psi}(w; x, t) := \left(\frac{t}{x}\right)^{\frac{\sigma_3}{4}} \Psi(z = w\frac{x}{t}; x, t).$$

Then, the RH problem for Ψ transforms into the following conditions for $\tilde{\Psi}$.

RH problem for $\tilde{\Psi}$

- (a) $\tilde{\Psi} : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- (b) $\tilde{\Psi}$ has continuous boundary values on $\mathbb{R}$, and they satisfy the jump relation

$$\tilde{\Psi}_+(w; x, t) = \tilde{\Psi}_-(w; x, t) \begin{pmatrix} 1 & \frac{1}{F(\frac{x}{t}w)} \\ 0 & 1 \end{pmatrix}, \quad w \in \mathbb{R},$$

where we recall that $F = \frac{1}{1-\sigma}$.

- (c) As $w \rightarrow \infty$, we have

$$\begin{aligned} \tilde{\Psi}(w; x, t) &= \left(I + \frac{1}{w} \tilde{\Psi}_1(x, t) + \mathcal{O}(w^{-2}) \right) w^{\frac{1}{4}\sigma_3} A^{-1} e^{x^{3/2}t^{-1/2}(-\frac{2}{3}w^{\frac{3}{2}}+w^{\frac{1}{2}})\sigma_3} \\ &\quad \times \begin{cases} I, & |\arg w| < \pi - \varepsilon, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \varepsilon < \pm \arg w < \pi, \end{cases} \end{aligned}$$

for any $\varepsilon \in (0, \frac{\pi}{2})$, where

$$\tilde{\Psi}_1(x, t) = \frac{t}{x} \begin{pmatrix} q(x, t) & -it^{1/2}x^{-1/2}r(x, t) \\ it^{-1/2}x^{1/2}p(x, t) & -q(x, t) \end{pmatrix}. \quad (4.1)$$

4.2 Construction of the g -function

Similarly as in [6], the key step in the asymptotic analysis of the RH problem is the construction of a g -function, which satisfies the following conditions.

Conditions for g

- (a) $g(w; x, t)$ is analytic in $w \in \mathbb{C} \setminus (-\infty, a(x, t)]$,
- (b) $g_+(w; x, t) + g_-(w; x, t) + \frac{t^{1/2}}{x^{3/2}} V_0(x, t) = \frac{t^{1/2}}{x^{3/2}} \log F(\frac{x}{t}w)$ for $w \in (-\infty, a(x, t))$,
- (c) $g(w; x, t) = -\frac{2}{3}w^{\frac{3}{2}} + w^{\frac{1}{2}} + g_0(x, t) + g_1(x, t)w^{-1/2} + \mathcal{O}(w^{-3/2})$ as $w \rightarrow \infty$,
- (d) $g_+(w; x, t) - g_-(w; x, t) = \mathcal{O}((w - a(x, t))^{3/2})$ as $w \rightarrow a(x, t)$, $w < a(x, t)$.

Here, $a(x, t) > 0$, $g_0(x, t)$, $g_1(x, t)$ and $V_0 = V_0(x, t)$ are to be determined. Similarly as in Section 3, it will be convenient to first construct g' , which satisfies the conditions below, and then to integrate.

Conditions for g'

- (a) $g'(w; x, t)$ is analytic in $w \in \mathbb{C} \setminus (-\infty, a(x, t)]$,
- (b) $g'_+(w; x, t) + g'_-(w; x, t) = \frac{1}{\sqrt{xt}}(\log F)'(\frac{x}{t}w)$ for $w \in (-\infty, a(x, t))$,
- (c) $g'(w; x, t) = -w^{1/2} + \frac{1}{2}w^{-\frac{1}{2}} - \frac{1}{2}g_1(x, t)w^{-\frac{3}{2}} + \mathcal{O}(w^{-5/2})$ as $w \rightarrow \infty$,
- (d) $g'_+(w; x, t) - g'_-(w; x, t) = \mathcal{O}((w - a(x, t))^{1/2})$ as $w \rightarrow a(x, t)$, $w < a(x, t)$.

We can construct g' directly in terms of a Cauchy integral: set

$$g'(w; x, t) = \frac{a(x, t) + 1 - 2w}{2(w - a(x, t))^{1/2}} + \frac{1}{2\pi(w - a(x, t))^{1/2}} \frac{1}{\sqrt{xt}} \int_{-\infty}^{a(x, t)} (\log F)'(\frac{x}{t}\zeta) \sqrt{a(x, t) - \zeta} \frac{d\zeta}{\zeta - w}, \quad (4.2)$$

with $(w - a(x, t))^{1/2}$ analytic in $\mathbb{C} \setminus (-\infty, a(x, t)]$ and positive for $w > a(x, t)$, and where $\sqrt{a(x, t) - \zeta}$ denotes the positive square root. Then, it is straightforward to verify that g' satisfies conditions (a), (b), and (c) (with $g_1(x, t)$ given by (4.16) below). In order to achieve also the endpoint condition (d), we need to take $a(x, t)$ such that

$$\int_{-\infty}^{a(x, t)} (\log F)'(\frac{x}{t}\zeta) \frac{d\zeta}{\sqrt{a(x, t) - \zeta}} = \pi \sqrt{xt}(1 - a(x, t)). \quad (4.3)$$

For later convenience we study in detail the behavior of the endpoint $a(x, t)$.

Proposition 4.1. *For any σ satisfying Assumptions 1.1 and for any $x, t > 0$ there exists a unique real solution $a(x, t)$ to the equation (4.3). There exists $K > 0$, depending on σ only, such that for all $\delta > 0$ there exist $a_{\min}, a_{\max} > 0$, depending on δ and σ only, such that $a_{\min} < a(x, t) < a_{\max}$ for all x, t satisfying $x \geq K, xt \geq \delta$. Moreover, for any $\delta > 0$ and $t_0 > 0$ we have*

$$a(x, t) = a_0 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{t^{1/2}}{x^{3/2}} a_1 \left(\frac{\pi^2}{c_+^2} xt \right) + \frac{t^{3/2}}{x^{5/2}} a_2 \left(\frac{\pi^2}{c_+^2} xt \right) + \mathcal{O}(t^{5/2} x^{-7/2}), \quad (4.4)$$

as $x \rightarrow +\infty$, uniformly in $xt \geq \delta$ and $t \leq t_0$, where a_0, a_1 and a_2 are as in Theorem 1.3.

