Aspects of Regular and Exact Completions

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Dedication

Στη μνήμη της γιαγιάς μου, Αγγελικής

In loving memory of:

My godmother, Marianne Spada

My grandparents, John & Anna Spada

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Abstract

Over the last 30 years, the constructions of regular and exact completions of weakly lex categories and of the exact completion of a regular category have been extensively studied in the context of Category Theory and its applications. In particular, many results have been obtained concerning characterizations of when these completions satisfy various properties of interest, leading often to characterizations of certain classes of regular and exact categories, such as varieties of Universal Algebra. In the first part of this thesis, we consider when the regular and exact completion of a weakly lex category satisfy a property called 2-star-permutability, which subsumes at the same time two important notions: that of Mal'tsev and that of subtractive category. The main result here is a characterization of those categories which occur as projective covers of such 2-star-permutable regular categories. Next, we turn our attention to the exact completion of a regular category. First, we present a characterization of this completion and some associated applications, in particular in Categorical Logic. Subsequently, most of the work in this second part of the thesis shifts to the context of categories enriched over the cartesian closed category of partially ordered sets (posets) and monotone functions. We provide a construction of the completion in this setting by employing a type of enriched calculus of internal relations and prove that the expected universal property is satisfied. Furthermore, we obtain a result which characterizes the universal functor from a regular poset-enriched category into its completion, much as in the ordinary case, and use it to deduce interesting examples, both in the realm of ordered Universal Algebra and of ordered topological spaces. In particular, we show that the exact completion in this enriched sense of the category of Priestley spaces is the category of compact ordered spaces introduced by L. Nachbin.

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Introduction

The notion of exact category was first introduced by M. Barr in 1971 in his joint monograph [10] with P. A. Grillet and D. van Osdol and, together with the weaker notion of regularity, has since occupied a central position within Category Theory and its applications to Algebra and Geometry. To a large extent, the aforementioned authors were guided by the desire to develop a non-abelian categorical algebra. Indeed, as Barr explains in his introduction to [10], "exact categories are, roughly speaking, categories which satisfy the equation

$$\text{(Abelian)} \quad = \quad \text{(Exact)} \quad + \quad \text{(Additive)."}$$

The naive above equation is a pictorial representation of a theorem of M. Tierney which, in the modern terminology, identified abelian categories as precisely those categories which are both exact and additive. However, one should note that exactness, hence also the weaker regularity, was present as well in the other realm of early success of Category Theory, namely the theory of Toposes. Indeed, every elementary topos is exact in the sense of Barr.

In 1982 A. Carboni and R. Celia-Magno [27], following a suggestion of A. Joyal, showed that there is a universal construction which produces an exact category $\mathcal{C}_{ex}$ from any given finitely complete category $\mathcal{C}$. The category $\mathcal{C}_{ex}$ is called the *exact completion* of $\mathcal{C}$. A few yeas later, Carboni together with E. M. Vitale [33] greatly extended this construction and at the same time showed that there is also a *regular completion*. More precisely, given any category $\mathcal{C}$ with *weak* finite limits (called *weakly lex*), there are categories $\mathcal{C}_{reg}$ and $\mathcal{C}_{ex}$ which are respectively regular and exact and satisfy a suitable universal property. The latter is expressed via so-called *left covering* functors and its details along with many applications were the subject of Vitale's thesis [98].

The regular and exact completions of a weakly lex category are indeed of great importance and have many applications. Already in the left exact context, Carboni & Celia-Magno [27] showed how the exact completion can be used to give a two-step construction of the free abelian category on an additive one. Subsequently, the exact completion of a weakly lex category was employed by Vitale to obtain a new proof of the characterization of monadic categories over Set. Furthermore, Vitale obtained characterization results for categories that occur as localizations or epireflections of monadic categories [99–101].

An important aspect of the regular and exact completions of a weakly lex category is its connection with projective objects. As Carboni & Vitale showed,

every weakly lex category $\mathcal{C}$ sits inside both $\mathcal{C}_{reg}$ and $\mathcal{C}_{ex}$ as a *projective cover*. The latter means that the objects of $\mathcal{C}$ are regular projective inside both completions and that they are sufficient to obtain projective presentations of every object in these completions. In particular, regular and exact completions always have *enough projectives*. In fact, the authors furthermore showed that an exact category which has enough projectives can be realized as the exact completion of any one of its projective covers. Thus, as a primary example, we have that any variety of universal algebras is the exact completion of its full subcategory of projectives.

This connection with projective objects just described has spurred great interest in the possibility of characterizing various interesting classes of regular or exact categories via simpler properties of their projective covers. In the varietal context, these properties can very often be related back to certain syntactic conditions in the corresponding algebraic theory. In this vein, various authors have obtained characterizations of the projective covers of regular categories which are: extensive [52, 67], Mal'tsev [90], subtractive and (strongly) unital [49], protomodular and semi-abelian [43], Goursat [88]. In this thesis we add to this list the context of *2-star-permutable* categories, which generalizes at the same time the Mal'tsev and subtractive cases. Our results recover in particular the syntactic characterization of the so-called E-subtractive varieties in the sense of A. Ursini [97].

On the other hand, somewhat complementary to the above completions, there is another completion construction which connects regularity and exactness and which is actually historically older. This is the *exact completion of a regular category* or *ex/reg completion*, which produces an exact category $\mathcal{C}_{ex/reg}$ out of any regular category $\mathcal{C}$ and satisfies the expected universal property: any regular functor $\mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ an exact category extends uniquely to a regular functor $\mathcal{C}_{ex/reg} \to \mathcal{E}$. The construction of the ex/reg completion was first briefly described by Lawvere in his famous Perugia lectures [69] on topos theory, while a more detailed account of it is contained in a beautiful paper by R. Succi-Cruciani [95]. Much later, an alternative construction in the case when $\mathcal{C}$ is small has been given by S. Lack [66], which exhibits $\mathcal{C}_{ex/reg}$ as a subcategory of a category of sheaves on $\mathcal{C}$.

Just as the ex/wlex and reg/wlex completions, so too the ex/reg completion is of great importance. In general, whenever one wants to realize a certain regular category $\mathcal{C}$ as a subcategory of a suitable exact category, then the natural candidate for an embedding is the completion $\mathcal{C} \hookrightarrow \mathcal{C}_{ex/reg}$. For example, this general strategy can be used to characterize categories which occur as particular types of (epi)reflections of certain classes of exact categories. A particular instance of this are the results of Carboni & Mantovani [31], which characterize those categories which occur as categories of *separated objects* inside pretoposes or abelian categories. Mantovani later [75] used the ex/reg completion in order to obtain abstract characterizations of those categories which are *semi-localizations* of exact categories or of pretoposes. Building on the latter work, Gran & Lack [48] have obtained similar results for semi-localizations of exact Mal'tsev, exact protomodular and semi-abelian categories. In particular, these

results allow for a characterization of torsion-free classes inside (semi)abelian categories. We should note that, in all these examples, a fundamental part of the argument is always to show that certain properties of $\mathcal{C}$ translate to corresponding properties of the exact completion $\mathcal{C}_{ex/reg}$.

Since the introduction of regularity and exactness, there has also been some work done on enriched versions of these notions, i.e. for $\mathcal{V}$-categories where $\mathcal{V}$ is some sufficiently nice monoidal closed category. A case which has received attention early on is when $\mathcal{V} = \mathsf{Cat}$ is the category of small categories, or in other words the context of *2-categories*, which goes back to work of R. Street [94]. More recently, J. Bourke & R. Garner [21] have considered multiple different notions of regularity and exactness for 2-categories. In fact, these authors have shown that there is a general framework which allows one to define notions of regularity and exactness for categories enriched over very general base categories $\mathcal{V}$. This is based on earlier unpublished work of Street and others, as well as on the theory of *lex colimits* of Garner & Lack [40].

Another particular instance of enrichment, one which can in fact be viewed as a simplified version of the 2-categorical context, is when $\mathcal{V} = \mathsf{Pos}$ is the cartesian closed category of partially ordered sets (or *posets*) and order-preserving functions. This context has received interest in recent years, beginning in the work of A. Kurz & J. Velebil [65] where Pos-enriched versions of regularity and exactness were introduced and studied. The central theme of the latter article is to initiate a categorical treatment of *ordered universal algebra* in the sense of Bloom & Wright [14, 15] via the study of categories enriched in Pos. Its main results are categorical characterizations of quasivarieties and varieties of ordered algebras based on the enriched notions of regularity and exactness respectively. In this thesis, we continue this line of research by providing a construction of the ex/reg completion in the Pos-enriched context, based on a suitable calculus of relations. We also present applications concerning the connections with (quasi)varieties of ordered algebras and the interaction between the exact completion and categories of internal posets in ordinary categories.

Overview of Contents

In Chapter 1, we collect those categorical notions which are central for the thesis. First, we recall some basic notions of epimorphism and then move on to recall the definitions of regular and exact category. We also recall the fundamentals of the calculus of internal relations, which is available in every regular category, together with any basic properties of said calculus which will be needed later. Finally, we recall the definitions and fundamental properties of the ex/wlex, reg/wlex and ex/reg completions.

In Chapter 2, we begin by recalling the definitions and some characterizations of Mal'tsev categories on one hand and subtractive categories on the other. We continue by recalling the context of multi-pointed categories and some fundamental notions which are afforded therein. We then introduce a notion of star-symmetry for relations and use it to obtain a new characterization of 2-

star-permutable regular multi-pointed categories. With this characterization in hand, we introduce a class of weakly lex multi-pointed categories which we prove are precisely the projective covers of regular 2-star-permutable categories. As an application, we relate this result with the syntactic conditions defining the E-subtractive varieties [97]. These results have been published in [6].

In Chapter 3, we present joint work with P. Karazeris concerning a characterization of the ex/reg completion and some applications thereof. The first two applications are in the realm of Categorical Logic, one concerning a proof of *conceptual completeness* for regular logic and another the characterization of so-called *quotient functors*. A third and final application is a commutation result between the ex/reg completion and the filtered colimit completion. With the exception of the last section, the results presented here have been published in [8]. This chapter is also partially intended as a segue into part of the next one, where we shall obtain an entirely analogous result characterizing the Pos-enriched ex/reg completion. In particular, we note that the proofs in the first section of this chapter have been reworked from [8] so that they can be directly adapted to the enriched context of Chapter 4.

In Chapter 4, we construct the exact completion of a regular category in the Pos-enriched context. We begin the chapter by recalling some basic definitions and properties concerning finite limits in enriched categories, mostly specialized to the case of base $\mathcal{V} = \mathsf{Pos}$. We then state the definition of regularity, as introduced by Kurz & Velebil, and observe that one of the four defining conditions is in fact redundant. In the third section we consider the calculus of relations for Pos-enriched categories and recall the notion of exactness. The main result of the section is the identification of the morphisms of a regular Pos-category as maps in a suitable bicategory of relations. The fourth section contains the main result and the bulk of the work in this chapter, which is a description of the exact completion of a regular Pos-category and a proof that it satisfies the required universal property. Then in Section 5 we obtain a characterization of the completion functor $\mathcal{C} \to \mathcal{C}_{ex/reg}$ which is a suitable adaptation of the corresponding result in the ordinary context, which was presented in Chapter 3. As applications, we deduce examples of Pos-categories which are exact completions of regular ones, in particular identifying the category of *compact ordered spaces* as the exact completion of *Priestley spaces*. Finally, in the last section we consider categories of internal posets in ordinary categories and their relationship with the ordinary and Pos-enriched ex/reg completions. The results of this chapter will be published in [7].

We conclude the thesis with the brief Chapter 5, where we quickly and somewhat vaguely describe potential directions for our future research.

Chapter 1

Preliminaries

In this chapter we collect various notions and results which are needed throughout the thesis. The central themes here are the concepts of regular and exact categories and some completion constructions associated with these. For most of the basic material the reader can consult the first two volumes of F. Borceux's Handbook of Categorical Algebra [16], [17].

1.1 Notions of Epimorphism

We begin with a short section on some of the notions of epimorphism that feature prominently in Category Theory.

1.1.1 Definition. A morphism $e\colon Z \to X$ in a category $\mathcal{C}$ is called an *epimorphism* if, given any two morphisms $f, g\colon X \to Y \in \mathcal{C}$, the equality $fe = ge$ implies that $f = g$.

The epimorphisms in the category Set of sets are precisely the surjective functions. In categories such as those of abelian groups AbGrp or all groups Grp they are the surjective homomorphisms. In the category of topological spaces Top the epimorphisms are the continuous surjections.

It is not however true that the epimorphisms always coincide with the appropriate surjective functions in concrete categories. A standard example of this behavior is given by the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ in the category Mon of monoids, which can indeed be seen to be an epimorphism. This might already hint at the need for a stronger concept of epimorphism, while it is also an example of a morphism which is both an epimorphism and a monomorphism but not an isomorphism.

1.1.2 Definition. An epimorphism $e\colon X \to Y$ in a category $\mathcal{C}$ is said to be an *extremal* epimorphism if, whenever there is a factorization $e = mf$ with $m\colon M \rightarrowtail Y$ a monomorphism in $\mathcal{C}$, then m must be an isomorphism.

Before the next notion of epimorphism we need to recall what it means for two morphisms in a category to be orthogonal. Given $f\colon X \to Y$ and $g\colon A \to B$ in a category $\mathcal{C}$, we say that f is *left orthogonal* to g and write $f \perp g$ if, given any commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[g]{} & B \end{array}$$

there exists a unique diagonal morphism $d\colon Y \to A$ such that $df = u$ and $gd = v$.

1.1.3 Definition. An epimorphism $e\colon X \to Y$ in a category $\mathcal{C}$ is called a *strong* epimorphism if $e \perp m$ for all monomorphisms m in $\mathcal{C}$.

1.1.4 Definition. A morphism $e\colon X \to Y$ in a category $\mathcal{C}$ is called a *regular* epimorphism if it occurs as a coequalizer, i.e. if there is a diagram

$$Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{e} Y$$

exhibiting e as the coequalizer of f, g in $\mathcal{C}$.

1.1.5 Definition. A morphism $e\colon X \to Y$ in a category $\mathcal{C}$ is called a *split* epimorphism if there exists an $s\colon Y \to X \in \mathcal{C}$ such that $es = 1_Y$.

1.1.6 Proposition. *In any category $\mathcal{C}$ the following hold:*

1. *Every split epimorphism is regular.*
2. *Every regular epimorphism is strong.*
3. *Every strong epimorphism is extremal.*

In other words, we have a hierarchy of notions of epimorphism which we can summarize in the following sequence of implications

$$\text{split} \implies \text{regular} \implies \text{strong} \implies \text{extremal} \implies \text{epimorphism}$$

Examples can be given to show that all of the above implications are not reversible in general. We note here however the following easy observation.

1.1.7 Lemma. *If $\mathcal{C}$ has pullbacks, then strong and extremal epimorphisms coincide in $\mathcal{C}$.*

Since we will mostly be working with categories that are at least finitely complete, we will not be interested in the distinction between strong and extremal epimorphisms. On the other hand, the gap between regular and strong ones will be considered in the next section. In some sense, the regular context is the setting where this distinction also vanishes. The latter is important because strong epimorphisms in general have nicer properties than regular epimorphims, such as those recorded below.

1.1.8 Proposition. *In any category $\mathcal{C}$, the following hold:*

1. *Strong epimorphisms are closed under composition.*
2. *If $g \circ f$ is a strong epimorphism, then so is g.*

1.2 Regular and Exact Categories

In this section we recall some of the basic definitions and results concerning regular categories. One way in which one can introduce the notion of regularity is the following.

1.2.1 Definition. A finitely complete category $\mathcal{C}$ is called *regular* [10] if

1. $\mathcal{C}$ has coequalizers of kernel pairs.
2. Regular epimorphisms are stable under pullback in $\mathcal{C}$.

1.2.2 Example.
1. Set is a regular category and so is any presheaf category $[\mathcal{C}, \mathsf{Set}]$.
2. Generalizing the previous example, any elementary topos $\mathcal{E}$ is in particular a regular category.
3. Every abelian category is regular. The non-abelian category $\mathsf{AbGrp}_{t.f.}$ of torsion-free abelian groups is also regular.
4. Every *quasi-variety of universal algebras* is a regular category. In fact, quasi-varieties can be characterized as precisely those categories which are regular and possess a suitably "nice" generator (see e.g. [82]).
5. The category $\mathsf{Grp}(\mathsf{Top})$ of topological groups is regular (see e.g. [19]).
6. Prototypical examples of non-regular categories are given by Top, Cat and Pos (see [17]).

A very important aspect of regularity is the fact that it implies the existence of factorizations of morphisms into an epimorphism followed by a monomorphism. In fact, regularity can be equivalently rephrased in terms of this factorization property as follows.

1.2.3 Proposition. *A finitely complete category $\mathcal{C}$ is regular if and only if*

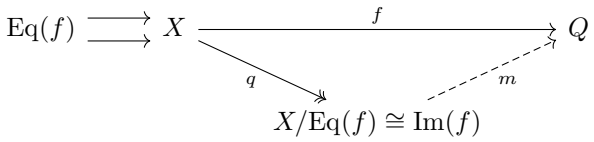
1. *Every morphism in $\mathcal{C}$ factors as a regular epimorphism followed by a monomorphism.*
2. *Regular epimorphisms are stable under pullback in $\mathcal{C}$.*

In other words, condition (1) above says that in every regular category there exists a (regular epi, mono) *factorization system*. Moreover, condition (2) says that this factorization system is *stable*.

Given any morphism $f\colon X \to Y$ in a regular category $\mathcal{C}$, its factorization $X \xrightarrow{\ p\ } I \rightarrowtail^{m} Y$ is unique up to isomorphism in the appropriate sense. Thus, I is uniquely determined as a subobject of Y. This subobject is called the *(regular) image* of the morphism f and is often denoted by $\mathrm{Im}(f)$.

1.2.4 Remark. We would like to record here how the factorization of morphisms in a regular category is obtained from the axioms of Definition 1.2.1. Given $f\colon X \to Y \in \mathcal{C}$, we can consider its kernel pair $\mathrm{Eq}(f)$ and its coequalizer $\mathrm{Eq}(f) \rightrightarrows X \xrightarrow{\ q\ } Q$. Then there is a unique $m\colon Q \to Y$ such that $mq = f$. One can then use the stability of regular epimorphisms under pullback to prove that m is in fact a monomorphism.

It is easy to see that in many familiar categories such as Set, Mon and Grp this produces the factorization of a function or homomorphism through its image in the usual sense.



Now it is an easy consequence of the existence of the factorizations in Proposition 1.2.3 and the properties of strong epimorphisms mentioned in the previous section that the following holds.

1.2.5 Corollary. *In every regular category $\mathcal{C}$ strong epimorphisms coincide with regular ones.*

In fact, it is an observation of A. Joyal that regularity can be equivalently defined with strong epimorphisms in place of regular ones, as contained in the following.

1.2.6 Proposition. *[A. Joyal] A finitely complete category $\mathcal{C}$ is regular if and only if*

1. *Every morphism in $\mathcal{C}$ factors as a strong epimorphism followed by a monomorphism.*

2. *Strong epimorphisms are stable under pullback in $\mathcal{C}$.*

We record also that the identification of regular epimorphisms with strong ones and Proposition 1.1.8 immediately imply the following properties of regular epimorphisms in a regular category.

1.2.7 Proposition. *In any regular category $\mathcal{C}$, the following hold:*

1. *Regular epimorphisms are closed under composition.*
2. *If $g \circ f$ is a regular epimorphism, then so is g.*

In the context of a regular category we also have a notion of exact sequence or "exact fork".

1.2.8 Definition. A diagram of the form $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{f} Z$ in a regular category $\mathcal{C}$ is said to be an *exact sequence* if the following two conditions hold:

1. (f_0, f_1) is the kernel pair of f.
2. f is the coequalizer of (f_0, f_1).

1.2.9 Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be regular categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ will be called a *regular functor* if it preserves finite limits and regular epimorphisms.

The stronger notion of exactness was introduced by Barr in [10]. The initial motivation comes from a result of Myles Tierney which essentially identified exactness as that which remains after removing additivity from the axioms defining an abelian category. Since the definition speaks about internal equivalence relations, we first recall some notions regarding relations on an object in a category.

A *relation* on an object X in a finitely complete category $\mathcal{C}$ is a subobject $\langle r_0, r_1\rangle\colon R \rightarrowtail X \times X$.

1.2.10 Definition. Let $\mathcal{C}$ be a finitely complete category. A relation $\langle r_0, r_1\rangle\colon R \rightarrowtail X \times X$ on an object $X \in \mathcal{C}$ is called:

- *reflexive* if there is a morphism $\delta\colon X \to R$ such that $r_0\delta = 1_X = r_1\delta$

$$\begin{array}{ccc} X & \xrightarrow{\quad\delta\quad} & R \\ & {\scriptstyle\langle 1_X, 1_X\rangle}\searrow \quad \swarrow{\scriptstyle\langle r_0, r_1\rangle} & \\ & X \times X & \end{array}$$

- *symmetric* if there exists a $\sigma\colon R \to R$ such that $r_1\sigma = r_0$ and $r_0\sigma = r_1$.

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & R \\ & {\scriptstyle\langle r_1, r_0\rangle}\searrow \quad \swarrow{\scriptstyle\langle r_0, r_1\rangle} & \\ & X \times X & \end{array}$$

- *transitive* if, given the pullback square

$$\begin{array}{ccc} R \times_X R & \xrightarrow{d_1^R} & R \\ {\scriptstyle d_0^R}\downarrow & & \downarrow{\scriptstyle r_0} \\ R & \xrightarrow[r_1]{} & X \end{array}$$

there exists a morphism $\tau\colon R \times_R R \to R$ such that $r_1\tau = r_1 d_1^R$ and $r_0\tau = r_0 d_0^R$.

- an *equivalence relation* if it is reflexive, symmetric and transitive.

Given any morphism $f\colon X \to Y \in \mathcal{C}$, the kernel pair $\mathrm{Eq}(f)$ is easily seen to be an equivalence relation on X in $\mathcal{C}$. An equivalence relation $R \rightarrowtail X \times X$ is said to be *effective* if it is the kernel pair of some morphism in $\mathcal{C}$.

1.2.11 Definition. [10] A category $\mathcal{E}$ is called *exact* if it is regular and every equivalence relation in $\mathcal{E}$ is effective.

1.2.12 Example.

1. Set and any presheaf category $[\mathcal{C}, \mathsf{Set}]$ are exact. More generally, every elementary topos is exact.
2. Every abelian category is exact. In fact, Tierney's theorem says that a category $\mathcal{A}$ is abelian if and only if it is exact and additive.
3. Every *variety* of universal algebras, such as Grp and Mon, is exact. As first proved by Lawvere in [70], varieties are precisely the (locally small) exact categories which possess a "nice" generator.
4. More generally, every category which is monadic over Set is exact. An example of this kind not already included in the above is the category CHaus of compact Hausdorff spaces and continuous maps.
5. Quasi-varieties on the other hand are generally not exact. For example, the category $\mathsf{AbGrp}_{t.f.}$ is not exact (see [17]). Similarly, the category Mon_{can} of cancellative monoids is not exact.
6. Finally, the category Stone of Stone topological spaces is regular but not exact.

1.3 Calculus of Relations in Regular Categories

The existence of the stable (regular epi, mono) factorization system in any regular category $\mathcal{C}$ implies the existence in $\mathcal{C}$ of a well-behaved *calculus of relations*. In fact, there is a sense in which regularity is the minimal context which affords this calculus.

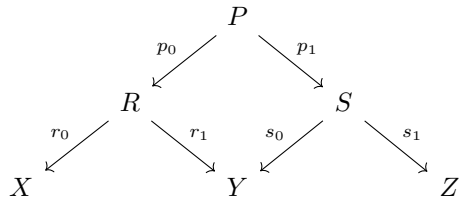
Recall that a relation R from an object X to an object Y in a category $\mathcal{C}$ is a subobject $R \rightarrowtail X \times Y$. We will indicate this situation by writing $R\colon X \nrightarrow Y$.

Now given relations $R : X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$, in a category such as $\mathcal{C} = \mathsf{Set}$ we would normally define the composition $SR\colon X \nrightarrow Z$ by the formula

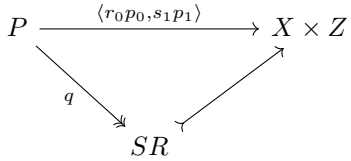
$$SR = \{(x, z) \in X \times Z | (\exists y \in Y)((x, y) \in R \wedge (y, z) \in S)\}$$

In the context of an arbitrary regular category $\mathcal{C}$ we can internalize this construction as follows:

First construct the pullback square in the center of the diagram below



Then we define the composite relation SR as the image of the morphism $\langle r_0 p_0, s_1 p_1 \rangle$ in $\mathcal{C}$.



Notice that the above definition only really used the existence in $\mathcal{C}$ of image factorizations of morphisms. The stability of regular epimorphisms under pullback is then exactly what is needed for this composition of relations to be well-behaved, in the following sense.

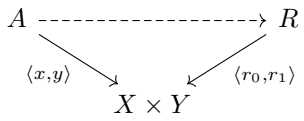
1.3.1 Lemma. *[57] Let $\mathcal{C}$ be a finitely complete category with image factorizations of morphisms. Then the composition of relations in $\mathcal{C}$ is associative if and only if $\mathcal{C}$ is regular.*

It follows that to any regular category $\mathcal{C}$ one can associate a bicategory $\mathrm{Rel}(\mathcal{C})$ whose objects are those of $\mathcal{C}$, whose morphisms are relations $R\colon X \nrightarrow Y$ in $\mathcal{C}$ and 2-cells are inclusions of relations $R \subseteq S$ as subobjects in $\mathcal{C}$. Since the 2-cell structure is simply that of a poset, $\mathrm{Rel}(\mathcal{C})$ is a type of bicategory which is often referred to as *locally posetal* (or *locally ordered*) bicategory. The identity on an object X in $\mathrm{Rel}(\mathcal{C})$ is the *diagonal* relation Δ_X represented by the monomorphism $\langle 1_X, 1_X \rangle\colon X \rightarrowtail X \times X$. We call $\mathrm{Rel}(\mathcal{C})$ the *bicategory of relations* of $\mathcal{C}$.

Before proceeding with a discussion of the structure of $\mathrm{Rel}(\mathcal{C})$, we would like to record here how one can adapt some set-theoretic arguments for use in an arbitrary regular category. On one hand, this will be of value because we shall have occasion to employ such arguments in various proofs in this thesis. On the other hand, it will also serve the reader who would like to check for themselves some of the facts that we leave without proof in this section.

Recall that, in any category $\mathcal{C}$, a morphism $x\colon A \to X$ is often called a *generalized element* of X of type A, or an A-element of X. Then, given a subobject $m\colon M \rightarrowtail X$ in $\mathcal{C}$, we say that x belongs to M and write $x \in_A M$ if x factors through m. Note also that, if $e\colon B \to A$ is a strong epimorphism, then we clearly have $x \in_A M \iff xe \in_B M$.

In the same vein, given a relation $\langle r_0, r_1 \rangle\colon R \rightarrowtail X \times Y$ and elements $x\colon A \to Y$ and $y\colon A \to Y$, we say that the pair (x, y) *belongs to* R and write $(x, y) \in_A R$ if $\langle x, y \rangle$ factors through $\langle r_0, r_1 \rangle$.



We can then argue with these generalized elements to prove that one relation is contained in another in the context of a regular category $\mathcal{C}$. Indeed, for any two relations $R, S\colon X \nrightarrow Y$, the inclusion $R \subseteq S$ is clearly equivalent to the validity of the implication $(x,y) \in_A R \implies (x,y) \in_A S$ for all generalized elements $x\colon A \to X$ and $y\colon A \to Y$. For example, associativity of the composition of relations can be proved this way. Such arguments will most often rely on the following fundamental observation, which internalizes the set-theoretic description of relational composition to any regular category.

1.3.2 Lemma. *Let $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$ be relations in the regular category $\mathcal{C}$ and consider any generalized elements $x\colon A \to X$ and $z\colon A \to Z$. Then $(x,z) \in_A SR$ if and only if there exist a regular epimorphism $e\colon B \twoheadrightarrow A$ and a generalized element $y\colon B \to Y$ such that $(xe,y) \in_B R$ and $(y,ze) \in_B S$.*

The locally posetal bicategory $\mathrm{Rel}(\mathcal{C})$ has a very rich structure. First of all, binary infima of morphisms exist in $\mathrm{Rel}(\mathcal{C})$, where for any $R, S\colon X \nrightarrow Y$ we have that $R \wedge S$ is given by the intersection of relations $R \cap S$ in $\mathcal{C}$. Second, for any relation $R\colon X \nrightarrow Y$ represented by $\langle r_0, r_1\rangle$, we can define the *opposite* relation $R^\circ\colon Y \nrightarrow X$ which is represented by $\langle r_1, r_0\rangle$. This equips $\mathrm{Rel}(\mathcal{C})$ with an order-preserving anti-involution, since trivially $(R^\circ)^\circ = R$ and $(SR)^\circ = R^\circ S^\circ$. Finally, and more importantly, $\mathrm{Rel}(\mathcal{C})$ satisfies the so-called *Freyd Modular Law*: given relations $R\colon X \nrightarrow Y$, $S\colon Y \nrightarrow Z$ and $T\colon X \nrightarrow Z$ the following inclusion holds

$$SR \cap T \subseteq S(R \cap S^\circ T) \tag{ML 1}$$

All of this can be summarized by saying that $\mathrm{Rel}(\mathcal{C})$ is an *allegory*, for every regular category $\mathcal{C}$ (see [38, 57]). Observe also that, as in any allegory, the existence of the involution $(-)^\circ$ implies that the modular law is equivalent to its dual form

$$SR \cap T \subseteq (S \cap TR^\circ)R \tag{ML 2}$$

Note that every morphism $f\colon X \to Y \in \mathcal{C}$ determines a relation $X \nrightarrow Y$ via its *graph* $\langle 1_X, f\rangle\colon X \rightarrowtail X \times Y$. We shall henceforth denote a morphism and the corresponding relation by the same symbol. The assignment $f \mapsto f$ defines a functor $\mathcal{C} \to \mathrm{Rel}(\mathcal{C})$ which is the identity on objects. Furthermore, it is easy to see that for any $f\colon X \to Y$ in the regular category $\mathcal{C}$ the following hold:

- $f^\circ f \supseteq \Delta_X$, with equality if and only if f is a monomorphism.
- $ff^\circ \subseteq \Delta_Y$, with equality if and only if f is a regular epimorphism.

The two inclusions above say that every morphism of $\mathcal{C}$ is a *map* in $\mathrm{Rel}(\mathcal{C})$, meaning that it is a morphism possessing a right adjoint. In fact, every map in $\mathrm{Rel}(\mathcal{C})$ is of this form.

1.3.3 Proposition. *(see [38, 57]) Let $\mathcal{C}$ be a regular category. If $R\colon X \nrightarrow Y$ is a map in* $\mathrm{Rel}(\mathcal{C})$*, then there exists a morphism $f\colon X \to Y \in \mathcal{C}$ such that $R = f$.*

1.3.4 Remark. We note here a couple of properties of maps that are extremely useful when arguing in the calculus of relations and which are just reformulations of the adjunction property. For any $f\colon X \to Y \in \mathcal{C}$, the adjunction $f \dashv f^\circ$ in $\mathrm{Rel}(\mathcal{C})$ implies the following: given relations $R\colon Z \nrightarrow X$ and $S\colon Z \nrightarrow Y$, we have $fR \subseteq S \iff R \subseteq f^\circ S$. Similarly, if $R\colon X \nrightarrow Z$ and $S\colon Y \nrightarrow Z$, then $Rf^\circ \subseteq S \iff R \subseteq Sf$. Finally, the adjunction also implies the triangular identities $ff^\circ f = f$ and $f^\circ ff^\circ = f^\circ$.

Thus, we can recover the morphisms of $\mathcal{C}$ from the bicategory of relations. To be a little more precise, for every regular category $\mathcal{C}$ there is an equivalence of categories $\mathcal{C} \simeq \mathrm{Map}(\mathrm{Rel}(\mathcal{C}))$. Incidentally, note that in every allegory $\mathcal{A}$ the existence of the anti-involution $(-)^\circ$ forces the order between maps to be discrete, i.e. $f \subseteq g \implies f = g$ whenever $f, g \in \mathrm{Map}(\mathcal{A})$. Hence, $\mathrm{Map}(\mathcal{A})$ is simply an ordinary category.

Furthermore, there is certain precise sense in which the maps generate all morphisms in $\mathrm{Rel}(\mathcal{C})$. Specifically, for every $R\colon X \nrightarrow Y$ there exist maps $X \xleftarrow{r_0} Z \xrightarrow{r_1} Y$ with the following two properties:

- $R = r_1 r_0^\circ$.
- $r_0^\circ r_0 \cap r_1^\circ r_1 = \Delta_Z$.

In the context of a general allegory, such a pair (r_0, r_1) of maps is called a *tabulation* of R. An allegory $\mathcal{A}$ in which all morphisms have tabulations is called *tabular* [38, 57]. In $\mathrm{Rel}(\mathcal{C})$ these two properties say precisely that $\langle r_0, r_1 \rangle\colon Z \to X \times Y$ is a monomorphism representing the relation R as a subobject of $X \times Y$. In particular, in any allegory $\mathcal{A}$, the second condition means that r_0, r_1 are jointly monomorphic in $\mathrm{Map}(\mathcal{A})$.

1.3.5 Remark. We should remark here that the notion of tabular allegory can be used to characterize bicategories of relations in regular categories. Precisely, an allegory $\mathcal{A}$ is equivalent to one of the form $\mathrm{Rel}(\mathcal{C})$ for a regular category $\mathcal{C}$ if and only if $\mathcal{A}$ is tabular and *unital*. The latter property is essentially the requirement that $\mathrm{Map}(\mathcal{A})$ have a terminal object. Thus, the assignments $\mathcal{C} \mapsto \mathrm{Rel}(\mathcal{C})$ and $\mathcal{A} \mapsto \mathrm{Map}(\mathcal{A})$ provide a 1-1 correspondence between regular categories and tabular unital allegories. For details the reader can consult [38] and [57].

Now tabulations have a certain universal property which allows one to express various limit or exactness properties in a regular category $\mathcal{C}$ in terms of relations. Similarly, for any allegory $\mathcal{A}$, one can formulate limit properties of the category $\mathrm{Map}(\mathcal{A})$ in terms of $\mathcal{A}$ itself. The universal property is contained in the following.

1.3.6 Proposition. *Suppose that* $X \xleftarrow{r_0} Z \xrightarrow{r_1} Y$ *tabulate* $R\colon X \nrightarrow Y$. *Then, for any pair of maps* $X \xleftarrow{f} W \xrightarrow{g} Y$ *, we have* $gf^\circ \subseteq R$ *if and only if there exists a (necessarily unique)* $h\colon W \to Z$ *such that* $r_0 h = f$ *and* $r_1 h = g$.

For example, the properties related to a certain square being a pullback can be formulated as follows.

1.3.7 Proposition. *Consider the following square in a regular category* $\mathcal{C}$.

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Then the square is

1. *commutative if and only if* $qp^\circ \subseteq g^\circ f$.
2. *commutative and such that the induced* $P \to X \times_Z Y$ *is a regular epimorphism if and only if* $qp^\circ = g^\circ f$.
3. *a pullback if and only if* $qp^\circ = g^\circ f$ *and* $p^\circ p \cap q^\circ q = \Delta_P$.

Finally, exactness of a regular category $\mathcal{C}$ can also be formulated in terms of $\mathrm{Rel}(\mathcal{C})$. This is then the content of the next proposition, which actually characterizes when any given equivalence relation is effective.

1.3.8 Proposition. *Let* $\mathcal{C}$ *be a regular category and consider an equivalence relation* $\langle e_0, e_1\rangle\colon E \rightarrowtail X \times X$ *on* X *in* $\mathcal{C}$. *Then* E *is effective if and only if there exist relations* $R\colon X \nrightarrow Y$ *and* $S\colon Y \nrightarrow X$ *such that* $E = SR$ *and* $RS = \Delta_Y$.

Proof. Assume that E is effective, in which case it must be the kernel pair of its coequalizer $q\colon X \twoheadrightarrow Y$. Then plainly $E = q^\circ q$ and $qq^\circ = \Delta_Y$, so we can take $R = q$ and $S = q^\circ$.

Conversely, assume that such relations R and S exist. Because E is reflexive, we have that $SR \supseteq \Delta_X$. Since also $RS \subseteq \Delta_Y$, we have that R is a map in $\mathrm{Rel}(\mathcal{C})$. It follows from Proposition 1.3.3 that there exists a morphism $q\colon X \to Y \in \mathcal{C}$ such that $R = q$ and $S = q^\circ$. Then the equality $E = SR = q^\circ q$ means that E is the kernel pair of q. □

In other words, the above proposition says that an equivalence relation E in a regular category $\mathcal{C}$ is effective precisely if it is a *split idempotent* in $\mathrm{Rel}(\mathcal{C})$. This observation is the main motivation behind one construction of the exact completion of a regular category. We shall say more on this matter in Section 1.5.

To end this section, we record some facts on how direct and inverse images of relations in a regular category can be expressed via the calculus of relation.

Recall that, given a relation $S \rightarrowtail Y \times Y$ and a morphism $f\colon X \to Y$, the *inverse image* $f^{-1}(S)$ is the relation on X which as a subobject of $X \times X$ is the pullback of S along $f \times f\colon X \times X \to Y \times Y$. On the other hand, for a relation $R \rightarrowtail X \times X$ on X, the *direct image* $f(R)$ is the relation on Y which is the image of the subobject R along $f \times f$, i.e. is defined by the existence of a commutative square

$$\begin{array}{ccc} R & \twoheadrightarrow & f(R) \\ \downarrow & & \downarrow \\ X \times X & \xrightarrow[f\times f]{} & Y \times Y \end{array}$$

with $R \twoheadrightarrow f(R)$ a regular epimorphism. We then have the following basic observation, which can be found in [28].

1.3.9 Lemma. *Let $f\colon X \to Y$ be a morphism in the regular category $\mathcal{C}$. Then for any relations $R \rightarrowtail X \times X$ and $S \rightarrowtail Y \times Y$ in $\mathcal{C}$ we have that in* $\mathrm{Rel}(\mathcal{C})$ *the following hold:*

1. $f^{-1}(S) = f^{\circ} S f$.
2. $f(R) = fRf^{\circ}$.

1.4 Projective Covers and Exact Completions of Weakly Lex Categories

In this section we recall the notions of projective object and of projective cover of a category and their relationship with the regular and exact completion of weakly lex categories.

1.4.1 Definition. An object P of a category $\mathcal{C}$ is called a (regular) *projective* if, given any regular epimorphism $e\colon X \twoheadrightarrow Y$ and any morphism $f\colon P \to Y$, there exists a $g\colon P \to X$

$$\begin{array}{ccc} P & & \\ {\scriptstyle g}\downarrow & \searrow^{f} & \\ X & \xrightarrow[e]{} & Y \end{array}$$

such that $eg = f$.

If $\mathcal{C}$ is a category with a full subcategory $\mathcal{P}$, then we say that $\mathcal{P}$ is a *projective cover* of $\mathcal{C}$ if the following two conditions hold:

- Every object of $\mathcal{P}$ is a regular projective in $\mathcal{C}$.

- For every object $X \in \mathcal{C}$ there exists a regular epimorphism $P \twoheadrightarrow X$ with $P \in \mathcal{P}$.

A regular epimorphism $P \twoheadrightarrow X$ with $P \in \mathcal{P}$ is often called a $\mathcal{P}$-cover of X. When a projective cover of $\mathcal{C}$ exists, we say that $\mathcal{C}$ has *enough projectives*.

The property of having enough projectives is of course familiar from and of great importance in Homological Algebra. Varieties $\mathbb{V}$ of universal algebras always have this property and a projective cover $\mathcal{P}$ can be obtained by taking the full subcategory on the free algebras. More generally, for any monad $\mathbb{T}$ on Set, the category of Eilenberg-Moore algebras $\mathsf{Set}^{\mathbb{T}}$ has enough projectives (assuming the Axiom of Choice), where again the free algebras are projective [99].

1.4.2 Remark. If $\mathcal{P}$ is a projective cover of a finitely complete category $\mathcal{C}$, in fact even if $\mathcal{C}$ is regular, then $\mathcal{P}$ does not in general have finite limits. It does however have *weak* finite limits. Recall that the notion of weak limit of a diagram $D : \mathbb{I} \to \mathcal{C}$ in a category $\mathcal{C}$ is obtained by removing the uniqueness requirement from the definition of limit. Of course a weak limit of a diagram is no longer unique up to isomorphism.

It is useful also to remark on how these weak limits in the projective cover $\mathcal{P}$ are obtained from the actual limits in $\mathcal{C}$. In general, to construct a weak limit for a diagram in $\mathcal{P}$ one first considers the limit of this diagram in the ambient category $\mathcal{C}$ and then covers the latter with an object of $\mathcal{P}$. For example, if $P \overset{f}{\underset{g}{\rightrightarrows}} Q$ is a pair of arrows in $\mathcal{P}$, to construct a weak equalizer we first consider the honest equalizer $E \overset{e}{\rightarrowtail} P \overset{f}{\underset{g}{\rightrightarrows}} Q$ in $\mathcal{C}$. Then we pick a regular epimorphism $r\colon R \twoheadrightarrow E$ with $R \in \mathcal{P}$ and a weak equalizer is given by $R \xrightarrow{er} P \overset{f}{\underset{g}{\rightrightarrows}} Q$.

From a more categorical perspective, projective covers are also significant due to their intimate connection to the *regular completion* and *exact completion* of a weakly lex category.

