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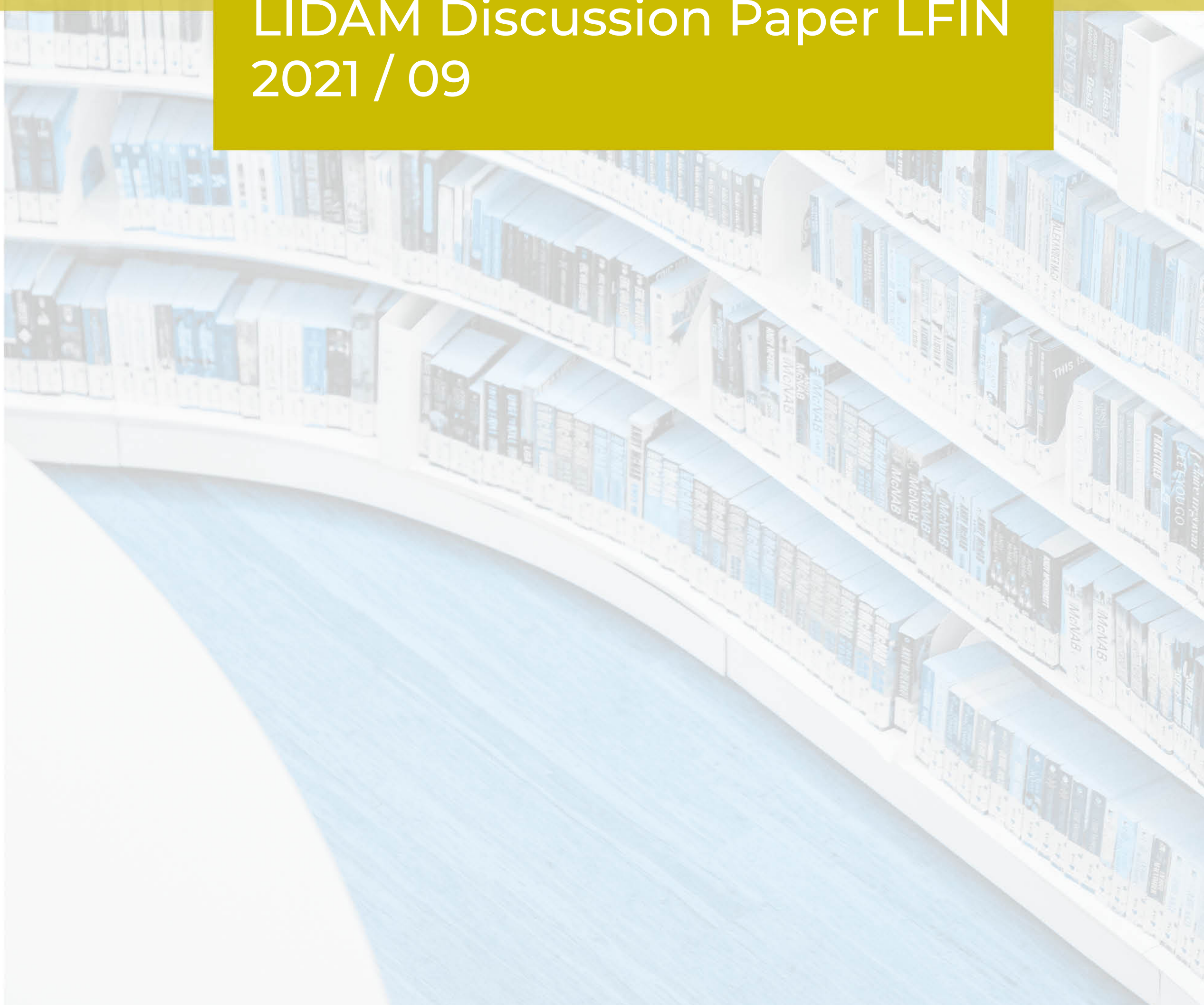

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Asymptotic Single Risk Factor Models with Stochastic and Correlated Loss Given Default

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Abstract

In line with the recent steer of the Basel Committee to foster a regulatory framework balancing greater risk-sensitivity, simplicity and comparability, we propose two extensions to the asymptotic single risk factor (ASRF) model in order to account for stochastic and correlated losses given default. In either setups, the strength of the PD-LGD link is controlled via a single additional risk parameter, as for the default dependence in the standard Basel framework. This parameter is connected to the correlation between the default rate and average observed loss given default, which is an observable statistic. We provide portfolio-invariant semi-analytical formulas for computing value-at-risk, solely by supplying new regulatory mapping functions translating unconditional LGDs into conditional LGDs. These ASRF extensions provide control on the PD-LGD link and give full flexibility regarding the choice of the marginal LGD distributions without disrupting the standard Basel machinery. This contributes to enhance regulatory capital computations by considering well-documented empirical evidence while maintaining computational tractability.

Keywords: Credit Risk; Factor Model; Value-at-risk; Basel Banking Regulations; Capital requirement

JEL classification: G21: G28; G32

Declaration of interest: none

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1. Introduction

Since the 2008 financial crisis, credit risk quantification for regulatory capital purposes has become a key concern for Basel compliant regulators and supervisors wanting to strengthen the stability and resilience of the banking sector worldwide (Mariathasan and Merrouche, 2014; Juelsrud and Wold, 2020; Thakor, 2021). Credit value-at-risk (VaR) portfolio models and their regulatory implementations are the tools by which global systemically important banks (G-SIBs) compute the amount of economic capital needed to

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cover against a pre-defined level of insolvency risk (Rossignolo et al., 2012; Puzanova et al., 2009). The goal of any credit VaR portfolio model is to compute a loss quantile, which is typically achieved by characterising the shape of the probability distribution of the portfolio credit losses. The statistics of the portfolio loss distribution depend on the joint probability law of the individual losses which, in turn, depends on both the marginal distributions of the default indicators, on the losses given default and on their dependence structure.