Remark 4.2. *The leading order in (4.4) satisfies $a_0(\frac{\pi^2}{c_+^2} xt) = xt\alpha(xt)$, where $\alpha(xt)$ is the endpoint for the RH analysis of Section 3, see (3.7).*

Proof. Let us write the endpoint equation (4.3) as

$$h(a; x/t) = \pi\sqrt{xt}(1-a), \quad h(a; \lambda) := \int_{-\infty}^a (\log F)'(\lambda\zeta) \frac{d\zeta}{\sqrt{a-\zeta}}. \quad (4.5)$$

Since F is log-convex (see Assumptions 1.1), the function $h = h(a; x/t)$ is increasing in a , while $a \mapsto \pi\sqrt{xt}(1-a)$ is a decreasing affine function of a , thus there exists exactly one real solution $a = a(x, t)$, proving the first statement. Next, let us define $v := (\log F)' - c_+ 1_{(0, +\infty)}$. By Assumptions 1.1, see in particular (1.23) and (1.24), $v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ so that for all $a > 0$ we have

$$\begin{aligned} |h(a; x/t) - 2c_+\sqrt{a}| &= \left| \int_{-\infty}^a v\left(\frac{x}{t}\zeta\right) \frac{d\zeta}{\sqrt{a-\zeta}} \right| \leq \sqrt{\frac{t}{x}} \int_{-\infty}^{\frac{x}{t}a} |v(r)| \frac{dr}{\sqrt{\frac{x}{t}a-r}} \\ &\leq \sqrt{\frac{t}{x}} \int_{-\infty}^{\frac{x}{t}a-1} |v(r)| dr + \sqrt{\frac{t}{x}} \|v\|_\infty \int_{-1}^0 \frac{ds}{\sqrt{-s}} \\ &\leq (\|v\|_1 + 2\|v\|_\infty) \sqrt{\frac{t}{x}}. \end{aligned}$$

Setting $M := \|v\|_1 + 2\|v\|_\infty$ it then follows by the inequalities $2c_+\sqrt{a} - M\sqrt{t/x} \leq h(a; x/t) \leq 2c_+\sqrt{a} + M\sqrt{t/x}$, which we have just shown, that $a_-(x, t) \leq a(x, t) \leq a_+(x, t)$ where $a_\pm(x, t)$ are the solutions to

$$2c_+\sqrt{a_\pm} \mp M\sqrt{\frac{t}{x}} = \pi\sqrt{xt}(1-a_\pm) \Rightarrow a_\pm(x, t) = 1 + \frac{2c_+^2}{\pi^2 tx} \pm \frac{M}{\pi x} - \frac{2c_+\sqrt{c_+^2 + \pi t(\pi x \pm M)}}{\pi^2 tx}.$$

Here we assume $x > M/\pi$. It is straightforward to check that $a_-(x, y/x)$ is an increasing function of both x and y provided that $x > M/\pi$, and that $a_-(M/\pi, t) = 0$ for all $t > 0$. Fixing $K > M/\pi$, for all $x, t > 0$ such that $x \geq K, xt \geq \delta$ we have $a(x, t) \geq a_{\min} := a_-(K, \delta/K)$. A completely analogous argument proves the existence of $a_{\max}$ such that $a(x, t) \leq a_{\max}$ for all $x, t > 0$ such that $x \geq K, xt \geq \delta$. (In fact, we can set $a_{\max} := 1 + \frac{M}{\pi K}$.)

Next, we claim that $h(a; \lambda)$ has an expansion

$$h(a; \lambda) = h_0(a) + \lambda^{-1}h_1(a) + \lambda^{-2}h_2(a) + \mathcal{O}(\lambda^{-3}), \quad \lambda \rightarrow +\infty, \quad (4.6)$$

uniform for a in compact subsets of $(0, +\infty)$, where

$$h_0(a) = 2c_+\sqrt{a}, \quad h_1(a) = \frac{\log c_+}{\sqrt{a}}, \quad h_2(a) = \frac{\pi j_\sigma}{a^{3/2}}, \quad (4.7)$$

where j_σ is defined in (3.22). Postponing for a moment the proof of (4.6), let us note that the Implicit Function Theorem then implies an expansion of the form

$$a(x, t) = \widehat{a}_0(xt) + \frac{1}{\sqrt{xt}} \left(\frac{t}{x} \widehat{a}_1(xt) + \frac{t^2}{x^2} \widehat{a}_2(xt) + \mathcal{O}(t^3 x^{-3}) \right) \quad (4.8)$$

as $x \rightarrow +\infty$, where $\widehat{a}_i(xt)$ are found from the equations

$$\begin{aligned} h_0(\widehat{a}_0(xt)) &= \pi\sqrt{xt}(1-\widehat{a}_0(xt)) \\ h_1(\widehat{a}_0(xt)) + \frac{\widehat{a}_1(xt)}{\sqrt{xt}} h_0'(\widehat{a}_0(xt)) &= -\pi\widehat{a}_1(xt) \\ h_2(\widehat{a}_0(xt)) + \frac{\widehat{a}_1(xt)}{\sqrt{xt}} h_1'(\widehat{a}_0(xt)) + \frac{1}{2} \frac{\widehat{a}_1(xt)^2}{xt} h_0''(\widehat{a}_0(xt)) + \frac{\widehat{a}_2(xt)}{\sqrt{xt}} h_0'(\widehat{a}_0(xt)) &= -\pi\widehat{a}_2(xt) \end{aligned}$$

and they are in particular bounded for $xt \geq \delta$, and the $\mathcal{O}(t^{5/2}x^{-7/2})$ is also easily checked to be uniform for $xt \geq \delta$ and $t \leq t_0$ for any $\delta > 0$ and $t_0 > 0$. These equations imply

$$\begin{aligned}\widehat{a}_0(xt) &= \left(\sqrt{1 + \frac{c_+^2}{\pi^2 xt}} - \frac{c_+}{\pi \sqrt{xt}} \right)^2, & \widehat{a}_1(xt) &= -\frac{\log c'_+}{\pi \sqrt{\widehat{a}_0(xt)} + \frac{c_+}{\sqrt{xt}}}, \\ \widehat{a}_2(xt) &= \frac{c_+ \widehat{a}_1(xt)^2 + 2\widehat{a}_1(xt) \sqrt{xt} \log c'_+ - 4\pi xt j_\sigma}{4\widehat{a}_0(xt) \left(\pi \sqrt{\widehat{a}_0(xt)} + \frac{c_+}{\sqrt{xt}} \right) xt},\end{aligned}$$

and then (4.4) is obtained by simple algebra.

It only remains to prove (4.6); to this end we write

$$h(a; \lambda) - 2c_+ \sqrt{a} = \int_{-\infty}^a v(\lambda \zeta) \frac{d\zeta}{\sqrt{a - \zeta}} = \frac{1}{\lambda} \left[\int_{-\infty}^{-a\lambda/2} \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} + \int_{-a\lambda/2}^{a\lambda/2} \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} + \int_{a\lambda/2}^a \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} \right] \quad (4.9)$$

and we reason for each term separately. For the first one we have

$$\left| \int_{-\infty}^{-a\lambda/2} \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} \right| \leq \sqrt{\frac{2}{3a}} \left(\sup_{r < -\frac{a\lambda}{2}} |v(r)|^{1/2} \right) \int_{-\infty}^{-a\lambda/2} |v(r)|^{1/2} d\zeta \quad (4.10)$$

which is exponentially small as $\lambda \rightarrow +\infty$ by Assumptions 1.1, see (1.24). For the second term we use the expansion

$$\frac{1}{\sqrt{a - \frac{r}{\lambda}}} = \frac{1}{\sqrt{a}} + \lambda^{-1} \frac{r}{2a^{3/2}} + \mathcal{O}(r^2 \lambda^{-2}), \quad \lambda \rightarrow +\infty, \quad (4.11)$$

which is uniform for $|r| < a\lambda/2$, to obtain

$$\int_{-a\lambda/2}^{a\lambda/2} \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} = \frac{1}{\sqrt{a}} \int_{-a\lambda/2}^{a\lambda/2} v(r) dr + \lambda^{-1} \frac{1}{2a^{3/2}} \int_{-a\lambda/2}^{a\lambda/2} r v(r) dr + \mathcal{O}(\lambda^{-2}) \quad (4.12)$$

and so, since $v(r)$ has exponential decay as $r \rightarrow \pm\infty$, see (1.23) and (1.24), we have

$$\int_{-a\lambda/2}^{a\lambda/2} \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} v(r) dr + \lambda^{-1} \frac{1}{2a^{3/2}} \int_{-\infty}^{+\infty} r v(r) dr + \mathcal{O}(\lambda^{-2}). \quad (4.13)$$