The free exact category $\mathcal{C}_{ex}$ over a finitely complete category $\mathcal{C}$ was first described by Carboni & Celia-Magno in [27], following a suggestion of Joyal. It was later observed that in the construction of $\mathcal{C}_{ex}$ the finite limits of $\mathcal{C}$ were only ever used for the existence clause in the universal property defining them. This suggested the possibility of performing the completion construction for categories $\mathcal{C}$ which only have weak finite limits, called *weakly lex* categories, and led to the work of Carboni & Vitale [33] (see also [98]), where both the regular completion $\mathcal{C}_{reg}$ and exact completion $\mathcal{C}_{ex}$ of any weakly lex category $\mathcal{C}$ were constructed.

We shall not need the explicit description of these completions or their universal property in this thesis. The fundamental fact that we shall refer to is the following.

1.4.3 Proposition. *If $\mathcal{C}$ is a weakly lex category, then $\mathcal{C}$ is embedded as a projective cover inside both $\mathcal{C}_{reg}$ and $\mathcal{C}_{ex}$.*

For completeness, we also recall the characterizations of free regular and exact categories, since these also exhibit the close relationship to projective objects.

1.4.4 Theorem. *[33, 98] An exact category $\mathcal{E}$ is equivalent to one of the form $\mathcal{C}_{ex}$ for some weakly lex category $\mathcal{C}$ if and only if $\mathcal{E}$ has enough projectives. In the latter case, for any projective cover $\mathcal{P}$ of $\mathcal{E}$ we have $\mathcal{P}_{ex} \simeq \mathcal{E}$.*

1.4.5 Theorem. *[33, 98] A regular category $\mathcal{E}$ is equivalent to one of the form $\mathcal{C}_{reg}$ for some weakly lex category $\mathcal{C}$ if and only if it satisfies the following two properties:*

1. *$\mathcal{E}$ has enough projectives.*
2. *For every $X \in \mathcal{E}$, there exist a monomorphism $X \rightarrowtail P_1 \times ... \times P_n$ with $P_1, ..., P_n$ projective objects in $\mathcal{E}$.*

Because of all the above, any result which implies a relationship between a regular category and its projective covers contains also a result on regular and exact completions of weakly lex categories. We shall return to this point in Chapter 2.

1.4.6 Remark. The regular and exact completion for weakly lex categories are powerful tools in category theory. For example, as shown by Vitale [98–101], they can be used to obtain characterizations of categories which are either themselves monadic over Set or appear as various types of reflections of such categories.

1.5 Exact Completion of a Regular Category

In this section we recall the construction of the exact completion of a regular category.

In the previous section we recalled the constructions which take as input a weakly lex category and in a universal way complete it to a regular or an exact category. Of at least equal significance is another type of completion, which embeds a given regular $\mathcal{C}$ category into an exact one $\mathcal{C}_{ex/reg}$ in a universal manner. This is called the *exact completion of a regular category* or ex/reg completion for short. The term *effectivization* is sometimes also used.

The construction of the exact completion of a regular category was briefly summarized in Lawvere's Perugia lectures [69], although it seems to first have been published in detail by R. Succi-Cruciani in [95]. The general idea of the construction makes use of the correspondence between regular categories and certain allegories mentioned in Section 1.3. More precisely, it proceeds by first

constructing a locally posetal bicategory and showing that the latter in fact satisfies the axioms of a tabular unital allegory. This will be the allegory of relations of the desired completion and then the completion itself is naturally obtained by taking the associated category of maps. In essence, the main motivation is derived from Proposition 1.3.8.

Recall then that an equivalence relation E on an object X in a regular category $\mathcal{C}$ is a relation $E\colon X \nrightarrow X$ which is reflexive, symmetric and transitive. In particular, any such E is *idempotent* when considered as a morphism in $\mathrm{Rel}(\mathcal{C})$. The aforementioned proposition then says precisely that E is effective if and only if it is a *split* idempotent in the sense of the following definition.

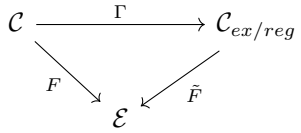
1.5.1 Definition. Let $e\colon X \to X$ be an idempotent morphism in a category $\mathcal{A}$. We say that e is *split* if there exist morphisms $f\colon X \to Y$ and $g\colon Y \to X$ in $\mathcal{A}$ such that $e = gf$ and $fg = 1_Y$.

If $\mathcal{A}$ is any category and $\mathcal{S}$ is a class of idempotent morphisms in $\mathcal{A}$, then one can construct a new category $\mathcal{A}[\check{\mathcal{S}}]$ in which all elements of $\mathcal{S}$ become split idempotents and which is universal with this property. Explicitly, $\mathcal{A}[\check{\mathcal{S}}]$ has objects (X, e) where $e\colon X \to X \in \mathcal{S}$ and morphisms $h\colon (X, e) \to (Y, f)$ are morphisms $h\colon X \to Y \in \mathcal{A}$ such that $he = h = fh$, or equivalently $h = fhe$. There is an obvious canonical functor $\Gamma\colon \mathcal{A} \to \mathcal{A}[\check{\mathcal{S}}]$ mapping $X \mapsto (X, 1_X)$

In the case where $\mathcal{A}$ is an allegory and $\mathcal{S}$ is a class of *symmetric* idempotents, then it can be shown (see e.g. [57]) that $\mathcal{A}[\check{\mathcal{S}}]$ also acquires a canonical structure of allegory. Furthermore, if $\mathcal{A}$ has a unit, then so does $\mathcal{A}[\check{\mathcal{S}}]$. Finally, if $\mathcal{S}$ is the class of *equivalences* in $\mathcal{A}$, i.e. of those morphisms which are reflexive, symmetric and transitive, then $\mathcal{A}[\check{\mathcal{S}}]$ is also tabular.

Considering the case of interest to us here, i.e. when $\mathcal{A} = \mathrm{Rel}(\mathcal{C})$ for a regular category $\mathcal{C}$, the general theory just described implies that $\mathcal{A}[\check{\mathcal{S}}]$ is a tabular allegory with a unit. Thus, we have $\mathcal{A}[\check{\mathcal{S}}] \simeq \mathrm{Rel}(\mathcal{C}_{\mathrm{ex/reg}})$ for a regular category $\mathcal{C}_{ex/reg}$. By the manner in which it has been constructed, it is immediate that $\mathcal{C}_{ex/reg}$ is in fact exact, as all equivalence relations are split idempotents in $\mathrm{Rel}(\mathcal{C}_{\mathrm{ex/reg}})$. Moreover, the universal property of the splitting of idempotents construction translates to the expected universal property of the exact completion, namely the following.

1.5.2 Theorem. *Given a regular functor $F\colon \mathcal{C} \to \mathcal{E}$ to an exact category $\mathcal{E}$, there is a unique (up to isomorphism) regular functor $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$ such that $\tilde{F} \circ \Gamma = F$.*



Proof. (Sketch.) Every object (X, E) appears in an exact sequence in $\mathcal{C}_{ex/reg}$ of the form $\Gamma E \underset{\Gamma e_1}{\overset{\Gamma e_0}{\rightrightarrows}} \Gamma X \xrightarrow{E} (X, E)$. Then $\tilde{F}(X, E)$ is defined as the

coequalizer $FE \underset{Fe_1}{\overset{Fe_0}{\rightrightarrows}} FX \xrightarrow{p_{(X,E)}} \tilde{F}(X,E)$ in $\mathcal{E}$ of the equivalence relation $FE \underset{Fe_1}{\overset{Fe_0}{\rightrightarrows}} FX$. □

While the theory of allegories streamlines the construction of the exact completion nicely, for the purposes of this thesis it will be convenient to have a more concrete handle on its description as well as on the motivation behind it. First of all, following through the two steps of splitting the idempotents which are equivalence relations and then taking maps, we see that the category $\mathcal{C}_{ex/reg}$ is defined as follows:

- Objects: Pairs (X, E) where E is an equivalence relation on the object X in $\mathcal{C}$.
- Morphisms: A morphism $R\colon (X,E) \to (Y,G)$ is a relation $R\colon X \nrightarrow Y$ in $\mathcal{C}$ such that
  1. $GRE = R$.
  2. $R^\circ R \supseteq E$.
  3. $RR^\circ \subseteq G$.

Composition is given by the composition of relations in $\mathcal{C}$ and the identity morphism on (X,E) is the relation $E\colon X \nrightarrow X$. The canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ maps an object X to the pair (X, Δ_X) and a morphism $f\colon X \to Y$ to its graph $f\colon X \nrightarrow Y$. In addition, it is easy to see that Γ is fully faithful and regular.

We would at this point like to try and provide a type of heuristic to convince the reader that they could have come up with this description of $\mathcal{C}_{ex/reg}$ without knowing anything about allegories or splitting idempotents. In Chapter 4 we aim to build an exact completion for regular categories in a certain enriched context by adapting the construction for ordinary categories described here. In that setting, however, we cannot simply fall back on the correspondence between allegories and regular categories, as now regularity will be in an enriched sense and the appropriate bicategory of relations will not be an allegory anymore. Nevertheless, a similar motivation and copious use of the calculus of relations will lead us to success.

So given a regular category $\mathcal{C}$, let us consider what might be preventing it from being exact. For one, an exact category has not only coequalizers of kernel pairs, but more generally coequalizers of all equivalence relations. Hence, one could imagine formally adding, for every equivalence relation $E \rightarrowtail X \times X$ in $\mathcal{C}$, an object (X,E) to serve as its coequalizer. In other words, we are thinking of (X,E) as X/E. But then what should the morphisms $(X,E) \to (Y,G)$ be?

On the other hand, we know that exactness is not just the existence of quotients of all equivalence relations. After all, there are many examples of categories, such as $\mathsf{Ab}_{t.f.}$, which are cocomplete but not exact. At the very

least, we would like for the new object (X, E) to appear in an exact sequence of the form $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X \twoheadrightarrow (X, E)$. Now if we engage in some wishful thinking and assume that such a sequence exists, we can actually determine what the missing morphisms should be, all in terms of the category $\mathcal{C}$ itself, as the following lemma shows us.

1.5.3 Lemma. *Consider two exact sequences* $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X \overset{p}{\twoheadrightarrow} X/E$ *and* $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} Y \overset{q}{\twoheadrightarrow} Y/G$ *in some regular category* $\mathcal{C}$*. Then there is a bijective correspondence between morphisms* $X/E \to Y/G$ *and relations* $R\colon X \nrightarrow Y$ *satisfying* $GRE = R$*,* $R^\circ R \supseteq E$ *and* $RR^\circ \subseteq G$*.*

Proof. First, consider a morphism $r\colon X/E \to Y/G$ in $\mathcal{C}$. We define the relation $R\colon X \nrightarrow Y$ by setting $R \coloneqq q^\circ rp$. Then we have $GRE = q^\circ qq^\circ rpp^\circ p = q^\circ rp = R$, $R^\circ R = p^\circ r^\circ qq^\circ rp = p^\circ r^\circ rp \supseteq p^\circ p = E$ and $RR^\circ = q^\circ rpp^\circ r^\circ q = q^\circ rr^\circ q \subseteq q^\circ q = G$.

Conversely, assume we have a relation $R\colon X \nrightarrow Y$ with the properties in the statement. We then consider the relation $r\colon X/E \nrightarrow Y/G$ in $\mathcal{C}$ defined by $r \coloneqq qRp^\circ$. We now have that $r^\circ r = pR^\circ q^\circ qRp^\circ \supseteq pR^\circ Rp^\circ \supseteq pEp^\circ = pp^\circ pp^\circ = \Delta_{X/E}$ and $rr^\circ = qRp^\circ pR^\circ q^\circ = qRER^\circ q^\circ = qRR^\circ q^\circ \subseteq qq^\circ = \Delta_{Y/G}$. Hence, the relation r is actually a map $r\colon X/E \to Y/G$ in $\mathcal{C}$.

Finally, the fact that these two assignments are inverse to each other is contained in the two equalities $qq^\circ rpp^\circ = \Delta_{Y/G} r \Delta_{X/E} = r$ and $q^\circ qRp^\circ p = GRE = R$. □

Thus, in principle, one can be led to the definition of $\mathcal{C}_{ex/reg}$ and then prove that it satisfies all required properties by relying entirely on the calculus of relations inside a regular category 1.3. This is, to some extent, what we shall do in Chapter 4 for categories enriched in the category of posets.

1.5.4 Remark. Lemma 1.5.3 is also what tells how to define the unique extension $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$ of a regular functor $F\colon \mathcal{C} \to \mathcal{E}$ into an exact category $\mathcal{E}$ on morphisms. Indeed, given a morphism $R\colon (X, E) \to (Y, G)$ in $\mathcal{C}_{ex/reg}$, we have a relation $F(R) : FX \nrightarrow FY$ in $\mathcal{E}$ which satisfies $F(G)F(R)F(E) = F(R)$, $F(R)^\circ F(R) \supseteq F(E)$ and $F(R)F(R)^\circ \subseteq F(G)$, all because F is a regular functor. Then $\tilde{F}(R)$ is the unique morphism induced between the corresponding quotients $\tilde{F}(X, E) \to \tilde{F}(Y, G)$ in $\mathcal{E}$.

Since we will need its concrete description later on, we note that by the proof of Lemma 1.5.3 we have $\tilde{F}(R) = p_{(Y,G)} F(R) p^\circ_{(X,E)}$ as relations in $\mathcal{E}$, where $FE \underset{Fe_1}{\overset{Fe_0}{\rightrightarrows}} FX \overset{p_{(X,E)}}{\twoheadrightarrow} \tilde{F}(X, E)$ and $FG \underset{Fg_1}{\overset{Fg_0}{\rightrightarrows}} FY \overset{p_{(Y,G)}}{\twoheadrightarrow} \tilde{F}(Y, G)$ are the corresponding exact sequences in $\mathcal{E}$.

We should note also that the exact completion of a regular category is "conservative", in the sense of the following proposition. This is in contrast to the ex/wlex and reg/wlex completions of Section 1.4, which are more appropriately called "free" completions.

1.5.5 Proposition. *Let $\mathcal{C}$ be a regular category. The canonical regular functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ is an equivalence if and only if $\mathcal{C}$ is exact.*

The construction of the exact completion can be promoted to a 2-functor $(-)_{ex/reg} : \mathsf{REG} \to \mathsf{EX}$, where REG and EX denote the 2-categories whose objects are respectively regular and exact categories, with 1-cells the regular functors in both instances. Indeed, for every $F\colon \mathcal{C} \to \mathcal{D} \in \mathsf{REG}$ we have a regular functor $\Gamma_{\mathcal{D}} \cdot F\colon \mathcal{C} \to \mathcal{D}_{ex/reg}$ into an exact category, hence we can define $F_{ex/reg}\colon \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$ by $F_{ex/reg} \coloneqq \widetilde{\Gamma_{\mathcal{D}} \cdot F}$. The universal property then says that $(-)_{ex/reg}$ is the left 2-adjoint to the inclusion $\mathsf{EX} \hookrightarrow \mathsf{REG}$.

We postpone the discussion of some examples of exact completions to Chapter 3. There, we will obtain a result characterizing the canonical functor $\mathcal{C} \to \mathcal{C}_{ex/reg}$ and this will allow us to deduce some interesting examples more easily.

1.5.6 Remark. In the case when the regular category $\mathcal{C}$ is small, there is another description of the ex/reg completion due to S. Lack [66]. Therein, $\mathcal{C}_{ex/reg}$ is constructed by taking the closure of the representable functors under coequalizers of equivalence relations in a suitable category of sheaves.

1.5.7 Remark. The ex/reg completion can be composed with the reg/wlex completion to yield the ex/wlex completion. In other words, given any weakly lex category $\mathcal{C}$, there is an equivalence of categories $(\mathcal{C}_{reg})_{ex/reg} \simeq \mathcal{C}_{ex}$.

Chapter 2

Projective Covers of 2-star-permutable Categories

The notion of *multi-pointed category* has in recent years been studied as a setting where certain pointed and non-pointed contexts of interest in Categorical and Universal Algebra can be treated simultaneously. It was first considered under this prism in the article [46] by M. Gran, Z. Janelidze and A. Ursini, where the context of a multi-pointed category with a *good theory of ideals* was used to unify results from the realm of *ideal determined* categories, on one hand, and *Barr-exact Goursat* categories on the other. Next, in [47], notions of permutability of equivalence relations in multi-pointed categories were introduced and used to extend certain diagrammatic characterizations known for regular *subtractive* categories and *Goursat* categories. Similarly, generalizations of homological lemmas, such as the *3 × 3 Lemma* and the *Short Five Lemma* were established in [45]. Of greater interest to us here is the notion of 2-star-permutable category, which was studied in [51] as a common extension of both regular subtractive and regular Mal'tsev categories and used to generalize characterizations of these categories via diagrams such as regular pushouts.

In the present chapter, specifically in its last section, we will add to the above list of results a characterization of the *projective covers* of regular 2-star-permutable multi-pointed categories (Theorem 2.4.6), generalizing the corresponding results for regular Mal'tsev categories, due to J. Rosický and E. Vitale [90], and regular subtractive categories, due to M. Gran and D. Rodelo [49]. On the way to accomplishing this we will first introduce a notion of *star-symmetry* for relations and use it to obtain a characterization of 2-star-permutable categories (Proposition 2.3.4). Also, as a small application, we will show how the characterization in terms of star-symmetry can be used to obtain the syntactic conditions defining the *E-subtractive* varieties of Ursini [97]. These results have been published in [6].

2.1 Mal'tsev and Subtractive Categories

2.1.1 Mal'tsev Categories

In this section we briefly recall the notions of Mal'tsev and subtractive categories, since these two comprise the main motivation for the 2-star-permutable categories which will be central in the present chapter.

The notion of a Mal'tsev variety has its roots in a 1954 paper of A. I. Mal'tsev, where the following fundamental result of Universal Algebra was proved.

2.1.1 Theorem. *[Mal'tsev [74]] For a variety $\mathbb{V}$ of universal algebras, the following are equivalent:*

1. *For any two congruences R, S on an algebra $A \in \mathbb{V}$, the equality $R \circ S = S \circ R$ holds.*
2. *There exists a ternary term $p(x, y, z)$ in the theory of $\mathbb{V}$ which satisfies the following two equalities:*
   - $p(x, x, y) = y$.
   - $p(x, y, y) = x$.

Notice how the result connects a semantical condition with a syntactical one. An operation $p(x, y, z)$ with the two properties above is called a *Mal'tsev operation* and any variety which satisfies the above equivalent conditions is called a *Mal'tsev variety*. This terminology is due to J. D. Smith [92].

Mal'tsev varieties are ubiquitous in mathematics. Indeed, many familiar varieties enjoy this property, such as groups and rings, but also Heyting algebras. The notion proved to be a very fruitful context in which to prove homological lemmas (e.g. [68]), as well as to develop various notions such as the theory of centrality and commutators for congruences [92].

The notion of Mal'tsev variety was then extended to the context of an exact category by A. Carboni, J. Lambek and M.C. Pedicchio in [30], where they showed that homological lemmas such as the Snake Lemma remain valid in this context. Later, Carboni and Pedicchio together with G. M. Kelly [28] considered the Mal'tsev condition as well as the weaker Goursat condition in the weaker context of regular categories. In these two papers we find the following result, which generalizes equivalences which were already familiar for varieties of universal algebras to arbitrary regular categories.

2.1.2 Lemma. *[28, 30, 78] In any regular category $\mathcal{C}$ the following are equivalent:*

1. *Every reflexive relation in $\mathcal{C}$ is an equivalence relation.*
2. *Every reflexive relation in $\mathcal{C}$ is symmetric.*
3. *Every reflexive relation in $\mathcal{C}$ is transitive.*

4. *If R, S are equivalence relations on an object $X \in \mathcal{C}$, then SR is an equivalence relation.*

5. *For any two equivalence relations R, S on the same object $X \in \mathcal{C}$, the equality $SR = RS$ holds (or equivalently $SR \subseteq RS$).*

6. *For any two effective equivalence relations R, S on the same object $X \in \mathcal{C}$, the equality $SR = RS$ holds.*

7. *Every relation $R : X \nrightarrow Y \in \mathcal{C}$ is* difunctional*, that is, $RR^{\circ}R = R$.*

The condition expressed in (5) above is referred to as *permutability of equivalence relations*. We also note that the first three conditions make sense in a context much weaker than a regular category, namely in a category which simply has finite limits. Condition (7) can also be reformulated in this weaker context and remains equivalent to (1)-(3). This aspect is in fact sufficiently important that the notion is now most often considered in this broader sense. We accordingly officially adopt the following definition.

2.1.3 Definition. [83] A category $\mathcal{C}$ which has finite limits is called a *Mal'tsev* category if every internal reflexive relation in $\mathcal{C}$ is an equivalence relation.

As we have already mentioned, there are many varieties which are Mal'tsev categories, such as the categories Grp of groups, Rings of unital rings and Heyt of Heyting algebras. There are however many interesting classes of non-varietal examples, including every abelian category, the category of C*-algebras and the category $\mathsf{Grp}(\mathsf{Top})$ of topological groups and the dual of any elementary topos. In addition, given any Mal'tsev algebraic theory $\mathbb{T}$, the category $\mathsf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras is a regular Mal'tsev category [19, 53].

In the decades since their introduction, Mal'tsev categories have been extensively studied in connection with notions such as centrality, the theory of commutators, central extensions, internal groupoids and categories, nilpotence, homological lemmas and others. The interested reader can consult the vast literature for more details, for example any one of [18,24,25,28,30,41,42,50,54,81,83] and their respective references.

2.1.2 Subtractive Categories

The notion of subtractive category is another abstraction stemming from Universal Algebra. It was first introduced by Z. Janelidze [55] as a pointed categorical version of the subtractive varieties of A.Ursini [96]. A variety $\mathbb{V}$ of universal algebras is said to be subtractive if there exists a constant 0 in the theory and a binary term $s(x, y)$ such that $s(x, x) = 0$ and $s(x, 0) = x$.

In order to state the categorical definition, recall [23] that a relation $\langle r_0, r_1 \rangle \colon R \rightarrowtail X \times Y$ in a pointed category $\mathcal{C}$ is said to be *left* (resp. *right*) *punctual* if $\langle 1_X, 0 \rangle \colon X \to X \times Y$ (resp. $\langle 0, 1_Y \rangle \colon Y \to X \times Y$) factors through R.

2.1.4 Definition. A pointed and finitely complete category $\mathcal{C}$ is called *subtractive* if every right punctual reflexive relation in $\mathcal{C}$ is also left punctual.

We then have the following theorem of Z. Janelidze.

2.1.5 Theorem. *Z. Janelidze [55] A variety $\mathbb{V}$ of universal algebras is a subtractive category if and only if it is a pointed (i.e. has a unique constant 0) subtractive variety in the sense of Ursini [96].*

Subtractive categories are in close relationship with various other interesting classes of categories. For example, a category is additive if and only if it is *linear* [18] (also called *half-additive* in [38]) and subtractive. Furthermore, there is a precise connection to the *unital* and *strongly unital* categories of [18]. More precisely, as shown in [55], a category is strongly unital if and only if it is unital and subtractive. This in particular says that every pointed Mal'tsev category is subtractive. There are also examples of subtractive categories which are not Mal'tsev and not even unital, such as the category of non-empty implication algebras [1].

2.1.6 Remark. Subtractive categories are also connected to Mal'tsev categories via the so-called *fibration of points* [22], which has been extensively used by D. Bourn to classify various properties of categories. Specifically, as proved by Z. Janelidze in [55], a finitely complete category $\mathcal{C}$ is Mal'tsev if and only if for every $X \in \mathcal{C}$ the fiber $\mathrm{Pt}_X(\mathcal{C})$ of the fibration of points is a subtractive category. Since we shall not really have occasion to invoke the fibration of points, we do not go into details about these matters in this thesis but simply refer the interested reader to the book of F. Borceux & D. Bourn [18].

Up to this point we have already pointed out some interconnections of subtractivity with the Mal'tsev condition. However, we would also like to present a reformulation of the notion of subtractive category which rather exhibits it as a pointed analogue of Mal'tsev categories and will serve as motivation later in this chapter. To this effect, we recall the following definition.

2.1.7 Definition. A relation $R \rightarrowtail X \times X$ in a pointed category $\mathcal{C}$ is called *left* (resp. *right*) *0-symmetric*, if for every generalized element $x\colon A \to X \in \mathcal{C}$ the implication $(0,x) \in_A R \implies (x,0) \in_A R$ (resp. $(x,0) \in_A R \implies (0,x) \in_A R$) holds in $\mathcal{C}$. If R is both left and right 0-symmetric then we simply say that it is *0-symmetric*.

We then have the following characterization of subtractive categories, which follows from the general theory of [56] and in the varietal context was already in [5]. For the sake of the reader, let us also provide a proof here.

2.1.8 Proposition. *For a pointed and finitely complete category $\mathcal{C}$ the following are equivalent:*

1. *$\mathcal{C}$ is subtractive.*
2. *Every reflexive relation $R \rightarrowtail X \times X$ in $\mathcal{C}$ is left 0-symmetric.*
3. *Every reflexive relation $R \rightarrowtail X \times X$ in $\mathcal{C}$ is 0-symmetric.*

Proof. It is clear that (2) and (3) are equivalent, since left 0-symmetry of the opposite relation R° is precisely right 0-symmetry of R.

Now assume $\mathcal{C}$ is subtractive and consider any $x\colon A \to X \in \mathcal{C}$ such that $(0,x) \in_A R$. This means that there exists a $u : A \to R$ such that $\langle r_0, r_1\rangle u = \langle 0, x\rangle$. We then form the following pullback in $\mathcal{C}$

$$\begin{array}{ccc} S & \xrightarrow{v} & R \\ {\scriptstyle \langle s_0,s_1\rangle}\downarrow & & \downarrow{\scriptstyle \langle r_0,r_1\rangle} \\ A \times A & \xrightarrow[x\times x]{} & X \times X \end{array}$$

i.e. S is the inverse image relation $x^{-1}(R)$ on A. Now we have that R being reflexive implies the same for S and $\langle r_0, r_1\rangle u = \langle 0, x\rangle = (x \times x)\langle 0, 1_A\rangle$ implies that S is right punctual. By subtractivity we have that S must also be left punctual, which means that $(1_A, 0) \in_A S$ and consequently that $(x, 0) \in_A R$.

Conversely, assume (2) holds and let $R \rightarrowtail X \times X$ be reflexive and right punctual. Then $(0, 1_A) \in_A R \implies (1_A, 0) \in_A R$ and the latter says precisely that R is left punctual. □

Note how this characterizes subtractive categories in terms of a condition which is a pointed version of condition (2) in Lemma 2.1.2 for Mal'tsev categories. In the regular context, we can also add a corresponding version of condition (5) of Lemma 2.1.2, expressing now a certain restricted permutability property for equivalence relations which is called *0-permutability*.

To state it more compactly let us at the same time introduce some notation and recall terminology. Given a relation $\langle r_0, r_1\rangle\colon R \rightarrowtail X \times Y$ in a pointed category $\mathcal{C}$, the *0-class* of R is the subobject $R[0] \subseteq Y$ defined by $R[0] \coloneqq r_1(r_0^{-1}(0))$. In other words, for any generalized element $y : A \to Y \in \mathcal{C}$, we have $y \in_A R[0] \iff (0, y) \in_A R$. Note that the object $R[0]$ is also defined via the pullback

$$\begin{array}{ccc} R[0] & \longrightarrow & R \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\langle 0,1_Y\rangle]{} & X \times Y \end{array}$$

Observe also that 0-symmetry of any $R \rightarrowtail X \times X$ is precisely the equality $R^\circ[0] = R[0]$.

2.1.9 Proposition. *For a pointed regular category $\mathcal{C}$, the following are equivalent:*

1. *$\mathcal{C}$ is subtractive.*
2. *For every two equivalence relations R, S on any $X \in \mathcal{C}$, we have the equality $SR[0] = RS[0]$.*
3. *For every two effective equivalence relations R, S on any $X \in \mathcal{C}$, we have the equality $SR[0] = RS[0]$.*

Proof. To see that (1) implies (2), note that for any two equivalence relations R, S on an object X the relation $E \coloneqq RS$ is reflexive. Then by the previous proposition we have that it must be 0-symmetric and the latter is precisely the desired equality.

Finally, assuming (3) holds, let us consider any given reflexive relation $\langle e_0, e_1\rangle \colon E \rightarrowtail X \times X$, with diagonal $\delta \colon X \to E$, and assume $x \colon A \to X$ is such that $(0, x) \in_A E$, so that $\langle e_0, e_1\rangle u = \langle 0, x\rangle$ for some $u \colon A \to E$. Let $R \coloneqq \mathrm{Eq}(e_0)$, $S \coloneqq \mathrm{Eq}(e_1)$ be the corresponding kernel pairs, which are relations on E. Then we have that $(0, u) \in_A R$ and $(u, \delta x) \in_A S$, so that $(0, \delta x) \in_A SR$. By our assumption this yields $(0, \delta x) \in_A RS$, which by Lemma 1.3.2 means that there exists a regular epimorphism $q \colon B \twoheadrightarrow A$ and a $v \colon B \to E$ such that $(0, v) \in_B S$ and $(v, \delta xq) \in_B R$. From these two equalities we respectively obtain $e_1 v = 0$, $e_0 v = e_0 \delta xq = xq$. Thus, $\langle xq, 0\rangle = \langle e_0 0, e_1\rangle v$, which says that $(xq, 0) \in_B E$, whence we have $(x, 0) \in_A E$ because q is a strong epimorphism. Hence, E is left 0-symmetric and so we conclude by the previous proposition. □

2.1.10 Remark. We remark that subtractivity can also be characterized via suitable pointed versions of conditions (3) and (7) of Lemma 2.1.2. Since we will not have any use for these we do not go into further details but simply refer to the literature, [5] for the varietal case and [56] for the pointed categorical one.

2.2 Multi-pointed Categories

The context of a category equipped with an ideal of morphisms had been used extensively by M. Grandis in his categorical foundation of homological and homotopical algebra. Much later, it was introduced into categorical algebra by M. Gran, Z. Janelidze and A. Ursini in [46] as a way of unifying and extending various notions and results which have both a pointed and non-pointed incarnation. It is in this second aspect that these multi-pointed categories are of interest to us in this thesis, so let us begin by recalling their definition.

2.2.1 Definition. A *multi-pointed category* is a pair $(\mathcal{C}, \mathcal{N})$ consisting of a category $\mathcal{C}$ and a distinguished class $\mathcal{N}$ of morphisms in $\mathcal{C}$ which is an *ideal*, i.e. such that whenever $f \in \mathcal{N}$ or $g \in \mathcal{N}$ we also have $gf \in \mathcal{N}$.

The elements of $\mathcal{N}$ are usually referred to as *null* morphisms. We will also by abuse say that $\mathcal{C}$ is a multi-pointed category and suppress the ideal $\mathcal{N}$ if there is no possibility of confusion.

The main motivating examples of multi-pointed categories are the following:

- A trivial first example of multi-pointed category is obtained by considering any category $\mathcal{C}$ and taking $\mathcal{N}$ to be the collection of all morphisms in $\mathcal{C}$. This class of examples is known as the *total context*.
- A second example of importance arises when $\mathcal{C}$ is pointed and $\mathcal{N}$ is defined as the collection of zero morphisms, i.e. the morphisms that factor through the zero object. This general class of examples is referred to as the *pointed context*.

- In the regular context, the previous example can in fact be seen as a special case of a more general class of multi-pointed categories, the so called *proto-pointed context* introduced in [46]. This refers to a regular category $\mathcal{C}$ in which every object has a smallest subobject and where $\mathcal{N}$ is defined as the collection of those $f : X \to Y$ whose image is the smallest subobject of Y. In the case of a variety $\mathbb{V}$ of universal algebras these morphisms are exactly those whose image is the subalgebra E_Y of Y generated by the constants. This latter situation has been called the *algebraic proto-pointed context* in [47] and is actually the motivation for the term "multi-pointed". Indeed, a proto-pointed category is the category-theoretic notion that corresponds to varieties with potentially more than one constant, such as unital rings and Heyting algebras, just as that of pointed category corresponds to varieties possessing a unique constant.

In the context of multi-pointed categories there is a notion of kernel with respect to the ideal $\mathcal{N}$ which is of central importance.

2.2.2 Definition. Let $f\colon X \to Y$ be a morphism in the multi-pointed category $\mathcal{C}$. An $\mathcal{N}$-*kernel* of f is a morphism $k\colon K \to X$ such that $fk \in \mathcal{N}$ and which is universal with this property, i.e. whenever $fg \in \mathcal{N}$ there is a unique morphism u such that $ku = g$.

It is easy to see that the $\mathcal{N}$-kernel k of any morphism $f\colon X \to Y$ is unique up to isomorphism and is necessarily a monomorphism. Furthermore, it is clear that in the total context the $\mathcal{N}$-kernels are always isomorphisms, while in the pointed context we obtain the usual notion of kernel.

If $\mathcal{N}$-kernels exist for all morphisms in $\mathcal{C}$, then we shall say that $\mathcal{C}$ is a *multi-pointed category with kernels*.

In the more general proto-pointed setting the kernel of f will be the inverse image of the smallest subobject of Y, which in the algebraic case becomes precisely the subalgebra of X consisting of those elements that map to the subalgebra generated by the constants in Y. For example, if $f\colon X \to Y$ is a morphism in the proto-pointed category Heyt of Heyting algebras, then the $\mathcal{N}$-kernel of f is the subalgebra $\{x \in X | f(x) = 0 \vee f(x) = 1\}$. Similarly, if f lives in the category Ring of unitary rings, then its kernel in the above sense is $\{x \in X | (\exists n \in \mathbb{Z}) f(x) = n \cdot 1\}$. Note how the latter is indeed a subring of X and hence defines a subobject in the category Ring, whereas the ordinary kernel does not.

We record here an observation on the stability of $\mathcal{N}$-kernels under pullback, as we will need it later.

2.2.3 Lemma. *Consider the following pullback square in a multi-pointed category* $(\mathcal{C}, \mathcal{N})$.

$$\begin{array}{ccc} K' & \xrightarrow{k'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow[k]{} & Y \end{array}$$

If k is the $\mathcal{N}$-kernel of some $f\colon Y \to Z$, then k' is the $\mathcal{N}$-kernel of $fg\colon X \to Z$.

Proof. Let $h\colon A \to X$ be such that $fgh \in \mathcal{N}$. Then, since k is the $\mathcal{N}$-kernel of f, there exists a unique $u\colon A \to K$ such that $ku = gh$. Now the universal property of the pullback gives a unique $v\colon A \to K'$ such that $g'v = u$ and $k'v = h$. Finally, note that k' is monomorphic because k is monomorphic. □

2.2.4 Remark. We would like to note here that multi-pointed categories could also be approached via enriched category theory. To be more precise, a multi-pointed category can be equivalently defined as a category enriched in the monoidal closed category of multi-pointed sets. The latter has objects pairs (X, A), where X, A are sets with $A \subseteq X$, and morphisms $f\colon (X, A) \to (Y, B)$ the functions $f\colon X \to Y$ such that $f[A] \subseteq B$. Under that prism, for example, the notion of $\mathcal{N}$-kernel is an instance of a *weighted* limit. Since this point of view is not the one pursued in the literature and since for the purposes of this thesis it would not necessarily simplify matters, we chose to not follow it here.

The notion of kernel in a multi-pointed category in turn leads to what has been called a star relation in the literature.

2.2.5 Definition. A pair of morphisms $r = (r_0, r_1)\colon R \rightrightarrows X$ in a multi-pointed category $(\mathcal{C}, \mathcal{N})$ is called a *star* if $r_0 \in \mathcal{N}$. When it is moreover jointly monomorphic, we say that it is a *star relation*.

In the total context a star relation is simply a relation in the ordinary sense, while in the pointed case it is a relation whose first projection is zero. A more motivating example is given in the proto-pointed setting of the category Ring. Given any ideal $I \subseteq A$ inside the unitary ring A, we have an associated star relation R_I on A defined by $R_I \coloneqq \bigcup_{n\in\mathbb{Z}} \{n\} \times (n + I)$. This star relation clearly uniquely determines the ideal I. However, similarly to the $\mathcal{N}$-kernel of a ring homomorphism, it has the advantage that it is a subalgebra of $A \times A$ and hence lives in the category Ring, while the ideal I itself generally does not.

The existence of $\mathcal{N}$-kernels in a multi-pointed category $\mathcal{C}$ will allow us to associate a star relation to any given relation. This construction will be seen to be a generalization of the 0-class of a relation in the pointed context.

2.2.6 Definition. Let $(\mathcal{C}, \mathcal{N})$ be a multi-pointed category with kernels. Given a relation $R\colon X \nrightarrow Y$ in $\mathcal{C}$ which is represented by the jointly monomorphic pair $\langle r_0, r_1\rangle\colon R \rightarrowtail X \times Y$, we define the *star* of R to be the relation $R^* \rightarrowtail X \times Y$ represented by the pair (r_0k_0, r_1k_0), where $k_0\colon K_0 \to R$ is the $\mathcal{N}$-kernel of r_0.

Equivalently, we could define R^* by saying that it is the largest star relation which is contained in R. Also, in the particular case when $R = \mathrm{Eq}(f)$ is the kernel pair of a morphism $f\colon X \to Y \in \mathcal{C}$, we call $R^* = \mathrm{Eq}(f)^*$ the *star-kernel* of f.

In the total context the star construction is trivial, in the sense that $R^* = R$ for every relation R. On the other hand, in the pointed context we have that R^* is the 0-class $R[0]$ of R, but simply considered as a subobject of $X \times Y$. In terms of elements, it is the subrelation $R^* = \{(0, y) | (0, y) \in R\}$. In the algebraic proto-pointed context it is the set $R^* = \{(e, y) | (e, y) \in R, e \in E_X\}$, where E_X is the subalgebra of X generated by the constants. In the general case, given generalized elements $x\colon A \to X$ and $y\colon A \to Y$ in $\mathcal{C}$, we have that $(x, y) \in_A R^*$ precisely when $(x, y) \in_A R$ and $x \in \mathcal{N}$.

The similarity of stars with 0-classes is also present in the following lemma, which is an observation on how the star of a relation on an object X can be computed as a certain pullback involving the $\mathcal{N}$-kernel $\kappa_X\colon K_X \rightarrowtail X$ of the identity morphism $1_X\colon X \to X$. This is essentially a generalization of the fact that the 0-class of a relation $R \rightarrowtail X \times X$ in a pointed category can be computed as the pullback of that relation along $\langle 0, 1\rangle\colon X \to X \times X$. Since the more general statement does not appear in the literature, we also provide a proof.

2.2.7 Lemma. *Consider a relation* $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X$ *in the multi-pointed category* $(\mathcal{E}, \mathcal{N})$ *with kernels. Then the* $\mathcal{N}$*-kernel* k_0 *of* r_0 *is obtained as the following pullback.*

$$\begin{array}{ccc} K_0 & \overset{k_0}{\rightarrowtail} & R \\ {\scriptstyle \langle \overline{r_0}, r_1 k_0\rangle}\downarrow & & \downarrow{\scriptstyle \langle r_0, r_1\rangle} \\ K_X \times X & \underset{\kappa_X \times 1_X}{\rightarrowtail} & X \times X \end{array}$$

where $\kappa_X\colon K_X \rightarrowtail X$ *is the* $\mathcal{N}$*-kernel of the identity* $1_X\colon X \to X$*.*

Proof. We consider the $\mathcal{N}$-kernel $k_0\colon K_0 \rightarrowtail R$ of r_0 and we will show that there is a pullback square as indicated.

First, observe that $r_0 k_0 \in \mathcal{N}$ implies that there is a $\overline{r_0}\colon K_0 \to K_X$ such that $\kappa_X \overline{r_0} = r_0 k_0$, giving the indicated morphism $K_0 \to K_X \times X$ in the above commutative diagram.

Now assume that $f = \langle f_0, f_1\rangle\colon Z \to K_X \times X$ and $g\colon Z \to R$ are such that $(\kappa_X \times 1_X) f = \langle r_0, r_1\rangle g$. Then $r_0 g = \kappa_X f_0$ and $r_1 g = f_1$. Since $\kappa_X \in \mathcal{N}$, the first of these implies that $r_0 g \in \mathcal{N}$ and hence there exists a unique $h\colon Z \to K_0$ such that $k_0 h = g$. Then also $\langle \overline{r_0}, r_1 k_0\rangle h = \langle \overline{r_0} h, r_1 k_0 h\rangle = \langle \overline{r_0} h, r_1 g\rangle = \langle f_0, f_1\rangle = f$, where $\overline{r_0} h = f_0$ follows because $\kappa_X \overline{r_0} h = r_0 k_0 h = r_0 g = \kappa_X f_0$ and κ_X is monomorphic. □

2.3 2-star-permutability and Star-symmetry

The multi-pointed context allows one to define a class of regular categories which specializes to both the Mal'tsev and subtractive contexts. Below we call

a multi-pointed category $(\mathcal{C}, \mathcal{N})$ regular if the underlying category $\mathcal{C}$ is regular in the ordinary sense. We thus recall the following notion which first appeared in [47] and was later studied more in [51].

2.3.1 Definition. Let $\mathcal{C}$ be a regular multi-pointed category with kernels. We say that $\mathcal{C}$ is *2-star-permutable* if for any two effective equivalence relations R, S on an object $X \in \mathcal{C}$ we have $RS^* = SR^*$.

To see how this generalizes the regular Mal'tsev and regular subtractive cases, recall that these classes of categories can be defined via permutability (Lemma 2.1.2) and 0-permutability (Proposition 2.1.9) respectively of equivalence relations. More generally, 2-star-permutability indeed expresses the fact that effective equivalence relations *permute up to taking stars*. To see this it suffices to observe that, for any two relations R, S on an object X, one has $(SR)^* = SR^*$. Thus, the condition in the above definition can be equivalently stated as $(RS)^* = (SR)^*$.

The goal of this chapter is to provide a characterization of those categories which occur as projective covers of regular 2-star-permutable categories, thus generalizing the corresponding results for regular Mal'tsev [90] and regular subtractive [49] categories.