The asymptotic single risk factor model (ASRF) is the credit VaR portfolio model underlying the risk-weighted assets (RWA) formulas involved in the prevailing internal ratings-based (IRB) approach (BCBS, 2019b) of the Basel Committee on Banking Supervision (BCBS). The RWA framework exploits an appealing feature of the ASRF model, which delivers a closed form and portfolio-invariant expression for the portfolio loss quantiles. The latter stems from its core two assumptions (BCBS, 2005, 2006), namely (i) the credit portfolio is asymptotically fine-grained and (ii) there is a single systematic normal risk-factor driving the default dependence between the obligor's individual losses (Gordy, 2003). Those assumptions have been widely discussed in the literature, see for example Gordy (2000); Gourieroux et al. (2000); Emmer and Tasche (2003); Gordy and Lütkebohmert (2007) and Frey et al. (2001); Pykhtin (2004); Laurent et al. (2016), respectively. To some extent, Basel regulation mitigates the impact of the latter by adopting a conservative view about the confidence level, considering a 99.9% quantile for the VaR.

Although the above assumptions (i) and (ii) remain silent about how to model losses given default, the derivation of the Basel RWA formula for credit risk is compatible with two assumptions regarding the LGD: it can be modelled either as a constant (under the foundation IRB approach, F-IRB) or, more generally, as a latent variable *independent* from the systematic factor (under the advanced IRB approach, A-IRB). In both cases, the LGD value appearing in the RWA formula coincides with the expected LGDs (see eq. (5) in Tarashev and Zhu (2008)). However, multiple empirical studies from academia (Altman et al., 2005; Khieu et al., 2012; Jokivuolle and Virén, 2013; Yao et al., 2017), industry participants (Perraudin and Hu, 2006; Hu, 2007), central banks (Düllmann and Gehde-Trapp, 2004; Mora, 2012) and rating agencies (Varma and Cantor, 2005; Cantor et al., 2007) provide empirical evidence of a negative correlation existing between recovery rates (RRs) and default rates (DRs). Clearly, discarding this reality can lead to serious underestimation of the capital requirements. In the current Basel regulatory framework, this positive relationship between PDs and LGDs is tackled indirectly, by considering *downturn LGDs* (BCBS, 2017). Nevertheless, handling the "PD-LGD dependency" in such a way may seem somewhat artificial because this procedure actually ends up using "stressed" LGD estimates (similar to quantiles), hence contradicting the LGD assumptions underlying the RWA formula. Another issue inherent to the downturn LGD methodology is put forward by banking authorities: "the lack of explicit guidance and limited supervisory and industry consensus on how to incorporate economic downturn component in model estimation has led to significant differences in practices and [...] variability in risk-weighted exposure amounts [...]" (EBA, 2019). For these reasons, the PD-LGD dependence should be accounted for within the regulatory formula, calling for transparent extensions of the ASRF model whereby the stochastic link between the LGDs and the default events is explicit, just as for the dependency between the defaults themselves.

Several approaches have been previously proposed to tackle the PD-LGD dependency; they can be grouped in two main classes of models. The first stream of literature actually focuses on the computation of the "stressed" LGDs that will serve as "downturn" LGDs to plug in the current ASRF. Those stressed values are deemed to represent LGDs computed in adverse economic conditions (Miu and Ozdemir (2006); Calabrese (2014); Buncic and Melecky (2013)), during which LGDs and DRs are typically higher than average. This methodology is inexpensive, both computationally-wise and modelling-wise. Indeed, it amounts to treat the LGDs as constants; the dependency is accounted for by using stressed values, instead of average values, as it should be the case under PD-LGD independence. However, this approach does not allow to control the strength of the PD-LGD relationship. Moreover, it does not address the lack of transparency regarding the modelling of the underlying dependence structure.

A second stream of the literature focuses on the explicit modelling of the PD-LGD dependence. Even though these models differ in their specification, they are all based on the seminal work of Merton (1974). Giese (2005) models the PD-LGD dependency via a re-parametrization of the expected LGD conditional upon the systematic factor using the PD. The function mapping PD to LGD is then obtained by regression. However, the PD-LGD dependency is controlled by a functional parameter (the shape of the conditional LGD as a function of the PD), and it is not clear what is the corresponding factor model hidden behind. Among the approaches specifying the factor model explicitly, Frye (2000) defines the RR as a floored normal variable representing the collateral value, modelled as a linear transform of the systematic risk factor driving the default event. Chabaane et al. (2004) and Pykhtin (2003) extend the work of Frye (2000) and feature either the systematic factor only, or both the systematic and idiosyncratic factors driving default event, respectively. Unfortunately, no guidance is provided regarding how to calibrate the additional parameters involved. Moreover, both approaches postulate a specific shape for the LGD distribution whose parameters are controlled indirectly, via lognormal collateral values. In Tasche (2004), the author directly models the loss as a mixture distribution with a mass at 0 (if no default) and a continuous distribution on $(0, 1)$. The losses are then correlated using a Gaussian copula that substitutes the copula controlling the default correlation; default correlation is thus a consequence of loss (not LGD) correlation. The expression of the loss correlation parameter is then simply set equal to the formula prescribed by Basel for the default correlation. Han (2017) proposes a more complex modelling framework that is workable using appropriate numerical methods, but is losing the analytical tractability and portfolio invariance regulatory features. Very recently, García-Céspedes and Moreno (2020) extended the multi-factor approach of Pykhtin (2004) to stochastic LGDs. The authors consider various LGD distributions constructed from a factor model. However, the featured "LGDs" are in fact not losses given default strictly speaking because they are defined independently from the default status. As a consequence, it is difficult to connect the statistics of the "LGD" variables to that of observed realised losses, which substantially complicates the calibration procedure. The generic framework proposed in Van Damme (2011) is also worth mentioning: the LGD is modelled as a function of the latent credit worthiness variable governing the default event of the corresponding obligor, and is a "true loss-given-default", as the LGD variables are only defined *conditional upon default*. This provides a clean mathematical setup giving full flexibility with respect to the choice of the marginal LGD distributions in a homogeneous pool environment. Unfortunately, there is no control parameter available to tune the strength of the PD-LGD dependency (the same issue arises in Frye (2014), where the form of the conditional loss rate is set identical to that of the conditional default probability). Moreover, as we will show in the paper, the strength of this link does not comply with empirical evidence.