For the third term we estimate

$$\int_{a\lambda/2}^a \frac{v(r) dr}{\sqrt{a - \frac{r}{\lambda}}} \leq \lambda \left(\sup_{r > \frac{a\lambda}{2}} |v(r)| \right) \int_{a/2}^a \frac{d\zeta}{\sqrt{a - \zeta}} = \lambda \sqrt{2a} \left(\sup_{r > \frac{a\lambda}{2}} |v(r)| \right) \quad (4.14)$$

which is exponentially small as $\lambda \rightarrow +\infty$ by (1.23). Uniformity for a in compact subsets of $(0, +\infty)$ is clear in all these estimates. The proof is complete by the following integrations by parts;

$$\begin{aligned}\int_{-\infty}^{+\infty} v(r) dr &= \int_{-\infty}^0 (\log F)'(r) dr + \int_0^{+\infty} (\log F(r) - c_+ r)' dr = \lim_{r \rightarrow +\infty} (\log F(r) - c_+ r) = \log c'_+ \\ \int_{-\infty}^{+\infty} r v(r) dr &= \int_{-\infty}^0 r (\log F)'(r) dr + \int_0^{+\infty} r (\log F(r) - c_+ r - \log c'_+)' dr \\ &= - \int_{-\infty}^{+\infty} [\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)] dr = 2\pi j_\sigma\end{aligned}$$

where in the last equality we use (3.22). $\square$

Remark 4.3. We can construct g' equivalently as

$$g'(w; x, t) = -\sqrt{w - a(x, t)} \left(1 + \frac{1}{2\pi\sqrt{xt}} \int_{-\infty}^{a(x, t)} (\log F)' \left(\frac{x}{t} \zeta \right) \frac{1}{\sqrt{a(x, t) - \zeta}} \frac{d\zeta}{\zeta - w} \right), \quad (4.15)$$

where $a(x, t)$ is found by requiring that $g'(w; x, t) = -w^{1/2} + \frac{1}{2}w^{-\frac{1}{2}} + \mathcal{O}(w^{-\frac{3}{2}})$ as $w \rightarrow \infty$. Here too, we see that $a(x, t)$ must satisfy (4.3). Using (4.3), we easily show that the right-hand sides of (4.2) and (4.15) are indeed equal.

Let us define

$$g(w; x, t) = \int_{a(x, t)}^w g'(s; x, t) ds.$$

Since $g(a(x, t); x, t) = 0$, we must choose

$$V_0(x, t) = -\log(1 - \sigma(\frac{x}{t}a(x, t))) = \log F(\frac{x}{t}a(x, t)).$$

By (4.2), since $(\log F)'$ decays rapidly at $-\infty$, it is straightforward to check that the coefficient g_1 in the large w expansion of g and g' is given by

$$g_1(x, t) = \frac{a(x, t)^2}{4} - \frac{a(x, t)}{2} + \frac{1}{\pi\sqrt{xt}} \int_{-\infty}^{a(x, t)} (\log F)' \left(\frac{x}{t} \zeta \right) \sqrt{a(x, t) - \zeta} d\zeta \quad (4.16)$$

Also, in view of condition (b) for g , $g_0(x, t)$ is given by

$$g_0(x, t) = -\frac{t^{1/2}}{2x^{3/2}} V_0(x, t).$$

4.3 Normalization of the RH problem

Let us define

$$T(w; x, t) := \begin{pmatrix} 1 & ig_1(x, t)x^{3/2}t^{-1/2} \\ 0 & 1 \end{pmatrix} \tilde{\Psi}(w; x, t) e^{-x^{3/2}t^{-1/2}(g(w; x, t) - g_0(x, t))\sigma_3},$$

and

$$\phi(w; x, t) := 2g(w; x, t) + \frac{t^{1/2}}{x^{3/2}} V_0(x, t) - \frac{t^{1/2}}{x^{3/2}} \log F\left(\frac{x}{t}w\right). \quad (4.17)$$

This function satisfies

$$\phi_{\pm}(w; x, t) = \pm (g_+(w; x, t) - g_-(w; x, t)), \quad w \in (-\infty, a(x, t)).$$

Then, T satisfies the following RH problem.

RH problem for T

(a) $T : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) T has the jumps

$$T_+(w; x, t) = T_-(w; x, t) \begin{pmatrix} e^{-x^{3/2}t^{-1/2}\phi_+(w; x, t)} & 1 \\ 0 & e^{-x^{3/2}t^{-1/2}\phi_-(w; x, t)} \end{pmatrix}, \quad w \in (-\infty, a(x, t)),$$

$$T_+(w; x, t) = T_-(w; x, t) \begin{pmatrix} 1 & e^{x^{3/2}t^{-1/2}\phi(w; x, t)} \\ 0 & 1 \end{pmatrix}, \quad w \in (a(x, t), +\infty).$$

(c) We have

$$T(w; x, t) = \left(I + \frac{1}{w} T_1(x, t) + \mathcal{O}(w^{-2}) \right) w^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } w \rightarrow \infty, \quad (4.18)$$

where

$$T_1(x, t) = \begin{pmatrix} -\frac{x^3}{2t} g_1^2 - x g_1 p + \frac{t}{x} q & i \frac{x^{5/2}}{t^{1/2}} g_1^2 p - 2i(xt)^{1/2} g_1 q + i \frac{x^{9/2}}{3t^{3/2}} g_1^3 - i \frac{x^{3/2}}{t^{1/2}} g_2 - i \left(\frac{t}{x} \right)^{3/2} r \\ i \frac{x^{3/2}}{t^{1/2}} g_1 + i \frac{t^{1/2}}{x^{1/2}} p & \frac{x^3}{2t} g_1^2 + x g_1 p - \frac{t}{x} q \end{pmatrix}, \quad (4.19)$$

where $g_i = g_i(x, t)$, $p = p(x, t)$, $q = q(x, t)$, and $r = r(x, t)$.

(d) As $w \rightarrow a(x, t)$, $T(w; x, t) = \mathcal{O}(1)$.

4.4 Opening of the lenses

For $w \in (-\infty, a(x, t))$, note that the jump matrix can be factorized as follows:

$$\begin{pmatrix} e^{-x^{3/2}t^{-1/2}\phi_+(w;x,t)} & 1 \\ 0 & e^{-x^{3/2}t^{-1/2}\phi_-(w;x,t)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-x^{3/2}t^{-1/2}\phi_-(w;x,t)} & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-x^{3/2}t^{-1/2}\phi_+(w;x,t)} & 1 \end{pmatrix}. \quad (4.20)$$

This factorization will allow us to split the jump for T on $(-\infty, a(x, t))$ into three jumps for a new matrix function S with jump contour $\mathbb{R} \cup (i\mathbb{R} + a(x, t))$, exactly as in Section 3. To this end, we define S as

$$S(w; x, t) := \begin{cases} T(w; x, t) \begin{pmatrix} 1 & 0 \\ \mp e^{-x^{3/2}t^{-1/2}\phi(w;x,t)} & 1 \end{pmatrix}, & \text{if } \operatorname{Re} w < a(x, t) \text{ and } \pm \operatorname{Im} w > 0, \\ T(w; x, t), & \text{if } \operatorname{Re} w > a(x, t). \end{cases} \quad (4.21)$$

The following RH conditions are direct consequences of this definition and of the RH problem for T .