The literature contains results characterizing the projective covers of many classes of regular categories of interest. For example, we have such results for categories which are abelian [37], extensive [67], Mal'tsev [90], (strongly) unital and subtractive [49], protomodular and semi-abelian [43] and (locally) cartesian closed [32, 36]. If one peruses theses results, a certain theme becomes apparent. The property which characterizes the projective covers is a weak version of the property which defines the corresponding class of regular categories. The adjective "weak" here is not intended to mean that the property in the regular case implies its weakened version. Rather, it means a version of the property adapted to the context of a weakly lex category as opposed to a finitely complete or regular one. Accordingly, the work in many cases consists in identifying which property amongst a host of equivalent ones in the regular context can be appropriately weakened.

Now it is clear that the definition of 2-star-permutability as stated above is not amenable to a suitable weakening to the weakly lex setting. Indeed, since it involves the composition of relations, it makes full use of regularity of the category and there is no obvious way to transform this into something which would make sense in the presence of only weak finite limits. To work around this we will characterize 2-star-permutability by an equivalent condition involving a certain symmetry property of internal relations, essentially generalizing the similar known characterization of Mal'tsev and subtractive categories which we recalled in Section 2.1. This will be the content of the remainder of this section.

2.3.2 Definition. Let $\mathcal{E}$ be a multi-pointed category with kernels and suppose that $R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X$ is a relation in $\mathcal{E}$. We say that R is *left star-symmetric* if $R^* \subseteq (R^\circ)^*$. We say that it is *star-symmetric* if $R^* = (R^\circ)^*$, i.e. if both R and R° are left star-symmetric.

In the total context, R being left star-symmetric just means that $R \subseteq R^\circ$, which is to say that R is a symmetric relation in the ordinary sense. In particular, in this case left star-symmetry and star-symmetry become equivalent. In the pointed context, on the other hand, it becomes precisely the notion of left 0-symmetry Definition 2.1.7. In the algebraic proto-pointed setting, left star symmetry is the validity of the implication $(e, x) \in R \implies (x, e) \in R$, *for every* $e \in E_X$, where as usual E_X is the subalgebra of X generated by the constants.

Generally, R being left star-symmetric means that, given any generalized elements $n\colon A \to X$ and $x\colon A \to X$ with $n \in \mathcal{N}$, the implication $(n, x) \in_A R \implies (x, n) \in_A R$ holds.

We will need the following lemma, from [47], for the proof of our next proposition.

2.3.3 Lemma. *For any morphism $f\colon X \to Y$ and every relation S on Y in a multi-pointed category with kernels, we have $(f^{-1}(S))^* = (f^{-1}(S^*))^*$.*

We can now present new equivalent characterizations of 2-star-permutability using the notion of star-symmetry. As a byproduct, this allows us to also deduce that 2-star-permutability is equivalent to having the equality $RS^* = SR^*$ for *all* equivalence relations R, S on the same object, not just effective ones as initially required by Definition 2.3.1. This does not seem to have appeared in the literature before. The following then is the first original contribution of this chapter.

2.3.4 Proposition. *[6] For a regular multi-pointed category $\mathcal{C}$ with kernels the following are equivalent:*

1. *$\mathcal{C}$ is 2-star-permutable.*
2. *For any two equivalence relations R, S on an object $X \in \mathcal{C}$ we have $RS^* = SR^*$.*
3. *Every reflexive relation E in $\mathcal{C}$ is left star-symmetric, i.e. $E^* \subseteq (E^\circ)^*$.*
4. *Every reflexive relation E in $\mathcal{C}$ is star-symmetric, i.e. $E^* = (E^\circ)^*$.*

Proof. 1. $\implies$ 4. Let $E \overset{e_0}{\underset{e_1}{\rightrightarrows}} X$ be a reflexive relation with diagonal $\delta\colon X \to E$. Set $R \coloneqq \mathrm{Eq}(e_0) = e_0^\circ e_0$ and $S \coloneqq \mathrm{Eq}(e_1) = e_1^\circ e_1$, so that both R and S are effective equivalence relations on E. Observe that $\delta^{-1}(SR) = \delta^\circ e_1^\circ e_1 e_0^\circ e_0 \delta =$

$e_1e_0^\circ = E$ and $\delta^{-1}(RS) = \delta^\circ e_0^\circ e_0 e_1^\circ e_1 \delta = e_0 e_1^\circ = E^\circ$. Now using the assumption (1) and Lemma 2.3.3 we have

$$\begin{aligned} RS^* = SR^* \quad &\Longrightarrow \quad (RS)^* = (SR)^* \\ &\Longrightarrow \quad \delta^{-1}((RS)^*) = \delta^{-1}((SR)^*) \\ &\Longrightarrow \quad \delta^{-1}((RS)^*)^* = \delta^{-1}((SR)^*)^* \\ &\Longrightarrow \quad \delta^{-1}(RS)^* = \delta^{-1}(SR)^* \\ &\Longrightarrow \quad (E^\circ)^* = E^*. \end{aligned}$$

4. $\Longrightarrow$ 2. Consider the reflexive relation $E \coloneqq SR$ on X. Then we have $E^* = (E^\circ)^*$, which by symmetry of R, S is to say $(SR)^* = (RS)^*$, or equivalently $SR^* = RS^*$.

2. $\Longrightarrow$ 1. Clear.

3. $\Longleftrightarrow$ 4. Clear by considering both reflexive relations E and E°. □

The importance of conditions (3) and (4) above is the fact that they can be formulated in any finitely complete multi-pointed category $(\mathcal{C},\mathcal{N})$ with kernels, thus enlarging the class of categories for which the notion of 2-star-permutability can be considered to include non-regular ones. It generalizes and should be compared to the corresponding statements for Mal'tsev (Definition 2.1.3) and subtractive categories (Proposition 2.1.8). The following definition therefore appears pertinent.

2.3.5 Definition. A multi-pointed category with kernels $(\mathcal{C},\mathcal{N})$ is said to be *star-Mal'tsev* if every reflexive relation in $\mathcal{C}$ is left star-symmetric. Equivalently, if every reflexive relation in $\mathcal{C}$ is star-symmetric.

In this terminology, Proposition 2.3.4 says that a regular multi-pointed category with kernels is 2-star-permutable if and only if it is star-Mal'tsev.

2.4 Projective Covers of 2-star-permutable Categories

This section contains the main result of this chapter, namely a characterization of the projective covers of 2-star-permutable categories.

To begin with, since we will be passing between a regular category $\mathcal{E}$ and one of its projective covers $\mathcal{C}$, we should briefly comment on how ideals of morphisms transfer between the two. An analysis of this situation is contained in [44], from which we will borrow what we need here.

In the more obvious direction, if $\mathcal{N}$ is an ideal of morphisms in the regular category $\mathcal{E}$, then we denote by $\mathcal{N}_\mathcal{C}$ the restriction of $\mathcal{N}$ to $\mathcal{C}$. It is then clear that $\mathcal{N}_\mathcal{C}$ is an ideal in $\mathcal{C}$.

In the reverse direction, given an ideal $\mathcal{N}$ in the projective cover $\mathcal{C}$, then we define $\mathcal{N}^\mathcal{E}$ to be the collection of morphisms $f\colon X \to Y \in \mathcal{E}$ for which there exists a commutative square

$$\begin{array}{ccc} P & \xrightarrow{n} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where p and q are regular epimorphisms and $n \in \mathcal{N}$. It is again not hard to check that $\mathcal{N}^{\mathcal{E}}$ is an ideal in $\mathcal{E}$.

Now just as for ordinary limits, the existence of $\mathcal{N}$-kernels in a multi-pointed regular category $\mathcal{E}$ gives the existence of weak $\mathcal{N}_{\mathcal{C}}$-kernels in the projective cover $\mathcal{C}$. This and the other basic observations we shall need are contained in the following, all of which is contained in [44].

2.4.1 Lemma. *[44] Let $\mathcal{E}$ be a regular category having a projective cover $\mathcal{C}$.*

1. *For any ideal $\mathcal{N}$ in $\mathcal{E}$, if $\mathcal{E}$ has $\mathcal{N}$-kernels, then $\mathcal{C}$ has weak $\mathcal{N}_{\mathcal{C}}$-kernels, which can be computed by taking a projective cover of the domain of the $\mathcal{N}$-kernel in $\mathcal{E}$.*
2. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, the category $\mathcal{C}$ has weak $\mathcal{N}$-kernels if and only if $\mathcal{E}$ has $\mathcal{N}^{\mathcal{E}}$-kernels.*
3. *For any ideal $\mathcal{N}$ in $\mathcal{C}$ we have $(\mathcal{N}^{\mathcal{E}})_{\mathcal{C}} = \mathcal{N}$.*
4. *For any ideal $\mathcal{N}$ in $\mathcal{C}$, regular epimorphisms are $\mathcal{N}^{\mathcal{E}}$-saturating in $\mathcal{E}$ (see Definition 2.4.5).*

Coming back to the main goal, recall that in the previous section we reformulated 2-star-permutability as a symmetry property Definition 2.3.2 of internal relations Proposition 2.3.4. In order to characterize the projective covers of 2-star-permutable or, in other words, of regular star-Mal'tsev categories, we will formulate this property in a weakly lex context. But, as with the Mal'tsev case [90], when passing to this context the symmetry property must now be required more generally for *graphs*, i.e. for pairs $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ which are not necessarily jointly monomorphic. We therefore must introduce the following definition.

2.4.2 Definition. Let $(\mathcal{C}, \mathcal{N})$ be a weakly lex multi-pointed category with weak $\mathcal{N}$-kernels. A graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ is said to be *left star-symmetric* if, given weak $\mathcal{N}$-kernels $k_0 \colon K_0 \to G$ and $k_1 \colon K_1 \to G$ of g_0 and g_1 respectively, there exists a $\sigma \colon K_0 \to K_1$ such that the following diagram serially commutes

$$\begin{array}{ccccc} K_0 & & \xrightarrow{\quad\sigma\quad} & & K_1 \\ & {\scriptstyle g_0k_0}\searrow & & \swarrow{\scriptstyle g_0k_1} & \\ {\scriptstyle g_1k_0}\searrow & & X & & \swarrow{\scriptstyle g_1k_1} \end{array}$$

i.e. such that $g_1k_1\sigma = g_0k_0$ and $g_0k_1\sigma = g_1k_0$ both hold. We say that it is *star-symmetric* if both G and its opposite graph $G \underset{g_0}{\overset{g_1}{\rightrightarrows}} X$ are left star-symmetric.

We must note that the above definition does not depend on the chosen weak $\mathcal{N}$-kernels because any two weak $\mathcal{N}$-kernels of the same morphism factor through each other. Hence, if the stated factorization σ exists for one choice of weak kernels then it exists for the other as well. Furthermore, it should be clear that when G is a relation and $\mathcal{N}$-kernels exist the definition says precisely that $G^* \subseteq (G^\circ)^*$, i.e. that G is a left star-symmetric relation.

We can now officially introduce the categories that will appear in our characterization theorem for projective covers. These are the multi-pointed categories with weak finite limits and weak kernels which satisfy the appropriate "weakening" of the star-Mal'tsev property. Our terminology is inspired by that of Rosický & Vitale in [90] for the total context.

2.4.3 Definition. We will say that a weakly lex multi-pointed category $\mathcal{C}$ with weak kernels is *star-G-Mal'tsev* if every reflexive graph in $\mathcal{C}$ is left star-symmetric. Equivalently, if every reflexive graph is star-symmetric.

Before presenting our main result, it will be convenient to have isolated the following fundamental observation, which relates the star-symmetry of graphs in the projective cover with star-symmetry of relations in the ambient regular category.

2.4.4 Lemma. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category $(\mathcal{E},\mathcal{N})$ with kernels. Consider a graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ with its image factorization $\langle g_0, g_1\rangle = G \xrightarrow{q} R \overset{\langle r_0,r_1\rangle}{\rightarrowtail} X \times X$ in $\mathcal{E}$. Then G is a left star-symmetric graph if and only if R is a left star-symmetric relation.*

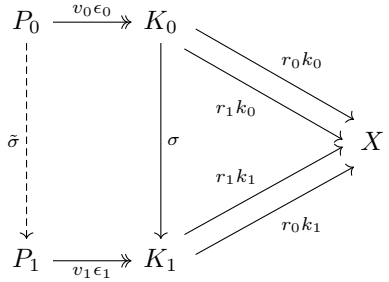
Proof. Consider $\mathcal{N}$-kernels $k_i\colon K_i \to R$ of r_i, for $i = 0, 1$. Form the pullbacks below for $i = 0, 1$ and then take $\mathcal{C}$-covers $\epsilon_i\colon P_i \twoheadrightarrow K_i'$.

$$\begin{array}{ccc} K_i' & \xrightarrow{v_i} & K_i \\ {\scriptstyle k_i'}\downarrow & & \downarrow{\scriptstyle k_i} \\ G & \xrightarrow[q]{} & R \end{array}$$

By Lemma 2.2.3 we know that $k_i'\colon K_i' \rightarrowtail G$ is the $\mathcal{N}$-kernel of $r_iq = g_i$. Thus, we have that $u_i \coloneqq k_i'\epsilon_i\colon P_i \to G$ is a weak $\mathcal{N}$-kernel of g_i in $\mathcal{C}$ for $i = 0, 1$ by Lemma 2.4.1.

Assume first that R is left star-symmetric, so that there exists a morphism $\sigma\colon K_0 \to K_1$ such that $r_1k_1\sigma = r_0k_0$ and $r_0k_1\sigma = r_1k_0$. By projectivity of

P_0 and the fact that $v_1\epsilon_1$ is a regular epimorphism, there exists a morphism $\tilde{\sigma}\colon P_0 \to P_1$ making the following diagram commute.



Now we have

$$\begin{aligned} g_1u_1\tilde{\sigma} &= r_1qk_1'\epsilon_1\tilde{\sigma} \\ &= r_1k_1v_1\epsilon_1\tilde{\sigma} \\ &= r_1k_1\sigma v_0\epsilon_0 \\ &= r_0k_0v_0\epsilon_0 \\ &= r_0qk_0'\epsilon_0 \\ &= g_0u_0 \end{aligned}$$

and similarly

$$\begin{aligned} g_0u_1\tilde{\sigma} &= r_0qk_1'\epsilon_1\tilde{\sigma} \\ &= r_0k_1v_1\epsilon_1\tilde{\sigma} \\ &= r_0k_1\sigma v_0\epsilon_0 \\ &= r_1k_0v_0\epsilon_0 \\ &= r_1qk_0'\epsilon_0 \\ &= g_1u_0 \end{aligned}$$

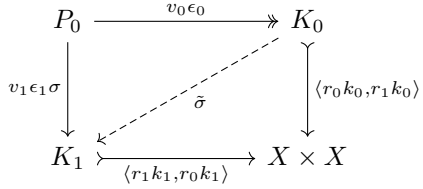
proving that G is left star-symmetric.

Conversely, assume that G is left star-symmetric. This means that there exists a $\sigma\colon P_0 \to P_1$ such that $g_1u_1\sigma = g_0u_0$ and $g_0u_1\sigma = g_1u_0$. We can then again calculate as follows:

$$\begin{aligned} r_1k_1v_1\epsilon_1\sigma &= r_1qk_1'\epsilon_1\sigma \\ &= g_1k_1'\epsilon_1\sigma \\ &= g_1u_1\sigma \\ &= g_0u_0 \\ &= r_0qk_0'\epsilon_0 \\ &= r_0k_0v_0\epsilon_0 \end{aligned}$$

$$\begin{aligned} r_0 k_1 v_1 \epsilon_1 \sigma &= r_0 q k_1' \epsilon_1 \sigma \\ &= g_0 k_1' \epsilon_1 \sigma \\ &= g_0 u_1 \sigma \\ &= g_1 u_0 \\ &= r_1 q k_0' \epsilon_0 \\ &= r_1 k_0 v_0 \epsilon_0 \end{aligned}$$

This means that the square below commutes and so we obtain the indicated morphism $\tilde{\sigma}$ because $v_0\epsilon_0$ is a regular epimorphism and $\langle r_1 k_1, r_0 k_1 \rangle$ is monomorphic, being the star of the relation R°.

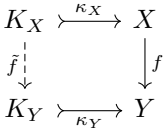


The commutation of the bottom triangle is precisely left star-symmetry of R. □

Now as the reader probably has anticipated, we would like to prove that the star-G-Mal'tsev categories are precisely the projective covers of star-Mal'tsev categories. Unfortunately, we cannot prove that statement in complete generality. Nevertheless, we will be able to prove it modulo a small additional technical assumption, which regards the behavior of regular epimorphisms with respect to $\mathcal{N}$-kernels. This condition is familiar from the literature (see [44, 47, 51]) and often appears when one wants to prove more significant results. More importantly, it is mild enough that it includes all examples of interest.

Consider any object X in the multi-pointed category $(\mathcal{E}, \mathcal{N})$. We will denote by $\kappa_X \colon K_X \rightarrowtail X$ the $\mathcal{N}$-kernel of the identity morphism 1_X. Observe that by definition the generalized elements of K_X correspond precisely to the generalized elements of X that are in $\mathcal{N}$. Hence, K_X should be thought of as consisting of the "trivial elements" of the object X. Indeed, in the algebraic proto-pointed setting K_X is exactly what we have earlier in the text denoted by E_X, namely the subalgebra of X generated by the constants of the theory.

Now given any morphism $f \colon X \to Y$ in $\mathcal{E}$, we have a uniquely induced morphism $\tilde{f} \colon K_X \to K_Y$ making the following square commute.



We then have the following definition.

2.4.5 Definition. [47] A morphism $f\colon X \to Y$ in a multi-pointed category $(\mathcal{E},\mathcal{N})$ is called *saturating* if the induced morphism $\tilde{f}\colon K_X \to K_Y$ is a regular epimorphism.

Note that in the pointed context all morphisms are saturating, since $K_X = 0$ for any object X. The same holds in any algebraic proto-pointed setting, since every element $e \in X$ which is generated by constants is preserved under all homomorphisms $f\colon X \to Y$. Furthermore, it is not hard to see that regular epimorphisms are saturating in any proto-pointed context, not just the varietal one. In the total context, on the other hand, it is clear that the saturating morphisms are exactly the regular epimorphisms. In fact, that all regular epimorphisms are saturating is precisely what we shall require below.

Now we finally come to the main result of this chapter.

2.4.6 Theorem. *Let $\mathcal{C}$ be a projective cover of the regular multi-pointed category with kernels $(\mathcal{E},\mathcal{N})$ and assume regular epimorphisms in $\mathcal{E}$ are saturating. Then $(\mathcal{E},\mathcal{N})$ is 2-star-permutable if and only if $(\mathcal{C},\mathcal{N}_\mathcal{C})$ is star-G-Mal'tsev.*

Proof. Assume first that $\mathcal{E}$ is 2-star-permutable and consider any reflexive graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} X$ in $\mathcal{C}$ with splitting $\delta : X \to G$. Consider its image factorization $\langle g_0, g_1\rangle = G \overset{q}{\twoheadrightarrow} R \overset{\langle r_0,r_1\rangle}{\rightarrowtail} X \times X$ in the regular category $\mathcal{E}$. Then the relation R on X is reflexive as well, since $r_i q\delta = g_i\delta = 1_X$ for $i = 0, 1$. Since $\mathcal{E}$ is 2-star-permutable, we know by Proposition 2.3.4 that R must be star-symmetric. Now Lemma 2.4.4 implies that G is (left) star-symmetric.

Conversely, assume that $(\mathcal{C},\mathcal{N}_\mathcal{C})$ is star-G-Mal'tsev. Consider any reflexive relation $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X$ in $\mathcal{E}$. We want to show that E is left star-symmetric.

Take a $\mathcal{C}$-cover $p\colon \tilde{X} \twoheadrightarrow X$ of X and consider the inverse image relation $E' \coloneqq p^{-1}(E)$. i.e. form the following pullback

$$\begin{array}{ccc} E' & \overset{\langle e'_0,e'_1\rangle}{\rightarrowtail} & \tilde{X} \times \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p\times p} \\ E & \overset{\langle e_0,e_1\rangle}{\rightarrowtail} & X \times X \end{array}$$

Now again take a $\mathcal{C}$-cover $\epsilon\colon G \twoheadrightarrow E'$ and set $g_0 \coloneqq e'_0\epsilon$ and $g_1 \coloneqq e'_1\epsilon$, so that we have a graph $G \underset{g_1}{\overset{g_0}{\rightrightarrows}} \tilde{X}$ in $\mathcal{C}$. Observe that the relation E' is reflexive, being the inverse image of a reflexive relation. It follows that the graph G is also reflexive. Indeed, if $\delta'\colon \tilde{X} \to E'$ is the diagonal of E', then by projectivity of $\tilde{X}$

we can lift to a $\tilde{\delta}\colon \tilde{X} \to G$ such that $\epsilon\tilde{\delta} = \delta'$ and then $g_i\tilde{\delta} = e'_i\epsilon\tilde{\delta} = e'_i\delta' = 1_{\tilde{X}}$ for $i = 0, 1$.

Now consider $\mathcal{N}$-kernels $k_i\colon K_i \to E$ of e_i and $k'_i\colon K'_i \to E'$ of e'_i in $\mathcal{E}$ for $i = 0, 1$. We then have induced morphisms $u_i\colon K'_i \to K_i$ such that $k_iu_i = qk'_i$. We claim that the u_i are regular epimorphisms.

$$\begin{array}{ccc} K'_i & \overset{k'_i}{\rightarrowtail} & E' \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle q} \\ K_i & \underset{k_i}{\rightarrowtail} & E \end{array}$$

To see this for u_0 we consider the following two commutative diagrams. In the first one the right-hand square is a pullback by construction, while the left-hand square is a pullback by Lemma 2.2.7. In the second diagram we know only that the right-hand square is a pullback, again by Lemma 2.2.7.

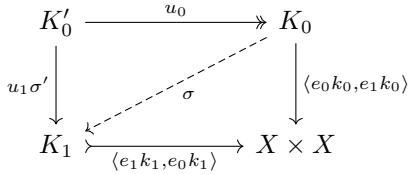
$$\begin{array}{ccccc} K'_0 & \overset{k'_0}{\rightarrowtail} & E' & \overset{q}{\twoheadrightarrow} & E \\ {\scriptstyle \langle\overline{e'_0}, e'_1k'_0\rangle}\downarrow & & {\scriptstyle \langle e'_0, e'_1\rangle}\downarrow & & \downarrow{\scriptstyle \langle e_0, e_1\rangle} \\ K_{\tilde{X}} \times \tilde{X} & \underset{\kappa_{\tilde{X}}\times 1_{\tilde{X}}}{\longrightarrow} & \tilde{X} \times \tilde{X} & \underset{p\times p}{\twoheadrightarrow} & X \times X \end{array}$$

$$\begin{array}{ccccc} K'_0 & \overset{u_0}{\twoheadrightarrow} & K_0 & \overset{k_0}{\rightarrowtail} & E \\ {\scriptstyle \langle\overline{e'_0}, e'_1k'_0\rangle}\downarrow & & {\scriptstyle \langle\overline{e_0}, e_1k_0\rangle}\downarrow & & \downarrow{\scriptstyle \langle e_0, e_1\rangle} \\ K_{\tilde{X}} \times \tilde{X} & \underset{\tilde{p}\times p}{\twoheadrightarrow} & K_X \times X & \underset{\kappa_X\times 1_X}{\longrightarrow} & X \times X \end{array}$$

Since $(p\times p)(\kappa_{\tilde{X}}\times 1_{\tilde{X}}) = (\kappa_X \times 1_X)(\tilde{p}\times p)$, we deduce that the outer rectangle in the second diagram is a pullback. Then by the usual pullback-cancellation property we have that the left-hand square is a pullback as well. But since both p and $\tilde{p}$ are regular epimorphisms (the second one because regular epimorphisms are assumed saturating) and $\mathcal{E}$ is a regular category, we deduce that $\tilde{p}\times p$ is also a regular epimorphism and in turn that the pullback u_0 is a regular epimorphism.

By the assumption that $\mathcal{C}$ is star-G-Mal'tsev, the reflexive graph G is left star-symmetric and so by Lemma 2.4.4 its image relation E' is also left star-symmetric. Thus, there exists a $\sigma'\colon K'_0 \to K'_1$ such that $e'_1k'_1\sigma' = e'_0k'_0$ and $e'_0k'_1\sigma' = e'_1k'_0$.

Finally, consider the commutative square that follows.



Since u_0 is a regular epimorphism and $\langle e_1k_1, e_0k_1\rangle$ is monomorphic (being the star of the relation E°), we get the indicated factorization $\sigma\colon K_0 \to K_1$, which shows that E is left star-symmetric. Thus, by Proposition 2.3.4 it follows that $\mathcal{E}$ is 2-star-permutable. □

As with all results on projective covers, we immediately deduce corresponding results for regular and exact completions of weakly lex categories. Namely, we have characterizations of when the regular or exact completion of a weakly lex multi-pointed category with weak kernels is 2-star-permutable.

2.4.7 Corollary. *Let $(\mathcal{C}, \mathcal{N})$ be a weakly lex multi-pointed category with weak kernels. Then $(\mathcal{C}_{reg}, \mathcal{N}^{\mathcal{C}_{reg}})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N})$ is star-G-Mal'tsev.*

Proof. $\mathcal{C}$ appears as a projective cover inside $\mathcal{C}_{reg}$. Then Theorem 2.4.6 indeed applies to give the result because by Lemma 2.4.1 we know that $\mathcal{C}_{reg}$ has $\mathcal{N}^{\mathcal{C}_{reg}}$-kernels, regular epimorphisms in $\mathcal{C}_{reg}$ are $\mathcal{N}^{\mathcal{C}_{reg}}$-saturating and $(\mathcal{N}^{\mathcal{C}_{reg}})_{\mathcal{C}} = \mathcal{N}$. □

2.4.8 Corollary. *Let $(\mathcal{C}, \mathcal{N})$ be a weakly lex multi-pointed category with weak kernels. Then $(\mathcal{C}_{ex}, \mathcal{N}^{\mathcal{C}_{ex}})$ is 2-star-permutable if and only if $(\mathcal{C}, \mathcal{N})$ is star-G-Mal'tsev.*

We end this chapter with a small application. Namely, we show how the characterization in terms of the star-symmetry property can be used in the algebraic proto-pointed context to deduce the syntactic conditions which define the *E-subtractive* varieties of A Ursini. Note that it was already proved in [47] that in this case 2-star-permutability is equivalent to a priori more general properties such as *3-star-permutability* and the *symmetric saturation property.*

2.4.9 Corollary. *Let $\mathbb{V}$ be a variety of universal algebras and let $E_{\mathbb{V}} \neq \emptyset$ be its algebra of constants (i.e. the free $\mathbb{V}$-algebra on the empty set). Then $\mathbb{V}$ is 2-star-permutable (in the algebraic proto-pointed context) if and only if the following syntactic condition holds:*

For every $e \in E_{\mathbb{V}}$ there exists a binary term $s_e(x, y)$ such that $s_e(x, x) = e$ and $s_e(x, e) = x$.

Proof. Suppose 2-star-permutability holds and fix any $e \in E_{\mathbb{V}}$. We then consider a graph $F(x, y) \underset{g_1}{\overset{g_0}{\rightrightarrows}} F(x)$ between free algebras on 2 and 1 generator respectively, where g_0, g_1 are defined by setting $g_0(x) = x$, $g_0(y) = e$ and

$g_1(x) = g_1(y) = x$. This graph is reflexive, since it is clearly split by the map $\delta\colon F(x) \to F(x,y)$ defined by $\delta(x) = x$. Since free algebras are projective, we can apply Theorem 2.4.6 (i.e. $\mathcal{C}$ here is the full subcategory of free algebras) to deduce that this graph must be star-symmetric.

Now we have $(e, x) = (g_0(y), g_1(y))$, so by the star-symmetry we must also have $(x, e) = (g_0(s_e), g_1(s_e))$ for some $s_e(x, y) \in F(x, y)$. Thus, $x = g_0(s_e(x, y)) = s_e(x, e)$ and $e = g_1(s_e(x, y)) = s_e(x, x)$.

Conversely, suppose we have binary terms $s_e(x, y)$ for all $e \in E_{\mathbb{V}}$ with the indicated properties. We will show that any reflexive relation $R \rightarrowtail X \times X$ in the variety $\mathbb{V}$ is left star-symmetric.

Indeed, assume that $(e, x) \in R$ for some $e \in E_X$ and $x \in X$. Since R is reflexive, we also have $(x, x) \in R$. By compatibility with the operations we then must have $(s_e(x, e), s_e(x, x)) \in R$, i.e. that $(x, e) \in R$. This concludes the proof. □

As particular examples of E-subtractive varieties we have the categories Bool and Heyt of Boolean and Heyting algebras respectively, both with $E = \{0, 1\}$. Similarly, we have the category Ring of unitary rings, with E the prime subring, i.e. the image of $\mathbb{Z}$ in each ring. A non-Mal'tsev example is given by the category Imp of *implication algebras* [1].

2.4.10 Remark. Since our main result applies to any regular category and does not require exactness, it can equally well be applied to *quasi-varieties* of universal algebras, since these are still regular categories. This allows to subsume more interesting examples, such as the category RedRng of *reduced rings*, i.e. unitary rings R satisfying $(\forall x \in R)(\forall n \geq 1)(x^n = 0 \implies x = 0)$, or the quasi-variety of *BCK algebras* [13], for example).

Chapter 3

A Property of the Exact Completion and Applications

This chapter presents joint work of the author with P. Karazeris. The first section presents a property of the ex/reg completion and the subsequent three sections concern applications of this property. The material in the first three sections has been published in [8]. However, some of the proofs involved in the main result, contained in the first section of this chapter, have been reworked by the present author to use the relational construction of $\mathcal{C}_{ex/reg}$ as recalled in Section 1.5, while in loc. cit. the authors used the sheaf-theoretic construction of S. Lack from [66]. This has been done so that the first section of this chapter can also serve as somewhat of a segue into the next chapter, where we shall directly adapt some of the arguments to the context of categories enriched in posets.

3.1 Characterizing the ex/reg Completion

In this section we present a result which characterizes the exact completion $\mathcal{C}_{ex/reg}$ of a regular category $\mathcal{C}$. More appropriately, we should say that it characterizes the canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ as a regular functor from $\mathcal{C}$ into an exact category which satisfies certain simple properties. The essence of our work here consists of some observations on how certain properties of a regular functor $F\colon \mathcal{C} \to \mathcal{E}$ into an exact category $\mathcal{E}$ translate to properties of the extension $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$. The concrete description of the latter functor was recalled in Remark 1.5.4.

3.1.1 Proposition. *Let $F : \mathcal{C} \to \mathcal{E}$ be a regular functor into the exact category $\mathcal{E}$. If F is fully faithful, then $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$ is faithful.*

Proof. Let $(X,E) \underset{S}{\overset{R}{\rightrightarrows}} (Y,G)$ be two morphisms in $\mathcal{C}_{ex/reg}$ such that we have $\tilde{F}(R) = \tilde{F}(S)$ in $\mathcal{E}$. Then using the definition of $\tilde{F}$ we have

$$\begin{aligned}
\tilde{F}(R) = \tilde{F}(S) \quad &\Longrightarrow \quad p^{\circ}_{(Y,G)}\tilde{F}(R)p_{(X,E)} = p^{\circ}_{(Y,G)}\tilde{F}(S)p_{(X,E)} \\
&\Longrightarrow \quad p^{\circ}_{(Y,G)}p_{(Y,G)}F(R)p^{\circ}_{(X,E)}p_{(X,E)} = \\
&= \quad p^{\circ}_{(Y,G)}p_{(Y,G)}F(S)p^{\circ}_{(X,E)}p_{(X,E)} \\
&\Longrightarrow \quad F(G)F(R)F(E) = F(G)F(S)F(E) \\
&\Longrightarrow \quad F(R) = F(S)
\end{aligned}$$

But since F is fully faithful, it reflects inclusions of subobjects and hence of relations. Thus, we obtain $R = S$. □

In particular, given a regular functor $F : \mathcal{C} \to \mathcal{D}$ we deduce the following result about the induced functor between exact completions.

3.1.2 Corollary. *[8] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is fully faithful, then $F_{ex/reg}\colon \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$ is faithful.*

We introduce some further properties of functors that will be of interest. Our choice of terminology here follows the literature of Categorical Logic (e.g. [73]).

3.1.3 Definition. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *covering* if, for every object $Y \in \mathcal{D}$, one can find an object $X \in \mathcal{C}$ and a regular epimorphism $FX \twoheadrightarrow Y$.

We say that F is *full on subobjects* if, for every monomorphism $B \rightarrowtail FX$ in $\mathcal{D}$, there exists a monomorphism $A \rightarrowtail X$ in $\mathcal{C}$ such that $FA \cong B$ in $\mathrm{Sub}_{\mathcal{D}}(\mathrm{FX})$.

We now have the following observation connecting the properties of being covering and full on subobjects for certain regular functors. Since this seems to not have appeared explicitly in the literature, we also give its proof.

3.1.4 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is full and covering, then F is full on subobjects.*

Proof. Consider a monomorphism $v : D \rightarrowtail FY$ in $\mathcal{D}$. Since F is covering, there exists some regular epimorphism $q : FX \twoheadrightarrow D$ in $\mathcal{D}$. Now consider the composite morphism $FX \overset{q}{\twoheadrightarrow} D \overset{v}{\rightarrowtail} FY$. Since F is full, there is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $Ff = vq$.

Since $\mathcal{C}$ is regular, we can factor f as a regular epimorphism followed by a monomorphism, say $f = X \overset{p}{\twoheadrightarrow} I \overset{u}{\rightarrowtail} Y$. But F is a regular functor, so $FX \overset{Fp}{\twoheadrightarrow} FI \overset{Fu}{\rightarrowtail} FY$ is the corresponding factorization of $Ff = vq$ in $\mathcal{D}$. By uniqueness of image factorizations in the regular category $\mathcal{D}$, we deduce that $FI \cong D$ as subobjects of FY. □

Next, we add the covering property to a regular $F\colon \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ exact and see what this implies about $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$.

3.1.5 Proposition. *Let $F\colon \mathcal{C} \to \mathcal{E}$ be a regular functor with $\mathcal{E}$ an exact category. If F is fully faithful and covering, then $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$ is fully faithful and covering.*

Proof. We saw earlier that F being fully faithful implies that $\tilde{F}$ is faithful. Furthermore, it is immediate that F being covering implies the same property for $\tilde{F}$, since we have $\tilde{F}\Gamma \cong F$.

Now consider any morphism $g\colon \tilde{F}(X,E) \to \tilde{F}(Y,G) \in \mathcal{E}$. Let $S\colon FX \nrightarrow FY$ denote the relation corresponding to this morphism via the bijection of Lemma 1.5.3, i.e. the relation $S = p^{\circ}_{(Y,G)} g p_{(X,E)}$. Now since F is a full and covering regular functor, we know by the previous lemma that it is also full on subobjects and so there exists a relation $R\colon X \nrightarrow Y$ in $\mathcal{C}$ such that $F(R) = S$. Furthermore, we have the following:

$$F(GRE) = F(G)F(R)F(E) = F(G)SF(E) = S = F(R)$$

$$F(R^{\circ}R) = F(R)^{\circ}F(R) = S^{\circ}S \supseteq F(E)$$

$$F(RR^{\circ}) = F(R)F(R)^{\circ} = SS^{\circ} \subseteq F(G)$$

But now because F is fully faithful it reflects inclusions of subobjects. Thus, we deduce that $GRE = R$, $R^{\circ}R \supseteq E$ and $RR^{\circ} \subseteq G$, so that R is a morphism $(X,E) \to (Y,G) \in \mathcal{C}_{ex/reg}$.

Finally, we have by definition of the functor $\tilde{F}$ that

$$\tilde{F}(R) = p_{(Y,G)} F(R) p^{\circ}_{(X,E)} = p_{(Y,G)} S p^{\circ}_{(X,E)} = p_{(Y,G)} p^{\circ}_{(Y,G)} g p_{(X,E)} p^{\circ}_{(X,E)} = g$$

and hence that $\tilde{F}(R) = g$. □

Again, we can restate this result in the following form.

3.1.6 Corollary. *[8] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is fully faithful and covering, then $F_{ex/reg}\colon \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$ is fully faithful and covering.*

The final ingredient we shall need is the following result, which appears as Lemma 1.4.9 in [73].

3.1.7 Lemma. *[73] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is fully faithful and covering and $\mathcal{C}$ is exact, then F is an equivalence of categories.*

Thus, combining all the above we immediately deduce the following theorem characterizing the exact completion of $\mathcal{C}$.

3.1.8 Theorem. *Let $F\colon \mathcal{C} \to \mathcal{E}$ be a regular functor with $\mathcal{E}$ an exact category. Then $\tilde{F}\colon \mathcal{C}_{ex/reg} \to \mathcal{E}$ is an equivalence if and only if F is fully faithful and covering.*

In other words, the canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ is the unique up to equivalence regular functor from $\mathcal{C}$ into an exact category which is fully faithful and covering. The above result is essentially equivalent to its following form, which appeared as Theorem 2.4 in [8].

3.1.9 Corollary. *[8] Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is fully faithful and covering, then $F_{ex/reg}\colon \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$ is fully faithful and covering, hence an equivalence of categories.*

One pleasant aspect of the characterization theorem is that it allows us to more easily identify the exact completion of a regular category in some examples of interest. Indeed, in many cases a given regular category $\mathcal{C}$ is known to be a regular subcategory of an exact category $\mathcal{E}$. If we can then show that every object of $\mathcal{E}$ is covered by an object of $\mathcal{C}$, we will know that we have successfully identified the exact completion of $\mathcal{C}$. Here are some examples of this situation.

3.1.10 Example.

1. $\mathsf{Ab} \simeq (\mathsf{Ab}_{t.f.})_{ex/reg}$: Indeed, we know that any abelian group can be covered by a free group, which is in particular torsion-free. Hence, Theorem 3.1.8 applies to the inclusion $\mathsf{Ab}_{t.f.} \hookrightarrow \mathsf{Ab}$.

   In fact, this generalizes to any torsion theory. More precisely, if $\mathcal{F}$ is a torsion-free class in Ab, then we have $\mathcal{F}_{ex/reg} \simeq \mathsf{Ab}$.

2. $\mathsf{Stone}_{ex/reg} \simeq \mathsf{CHaus}$. Indeed, every compact Hausdorff space X admits a continuous surjection from a Stone space. For example, we have the cover $\beta X \twoheadrightarrow X$, where βX is the Stone-Čech compactification of the discrete space on X.

3.2 Conceptual Completeness for Regular Logic

The initial motivation of the authors of [8] for proving the properties of the ex/reg completion contained in the previous section came from the study of conceptual completeness in categorical logic. A "conceptual completeness" result says, in some sense, that certain types of logical theories can be recovered from their models.

Recall that within the paradigm of categorical logic we think of small categories $\mathcal{C}$ as certain types of logical theories. The more structure $\mathcal{C}$ has, the more logical connectives one can express inside it. For example, regular categories correspond to the fragment of first-order logic which allows the connectives $\wedge$, $\top$ and $\exists$ in the construction of formulae. Similarly, under this correspondence between categories and theories, the structure preserving functors $\mathcal{C} \to \mathsf{Set}$ are models of the theory of $\mathcal{C}$. More generally, regular functors $F\colon \mathcal{C} \to \mathcal{D}$ are interpretations of one regular theory into another.

Now a natural way in which one could ask if conceptual completeness holds for regular theories would be the following: given a morphism $F\colon \mathcal{C} \to \mathcal{D}$ between regular categories, if the induced functor between categories of (Set-valued) models

$$F^*\colon \mathsf{REG}(\mathcal{D}, \mathsf{Set}) \to \mathsf{REG}(\mathcal{C}, \mathsf{Set})$$

is an equivalence, does it follow that F itself must be an equivalence?

The answer to the question in this form is, however, clearly negative. Indeed, for any regular category $\mathcal{C}$ the canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ induces an equivalence $- \cdot \Gamma\colon \mathsf{REG}(\mathcal{C}_{ex/reg}, \mathsf{Set}) \to \mathsf{REG}(\mathcal{C}, \mathsf{Set})$ simply by the universal property of the exact completion and the fact that Set is an exact category. Nevertheless, it becomes positive if one moves from the regular categories $\mathcal{C}$, $\mathcal{D}$ to their respective exact completions. Precisely, if $F\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence between the categories of models, then $F_{ex/reg}\colon \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$ is an equivalence. It is then this result which is referred to as "conceptual completeness" for regular logic and it is an immediate consequence of the work of Makkai in [72]. The argument given there, however, is heavily model-theoretic and non-constructive. The main contribution of [8], which we would now like to present, is a purely categorical and intuitionistically valid argument for conceptual completeness of regular logic, exploiting the results of the previous section.

3.2.1 Remark. The appearance of the exact completion here might at first seem peculiar, but it turns out that it corresponds to something familiar to logicians. If $\mathbb{T}$ is a regular theory and $\mathcal{C}_{\mathbb{T}}$ is its syntactic category, then $(\mathcal{C}_{\mathbb{T}})_{ex/reg}$ is the syntactic category of the theory $\mathbb{T}^{eq}$, introduced by S. Shelah, which is obtained by adding new sorts for definable equivalence relations.

Our strategy will be to deduce the result from the conceptual completeness of *coherent* logic. The latter is the fragment of first-order logic obtained by adding the connective $\vee$ to regular logic. While the natural categorical analogue here is that of coherent category, we similarly have that the corresponding conceptual completeness theorem requires a completion step, so that in this case it is stated in terms of pretoposes. We briefly recall the associated definitions.

3.2.2 Definition. A category $\mathcal{C}$ is called *coherent* if it is regular and it admits finite unions of subobjects which are stable under pullback.