Overall, it seems that none of the existing attempts to introduce a proper PD-LGD stochastic link within an ASRF framework succeeded to comply with the 2008 post financial crisis Basel regulatory philosophy promoting simultaneously greater *simplicity, comparability and risk-sensitivity* of the capital requirements (BCBS, 2014).

Following the most recent steer in the Committee's regulatory strategy, our aim is to fill this gap by providing a model addressing the PD-LGD dependence explicitly in a transparent and consistent way while maintaining the balance between risk-sensitivity and simplicity required for regulatory purposes (Ahnert et al., 2021). We make the following contributions to the literature on economic capital.

First, combining the flexible general setup proposed in Van Damme (2011) with the insights provided by the seminal works of Frye (2000) and Pykhtin (2003), we propose two extensions to the ASRF framework, able to deal with the PD-LGD dependence. The first approach can be regarded as a standard extension to the original Van Damme (2011) setup, modelling a PD-LGD link based on the latent random variable driving the default event. In the second model specification, we introduce a PD-LGD link based on the single systematic risk-factor driving the default event, a dependence structure more in line with the current ASRF methodology. We show that those models preserve the standard Basel machinery. Interestingly, we are able to control the strength of the PD-LGD link in both model setups, possibly via a single dedicated parameter. Indeed, given a specific portfolio credit risk profile, the strength of the PD-LGD link relies solely in the choice of the value for this new factor loading, which does not impact the marginal distributions of the LGDs. In addition, semi-analytical formulae for computing the value-at-risk under the large homogeneous pool approximation are available in either setups, thereby avoiding to rely on computationally intensive numerical methods such as Monte Carlo simulations. Moreover, those formulae exhibit portfolio-invariance, that is, the individual contribution of each exposure to the total portfolio VaR only depends on the individual risk-profile of the single exposure and not on the portfolio it is added to. To the best of our knowledge, these are the first setups able to model and control, within an ASRF framework, the strength of the PD-LGD relationship via an additional parameter, while at the same time providing full flexibility regarding the LGD marginal distributions and maintaining the appealing regulatory feature of delivering a portfolio-invariant semi-analytical formula for the computation of value-at-risk.

Second, we provide insights for calibrating this new parameter to a given *empirical correlation*, which is an observable statistic defined as the correlation between the DR and the average LGD observed in a given period of time. By doing so, we fill the gap between the ASRF framework and the empirical evidence provided by Altman et al. (2005). In particular, we derive the analytical expression of the empirical correlation as a function of the PD-LGD strength in a fully homogeneous portfolio setting.

Third, we investigate the range of attainable empirical correlations that can be generated in either setups, and conclude that the second model – which is also the simpler, mathematically speaking – displays superior performance over the other, in the sense that it is able to capture larger correlation spans. This proves to be key in practice.

Finally, we provide numerical experiments which allow to validate our formulae, to understand the impact of the PD-LGD dependence parameter on the VaR, to illustrate the convergence speed at which the actual VaR converges to our large pool approximation as the size of the pool increases, and to compare our models with the downturn LGD methodology inherent to the IRB Basel approach.

This article is structured as follows. In Section 2 , we provide a brief overview of the credit VaR portfolio model, underlying the risk-weighted assets formulas available under the IRB approach of Basel III. In Section 3, we introduce two extensions to the Gaussian ASRF model, providing control on the correlation between the default rate and the loss given default. In Section 4, we derive the formulae to connect the model parameter controlling the PD-LGD dependence to the empirical correlation and compare the results associated with both extensions. The empirical analysis is performed in Section 5. Section 6 concludes.

2. Capital requirements in Basel regulation

Under the Basel II accords (IRB), capital requirements on the banking book are defined through a one-year VaR paradigm. For a given credit portfolio, it is defined as the amount of capital a financial institution needs to absorb the unexpected losses (UL) on the portfolio over a one year horizon, at confidence level α:

$$K_\alpha := \mathbb{Q}_\alpha[L] - \mathbb{E}[L] \ . \tag{1}$$

In this expression, L stands for the random variable representing the total loss on the portfolio, $\mathbb{E}[L]$ is the expected loss (EL) on the pool and $\mathbb{Q}_\alpha[L] = \mathrm{VaR}_\alpha(L)$ defines the portfolio loss quantile at confidence level α, that is, the value-at-risk.

The computation of the EL is relatively straightforward, it is the sum of the expected loss on each constituent of the portfolio. The computation of the quantile, however, is more difficult because it strongly relies on the dependence structure between the individual losses. The formula associated with the IRB approach relies on the ASRF model which postulates a one-factor Gaussian copula dependency structure between the defaults. Since Basel II (BCBS, 2006), the latter idea has remained at the core of the RWA formulas found under the IRB approach, translating equation (1) into specific capital requirements for a series of credit portfolio asset classes defined in the accords (BCBS, 2017, 2019a,b).