RH problem for S

(a) $S : \mathbb{C} \setminus (\mathbb{R} \cup (i\mathbb{R} + a(x, t))) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) S has the jumps

$$\begin{aligned} S_+(w; x, t) &= S_-(w; x, t) \begin{pmatrix} 1 & 0 \\ e^{-x^{3/2}t^{-1/2}\phi(w;x,t)} & 1 \end{pmatrix}, & w \in i\mathbb{R} + a(x, t), \\ S_+(w; x, t) &= S_-(w; x, t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & w \in (-\infty, a(x, t)), \\ S_+(w; x, t) &= S_-(w; x, t) \begin{pmatrix} 1 & e^{x^{3/2}t^{-1/2}\phi(w;x,t)} \\ 0 & 1 \end{pmatrix}, & w \in (a(x, t), +\infty), \end{aligned}$$

where $\mathbb{R}$ is oriented from left to right, and the vertical half-lines $a(x, t) \pm i\mathbb{R}^+$ are oriented towards $a(x, t)$. As usual, $+$ and $-$ denote the boundary values from the left and right sides of the contour, respectively.

(c) We have

$$S(w; x, t) = \left(I + \frac{1}{w} T_1(x, t) + \mathcal{O}(w^{-2}) \right) w^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } w \rightarrow \infty,$$

where $T_1(x, t)$ is the same as in (4.19).

(d) As $w \rightarrow a(x, t)$, $S(w; x, t) = \mathcal{O}(1)$.

The next proposition shows that the jump matrix for S is close to the identity except on $(-\infty, a(x, t))$ and near $a(x, t)$.

Proposition 4.4. *Let σ satisfy Assumptions 1.1. Then, we have the inequalities*

$$\begin{aligned} \exp \left(x^{3/2} t^{-1/2} \phi(w; x, t) \right) &\leq \exp \left(-\frac{4}{3} x^{3/2} t^{-1/2} (w - a(x, t))^{3/2} \right), & \text{for } w > a(x, t), \\ \exp \left(-x^{3/2} t^{-1/2} \phi(w; x, t) \right) &\leq \exp \left(-\frac{2\sqrt{2}}{3} x^{3/2} t^{-1/2} |w - a(x, t)|^{3/2} \right), & \text{for } w \in a(x, t) + i\mathbb{R}. \end{aligned}$$

Proof. The proof is identical to that of [6, Proposition 3.5], and relies on Assumptions 1.1, part 4. $\square$

4.5 Global parametrix

Ignoring the exponentially small jumps of S on the vertical line and a small, but fixed, disk centered at $a(x, t)$, we obtain a RH problem with jumps only on $(-\infty, a(x, t))$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, a(x, t)) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) $P^{(\infty)}$ has the jumps

$$P_+^{(\infty)}(w; x, t) = P_-^{(\infty)}(w; x, t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w \in (-\infty, a(x, t)). \quad (4.22)$$

(c) There exists a matrix $P_1^{(\infty)}(x, t)$ such that

$$P^{(\infty)}(w; x, t) = \left(I + \frac{1}{w} P_1^{(\infty)}(x, t) + \mathcal{O}(w^{-2}) \right) w^{\frac{1}{4}\sigma_3} A^{-1}, \quad \text{as } w \rightarrow \infty. \quad (4.23)$$

If we impose in addition the condition

(d) As $w \rightarrow a(x, t)$, $P^{(\infty)}(w; x, t) = \mathcal{O}(|w - a(x, t)|^{-\frac{1}{4}})$,

there is a unique solution to this RH problem, given by

$$P^{(\infty)}(w; x, t) = (w - a(x, t))^{\frac{1}{4}\sigma_3} A^{-1}, \quad (4.24)$$

and then the matrix $P_1^{(\infty)}(x, t)$ is given by

$$P_1^{(\infty)}(x, t) = -\frac{1}{4} a(x, t) \sigma_3. \quad (4.25)$$

4.6 Local Airy parametrix near $a(x, t)$

As in Section 3.6, we now want to construct a local parametrix $P^{(a)}$ which has exactly the same jumps as S in a disk $|w - a(x, t)| < \rho$. The local parametrix is constructed again in terms of the Airy model RH problem (see Appendix A), and it is given by

$$P^{(a)}(w; x, t) := E(w; x, t) \Phi_{\text{Ai}}^{[j]}(xt^{-1/3}f(w; x, t)) e^{-\frac{x^{3/2}t^{-1/2}}{2}\phi(w; x, t)\sigma_3},$$

where $j = 1$ for $0 < \arg(w - a(x, t)) < \pi/2$, $j = 2$ for $\pi/2 < \arg(w - a(x, t)) < \pi$, $j = 3$ for $-\pi < \arg(w - a(x, t)) < -\pi/2$, and $j = 4$ for $-\pi/2 < \arg(w - a(x, t)) < 0$, with $\Phi_{\text{Ai}}^{[1]}, \dots, \Phi_{\text{Ai}}^{[4]}$ the entire functions defined in Appendix A which together form the solution to the Airy model RH problem. The functions f and E are given by

$$f(w; x, t) := \left(-\frac{3}{4}\phi(w; x, t)\right)^{2/3}, \quad E(w; x, t) := P^{(\infty)}(w; x, t) A^{-1}(xt^{-1/3}f(w; x, t))^{\frac{\sigma_3}{4}}. \quad (4.26)$$

Similarly as in Section 3.6, we will now show that there exists a sufficiently small $\rho > 0$ such that $f(w; x, t)$ is a conformal transformation of $w \in \mathcal{D} := \{w \in \mathbb{C} : |w - a(x, t)| < \rho\}$ onto a neighborhood of 0, satisfying $f'(a(x, t); x, t) > 0$, and that $E(w; x, t)$ is a holomorphic function of $w \in \mathcal{D}$. For the first statement we start by rewriting (4.15) as

$$\begin{aligned} g'(w; x, t) = & -\sqrt{w - a(x, t)} \left(1 + \frac{1}{2\pi\sqrt{xt}} \int_{-\infty}^{a(x, t)} \frac{(\log F)'(\frac{x}{t}\zeta) - (\log F)'(\frac{x}{t}w)}{\zeta - w} \frac{d\zeta}{\sqrt{a(x, t) - \zeta}}\right) \\ & + \frac{1}{2\sqrt{xt}} (\log F)'(\frac{x}{t}w). \end{aligned}$$