Any regular category which has finite coproducts which are *universal* (i.e. stable under pullback) is automatically coherent. Indeed, given subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$, the union $A \cup B$ can be constructed as the image of the morphism $(a, b)\colon A \sqcup B \to X$ in $\mathcal{C}$. A category with finite coproducts which are both universal and disjoint is called extensive [29].

3.2.3 Definition. A category which is both exact and extensive is called a *pretopos*.

It was known since [73] that whenever a morphism of pretoposes $F\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence

$$F^*\colon \mathsf{PRETOP}(\mathcal{D}, \mathsf{Set}) \to \mathsf{PRETOP}(\mathcal{C}, \mathsf{Set})$$

between categories of models, then F itself is an equivalence. The proof was model-theoretic, by an argument involving the compactness theorem and the

method of diagrams. An improvement was later provided by A. Pitts in [84], where a categorical proof was given which moreover is valid over any base topos with a natural number object, but when allowing models to take values in a *suitable class of toposes* and equivalence to mean a fully faithful functor which is essentially surjective on objects.

Now given any regular category $\mathcal{C}$, there exists a category $P(\mathcal{C})$ which is the *pretopos completion* of $\mathcal{C}$. The defining property of this completions is given by the equivalence $\mathsf{REG}(\mathcal{C},\mathcal{E}) \simeq \mathsf{PRETOP}(P(\mathcal{C}),\mathcal{E})$, for every pretopos $\mathcal{E}$. We denote by $\eta_{\mathcal{C}}\colon \mathcal{C} \to P(\mathcal{C})$ the embedding of $\mathcal{C}$ into its completion. Then the first observation we make is the following.

3.2.4 Lemma. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a regular functor such that the induced functor $P(F)\colon P(\mathcal{C}) \to P(\mathcal{D})$ between the associated pretopos completions is covering, then F itself is covering.*

Proof. Any $D \in \mathcal{D}$ is taken by $\eta_{\mathcal{D}}$ to an object that is covered by an object in the image of $P(F)$, $P(F)X \to \eta_{\mathcal{D}}D$. Such an X is constructed by objects in the image of $\eta_{\mathcal{C}}$ by taking finite unions of subobjects of objects in $\mathcal{C}$, coproducts and quotients of such objects. The functor $P(F)$ is a pretopos morphism and hence preserves all aforementioned constructions, whence $\eta_{\mathcal{D}}D$ is covered by an object that is coproduct of finite unions of subobjects of objects in the image of $P(F) \circ \eta_{\mathcal{C}} \cong \eta_{\mathcal{D}} \circ F$. But objects in the image of $\eta_{\mathcal{D}}$ are *super-compact* (see [58] D3.3.11), which implies that $\eta_{\mathcal{D}}D$ is covered by one of them, i.e. there is a regular epimorphism $FC \twoheadrightarrow D$. □

We finally come to the main result of this section.

3.2.5 Theorem. *(Conceptual Completeness for Regular Logic) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor such that, for all toposes $\mathcal{V}$ in a sufficient class (in the sense of [84], Definition 2.3), the induced functors between the categories of models*

$$- \cdot F\colon \mathsf{REG}(\mathcal{D},\mathcal{V}) \to \mathsf{REG}(\mathcal{C},\mathcal{V})$$

are equivalences. Then

$$F_{ex/reg} : \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$$

is an equivalence of categories.

Proof. Since we have $\mathsf{REG}(\mathcal{C},\mathcal{V}) \simeq \mathsf{PRETOP}(P(\mathcal{C}),\mathcal{V})$ and similarly for $\mathcal{D}$, we have that the pretopos morphism $P(F)$ induces an equivalence on models

$$- \circ P(F)\colon \mathsf{PRETOP}(P(\mathcal{D}),\mathcal{V}) \simeq \mathsf{PRETOP}(P(\mathcal{C}),\mathcal{V}),$$

which from the conceptual completeness result for pretoposes of A. Pitts ([84], Theorem 2.13) gives an equivalence $P(F)\colon P(\mathcal{C}) \to P(\mathcal{D})$. Now $P(F)$ is fully faithful, hence from the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \eta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{D}}} \\ P(\mathcal{C}) & \xrightarrow[P(F)]{} & P(\mathcal{D}) \end{array}$$

which commutes up to natural isomorphism, we get that F is fully faithful. By the previous lemma it is also covering, hence by Corollary 3.1.9 we have that $F_{ex/reg}$ is an equivalence. □

3.2.6 Remark. We should note that regular theories arise rather frequently in mathematics. In fact, their categorical treatment and the process of exact completion itself can be used to illuminate quite complicated constructions in areas such as Algebraic Geometry. For example, when applied to the regular theory of representations of quivers inside finitely presented modules over some ring R, it can be used to describe the category of motives of M. Nori (see [9] for details).

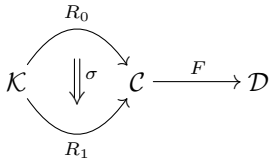
Furthermore, the intuitionistically valid nature of the conceptual completeness theorem proved above is also of some value. In particular, any such statement can be internalized to any elementary topos. For example, if R is an internal ring in a topos which is internally coherent, then the theory of flat R-modules is regular. This observation can noticeably simplify quite complicated proofs in Algebraic Geometry which involve sheaves of modules, by essentially reducing them to the case of plain modules over a ring. This and other examples can be viewed in [12].

3.3 Quotient Morphisms for Exact Categories and Pretoposes

In this section we present an application of our results from Section 3.1 to so-called *quotient morphisms* in categorical logic, once again drawn from [8] . While quotient morphisms in 2-categories such as REG, EX and PRETOP play a prominent role in the subject, the notions of quotient that appear in the literature on first-order categorical logic are usually defined in an ad hoc manner. Our purpose here is to give a categorical justification for these definitions, starting from an appropriate 2-categorical definition of quotient.

One such definition which is more suitable for our purposes is contained in [11]. The latter work falls into the much more general framework of Bourke & Garner [21], where various notions of regularity and exactness for 2-categories are considered. Adapting the general definition of [21], let us then define a 1-cell $F\colon \mathcal{C} \to \mathcal{D}$ in the 2-category REG (resp. EX and PRETOP) to be a *quotient* if

there is a diagram

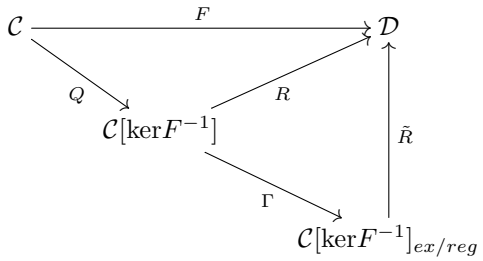


so that $F\sigma$ is invertible and F is universal with this property. Then by results of [11] we know that every regular functor factors in REG as a quotient morphism followed by a conservative one. Specifically, if one denotes by $\ker F^{-1}$ the class of morphisms which are mapped to isomorphisms by F, then the factorization has as interpolating object the *category of fractions* $\mathcal{C}[\ker F^{-1}]$ (see [39]). The latter category as well as the canonical functor $Q\colon \mathcal{C} \to \mathcal{C}[\ker F^{-1}]$ are regular precisely because the class $\ker F$ is in this case a *regular congruence* in the sense of [11].

In order to obtain the corresponding factorization in EX, it suffices to consider the factorization in REG and apply the exact completion to the category of fractions described above. This allows us then to characterize the quotient morphisms in EX.

3.3.1 Proposition. *Let $F\colon \mathcal{C} \to \mathcal{D} \in \mathsf{EX}$ be a regular functor which is covering and full on subobjects. Then the conservative part $R\colon \mathcal{C}[ker F^{-1}] \to \mathcal{D}$ of its quotient-conservative factorization in REG is fully faithful and covering, so that the induced regular functor $\tilde{R}\colon \mathcal{C}[ker F^{-1}]_{ex/reg} \to \mathcal{D}$ is an equivalence.*

Proof. Consider the factorization of F in EX



which is obtained by first taking its factorization $F \cong R \circ Q$ in REG. We have that R is faithful, since it is a conservative functor that preserves equalizers. It is also covering, because F is covering by assumption.

Finally, to show that R is full, consider any morphism $g\colon RX \to RY \in \mathcal{D}$. Since $RX = FX$, $RY = FY$ and F is full on subobjects, there is a relation $\phi\colon X \nrightarrow Y$ in $\mathcal{C}$ such that $F(\phi) = g$. The fact that R is faithful implies that it reflects inclusions of subobjects and hence $Q(\phi)$ is a morphism $f\colon X \to Y \in \mathcal{C}[\ker F^{-1}]$. Then clearly $Rf = g$.

Since R is a fully faithful and covering regular functor, by Theorem 3.1.8 we deduce that $\tilde{R}$ is an equivalence of categories. □

3.3.2 Corollary. *A 1-cell $F\colon \mathcal{C} \to \mathcal{D}$ in* EX *is a quotient if and only if it is covering and full on subobjects.*

One can now use similar arguments to extend this characterization to one of quotient morphisms in PRETOP.

To this end, let us recall further from [71, Proposition 2.4.7] that, if $F\colon \mathcal{C} \to \mathcal{D}$ is a morphism of pretoposes, then the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[\mathrm{ker}F^{-1}]$ gives the quotient part of its factorization in the 2-category COH of coherent categories and that the quotient part of its factorization in PRETOP is obtained as the composition

$$\mathcal{C} \to \mathcal{C}[\mathrm{ker}F^{-1}] \to P(\mathcal{C}[\mathrm{ker}F^{-1}]),$$

where $P(\mathcal{C}[\mathrm{ker}F^{-1}])$ denotes the pretopos freely generated by the coherent category $\mathcal{C}[\mathrm{ker}F^{-1}]$. In general, by [57] we know that the pretopos $P(\mathcal{C})$ freely generated by a coherent category $\mathcal{C}$ is given by first obtaining its *positivization* and then its exact completion as a regular category. The observation we shall make is that in our particular setting the category $\mathcal{C}[\mathrm{ker}F^{-1}]$ is already extensive (hence positive), so that consequently the quotient part of the factorization of F in PRETOP arises more simply as

$$\mathcal{C} \to \mathcal{C}[\mathrm{ker}F^{-1}] \to \mathcal{C}[\mathrm{ker}F^{-1}]_{ex/reg}.$$

In particular, it will follow that $\mathcal{C}[\mathrm{ker}F^{-1}]_{ex/reg}$ is extensive and $\Gamma\colon \mathcal{C}[\mathrm{ker}F^{-1}] \to \mathcal{C}[\mathrm{ker}F^{-1}]_{ex/reg}$ preserves coproducts by [67, Theorem 2.3].

We first need an observation on the behavior of coproducts in categories of fractions. The following is an immediate corollary of more general results concerning reflexive coinverters given in [63], but we nevertheless indicate here a more direct proof.

3.3.3 Proposition. *Let $\mathcal{C}$ be a category with finite coproducts and Σ a class of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Assume that, for all $s\colon A \to B$, $t\colon C \to D$ in Σ we have that $s \sqcup t\colon A \sqcup C \to B \sqcup D$ is in Σ. Then the category of fractions $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $P_\Sigma\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves them.*

Proof. Given objects A, B in $\mathcal{C}[\Sigma^{-1}]$, their coproduct is given by $A \sqcup B$, because if $A \xleftarrow{s} I \xrightarrow{f} C$ and $B \xleftarrow{t} J \xrightarrow{g} C$ are two arrows from A to C and from B to C, respectively, in $\mathcal{C}[\Sigma^{-1}]$, then it is not hard to see that the fraction $A \sqcup B \xleftarrow{s \sqcup t} I \sqcup J \xrightarrow{[f,g]} C$ is the required unique factorization through $A \sqcup B$ making the two triangles commutative. □

In particular, we have

3.3.4 Corollary. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a morphism of pretoposes then the category of fractions $\mathcal{C}[\mathrm{ker}F^{-1}]$ is regular extensive and the quotient functor $Q\colon \mathcal{C} \to \mathcal{C}[\mathrm{ker}F^{-1}]$ is regular and preserves finite coproducts.*

Proof. The class $\ker F^{-1}$ is a regular congruence which moreover, because F preserves finite coproducts, has the property of the previous proposition. Thus, $\mathcal{C}[\ker F^{-1}]$ has finite coproducts which are preserved by Q. Furthermore, since Q preserves pullbacks, $\mathcal{C}[\ker F^{-1}]$ inherits extensivity from $\mathcal{C}$. □

Finally, we note that $\tilde{R}$ also preserves finite coproducts (essentially by [67, Proposition 2.5]), because $R\colon \mathcal{C}[\ker F^{-1}] \to \mathcal{D}$ does, and hence it is indeed a morphism of pretoposes. We can thus conclude that

3.3.5 Corollary. *A 1-cell $F\colon \mathcal{C} \to \mathcal{D}$ in* PRETOP *is a quotient if and only if it is covering and full on subobjects.*

3.4 A Commutation Result for ex/reg and the Ind-completion

In the final section of this chapter, we present another application of Theorem 3.1.8 which shows that, under certain conditions, the exact completion of a regular category commutes with the Ind-completion.

Recall that in general $\mathrm{Ind}(\mathcal{C})$ denotes the completion of a category $\mathcal{C}$ under filtered colimits. In the case when $\mathcal{C}$ is small and has finite colimits, $\mathrm{Ind}(\mathcal{C})$ is equivalent to the category $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ of presheaves $F\colon \mathcal{C}^{op} \to \mathsf{Set}$ which preserve finite limits. We now recall the following condition which appeared in [20] as part of the characterization of those categories $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ which are exact. The terminology is taken from [59], where the same notion appeared in a different context.

3.4.1 Definition. Let $\mathcal{C}$ be a regular category. We say that $\mathcal{C}$ satisfies condition *BTH* (*bounded transitive hulls*) if, for every reflexive and symmetric relation $R \rightarrowtail X \times X$ in $\mathcal{C}$ the ascending chain

$$R \subset R \circ R \subset R \circ R \circ R \subset \ldots \subset R^n \subset \ldots$$

of iterated compositions stabilizes.

We first record the following lemma containing two easy observations involving the calculus of relations that will be used in the proof of the next result.

3.4.2 Lemma. *Let $f\colon X \twoheadrightarrow Y$ be a regular epimorphism in the regular category $\mathcal{C}$. Then for any relation $S \rightarrowtail Y \times Y$ we have:*

1. $f(f^{-1}(S)) = S$.
2. $f^{-1}(S \circ S) = f^{-1}(S) \circ f^{-1}(S)$.

Proof. Recall that in the calculus of relations in $\mathcal{C}$ we have $f^{-1}(S) = f^\circ S f$, for any morphism $f\colon X \to Y$. Similarly, for any relation $R \rightarrowtail X \times X$ the direct image $f(R)$ is given as $f(R) = fRf^\circ$. Now we simply calculate

1. $f(f^{-1}(S)) = f(f^\circ S f)f^\circ = ff^\circ Sff^\circ = \Delta_Y S \Delta_Y = S$.
2. $f^{-1}(S) \circ f^{-1}(S) = f^\circ Sff^\circ Sf = f^\circ S\Delta_Y Sf = f^\circ SSf = f^{-1}(SS)$. □

Now we make a small observation that will be needed shortly, concerning the passage of BTH between a regular category and its exact completion.

3.4.3 Lemma. *Let $\mathcal{C}$ be a regular category. Then $\mathcal{C}$ satisfies BTH if and only if $\mathcal{C}_{ex/reg}$ satisfies BTH.*

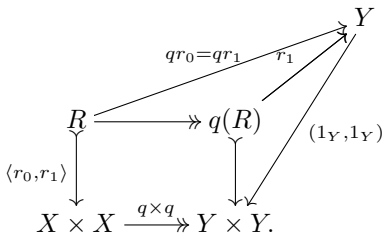
Proof. Suppose $\mathcal{C}$ satisfies BTH and consider a reflexive and symmetric relation $S \rightarrowtail Q \times Q$ in $\mathcal{C}_{ex/reg}$. There exists a regular epimorphism $q\colon X \twoheadrightarrow Q$ such that $X \in \mathcal{C}$. Now set $R \coloneqq q^{-1}(S)$. Since the inclusion of $\mathcal{C}$ into $\mathcal{C}_{ex/reg}$ is full on subobjects, we have that the relation R is in $\mathcal{C}$. Moreover, it is easy to see that R is reflexive and symmetric, since S is.

Then, by assumption, there exists an $n \in \mathbb{N}$ such that $R^m = R^n$ for all $m \geq n$. Now observe that, for any $m \in \mathbb{N}$ we have $R^m = (q^{-1}(S))^m = q^{-1}(S^m)$ and so $S^m = q(q^{-1}(S^m)) = q(R^m)$, both by the lemma above. This shows that $S^m = S^n$ for all $m \geq n$. □

We will also need some information about the existence of finite colimits in exact categories that satisfy BTH. First, we make a couple of easy observation, stated as lemmas.

3.4.4 Lemma. *Let $\mathcal{C}$ be a regular category with finite coproducts and R, S relations on an object X of $\mathcal{C}$. If $q\colon X \to Y$ coequalizes both R and S in $\mathcal{C}$, then q coequalizes the relation $R \cup S$.*

Proof. Observe that q coequalizes R if and only if $q(R) \subseteq \Delta_Y$:



Also, since the map

$$q(-)\colon \mathrm{Sub}_{\mathcal{C}}(X \times X) \to \mathrm{Sub}_{\mathcal{C}}(Y \times Y)$$

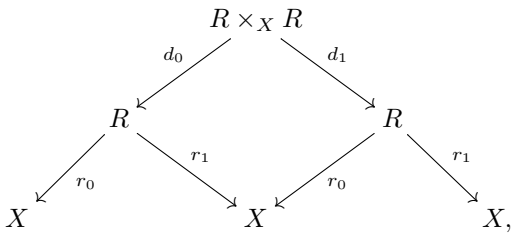
between the partially ordered sets of subobjects is a left adjoint, it preserves unions. Thus

$$q(R \cup S) = q(R) \cup q(S) \subseteq \Delta_Y \cup \Delta_Y = \Delta_Y,$$

hence q coequalizes $R \cup S$. □

3.4.5 Lemma. *Let $\mathcal{C}$ be a regular category with finite coproducts, R a relation on an object X and $q\colon X \to Y$ any arrow in $\mathcal{C}$. If q coequalizes R, then q coequalizes the relation $R \circ R$.*

Proof. If R is represented by $\langle r_0, r_1\rangle\colon R \rightarrowtail X \times X$ we construct the composite $R \circ R$ in the familiar way as the pullback in



$$\langle r_0 d_0, r_1 d_1\rangle = R \times_X R \xrightarrow{e} R \circ R \xrightarrowtail{\langle s_0, s_1\rangle} X \times X$$

Assuming that q coequalizes (r_0, r_1), we have

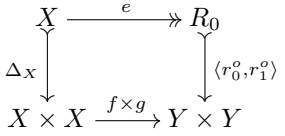
$$qs_0 e = qr_0 d_0 = qr_1 d_0 = qr_0 d_1 = qr_1 d_1 = qs_1 e,$$

so that cancelling out the regular epi e we get $qs_0 = qs_1$. □

3.4.6 Proposition. *Let $\mathcal{C}$ be a regular category with finite coproducts, satisfying the condition BTH. If $\mathcal{C}$ has coequalizers of equivalence relations then it has all finite colimits.*

Proof. It suffices to construct arbitrary coequalizers, so consider any pair of morphisms $f, g\colon X \to Y$ in $\mathcal{C}$. Let R_0 be the relation on Y which is the image of the morphism $\langle f, g\rangle\colon X \to Y \times Y$ (equivalently, $R_0 = (f \times g)(\Delta_X)$). Then set $R := R_0 \cup R_0^o \cup \Delta_Y$. The relation R is now reflexive and symmetric, hence for some $n \geq 1$, $R^{(n)}$ is an equivalence relation.

Consider the coequalizer $Y \xrightarrow{e} Q$ of this $R^{(n)}$. We claim that q is also the coequalizer of (f, g). To prove this it suffices to show that any $h\colon Y \to Z$ which coequalizes (f, g) also coequalizes the equivalence relation $R^{(n)}$. So assume that $hf = hg$. Then, with reference to the diagram below,



we have

$$hr_0^0 e = hf = hg = hr_1^o,$$

hence $hr_0^o = hr_1^o$. This means that h coequalizes R_0. This clearly implies that it also coequalizes R_0^o, while it trivially coequalizes Δ_X, so by Lemma 3.4.4 we have that h coequalizes R. Finally, by Lemma 3.4.5 we inductively deduce that h coequalizes $R^{(n)}$. □

If $\mathcal{C}$ is a small, regular category with finite colimits, then the category $\mathrm{Ind}(\mathcal{C}) = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ is regular. A proof of this can be found in [35]. In the next result, we show that similarly regularity of a functor between such categories is preserved by the Ind-completion. Note that the abelian case of this appears essentially as [91, Corollary 8.6.8].

3.4.7 Proposition. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor, where $\mathcal{C}$ and $\mathcal{D}$ are regular and finitely cocomplete. Then $\mathrm{Ind}(F)\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is also regular.*

Proof. Let $X \twoheadrightarrow Y$ be a regular epimorphism in $\mathrm{Ind}(\mathcal{C})$ with kernel pair $K \rightrightarrows X \twoheadrightarrow Y$. By the *uniformity lemma* of [34] applied to the diagram consisting of the morphism $X \twoheadrightarrow Y$, we find that there exists a filtered category $\mathcal{I}$ such that

$$\begin{array}{ccc} \operatorname*{colim}_{\mathcal{I}} A_i & \xrightarrow{(\phi_i)} & \operatorname*{colim}_{\mathcal{I}} B_i \\ \cong\downarrow & & \downarrow\cong \\ X & \twoheadrightarrow & Y \end{array}$$

where for each $i \in \mathcal{I}$ the arrow $\phi_i\colon A_i \to B_i$ is in the image of the embedding $y\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$. For each $i \in \mathcal{I}$ we consider the kernel pair K_i and the image Q_i of ϕ_i in $\mathcal{C}$.

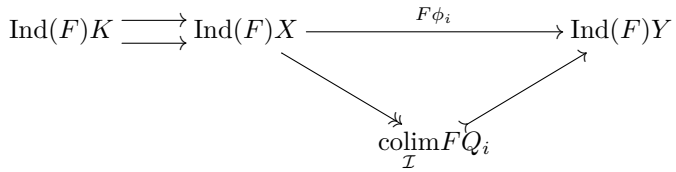
$$\begin{array}{ccccc} K_i & \rightrightarrows & A_i & \xrightarrow{\phi_i} & B_i \\ & & & \searrow & \nearrow \\ & & & Q_i & \end{array}$$

Note that $\mathcal{C}$ is closed under finite limits in $\mathrm{Ind}(\mathcal{C})$ and hence $K_i \rightrightarrows A_i$ is the kernel pair of $\phi_i\colon A_i \to B_i$ also in $\mathrm{Ind}(\mathcal{C})$. Then, because in the latter category finite limits commute with filtered colimits, we have that $K \coloneqq \operatorname*{colim}_{\mathcal{I}} K_i \rightrightarrows \operatorname*{colim}_{\mathcal{I}} A_i = X$ is the kernel pair of $X \twoheadrightarrow Y$ in $\mathrm{Ind}(\mathcal{C})$. Note here also that, since colimits commute with colimits, we have that $\operatorname*{colim}_{\mathcal{I}} A_i \twoheadrightarrow \operatorname*{colim}_{\mathcal{I}} Q_i$ is the coequalizer of $K \rightrightarrows A$. Thus, we deduce that $Y \cong \operatorname*{colim}_{\mathcal{I}} Q_i$.

Now by regularity of the functor F we know that the following diagram exhibits the kernel pair and (regular epi, mono) factorization of $F\phi_i$ in $\mathcal{D}$, hence also in $\mathrm{Ind}(\mathcal{D})$, for every $i \in \mathcal{I}$.

$$\begin{array}{ccccc} FK_i & \rightrightarrows & FA_i & \xrightarrow{F\phi_i} & FB_i \\ & & & \searrow & \nearrow \\ & & & FQ_i & \end{array}$$

Then, as above and after taking filtered colimits in $\mathrm{Ind}(\mathcal{D})$ and recalling that $\mathrm{Ind}(F)$ preserves these colimits, we obtain a diagram of the same type



But now we observe that $\operatorname{colim}_{\mathcal{I}} FQ_i \cong \operatorname{Ind}(F)(\operatorname{colim}_{\mathcal{I}} Q_i) \cong \operatorname{Ind}(F)Y$, because $\operatorname{Ind}(F)$ preserves filtered colimits and $Y \cong \operatorname{colim}_{\mathcal{I}} Q_i$. This proves that the morphism $\operatorname{Ind}(F)X \to \operatorname{Ind}(F)Y$ is the coequalizer of $\operatorname{Ind}(F)K \rightrightarrows \operatorname{Ind}(F)X$ and concludes the proof. □

Next, we determine conditions on $F\colon \mathcal{C} \to \mathcal{D}$ under which the induced $\operatorname{Ind}(F)\colon \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{D})$ is a covering regular functor.

3.4.8 Proposition. *Let $\mathcal{C}$,$\mathcal{D}$ be small, regular and finitely cocomplete categories and $F : \mathcal{C} \to \mathcal{D}$ a regular functor which preserves finite coproducts. If F is covering, then so is* $\operatorname{Ind}(F)\colon \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{D})$.

Proof. Consider an object $Y \in \operatorname{Ind}(\mathcal{D})$. Write it as a filtered colimit $Y = \operatorname{colim}_{\mathcal{J}} D_j$, where each D_j is in $\mathcal{D}$. Since F is covering, for each $j \in \mathcal{J}$ there exists an object $C_j \in \mathcal{C}$ and a regular epimorphism $FC_j \twoheadrightarrow D_j$ in $\mathcal{D}$. Then we can take the coproduct of these regular epimorphisms in $\operatorname{Ind}(\mathcal{D})$ to get another regular epimorphism $\bigsqcup_{j\in\mathcal{J}} FC_j \twoheadrightarrow \bigsqcup_{j\in\mathcal{J}} D_j$. Note also that, by the construction of colimits via coproducts and coequalizers, there exists a regular epimorphism $\bigsqcup_{j\in\mathcal{J}} D_j \twoheadrightarrow \operatorname{colim}_{\mathcal{J}} D_j$. Hence, by composition we obtain a regular epimorphism $\bigsqcup_{j\in\mathcal{J}} FC_j \twoheadrightarrow \operatorname{colim}_{\mathcal{J}} D_j = Y$.

Now we can write the coproduct $\bigsqcup_{j\in\mathcal{J}} FC_j$ as a filtered colimit of finite coproducts, $\bigsqcup_{j\in\mathcal{J}} FC_j \cong \operatorname*{colim}_{I\subseteq\mathcal{J}_0, I\text{finite}} (\bigsqcup_{j\in I} FC_j)$. But since F preserves finite coproducts, $\operatorname*{colim}_{I\subseteq\mathcal{J}_0, I\text{finite}} (\bigsqcup_{j\in I} FC_j) \cong \operatorname*{colim}_{I\subseteq\mathcal{J}_0, I\text{finite}} F(\bigsqcup_{j\in I} C_j) \cong \operatorname{Ind}(F)(\operatorname*{colim}_{I\subseteq\mathcal{J}_0, I\text{finite}} \bigsqcup_{j\in I} C_j)$. □

We can now obtain the following commutation result between the ex/reg and Ind completions. We need to assume the preservation of finite coproducts by the canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$, because this does not seem to be true in general. However, this assumption still includes interesting classes of examples, such as when $\mathcal{C}$ is either extensive or additive.

3.4.9 Corollary. *Let $\mathcal{C}$ be a small regular, finitely cocomplete category with BTH. If the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}_{ex/reg}$ preserves finite coproducts, then there is an equivalence of categories* $\operatorname{Ind}(\mathcal{C})_{ex/reg} \simeq \operatorname{Ind}(\mathcal{C}_{ex/reg})$.

Proof. The canonical functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{ex/reg}$ is regular, covering and preserves finite coproducts. By the preceding results of this section we have that the induced $\mathrm{Ind}(\Gamma)\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}_{ex/reg})$ is a covering regular functor and furthermore that $\mathcal{C}_{ex/reg}$ is finitely cocomplete and satisfies BTH. We can then deduce that the category $\mathrm{Ind}(\mathcal{C}_{ex/reg})$ is exact by [20, Theorem 1.3]. It is also easy to see that Γ being fully faithful implies the same property for $\mathrm{Ind}(\Gamma)$. The result now follows from Theorem 3.1.8. □

Chapter 4

Exact Completion of Regular Categories Enriched in Posets

In recent years there has been a notable interest in the literature in pursuing the categorical treatment of ordered algebras. The term "ordered algebra" here will always be used in the sense of Bloom & Wright [14, 15], i.e. it essentially refers to Σ-algebras of Universal Algebra with an underlying poset structure with respect to which all operations $\sigma \in \Sigma$ must be monotone. This pursuit leads naturally to the realm of categories which are enriched in the cartesian closed category Pos of posets and monotone functions. An important starting point in the development of a categorical ordered universal algebra is the paper [65] of A. Kurz & J.Velebil. Therein, the authors consider notions of regularity and Barr-exactness in the context of categories enriched in Pos. They then use these notions to obtain categorical characterizations of quasivarieties and varieties of ordered algebras, results which bare obvious similarities to the corresponding characterizations of ordinary (quasi)varieties. In fact, ordered quasivarieties are precisely those categories which are regular in the Pos-enriched sense and admit a sufficiently "nice" generator, while varieties are characterized in a similar manner by simply replacing regularity with the stronger exactness. On the other hand, more recently various authors have studied monads on Pos and their relationship to varieties of ordered algebras [2, 3].

In this chapter we aim to contribute to this line of research by providing a construction of the exact completion of a regular category in the context of categories enriched in Pos. More accurately, we want to describe a construction which employs the calculus of relations and is similar to the ordinary case as recalled in Section 1.5. This similarity will allow us to deduce the obvious enriched versions of the results in Section 3.1 by almost copying the arguments given there for the ordinary case. We can then apply these to deduce interesting

examples of categories which are exact completions of regular ones. Many of these involve varieties of ordered algebras which can be identified as the exact completion of corresponding quasivarieties. However, we also have an interesting example involving ordered topological spaces. Namely, we show that the exact completion of the category of *Priestley* spaces [85,86] is the category of *compact ordered* spaces of L. Nachbin [80]. Finally, we also examine the interactions between the ordinary and enriched exact completions and categories of internal posets in ordinary categories. The results of this chapter are the subject of the paper [7].

4.1 Poset-enriched Categories and Weighted Limits

Throughout this chapter we will be working with categories $\mathcal{C}$ which are enriched in the cartesian closed category $\mathcal{V} = \mathsf{Pos}$ of posets and order-preserving functions. Thus, our categories will generally be $\mathcal{V}$-categories and similarly our functors will be $\mathcal{V}$-functors in the sense of [62], for $\mathcal{V} = \mathsf{Pos}$. In more concrete terms this means the following:

- A Pos-category $\mathcal{C}$ is an ordinary category such that for every two objects $X, Y \in \mathcal{C}$ the morphisms $\mathcal{C}(X, Y)$ form a poset and such that composition on either side preserves the order of morphisms, i.e. the inequality $f \leq f'$ implies $gfh \leq g'f'h'$, whenever these compositions make sense.
- A Pos-functor $F\colon \mathcal{C} \to \mathcal{D}$ is an ordinary functor which also preserves the order of morphisms, i.e. $f \leq g$ in $\mathcal{C}(X, Y)$ implies $Ff \leq Fg$ in $\mathcal{D}(FX, FY)$.

Convention: Throughout this chapter the terms "category" and "functor" will always respectively mean Pos-category and Pos-functor. If we wish to refer to the non-enriched notions, we will explicitly adjoin the adjective "ordinary".

4.1.1 Example. 1. Besides Pos itself, we can consider the category Set of sets and functions with the hom-sets equipped with the discrete order. More generally, any ordinary (locally small) category $\mathcal{C}$ can be considered as a category with the order on every $\mathcal{C}(X, Y)$ being the discrete one. We will refer to such examples as *locally discrete* categories.

2. Any category of ordered algebras in the sense of Bloom & Wright is a category in our sense.

3. Given any category $\mathcal{C}$, we also have the *opposite* category $\mathcal{C}^{op}$, where we reverse the direction of morphisms and the order stays the same.

4. Given categories $\mathcal{C}$ and $\mathcal{D}$, we have a category $[\mathcal{C}, \mathcal{D}]$ of functors and natural transformations, where the order relation is defined component-wise. In particular, for every category $\mathcal{C}$ we have functor categories $[\mathcal{C}^{op}, \mathsf{Pos}]$ and $[\mathcal{C}, \mathsf{Pos}]$.

When working in enriched category theory, the more appropriate notion of (co)limit is that of *weighted* (co)limit (also called *indexed* (co-)limits in [62]). In this context, the limit (resp. colimit) of a *diagram* $D\colon \mathcal{D} \to \mathcal{C}$ is defined with respect also to a fixed functor $W\colon \mathcal{D} \to \mathsf{Pos}$ (resp. $W\colon \mathcal{D}^{op} \to \mathsf{Pos}$), called a *weight*. More precisely, these are defined as follows:

- The *limit* of the diagram $D\colon \mathcal{D} \to \mathcal{C}$ weighted by $W\colon \mathcal{D} \to \mathsf{Pos}$ is an object $\{W, D\}$ with an isomorphism

$$\mathcal{C}(X, \{W, D\}) \cong [\mathcal{D}, \mathsf{Pos}]^{op}(\mathcal{C}(X, D-), W)$$

  which is natural in X.

- The *colimit* of the diagram $D\colon \mathcal{D} \to \mathcal{C}$ weighted by $W\colon \mathcal{D}^{op} \to \mathsf{Pos}$ is an object $W \star D$ with an isomorphism

$$\mathcal{C}(W \star D, X) \cong [\mathcal{D}^{op}, \mathsf{Pos}](W, \mathcal{C}(D-, X))$$

  which is natural in X.

A category $\mathcal{C}$ is said to be *complete* if the weighted limit $\{W, D\}$ exists for every diagram $D\colon \mathcal{D} \to \mathcal{C}$ with $\mathcal{D}$ a small category and every weight $W\colon \mathcal{D} \to \mathsf{Pos}$. Dually, we have the notion of *cocomplete* category.

Let us now look at some particular classes of limits and colimits which will be used throughout this chapter.

First, suppose that the weight $W\colon \mathcal{D} \to \mathsf{Pos}$ is taken to be the constant functor $\Delta 1\colon \mathcal{D} \to \mathsf{Pos}$ on the one element poset. Then we say that the limit $\{\Delta 1, D\}$ is *conical* and often denote it simply by $\lim D$. Note that it is defined by the natural isomorphism $\mathcal{C}(X, \lim D) \cong [\mathcal{D}, \mathcal{C}](\Delta X, D)$.

The conical limits are in some sense the incarnation of the limits of ordinary category theory in enriched category theory. However, their universal property is not simply the same as in the ordinary case. To see this, let us look at the example of conical limit which is the *product* $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ of two objects X, Y in a category $\mathcal{C}$. The universal property which defines this limit amounts to the following two requirements:

1. Given any $f\colon Z \to X$ and $g\colon Z \to Y$ in $\mathcal{C}$, there exists a unique $h\colon Z \to X \times Y$ such that $\pi_X h = f$ and $\pi_Y h = g$.

2. For every $h, h'\colon Z \to X \times Y \in \mathcal{C}$, if both $\pi_X h \leq \pi_X h'$ and $\pi_Y h \leq \pi_Y h'$ hold, then $h \leq h'$.

In other words, along with the "ordinary" unique factorization property, there is a 2-dimensional part of the universal property. This amounts to strengthening the requirement that π_X, π_Y be jointly monomorphic to the statement that they jointly reflect the order of morphisms, which essentially follows from the fact that the defining isomorphism $\mathcal{C}(Z, X \times Y) \cong \mathcal{C}(Z, X) \times \mathcal{C}(Z, Y)$ lives in Pos as

opposed to Set. Similar remarks apply to other conical limits such as *equalizers* and *pullbacks*.

A second important type of limit is the *power* (also called *cotensor*) of an object $C \in \mathcal{C}$ to a poset $P \in \mathsf{Pos}$. This is an object C^P defined by the natural isomorphism

$$\mathcal{C}(X, C^P) \cong \mathrm{Hom}_{\mathsf{Pos}}(\mathrm{P}, \mathcal{C}(\mathrm{X}, \mathrm{C}))$$

For example, if $\mathcal{C} = \mathsf{Pos}$, then C^P is the poset of order-preserving functions $P \to C$. The dual notion is that of *copower* (or *tensor*) and is denoted by $P \bullet C$.

An important fact concerning conical (co)limits and (co)powers is that they can be used together to construct all other (co)limits. This is the content of the following well known result.

4.1.2 Theorem. *[62] A category $\mathcal{C}$ is (co)complete if and only if it has all small conical (co)limits and all (co)powers. A functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves all limits if and only if it preserves all small conical (co)limits and (co)powers.*

The *comma object* of an ordered pair of morphisms $\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$ with common codomain is a square

$$\begin{array}{ccc} C & \xrightarrow{c_1} & Y \\ {\scriptstyle c_0}\downarrow & \leq & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

such that $fc_0 \leq gc_1$ and which is universal with this property, in the following sense:

1. Given $u_0 : W \to X$ and $u_1 \colon W \to Y$ in $\mathcal{C}$ such that $fu_0 \leq gu_1$, there exists a $u\colon W \to C \in \mathcal{C}$ such that $c_0 u = u_0$ and $c_1 u = u_1$.
2. If $h, h' \colon A \to C$ are such that $c_0 h \leq c_0 h'$ and $c_1 h \leq c_1 h'$, then $h \leq h'$.

Note in particular that the factorization given in (1) will be unique by (2). We will usually denote the comma object C by f/g. In Pos or more generally in any category of ordered algebras, the comma object is given by $f/g = \{(x, y) \in X \times Y | f(x) \leq g(y)\}$ with the order induced from the product.

The *inserter* of an ordered pair (f, g) of parallel morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a morphism $e\colon E \to X \in \mathcal{C}$ such that $fe \leq ge$ and universal in the following sense:

1. If $h\colon Z \to X \in \mathcal{C}$ is such that $fh \leq gh$, then there exists a $u\colon Z \to E$ such that $eu = h$.

2. For any $u, v\colon Z \to E$, if $eu \leq ev$, then $u \leq v$.

Again, note that the factorization posited in (1) is unique by property (2). In Pos and in categories of ordered algebras, the inserter is precisely $E = \{x \in X | f(x) \leq g(x)\}$ with the order induced from X.

Since it will play a significant role later, let us also explicitly mention here the dual notion to the above. The *coinserter* of an ordered pair (f, g) of parallel morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a morphism $q\colon Y \to Q \in \mathcal{C}$ such that $qf \leq qg$ and universal in the following sense:

1. If $h\colon Y \to Z \in \mathcal{C}$ is such that $hf \leq hg$, then there exists a $u\colon Q \to Z$ such that $uq = h$.

2. For every $u, v\colon Q \to Z$, if $uq \leq vq$, then $u \leq v$.

The construction of coinserters is in general slightly more complicated than that of inserters, just as in the ordinary context coequalizers are more complicated than equalizers in Set. For example, in Pos the coinserter $q\colon Y \to Q$ in the above notation is constructed as follows:

- First, define a relation R on Y by specifying that $(y, y') \in R$ if and only if there is a finite sequence of elements $x_1, ..., x_n \in X$ such that the following inequalities hold in Y:

  $$y \leq f(x_1), \;\; g(x_1) \leq f(x_2), \;\; g(x_2) \leq f(x_3), \;\; \ldots \;\; , g(x_n) \leq y'$$

  Then it is easy to see that R is a preorder (i.e. reflexive and transitive) relation on Y.

- Then define Q to be the quotient $Y/\sim$, where $\sim := R \cap R^{\circ}$ is the associated equivalence relation. The partial order on Q is the one canonically induced by R and $q\colon Y \to Q$ is the canonical quotient map.

The last three examples of (co)limits which we mentioned are instances of what one might reasonably like to call *finite* (co)limits. It turns out there is a way that this can actually be formulated precisely.

When the base of enrichment $\mathcal{V}$ is locally finitely presentable as a monoidal category in the sense of [60], as in our case with Pos, there is a useful notion of *finite* weighted limit. Recall then that a limit $\{W, D\}$ is called finite if the weight $W\colon \mathcal{D} \to \mathcal{C}$ satisfies the following:

1. $\mathcal{D}$ has finitely many (isomorphism classes of) objects.

2. $\mathcal{D}(d, d')$ is a finitely presentable object in $\mathcal{V}$, for all $d, d' \in \mathcal{D}$.

3. Wd is a finitely presentable object in $\mathcal{V}$, for all $d \in \mathcal{D}$.

Since the finitely presentable objects in $\mathcal{V} = \mathsf{Pos}$ are precisely the finite posets, conditions (2) and (3) in our context simply say that $\mathcal{D}(d, d')$ and Wd are finite posets for all $d, d' \in \mathcal{D}$.

Then by the work of Kelly [60] we have the following construction theorem for finite (co)limits. By *finite power* here we mean a power object X^P where P is a finitely presentable object in Pos, i.e. a finite poset. Similarly for finite copowers.

4.1.3 Theorem. *A category $\mathcal{C}$ is finitely (co)complete if and only if it has all finite conical (co)limits and all finite (co)powers. A functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves all limits if and only if it preserves all small conical (co)limits and (co)powers.*

Despite the above result, it will be convenient in our subsequent work to have an alternative way of constructing all finite weighted limits using inserters along with some conical limits. This is then the content of the following proposition, which is probably part of the folklore and in any case follows from specializing known facts about 2-categorical limits. We nevertheless include a proof for the sake of completeness.