2.1. One-factor Gaussian default model

We consider a portfolio made of I loans and define the weight of loan i by w_i, representing the exposure of loan i relative to the total portfolio exposure. We note by B_i the default indicator of loan i. In case of default, the loss severity on loan i is modelled as a random variable Λ_i with support in $[0, 1]$. Mathematically, the loss on the pool relative to the total exposure can be modelled as

$$L = \sum_{i=1}^{I} w_i L_i \quad \text{where} \quad L_i := \begin{cases} \Lambda_i & \text{if} \quad B_i = 1 \\ 0 & \text{if} \quad B_i = 0 \end{cases} . \tag{2}$$

Clearly, $B_i \sim Ber(p_i)$ is a Bernoulli random variable whose parameter p_i corresponds to the default probability and Λ_i is the associated loss-given-default. Note that the latter is only defined on the outcomes ω for which $B_i(\omega) = 1$, so that the expectation of the LGDs are defined as

$$\lambda_i := \mathbb{E}[\Lambda_i | B_i = 1]$$

The total expected loss only depends on the expected value of the individual losses,

$$\mathbb{E}[L] = \sum_{i=1}^{I} w_i \, \mathbb{E}[L_i] \, .$$

However, the computation of $\mathbb{Q}_\alpha[L]$ depends on the joint distribution of the L_is. In line with the IRB approach, we rely on the Gaussian one-factor model, first introduced by Vasicek (1991, 2002) for homogeneous portfolios, and later extended by Gordy (2003) to fit heterogeneous portfolios. Those specifications are based on the seminal work of Merton (1974). Precisely, the default events are modelled using correlated latent credit worthiness variables $X_1, \dots, X_I$:

$$B_i = 1_{\{X_i \leq c_i\}} \quad \text{where} \quad X_i := \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i \tag{3}$$

and $Y, Z_1, \dots, Z_I$ are i.i.d. standard normal random variables. The dependence between the default events results from the common systematic risk factor Y and is controlled by the factor loadings $\sqrt{\rho_i}$'s. Noting $\Phi(\cdot)$ the cumulative distribution function (cdf) of the standard normal variable, setting the default threshold to $c_i := \Phi^{-1}(p_i)$ ensures that

$$\mathbb{P}(B_i = 1) = \mathbb{P}(X_i \leq c_i) = \Phi(c_i) = p_i \, .$$

Yet, a quantile of the loss depends on the distribution of L, which is a complicated function of the model parameters. However, a simple expression for the asymptotic distribution of L can be obtained when $I \to \infty$; this is the *large pool* (LP) approximation.

2.2. Large pool approximation

Following the work of Gordy (2003), a closed form asymptotic approximation for quantiles on the portfolio loss can be found based on two core assumptions: (i) the portfolio must be asymptotically fine grained, such that when $I \to \infty$ the w_i tend towards 0 and (ii) conditionally on the single systematic risk factor Y, the L_is are independent random variables. Under these two conditions, it follows from the strong law of large numbers that the idiosynchratic risks Z_i's can be fully diversified away, and that the relative portfolio loss distribution degenerates to its expectation conditional upon Y, i.e., it is a function of Y only. In particular,

$$\mathbb{Q}_\alpha[L] \xrightarrow{I\to\infty} \mathbb{Q}_\alpha[\mathbb{E}[L|Y]] = \mathbb{Q}_\alpha\left[\sum_{i=1}^{I} w_i\, \mathbb{E}[L_i|Y]\right] .$$

Using the tower law, the conditional expectation of L_i takes the general form

$$\mathbb{E}[L_i|Y] = \mathbb{E}[L_i|Y, B_i = 0] \times \mathbb{P}(B_i = 0|Y) + \mathbb{E}[L_i|Y, B_i = 1] \times \mathbb{P}(B_i = 1|Y) .$$

Because $L_i = 0$ whenever $B_i = 0$ and $L_i = \Lambda_i$ otherwise, one gets

$$\mathbb{E}[L_i|Y] = \lambda_i(Y) \times p_i(Y) \tag{4}$$

where the two factors stand for the conditional probability and conditional expected LGD of loan i, respectively:

$$\lambda_i(Y) := \mathbb{E}[\Lambda_i|B_i = 1, Y]$$

and

$$p_i(Y) := \mathbb{P}(B_i = 1|Y) = \mathbb{P}(X_i \leq c_i|Y) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right) . \tag{5}$$

Using this notation, the conditional expected loss on the pool becomes

$$\mathbb{E}[L|Y] = \sum_{i=1}^{I} w_i \lambda_i(Y) p_i(Y) . \tag{6}$$

Observe that, from the tower law,

$$\mathbb{E}[\lambda_i(Y)|B_i = 1] = \mathbb{E}[\Lambda_i|B_i = 1] = \lambda_i \quad \text{and} \quad \mathbb{E}[p_i(Y)] = \mathbb{E}[B_i] = p_i .$$

Hence, if the Λ_is are independent from the systematic factor on the set $\{B_i = 1\}$, the EL simply reads as

$$\mathbb{E}[L] = \mathbb{E}[\mathbb{E}[L|Y]] = \sum_{i=1}^{I} w_i\, \mathbb{E}[\Lambda_i|B_i = 1]\, \mathbb{E}[p_i(Y)] = \sum_{i=1}^{I} w_i \lambda_i p_i . \tag{7}$$

We have the following result, which proof is given in Appendix A.1 and is a rather immediate consequence of Proposition 4 in Gordy.

Lemma 1. *If the λ_i's are decreasing functions, then*

$$\mathbb{Q}_\alpha[\mathbb{E}[L|Y]] = \sum_{i=1}^{I} w_i \lambda_i(\mathbb{Q}_{1-\alpha}[Y]) p_i(\mathbb{Q}_{1-\alpha}[Y]) \ . \tag{8}$$

Note that in the standard case where $Y \sim \mathcal{N}(0,1)$, $\mathbb{Q}_{1-\alpha}[Y] = -z_\alpha$ where $z_\alpha := \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal variable.