The Taylor series at $w = a(x, t)$

$$\begin{aligned} \frac{(\log F)'(\frac{x}{t}\zeta) - (\log F)'(\frac{x}{t}w)}{\zeta - w} &= \sum_{\ell=0}^{+\infty} \frac{\varphi_{\ell}(\zeta; x, t)}{(\zeta - a(x, t))^{\ell+1}} (w - a(x, t))^{\ell}, \\ \varphi_{\ell}(\zeta; x, t) &= (\log F)' \left(\frac{x}{t}\zeta\right) - \sum_{j=0}^{\ell} \left(\frac{x}{t}\right)^j (\log F)^{(j+1)} \left(\frac{x}{t}a(x, t)\right) \frac{(\zeta - a(x, t))^j}{j!}, \end{aligned}$$

implies the identity

$$\begin{aligned} g'(w; x, t) = & -\sqrt{w - a(x, t)} + \frac{(\log F)'(\frac{x}{t}w)}{2\sqrt{xt}} \\ & - \frac{1}{2\pi\sqrt{xt}} \sum_{\ell=0}^{+\infty} (w - a(x, t))^{\ell+\frac{1}{2}} \int_{-\infty}^{a(x, t)} \frac{\varphi_{\ell}(\zeta; x, t) d\zeta}{(\zeta - a(x, t))^{\ell+1} \sqrt{a(x, t) - \zeta}}, \end{aligned} \quad (4.27)$$

where the last term is a Puiseux series as $w \rightarrow a(x, t)$ away from the cut $(-\infty, a(x, t))$.

To simplify the coefficients let us integrate by parts

$$\begin{aligned} (-1)^{\ell+1} \int_{-\infty}^{a(x, t)} \frac{\varphi_{\ell}(\zeta; x, t) d\zeta}{(a(x, t) - \zeta)^{\ell+\frac{3}{2}}} &= \frac{2^{\ell+1}}{(2\ell+1)!!} \int_{-\infty}^{a(x, t)} \varphi_{\ell}(\zeta; x, t) \left[(-\partial_{\zeta})^{\ell+1} \frac{1}{\sqrt{a(x, t) - \zeta}} \right] d\zeta \\ &= \frac{2^{\ell+1}}{(2\ell+1)!!} \left(\frac{x}{t}\right)^{\ell+1} \int_{-\infty}^{a(x, t)} (\log F)^{(\ell+2)} \left(\frac{x}{t}\zeta\right) \frac{d\zeta}{\sqrt{a(x, t) - \zeta}}. \end{aligned}$$

Using this relation, integrating (4.27) in w (by construction $g(a(x, t); x, t) = 0$) and applying the definition (4.17) of $\phi(w; x, t)$ we finally obtain

$$\begin{aligned} \phi(w; x, t) &= -\frac{4}{3}(w - a(x, t))^{3/2} \\ &\times \left[1 + \frac{3}{2\pi\sqrt{xt}} \sum_{\ell=0}^{+\infty} \left(\frac{2x}{t} \right)^{\ell+1} \frac{(w - a(x, t))^\ell}{(2\ell + 3)!!} \int_{-\infty}^{a(x, t)} (\log F)^{(\ell+2)} \left(\frac{x}{t}\zeta \right) \frac{d\zeta}{\sqrt{a(x, t) - \zeta}} \right]. \end{aligned} \quad (4.28)$$

Consequently, f defined in (4.26) is holomorphic in a neighborhood of $a(x, t)$, maps $a(x, t)$ to the origin, and $f'(a(x, t); x, t) > 0$. The first few terms in the Taylor expansion of f near $w = a(x, t)$ are

$$\begin{aligned} f(w; x, t) &= f_1(x, t)(w - a(x, t)) + f_2(x, t)(w - a(x, t))^2 + \mathcal{O}((w - a(x, t))^3), \\ f_1(x, t) &= \left(1 + \frac{x^{1/2}}{\pi t^{3/2}} \int_{-\infty}^{a(x, t)} (\log F)'' \left(\frac{x}{t}\zeta \right) \frac{d\zeta}{\sqrt{a(x, t) - \zeta}} \right)^{2/3}, \\ f_2(x, t) &= \frac{4x^{3/2}}{15\pi t^{5/2} f_1(x, t)^{1/2}} \int_{-\infty}^{a(x, t)} (\log F)''' \left(\frac{x}{t}\zeta \right) \frac{d\zeta}{\sqrt{a(x, t) - \zeta}}. \end{aligned} \quad (4.29)$$

Finally, the proof that E is holomorphic in $\mathcal{D}$ is a straightforward adaptation of the argument in Section 3.6.

4.7 Small norm RH problem

Define

$$R(w; x, t) := \begin{cases} S(w; x, t)P^{(\infty)}(w; x, t)^{-1}, & w \in \mathbb{C} \setminus \overline{\mathcal{D}}, \\ S(w; x, t)P^{(a)}(w; x, t)^{-1}, & w \in \mathcal{D}. \end{cases} \quad (4.30)$$

The RH conditions for R are detailed below and follow directly from those of S , $P^{(\infty)}$, and $P^{(a)}$. Let us denote $\Gamma_R := \partial\mathcal{D} \cup \Gamma_- \cup \Gamma_0 \cup \Gamma_+$ where $\Gamma_0 := (a(x, t) + \rho, +\infty)$, $\Gamma_\pm := (a(x, t) \pm i\rho, a(x, t) \pm i\infty)$; we consider Γ_0 oriented from left to right, $\Gamma_\pm$ oriented towards $a(x, t)$, and $\partial\mathcal{D}$ oriented clockwise (see Figure 3 for the analogous contour of Section 3). As usual, boundary values are labeled by $+$ (resp. $-$) when we approach the oriented contour from the left (resp. right).

RH problem for R

(a) $R : \mathbb{C} \setminus \Gamma_R \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) R has the jumps

$$R_+(w; x, t) = R_-(w; x, t)P^{(\infty)}(w; x, t) \begin{pmatrix} 1 & 0 \\ e^{-x^{3/2}t^{-1/2}\phi(w; x, t)} & 1 \end{pmatrix} P^{(\infty)}(w; x, t)^{-1}, \quad w \in \Gamma_\pm, \quad (4.31)$$

$$R_+(w; x, t) = R_-(w; x, t)P^{(\infty)}(w; x, t) \begin{pmatrix} 1 & e^{x^{3/2}t^{-1/2}\phi(w; x, t)} \\ 0 & 1 \end{pmatrix} P^{(\infty)}(w; x, t)^{-1}, \quad w \in \Gamma_0, \quad (4.32)$$

$$R_+(w; x, t) = R_-(w; x, t)P^{(a)}(w; x, t)P^{(\infty)}(w; x, t)^{-1}, \quad w \in \partial\mathcal{D}. \quad (4.33)$$

(c) We have

$$R(w; x, t) = I + \frac{1}{w} R_1(x, t) + \mathcal{O}(w^{-2}), \quad \text{as } w \rightarrow \infty. \quad (4.34)$$

(d) As $w \rightarrow a(x, t) + \rho$ and as $w \rightarrow a(x, t) \pm i\rho$, $R(w; x, t) = \mathcal{O}(1)$.

In particular, by construction of the global parametrix $P^{(\infty)}$ there is no jump on $(-\infty, a(x, t))$, and by construction of the local parametrix $P^{(a)}$ there is no jump inside $\mathcal{D}$. Furthermore, since both S and $P^{(a)}$ remain bounded near $a(x, t)$, R has no pole at $a(x, t)$.