4.1.4 Proposition. *A category $\mathcal{C}$ is finitely complete if and only if it has finite products and inserters.*

Proof. Suppose that $\mathcal{C}$ has finite products and inserters. To have all finite conical limits it suffices to construct equalizers. So consider any parallel pair of morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{C}$. Let $m\colon M \to X$ be the inserter of the pair (f, g) and then let $n : E \to M$ be the inserter of (gm, fm). We claim that now $e \coloneqq mn\colon E \to X$ is the desired equalizer. Indeed, note first that $gmn \leq fmn$ and also $fm \leq gm \implies fmn \leq gmn$, so that $fmn = gmn$. Then suppose that $h\colon Z \to X$ is such that $fh = gh$. Since in particular $fh \leq gh$, there exists a unique $u\colon Z \to M$ such that $mu = h$. Now u is such that $gmu = gh \leq fh = fmu$, so we have a unique $v\colon Z \to E$ such that $nv = u$. So $ev = mnv = mu = h$. Finally, it is clear that e is order-monic, since both m, n are so.

Second, we need to construct finite powers, so let P be a finite poset and consider any $X \in \mathcal{C}$. Consider the product $\prod_{a \in P} X$ (i.e. the ordinary power) and for every pair of elements $a, b \in P$ with $a \leq b$ form the inserter of (π_a, π_b), say $E_{ab} \overset{e_{ab}}{\rightarrowtail} \prod_{a \in P} X \underset{\pi_b}{\overset{\pi_a}{\rightrightarrows}} X$ in $\mathcal{C}$. Set $E \coloneqq \prod_{a \leq b} E_{ab}$. We claim the E is the power X^P.

To show this, consider any family of morphisms $(f_a\colon C \to X)_{a \in P}$ with $a \leq b \implies f_a \leq f_b$, i.e. a homomorphism $P \to \mathcal{C}(C, X)$ in Pos. There is then a unique $f\colon C \to \prod_{a \in P} X$ such that $\pi_a f = f_a$ for all $a \in P$. Now for each pair of elements $a, b \in P$ with $a \leq b$ we have $f_a \leq f_b$, which is to say $\pi_a f \leq \pi_b f$.

Hence, there is a unique $u_{ab}\colon C \to E_{ab}$ with $e_{ab}u_{ab} = f$. This in turn induces a unique $u\colon C \to E$ such that $\pi_{ab}u = u_{ab}$ whenever $a, b \in P$ with $a \leq b$, where $\pi_{ab}\colon E \to E_{ab}$ denotes the corresponding projection from the product.

Finally, let's show that this assignment is order-preserving and order reflecting. So consider another family $(g_a\colon C \to X)_{a\in P}$, with $g\colon C \to \prod_{a\in P} X$ and $v\colon C \to E$ corresponding to f and u as defined above for $(f_a)_{a\in P}$. If $(f_a)_{a\in P} \leq (g_a)_{a\in P}$, then $f_a \leq g_a$ for all $a \in P$, i.e. $\pi_a f \leq \pi_a g$ for all $a \in P$ and hence $f \leq g$ by the universal property of the product. This in turn means that whenever $a \leq b$ we have $e_{ab}u_{ab} = f \leq g = e_{ab}v_{ab}$ and so $u_{ab} \leq v_{ab}$, hence $u \leq v$. It is clear that these implications can also be reversed. □

4.2 Regularity for Pos-enriched Categories

In this section we recall the notion of regularity for Pos-enriched categories by mostly following Kurz & Velebil [65]. However, as the first contribution of this chapter, we also present a simplification of their definition. Indeed, by straightforwardly adapting arguments from the context of ordinary categories, we observe that one of the four conditions in the initial definition follows for free from the remaining three.

We begin by recalling some basic notions, most of which can also be found in [65]. First, the notion of monomorphism that is more appropriate in the ordered context and which will form part of the factorization system leading to the notion of regularity for Pos-categories.

4.2.1 Definition. A morphism $m\colon X \to Y$ in a category $\mathcal{C}$ is called an ff-*morphism* (or *representably fully faithful*, or an *order-monomorphism*) if for every $Z \in \mathcal{C}$ the monotone map $\mathcal{C}(Z, m)\colon \mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$ in Pos also reflects the order.

We shall use the terms "ff-morphism" and "order-monomorphism" interchangeably throughout the text. Furthermore, we will use the term "order-epimorphism" for the dual notion.

For future reference, we record here a few very basic facts concerning ff-morphisms. These are entirely analogous to properties of monomorphisms in ordinary categories.

4.2.2 Lemma. *Consider morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in a category $\mathcal{C}$. Then:*

1. *If f, g are* ff*-morphisms, then so is gf.*
2. *If gf is an* ff*-morphism, then so is f.*
3. ff*-morphisms are stable under pullback.*

Proof. Perhaps only item (3) needs some details, so consider the following pullback square in $\mathcal{C}$ where f is an ff-morphism and assume that $u, v\colon A \to P$ are such that $qu \leq qv$.

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

We then have $gqu \leq gqv \implies fpu \leq fpv \implies pu \leq pv$, since f is an ff-morphism. Then, because we have both $pu \leq pv$ and $qu \leq qv$, we conclude by the limit property of the pullback that $u \leq v$. □

If a category $\mathcal{C}$ has comma objects, then in particular for any morphism $f\colon X \to Y \in \mathcal{C}$ we can form the comma of the pair (f, f). This comma measures the extent to which f fails to be an ff-morphism. Furthermore, it will ultimately be connected with the notions of regularity and exactness to be introduced shortly.

4.2.3 Definition. Given any morphism $f\colon X \to Y$ in a category $\mathcal{C}$, the comma object f/f is called the *kernel congruence* of f.

4.2.4 Lemma. *For any morphism $f\colon X \to Y$ with kernel congruence $f/f \overset{f_0}{\underset{f_1}{\rightrightarrows}} X$ in a category $\mathcal{C}$, the following are equivalent:*

1. *f is an ff-morphism.*
2. *$f_0 \leq f_1$.*
3. *The canonical morphism $\iota_f\colon 1_X/1_X \to f/f$ is an iso.*

Proof. If f is an ff-morphism, then $ff_0 \leq ff_1 \implies f_0 \leq f_1$.

If $f_0 \leq f_1$, then $1_X f_0 \leq 1_X f_1$ implies that (f_0, f_1) must factor through $1_X/1_X$ via a morphism which is then easily seen to be inverse to $1_X/1_X \to f/f$.

Finally, assume that $f/f \cong 1_X/1_X$ and let $u_0, u_1\colon Z \to X$ be such that $fu_0 \leq fu_1$. Then (u_0, u_1) factors through f/f and so through $1_X/1_X$. But the latter means precisely that $u_0 \leq u_1$. □

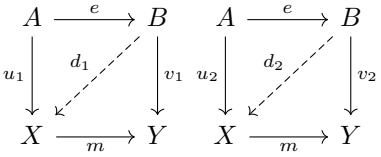
As we have mentioned already, the class of ff-morphisms will be the "mono part" of a factorization system for regular categories. The other class of morphisms is taken to be the class of morphisms orthogonal to all ff-morphisms, as has to be the case in any orthogonal factorization system. However, as with everything so far, the notion of orthogonality is in a suitable enriched sense.

4.2.5 Definition. Given morphisms $e\colon A \to B$ and $m\colon X \to Y$ in a category $\mathcal{C}$, we say that e is *left orthogonal* to m and write $e \perp m$ if the square

$$\begin{array}{ccc} \mathcal{C}(B,X) & \xrightarrow{-\circ e} & \mathcal{C}(A,X) \\ {\scriptstyle m\circ -}\downarrow & & \downarrow{\scriptstyle m\circ -} \\ \mathcal{C}(B,Y) & \xrightarrow[-\circ e]{} & \mathcal{C}(A,Y) \end{array}$$

is a pullback in Pos.

To make things more explicit, the statement $e \perp m$ means two things:

1. The usual diagonal fill-in property.
2. Given two commutative squares  in $\mathcal{C}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, the diagonal fill-ins must also satisfy $d_1 \leq d_2$.

Then, as in [65] we introduce the following class of morphisms, which in some sense are the Pos-enriched analogue of strong epimorphisms for ordinary categories.

4.2.6 Definition. A morphism $e\colon A \to B$ is called an so-*morphism* (or *surjective on objects*) if $e \perp m$ for every ff-morphism $m\colon X \to Y \in \mathcal{C}$.

4.2.7 Remark. We note here that, in the special case of checking a condition $e \perp m$ with m an ff-morphism, the 2-dimensional part of the definition of orthogonality (property 2 above) follows for free. Indeed, in the notation of the above diagrams, the fact that $md_1 = v_1 \leq v_2 = md_2$ then implies $d_1 \leq d_2$.

As with any class of morphisms which is defined by a left orthogonality property, it is easy to observe that so-morphisms satisfy the following two basic properties, which we record here for future use.

4.2.8 Lemma. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in the category $\mathcal{C}$. Then the following hold:*

1. *If both f, g are* so*-morphisms, then gf is an* so*-morphism.*
2. *If gf is an* so*-morphism, then g is an* so*-morphism.*

The so-morphisms will be the "epi part" of the factorization system on regular categories. Indeed, given the existence of some limits, every so-morphism is an order-epimorphism.

4.2.9 Lemma. *If $\mathcal{C}$ has inserters, then every* so*-morphism in $\mathcal{C}$ is an order-epimorphism.*

Proof. Let $e\colon A \to B$ be an so-morphism and consider $f, g\colon B \to C$ such that $fe \leq ge$. Let $m\colon M \to B$ be the inserter of (f, g). Then there exists a unique $u\colon A \to M$ such that $mu = e$. But we have $e \perp m$ and so we obtain a $v\colon B \to M$ such that $ve = u$, $mv = 1_B$. Now m is both a split epimorphism and a monomorphism, hence an iso and so $fm \leq gm \implies f \leq g$. □

The notion of regularity for ordinary categories, as is well known, can be defined either in terms of strong epimorphisms or in terms of regular epimorphisms. In fact, one can argue that a significant part of the power of regularity is that it forces these two classes of epimorphisms to coincide. From another perspective, the regular epimorphisms are the categorical notion of quotient that is usually appropriate in ordinary categories. For Pos-categories the corresponding notion is that of coinserter. Intuitively, whereas taking a coequalizer corresponds to adding new equalities, constructing a coinserter should be thought of as adding new *inequalities*.

Now, just as every regular epimorphism is always strong, so too we have the following.

4.2.10 Lemma. *Every coinserter is an* so*-morphism.*

Proof. Consider the following commutative diagram, where e is assumed to be the coinserter of (f_0, f_1) and m is an ff-morphism.

$$\begin{array}{ccccc} C & \overset{f_0}{\underset{f_1}{\rightrightarrows}} & A & \xrightarrow{e} & B \\ & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ & & X & \xrightarrow[m]{} & Y \end{array}$$

Now we have $muf_0 = vef_0 \leq vef_1 = muf_1$, so by virtue of m being an ff-morphism we get $uf_0 \leq uf_1$. Then by the coinserter property there exists a unique $d\colon B \to X$ such that $de = u$. It follows also that $md = v$ because e is an order-epimorphism. □

4.2.11 Definition. [65] A morphism $q\colon X \to Y$ is called an *effective* epimorphism if it is the coinserter of some pair of parallel morphisms.

The notion of regularity of a category is essentially an exactness property relating kernel congruences and coinserters. There are however some exactness properties involving these notions that always hold, regardless of regularity. This is the content of the following proposition, which again should be compared to the corresponding facts involving kernel pairs and coequalizers in an ordinary category (see Proposition 2.5.7 and Proposition 2.5.8 in [16]).

4.2.12 Proposition. *1. If an effective epimorphism has a kernel congruence, then it must be the coinserter of that kernel congruence.*

2. If a kernel congruence has a coinserter, then it is also the kernel congruence of that coinserter.

Proof. 1. Suppose $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{q} Q$ is a coinserter diagram and assume q has a kernel congruence $q/q \underset{q_1}{\overset{q_0}{\rightrightarrows}} Y$ in $\mathcal{C}$. Then $qf_0 \leq qf_1$ implies that $(\exists! u\colon X \to q/q)\ q_0 u = f_0, q_1 u = f_1$, by the universal property of kernel congruence.

Now let $g\colon Y \to Z$ be such that $gq_0 \leq gq_1$. Then $gq_0 u \leq gq_1 u$ implies $gf_0 \leq gf_1$ and so $(\exists! v\colon Q \to Z)\ vq = g$. Finally, q is already an order-epimorphism by virtue of being a coinserter of some pair of morphisms Lemma 4.2.10.

2. Suppose that $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \xrightarrow{q} Q$ is a coinserter diagram and that (r_0, r_1) is the kernel congruence of some $f\colon X \to Y$. Then $fr_0 \leq fr_1$ implies the existence of a unique $u : Q \to Y$ such that $uq = f$.

Now let $a, b\colon A \to X$ be such that $qa \leq qb$. Then also $uqa \leq uqb$, i.e. $fa \leq fb$ and so $(\exists! v\colon A \to R)\ r_0 v = a, r_1 v = b$.

□

We can now state the definition of regularity for Pos-enriched categories as presented by Kurz & Velebil in [65]:

A category $\mathcal{C}$ is *regular* if it satisfies the following:
(R1) $\mathcal{C}$ has all finite (weighted) limits.
(R2) $\mathcal{C}$ has (so,ff)-factorizations.
(R3) so-morphisms are stable under pullback in $\mathcal{C}$.
(R4) Every so-morphism is effective in $\mathcal{C}$.

A main feature of this definition is that it posits the existence of a stable (so,ff) factorization system in $\mathcal{C}$. The authors go on to state that the "gist of the definition" is property (R4), i.e. the assumption that so-morphisms and effective morphisms coincide, which essentially states that it is equivalent to require the stable factorization system to be (effective,ff). However, we shall show that condition (R4) in fact follows from the first three conditions, much like in the case of ordinary regularity, where one can state the definition equivalently either in terms of regular epimorphisms or strong epimorphisms. In fact, the proof is essentially a direct adaptation of the corresponding one in the ordinary context (see section 2.2 in [17]).

Before making good on our claim, we need a preparatory result on the pasting of a pullback square with a comma square.

4.2.13 Lemma. *Consider the following diagram in a category $\mathcal{C}$, where the right-hand square is a comma square and the left-hand square commutes.*

$$\begin{array}{ccccc} P & \xrightarrow{p_1} & Q & \xrightarrow{q_1} & Z \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle q_0}\downarrow & \leq & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[x]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Then the outer rectangle is a comma square if and only if the left-hand square is a pullback.

Proof. Assume first that the left-hand square is a pullback. Note first that by our assumptions on the diagram we have $fxp_0 = fq_0p_1 \leq gq_1p_1$. Now suppose that $u\colon A \to X'$ and $v\colon A \to Z$ are such that $fxu \leq gv$. Since the right-hand square is a comma, $(\exists! w\colon A \to Q)\ q_0w = xu, q_1w = v$. The first of these equalities by virtue of the pullback property gives that $(\exists! z\colon A \to P)\ p_0z = u, p_1z = w$. Then we have $q_1p_1z = q_1w = v$ as well.

Finally, assume that $z, z'\colon A \to P$ are such that $p_0z \leq p_0z'$ and $q_1p_1z \leq q_1p_1z'$. Then we also have $xp_0z \leq xp_0z'$, or equivalently $q_0p_1z \leq q_0p_1z'$, so that by the universal property of the comma square we get $p_1z \leq p_1z'$. The latter inequality together with $p_0z \leq p_0z'$ yield $z \leq z'$ by the universal property of the pullback.

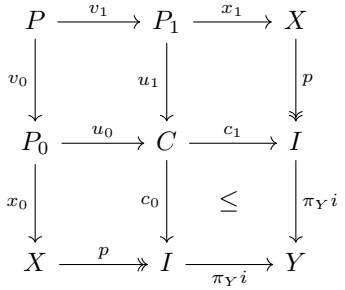
Conversely, assume that the outer square is a comma and let $u\colon A \to X'$ and $v\colon A \to Q$ be such that $xu = q_0v$. Then we have $fxu = fq_0v \leq gq_1v$, so the outer rectangle being a comma says that $(\exists! w\colon A \to P)\ p_0w = u, q_1p_1w = q_1v$. But since also $q_0p_1w = xp_0w = xu = q_0v$ and q_0, q_1 are jointly monomorphic, we obtain also that $p_1w = v$. Finally, it is clear that p_0, p_1 are jointly order-monomorphic because p_0, q_1p_1 are so. □

Now we can prove that condition (R4) in the definition of regularity is superfluous. The proof that follows is almost identical to that of Proposition 2.2.2 in [17], concerning ordinary regularity, where we replace some uses of the familiar lemma on pasting of pullback squares with Lemma 4.2.13.

4.2.14 Proposition. *If $\mathcal{C}$ is a category satisfying conditions (R1),(R2) and (R3) above, then it also satisfies (R4).*

Proof. Let $f\colon X \to Y$ be an so-morphism and consider its kernel congruence $f/f \xrightarrow[k_1]{k_0} X$, which exists in $\mathcal{C}$ by (R1). We want to show that f is the coinserter of (k_0, k_1).

So let $g\colon X \to Z \in \mathcal{C}$ be such that $gk_0 \leq gk_1$. We can consider then the induced morphism $\langle f, g\rangle\colon X \to Y \times Z$. By (R2) we can factor this morphism as an so-morphism followed by an ff-morphism, say $X \xrightarrow{p} I \rightarrowtail^{i} Y \times Z$. We then form the following diagram in $\mathcal{C}$, where we begin by forming the bottom right-hand square as a comma square and then the remaining three squares are pullbacks.



By an application of Lemma 4.2.13 and its order-dual as well as the usual pullback gluing lemma we deduce that the big outer square resulting from the pasting of all four smaller ones is also a comma square. Then, since $\pi_Y ip = f$, we have $P \cong f/f$ and we can assume that $x_0v_0 = k_0$ and $x_1v_1 = k_1$. Also, observe that by (R3) we have that u_0, u_1 and then also v_0, v_1 are so-morphisms.

Now we want to show that $\pi_Y i$ is an iso. Since $(\pi_Y i)p = f$ is an so-morphism, we already know that $\pi_Y i$ is an so-morphism as well. Thus, it suffices to show that it is also an ff-morphism, which is equivalent to showing $c_0 \leq c_1$. Since $u_0v_0 = u_1v_1$ is an so-morphism, so in particular an order-epimorphism, the latter inequality is equivalent to having $c_0u_0v_0 \leq c_1u_0v_0$. This is in turn equivalent to $ic_0u_0v_0 \leq ic_1u_0v_0$, because i is an ff-morphism. To prove this last inequality we now observe the following:

$$\begin{aligned}\pi_Y ic_0u_0v_0 &= \pi_Y ipx_0v_0 = fx_0v_0 = fk_0 \leq fk_1 = fx_1v_1 = \pi_Y ipx_1v_1\\ &= \pi_Y ic_1u_0v_0,\end{aligned}$$

$$\begin{aligned}\pi_Z ic_0u_0v_0 &= \pi_Z ipx_0v_0 = gx_0v_0 = gk_0 \leq gk_1 = gx_1v_1 = \pi_Z ipx_1v_1\\ &= \pi_Z ic_1u_0v_0.\end{aligned}$$

Then the universal property of the product yields the desired inequality.

Finally, we now have a morphism $\pi_Z i(\pi_Y i)^{-1} \colon Y \to Z$ such that

$$\pi_Z i(\pi_Y i)^{-1} f = \pi_Z i(\pi_Y i)^{-1}\pi_Y ip = \pi_Z ip = g.$$

Furthermore, we know already that f is an order-epimorphism because it is an so-morphism by assumption. □

Thus, we can now officially introduce regular categories in the following definition.

4.2.15 Definition. A category $\mathcal{C}$ will be called *regular* if it satisfies the following:
(R1) $\mathcal{C}$ has all finite (weighted) limits.
(R2) $\mathcal{C}$ has (so,ff)-factorizations.
(R3) so-morphisms are stable under pullback in $\mathcal{C}$.

Furthermore, we can now give another equivalent characterization of regularity in terms of the existence of quotients for kernel congruences.

4.2.16 Proposition. *A finitely complete category $\mathcal{C}$ is regular if and only if the following hold:*

1. *Every kernel congruence in $\mathcal{C}$ has a coinserter.*
2. *Effective morphisms are stable under pullback in $\mathcal{C}$.*

Proof. If $\mathcal{C}$ is regular, then it is easy to see that it satisfies the two conditions above by definition and be an appeal to part (2.) of Proposition 4.2.12.

Conversely, let us assume that $\mathcal{C}$ satisfies the two conditions in the statement. Consider any $f\colon X \to Y \in \mathcal{C}$, its kernel congruence $f/f \overset{f_0}{\underset{f_1}{\rightrightarrows}} X$ and the coinserter $q\colon X \to Q$ in $\mathcal{C}$ of the latter, which exists by condition 1. Since $ff_0 \leq ff_1$, there exists a unique $m\colon Q \to Y$ such that $f = mq$. It now suffices to show that m is an ff-morphism.

For this, consider the kernel congruence $m/m \overset{m_0}{\underset{m_1}{\rightrightarrows}} Q$ and form the following diagram where the bottom right-hand square is a comma and the other three are pullbacks.

$$\begin{array}{ccccc}
P & \xrightarrow{v_1} & P_1 & \xrightarrow{x_1} & X \\
{\scriptstyle v_0}\downarrow & & {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle q} \\
P_0 & \xrightarrow{u_0} & m/m & \xrightarrow{m_1} & Q \\
{\scriptstyle x_0}\downarrow & & {\scriptstyle m_0}\downarrow & \leq & \downarrow{\scriptstyle m} \\
X & \underset{q}{\twoheadrightarrow} & Q & \underset{m}{\longrightarrow} & Y
\end{array}$$

Similarly to the proof of Proposition 4.2.14, we have $P \cong f/f$ and we can assume that $x_0v_0 = f_0$ and $x_1v_1 = f_1$. Furthermore, by the assumed stability of effective morphisms under pullback, we can deduce that $u_0v_0 = u_1v_1$ is an order-epimorphism. Now we have that

$$m_0u_0v_0 = qx_0v_0 = qf_0 \leq qf_1 = qx_1v_1 = m_1u_1v_1$$

whence we deduce that $m_0 \leq m_1$ and so that m is an ff-morphism. □

Before ending this section, we would be remiss not to mention examples of regular Pos-categories. For more details on a lot of these one can consult [65].

4.2.17 Example. 1. Pos is regular as a Pos-category and so is any enriched presheaf category $[\mathcal{C}^{op}, \mathsf{Pos}]$ for $\mathcal{C}$ a small category. Recall here that Pos is not regular as an ordinary category.

2. Any ordinary regular category $\mathcal{C}$ is also regular in the Pos-enriched sense when equipped with the discrete order on its Hom-sets. Indeed, in this case ff-morphisms coincide with monomorphisms and so-morphisms with strong epimorphisms, while limits are just ordinary limits.

3. *Quasivarieties of ordered algebras* in the sense of Bloom and Wright [15] are regular categories [65]. As particular examples here we have the categories OrdMon of ordered monoids, $\mathsf{OrdSGrp}$ of ordered semi-groups, $\mathsf{OrdCMon}$ of commutative ordered monoids and OrdMon_0 of ordered monoids with the neutral element 0 of the monoid operation as the minimum element for the order. These are all in fact varieties. An example of a quasivariety which is not a variety is the category $\mathsf{OrdCMon}_{\mathsf{t.f.}}$ of torsion-free commutative monoids, i.e. those ordered commutative monoids $(M, +, \leq)$ satisfying the implication $nx \leq ny \implies x \leq y$ for all $x, y \in M$ and $n \in \mathbb{N}$.

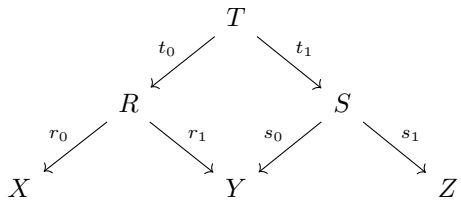
4. The categories Nach of *Nachbin* spaces (or compact ordered spaces) and Pries of *Priestley* spaces with continuous order-preserving functions in both instances are examples of regular categories. We shall have more to say on these in a later section.

5. If $\mathcal{C}$ is monadic over Pos for a monad $T\colon \mathsf{Pos} \to \mathsf{Pos}$ which preserves so-morphisms (i.e. surjections), then $\mathcal{C}$ is regular. An example of this kind is given by the category S-Pos of S-posets for any ordered monoid S (see [26]). The objects in the latter category are posets X with a monoid action $S \times X \to X$ which is monotone in both variables, while the morphisms are the monotone equivariant functions.

4.3 Calculus of Relations and Exact Categories

Recall from Section 1.3 that in an ordinary regular category $\mathcal{E}$ the existence of a stable (regular epi,mono) factorization system allows for a well-behaved calculus of relations in $\mathcal{E}$. In an entirely analogous manner, we can develop a corresponding calculus of relations in our Pos-enriched setting. On a first level, this is will not be very different at all from the ordinary case itself. More precisely, it will in essence be the calculus of relations *relative to a proper factorization system*, as has been presented in the literature by authors such as Meisen [79], Richter [87] and Kelly [61]. In some sense however this will only capture the ordinary part of the category $\mathcal{E}$. Thus, on a second level, we will also have an aspect of the calculus of relations which is unique to and more accurately describes the Pos-enriched context. The central result of this section will be Theorem 4.3.8, which shows that by choosing an appropriate bicategory of relations in $\mathcal{E}$ one can again characterize the morphisms of $\mathcal{E}$ as the maps therein. This will in turn imply in the usual way that *exactness*, which we will also define in this section, can be characterized via splitting of idempotents.

If $\mathcal{E}$ is a regular category, then by a *relation* in $\mathcal{E}$ we shall mean an order-subobject $R \rightarrowtail X \times Y$, i.e. a subobject of a product represented by an ff-morphism. We shall write $R\colon X \nrightarrow Y$ to denote that R is a relation from X to Y in $\mathcal{E}$. The existence of (so,ff) factorization in $\mathcal{E}$ allow us to define a composition of relations in $\mathcal{E}$ in the usual way: given $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$ in $\mathcal{E}$, the relation $S \circ R\colon X \nrightarrow Z$ is defined by first constructing the pullback square below



and then taking the $(\mathsf{so}, \mathsf{ff})$ factorization of $\langle r_0 t_0, s_1 t_1 \rangle\colon T \to X \times Z$, say

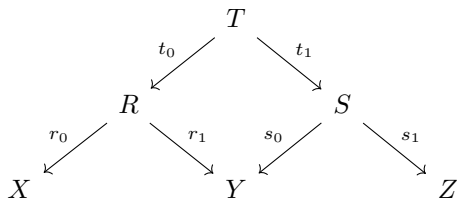
$$T \xrightarrow{\quad q \quad} M \xrightarrow{\langle m_0, m_1 \rangle} X \times Z$$

In other words, in our above notation $S \circ R$ is the relation represented by the ff-morphism $\langle m_0, m_1 \rangle$. In the interest of easing the notation, we shall often denote such a composition simply by SR.

Then we have the following observation concerning generalized elements of a composition of relations, much as in the ordinary context Lemma 1.3.2. We go through the proof again in detail in order to exhibit that nothing changes in the argument.

4.3.1 Lemma. *Let $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow Z$ be relations in a regular category $\mathcal{E}$ and consider any generalized elements $x\colon P \to X$ and $z\colon P \to Z$. Then $(x, z) \in_P SR$ if and only if there exists an effective epimorphism $q\colon Q \twoheadrightarrow P$ and a generalized element $y\colon Q \to Y$ such that $(xq, y) \in_Q R$ and $(y, zq) \in_Q S$.*

Proof. Consider the diagram below, where the square is a pullback.



Then SR is given by the image factorization

$$\langle r_0 t_0, s_1 t_1 \rangle = \; T \xrightarrow{\quad e \quad} SR \xrightarrow{\langle i_0, i_1 \rangle} X \times Z$$

.

Assume first that $(x, z) \in_P SR$, i.e. there exists a morphism $u\colon P \to SR$ such that $\langle i_0, i_1 \rangle u = \langle x, z \rangle$. We then form the pullback square below.

$$\begin{array}{ccc} Q & \xrightarrow{q} & P \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ T & \xrightarrow[e]{} & SR \end{array}$$

Note that q is an effective epimorphism because e is such. Now set $y \coloneqq r_1 t_0 v = s_0 t_1 v : Q \to Y$. We then have

$$\langle xq, y\rangle = \langle i_0 uq, r_1 t_0 v\rangle = \langle i_0 ev, r_1 t_0 v\rangle = \langle r_0 t_0 v, r_1 t_0 v\rangle = \langle r_0, r_1\rangle t_0 v$$

and

$$\langle y, zq\rangle = \langle s_0 t_1 v, i_1 uq\rangle = \langle s_0 t_1 v, i_1 ev\rangle = \langle s_0 t_1 v, s_1 t_1 v\rangle = \langle s_0, s_1\rangle t_1 v,$$

so that $(xq, y) \in_Q R$ and $(y, zq) \in_Q S$.

Conversely, assume that $(xq, y) \in_Q R$ and $(y, zq) \in_Q S$ for some $y\colon Q \to Y$ and effective epimorphism $q\colon Q \twoheadrightarrow P$. This means that there exist morphisms $u\colon Q \to R$ and $v\colon Q \to S$ such that $\langle r_0, r_1\rangle u = \langle xq, y\rangle$ and $\langle s_0, s_1\rangle v = \langle y, zq\rangle$. Since $r_1 u = y = s_0 v$, there exists a unique $w\colon Q \to T$ such that $t_0 w = u$ and $t_1 w = v$. Then $\langle i_0, i_1\rangle ew = \langle r_0 t_0, s_1 t_1\rangle w = \langle r_0 u, s_1 v\rangle = \langle xq, zq\rangle$, so that $(xq, zq) \in_Q SR$. Since q is an effective epimorphism, we can conclude that also $(x, z) \in_P SR$. □

In particular, the above lemma can be used to show that the composition of relations in any regular category $\mathcal{E}$ is associative. Thus, we can define the locally posetal bicategory $\mathrm{Rel}(\mathcal{E})$, whose objects are those of $\mathcal{E}$ and whose morphisms are the relations $R\colon X \nrightarrow Y$ in $\mathcal{E}$, while the identity morphism on the object X in $\mathrm{Rel}(\mathcal{E})$ is the diagonal relation $\langle 1_X, 1_X\rangle\colon \Delta_X \rightarrowtail X \times X$.

Furthermore, Lemma 4.3.1 can now be used to prove many of the fundamental properties of $\mathrm{Rel}(\mathcal{E})$ which are familiar for ordinary regular categories (see Section 1.3). More precisely, $\mathrm{Rel}(\mathcal{E})$ is a *tabular allegory with a unit* in the sense of [57], [38], where the anti-involution $(-)^\circ\colon \mathrm{Rel}(\mathcal{E})^{\mathrm{op}} \to \mathrm{Rel}(\mathcal{E})$ is given by taking the *opposite* relation. In particular, we have that Freyd's *Modular Law* holds in $\mathrm{Rel}(\mathcal{E})$, i.e. the following inclusions hold

$$SR \cap T \subseteq S(R \cap S^\circ T) \tag{ML}$$

$$SR \cap T \subseteq (S \cap TR^\circ)R \tag{ML*}$$

for any relations $R : X \nrightarrow Y$, $S : Y \nrightarrow Z$ and $T : X \nrightarrow Z$.

Every morphism $f\colon X \to Y \in \mathcal{E}$ defines a relation $X \nrightarrow Y$ represented by the ff-morphism $\langle 1_X, f\rangle\colon X \to X \times Y$, which we call its *graph* and denote by the same letter. This assignment defines a faithful ordinary functor $\mathcal{E}_0 \to \mathrm{Rel}(\mathcal{E})$ on the underlying ordinary category of $\mathcal{E}$ which is the identity on objects. Furthermore, in $\mathrm{Rel}(\mathcal{E})$ we have an adjunction $f \dashv f^\circ$, which means that the inclusions $f^\circ f \supseteq \Delta_X$ and $ff^\circ \subseteq \Delta_Y$ hold. In other words, the morphisms of $\mathcal{E}$ are *maps* in the bicategory $\mathrm{Rel}(\mathcal{E})$. We should perhaps stress here that taking the graph of a morphism does not define a functor $\mathcal{E} \to \mathrm{Rel}(\mathcal{E})$ because the order

of morphisms is not preserved. In fact, precisely as in the ordinary setting, the existence of the anti-involution $(-)^\circ$ in the allegory $\mathrm{Rel}(\mathcal{E})$ forces any inclusion $f \subseteq g$ for morphisms $f, g\colon X \to Y$ to be an equality (see e.g. Lemma 3.2.3 in [57]).

The modular law also implies some restricted versions of distributivity of composition over binary intersections in $\mathrm{Rel}(\mathcal{E})$ (see e.g. A3.1.6 of [57]). Two particular instances of this which we would like to explicitly record for future reference are the following distributivity properties of graphs of morphisms over arbitrary binary intersections:

$$(R \cap S)f = Rf \cap Sf \tag{MD}$$

$$g^\circ(R \cap S) = g^\circ R \cap g^\circ S \tag{MD*}$$

where R, S are relations $Y \nrightarrow Z$ and $f : X \to Y$ and $g : X \to Z$ are morphisms in $\mathcal{E}$.

4.3.2 Remark. The term "locally posetal bicategory" of course means exactly the same as "Pos-category". However, when dealing with categories of relations such as $\mathrm{Rel}(\mathcal{E})$ we often prefer to use the word "bicategory", both to follow tradition and also to explicitly distinguish it from the generic Pos-category $\mathcal{E}$.

While $\mathrm{Rel}(\mathcal{E})$ is a very useful category in terms of performing calculations with relations in $\mathcal{E}$, it does not really capture the enriched nature of $\mathcal{E}$. As we have mentioned earlier, $\mathrm{Rel}(\mathcal{E})$ in some sense only involves an ordinary category with a stable factorization system. In order to also capture the Pos-enriched aspects of a regular category $\mathcal{E}$ we will need to in addition introduce a different bicategory of relations. In terms of our broader goals in this chapter, this need is related to our desire to identify the morphisms of $\mathcal{E}$ as the maps in a certain bicategory of relations, thus generalizing the familiar corresponding fact for ordinary regular categories. Indeed, while certainly any morphism $f\colon X \to Y \in \mathcal{E}$ has as its right adjoint in $\mathrm{Rel}(\mathcal{E})$ the opposite relation f°, this is not a complete characterization of (graphs of) morphisms. It is not too hard to see that a relation $R\colon X \nrightarrow Y \in \mathcal{E}$ represented by the ff-morphism $\langle r_0, r_1\rangle$ is a map in $\mathrm{Rel}(\mathcal{E})$ if and only if r_0 is both a monomorphism and an so-morphism. Since this does not force r_0 to be an isomorphism, being a map in $\mathrm{Rel}(\mathcal{E})$ is a strictly weaker property.

Furthermore, the requirement for moving a different bicategory of relations will be dictated also by the notion of *congruence* which is appropriate in this enriched setting and the manner in which exactness will be defined. This will become apparent a little bit later.

4.3.3 Definition. A relation $R\colon X \nrightarrow Y$ in the regular category $\mathcal{E}$ is called *weakening* or *weakening-closed* if, whenever $x, x'\colon A \to X$ and $y, y'\colon A \to Y$ are generalized elements in $\mathcal{E}$, the following implication holds

$$x' \leq x \ \wedge\ (x, y) \in_A R \ \wedge\ y \leq y' \implies (x', y') \in_A R$$

The above terminology has been borrowed from [64]. The condition that a relation be weakening-closed is a type of compatibility condition. Intuitively, if we think of a relation $R\colon X \nrightarrow Y$ in this enriched context as specifying that certain elements of X are less than or equal to certain elements of Y, then R being weakening-closed says that these specified inequalities are compatible with the order relations that already exist on the elements of X and Y.

For any given object $X \in \mathcal{E}$, there is a canonical weakening-closed relation $I_X\colon X \nrightarrow X$ given by the comma $I_X \coloneqq 1_X/1_X$. Notice that I_X is a partial order relation, i.e. it is reflexive, transitive and antisymmetric. Then it is easy to see that an arbitrary relation $R\colon X \nrightarrow Y$ is weakening-closed if and only if we have $I_Y R I_X \subseteq R$ or equivalently $R = I_Y R I_X$ in $\mathrm{Rel}(\mathcal{E})$. In particular, the relations I_X act as identity elements for composition of weakening-closed relations. Thus, we can define a bicategory $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ where the objects are again those of $\mathcal{E}$ but now the morphisms are the weakening-closed relations.

4.3.4 Remark. We should note here that, although every morphism of $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ is also a morphism in $\mathrm{Rel}(\mathcal{E})$ and composition in both categories is the same, this is not a functorial inclusion $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}) \hookrightarrow \mathrm{Rel}(\mathcal{E})$ because identity morphisms are not preserved.

Now to any given morphism $f\colon X \to Y \in \mathcal{E}$ we can canonically associate two weakening-closed relations $f_*\colon X \nrightarrow Y$ and $f^*\colon Y \nrightarrow X$ via the following commas:

$$f_* \coloneqq f/1_Y \quad f^* \coloneqq 1_Y/f.$$

We sometimes call f_* and f^* the *hypergraph* and *hypograph* of f respectively. In terms of generalized elements $x\colon A \to X$ and $y\colon A \to Y$ in $\mathcal{E}$ we have $(x, y) \in_A f_* \iff fx \leq y$ and $(y, x) \in_A f^* \iff y \leq fx$.

The following are then easy to see, for example by arguing with generalized elements and using the above observations and Lemma 4.3.1.

4.3.5 Lemma. *Let $\mathcal{E}$ be a regular category. Then for any $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{E}$ we have*

1. $(gf)_* = g_* f_*$.
2. $(gf)^* = f^* g^*$.

Proof. We only prove (1), as (2) is entirely analogous.

Consider any generalized elements $x\colon A \to X$ and $z\colon A \to Z$ in $\mathcal{E}$. We then have $(x, z) \in_A g_* f_*$ if and only if there exist an so-morphism $q\colon B \twoheadrightarrow A$ and a generalized element $y\colon B \to Y$ such that $(xq, y) \in_B f_*$ and $(y, zq) \in_B g_*$. The latter two say exactly that $fxq \leq y$ and $gy \leq zq$, so that $gfxq \leq gy \leq zq$. Since q is an order-epimorphism, this is equivalent to $gfx \leq z$, i.e. to $(x, z) \in_A (gf)_*$. □

It is also easy to see that, for any $f, g\colon X \to Y \in \mathcal{E}$, we have

$$f \leq g \iff g_* \subseteq f_* \iff f^* \subseteq g^*$$

We thus have two fully order-faithful functors $(-)_*\colon \mathcal{E}^{co} \hookrightarrow \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ and $(-)^*\colon \mathcal{E}^{op} \hookrightarrow \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$, where $\mathcal{E}^{co}$ denotes the "order-dual" category, i.e. the category obtained from $\mathcal{E}$ by reversing the order on morphisms. Similarly, arguing with generalized elements we easily deduce the following.

4.3.6 Lemma. *For any $f\colon X \to Y$ in a regular category $\mathcal{E}$ we have $f^*f_* = f/f$ as relations in $\mathcal{E}$.*

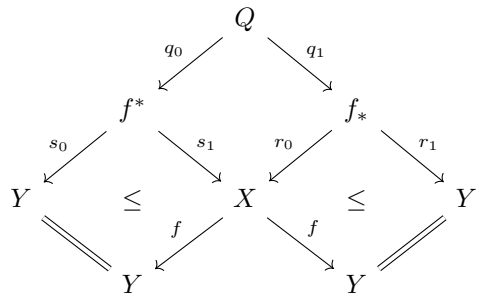
In particular, we see that $f\colon X \to Y$ is an ff-morphism in $\mathcal{E}$ if and only if $f^*f_* = I_X$ in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$. Similarly, a pair of morphisms $Y \xleftarrow{f} X \xrightarrow{g} Z$ is jointly ff precisely when $f^*f_* \cap g^*g_* = I_X$.

Next, we observe that just as the graph of every morphism $f\colon X \to Y$ induces an adjunction $f \dashv f^\circ$ in $\mathrm{Rel}(\mathcal{E})$, so do the hypergraph and hypograph of f form an adjunction in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$.

4.3.7 Lemma. *For any $f\colon X \to Y$ in a regular category $\mathcal{E}$, there is an adjunction $f_* \dashv f^*$ in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$.*

Proof. We recorded earlier that f^*f_* is precisely the kernel congruence of f, so clearly we have $I_X \subseteq f^*f_*$. For the other required inclusion, we can argue again with generalized elements, or alternatively as follows:

To form the composition f_*f^* we consider the diagram below, where the top square is a pullback and the other two are commas.

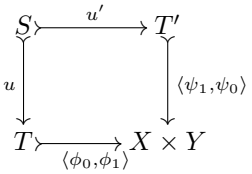


Now by definition we have that f_*f^* is given by the image of $\langle s_0q_0, r_1q_1\rangle$. But $s_0q_0 \leq fs_1q_0 = fr_0q_1 \leq r_1q_1$, so that $\langle s_0q_0, r_1q_1\rangle$ factors through $I_Y = 1_Y/1_Y$. This yields the inclusion $f_*f^* \subseteq I_Y$. □

We have thus established that every morphism $f\colon X \to Y$ in a regular category $\mathcal{E}$ is a map in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ via its hypergraph f_*. Our first result in this chapter shows that in any regular category $\mathcal{E}$ this is now indeed a complete characterization of morphisms. In other words, every map in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ is of the form f_* for a (necessarily unique) morphism $f\colon X \to Y \in \mathcal{E}$.