Interestingly, the quantile of $\mathbb{E}[L|Y]$ in eq. (8) takes the same form as the expectation in eq. (7) in the sense that it is a simple sum of terms associated with the individual loans, each of which being the product of three factors: the relative exposure, a (downturn[1]) LGD and a (downturn) PD. As a result, under the LP approximation, $L \to \mathbb{E}[L|Y]$ and the UL on the pool can be decomposed as the sum of the unexpected loss on each constituent, i.e., assuming independence between Λ_i and Y conditional upon default in the expected loss term. If Y is normally distributed,

$$K_\alpha \xrightarrow{I \to \infty} \mathbb{Q}_\alpha[\mathbb{E}[L|Y]] - \mathbb{E}[\mathbb{E}[L|Y]]] = \sum_{i=1}^{I} w_i \left(\lambda_i(-z_\alpha) p_i(-z_\alpha) - \lambda_i p_i\right) \tag{9}$$

2.3. Basel IRB formula

The formula to compute the minimum capital requirements in the Basel regulation[2] can be found in p. 4/14, paragraph 31.4 of BCBS (2019b). Adapted to our notation system, it reads as

$$K = \sum_{i=1}^{I} w_i \eta_i \left(\Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho_i} z_{0.999}}{\sqrt{1-\rho_i}} \right) - p_i \right) \ . \tag{10}$$

All parameters have been previously defined and have the same interpretation, except η_i which stands for the downturn LGD of loan i. Evidently, formula (10) corresponds to formula (9) with $Y \sim \mathcal{N}(0,1)$, $\alpha = 99.9\%$, replacing both $\lambda_i(-z_\alpha)$ and λ_i by η_i.

This formula has specific features that makes it quite practical for regulatory purposes. First, it exhibits the portfolio invariance feature, in the sense that the capital requirement of the pool is decomposed as a sum of marginal contributions. The linearity of the expectation operator guarantees that the capital charge for loan i only depends on its own risk-profile, and not on the risk-profile of the portfolio it is added to. Second, the LP approximation works quite well even for relatively small portfolio sizes compared to actual pools. Finally, all the terms have a clear interpretation, possibly except the factor loadings ρ_i, controlling the default dependence.

[1]Equivalently, these downturn LGDs and PDs, can be understood as "worst-case" LGDs and PDs.

[2]We ignore the maturity adjustment (MA) factor within the scope of this analysis. Indeed, the latter is simply added "on top" of the resulting analytical formula coming from eq. (9) and does not derive from the detailed Gaussian factor model setup.

The form of the conditional PD $p_i(y)$ featured in the Basel formula follows from the Gaussian one-factor model. By contrast, there is no specific model proposed for the conditional LGD $\lambda_i(y)$ (BCBS, 2005, 2006), only the term "LGD" is encountered in the risk-weighted assets formulas (BCBS, 2019b). Hence, as far as Basel regulation is concerned, the validity of the LP approximation, relies either on the assumption that the LGD is deterministic (under the F-IRB approach), or independent from the systematic factor Tarashev and Zhu (2008) under the A-IRB approach. Nevertheless, in line with the seminal works of Altman et al. (2005), numerous studies from industry participants (Perraudin and Hu, 2006; Hu, 2007), central banks (Düllmann and Gehde-Trapp, 2004; Mora, 2012) and rating agencies (Varma and Cantor, 2005; Cantor et al., 2007), provide empirical evidence of a negative correlation existing between RRs and DRs. It is widely admitted that failing to include in the credit risk model the above dependency leads to underestimate regulatory capital requirements (Altman, 2006, 2009).

3. Extended Gaussian frameworks with stochastic and correlated LGD

Hence, without adequate counter-measures, the lack of a proper stochastic link inherent to the credit VaR portfolio model would lead to underestimate the regulatory capital charges (Altman, 2006, 2009). As detailed earlier, the Basel framework circumvents this issue by relying on the notion of "downturn LGD": the LGDs plugged in (10) are considered as constants η_i's, but the latter are estimated in adverse scenarii. By doing so, Basel regulations implicitly acknowledge a dependence between the default likelihood and the loss severity, but without explicitly modelling this relationship. By comparing the first terms of (9) and (10), it is clear that the downturn LGD η_i could be interpreted as the worst-case LGD at confidence level 99.9%, $\lambda_i(-z_{0.999})$, thereby justifying the "downturn LGD" procedure. However, in contrast to the conditional PD, no functional form is provided for the conditional LGD, and this way to proceed remains obscure for at least two reasons. First, it is not clear whether the estimated downturn LGDs would indeed correspond to figures associated to the 99.9% worst-case scenario for the systematic factor, as the concept of "downturn LGD" is not present in the ASRF model, where the LGDs are considered as constants. Second, the comparison of the second terms of (9) and (10) suggests that the IRB formula replaces the *expected* LGD by the *downturn* LGD also in the expected loss term. Again, this seems questionable because using η_i instead of λ_i in the expected loss term tends to increase the EL, hence, to decrease the UL, that is, to lower capital requirements.[3] Overall, it seems that making the economy of specifying the conditional LGDs $\lambda_i(y)$'s comes at a cost of a PD-LGD dependence scheme that is hard to grasp, which is not desirable.

[3] A possible motivation for replacing the expected LGD (λ_i) in (9) by the downturn LGD (η_i) in (10) in the EL term could result from (6), that is, from a potential dependency between PD and LGD resulting from Y. Such a dependency could explain why $\mathbb{E}[\mathbb{E}[\Lambda_i|B_i = 1, Y]p_i(Y)]$ could not be replaced by $\mathbb{E}[\Lambda_i|B_i = 1]\,\mathbb{E}[p_i(Y)] = \lambda_i p_i$ as done in (7) when independence holds. However, we could not find any justification of this in the official Basel literature, and there is clearly no reason for $\mathbb{E}[\mathbb{E}[\Lambda_i|B_i = 1, Y]p_i(Y)]$ to be equal to $\eta_i p_i$ when interpreting η_i as $\lambda_i(-z_{0.999})$.