We now show that the jump for R satisfies a small norm RH problem; to this end we introduce the matrix function $J_R : \Gamma_R \rightarrow \mathbb{C}^{2 \times 2}$, piece-wise defined according to (4.31)–(4.33) so that $R_+ = R_- J_R$ on Γ_R .

Lemma 4.5. *We have $\|J_R - I\|_p = \mathcal{O}(x^{-3/2}t^{1/2})$ for $p = 1, 2, \infty$, uniformly in $x \geq K$, $xt \geq \delta$, $t \leq t_0$; here $\|\cdot\|_p$ denotes the norm in $L^p(\Gamma_R, \mathbb{C}^{2 \times 2})$ with respect to any matrix norm on $\mathbb{C}^{2 \times 2}$.*

Proof. By construction of the global parametrix $P^{(\infty)}$ we see that $P^{(\infty)}(w; x, t)$ and $P^{(\infty)}(w; x, t)^{-1}$ are $\mathcal{O}(\sqrt[4]{|w|+1})$ for $|w - a(x, t)| > \rho$, uniformly in $xt \geq \delta$, $x \geq K$, $t \leq t_0$. From this and Proposition 4.4, it follows that $J_R(w) = I + \mathcal{O}(\sqrt{|w|+1} e^{-\frac{2\sqrt{2}}{3}x^{3/2}t^{-1/2}|w-a(x,t)|^{3/2}})$ for $|w - a(x, t)| > \rho$, uniformly in $xt \geq \delta$, $x \geq K$, $t \leq t_0$. It follows that the $L^p(\Gamma_R, \mathbb{C}^{2 \times 2})$ -norms of $J_R - I$ for $p = 1, 2, \infty$ are $\mathcal{O}(e^{-x^{3/2}t^{-1/2}\rho^{3/2}})$ uniformly in $xt \geq \delta$, $x \geq K$, $t \leq t_0$. On the remaining part of the contour we have, as $x \rightarrow +\infty$,

$$\begin{aligned} J_R(w; x, t) - I &= P^{(a)}(w; x, t)P^{(\infty)}(w; x, t)^{-1} - I \\ &= \frac{1}{x^{3/2}t^{-1/2}f(w; x, t)^{3/2}} P^{(\infty)}(w; x, t)\Phi_{\text{Ai},1}P^{(\infty)}(w; x, t)^{-1} + \mathcal{O}(x^{-3}t) \\ &= \frac{i}{48x^{3/2}t^{-1/2}f(w; x, t)^{3/2}} \begin{pmatrix} 0 & 7(w - a(x, t))^{1/2} \\ 5(w - a(x, t))^{-1/2} & 0 \end{pmatrix} + \mathcal{O}(x^{-3}t), \end{aligned} \quad (4.35)$$

uniformly for $w \in \partial\mathcal{D}$, where we use (4.25) and $\Phi_{\text{Ai},1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$, see Appendix A. Moreover this asymptotic relation is uniform for $x \geq K$, $t \leq t_0$ and $xt \geq \delta$; indeed, it is clear by (4.28) and (4.26) that $f(w; x, t)$ is bounded away from zero for $w \in \partial\mathcal{D}$, uniformly for the relevant values of x, t . $\square$

Similarly as in Section 3.7, it should be noted that the contour Γ_R depends on x and t through $a(x, t)$. This is not so convenient to apply the standard theory of small-norm RH problems; however one observes easily that the jump contour is independent of x and t after the shift $\tilde{w} = w - a(x, t)$ which is uniformly bounded by Proposition 4.1, so that the standard theory applies to the RH problem in that variable. Back in our variable w , we obtain that, as $x \rightarrow +\infty$,

$$R(w; x, t) = I + \frac{1}{x^{3/2}t^{-1/2}} R^{(1)}(w; x, t) + \mathcal{O}(x^{-3}t), \quad R^{(1)}(w; x, t) = \frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{J_R^{(1)}(s; x, t)}{s - w} ds \quad (4.36)$$

uniformly in w , where (see (4.35))

$$J_R^{(1)}(w; x, t) := \frac{i}{48f(w; x, t)^{3/2}} \begin{pmatrix} 0 & 7(w - a(x, t))^{1/2} \\ 5(w - a(x, t))^{-1/2} & 0 \end{pmatrix}. \quad (4.37)$$

It is straightforward to check, by using the definition of $f(w; x, t)$ in (4.26), that $J_R^{(1)}(w; x, t)$ is meromorphic for $w \in \mathcal{D}$ with a double pole at $w = a(x, t)$ only. Thus we can compute $R^{(1)}$ in (4.36) for $w \in \mathbb{C} \setminus \overline{\mathcal{D}}$ by extracting the polar part of $J_R^{(1)}$ at $w = a(x, t)$; namely, if $|w - a(x, t)| > \rho$,

$$R^{(1)}(w; x, t) = \frac{i}{16f_1^{5/2}} \begin{pmatrix} 0 & \frac{7}{3}f_1 \\ -\frac{5}{2}f_2 & 0 \end{pmatrix} \frac{1}{w - a(x, t)} + \frac{5i}{48f_1^{3/2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{(w - a(x, t))^2}, \quad (4.38)$$

where $f_1 = f_1(x, t)$ and $f_2 = f_2(x, t)$ are given in (4.29). Since (4.36) is uniform in w we also obtain

$$R_1(x, t) = \frac{it^{1/2}}{16x^{3/2}f_1(x, t)^{5/2}} \begin{pmatrix} 0 & \frac{7}{3}f_1(x, t) \\ -\frac{5}{2}f_2(x, t) & 0 \end{pmatrix} + \mathcal{O}(x^{-3}t). \quad (4.39)$$

4.8 Asymptotics for $u_\sigma(x, t)$ and $\partial_x \log Q_\sigma(x, t)$

Proposition 4.6. *As $x \rightarrow +\infty$ we have, uniformly in $xt \geq \delta$, $t \leq t_0$ for any $\delta > 0$ and $t_0 > 0$,*

$$u_\sigma(x, t) = \frac{x}{2t}a(x, t) + \mathcal{O}(x^{-2}). \quad (4.40)$$

where $a(x, t)$ is the unique solution to (4.3). In particular we have

$$u_\sigma(x, t) = \frac{x}{2t}a_0 \left(\frac{\pi^2}{c_+^2}xt \right) + \frac{1}{2\sqrt{xt}}a_1 \left(\frac{\pi^2}{c_+^2}xt \right) + \frac{t^{1/2}}{2x^{3/2}}a_2 \left(\frac{\pi^2}{c_+^2}xt \right) + \mathcal{O}(x^{-2}) \quad (4.41)$$

as $x \rightarrow +\infty$, uniformly in $xt \geq \delta$, $t \leq t_0$, where a_i are as in (4.4) and in Theorem 1.3.