4.3.8 Theorem. *If $\phi\colon X \nrightarrow Y \in \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ has a right adjoint in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$, then there exists a (necessarily unique) morphism $f\colon X \to Y \in \mathcal{E}$ such that $\phi = f_*$.*

Proof. Let $\psi\colon Y \nrightarrow X \in \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ be the right adjoint, so that we have $I_X \subseteq \psi\phi$ and $\phi\psi \subseteq I_Y$. Suppose also that ϕ and ψ are represented respectively by the ff-morphisms $\langle \phi_0, \phi_1\rangle\colon T \rightarrowtail X \times Y$ and $\langle \psi_0, \psi_1\rangle\colon T' \rightarrowtail Y \times X$. We next form the pullback square below, so that $\langle \phi_0 u, \phi_1 u\rangle = \langle \psi_1 u', \psi_0 u'\rangle : S \rightarrowtail X \times Y$ represents the relation $\phi \cap \psi^\circ : X \nrightarrow Y \in \mathrm{Rel}(\mathcal{E})$. We first want to show that $\phi_0 u = \psi_1 u'$ is an isomorphism, in which case we will have $\phi \cap \psi^\circ = f$ for the morphism $f \coloneqq \phi_1 u(\phi_0 u)^{-1}\colon X \to Y$.



First of all, since $I_X \subseteq \psi\phi$, we have $(1_X, 1_X) \in_X \psi\phi$ and hence there exist an effective epi $e\colon P \twoheadrightarrow X$ and a $y\colon P \to Y$ such that $(e, y) \in_P \phi$ and $(y, e) \in_P \psi$. Then we have $(e, y) \in_P \phi \cap \psi^\circ$ and so there exists a $v\colon P \to S$ such that $\langle \phi_0 u, \phi_1 u\rangle v = \langle e, y\rangle$. In particular, since $\phi_0 uv = e$ is an effective epi, we deduce that $\phi_0 u$ is an effective epi as well.

Now it suffices to show that $\phi_0 u$ is also an ff-morphism. So let $a, b\colon A \to S$ be such that $\phi_0 ua \leq \phi_0 ub$. Then $(\phi_0 ua, \phi_0 ub) \in_A I_X$ and hence $(\phi_0 ua, \phi_0 ub) \in_A \psi\phi$, so there is an effective epimorphism $e\colon P \twoheadrightarrow A$ and a $z\colon P \to Y$ such that $(\phi_0 uae, z) \in_P \phi$ and $(z, \phi_0 ube) \in_P \psi$. Since we also clearly have

$$(\phi_1 uae, \phi_0 uae) = (\psi_0 u'ae, \psi_1 u'ae) \in_P \psi,$$

we get $(\phi_1 uae, z) \in_P \phi\psi$, which implies $(\phi_1 uae, z) \in_P I_Y$, i.e. that $\phi_1 uae \leq z$.

Similarly, since $(\phi_0 ube, \phi_1 ube) \in_P \phi$ and $(z, \phi_0 ube) \in \psi$, we have that $(z, \phi_1 ube) \in_P \phi\psi$ and hence $(z, \phi_1 ube) \in_P I_Y$, which is to say that $z \leq \phi_1 ube$. We now have $\phi_1 uae \leq z \leq \phi_1 ube$, hence $\phi_1 uae \leq \phi_1 ube$, which in turn implies that $\phi_1 ua \leq \phi_1 ub$, because e is an order-epimorphism. Since also $\phi_0 ua \leq \phi_0 ub$, we obtain $a \leq b$ because $\langle \phi_0 u, \phi_1 u\rangle$ is an ff-morphism.

We now claim that $\phi = f_*$. To see this, observe first that $f_* = I_Y f = I_Y(\phi \cap \psi^\circ) \subseteq I_Y \phi = \phi$, where the last equality follows because ϕ is weakening-closed. Thus, $f_* \subseteq \phi$. But by an application of the modular law ML in $\mathrm{Rel}(\mathcal{E})$ we also have $\psi f = \psi(\phi \cap \psi^\circ) \supseteq \psi\phi \cap \Delta_X \supseteq I_X \cap \Delta_X = \Delta_X$ and so that $\phi\psi f \supseteq \phi$. Then $f_* = I_Y f \supseteq \phi\psi f \supseteq \phi$ and so we conclude that $f_* = \phi$.

Finally, for the uniqueness claim let us assume that $f, g\colon X \to Y$ are such that $f_* = g_*$. By an earlier observation we know that the inclusion $f_* \subseteq g_*$ is equivalent to the inequality $g \leq f$. Then we have both $f \leq g$ and $g \leq f$, whence $f = g$. □

Next, let us discuss here a little bit how the calculus of relations in a regular category $\mathcal{E}$ can be used to express various limit properties therein. This is similar to and extends the ordinary case which we talked about in Section 1.3.

First, the statement that a pair of morphisms $X \xleftarrow{f} Z \xrightarrow{g} Y$ represents a given relation $R\colon X \nrightarrow Y$ is equivalent to the following two equalities between relations:

1. $R = gf^\circ$.
2. $f^*f_* \cap g^*g_* = I_Z$.

Note here that we cannot replace condition (1) by $R = g_*f^*$, even if R is weakening-closed. Similarly, we cannot replace (2) by $f^\circ f \cap g^\circ g = I_Z$, because the latter means that (f, g) are only jointly monomorphic instead of jointly order-monomorphic.

Based on the above observation, now consider the statement that a commutative square

$$\begin{array}{ccc} P & \xrightarrow{p_1} & Y \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a pullback. It is easy to see that this is equivalent to the following pair of equalities:

1. $g^\circ f = p_1 p_0^\circ$.
2. $p_1^* p_{1*} \cap p_0^* p_{0*} = I_P$.

Again, the second condition is there to specify that p_0, p_1 are jointly order-monomorphic, since the limits that we are discussing are always in the enriched sense.

Similarly, the statement that the square

$$\begin{array}{ccc} P & \xrightarrow{p_1} & Y \\ {\scriptstyle p_0}\downarrow & \leq & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

is a comma is equivalent to:

1. $g^*f_* = p_1 p_0^\circ$.
2. $p_1^* p_{1*} \cap p_0^* p_{0*} = I_P$.

Now we turn to discussing exactness for Pos-categories. First, let us recall the definition of congruence relation from [65]. This can be seen as the ordered analogue of equivalence relations in an ordinary category. It is also a special case of a more general notion of congruence for 2-categories.

4.3.9 Definition. Let X be an object of the regular category $\mathcal{E}$. A *congruence* on X is a relation $E\colon X \nrightarrow X \in \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ which is reflexive and transitive.

We say that the congruence E is *effective* if there exists a morphism $f\colon X \to Y \in \mathcal{E}$ such that $E = f/f$.

Equivalently, we can say that $E\colon X \nrightarrow X$ is a congruence if it is a transitive relation such that $E \supseteq I_X$. In essence, a congruence is a pre-order relation on X which is compatible with the canonical order relation on X, the latter expressed by the requirement that it is weakening-closed. We think of a congruence as imposing additional inequalities on X, just as an equivalence relation corresponds to the idea of imposing new equalities.

With the notion of congruence in hand, we are led naturally to the notion of (Barr-)exactness for Pos-categories as considered by Kurz & Velebil.

4.3.10 Definition. [65] A regular category $\mathcal{E}$ is called *exact* if every congruence in $\mathcal{E}$ is effective.

4.3.11 Example.

1. Pos is exact and so is any presheaf category $[\mathcal{C}, \mathsf{Pos}]$ for any small category $\mathcal{C}$.

2. The locally discrete category Set is an example of a category which is regular but not exact (see [65]).

3. Generalizing the case of Pos, any *variety of ordered algebras*, always in the sense of Bloom & Wright [15], is an exact category. Particular examples here are furnished by the categories OrdSGrp, OrdMon, OrdMon_0 and OrdCMon. On the other hand, the quasivariety $\mathsf{OrdCMon}_{\mathrm{t.f.}}$ is not exact.

4. There are also examples of exact categories which are not varieties, but are monadic over Pos. One such, which will appear in more detail in section 5, is the category Nach of Nachbin spaces. Another is given by the category S-Pos for any ordered monoid S.

A congruence in a regular category $\mathcal{E}$, being a transitive relation, is an idempotent when considered as a morphism in either of $\mathrm{Rel}(\mathcal{E})$ and $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$. When it is moreover effective, then it actually is a *split* idempotent in the latter bicategory. Indeed, if $E = f/f$ for some $f\colon X \to Y$, then we can assume that f is actually the coinserter of E by Proposition 4.2.12. Then we have $f^* f_* = f/f = E$ and $f_* f^* = I_Y$. The next proposition shows that this splitting actually characterizes effective congruences. This is analogous to Proposition 1.3.8 in the ordinary context.

4.3.12 Proposition. *Let $E\colon X \nrightarrow X$ be a congruence in the regular category $\mathcal{E}$. Then E is effective if and only if it splits as an idempotent in* $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$.

Proof. Suppose that $E = \psi\phi$ for some $\phi\colon X \nrightarrow Y$ and $\psi\colon Y \nrightarrow X$ in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ with $\phi\psi = I_Y$. Since E is reflexive and weakening-closed, we have $\psi\phi = E \supseteq I_X$. Since also trivially $\phi\psi \subseteq I_Y$, we have $\phi \dashv \psi$ in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ and so by Theorem 4.3.8 we have that $\phi = f_*$ and $\psi = f^*$ for some (unique) morphism $f\colon X \to Y \in \mathcal{E}$. Thus, we obtain $E = \psi\phi = f^*f_* = f/f$. □

In the following section we will embark on the goal of constructing the exact completion $\mathcal{E}_{ex/reg}$ of a regular category $\mathcal{E}$ by a process of splitting idempotents in a bicategory of relations and then taking maps in the resulting bicategory. This is essentially an attempt to mimic the construction of the ordinary exact completion of a regular category, as initially described by Lawvere in [69] and then with more details for example in [95], [38], [57], motivated by the combination of the two results contained in Theorem 4.3.8 and Proposition 4.3.12.

However, even if the reader is not familiar with the construction of the splitting of idempotents, the next proposition can serve as a type of heuristic for coming up with the definition of morphisms in the completion $\mathcal{E}_{ex/reg}$. It will also be of practical use a little bit later.

We say here that a diagram of the form $E \underset{e_1}{\overset{e_0}{\rightrightarrows}} X \xrightarrow{\ p\ } P$ in a regular category is an *exact sequence* if (e_0, e_1) is the kernel congruence of p and also p is the coinserter of (e_0, e_1). By earlier observations, note that exactness of such a sequence can be formulated in the calculus of relations by the two equalities $p^*p_* = E$ and $p_*p^* = I_P$.

4.3.13 Proposition. *Let* $E \underset{}{\overset{}{\rightrightarrows}} X \xrightarrow{\ p\ } P$ *and* $F \underset{}{\overset{}{\rightrightarrows}} Y \xrightarrow{\ q\ } Q$ *be exact sequences in the regular category* $\mathcal{E}$*. Then there is an order-reversing bijection between the following:*

1. *Morphisms* $P \to Q$ *in* $\mathcal{E}$*.*
2. *Relations* $R_*\colon X \nrightarrow Y \in \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ *for which there exists another relation* $R^*\colon Y \nrightarrow X \in \mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ *such that the following are satisfied:*
   - $FR_*E = R_*$ *and* $ER^*F = R^*$*.*
   - $R^*R_* \supseteq E$ *and* $R_*R^* \subseteq F$*.*

Proof. Consider first a morphism $r\colon P \to Q \in \mathcal{E}$. Set $R_* \coloneqq q^*r_*p_*$ and $R^* \coloneqq p^*r^*q_*$. We have $FR_*E = q^*q_*q^*r_*p_*p^*p_* = q^*r_*p_* = R_*$ and $ER^*F = p^*p_*p^*r^*q_*q^*q_* = p^*r^*q_* = R^*$. In addition,

$$R^*R_* = p^*r^*q_*q^*r_*p_* = p^*r^*I_Qr_*p_* = p^*r^*r_*p_* \supseteq p^*p_* = E$$

$$R_*R^* = q^*r_*p_*p^*r^*q_* \subseteq q^*r_*r^*q_* \subseteq q^*q_* = F.$$

Conversely, consider a relation R_* as in (2.). Set $\phi \coloneqq q_*R_*p^*$ and then also $\psi \coloneqq p_*R^*q^*$. Then for these weakening-closed relations we have

$$\psi\phi = p_*R^*q^*q_*R_*p^* \supseteq p_*R^*R_*p^* \supseteq p_*Ep^* = I_P$$

$$\phi\psi = q_*R_*p^*p_*R^*q^* = q_*R_*ER^*q^* = q_*R_*R^*q^* \subseteq q_*Fq^* = I_Q$$

Thus, by Theorem 4.3.8 we have $\phi = r_*$ for a (unique) morphism $r\colon P \to Q$.

The fact that these two assignments are inverse to each other is expressed by the two equalities $q_*q^*r_*p_*p^* = r_*$ and $q^*q_*R_*p^*p_* = FR_*E = R_*$. □

Before ending this section, let us make a couple more observations on the assignment $R_* \mapsto r$ from the last proposition. Note that, as the notation suggests and the above proof exhibits, the relation R_* corresponds to the hypergraph r_* of a morphism $r\colon P \to Q$. Specifically, r is uniquely determined from R_* by the equality $r_* = q_*R_*p^*$. We would like to also record here a relation R that in some sense corresponds directly to the (graph of the) morphism r.

Given $R_* : X \nrightarrow Y$ as in Proposition 4.3.13, set $R \coloneqq R_* \cap (R^*)^\circ\colon X \nrightarrow Y$. Then we claim that $r = qRp^\circ$. To see this, observe first that qRp° is also a map in $\mathrm{Rel}(\mathcal{E})$ because

$$(qRp^\circ)^\circ qRp^\circ = pR^\circ q^\circ qRp^\circ \supseteq pR^\circ Rp^\circ \supseteq p(E \cap E^\circ)p^\circ = \Delta_P$$

$$\begin{aligned}(qRp^\circ)(qRp^\circ)^\circ = qRp^\circ pR^\circ q^\circ = qR(E \cap E^\circ)R^\circ q^\circ = qRR^\circ q^\circ \\ \subseteq q(F \cap F^\circ)q^\circ = \Delta_Q\end{aligned}$$

Here we used the fact that in an exact sequence $E \rightrightarrows X \xrightarrow{p} P$ we have $E \cap E^\circ = p^*p_* \cap (p^*p_*)^\circ = p^\circ p$, i.e. $E \cap E^\circ$ is precisely the kernel pair of p. In addition, the inclusions $R^\circ R \supseteq E \cap E^\circ$ and $RR^\circ \subseteq F \cap F^\circ$ were used, the second of which is clear and the first follows by the modular law (see the proof of Lemma 4.4.4 in the next section for details).

Now, finally, we have

$$r = r_* \cap (r^*)^\circ = q_*R_*p^* \cap (q^*)^\circ(R^*)^\circ(p_*)^\circ \supseteq q(R_* \cap (R^*)^\circ)p^\circ = qRp^\circ.$$

But since we have an inclusion between two maps in the allegory $\mathrm{Rel}(\mathcal{E})$, these maps must be equal. Hence, we conclude that $r = qRp^\circ$.

4.4 A Construction of the Exact Completion of a Regular Pos-category

This section contains the main result of this chapter, which is a construction of the exact completion of a regular Pos-category $\mathcal{E}$ via the calculus of relations.

The general idea here is to try to perform a construction along the lines of the one recalled in Section 1.5 for ordinary categories. Our hope that such an attempt can work stems from the validity of Proposition 4.3.12 and Theorem 4.3.8 established in the previous section, which are the enriched analogues of Proposition 1.3.8 and Proposition 1.3.3 respectively. Hence, in our present

context, given a regular category $\mathcal{E}$ we should first look at $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$, split the idempotents therein which are congruences in $\mathcal{E}$, then finally take the category of maps in the resulting bicategory.

The fact that this process must be applied to $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ rather than $\mathrm{Rel}(\mathcal{E})$ presents some issues. As we have mentioned earlier, $\mathrm{Rel}(\mathcal{E})$ has the structure of an allegory and it is this fact that facilitates many computations, especially via the validity of the modular law. Furthermore, the theory of allegories is well developed and in fact there is a precise correspondence between ordinary regular and exact categories on the one hand and certain classes of allegories on the other (see e.g. [57], [38]). The latter actually indicates that allegories are in some sense too rigid to fully capture our enriched context. For example, the only inclusions between maps in an allegory are equalities, so that we cannot hope for the category of maps to have a non-trivial enrichment in Pos. On the other hand, $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ can be used to capture the enriched aspects of $\mathcal{E}$ but its structure is not as rich. In particular, the process of taking the opposite of a relation cannot be defined on $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$, since it clearly breaks the weakening-closed property. Thus, in our quest to construct the Pos-enriched exact completion as indicated above, we will not be able to simply rely on the general theory of allegories and will have to do more work "by hand". Nevertheless, allegories and $\mathrm{Rel}(\mathcal{E})$ in particular will still figure in our work below, as we will actually need to simultaneously split idempotents therein as well in order to facilitate computations.

4.4.1 Remark. Since by our proposed construction the morphisms in the completion $\mathcal{E}_{ex/reg}$ will be certain relations in $\mathcal{E}$, we will need to impose a size restriction on $\mathcal{E}$ in order to obtain a legitimate category enriched in Pos. Thus, throughout this section, we assume that $\mathcal{E}$ *is a well-powered regular category.* Here *well-powered* is defined to mean that every object of $\mathcal{E}$ has only a set of subobjects. Recall also that a subobject of $X \in \mathcal{E}$ is an equivalence class of ff-morphisms $A \rightarrowtail X$ in $\mathcal{E}$.

Motivated by the above, we embark towards our goal by first defining a category $Q_w(\mathcal{E})$ by splitting the idempotents in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ which are congruences in $\mathcal{E}$. Explicitly, $Q_w(\mathcal{E})$ is defined as follows:

- Objects of $Q_w(\mathcal{E})$ are pairs (X, E), where X is an object of $\mathcal{E}$ and $E\colon X \nrightarrow X$ is a congruence relation in $\mathcal{E}$.
- Morphisms $\Phi\colon (X, E) \to (Y, F)$ in $Q_w(\mathcal{E})$ are (weakening-closed) relations $\Phi\colon X \nrightarrow Y$ in $\mathcal{E}$ such that $\Phi E = \Phi = F\Phi$ or equivalently $\Phi = F\Phi E$.

Composition in $Q_w(\mathcal{E})$ is composition of relations in $\mathcal{E}$, while the identity morphism on $(X, E) \in Q_w(\mathcal{E})$ is the relation E itself. The morphisms are locally ordered by inclusion and it is clear that $Q_w(\mathcal{E})$ also has binary infima of morphisms given by intersection of relations in $\mathcal{E}$. Then we define a category $\mathcal{E}_{ex/reg}$ by taking the maps in $Q_w(\mathcal{E})$.

4.4.2 Definition. $\mathcal{E}_{ex/reg} \coloneqq \mathrm{Map}(\mathrm{Q_w}(\mathcal{E}))$.

Explicitly, $\mathcal{E}_{ex/reg}$ has the same objects as $Q_w(\mathcal{E})$ and as morphisms those $R_*\colon (X,E) \to (Y,F) \in Q_w(\mathcal{E})$ for which there exists an $R^*\colon (Y,F) \to (X,E) \in Q_w(\mathcal{E})$ such that $R^*R_* \supseteq E$ and $R_*R^* \subseteq F$.

Now we want $\mathcal{E}_{ex/reg}$ to be not merely an ordinary category, but more appropriately a Pos-category. This is accomplished by defining for any two morphisms $R_*, S_*\colon (X,E) \to (Y,F)$ in $\mathcal{E}_{ex/reg}$

$$R_* \leq S_* :\Leftrightarrow R_* \supseteq S_*$$

the inclusion on the right-hand side being that of relations in $\mathcal{E}$. This is clearly a partial order relation on Homs that is preserved by composition. Observe furthermore that we can equivalently define the order $R_* \leq S_*$ by requiring $R^* \subseteq S^*$ for the right adjoints.

We also have a canonical functor $\Gamma\colon \mathcal{E} \to \mathcal{E}_{ex/reg}$ defined by mapping an object $X \in \mathcal{E}$ to (X, I_X) and a morphism $f\colon X \to Y \in \mathcal{E}$ to its hypergraph $f_*\colon X \nrightarrow Y$ considered as a morphism $(X, I_X) \to (Y, I_Y)$. Note that Γ is order-preserving and order-reflecting by definition of the order on morphisms in $\mathcal{E}_{ex/reg}$.

4.4.3 Remark. We will consistently denote morphisms of $\mathcal{E}_{ex/reg}$ by a capital letter with a lower asterisk and their right adjoint in $Q_w(\mathcal{E})$ by the same letter with an upper asterisk. This notation represents our intuition that R_* is the hyper-graph of the morphism R and in some sense we are working towards making this a precise statement. In particular, we will denote the identity morphism $(X,E) \to (X,E)$ by $1_{(X,E)*}$, where as relations in $\mathcal{E}$ we have $1_{(X,E)*} = E$ and $1^*_{(X,E)} = E$.

Our goal now in this section is to show that the category $\mathcal{E}_{ex/reg}$ as defined above is indeed the *exact completion* of $\mathcal{E}$ as a regular category. The bicategory $Q_w(\mathcal{E})$ will be seen in the end to be precisely the bicategory of weakening-closed relations in $\mathcal{E}_{ex/reg}$. Accordingly, the proofs of the various statements about $\mathcal{E}_{ex/reg}$ later on in this section will naturally have to rely on the description of the various limit and exactness properties in terms of the calculus of relations in a regular category. However, we know that such a description cannot be achieved with only weakening-closed relations. Thus, it will be necessary to at the same time construct another bicategory, which in turn will a posteriori be the bicategory of all relations in $\mathcal{E}_{ex/reg}$.

This leads us to define the bicategory $Q(\mathcal{E})$ as follows:

- Objects of $Q(\mathcal{E})$ are again pairs (X,E), where $E\colon X \nrightarrow X$ is a congruence relation in $\mathcal{E}$.
- Morphisms $(X,E) \to (Y,F)$ in $Q(\mathcal{E})$ are relations $\Phi\colon X \nrightarrow Y$ in $\mathcal{E}$ such that $\Phi(E \cap E^\circ) = \Phi = (F \cap F^\circ)\Phi$ or equivalently $(F \cap F^\circ)\Phi(E \cap E^\circ) = \Phi$.

The composition of morphisms is that of relations in $\mathcal{E}$ while the identity on $(X,E) \in Q(\mathcal{E})$ is $1_{(X,E)} = E \cap E^\circ$.

In other words, $Q(\mathcal{E})$ is the locally ordered bicategory obtained from $\mathrm{Rel}(\mathcal{E})$ by splitting those idempotents of the form $E \cap E^\circ$ for a congruence E in $\mathcal{E}$. Notice that idempotents of this form are equivalence relations in $\mathcal{E}$ and in particular are symmetric. It then follows (see for example Theorem 3.3.4 in [57]) that $Q(\mathcal{E})$ is an allegory, where the opposite of a morphism is given by taking the opposite relation in $\mathcal{E}$.

Now we make some important observations regarding the connection between morphisms of $Q_w(\mathcal{E})$ and $Q(\mathcal{E})$ and between maps in these two bicategories. If the reader keeps in mind the intuition that these two categories should respectively be $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg})$ and $\mathrm{Rel}(\mathcal{E}_{ex/reg})$, then these observations are to be expected.

First, note that every morphism $\Phi\colon (X,E) \to (Y,F)$ in $Q_w(\mathcal{E})$ can also be considered as a morphism in $Q(\mathcal{E})$, since

$$\Phi = \Delta_Y \Phi \Delta_X \subseteq (F \cap F^\circ)\Phi(E \cap E^\circ) \subseteq F\Phi E = \Phi$$

from which we see that $(F \cap F^\circ)\Phi(E \cap E^\circ) = \Phi$. However, it is important to note as well that this assignment is not functorial as it does not preserve the identity morphisms.

Second, to any map in $Q_w(\mathcal{E})$, i.e. to any morphism of $\mathcal{E}_{ex/reg}$, we can associate in a natural way a map in $Q(\mathcal{E})$ as described in the following lemma.

4.4.4 Lemma. *Consider any $R_*\colon (X,E) \to (Y,F) \in \mathcal{E}_{ex/reg}$ and define the relation $\mathfrak{gr}(R_*) \coloneqq R_* \cap (R^*)^\circ\colon X \nrightarrow Y$ in $\mathcal{E}$. Then $\mathfrak{gr}(R_*)$ is a map $(X,E) \to (Y,F)$ in $Q(\mathcal{E})$.*

Proof. First, it is easy to see that $(F \cap F^\circ)\mathfrak{gr}(R_*)(E \cap E^\circ) = \mathfrak{gr}(R_*)$. Indeed, we have

$$\begin{aligned}\mathfrak{gr}(R_*) &= \Delta_Y \mathfrak{gr}(R_*)\Delta_X \subseteq (F \cap F^\circ)\mathfrak{gr}(R_*)(E \cap E^\circ)\\ &\subseteq (F\mathfrak{gr}(R_*)E) \cap (F^\circ \mathfrak{gr}(R_*)E^\circ) \subseteq FR_*E \cap F^\circ (R^*)^\circ E^\circ\\ &= R_* \cap (R^*)^\circ = \mathfrak{gr}(R_*)\end{aligned}$$

Second, to see that $\mathfrak{gr}(R_*)$ is indeed a map in $Q(\mathcal{E})$ we argue as follows:

$$\mathfrak{gr}(R_*)\mathfrak{gr}(R_*)^\circ = (R_* \cap (R^*)^\circ)((R_*)^\circ \cap R^*) \subseteq R_*R^* \cap (R^*)^\circ (R_*)^\circ \subseteq F \cap F^\circ$$

$$\begin{aligned}\mathfrak{gr}(R_*)^\circ\mathfrak{gr}(R_*) &= (R_*^\circ \cap R^*)(R_* \cap (R^*)^\circ) = (R_*^\circ \cap R^*)((R_* \cap (R^*)^\circ) \cap R_*)\\ &= (R_*^\circ \cap R^*)((R_*^\circ \cap R^*)^\circ (E \cap E^\circ) \cap R_*)\\ &\overset{ML}{\supseteq} (E \cap E^\circ) \cap (R_*^\circ \cap R^*)R_* = (E \cap E^\circ) \cap (E^\circ R_*^\circ \cap R^*)R_*\\ &\overset{ML*}{\supseteq} (E \cap E^\circ) \cap E^\circ \cap R^*R_* \supseteq (E \cap E^\circ) \cap E^\circ \cap E\\ &= E \cap E^\circ\end{aligned}$$

where for establishing the first and second inclusions we used the modular law in $\mathrm{Rel}(\mathcal{E})$. □

Given a morphism $R_*\colon (X,E) \to (Y,F) \in \mathcal{E}_{ex/reg}$, we call the relation $\mathfrak{gr}(R_*)$ defined above the *graph* of R_*. Observe furthermore that $\mathfrak{gr}(R_*)$ satisfies the following two basic equalities:

$$F \circ \mathfrak{gr}(R_*) = R_*$$

$$\mathfrak{gr}(R_*)^\circ \circ F = R^*$$

Indeed, on one hand clearly $F \circ \mathfrak{gr}(R_*) \subseteq FR_* = R_*$. On the other hand we have

$$\begin{aligned} F \circ \mathfrak{gr}(R_*) &\supseteq R_* R^* \mathfrak{gr}(R_*) = R_* R^* (R_* \cap (R^*)^\circ) \overset{ML}{\supseteq} R_*(R^* R_* \cap E^\circ) \\ &\supseteq R_*(E \cap E^\circ) \supseteq R_* \end{aligned}$$

The second equality follows in a similar fashion.

In fact, these two equalities characterize $\mathfrak{gr}(R_*)$ in the following sense: if $\Phi\colon (X,E) \to (Y,F)$ is a map in $Q(\mathcal{E})$ with $F\Phi = R_*$ and $\Phi^\circ F = R^*$, then $\Phi = \mathfrak{gr}(R_*)$. Indeed,

$$\Phi \subseteq (F \cap F^\circ)\Phi \subseteq F\Phi \cap F^\circ \Phi = F\Phi \cap (\Phi^\circ F)^\circ = R_* \cap (R^*)^\circ = \mathfrak{gr}(R_*)$$

and so $\Phi = \mathfrak{gr}(R_*)$ because $Q(\mathcal{E})$ is an allegory and hence the inclusion of maps is discrete.

Finally, the assignment $R_* \mapsto \mathfrak{gr}(R_*)$ is functorial. Given $R_*\colon (X,E) \to (Y,F)$ and $S_*\colon (Y,F) \to (Z,G)$ we have

$$G(\mathfrak{gr}(S_*)\mathfrak{gr}(R_*)) = (G\mathfrak{gr}(S_*))\mathfrak{gr}(R_*) = S_*\mathfrak{gr}(R_*) = S_* F \mathfrak{gr}(R_*) = S_* R_*$$

$$(\mathfrak{gr}(S_*)\mathfrak{gr}(R_*))^\circ G = \mathfrak{gr}(R_*)^\circ \mathfrak{gr}(S_*)^\circ G = \mathfrak{gr}(R_*)^\circ S^* = \mathfrak{gr}(R_*)^\circ F S^* = R^* S^*,$$

so we conclude by the above observation that $\mathfrak{gr}(S_*)\mathfrak{gr}(R_*) = \mathfrak{gr}(S_* R_*)$. Also, $\mathfrak{gr}(1_{(X,E)*}) = 1_{(X,E)*} \cap (1^*_{(X,E)})^\circ = E \cap E^\circ = 1_{(X,E)}$ and the latter is an identity morphism in $\mathrm{Map}(\mathrm{Q}(\mathcal{E}))$.

4.4.5 Remark. (on notation) Henceforth, to ease the notation, we shall often denote the graph $\mathfrak{gr}(R_*)$ of a morphism $R_*\colon (X,E) \to (Y,E) \in \mathcal{E}_{ex/reg}$ simply by R, i.e. we will just drop the lower asterisk. This shall not present too much risk for confusion as R_* will always have appeared before R.

We now begin our work on proving that $\mathcal{E}_{ex/reg}$ is indeed the desired exact completion of the regular category $\mathcal{E}$. This is broken down into a sequence of more bite-sized pieces. First, we establish a fundamental result asserting the existence of certain canonical representations for morphisms in the bicategories $Q_w(\mathcal{E})$ and $Q(\mathcal{E})$. Following this, we repeatedly employ this representation to establish step by step that $\mathcal{E}_{ex/reg}$ has the desired finite limit and exactness properties making it an exact category. The arguments here are essentially motivated by the description of these properties in terms of the calculus of relations. Finally, we show that we have indeed constructed the exact completion by establishing the relevant universal property.

To begin with, we record a small observation on jointly order-monomorphic pairs of morphisms in $\mathcal{E}_{ex/reg}$.

4.4.6 Lemma. *If* $(Y,F) \xleftarrow{R_*} (X,E) \xrightarrow{S_*} (Z,G)$ *are two morphisms in* $\mathcal{E}_{ex/reg}$ *such that* $R^*R_* \cap S^*S_* = E$*, then this pair is jointly order-monomorphic in* $\mathcal{E}_{ex/reg}$*.*

Proof. Assume that $R^*R_* \cap S^*S_* = E$ and let $H_*, K_* \colon (A,T) \to (X,E) \in \mathcal{E}_{ex/reg}$ be such that $R_*H_* \leq R_*K_*$ and $S_*H_* \leq S_*K_*$ i.e. $R_*H_* \supseteq R_*K_*$ and $S_*H_* \supseteq S_*K_*$. Then we have

$$\begin{aligned} K_*H^* &= EK_*H^* = (R^*R_* \cap S^*S_*)K_*H^* \subseteq R^*R_*K_*H^* \cap S^*S_*K_*H^* \\ &\subseteq R^*R_*H_*H^* \cap S^*S_*H_*H^* \subseteq R^*R_* \cap S^*S_*, \end{aligned}$$

hence $K_*H^* \subseteq E$. But recall that by definition of morphisms we have an adjunction $H_* \dashv H^*$ in $Q_w(\mathcal{E})$. Thus, $K_*H^* \subseteq E$ is equivalent to $K_* \subseteq H_*$, so that we obtain $H_* \leq K_*$. □

The result that follows will be of central importance in establishing all the desired properties of $\mathcal{E}_{ex/reg}$ throughout the remainder of this section. It says that any morphism of $Q(\mathcal{E})$ (hence also of $Q_w(\mathcal{E})$) can be expressed in a suitable way via morphisms of $\mathcal{E}_{ex/reg}$. This should be compared to the fact that in any regular category $\mathcal{C}$ every relation $R \colon X \nrightarrow Y$ can be written as $R = gf^\circ$, where $f \colon Z \to X$ and $g \colon Z \to Y$ are morphisms in $\mathcal{C}$ with $f^*f_* \cap g^*g_* = I_Z$. Actually, our goal is to show in the end that it is *precisely* this fact, since we will prove that $Q(\mathcal{E})$ is exactly the bicategory of relations of $\mathcal{E}_{ex/reg}$, while $Q_w(\mathcal{E})$ will be that of weakening-closed relations.

4.4.7 Proposition. *Let* $\Phi \colon (X,E) \to (Y,F)$ *be a morphism of* $Q(\mathcal{E})$*. Then there exists a pair of morphisms* $(X,E) \xleftarrow{R_{0*}} (Z,T) \xrightarrow{R_{1*}} (Y,F)$ *in* $\mathcal{E}_{ex/reg}$ *such that*

1. $\Phi = R_1R_0^\circ$.
2. $R_0^*R_{0*} \cap R_1^*R_{1*} = T$.

Moreover, any pair (R_{0*}, R_{1*}) *with these two properties has the following universal property:*

Given any morphisms $(X,E) \xleftarrow{S_{0*}} (C,G) \xrightarrow{S_{1*}} (Y,F)$ *in* $\mathcal{E}_{ex/reg}$ *such that* $S_1S_0^\circ \subseteq \Phi$*, there exists a unique morphism* $H_* \colon (C,G) \to (Z,T) \in \mathcal{E}_{ex/reg}$ *with* $R_{0*}H_* = S_{0*}$ *and* $R_{1*}H_* = S_{1*}$*.*

Proof. Suppose that Φ is represented by the ff-morphism $\langle r_0, r_1 \rangle \colon Z \rightarrowtail X \times Y$ in $\mathcal{E}$. We set $T \coloneqq r_0^\circ E r_0 \cap r_1^\circ F r_1$ and $R_{0*} \coloneqq Er_0$, $R_{1*} \coloneqq Fr_1$. The relations $r_0^\circ E r_0$ and $r_1^\circ F r_1$ are inverse images along r_0, r_1 respectively of the congruences E, F, hence are themselves congruences. Thus, so is their intersection T.

Also, we claim that that R_{0*} and R_{1*} as defined above are morphisms $(X,E) \xleftarrow{R_{0*}} (Z,T) \xrightarrow{R_{1*}} (Y,F)$ in $\mathcal{E}_{ex/reg}$. Let's check this for R_{0*}: first, we have $ER_{0*} = EEr_0 = Er_0 = R_{0*}$. Furthermore,

$$R_{0*}T = Er_0(r_0^\circ Er_0 \cap r_1^\circ Fr_1) \subseteq Er_0 r_0^\circ Er_0 \subseteq E\Delta_X Er_0 = EEr_0 = Er_0 = R_{0*},$$

hence $R_{0*}T = R_{0*}$. So R_{0*} is at least a morphism in $Q_w(\mathcal{E})$. To show that it is actually a map, define $R_0^* \coloneqq r_0^\circ E$. We then similarly have

$$TR_0^* = (r_0^\circ Er_0 \cap r_1^\circ Fr_1)r_0^\circ E \subseteq r_0^\circ Er_0 r_0^\circ E \subseteq r_0^\circ EE = r_0^\circ E = R_0^* \implies TR_0^* = R_0^*$$

and $R_0^* E = r_0^\circ EE = r_0^\circ E = R_0^*$. And finally, $R_0^* R_{0*} = r_0^\circ EEr_0 = r_0^\circ Er_0 \supseteq T$ and $R_{0*}R_0^* = Er_0 r_0^\circ E \subseteq EE = E$.

Similarly, it follows that R_{1*} is a morphism $(Z,T) \to (Y,F) \in \mathcal{E}_{ex/reg}$ whose right adjoint in $Q_w(\mathcal{E})$ is $R_1^* \coloneqq r_1^\circ F$.

Just by the definitions, we have

$$R_0^* R_{0*} \cap R_1^* R_{1*} = r_0^\circ EEr_0 \cap r_1^\circ FFr_1 = r_0^\circ Er_0 \cap r_1^\circ Fr_1 = T.$$

In addition, $R_0 = R_{0*} \cap (R_0^*)^\circ = Er_0 \cap (r_0^\circ E)^\circ = Er_0 \cap E^\circ r_0 \overset{MD}{=} (E \cap E^\circ)r_0$ and similarly $R_1 = (F \cap F^\circ)r_1$, so that

$$R_1 R_0^\circ = (F \cap F^\circ)r_1 r_0^\circ (E \cap E^\circ) = (F \cap F^\circ)\Phi(E \cap E^\circ) = \Phi.$$

where the last equality holds because $\Phi \in Q(\mathcal{E})$.

Next, we have to prove the stated universality property, so let $(X,E) \xleftarrow{S_{0*}} (C,G) \xrightarrow{S_{1*}} (Y,F)$ in $\mathcal{E}_{ex/reg}$ be such that $S_1 S_0^\circ \subseteq \Phi$.

Set $H_* \coloneqq R_0^* S_{0*} \cap R_1^* S_{1*}$ and $H^* \coloneqq S_0^* R_{0*} \cap S_1^* R_{1*}$. Since, both H_*, H^* are binary intersections of compositions of morphisms in $Q_w(\mathcal{E})$, it is immediate that they are both themselves morphisms in that bicategory. We will show that H_* is moreover a map with H^* as its right adjoint.

First of all, we have that

$$\begin{aligned} H_* H^* &\subseteq R_0^* S_{0*} S_0^* R_{0*} \cap R_1^* S_{1*} S_1^* R_{1*} \subseteq R_0^* E R_{0*} \cap R_1^* F R_{1*} \\ &= R_0^* R_{0*} \cap R_1^* R_{1*} = T \end{aligned}$$

For the other inclusion we argue as follows:

$$\begin{aligned} H^* H_* &= (S_0^* R_{0*} \cap S_1^* R_{1*})(R_0^* S_{0*} \cap R_1^* S_{1*}) \\ &\supseteq (S_0^\circ R_0 \cap S_1^\circ R_1)(R_0^\circ S_0 \cap R_1^\circ S_1) \\ &= (S_0^\circ R_0 \cap S_1^\circ R_1)((R_0^\circ S_0 \cap R_1^\circ S_1)(G \cap G^\circ) \cap R_0^\circ S_0) \\ &\overset{ML}{\supseteq} (G \cap G^\circ) \cap (S_0^\circ R_0 \cap S_1^\circ R_1) R_0^\circ S_0 \\ &= (G \cap G^\circ) \cap ((G \cap G^\circ) S_0^\circ R_0 \cap S_1^\circ R_1) R_0^\circ S_0 \\ &\overset{ML*}{\supseteq} (G \cap G^\circ) \cap (G \cap G^\circ) \cap S_1^\circ R_1 R_0^\circ S_0 \end{aligned}$$

But now, using adjunction properties in $Q(\mathcal{E})$ together with the assumption that $S_1 S_0^\circ \subseteq \Phi$, we observe that

$$S_1 S_0^\circ \subseteq \Phi = R_1 R_0^\circ \implies S_1 \subseteq R_1 R_0^\circ S_0 \implies G \cap G^\circ \subseteq S_1^\circ R_1 R_0^\circ S_0,$$

from which we deduce $H^* H_* \supseteq (G \cap G^\circ) \cap S_1^\circ R_1 R_0^\circ S_0 = G \cap G^\circ$. Then, finally, $H^* H_* G \supseteq (G \cap G^\circ) G$, which implies $H^* H_* \supseteq G$.

Thus, H_* is indeed a morphism in $\mathcal{E}_{ex/reg}$. Furthermore,

$$R_{0*} H_* = R_{0*}(R_0^* S_{0*} \cap R_1^* S_{1*}) \subseteq R_{0*} R_0^* S_{0*} \subseteq E S_{0*} = S_{0*}$$

$$R_{0*} H_* = R_{0*}(R_0^* S_{0*} \cap R_1^* S_{1*}) \supseteq R_0(R_0^\circ S_0 \cap R_1^\circ S_1) \overset{ML}{\supseteq} S_0 \cap R_0 R_1^\circ S_1 = S_0$$

where for the last inclusion we made use of the fact that $S_1 S_0^\circ \subseteq \Phi = R_1 R_0^\circ$ implies $S_0 \subseteq R_0 R_1^\circ S_1$ by the adjunction property in $Q(\mathcal{E})$. Now we have $E R_{0*} H_* \supseteq E S_0$, which implies $R_{0*} H_* \supseteq S_{0*}$. Thus, we deduce that $R_{0*} H_* = S_{0*}$ and similarly one obtains $R_{1*} H_* = S_{1*}$. Finally, uniqueness is clear because, as we have proved in the previous lemma, the equality $R_0^* R_{0*} \cap R_1^* R_{1*} = T$ implies that R_{0*}, R_{1*} are jointly order-monomorphic in $\mathcal{E}_{ex/reg}$. □

A pair of morphisms (R_{0*}, R_{1*}) in $\mathcal{E}_{ex/reg}$ with the properties in Proposition 4.4.7 will be called a *tabulation* of $\Phi\colon (X,E) \to (Y,F) \in Q(\mathcal{E})$, terminology which is borrowed from the theory of allegories. Note incidentally that the latter theory could have been directly applied to $Q(\mathcal{E})$, but this approach would not work for us. The reason for that is that we can not identify the morphisms of the would be completion $\mathcal{E}_{ex/reg}$ as maps in this allegory, but rather only in $Q_w(\mathcal{E})$. Thus, we need both of these bicategories at the same time: $Q_w(\mathcal{E})$ to identify the morphisms of $\mathcal{E}_{ex/reg}$ and $Q(\mathcal{E})$ to express the existence of tabulations and perform calculations more freely.