We discuss two different ways to include within the Basel credit risk quantification framework stochastic LGDs correlated to the systematic factor. Precisely, we include a PD-LGD link by proposing two functional forms for $\lambda_i(y)$ which, by analogy with unconditional PD *vs.* conditional PD, can be called "mapping functions", translating unconditional LGDs into conditional LGDs (BCBS, 2005). These different functional forms arise from explicit factor models and will lead to different Gaussian coupling schemes between the Λ_is, and between the B_is and Λ_is. However, these approaches are similar in the sense that both:

(i) depict the same conditional independence property: conditionally upon Y, not only $B_i \perp B_{j\neq i}$, but also $L_i \perp L_{j\neq i}$ and $\Lambda_i \perp B_{j\neq i}$, for each pair of loans (i,j) in the pool,

(ii) can accommodate distributions for the LGDs that could be given exogenously and satisfy the decreasing assumption stated in Lemma 1.

In the sequel, we introduce two dependency models. For each of them, we provide the conditional moments $\mathbb{E}[\Lambda_i^k|B_i = 1, Y]$. We note the conditional mean and conditional variance as $\lambda_i(Y)$ and $\nu_i(Y)$:

$$\lambda_i(Y) := \mathbb{E}[\Lambda_i|B_i = 1, Y], \quad \gamma_i(Y) := \mathbb{E}[\Lambda_i^2|B_i = 1, Y] \quad \text{and} \quad \nu_i(Y) := \gamma_i(Y) - \lambda_i(Y)^2.$$

Those functions play a central role in the models presented below.

3.1. Approach 1: PD-LGD Dependency on X_i

Van Damme (2011) develops a generic framework to include a stochastic and correlated LGD into structural models for credit risk, postulating

$$\Lambda_i = F^{-1}\left(1 - \frac{\Phi(X_i)}{p}\right) , \tag{11}$$

where X_i is the latent random variable in (3) driving the default indicator, and F represents the underlying cdf of Λ_i (whose support is assumed to be a compact subset of $[0,1]$). It is easy to verify that the cdf of L_i upon default is indeed F (see Appendix A.2).

Two comments are worth to be made. First, the author focuses on a large homogeneous pool of loans, both in terms of the default risk profile ($p_i = p, \rho_i = \rho$) and LGD risk profile ($F_i = F$). Second, this model does not feature any extra parameter compared to the case where Λ_i and B_i are independent. Consequently, there is no degree of freedom to control the strength of the PD-LGD relation.

In order to allow for heterogeneity and, more importantly, to provide control over the PD-LGD link, we consider

$$\Lambda_i = F_i^{-1} \circ \Phi\left(\hat{X}_i\right) \quad \text{where} \quad \hat{X}_i := \sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(X_i)}{p_i}\right) + \sqrt{1-\hat{\rho}_i}\hat{Z}_i\ , \tag{12}$$

and $\hat{Z}_1, \ldots, \hat{Z}_I$ are i.i.d. standard normal random variables independent from $Y, Z_1, \ldots, Z_I$.

Eq. (12) can be viewed as a standard one-factor extension of the model proposed in Van Damme (2011, p. 2526, eq. 9). Indeed, the link between PDs and LGDs is still made via the creditworthiness random variable X_i, the initial expression is recovered by imposing $\hat{\rho}_i = 1$ and pool homogeneity, and F_i is the cdf of L_i conditional upon default:

Lemma 2. *The cdf of Λ_i in (12) is F_i for all values of $\hat{\rho}_i$, i.e.,*

$$\mathbb{P}(L_i \leq l | B_i = 1) := \mathbb{P}(\Lambda_i \leq l | B_i = 1) = F_i(l) \ .$$

The proof is provided in Appendix A.3 and consists of showing that $\hat{X}_i$ is a standard normal random variable conditional upon $B_i = 1$, combined with the probability integral transform (PIT) .

Within this model setup, the dependence between the default event and the loss given default is driven *both* by the systematic factor Y, and the idiosyncratic factor Z_i. This impacts the computation of the conditional expectation of Λ_i upon default, as for obligor i the independence of the default event and the LGD is achieved only conditionally on X_i, that is, on the pair (Y, Z_i).

Lemma 3. *The conditional k-th moment of Λ_i is given by*

$$\mathbb{E}[\Lambda_i^k | B_i = 1, Y] := \frac{1}{p_i(Y)} \int_{-\infty}^{c_i(Y)} \int_{-\infty}^{\infty} \left(F_i^{-1} \left[\Phi \left(\sqrt{\hat{\rho}_i} \Phi^{-1} \left(1 - \frac{\Phi(\sqrt{\rho_i} Y + \sqrt{1-\rho_i} z)}{p_i} \right) + \sqrt{1 - \hat{\rho}_i} \hat{z} \right) \right] \right)^k \phi(\hat{z}) \phi(z) d\hat{z} dz \ , \tag{13}$$

with $c_i(y) := \Phi^{-1}(p_i(y))$ and $\phi = \Phi'$ is the density of the standard normal variable.

The proof is given in Appendix A.4.

The assumptions of Lemma 1 are satisfied given that the conditional mean $\lambda_i(y)$ associated with (13) is a decreasing function of y. The proof is provided in Appendix B.