Proof. By combining (1.33), and (4.19) we obtain the identity

$$u_\sigma(x, t) = \frac{x}{t}(T_{1,21}(x, t)^2 - 2T_{1,11}(x, t)).$$

For w satisfying $\operatorname{Re} w > a(x, t) + \rho$, using (4.21) and (4.30) we have

$$T(\zeta; x, t) = S(w; x, t) = R(w; x, t)P^{(\infty)}(w; x, t).$$

Substituting the large w asymptotics (4.18) for T , (4.34) for R , and (4.23) for $P^{(\infty)}$, we get

$$T_1(x, t) = P_1^{(\infty)}(x, t) + R_1(x, t). \quad (4.42)$$

The proof is complete by (4.25) and (4.39). $\square$

Proposition 4.7. *As $x \rightarrow +\infty$ we have, uniformly in $xt \geq \delta$, $t \leq t_0$ for any $\delta > 0$ and $t_0 > 0$,*

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{t} \left(\frac{1}{4} + g_1(x, t) \right) - \frac{5}{32x} \frac{f_2(x, t)}{f_1(x, t)^{5/2}} + \mathcal{O}(x^{-5/2}t^{1/2}), \quad (4.43)$$

where $g_1(x, t)$ is given in (4.16) and $f_1(x, t), f_2(x, t)$ in (4.29).

Proof. By combining (1.32) and (4.19) we obtain the identity

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{4t} - \frac{x^2}{t}g_1(x, t) - i\sqrt{\frac{x}{t}}T_{1,21}(x, t). \quad (4.44)$$

By (4.42) we get

$$\partial_x \log Q_\sigma(x, t) = -\frac{x^2}{4t} - \frac{x^2}{t}g_1(x, t) - i\sqrt{\frac{x}{t}}P_{1,21}^{(\infty)}(x, t) - i\sqrt{\frac{x}{t}}R_{1,21}(x, t), \quad (4.45)$$

and it suffices to use (4.25) and (4.39) to obtain the result. $\square$

We now study the asymptotics of the two terms on the right-hand side of (4.43); this is done in the next two lemmas.

Lemma 4.8. *As $x \rightarrow +\infty$ we have, uniformly in $xt \geq \delta$ and $t \leq t_0$,*

$$\begin{aligned} \frac{x^2}{t} \left(\frac{1}{4} + g_1(x, t) \right) &= (a_0(y) - 1)^2 \frac{x^2}{4t} + \frac{2c_+}{3\pi} a_0(y)^{3/2} \frac{x^{3/2}}{t^{3/2}} + \left(\frac{(a_0(y) - 1)a_1(y)}{2} + \frac{\log c'_+}{\pi} \sqrt{a_0(y)} \right) \frac{x^{1/2}}{t^{1/2}} \\ &\quad + \frac{c_+}{\pi} a_1(y) \sqrt{a_0(y)} \frac{1}{t} + \left(\frac{(a_0(y) - 1)a_2(y)}{2} - \frac{j_\sigma}{\sqrt{a_0(y)}} \right) \frac{t^{1/2}}{x^{1/2}} \\ &\quad + \left(\frac{a_1(y)^2}{4} + \frac{c_+}{\pi} a_2(y) \sqrt{a_0(y)} + \frac{\log c'_+}{2\pi \sqrt{a_0(y)}} a_1(y) \right) \frac{1}{x} \\ &\quad + \frac{c_+}{4\pi} \frac{a_1(y)^2}{\sqrt{a_0(y)}} \frac{1}{t^{1/2} x^{3/2}} + \mathcal{O}(t^{3/2} x^{-3/2}) \end{aligned} \quad (4.46)$$

where $y = \pi^2 xt / c_+^2$, $a_0(y), a_1(y), a_2(y)$ are as in (4.4) and in Theorem 1.3, and j_σ is as in (3.22).

Proof. We start by rewriting the expression (4.16) as

$$\begin{aligned} g_1(x, t) &= \frac{a(x, t)^2}{4} - \frac{a(x, t)}{2} + \frac{1}{\pi \sqrt{xt}} \int_{-\infty}^{a(x, t)} (\log F)' \left(\frac{x}{t} \zeta \right) \sqrt{a(x, t) - \zeta} d\zeta \\ &= \frac{a(x, t)^2}{4} - \frac{a(x, t)}{2} + \frac{1}{2\pi} \sqrt{\frac{t}{x^3}} \int_{-\infty}^{a(x, t)} \frac{\log F \left(\frac{x}{t} \zeta \right)}{\sqrt{a(x, t) - \zeta}} d\zeta \\ &= \frac{a(x, t)^2}{4} - \frac{a(x, t)}{2} + \frac{2c_+}{3\pi \sqrt{xt}} a(x, t)^{3/2} + \frac{\log c'_+}{\pi} \sqrt{\frac{t}{x^3}} a(x, t) \\ &\quad + \frac{1}{2\pi} \frac{t^{3/2}}{x^{5/2}} \int_{-\infty}^{xa(x, t)/t} \frac{\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)}{\sqrt{a(x, t) - \frac{t}{x} r}} dr \end{aligned} \quad (4.47)$$

where in the first step we integrate by parts, and in the last step we also change integration variable $r = \frac{x}{t} \zeta$. Similar arguments to those used previously in the proof of Proposition 4.1 allow to estimate the last integral as

$$\frac{1}{2\pi} \int_{-\infty}^{xa(x, t)/t} \frac{\log F(r) - (c_+ r + \log c'_+) 1_{(0, +\infty)}(r)}{\sqrt{a(x, t) - \frac{t}{x} r}} dr = -\frac{j_\sigma}{\sqrt{a(x, t)}} + \mathcal{O}(t/x). \quad (4.48)$$

It then suffices to apply (4.4). □

Lemma 4.9. *We have the following asymptotics as $x \rightarrow +\infty$, uniformly for $xt \geq \delta$ and $t \leq t_0$:*

$$-\frac{5f_2(x, t)}{32f_1(x, t)^{5/2}} = \frac{1 + \sqrt{1+y}}{48(1+y)} + \mathcal{O}(t^{1/2} x^{-3/2}), \quad (4.49)$$

where $y = \pi^2 xt / c_+^2$.

Proof. For any $a > 0$ we have

$$\int_{-\infty}^a (\log F)'' \left(\frac{x}{t} \zeta \right) \frac{d\zeta}{\sqrt{a - \zeta}} = \frac{t}{x} \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} (\log F)''(r) dr + \mathcal{O}(t^2/x^2)$$

as $x \rightarrow +\infty$, where we use similar estimates as in the proof of Proposition 4.1. Moreover

$$\int_{-\infty}^{+\infty} (\log F)''(r) dr = \lim_{r_1, r_2 \rightarrow +\infty} [(\log F)'(r_1) - (\log F)'(-r_2)] = c_+.$$

Similarly, for any $a > 0$ we have

$$\int_{-\infty}^a (\log F)'''(\frac{x}{t}\zeta) \frac{d\zeta}{\sqrt{a-\zeta}} = \frac{1}{2a^{3/2}} \frac{t^2}{x^2} \int_{-\infty}^{+\infty} r (\log F)'''(r) dr + \mathcal{O}(t^3/x^3)$$

as $x \rightarrow +\infty$. Moreover we can integrate by parts to obtain

$$\int_{-\infty}^{+\infty} r (\log F)'''(r) dr = - \int_{-\infty}^{+\infty} (\log F)''(r) dr = -c_+.$$

The proof is complete by the explicit formulae (4.29) for f_1, f_2 and the expansion (4.4) for $a(x, t)$. $\square$

Corollary 4.10. *We have*

$$\partial_x \log Q_\sigma(x, t) = \partial_x \left[-\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + F_3 \left(\frac{\pi^2}{c_+^2} xt \right) \right] + \mathcal{O}(t^{3/2} x^{-3/2}),$$

where F_1, F_2, F_3 are defined in Theorem 1.3.