Furthermore, since as we've noted earlier every morphism $\Phi : (X,E) \to (Y,F) \in Q_w(\mathcal{E})$ can be considered as a morphism in $Q(\mathcal{E})$, we also have tabulations for morphisms of $Q_w(\mathcal{E})$. In this case the inclusion $S_1 S_0^\circ \subseteq \Phi$ in the universal property of tabulations is equivalent to $S_{1*} S_0^* \subseteq \Phi$. Indeed, $S_1 S_0^\circ \subseteq \Phi \implies F S_1 S_0^\circ E \subseteq F \Phi E \implies S_{1*} S_0^* \subseteq \Phi$. Similarly, for the tabulation (R_{0*}, R_{1*}) we have $\Phi = R_1 R_0^\circ = R_{1*} R_0^*$.

As we've already claimed, the existence of tabulations will be a fundamental tool for establishing results about $\mathcal{E}_{ex/reg}$. As a first example of this, we can now completely characterize what it means for a pair of morphisms to be jointly order-monomorphic in $\mathcal{E}_{ex/reg}$.

4.4.8 Corollary. *A pair of morphisms* $(Y,F) \xleftarrow{R_*} (X,E) \xrightarrow{S_*} (Z,G)$ *is jointly order-monomorphic in* $\mathcal{E}_{ex/reg}$ *if and only if* $R^* R_* \cap S^* S_* = E$.

Proof. We've already proven sufficiency in Lemma 4.4.6, so assume conversely that R_*, S_* are jointly order-monomorphic. By the proposition above, there

exists a tabulation
$(X,E) \xleftarrow{U_*} (A,T) \xrightarrow{V_*} (X,E)$ for the morphism $R^*R_* \cap S^*S_* \in Q_w(\mathcal{E})$. Then by using the adjunctions in $Q_w(\mathcal{E})$ we have

$$V_*U^* = R^*R_* \cap S^*S_* \subseteq R^*R_* \implies R_*V_*U^* \subseteq R_* \implies R_*V_* \subseteq R_*U_*$$

and similarly that $S_*V_* \subseteq S_*U_*$. Thus, we have $R_*V_* \geq R_*U_*$ and $S_*V_* \geq S_*U_*$ and hence $V_* \geq U_*$, which means that $V_* \subseteq U_*$. But now

$$R^*R_* \cap S^*S_* = V_*U^* \subseteq U_*U^* \subseteq E$$

Since the reverse inclusion always holds, we conclude that $R^*R_* \cap S^*S_* = E$. □

Before beginning to prove the basic finite limit and exactness properties of $\mathcal{E}_{ex/reg}$ we will need some information on ff-morphisms and so-morphisms therein. Once again, keeping in mind the intuition that R_* is the hypergraph and R^* the hypograph of a morphism R, these should not be surprising.

4.4.9 Lemma. *Let $R_* \colon (X,E) \to (Y,F)$ be a morphism in $\mathcal{E}_{ex/reg}$. Then:*

1. *R_* is an ff-morphism in $\mathcal{E}_{ex/reg}$ if and only if $R^*R_* = E$.*
2. *R_* is an isomorphism if and only if $R^*R_* = E$ and $R_*R^* = F$.*
3. *If $R_*R^* = F$, then R_* is an so-morphism in $\mathcal{E}_{ex/reg}$.*

Proof. 1. This is a special case of the previous corollary.

2. Clear.

3. Consider a commutative square in $\mathcal{E}_{ex/reg}$ as below, where M_* is an ff-morphism.

$$\begin{array}{ccc} (X,E) & \xrightarrow{R_*} & (Y,F) \\ {\scriptstyle V_*}\downarrow & & \downarrow{\scriptstyle S_*} \\ (Z,G) & \xrightarrow[M_*]{} & (W,H) \end{array}$$

By (1.), we know that $M^*M_* = G$. Set $P_* \coloneqq V_*R^*$. First, we claim that P_* is a morphism $(Y,F) \to (Z,G)$ in $\mathcal{E}_{ex/reg}$ with $P^* = R_*V^*$. Indeed, we have $P^*P_* = R_*V^*V_*R^* \supseteq R_*R^* = F$. Observe also that $P_* = M^*S_*$, because $M_*V_* = S_*R_* \implies M^*M_*V_* = M^*S_*R_* \implies V_* = M^*S_*R_* \implies V_*R^* = M^*S_*R_*R^* = M^*S_*$. Then we can argue that $P_*P^* = M^*S_*R_*V^* = M^*M_*V_*V^* = V_*V^* \subseteq G$.

Finally, clearly $P_*R_* = M^*S_*R_* = M^*M_*V_* = V_*$ and also $M_*P_* = M_*V_*R^* = S_*R_*R^* = S_*$.

□

4.4.10 Proposition. *$\mathcal{E}_{ex/reg}$ has finite limits and $\Gamma\colon \mathcal{E} \to \mathcal{E}_{ex/reg}$ preserves them.*

Proof. Let's first construct the inserter of a pair $(X,E) \overset{R_*}{\underset{S_*}{\rightrightarrows}} (Y,F)$. To this end, consider a tabulation $(X,E) \xleftarrow{\Phi_{0*}} (A,T) \xrightarrow{\Phi_{1*}} (X,E)$ of $S^*R_* \cap (E \cap E^\circ)$ as a morphism $(X,E) \to (X,E)$ in $Q(\mathcal{E})$. Then we observe that

$$\begin{aligned}\Phi_1 = \Phi_1(T \cap T^\circ) = \Phi_1(\Phi_0^\circ\Phi_0 \cap \Phi_1^\circ\Phi_1) \subseteq \Phi_1\Phi_0^\circ\Phi_0 = (S^*R_* \cap E \cap E^\circ)\Phi_0 \\ \subseteq (E \cap E^\circ)\Phi_0 = \Phi_0\end{aligned}$$

and hence $\Phi_1 = \Phi_0$ because the inclusion of maps in $Q(\mathcal{E})$ is discrete. So we have $\Phi_{1*} = \Phi_{0*}$, which we henceforth denote simply by Φ_*. To prove that $\Phi_*\colon (A,T) \to (X,E)$ is the inserter of (R_*, S_*) it now suffices, due to the universal property of tabulations, to show that for every $H_*\colon (Z,G) \to (X,E)$ we have $R_*H_* \leq S_*H_*$ if and only if $HH^\circ \subseteq S^*R_* \cap E \cap E^\circ$. Indeed, by using the various adjunctions in $Q_w(\mathcal{E})$ we have equivalences

$$\begin{aligned} R_*H_* \leq S_*H_* \quad &\iff \quad S_*H_* \subseteq R_*H_* \\ &\iff \quad H_* \subseteq S^*R_*H_* \\ &\iff \quad H_*H^* \subseteq S^*R_* \end{aligned}$$

But observe that the last inclusion is equivalent to $HH^\circ \subseteq S^*R_*$. One direction of this is clear, because $H \subseteq H_*$ and $H^\circ \subseteq H^*$. The converse follows because $HH^\circ \subseteq S^*R_*$ implies $EHH^\circ E \subseteq ES^*R_*E = S^*R_*$ and we know that $EH = H_*$ and $H^\circ E = H^*$. Finally, since $HH^\circ \subseteq E \cap E^\circ$ always holds, we have that $HH^\circ \subseteq S^*R_*$ is in turn equivalent to $HH^\circ \subseteq S^*R_* \cap E \cap E^\circ$.

Next, let us construct the product of a pair of objects (X,E) and (Y,F) in $\mathcal{E}_{ex/reg}$. Observe that the maximal relation $X \nrightarrow Y$ given by the product is clearly a morphism $(X,E) \to (Y,F)$ in $Q(\mathcal{E})$. Therefore, by Proposition 4.4.7 it has a tabulation. In fact, looking back at the proof of the latter proposition we can easily see that the tabulation thus constructed is given by the pair of morphisms $(X,E) \xleftarrow{\Pi_{(X,E)*}} (X \times Y, E \times F) \xrightarrow{\Pi_{(Y,F)*}} (Y,F)$, where $\Pi_{(X,E)*} = E\pi_X$ and $\Pi_{(Y,F)*} = F\pi_Y$. In any case, the universal property of tabulations gives precisely the universal property of a product diagram in $\mathcal{E}_{ex/reg}$.

Finally, it is easy to check that $\Gamma 1$ is a terminal object for $\mathcal{E}_{ex/reg}$. □

Even though the above proposition tells us that $\mathcal{E}_{ex/reg}$ inherits all finite weighted limits from $\mathcal{E}$, we shall need some more specific information as well, namely on the construction of comma squares and pullbacks.

Consider any morphisms $R_*\colon (X,E) \to (Z,G)$ and $S_*\colon (Y,F) \to (Z,G)$ in $\mathcal{E}_{ex/reg}$. First, we construct the comma square below:

$$\begin{array}{ccc} (W,T) & \xrightarrow{P_{1*}} & (Y,F) \\ {\scriptstyle P_{0*}}\downarrow & \leq & \downarrow{\scriptstyle S_*} \\ (X,E) & \xrightarrow[R_*]{} & (Z,G) \end{array}$$

For this, we take (P_{0*}, P_{1*}) to be a tabulation of $S^*R_*\colon (X,E) \to (Y,F) \in Q_w(\mathcal{E})$. To prove that this square is indeed a comma, it suffices to prove that, given any $U_*\colon (A,H) \to (X,E)$ and $V_*\colon (A,H) \to (Y,F)$, we have $V_*U^* \subseteq S^*R_*$ if and only if $R_*U_* \leq S_*V_*$. But indeed, using properties of adjunctions we have

$$R_*U_* \leq S_*V_* \iff S_*V_* \subseteq R_*U_* \iff V_* \subseteq S^*R_*U_* \iff V_*U^* \subseteq S^*R_*$$

To construct pullbacks let us take now (P_{0*}, P_{1*}) to be a tabulation of the relation $S^\circ R \in Q(\mathcal{E})$. Then we claim that the following square is a pullback.

$$\begin{array}{ccc} (W,T) & \xrightarrow{P_{1*}} & (Y,F) \\ {\scriptstyle P_{0*}}\downarrow & & \downarrow{\scriptstyle S_*} \\ (X,E) & \xrightarrow[R_*]{} & (Z,G) \end{array}$$

Indeed, in this case we have for any two morphisms $U_*\colon (A,H) \to (X,E)$ and $V_*\colon (A,H) \to (Y,F)$ that $R_*U_* = S_*V_*$ if and only if $RU = SV$, which in the allegory $Q(\mathcal{E})$ is equivalent to the inclusion $SV \subseteq RU$. By the familiar adjunctions in $Q(\mathcal{E})$, the latter is in turn equivalent to $SVU^\circ \subseteq R$ and then to $VU^\circ \subseteq S^\circ R$. Thus, the universal property of the pullback is identified with that of the tabulation.

Next, we prove that $\mathcal{E}_{ex/reg}$ admits the required factorization system.

4.4.11 Proposition. *$\mathcal{E}_{ex/reg}$ has (*so,ff*)-factorizations.*

Proof. Consider a morphism $R_*\colon (X,E) \to (Y,F) \in \mathcal{E}_{ex/reg}$. Then RR° is a morphism $(Y,F) \to (Y,F)$ in $Q(\mathcal{E})$ and so it admits a tabulation, say given by the span $(Y,F) \xleftarrow{S_{0*}} (Z,G) \xrightarrow{S_{1*}} (Y,F)$. Since tautologically R_* is such that the inclusion $RR^\circ \subseteq S_1S_0^\circ$ holds, there exists a unique $Q_*\colon (X,E) \to (Z,G) \in \mathcal{E}_{ex/reg}$ such that $S_{0*}Q_* = R_* = S_{1*}Q_*$.

Now observe that in $Q(\mathcal{E})$ we have $S_1 \subseteq S_1S_0^\circ S_0 = RR^\circ S_0 \subseteq S_0$ and so we deduce that $S_1 = S_0$ and hence $S_{1*} = S_{0*}$. We denote this morphism now simply by S_*. Then we have $S^*S_* = G$ by the tabulation property and this tells us that S_* is an ff-morphism. It suffices now to show that $Q_*Q^* = G$, so that Q_* will be an so-morphism. For this we argue as follows:

$$\begin{aligned} SQQ^\circ S^\circ &= RR^\circ = SS^\circ \implies \\ S^\circ SQQ^\circ S^\circ S &= S^\circ SS^\circ S \implies \\ (G \cap G^\circ)QQ^\circ(G \cap G^\circ) &= G \cap G^\circ \implies \\ QQ^\circ &= G \cap G^\circ \end{aligned}$$

Then $Q_*Q^* = GQQ^\circ G = G(G \cap G^\circ)G = G$. □

4.4.12 Remark. For a morphism $R_*\colon (X,E) \to (Y,F) \in \mathcal{E}_{ex/reg}$ we have $R_*R^* = F$ if and only if $RR^\circ = F \cap F^\circ$. We showed the "if" direction in the course of the above proof. For the converse, assume that $R_*R^* = F$ and argue as follows:

$$\begin{aligned} RR^\circ = R(R^\circ(F \cap F^\circ) \cap R^*) &\overset{ML}{\supseteq} (F \cap F^\circ) \cap RR^* = (F \cap F^\circ) \cap (R_* \cap (R^*)^\circ)R^* \\ &\overset{ML*}{\supseteq} (F \cap F^\circ) \cap R_*R^* \cap F^\circ = (F \cap F^\circ) \cap F \cap F^\circ = F \cap F^\circ \end{aligned}$$

The existence of factorizations established above now allows us to obtain the expected characterization of so-morphisms in $\mathcal{E}_{ex/reg}$.

4.4.13 Corollary. *A morphism $R_*\colon (X,E) \to (Y,F)$ is an* so*-morphism in $\mathcal{E}_{ex/reg}$ if and only if $R_*R^* = F$, if and only if $RR^\circ = F \cap F^\circ$.*

With this equational characterization of so-morphisms in hand, we are in a good position to prove their stability under pullback.

4.4.14 Proposition. so*-morphisms are stable under pullback in $\mathcal{E}_{ex/reg}$.*

Proof. Consider the following pullback square in $\mathcal{E}_{ex/reg}$ where we assume that R_* is an so-morphism, so that we have $R_*R^* = G$ or equivalently $RR^\circ = G \cap G^\circ$.

$$\begin{array}{ccc} (W,T) & \xrightarrow{Q_*} & (Y,F) \\ {\scriptstyle P_*}\downarrow & & \downarrow{\scriptstyle S_*} \\ (X,E) & \xrightarrow[R_*]{} & (Z,G) \end{array}$$

By the construction of pullbacks we know that (P_*, Q_*) is a tabulation of $S^\circ R$, so that $QP^\circ = S^\circ R$. Thus, we have

$$\begin{aligned} QQ^\circ &= Q(T \cap T^\circ)Q^\circ = Q(P^\circ P \cap Q^\circ Q)Q^\circ \overset{ML}{\supseteq} (QP^\circ P \cap Q)Q^\circ \\ &\overset{ML*}{\supseteq} QP^\circ PQ^\circ \cap (F \cap F^\circ) = S^\circ RR^\circ S \cap (F \cap F^\circ) \\ &= S^\circ(G \cap G^\circ)S \cap (F \cap F^\circ) = S^\circ S \cap (F \cap F^\circ) = F \cap F^\circ \end{aligned}$$

Hence, $QQ^\circ = F \cap F^\circ$ or equivalently $Q_*Q^* = F$ and hence Q_* is an so-morphism. □

Putting together what we have proved so far, we have the following.

4.4.15 Corollary. *$\mathcal{E}_{ex/reg}$ is a regular category and $\Gamma\colon \mathcal{E} \to \mathcal{E}_{ex/reg}$ is a fully order-faithful regular functor.*

We next would like to prove that $\mathcal{E}_{ex/reg}$ is exact. To accomplish this we first make good on promises made much earlier and make certain intuitions into precise statements. Namely, we identify $Q_w(\mathcal{E})$ as the bicategory of weakening-closed relations in $\mathcal{E}_{ex/reg}$ and, before that, $Q(\mathcal{E})$ as the bicategory of all relations in $\mathcal{E}_{ex/reg}$.

4.4.16 Proposition. *There is an equivalence* $\mathrm{Rel}(\mathcal{E}_{ex/reg}) \simeq Q(\mathcal{E})$.

Proof. We will define a functor $\mathfrak{F}\colon \mathrm{Rel}(\mathcal{E}_{ex/reg}) \to Q(\mathcal{E})$ by letting it be the identity on objects and mapping a relation represented by any jointly order-monomorphic pair $(X,E) \xleftarrow{R_{0*}} (Z,T) \xrightarrow{R_{1*}} (Y,F)$ in $\mathcal{E}_{ex/reg}$ to the morphism $R_1R_0^\circ\colon (X,E) \to (Y,F) \in Q(\mathcal{E})$.

To show that this assignment is functorial, consider first the diagonal relation on the object (X,E) in $\mathrm{Rel}(\mathcal{E}_{ex/reg})$, i.e. the relation represented by the jointly order-monomorphic pair $(X,E) \xleftarrow{1_{(X,E)*}} (X,E) \xrightarrow{1_{(X,E)*}} (X,E)$. Then the image of this relation under $\mathfrak{F}$ is $1_{(X,E)}1^\circ_{(X,E)} = (E \cap E^\circ)(E \cap E^\circ)^\circ = E \cap E^\circ$ and so $\mathfrak{F}$ preserves identity morphisms.

Next, we consider two relations $\mathscr{R}, \mathscr{S}$ in $\mathcal{E}_{ex/reg}$ represented by the jointly order-monomorphic pairs $(X,E) \xleftarrow{R_{0*}} (A,T) \xrightarrow{R_{1*}} (Y,F)$ and $(Y,F) \xleftarrow{S_{0*}} (B,T') \xrightarrow{S_{1*}} (Z,G)$ respectively. To calculate the composition of these two relations we form the following pullback square in $\mathcal{E}_{ex/reg}$

$$\begin{array}{ccc} (C,\Omega) & \xrightarrow{\Pi_{1*}} & (B,T') \\ {\scriptstyle \Pi_{0*}}\downarrow & & \downarrow{\scriptstyle S_{0*}} \\ (A,T) & \xrightarrow[R_{1*}]{} & (Y,F) \end{array}$$

and then the image factorization of $\langle R_{0*}\Pi_{0*}, S_{1*}\Pi_{1*}\rangle$, say

$$(C,\Omega) \xrightarrow{\quad Q_* \quad} (D,\Theta) \xrightarrow{\langle U_{0*},U_{1*}\rangle} (X,E)\times(Z,G)$$

By construction of pullbacks in $\mathcal{E}_{ex/reg}$ we know that $\Pi_1\Pi_0^\circ = S_0^\circ R_1$. Also, by definition $\mathfrak{F}$ maps the composition of the two relations to $\mathfrak{F}(\mathscr{S}\mathscr{R}) = U_1U_0^\circ$. But now we have that

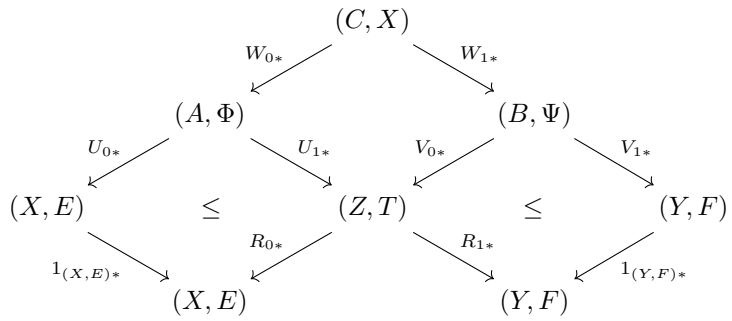
$$U_1U_0^\circ = U_1QQ^\circ U_0^\circ = S_1\Pi_1\Pi_0^\circ R_0^\circ = S_1S_0^\circ R_1R_0^\circ = \mathfrak{F}(\mathscr{S})\mathfrak{F}(\mathscr{R})$$

Finally, the fact that $\mathfrak{F}$ preserves the order of morphisms and is fully order-faithful is precisely the existence of tabulations proved in Proposition 4.4.7. Thus, $\mathfrak{F}$ is an equivalence of bicategories. □

4.4.17 Proposition. *There is an equivalence* $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg}) \simeq Q_w(\mathcal{E})$.

Proof. We define a functor $\mathfrak{F}\colon \mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg}) \to Q_w(\mathcal{E})$ exactly as in the proof of the previous proposition. Then the main observation to make here is the following: a relation $\mathscr{R}\colon (X,E) \nrightarrow (Y,F)$ represented by the jointly order-monomorphic pair $(X,E) \xleftarrow{R_{0*}} (Z,T) \xrightarrow{R_{1*}} (Y,F)$ in $\mathcal{E}_{ex/reg}$ is weakening-closed if and only if $R_{1*}R_0^* = R_1R_0^\circ$ as relations in $\mathcal{E}$.

Recall that $\mathscr{R}$ is a weakening-closed relation precisely if $I_{(Y,F)}\mathscr{R}I_{(X,E)} = \mathscr{R}$ in $\mathrm{Rel}(\mathcal{E}_{ex/reg})$. To compute the composition $I_{(Y,F)}\mathscr{R}I_{(X,E)}$ one has to form the following diagram in $\mathcal{E}_{ex/reg}$ where the top square is a pullback and the bottom two are commas and then take the image factorization of the morphism $\langle U_{0*}W_{0*}, V_{1*}W_{1*}\rangle$.



Note that by the various limit constructions in $\mathcal{E}_{ex/reg}$ we know that we must have $R_0^* = U_1U_0^\circ$, $R_{1*} = V_1V_0^\circ$ and $V_0^\circ U_1 = W_1W_0^\circ$.

If $I_{(Y,F)}\mathscr{R}I_{(X,E)} = \mathscr{R}$, then there is a factorization in $\mathcal{E}_{ex/reg}$

$$(C,X) \xrightarrow{\quad Q_* \quad} (Z,T) \xrightarrow{\langle R_{0*}, R_{1*}\rangle} (X,E) \times (Y,F)$$

with Q_* an so-morphism, so that $QQ^\circ = T \cap T^\circ$. Then we have that

$$R_{1*}R_0^* = V_1V_0^\circ U_1U_0^\circ = V_1W_1W_0^\circ U_0^\circ = R_1QQ^\circ R_0^\circ = R_1R_0^\circ.$$

Conversely, assume that $R_{1*}R_0^* = R_1R_0^\circ$ holds and consider morphisms $(X,E) \underset{U_*}{\overset{S_{0*}}{\leftleftarrows}} (C,G) \underset{V_*}{\overset{S_{1*}}{\rightrightarrows}} (Y,F)$ in $\mathcal{E}_{ex/reg}$ such that the pair (S_{0*}, S_{1*}) factors through the pair (R_{0*}, R_{1*}) and $U_* \leq S_{0*}$ and $S_{1*} \leq V_*$. Then we respectively have $S_{1*}S_0^* \subseteq R_{1*}R_0^*$ and $U^* \subseteq S_0^*$ and $V_* \subseteq S_{1*}$. Hence, $V_*U^* \subseteq S_{1*}S_0^* \subseteq R_{1*}R_0^* = R_1R_0^\circ$ and so (U_*, V_*) must also factor through (R_{0*}, R_{1*}) by the universal property of tabulations.

With this observation in hand, one can run the same proof as in the previous proposition to show that $\mathfrak{F}\colon \mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg}) \to Q_w(\mathcal{E})$ thus defined is an equivalence. The only point of minor difference is in the proof that $\mathfrak{F}$ preserves identity morphisms. For this, just recall that the identity on an object (X,E) in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg})$ is the relation given by the following comma square

$$\begin{array}{ccc} (A,T) & \xrightarrow{R_{1*}} & (X,E) \\ {\scriptstyle R_{0*}}\downarrow & \leq & \downarrow{\scriptstyle 1_{(X,E)*}} \\ (X,E) & \xrightarrow[1_{(X,E)*}]{} & (X,E) \end{array}$$

so that by construction of commas we have that $R_1 R_0^\circ = 1^*_{(X,E)} 1_{(X,E)*} = EE = E$. □

Now from this last proposition one can immediately deduce that $\mathcal{E}_{ex/reg}$ is indeed an exact category.

4.4.18 Corollary. *The category $\mathcal{E}_{ex/reg}$ is exact.*

Proof. Using the equivalence $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg}) \simeq Q_w(\mathcal{E})$, we see that a congruence R on an object $(X,E) \in \mathcal{E}_{ex/reg}$ corresponds precisely to a congruence R on the object $X \in \mathcal{E}$ with $R \supseteq E$.

Indeed, consider a congruence $\mathscr{R}$ on the object $(X,E) \in \mathcal{E}_{ex/reg}$, represented by the jointly order-monic pair $(X,E) \xleftarrow{R_{0*}} (Y,F) \xrightarrow{R_{1*}} (X,E)$. Consider the functor $\mathfrak{F}\colon \mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg}) \to Q_w(\mathcal{E})$ providing the equivalence and let $R \coloneqq \mathfrak{F}(\mathscr{R})$. Since $\mathfrak{F}$ is an equivalence, we have that $\mathscr{R}\mathscr{R} \subseteq \mathscr{R}$ if and only if $RR \subseteq R$ in $Q_w(\mathcal{E})$, i.e. that transitivity of $\mathscr{R}$ is equivalent to the same property for R as a relation in $\mathcal{E}$. Similarly, the inclusion $\mathscr{R} \supseteq I_{(X,E)}$ is equivalent to $R \supseteq \mathfrak{F}(I_{(X,E)}) = E$. In particular, $R \supseteq I_X$ and so R is a congruence on $X \in \mathcal{E}$.

The idempotent morphism $\mathfrak{F}(\mathscr{R}) = R : (X,E) \to (X,E)$ in $Q_w(\mathcal{E})$ now splits by construction, namely as $(X,E) \xrightarrow{R} (X,R) \xrightarrow{R} (X,E)$. Thus, $\mathscr{R}$ splits as an idempotent in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E}_{ex/reg})$ and hence we have shown that $\mathcal{E}_{ex/reg}$ is exact. □

It remains to prove that $\mathcal{E}_{ex/reg}$, or more precisely $\Gamma\colon \mathcal{E} \to \mathcal{E}_{ex/reg}$, satisfies the required universal property. Before doing this, we observe in the proposition that follows that every object of $\mathcal{E}_{ex/reg}$ appears as a quotient of a congruence coming from $\mathcal{E}$ in a canonical way.

4.4.19 Proposition. *For every object $(X,E) \in \mathcal{E}_{ex/reg}$ there exists an exact sequence $\Gamma E \underset{\Gamma e_1}{\overset{\Gamma e_0}{\rightrightarrows}} \Gamma X \xrightarrow{E} (X,E)$, where $\langle e_0, e_1\rangle\colon E \rightarrowtail X \times X$ is an* ff*-morphism representing the congruence E in $\mathcal{E}$.*

Proof. Observe that E indeed defines a morphism $E_* : \Gamma X \to (X,E)$ in $\mathcal{E}_{ex/reg}$ which is in fact effective because $EI_X = E = EE$, $E^*E_* = EE = E \supseteq I_X$, $E_*E^* = EE = E$. Now it suffices to show that the square below is a comma square in $\mathcal{E}_{ex/reg}$.

$$\begin{array}{ccc} \Gamma E & \xrightarrow{\Gamma e_1} & \Gamma X \\ {\scriptstyle \Gamma e_0}\downarrow & \leq & \downarrow{\scriptstyle E} \\ \Gamma X & \xrightarrow[E]{} & (X,E) \end{array}$$

But by the construction of comma squares in $\mathcal{E}_{ex/reg}$ this is equivalent to having $(\Gamma e_0)^* \Gamma e_0 \cap (\Gamma e_1)^* \Gamma e_1 = I_E$ and $\Gamma e_1 (\Gamma e_0)^\circ = E^* E_*$ i.e. $e_0^* e_{0*} \cap e_1^* e_{1*} = I_E$ and $e_1 e_0^\circ = EE$ as relations in $\mathcal{E}$, both of which hold. □

Now we at last come to the proof that $\mathcal{E}_{ex/reg}$ satisfies the universal property that exhibits it as the exact completion of the regular category $\mathcal{E}$. Before proceeding, let us make a couple of quick observations that will be used in the course of our calculations in the proof that follows below.

Consider an exact fork $E \rightrightarrows X \xrightarrow{p} P$ in the regular category $\mathcal{E}$. Recall that in the calculus of relations this means that $E = p^* p_*$ and $p_* p^* = I_P$, where the second equality can equivalently be replaced by $pp^\circ = \Delta_P$. In addition, the kernel pair $p^\circ p$ of p can be written as $p^* p_* \cap (p^* p_*)^\circ$ and hence we also have $p^\circ p = E \cap E^\circ$ We now observe that the following equalities must hold:

- $pE = p_*$.
- $Ep^\circ = p^*$.
- $p_* E p^* = I_P$.

Indeed, for the first of these we have $pE \subseteq p_* E = p_* p^* p_* = p_*$ and also $pE = pp^* p_* \supseteq pp^\circ p_* = \Delta_P p_* = p_*$. The second one follows similarly. Finally, for the last one we have $p_* E p^* = p_* p^* p_* p^* = I_P I_P = I_P$.

4.4.20 Theorem. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a regular functor with $\mathcal{F}$ an exact category. Then there is a unique (up to isomorphism) regular functor $\tilde{F}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}$ such that $\tilde{F} \circ \Gamma = F$.*

Proof. If $\tilde{F}$ is to both be regular and satisfy $\tilde{F}\Gamma \cong F$, then by the previous proposition we must define it on any object $(X, E) \in \mathcal{E}_{ex/reg}$ as the following coinserter in $\mathcal{F}$

$$F(E) \underset{Fe_1}{\overset{Fe_0}{\rightrightarrows}} FX \xrightarrow{p_{(X,E)}} \tilde{F}(X,E)$$

which exists because regular functors preserve congruences and $\mathcal{F}$ is exact. In particular, this shows that $\tilde{F}$ will be unique up to isomorphism, so long as it is a well-defined regular functor.

Also, given any morphism $R_*\colon (X,E) \to (Y,G) \in \mathcal{E}_{ex/reg}$, we have relations $F(R_*)\colon FX \nrightarrow FY$ and $F(R^*)\colon FY \nrightarrow FX$ in $\mathcal{F}$ satisfying the following:

$$\begin{aligned} F(G)F(R_*)F(E) &= F(GR_*E) = F(R_*) \\ F(E)F(R^*)F(G) &= F(ER^*G) = F(R^*) \\ F(R^*)F(R_*) &= F(R^*R_*) \supseteq F(E) \\ F(R_*)F(R^*) &= F(R_*R^*) \subseteq F(G) \end{aligned}$$

where we used the fact that regular functors preserve the compositions and inclusions of relations. Thus, by Proposition 4.3.13 we can define $\tilde{F}(R_*)$ to be the uniquely associated morphism between quotients $\tilde{F}(X,E) \to \tilde{F}(Y,G)$. More explicitly, $\tilde{F}(R_*)$ is the morphism uniquely determined by the equality $\tilde{F}(R_*)_* = p_{(Y,G)*}F(R_*)p^*_{(X,E)}$ in $\mathrm{Rel}(\mathcal{F})$. It is immediate that this defines a functor $\mathcal{E}_{ex/reg} \to \mathcal{C}$ and clearly $\tilde{F} \circ \Gamma \cong F$. Note that by the discussion following Proposition 4.3.13 we also have $\tilde{F}(R_*) = p_{(Y,G)}(F(R_*) \cap F(R^*)^\circ)p^\circ_{(X,E)} = p_{(Y,G)}F(R)p^\circ_{(X,E)}$ as relations in $\mathcal{F}$.

Now let's show that $\tilde{F}$ preserves finite limits. First, it is clear that it preserves the terminal object $\Gamma 1$, since $\tilde{F}\Gamma 1 = F1$ and F preserves the terminal object. Second, the preservation of binary products follows from the fact that exact sequences in any regular category are stable under binary products. It suffices then to prove the preservation of inserters.

So suppose that $(A,T) \rightarrowtail^{\Phi_*} (X,E) \underset{S_*}{\overset{R_*}{\rightrightarrows}} (Y,G)$ is an inserter diagram in $\mathcal{E}_{ex/reg}$. Recall from Proposition 4.4.10 that by construction of inserters in $\mathcal{E}_{ex/reg}$ this means that $\Phi\Phi^\circ = S^*R_* \cap (E \cap E^\circ)$ and $\Phi^*\Phi_* = T$ as relations in $\mathcal{E}$. Then first of all, using the second of these equalities, the definition of $\tilde{F}$ and the regularity of the functor F, we have the following

$$\begin{aligned} \tilde{F}(\Phi_*)^*\tilde{F}(\Phi_*)_* &= p_{(A,T)*}F(\Phi^*)p^*_{(X,E)}p_{(X,E)*}F(\Phi_*)p^*_{(A,T)} \\ &= p_{(A,T)*}F(\Phi^*)F(E)F(\Phi_*)p^*_{(A,T)} \\ &= p_{(A,T)*}F(\Phi^*E\Phi_*)p^*_{(A,T)} \\ &= p_{(A,T)*}F(\Phi^*\Phi_*)p^*_{(A,T)} \\ &= p_{(A,T)*}F(T)p^*_{(A,T)} = I_{\tilde{F}(A,T)} \end{aligned}$$

which tells us that $\tilde{F}(\Phi_*)$ is an ff-morphism in $\mathcal{F}$.

Second, we have the following sequence of calculations, which uses the definition of the functor $\tilde{F}$, the regularity of $F\colon \mathcal{E} \to \mathcal{F}$ and the defining properties of morphisms in $\mathcal{E}_{ex/reg}$, but also observations made before the statement of the theorem:

$$
\begin{aligned}
&\tilde{F}(S_*)^*\tilde{F}(R_*)_* \cap \Delta_{\tilde{F}(X,E)} = \\
&= p_{(X,E)}p^{\circ}_{(X,E)}(\tilde{F}(S_*)^*\tilde{F}(R_*)_* \cap \Delta_{\tilde{F}(X,E)})p_{(X,E)}p^{\circ}_{(X,E)} \\
&= p_{(X,E)}p^{\circ}_{(X,E)}(p_{(X,E)*}F(S^*)p^*_{(Y,G)}p_{(Y,G)*}F(R_*)p^*_{(X,E)} \cap \Delta_{\tilde{F}(X,E)})p_{(X,E)}p^{\circ}_{(X,E)} \\
&= p_{(X,E)}p^{\circ}_{(X,E)}(p_{(X,E)}F(E)F(S^*)F(G)F(R_*)F(E)p^{\circ}_{(X,E)} \cap \Delta_{\tilde{F}(X,E)})p_{(X,E)}p^{\circ}_{(X,E)} \\
&= p_{(X,E)}p^{\circ}_{(X,E)}(p_{(X,E)}F(ES^*GR_*E)p^{\circ}_{(X,E)} \cap \Delta_{\tilde{F}(X,E)})p_{(X,E)}p^{\circ}_{(X,E)} \\
&= p_{(X,E)}p^{\circ}_{(X,E)}(p_{(X,E)}F(S^*R_*)p^{\circ}_{(X,E)} \cap \Delta_{\tilde{F}(X,E)})p_{(X,E)}p^{\circ}_{(X,E)} \\
&\overset{MD,MD*}{=} p_{(X,E)}[p^{\circ}_{(X,E)}p_{(X,E)}F(S^*R_*)p^{\circ}_{(X,E)}p_{(X,E)} \cap p^{\circ}_{(X,E)}p_{(X,E)}]p^{\circ}_{(X,E)} \\
&= p_{(X,E)}[F(E \cap E^{\circ})F(S^*R_*)F(E \cap E^{\circ}) \cap F(E \cap E^{\circ})]p^{\circ}_{(X,E)} \\
&= p_{(X,E)}(F(S^*R_*) \cap F(E \cap E^{\circ}))p^{\circ}_{(X,E)} \\
&= p_{(X,E)}F(S^*R_* \cap E \cap E^{\circ})p^{\circ}_{(X,E)} \\
&= p_{(X,E)}F(\Phi\Phi^{\circ})p^{\circ}_{(X,E)} \\
&= p_{(X,E)}F(\Phi)F(T \cap T^{\circ})F(\Phi^{\circ})p^{\circ}_{(X,E)} \\
&= p_{(X,E)}F(\Phi)p^{\circ}_{(A,T)}p_{(A,T)}F(\Phi^{\circ})p^{\circ}_{(X,E)} \\
&= \tilde{F}(\Phi_*)\tilde{F}(\Phi_*)^{\circ}
\end{aligned}
$$

These two equalities taken together tell us precisely that the diagram $\tilde{F}(A,T) \overset{\tilde{F}(\Phi_*)}{\rightarrowtail} \tilde{F}(X,E) \underset{\tilde{F}(S_*)}{\overset{\tilde{F}(R_*)}{\rightrightarrows}} \tilde{F}(Y,G)$ is an inserter diagram in $\mathcal{F}$.

Next, consider an so-morphism $R_* \colon (X,E) \twoheadrightarrow (Y,G) \in \mathcal{E}_{ex/reg}$. This means that $R_*R^* = G$ and then we have

$$
\begin{aligned}
\tilde{F}(R_*)_*\tilde{F}(R_*)^* &= p_{(Y,G)*}F(R_*)p^*_{(X,E)}p_{(X,E)*}F(R^*)p^*_{(Y,G)} \\
&= p_{(Y,G)*}F(R_*)F(E)F(R^*)p^*_{(Y,G)} \\
&= p_{(Y,G)*}F(R_*R^*)p^*_{(Y,G)} = p_{(Y,G)*}F(G)p^*_{(Y,G)} \\
&= I_{\tilde{F}(Y,G)}
\end{aligned}
$$

from which we obtain that $\tilde{F}(R_*)$ is an so-morphism in $\mathcal{F}$. Thus, we have proved that $\tilde{F}$ is a regular functor. □

4.5 Characterizing the Completion, Examples and Priestley Spaces

In this section we obtain a characterization result for the ex/reg completion which is precisely the enriched analogue of the corresponding one in the ordinary context Theorem 3.1.8. Having achieved a construction of the completion which is very similar to that of the ordinary exact completion, the proofs in this section

will be almost verbatim restatements of those in Section 3.1, with the appropriate enriched notions substituted everywhere. Furthermore, the characterization result will be applied to deduce many interesting examples of categories which arise as exact completions of regular categories. A particularly interesting one is the identification of the category of Nachbin spaces as the exact completion of the category of Priestley spaces, but also of the locally discrete category of Stone spaces. The latter, as we shall observe, is a special case of a more general fact regarding exact categories with enough *discrete* objects.

But first, we need some preliminaries. We will say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *order-faithful* if, for every $f, g\colon X \to Y \in \mathcal{C}$ we have $Ff \leq Fg \implies f \leq g$. In other words, F is order-faithful if for every $X, Y \in \mathcal{C}$ the morphism $\mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is an ff-morphism in Pos. Note in particular that such a functor is faithful in the ordinary sense or, in more appropriate language, the underlying functor between ordinary categories is faithful. The converse is not true in general. However, we observe that if F is also full and $\mathcal{C}$ has inserters which are preserved by $F\colon \mathcal{C} \to \mathcal{D}$, then the two notions coincide.

4.5.1 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor and assume that $\mathcal{C}$ has inserters. If F preserves inserters, then F is order-faithful.*

Proof. Let $f, g\colon X \to Y \in \mathcal{C}$ be such that $Ff \leq Fg$ in $\mathcal{D}$ and consider the inserter $E \rightarrowtail X$ of (f,g) in $\mathcal{C}$. By assumption on F, we have that $Fe\colon FE \rightarrowtail FX$ is the inserter of (Ff, Fg) in $\mathcal{D}$. Hence, $Ff \leq Fg$ implies that there exists a unique $v\colon FX \to FE$ such that $Fe \circ v = 1_{FX}$. Since F is assumed to be fully faithful, we know that $v = Fu$ for a unique $u : X \to E \in \mathcal{C}$ and then we deduce that $eu = 1_X$. The latter now immediately yields $f \leq g$ in $\mathcal{C}$. □

Since in this section we will be dealing with regular functors, the terms "fully faithful" and "fully order-faithful" can be used interchangeably. In addition, notice that in the above lemma instead of being fully faithful we could have assumed more generally that F is *conservative*, i.e. that it reflects isomorphisms.

The crux of the work now, much as in Section 3.1, consists of establishing that certain properties of a regular functor $F\colon \mathcal{E} \to \mathcal{F}$ into an exact category $\mathcal{F}$ translate to corresponding properties of the induced $\tilde{F}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}$.

4.5.2 Lemma. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a regular functor with $\mathcal{F}$ an exact category. If F is fully order-faithful, then $\tilde{F}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}$ is order-faithful.*

Proof. Let $R_*, S_*\colon (X,E) \to (Y,G) \in \mathcal{E}_{ex/reg}$ be such that $\tilde{F}(R_*) \leq \tilde{F}(S_*)$ in $\mathcal{F}$, which is equivalent to $\tilde{F}(R_*)_* \supseteq \tilde{F}(S_*)_*$. Then using the definition of $\tilde{F}$ we have

$$\begin{aligned}
\tilde{F}(R_*)_* \supseteq \tilde{F}(S_*)_* \quad &\implies \quad p^*_{(Y,G)}\tilde{F}(R_*)_* p_{(X,E)*} \supseteq p^*_{(Y,G)}\tilde{F}(S_*)_* p_{(X,E)*} \\
&\implies \quad p^*_{(Y,G)} p_{(Y,G)*} F(R_*) p^*_{(X,E)} p_{(X,E)*} \supseteq \\
&\qquad\quad p^*_{(Y,G)} p_{(Y,G)*} F(S_*) p^*_{(X,E)} p_{(X,E)*} \\
&\implies \quad F(G)F(R_*)F(E) \supseteq F(G)F(S_*)F(E) \\
&\implies \quad F(R_*) \supseteq F(S_*)
\end{aligned}$$

But since F is fully (order-) faithful, it reflects inclusions of subobjects and hence we obtain $R_* \supseteq S_*$, i.e. $R_* \leq S_*$ in $\mathcal{E}_{ex/reg}$. □

We introduce some further terminology, copying from the ordinary context in Section 3.1. Note that subobjects in this context are equivalence classes of ff-morphisms rather than plain monomorphisms.