3.2. Approach 2: PD-LGD Dependency on Y

Similarly to the credit worthiness normal latent variables X_i's used to introduce correlation across default events B_i's, we introduce normal latent variables $\tilde{X}_i$'s to model the LGDs Λ_i's. They will take the form of a linear combination of the systematic factor and i.i.d. normal variables. Precisely, we postulate that

$$\Lambda_i = F_i^{-1} \circ H_i(\tilde{X}_i) \quad \text{where} \quad \tilde{X}_i := -\sqrt{\tilde{\rho}_i} Y + \sqrt{1 - \tilde{\rho}_i} \tilde{Z}_i \tag{14}$$

where $\tilde{Z}_1, \dots \tilde{Z}_I$ are i.i.d. standard normal random variables independent from $Y, Z_1, \dots Z_I$. As in Approach 1, this construction scheme allows to model LGDs having a specific cdf F_i whose domain is $[0, 1]$. The variable $\tilde{X}_i$ in (14) plays a similar role as $\hat{X}_i$ in (12). However, in contrast with the latter that features X_i (that is, both Y and Z_i), the former features Y only (not Z_i), leading to a different coupling scheme. As for Lemma 2, it results from the PIT that the distribution of Λ_i conditional upon $B_i = 1$ is indeed F_i provided that one chooses as H_i the cdf of $\tilde{X}_i$ given default.[4] As a consequence, the cdf of L_i for $l \in [0, 1]$ is the mixed distribution $\mathbb{P}(L_i \leq l) = (1 - p_i) + p_i \times F_i(l)$. The explicit expression of H_i is given in the next lemma.

[4]Recall that Λ_i is defined on the set $\{B_i = 1\}$ only.

Lemma 4. *The conditional cumulative distribution function H_i in (14) takes the form*

$$H_i(x) = \Phi_2\left(x, \Phi^{-1}(p_i), -\sqrt{\tilde{\rho}_i\rho_i}\right)/p_i \ , \tag{15}$$

where $\Phi_2(x, y, r)$ is the cumulative distribution function of the bivariate standard normal with correlation coefficient r.

The proof is given in Appendix A.5.

The negative sign of the factor loading $-\sqrt{\tilde{\rho}_i}$ in (14) guarantees that for higher values of Y, we observe fewer defaults, hence lower values of the LGD for these defaulted creditors. The correlation between the latent Gaussian random variables X_i and $\tilde{X}_i$ is

$$\mathrm{Corr}(X_i, \tilde{X}_i) = -\sqrt{\tilde{\rho}_i\rho_i} \ ,$$

and it is clear from (3) and (14) that this setup triggers a dependence between the conditional PD and the loss given default. However, because

$$\mathrm{Corr}(\tilde{X}_i, \tilde{X}_j) = \sqrt{\tilde{\rho}_i\tilde{\rho}_j} \ ,$$

this setup also generates a dependence between the stochastic LGDs themselves. These simple expressions result from the fact that in Approach 2, the variable $\tilde{X}_i$ is a linear function of the common factor Y. This is not the case in Approach 1, where the variable $\hat{X}_i$ is a non-linear function of X_i and Y. Consequently, the expressions of $\mathrm{Corr}(X_i, \hat{X}_i)$ and $\mathrm{Corr}(\hat{X}_i, \hat{X}_j)$ are much more involved.

We have the following result, proven in Appendix A.6.

Lemma 5. *The conditional k-th moment of the LGD on loan i is given by*

$$\mathbb{E}[\Lambda_i^k | B_i = 1, Y] = \int_{-\infty}^{\infty} \left(F_i^{-1}\left[H_i\left(-\sqrt{\tilde{\rho}_i}Y + \sqrt{1-\tilde{\rho}_i}\tilde{z}\right)\right]\right)^k \phi(\tilde{z})d\tilde{z} \tag{16}$$

where H_i is given in Lemma 4.

The conditional mean $\lambda_i(y)$ in Approach 2 is obtained by setting $k = 1$ in (16). As in Approach 1, it is a monotonic decreasing function of y (both the cdf H_i and the inverse cdf F_i^{-1} are monotonic increasing functions of their arguments). Therefore, the assumptions in Lemma 1 hold, and $\mathbb{E}[L_i|Y]$ remains a monotonic decreasing function of Y, too.

4. Empirical vs Copula correlation

One of the key features of our proposed ASRF extensions is to comply with the empirical evidences provided by Altman et al. (2005) via an explicit modelling of the PD-LGD relationship. Moreover, the strength of the latter is controlled via specific parameters, ρ_i^{LGD} being respectively $\hat{\rho}_i$ or $\tilde{\rho}_i$, whether we refer to the model in Section 3.1 or in Section 3.2.[5] Nevertheless, the choice of those factor loadings is not trivial because they control the dependency between *latent* variables, which cannot be directly observed from credit data. Therefore, we would like to provide guidance about how to choose the ρ_i^{LGD}'s in order to match a certain level of *empirical correlation*, that is, correlation between *observable quantities* within our pool of loans. This could be the default rate ($\hat{N}$) and the average observed loss given default ($\hat{L}$). In practice indeed, one observes a sample of obligors and can compute the default rate, $\hat{N}$, defined as the number N of defaults observed in a given pool (up to time T) relative to the size of the pool,

$$\hat{N} := \frac{N}{I} \quad \text{where} \quad N := \sum_{i=1}^{I} B_i \, . \tag{17}$$

Similarly, assuming $N > 0$, one observes $\hat{L}$, the sample average of the realised LGDs on the pool:

$$\hat{L} := \frac{1}{N} \sum_{i=1}^{I} L_i \quad \text{assuming} \quad N > 0 \, . \tag{18}$$

Observe that the denominator in (18) is the number of defaulted obligors (N), not the total number of observed obligors (I).