Proof. It is a direct verification based on (4.43) and the last two lemmas. $\square$

4.9 Proof of Theorem 1.3

The asymptotics (1.25) for u_σ follow from Proposition 3.8 if $0 < xt \leq \delta$. Indeed, in this regime, it is easy to see that the term $\frac{t^{1/2}}{2x^{3/2}} a_2 \left(\frac{\pi^2}{c_+^2} xt \right)$ is $\mathcal{O}(x^{-2})$, since $a_2(y) = \mathcal{O}(y^{-1/2})$ as $y \rightarrow 0$ by (1.28). If $xt \geq \delta$, (1.25) follows immediately from Proposition 4.6.

For Q_σ , we first observe that Corollary 3.10 and Corollary 4.10 together imply that

$$\partial_x \log Q_\sigma(x, t) = \partial_x \left[-\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + F_3 \left(\frac{\pi^2}{c_+^2} xt \right) \right] + \mathcal{O}(t^{3/2} x^{-3/2}) + \mathcal{O}(x^{-3}),$$

uniformly for $x \geq K, t \leq t_0$. Let us now integrate this expression in x , starting at the point $x = K$. Since the error terms are integrable in x between $x = K$ and $x = \infty$, we then obtain

$$\begin{aligned} \log Q_\sigma(x, t) &= \log Q_\sigma(K, t) + \frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} Kt \right) + \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} Kt \right) - F_3 \left(\frac{\pi^2}{c_+^2} Kt \right) \\ &\quad - \frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} xt \right) - \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} xt \right) + F_3 \left(\frac{\pi^2}{c_+^2} xt \right) + \mathcal{O}(1). \end{aligned} \quad (4.50)$$

Then, we distinguish two cases. First, if t does not tend to 0, then $\log Q_\sigma(K, t)$ remains bounded as well as the other terms that do not depend on x , and the result is proven. Secondly, if $t \rightarrow 0$, we know from [7, Theorem 1.14 (iii)], see (1.20), that

$$\log Q_\sigma(K, t) = -\frac{K^3}{12t} - \frac{1}{8} \log(Kt^{-1/3}) + \mathcal{O}(1) = -\frac{K^3}{12t} + \frac{1}{24} \log t + \mathcal{O}(1), \quad \text{as } t \rightarrow 0.$$

Furthermore, it is straightforward to see from (1.29)–(1.30), particular from the asymptotics as $y \rightarrow 0$, that

$$\frac{c_+^6}{\pi^6 t^4} F_1 \left(\frac{\pi^2}{c_+^2} Kt \right) + \frac{c_+^3 \log c'_+}{\pi^4 t^2} F_2 \left(\frac{\pi^2}{c_+^2} Kt \right) - F_3 \left(\frac{\pi^2}{c_+^2} Kt \right) = \frac{K^3}{12t} - \left(\frac{2c_+ j_\sigma}{\pi} + \frac{\log^2 c'_+}{2\pi^2} + \frac{1}{24} \right) \log t + \mathcal{O}(1),$$

and that yields the result also in this case.

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A Airy model RH problem

Introduce the following 2×2 -matrix valued entire functions:

$$\begin{aligned}\Phi_{\text{Ai}}^{[1]}(z) &:= \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-i\pi\sigma_3/6}, \\ \Phi_{\text{Ai}}^{[2]}(z) &:= \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} -\omega \text{Ai}(\omega z) & \text{Ai}(\omega^2 z) \\ -\omega^2 \text{Ai}'(\omega z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-i\pi\sigma_3/6}, \\ \Phi_{\text{Ai}}^{[3]}(z) &:= \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} -\omega^2 \text{Ai}(\omega^2 z) & -\omega^2 \text{Ai}(\omega z) \\ -\omega \text{Ai}'(\omega^2 z) & -\text{Ai}'(\omega z) \end{pmatrix} e^{-i\pi\sigma_3/6}, \\ \Phi_{\text{Ai}}^{[4]}(z) &:= \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \text{Ai}(z) & -\omega^2 \text{Ai}(\omega z) \\ \text{Ai}'(z) & -\text{Ai}'(\omega z) \end{pmatrix} e^{-i\pi\sigma_3/6},\end{aligned}$$

where Ai is the Airy function and $\omega := e^{2\pi i/3}$. Let the contour Σ_{Ai} be the union of four smooth arcs $\gamma_1, \dots, \gamma_4$ meeting at the origin only, each extending to infinity so that $\mathbb{C} \setminus \Sigma_{\text{Ai}}$ is the disjoint union of four domains $\Omega_1, \dots, \Omega_4$ (such that the Ω_j is bounded by the arcs γ_j, γ_{j+1}) satisfying

$$\begin{aligned}z \in \Omega_1 &\Rightarrow -\varepsilon < \arg z < \frac{\pi}{2} + \varepsilon, & z \in \Omega_2 &\Rightarrow \frac{\pi}{2} - \varepsilon < \arg z < \pi + \varepsilon, \\ z \in \Omega_3 &\Rightarrow -\pi - \varepsilon < \arg z < -\frac{\pi}{2} + \varepsilon, & z \in \Omega_4 &\Rightarrow -\frac{\pi}{2} - \varepsilon < \arg z < \varepsilon,\end{aligned}$$

for some $\varepsilon > 0$ sufficiently small ($\varepsilon < \pi/6$ is enough). It can be shown by the connection formula $\text{Ai}(z) + \omega \text{Ai}(\omega z) + \omega^2 \text{Ai}(\omega^2 z) = 0$ and standard asymptotic properties of the Airy function that the piecewise defined matrix

$$\Phi_{\text{Ai}}(z) := \Phi_{\text{Ai}}^{[j]}(z), \quad z \in \Omega_j, \quad (\text{A.1})$$

satisfies the following RH model problem [14]. For the formulation of the latter we agree that $\gamma_2, \gamma_3, \gamma_4$ are oriented towards the origin and γ_1 towards infinity, and we denote by the subscripts $+$ and $-$ the boundary values from, respectively, the left and right sides of Σ_{Ai} .

RH problem for Φ_{Ai}

- (a) $\Phi_{\text{Ai}} : \mathbb{C} \setminus \Sigma_{\text{Ai}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic
- (b) Φ_{Ai} has the jump relations

$$\begin{aligned}\Phi_{\text{Ai},+}(z) &= \Phi_{\text{Ai},-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \gamma_1, \\ \Phi_{\text{Ai},+}(z) &= \Phi_{\text{Ai},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_2 \cup \gamma_3, \\ \Phi_{\text{Ai},+}(z) &= \Phi_{\text{Ai},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \gamma_4.\end{aligned}$$

(c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{Ai}}$, we have

$$\Phi_{\text{Ai}}(z) = z^{-\frac{\sigma_3}{4}} A \left(I + \frac{1}{z^{3/2}} \Phi_{\text{Ai},1} + \mathcal{O}(z^{-3}) \right) e^{-\frac{2}{3} z^{3/2} \sigma_3}, \quad (\text{A.2})$$

where A is given in (1.34) and $\Phi_{\text{Ai},1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$.

(d) As $z \rightarrow 0$, $\Phi_{\text{Ai}}(z) = \mathcal{O}(1)$.

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