4.5.3 Definition. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *covering* if, for every object $Y \in \mathcal{D}$, one can find an object $X \in \mathcal{C}$ and an effective epimorphism $FX \twoheadrightarrow Y$.

We say that F is *full on subobjects* if, for every ff-morphism $B \rightarrowtail FX$ in $\mathcal{D}$, there exists an ff-morphism $A \rightarrowtail X$ in $\mathcal{C}$ such that $FA \cong B$ in $\mathrm{Sub}_{\mathcal{D}}(\mathrm{FX})$.

The next few statements and their accompanying proofs are almost identical to those in the ordinary case presented in Section 3.1, with only the natural alterations dictated by the enriched context.

4.5.4 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a regular functor between regular categories. If F is full and covering, then it is full on subobjects.*

Proof. Consider an ff-morphism $v\colon D \rightarrowtail FY$ in $\mathcal{D}$. Since F is covering, there exists some so-morphism $q\colon FX \twoheadrightarrow D$ in $\mathcal{D}$. Now consider the composition of the two, $FX \xrightarrow{q} D \xrightarrow{v} FY$. Since F is full, there is a morphism $f\colon X \to Y$ in $\mathcal{C}$ such that $Ff = vq$.

Since $\mathcal{C}$ is regular, we can consider the (so,ff) factorization of the morphism f, say $f = X \xrightarrow{p} I \xrightarrow{u} Y$. But F is a regular functor, so we know that $FX \xrightarrow{Fp} FI \xrightarrow{Fu} FY$ is the (so,ff) factorization of $Ff = vq$. By uniqueness of such factorizations in the regular category $\mathcal{D}$, we deduce that $FI \cong D$ as subobjects of FY. □

4.5.5 Proposition. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a regular functor with $\mathcal{F}$ an exact category. If F is fully order-faithful and covering, then $\tilde{F}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}$ is fully order-faithful and covering.*

Proof. We saw earlier that F being fully order-faithful implies that $\tilde{F}$ is order-faithful. Furthermore, it is immediate that F being covering implies the same property for $\tilde{F}$, since we have $\tilde{F}\Gamma \cong F$.

Now consider any morphism $g\colon \tilde{F}(X,E) \to \tilde{F}(Y,G) \in \mathcal{F}$. Let us denote by $S_*\colon FX \nrightarrow FY$ and $S^*\colon FY \nrightarrow FX$ the relations corresponding to this morphism via the bijection of Proposition 4.3.13, i.e. the relations $S_* = p^*_{(Y,G)} g_* p_{(X,E)*}$ and $S^* = p^*_{(X,E)} g^* p_{(Y,G)*}$. Now since F is a full and covering regular functor, we know by the previous lemma that it is also full on subobjects and so there exist relations $R_*\colon X \nrightarrow Y$ and $R^*\colon Y \nrightarrow X$ in $\mathcal{E}$ such that $F(R_*) = S_*$ and $F(R^*) = S^*$. Furthermore, we have the following:

$$F(GR_*E) = F(G)F(R_*)F(E) = F(G)S_*F(E) = S_* = F(R_*)$$

$$F(R^*R_*) = F(R^*)F(R_*) = S^*S_* \supseteq F(E)$$
$$F(R_*R^*) = F(R_*)F(R^*) = S_*S^* \subseteq F(G)$$

But now because F is fully (order-) faithful it reflects inclusions of subobjects. Thus, we deduce that $GR_*E = R_*$, $R^*R_* \supseteq E$ and $R_*R^* \subseteq G$, so that R_* is a morphism $(X,E) \to (Y,G) \in \mathcal{E}_{ex/reg}$.

Finally, we have by definition of the functor $\tilde{F}$ that

$$\begin{aligned}\tilde{F}(R_*)_* &= p_{(Y,G)*}F(R_*)p^*_{(X,E)} = p_{(Y,G)*}S_*p^*_{(X,E)}\\ &= p_{(Y,G)*}p^*_{(Y,G)}g_*p_{(X,E)*}p^*_{(X,E)} = g_*\end{aligned}$$

and hence $\tilde{F}(R_*) = g$. □

The final ingredient we need is the Pos-enriched analogue of Lemma 1.4.9 from [73] (see Lemma 3.1.7).

4.5.6 Lemma. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a regular functor between regular categories where moreover $\mathcal{E}$ is exact. If F is fully order-faithful and covering, then it is an equivalence.*

Proof. It suffices to show that F is essentially surjective on objects. So let $Y \in \mathcal{F}$. By the assumption that F is covering, we can find a coinserter $q\colon FX \twoheadrightarrow Y$ in $\mathcal{F}$ for some object $X \in \mathcal{E}$. Consider the kernel congruence $q/q \xrightarrow[q_1]{q_0} FX$ in $\mathcal{F}$. Since F is a covering and full regular functor, we know that it must also be full on subobjects. In particular, there is a relation $E \xrightarrow[e_1]{e_0} X$ in $\mathcal{E}$ such that $F(E) = q/q$ as relations in $\mathcal{F}$.

Now because F is order-faithful, $F(E) = q/q$ being a congruence in $\mathcal{F}$ implies that E is a congruence on X in $\mathcal{E}$. Since $\mathcal{E}$ is assumed to be exact, E has a coinserter, say $p\colon X \to P$, and is the kernel congruence of that coinserter. Now by regularity of the functor F we have that $Fp\colon FX \twoheadrightarrow FP$ is the coinserter of $F(E) = q/q$. Since $q\colon FX \to Y$ is also a coinserter of q/q, we deduce that there exists an isomorphism $FP \cong Y$. □

Now putting everything together we have proved the following result.

4.5.7 Theorem. *Let $F\colon \mathcal{E} \to \mathcal{F}$ be a regular functor with $\mathcal{F}$ an exact category. Then $\tilde{F}\colon \mathcal{E}_{ex/reg} \to \mathcal{F}$ is an equivalence if and only if F is fully order-faithful and covering.*

The above theorem can now be applied to deduce many examples of categories which arise as exact completions of a regular category. We begin below by listing some examples involving varieties of ordered algebras which occur as exact completions of corresponding quasi-varieties.

4.5.8 Example. 1. Consider Set viewed as a locally discrete Pos-category. We have seen that it is regular but not exact. The discrete poset functor $D : \mathsf{Set} \to \mathsf{Pos}$ is clearly fully order-faithful and regular. It is also trivially covering, since for any poset $(X, \leq)$ we have an order-preserving surjection $(X, =) \twoheadrightarrow (X, \leq)$. Thus, by Theorem 3.1.8 we have that $\mathsf{Set}_{ex/reg} \simeq \mathsf{Pos}$.

2. Consider the category $\mathsf{OrdCMon}$ of ordered commutative monoids, which is a variety of ordered algebras in the sense of Bloom & Wright [15] and hence is an exact category [65]. Recall that an ordered commutative monoid M is *torsion-free* if it satisfies $nx \leq ny \implies x \leq y$ for any $x, y \in M$ and $n \in \mathbb{N}^*$. Thus, the full subcategory on such monoids $\mathsf{OrdCMon}_{t.f.}$ is a regular category since it is an ordered quasivariety. Then by Theorem 3.1.8 we see that $(\mathsf{OrdCMon}_{t.f.})_{ex/reg} \simeq \mathsf{OrdCMon}$. Indeed, every ordered monoid admits a surjective homomorphism from a free one and clearly every free ordered monoid is torsion-free.

3. The ordinary category Mon of monoids is regular, hence also regular as a locally discrete Pos-category. The inclusion functor $\mathsf{Mon} \hookrightarrow \mathsf{OrdMon}$ is regular and any ordered monoid $(M, \leq)$ admits a surjective homomorphism $(M, =) \twoheadrightarrow (M, \leq)$. It follows then from Theorem 3.1.8 that $\mathsf{Mon}_{ex/reg} \simeq \mathsf{OrdMon}$.

4. It is easy to see that in the above example there is nothing special about the variety of monoids. Indeed, any ordinary variety gives rise to a corresponding variety of ordered algebras defined by the same set of axioms. The ordinary (unordered) version sits inside the ordered one as the discrete ordered algebras and as such is a regular subcategory, in fact an ordered quasivariety. It follows then that its exact completion qua Pos-category yields precisely the corresponding ordered variety. Thus, for example, in the case of semigroups we similarly have $\mathsf{SGrp}_{ex/reg} \simeq \mathsf{OrdSGrp}$.

The examples presented so far of exact completions have all been varieties of ordered algebras which appear as completions of certain corresponding quasivarieties. However, as we would now like to show, there is also an interesting example which is order-topological in nature and involves the category of Priestley spaces. Let us thus first recall some terminology.

We will say that a triple $(X, \tau, \leq)$ with τ a topology and $\leq$ a partial order relation on the set X is an *ordered topological space*. Morphisms between ordered topological spaces are simply functions which are both continuous with respect to the topologies and order-preserving. This defines the category of ordered topological spaces which we will denote by OrdTop.

A *compact ordered space* is an ordered topological space $(X, \tau, \leq)$ such that (X, τ) is compact and $\leq$ is closed as a subspace of $X \times X$. This class of spaces was introduced and developed by L. Nachbin in [80] as an ordered analogue of compact Hausdorff spaces, and so these are often also called *Nachbin* spaces. Together with the continuous order-preserving functions between them they form a subcategory of OrdTop which we denote by Nach. Note that, under the

assumption of compactness, the condition that the order relation be closed in the product space $X \times X$ means the following: whenever $x, y \in X$ with $x \not\leq y$, there exist a open up-set U and an open down-set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

In turn, within Nach sits the very interesting full subcategory Pries of *Priestley* spaces. This is a class of ordered topological spaces introduced by H. A. Priestley [85] in order to provide an extension of Stone duality from boolean algebras to distributive lattices. That is to say, there is an equivalence of categories $\mathsf{DLat}^{op} \simeq \mathsf{Pries}$, where DLat denotes the category of (bounded) distributive lattices and lattice homomorphisms.

Recall then than an ordered topological space $(X, \tau, \leq)$ is a *Priestley space* if (X, τ) is compact and the following is satisfied: whenever $x \not\leq y$, there exists a clopen up-set U such that $x \in U$ and $y \notin U$. Ordered spaces satisfying the latter condition are often called *totally order-disconnected*. It is immediate that every Priestley space is indeed a Nachbin space, since the complement of a clopen up-set is a clopen down-set. It is furthermore clear that the underlying topological space of a Priestley space is a Stone space. In fact, the category Stone of Stone spaces is embedded in Pries as the full subcategory on the objects for which the order relation is discrete. On the other hand, we should note that a Nachbin space $(X, , \tau, \leq)$ whose underlying space (X, τ) is Stone need not be a Priestley space, as exhibited by the counterexamples of Stralka [93].

In terms of regularity and exactness, the following can be said about the categories Nach and Pries.

4.5.9 Proposition. *The category* Nach *is exact, while the category* Pries *is regular.*

Proof. Observe that the coinserter $E \rightrightarrows X \xrightarrow{q} X/E \cap E^\circ$ of any internal congruence $E \rightarrowtail X \times X$ in Nach is constructed by equipping the set $X/E \cap E^\circ$ with the quotient topology and the induced order relation by the pre-order E. Since E is closed in $X \times X$, so is the equivalence relation $E \cap E^\circ$ and so at the level of spaces we know that the quotient will be a compact Hausdorff space. It is then a Nachbin space because the order relation is by definition equal to $(q \times q)[E]$ and the map $q \times q$ is closed.

It now follows that the effective epimorphisms in Nach are precisely the continuous monotone surjections. Indeed, if $f \colon X \to Y \in \mathsf{Nach}$ is surjective then on the level of spaces it is a continuous surjection between compact Hausdorff spaces and hence it is a quotient map. This means that the induced $\bar{f} \colon X/R \to Y$ is a homeomorphism, where $R = \{(x, x') | f(x) = f(x')\} = E \cap E^\circ$, for $E \coloneqq f/f$. But $\bar{f}$ also preserves and reflects the order because by definition we have

$$\bar{f}([x]) \leq \bar{f}([x']) \iff f(x) \leq f(x') \iff (x, x') \in E \iff [x] \leq [x'].$$

Thus, f is the coinserter of its kernel congruence.

This shows that Nach is regular, since the continuous monotone surjections are clearly stable under pullback. To see that Pries is regular, it suffices to observe that the latter is closed under finite limits and subobjects in Nach.

Finally, consider any internal congruence $E \rightarrowtail X \times X$ in Nach and construct its coinserter q as we did above. It is then immediate by the construction that $E = q/q$ and so we have proved that Nach is exact. □

We can now deduce an ordered version of the folklore result which identifies the exact completion of Stone as the category CHaus of compact Hausdorff spaces. The latter, to the best of the author's knowledge, seems to have first appeared explicitly in print in [8] where a similar argument involving the ordinary exact completion was invoked. The analogue here is that the exact completion of Pries is Nach. However, below we observe at the same time that the completion of Stone itself, viewed as a locally discrete Pos-category, is also Nach.

4.5.10 Corollary. $\mathsf{Pries}_{ex/reg} \simeq \mathsf{Nach} \simeq \mathsf{Stone}_{ex/reg}$.

Proof. The inclusions $\mathsf{Stone} \hookrightarrow \mathsf{Pries} \hookrightarrow \mathsf{Nach}$ are both regular functors. Furthermore, if $X \in \mathsf{Nach}$, then X is in particular a compact Hausdorff space and so admits a continuous surjection $\beta(X) \twoheadrightarrow X$ from a Stone space $\beta(X)$, the latter being the Stone-Cech compactification of the discrete set X. Equipping $\beta(X)$ with the equality relation this becomes a continuous monotone surjection in Nach. Thus, both inclusion functors are also covering and the result follows from Theorem 4.5.7. □

Before ending this section, we would like to state a final result which subsumes some observations that were made in the context of many of the examples mentioned above. To this effect, recall that some varieties of ordered algebras were described as exact completions of certain ordinary varieties which appeared as the objects with discrete order relation. As a prototypical example, we had $\mathsf{Mon}_{ex/reg} \simeq \mathsf{OrdMon}$ for the category of ordered monoids. Similarly, in the context of the above corollary we could have included the equivalence $\mathsf{CHaus}_{ex/reg} \simeq \mathsf{Nach}$, where CHaus denotes the locally discrete category of compact Hausdorff spaces. Also, as we had mentioned earlier, Stone spaces can be viewed as the objects of Pries with discrete order.

To generalize the theme present in these examples, we can introduce a notion of discrete object in any category.

4.5.11 Definition. An object $X \in \mathcal{E}$ in a category is said to be *discrete* if, for every $f, g \colon A \to X \in \mathcal{E}$, we have that $f \leq g$ implies $f = g$. In other words, X is discrete if $\mathcal{E}(A, X)$ is a discrete poset for all $A \in \mathcal{E}$.

We will denote by $\mathsf{Dis}(\mathcal{E})$ the full subcategory of $\mathcal{E}$ on the discrete objects. Then it is easy to see that $\mathsf{Dis}(\mathcal{E})$ is a locally discrete category which is closed under finite limits and subobjects in $\mathcal{E}$. In particular, this shows that if $\mathcal{E}$ is regular then so is $\mathsf{Dis}(\mathcal{E})$.

We will say that $\mathcal{E}$ has *enough discrete objects* if for every object $X \in \mathcal{E}$ there exists an so-morphism $D \twoheadrightarrow X$ in $\mathcal{E}$ with $D \in \mathsf{Dis}(\mathcal{E})$. By another application of Theorem 4.5.7 we now deduce the following:

4.5.12 Corollary. *Let $\mathcal{E}$ be an exact category with enough discrete objects. Then there is an equivalence $\mathcal{E} \simeq \mathsf{Dis}(\mathcal{E})_{ex/reg}$.*

4.5.13 Remark. In the above corollary it is important to notice that, although $\mathsf{Dis}(\mathcal{E})$ is a locally discrete category and hence could be considered simply as an ordinary category, the ex/reg completion which we apply to it is the *enriched* one. For example, $\mathsf{Dis}(\mathsf{Nach}) = \mathsf{CHaus}$ but of course the exact completion of CHaus as a mere ordinary category is itself.

4.6 Internal Posets and Exact Completion

In this final section we consider the process of taking internal posets in an ordinary category and how the ordinary and enriched notions of regularity and exactness are related through said process. In addition, we examine how the ordinary and enriched versions of the exact completion are connected with categories of internal posets.

To begin with, suppose that $\mathcal{C}$ is any finitely complete ordinary category. We can then define a category $\mathsf{Ord}(\mathcal{C})$ as follows:

- Objects: are pairs $(X, \leq_X)$, where X is an object of $\mathcal{C}$ and $\leq_X \rightarrowtail X \times X$ is a partial order relation in $\mathcal{C}$.

- Morphisms: A morphism $f\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$ is a morphism $f\colon X \to Y \in \mathcal{C}$ such that $f(\leq_X) \subseteq \leq_Y$.

The condition $f(\leq_X) \subseteq \leq_Y$ means that there is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} \leq_X & \dashrightarrow & \leq_Y \\ \downarrow & & \downarrow \\ X \times X & \xrightarrow[f \times f]{} & Y \times Y \end{array}$$

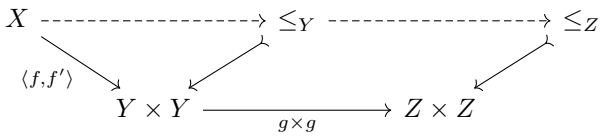
Composition of morphisms and identities are those of $\mathcal{C}$.

Furthermore, given morphisms $f, f'\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$, we define $f \leq f'$ to mean that there exists a commutative diagram

$$\begin{array}{ccc} X & \dashrightarrow & \leq_Y \\ & {\scriptstyle \langle f, f' \rangle} \searrow \quad \swarrow & \\ & Y \times Y & \end{array}$$

Now it is easy to see that this order relation on morphisms of $\mathsf{Ord}(\mathcal{C})$ is preserved by composition. For example, if $f, f'\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$ with $f \leq f'$ and $g\colon (Y, \leq_Y) \to (Z, \leq_Z)$, we have $gf \leq gf'$ by pasting the following

commutative diagrams



Thus, $\mathsf{Ord}(\mathcal{C})$ is enriched in Pos. Our first observation below is that finite completeness of the ordinary category $\mathcal{C}$ implies the existence of all finite weighted limits in $\mathsf{Ord}(\mathcal{C})$.

4.6.1 Proposition. *If $\mathcal{C}$ is finitely complete, then $\mathsf{Ord}(\mathcal{C})$ has finite weighted limits.*

Proof. It is easy to see that $(X, \leq_X) \xleftarrow{\pi_X} (X \times Y, \leq_X \times \leq_Y) \xrightarrow{\pi_Y} (Y, \leq_Y)$ is a product diagram for every $(X, \leq_X), (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$. Let us show how to construct the inserter of a pair of morphisms $(X, \leq_X) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leq_Y)$. For this, form the following pullback square in $\mathcal{C}$

$$\begin{array}{ccc} E & \xrightarrow{e'} & \leq_Y \\ {\scriptstyle e}\downarrow & & \downarrow \\ X & \xrightarrow[\langle f,g\rangle]{} & Y \times Y \end{array}$$

Let $\leq_E$ be the restriction of $\leq_X$ to the subobject $E \rightarrowtail X$, i.e. $\leq_E = (E \times E) \cap \leq_X$ as subobjects of $X \times X$. It is easy to see that $\leq_E$ is itself an internal partial order relation on $E \in \mathcal{C}$ so that we have a morphism $e\colon (E, \leq_E) \to (X, \leq_X) \in \mathsf{Ord}(\mathcal{C})$. Also, by commutativity of the pullback square above we have $fe \leq ge$.

Now let $h\colon (Z, \leq_Z) \to (X, \leq_X)$ be such that $fh \leq gh$. This means that $\langle fh, gh\rangle = \langle f, g\rangle h$ factors through $\leq_Y \rightarrowtail Y \times Y$, say via $u\colon Z \to \leq_Y$, so then by the pullback property there exists a unique $v\colon Z \to E$ satisfying $ev = h$, $e'v = u$.

Finally, e is an ff-morphism in $\mathsf{Ord}(\mathcal{C})$ by definition of $\leq_E$. Indeed, for any $(Z, \leq_Z) \underset{h'}{\overset{h}{\rightrightarrows}} (E, \leq_E)$, the inequality $eh \leq eh'$ means that $(e \times e)\langle h, h'\rangle$ factors through $\leq_X$, which implies $\langle h, h'\rangle$ factors through $\leq_E = (E \times E) \cap \leq_X$, i.e. that $h \leq h'$. □

Since it will be needed later, let us also record here how to construct comma squares

$$\begin{array}{ccc} (C, \leq_C) & \xrightarrow{c_1} & (Y, \leq_Y) \\ {\scriptstyle c_0}\downarrow & \leq & \downarrow{\scriptstyle g} \\ (X, \leq_X) & \xrightarrow[f]{} & (Z, \leq_Z) \end{array}$$

in $\mathsf{Ord}(\mathcal{C})$. This is accomplished by constructing the following pullback square in $\mathcal{C}$

$$\begin{array}{ccc} C & \longrightarrow & \leq_Z \\ {\scriptstyle \langle c_0, c_1 \rangle}\downarrow & & \downarrow \\ X \times Y & \xrightarrow[f \times g]{} & Z \times Z \end{array}$$

and then setting $\leq_C := (C \times C) \cap (\leq_X \times \leq_Y)$.

Before moving on, let us also discuss ff-morphisms in $\mathsf{Ord}(\mathcal{C})$. We saw in the course of the previous proof that an $m\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$ with $m\colon X \to Y \in \mathcal{C}$ monomorphic and $\leq_X = (X \times X) \cap \leq_Y$ as relations in $\mathcal{C}$ is an ff-morphism in $\mathsf{Ord}(\mathcal{C})$. It is in fact not too hard to see that this completely characterizes ff-morphisms in $\mathrm{Ord}(\mathcal{C})$. Indeed, assume that $m\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$ is an ff-morphism. If $f, g\colon Z \to X \in \mathcal{C}$ are such that $mf = mg$, then we can also consider them as morphisms $f, g\colon (Z, \Delta_Z) \to (X, \leq_X)$ in $\mathsf{Ord}(\mathcal{C})$ and hence deduce that $f = g$. This proves m must be a monomorphism in $\mathcal{C}$. Now arguing with generalized elements one can easily deduce that $\leq_X$ is indeed the restriction of $\leq_Y$ along $m\colon X \rightarrowtail Y$.

Now we can say something about so-morphisms in $\mathsf{Ord}(\mathcal{C})$.

4.6.2 Lemma. *If $f\colon (X, \leq_X) \to (Y, \leq_Y) \in \mathsf{Ord}(\mathcal{C})$ is such that the underlying $f\colon X \to Y$ is a strong epimorphism in $\mathcal{C}$, then f is an* so*-morphism in $\mathsf{Ord}(\mathcal{C})$.*

Proof. Consider the following commutative square in $\mathsf{Ord}(\mathcal{C})$, where we assume $m\colon (M, \leq_M) \to (Z, \leq_Z)$ is an ff-morphism.

$$\begin{array}{ccc} (X, \leq_X) & \xrightarrow{f} & (Y, \leq_Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ (M, \leq_M) & \xrightarrow[m]{} & (Z, \leq_Z) \end{array}$$

In particular, by the preceding discussion we know that m is monomorphic in $\mathcal{C}$ and so by the property of f as a strong epimorphism in the latter category we deduce the existence of a $u\colon Y \to M$ such that $uf = h$ and $mu = g$. The fact that u is actually a morphism in $\mathsf{Ord}(\mathcal{C})$ follows because $mu = g$ is such a morphism and because m being an ff-morphism also means that $\leq_M = (M \times M) \cap \leq_Z$. □

We can now prove that ordinary regularity of $\mathcal{C}$ implies (enriched) regularity for $\mathsf{Ord}(\mathcal{C})$. Note that this result is in some sense a special case of [21, Proposition 62], where the authors prove a form of 2-categorical regularity for the 2-category $\mathsf{Cat}(\mathcal{E})$ of internal categories in a regular (1-)category $\mathcal{E}$. Nevertheless, we include a proof here in order to make the text more self-contained.

4.6.3 Proposition. *If $\mathcal{C}$ is an ordinary regular category, then $\mathsf{Ord}(\mathcal{C})$ is regular.*

Proof. Consider any $f\colon (X,\leq_X) \to (Y,\leq_Y) \in \mathsf{Ord}(\mathcal{C})$ and consider the (regular epi,mono) factorization $X \xrightarrow{\;p\;} M \rightarrowtail Y$ of f in $\mathcal{C}$. Then by what we have already established earlier, upon setting $\leq_M := (M\times M)\cap \leq_Y$, we obtain an (so,ff) factorization $(X,\leq_X) \xrightarrow{\;p\;} (M,\leq_M) \rightarrowtail (Y,\leq_Y)$ in $\mathsf{Ord}(\mathcal{C})$.

Now the existence of these factorizations together with the previous lemma imply that any $f\colon (X,\leq_X) \to (Y,\leq_Y) \in \mathsf{Ord}(\mathcal{C})$ is an so-morphism in $\mathsf{Ord}(\mathcal{C})$ if and only if $f\colon X \to Y$ is a regular(=strong) epi in $\mathcal{C}$. For the "only if" direction, suppose that $f\colon (X,\leq_X) \to (Y,\leq_Y)$ is an so-morphism and consider its factorization $(X,\leq_X) \xrightarrow{\;p\;} (M,\leq_M) \rightarrowtail (Y,\leq_Y)$ as constructed above. Then we have that $m\colon (M,\leq_M) \to (Y,\leq_Y)$ is both an so and ff-morphism, hence is an isomorphism in $\mathsf{Ord}(\mathcal{C})$. In particular, m is an isomorphism in $\mathcal{C}$ and so p being a strong epimorphism in $\mathcal{C}$ implies the same for $f = mp$.

Finally, since pullbacks in $\mathsf{Ord}(\mathcal{C})$ are constructed by simply taking the pullback of the underlying morphisms in $\mathcal{C}$, pullback-stability of regular epimorphisms in $\mathcal{C}$ implies pullback-stability of so-morphisms in $\mathsf{Ord}(\mathcal{C})$. □

Similarly, ordinary exactness of $\mathcal{C}$ implies Pos-enriched exactness of $\mathsf{Ord}(\mathcal{C})$. Again, this should be compared to [21, Proposition 63].

4.6.4 Proposition. *If $\mathcal{C}$ is an ordinary exact category, then $\mathsf{Ord}(\mathcal{C})$ is exact.*

Proof. Suppose that $(E,\leq_E) \overset{\langle e_0,e_1\rangle}{\rightarrowtail} (X,\leq_X)\times(X,\leq_X)$ is a congruence on $(X,\leq_X) \in \mathsf{Ord}(\mathcal{C})$. Then we have a monomorphism $E \rightarrowtail X\times X \in \mathcal{C}$, i.e. a relation E on X in $\mathcal{C}$. Reflexivity and transitivity of the congruence in $\mathsf{Ord}(\mathcal{C})$ imply the same properties for the relation $E\colon X \nrightarrow X$ in $\mathcal{C}$. Then $R := E\cap E^\circ$ is an equivalence relation on X in $\mathcal{C}$.

Since $\mathcal{C}$ is exact, there exists an exact sequence $R \rightrightarrows X \xrightarrow{\;q\;} Q$ in $\mathcal{C}$, which means that q is the coequalizer of R and R is the kernel pair of q. Let $\leq_Q := q(E)$ be the image of E along q in $\mathcal{C}$. Note that in the calculus of relations in the ordinary regular category $\mathcal{C}$ we can write $q(E) = qEq^\circ$, while exactness of the sequence is equivalent to $q^\circ q = R$ and $qq^\circ = \Delta_Q$.

Now observe that $\leq_Q$ is indeed an internal partial order relation in $\mathcal{C}$. It is reflexive because E is so and q is a regular epimorphism. For transitivity we argue as follows

$$\leq_Q \circ \leq_Q = qEq^\circ qEq^\circ = qEREq^\circ = qE(E\cap E^\circ)Eq^\circ = qEq^\circ = \leq_Q$$

Finally, for anti-symmetry we have

$$\begin{aligned}\leq_Q \cap \leq_Q^\circ &= qq^\circ(\leq_Q \cap \leq_Q^\circ)qq^\circ = q(q^\circ \leq_Q q \cap q^\circ \leq_Q^\circ q)q^\circ = q(E\cap E^\circ)q^\circ \\ &= qq^\circ qq^\circ = \Delta_Q\end{aligned}$$

Now we claim that $(E,\leq_E) \overset{e_0}{\underset{e_1}{\rightrightarrows}} (X,\leq_X)$ is the kernel congruence of $q\colon (X,\leq_X) \to (Q,\leq_Q)$ in $\mathsf{Ord}(\mathcal{C})$. So let $g_0, g_1\colon (Z,\leq_Z) \to (X,\leq_X)$ be such that

$qg_0 \leq qg_1$. In terms of generalized elements in $\mathcal{C}$ this means that $(qg_0, qg_1) \in_Z \leq_Q$, which in turn is equivalent to $(g_0, g_1) \in_Z q^{-1}(\leq_Q)$. But now observe that

$$q^{-1}(\leq_Q) = q^{-1}(q(E)) = q^\circ q E q^\circ q = RER = (E \cap E^\circ)E(E \cap E^\circ) = E$$

Thus, $(g_0, g_1) \in_Z E$ as desired. □

Consider a regular functor $F\colon \mathcal{C} \to \mathcal{D}$ between ordinary regular categories. Since F preserves internal partial order relations, we have an induced functor $\mathsf{Ord}(F)\colon \mathsf{Ord}(\mathcal{C}) \to \mathsf{Ord}(\mathcal{D})$ defined on objects by $(X, \leq_X) \mapsto (FX, F(\leq_X))$. By the construction of finite weighted limits in $\mathsf{Ord}(\mathcal{C})$, the fact that F preserves finite limits implies the same for the enriched functor $\mathsf{Ord}(F)$. Similarly, F preserving regular epimorphisms translates to the fact that $\mathsf{Ord}(F)$ preserves so-morphisms. Thus, $\mathsf{Ord}(F)$ is a regular functor between regular Pos-categories.

We now turn to discussing how the properties of F being fully faithful and covering translate to properties of $\mathsf{Ord}(F)$.

4.6.5 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an ordinary regular functor which is fully faithful. Then $\mathsf{Ord}(F)\colon \mathsf{Ord}(\mathcal{C}) \to \mathsf{Ord}(\mathcal{D})$ is fully order-faithful.*

Proof. It is clear that $\mathsf{Ord}(F)$ is faithful, since its action on morphisms is that of F itself. Since $\mathsf{Ord}(\mathcal{C})$ has finite limits and $\mathsf{Ord}(F)$ preserves them, full faithfulness is equivalent to full order-faithfulness. So it remains to show that $\mathsf{Ord}(F)$ is full.

Now consider any $h\colon (FX, F(\leq_X)) \to (FY, F(\leq_Y))$. By fullness of F, there exists an $f\colon X \to Y \in \mathcal{C}$ such that $Ff = h$. It suffices then to show that f is order-preserving. For this, observe that $Ff = h$ being a morphism in $\mathsf{Ord}(\mathcal{D})$ means that there is a commutative diagram in $\mathcal{D}$ of the form

$$\begin{array}{ccc} F(\leq_X) & \longrightarrow & F(\leq_Y) \\ \downarrow & & \downarrow \\ F(X \times X) \cong FX \times FX & \xrightarrow[Ff \times Ff]{} & FY \times FY \cong F(Y \times Y) \end{array}$$

By full faithfulness of F this is then reflected to a commutative diagram in $\mathcal{C}$ which exhibits f as a morphism $(X, \leq_X) \to (Y, \leq_Y)$ in $\mathsf{Ord}(\mathcal{C})$. □

4.6.6 Lemma. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an ordinary regular functor which is covering. Then $\mathsf{Ord}(F)\colon \mathsf{Ord}(\mathcal{C}) \to \mathsf{Ord}(\mathcal{D})$ is covering.*

Proof. Consider any object $(Y, \leq_Y) \in \mathsf{Ord}(\mathcal{D})$. Since F is covering, we can find a regular epimorphism $q\colon FX \twoheadrightarrow Y$ in $\mathcal{D}$. We can then consider FX with the discrete order and so we have an object $(FX, \Delta_{FX}) \in \mathsf{Ord}(\mathcal{D})$. Since q is a regular epimorphism in $\mathcal{D}$, we have an so-morphism $q\colon (FX, \Delta_{FX}) \twoheadrightarrow (Y, \leq_Y)$ in $\mathsf{Ord}(\mathcal{D})$. That is to say, we have an so-morphism $q\colon \mathsf{Ord}(F)(X, \Delta_X) \twoheadrightarrow (Y, \leq_Y)$ and so $\mathsf{Ord}(F)$ is covering. □

Putting everything together we now obtain the main result of this section.

4.6.7 Proposition. *For any regular ordinary category $\mathcal{C}$ there is an equivalence of* Pos*-categories* $\mathsf{Ord}(\mathcal{C}_{oex/reg}) \simeq \mathsf{Ord}(\mathcal{C})_{ex/reg} \simeq \mathcal{C}_{ex/reg}$*, where $\mathcal{C}_{oex/reg}$ denotes the exact completion of $\mathcal{C}$ as an ordinary category.*

Proof. The ordinary regular functor $\Gamma\colon \mathcal{C} \to \mathcal{C}_{oex/reg}$ is fully faithful and covering. By the preceding lemmas we then have that the induced functor on categories of internal posets $\mathsf{Ord}(\Gamma)\colon \mathsf{Ord}(\mathcal{C}) \to \mathsf{Ord}(\mathcal{C}_{oex/reg})$ satisfies the same properties in the enriched sense. Furthermore, the category $\mathsf{Ord}(\mathcal{C}_{oex/reg})$ is exact by Proposition 4.6.4 and thus from Theorem 4.5.7 we deduce that $\mathsf{Ord}(\mathcal{C}_{oex/reg}) \simeq \mathsf{Ord}(\mathcal{C})_{ex/reg}$.

Similarly, the composite functor $\mathcal{C} \to \mathsf{Ord}(\mathcal{C}) \to \mathsf{Ord}(\mathcal{C}_{oex/reg})$ is regular, fully faithful and covering, being a composition of two functors satisfying these properties. Again by Theorem 4.5.7 we conclude that $\mathcal{C}_{ex/reg} \simeq \mathsf{Ord}(\mathcal{C}_{oex/reg})$. □

Chapter 5

Vistas

In this very short final chapter we would like to briefly talk about some potentially interesting perspectives for future research. The ideas considered here follow the general theme of exploring the notions of regularity and exactness together, potentially, with the associated completion constructions in various contexts.

5.1 Regular and Exact Completion for weakly lex Pos-categories

In this thesis we talked a lot about regular and exact completion constructions. In Chapter 4 we constructed the exact completion of a regular category in the Pos-enriched context. It could then be natural to ask if the regular and exact completions $\mathcal{C}_{reg}$ and $\mathcal{C}_{ex}$ for weakly lex ordinary categories $\mathcal{C}$ might also have an analogue in this enriched setting. If the answer is affirmative, then it would be interesting to explore the connection of these completions with varieties and quasivarieties of ordered algebras. This was actually one of the directions for future research suggested by Kurz & Velebil at the end of their paper [65].

Recall that, for any ordinary category $\mathcal{C}$, both $\mathcal{C}_{reg}$ and $\mathcal{C}_{ex}$ have enough projectives. If a similar construction of these completions were to work in the enriched context, it seems that this would still have to be true, where of course now projectivity is to be considered with respect to coinserters rather than regular epimorphisms. It might thus be prudent to first look for examples of exact Pos-categories which satisfy this property. In fact, as it appears to the present author, in varieties of ordered algebras but also in other categories such as Nach or S-Pos we do indeed have enough projectives. In all these cases it seems that every object can be covered by one which is free on a discrete poset and that the latter are projective objects.

Finally, one could also try to employ these reg/wlex and ex/wlex to investigate categories which are monadic over Pos or certain epireflections of such categories, as was done by Vitale [99–101]. Note, however, that we should ex-

pect the situation here to be more complicated. Indeed, monadic categories $\mathsf{Pos}^{\mathbb{T}}$ do not appear to be exact or even regular in general. In the ordinary case, $\mathsf{Set}^{\mathbb{T}}$ is always exact because of the validity of the Axiom of Choice in Set. Thus, we might need to impose additional conditions on the monad $T\colon \mathsf{Pos} \to \mathsf{Pos}$ in order to obtain interesting results.

5.2 Regularity and Exactness over other bases: the case $\mathcal{V} = \mathsf{Met}_\infty$

In recent years there has been some interest in categories which are enriched over the monoidal category Met_∞ of *generalized* metric spaces. For example, in the papers [4, 89] such metric-enriched categories were used among other things to provide largely categorical proofs of the existence and uniqueness of the Gurarii space.

The objects of Met_∞ are metric spaces (X, d) where $d : X \to [0, \infty]$, i.e. we allow the distance between two points to be infinite. Morphisms $f\colon (X, d) \to (X', d')$ are *non-expansive* maps, i.e. functions satisfying $d'(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. It is then easy to see that Met_∞ is complete and cocomplete. Furthermore, it acquires a monoidal structure by defining, for any two $(X, d), (Y, d') \in \mathsf{Met}_\infty$, a metric $d \otimes d'$ on the cartesian product $X \times Y$ of sets by

$$(d \otimes d')((x, y), (x', y')) \coloneqq d(x, x') + d'(y, y')$$

Then Met_∞ is in fact *closed* as a monoidal category, the internal hom being given by providing $\mathsf{Met}_\infty(X, Y)$ with the sup-metric $d(f, g) = \sup\{d'(fx, gx) | x \in X\}$.

This allows us to consider categories $\mathcal{K}$ which are enriched over Met_∞. In more concrete terms, such a category $\mathcal{K}$ is an ordinary category such that each $\mathcal{K}(X, Y)$ is a (generalized) metric space and such that we have

$$d(gf, gf') \leq d(f, f') \quad \text{and} \quad d(fh, f'h) \leq d(f, f')$$

whenever these compositions make sense, and where we now denote the metric on every hom-set by the same letter d.

A perhaps interesting question for us here is whether there exist useful notions of regularity and exactness for Met_∞-categories. Some preliminary calculations indicate that perhaps we could define a notion of regularity by starting from the notion of isometry as the metric analogue of monomorphism. Recall [89] that an $f\colon X \to Y \in \mathcal{K}$ is an *isometry* if $d(fx, fx') = d(x, x')$, for every $x, x'\colon A \to X \in \mathcal{K}$. Then we can consider the class of morphisms which are left orthogonal (in the enriched sense) to all isometries, which we might call *strong coisometries*. A definition of regularity could then naturally be based on the requirement of a stable (strong coisometry, isometry) factorization system.

There should also be an analogue of kernel pair which should be involved in both the notions of regularity and exactness. In our view, this should be what we could call the *kernel family* of a morphism $f\colon X' \to Y \in \mathcal{K}$: the family $(K(f, \epsilon))_{\epsilon \geq 0}$, where for every $\epsilon \geq 0$ the square

$$\begin{array}{ccc} K(f,\epsilon) & \xrightarrow{\ q\ } & X \\ {\scriptstyle p}\downarrow & \sim_\epsilon & \downarrow{\scriptstyle f} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

is an ϵ-*pullback* (see [4,89]). The latter means that every pair $u, v\colon Z \to X$ such that $d(fu, fv) \leq \epsilon$ factors uniquely through $K(f,\epsilon)$ and also that p, q are jointly isometric (note that this is stronger than the definition of ϵ-pullbacks in [89]).

Assuming regularity has been defined and works sufficiently well, we could propose a notion of exactness which asserts effectiveness in an appropriate sense of certain families of relations. More precisely, let us say that a family $(R_\epsilon)_{\epsilon\geq 0}$ of relations on $X \in \mathcal{K}$ (=equivalence classes of isometries $R_\epsilon \rightarrowtail X \times X$) is a *familial congruence* if:

1. $R_\epsilon \supseteq K(1_X, \epsilon)$ for all $\epsilon \geq 0$.
2. Every R_ϵ is symmetric.
3. $R_\epsilon R_{\epsilon'} \subseteq R_{\epsilon+\epsilon'}$, for all $\epsilon, \epsilon' \geq 0$.

Then we would say that a regular Met_∞-category $\mathcal{K}$ is *exact* if every familial congruence is the kernel family of some morphism.

In the case of Met_∞, a familial congruence is the same as a *pseudometric* p on X which is compatible with the existing metric d on X, in the sense that $p(x,x') \leq d(x,x')$ for all $x, x' \in X$. Then Met_∞ satisfies the exactness property because one can construct the quotient metric space of X under the pseudometric p in the usual manner.

Of course one would need to have good examples of regular and exact Met_∞-categories. A good testing ground should be the context of *quantitative algebras* introduced in [76]. These are a metric version of universal algebras which indeed form Met_∞-enriched categories, in the same way as the categories of ordered algebras of Bloom & Wright are enriched in Pos. In the subsequent paper [77], various notions of *quantitative variety* were introduced and characterized by suitable syntactic conditions. It would thus be interesting to explore whether these quantitative varieties can be characterized as exact Met_∞-categories with some other nice properties. If this turns out to be the case, one could of course also ask about exact completions in the Met_∞-enriched context.

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