In order to achieve the aforementioned goals, we switch from an obligor dependent definition of ρ_i^{LGD} to assigning the same factor loading ρ^{LGD} to all loans. This single ρ^{LGD} parameter has thus control over the PD-LGD strength, at the level of the entire pool, quantified via its impact on the value of $\mathrm{Corr}(\hat{N}, \hat{L})$. Considering a single parameter provides a good trade-off between better risk-sensitivity, simplicity, and tractability. It is therefore a significant improvement to the current framework while complying with latest call of the Committee to foster the enhancement of the regulatory framework along these lines (BCBS, 2013). Additionally, for G-SIBs the likelihood of observing no default over a one-year credit cycle on their entire credit portfolio is negligible, thus we will focus on calibrating ρ^{LGD} on

$$\mathrm{Corr}(\hat{N}, \hat{L} | N > 0) = \frac{\mathrm{Cov}(\hat{N}, \hat{L} | N > 0)}{\sqrt{\mathbb{V}[\hat{N} | N > 0]} \sqrt{\mathbb{V}[\hat{L} | N > 0]}} \, . \tag{19}$$

[5] Note that the distribution of the LGDs are not impacted by the $\hat{\rho}_i$ and $\tilde{\rho}_i$ parameters. In addition, under the large pool approximation, setting all ρ_i^{LGD}'s to 0 yields the same unexpected losses (hence, capital requirements) to those that would be obtained by using deterministic LGDs whose values are set to the expectation of the F_i's hence, to the same expression as the Basel formula when setting η_i to λ_i.

The proposed calibration procedure thus consists in finding the mapping between the model parameter ρ^{LGD} and the empirical correlation (19). For analytical tractability purposes, we consider the case of a *fully homogeneous credit portfolio*, i.e., $p_i = p, \rho_i = \rho, w_i = w = 1/I, F_i = F, \rho_i^{LGD} = \rho^{LGD}, \forall i \in \{1, ..., I\}$.

Theorem 1. *For a fully homogeneous credit portfolio, the conditional covariance between the default rate and the average loss is*

$$\mathrm{Cov}(\hat{N}, \hat{L}|N > 0) = \frac{\mathbb{E}[\lambda(Y)p(Y)] - \mathbb{E}[\lambda(Y)|N > 0] \times p}{\mathbb{P}(N > 0)} ,$$

where the conditional density of Y given $N > 0$ is

$$f_{Y|N>0}(y) = \frac{(1 - q(y)^I)\phi(y)}{\mathbb{P}(N > 0)}$$

with $q(y) := 1 - p(y)$ and

$$\mathbb{P}(N > 0) = 1 - \int_{-\infty}^{+\infty} q(x)^I \phi(x) dx .$$

The conditional variances of the default rate and the average loss are given by

$$\mathbb{V}[\hat{N}|N > 0] = \frac{1}{I\mathbb{P}(N > 0)} \times \left(p - \frac{Ip^2}{\mathbb{P}(N > 0)} + (I - 1)\Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \rho) \right)$$
$$\mathbb{V}[\hat{L}|N > 0] = \frac{1}{\mathbb{P}(N > 0)} \times \mathbb{E}\left[\nu(Y)\Psi(Y)\right] + \mathbb{V}\left[\lambda(Y)|N > 0\right]$$

where $\nu(Y) = \nu_i(Y)$ is the conditional variance of the LDG, and

$$\Psi(y) := \int_0^1 t^{-1} \left((q(y) + p(y)t)^I - q(y)^I \right) dt .$$

Proof. See Appendix C. □

This general formula applies to both approaches, but the $\lambda(y)$ and $\nu(y)$ functions depend on the chosen model. In addition, it is important to note that this formula has not been obtained under the LP approximation, and is applicable to smaller credit portfolios. The homogeneity restriction imposed in Theorem 1 is needed for the sake of getting a semi-analytical expression for the empirical correlation[6] but does not entail the applicability of our approaches. In particular, inhomogeneous pools can be considered once a value of ρ^{LGD} is specified. When confronting heterogeneous portfolios, several routes can be taken to calibrate ρ^{LGD} from a given empirical correlation level. For instance, compute the mapping function using simulations, or replace the heterogeneous pool by a comparable homogeneous one combined with Theorem 1.

[6] We can easily extend this analysis to a heterogeneous pool setting. The only required homogeneity is at the level of ρ^{LGD}.

4.1. Validating the empirical correlation formula

Technically speaking, estimating credit porftolio VaR in factor models such as those considered in this paper is easy to do via Monte Carlo simulations. In practice, however, it is rather problematic in a LP environment because of memory and computation time limitations. Indeed, estimating extreme quantiles of rare events requires a huge number of simulations to reach convergence. This becomes even more problematic when iterations are needed, for example, to calibrate the model in absence of analytical formula. Although dedicated techniques such as importance sampling could help mitigating the computational and memory burden, this stresses the importance of having a tractable model for regulatory purposes. Nevertheless, Monte Carlo simulations can be helpful to validate analytical expressions, or to assess the quality of approximations such as LP. This can be achieved easily by letting the computations run on a grid computing with high memory capacity. Moreover, boxplots can be used to assess the numerical error inherent to Monte Carlo simulations featuring a limited number of drawings. This is the procedure we follow to illustrate the validity of the empirical correlation formula in Theorem 1.

We simulate a realistic Basel III compliant[7] fully homogeneous credit portfolio, made of $I = 1,500$ loans and characterised by a marginal default probability of $p = 0.925\%$. Following the IRB approach, this parameter and the type of loans uniquely determines the default factor loading ρ. Assuming that the pool is made of exposures classified as corporate, sovereign and bank exposure, the standard Basel expression to consider is

$$\rho = \rho(p) := 0.12 \times \frac{(1 - e^{-50\times p})}{(1 - e^{-50})} + 0.24 \times \left(1 - \frac{(1 - e^{-50\times p})}{(1 - e^{-50})}\right) , \tag{20}$$

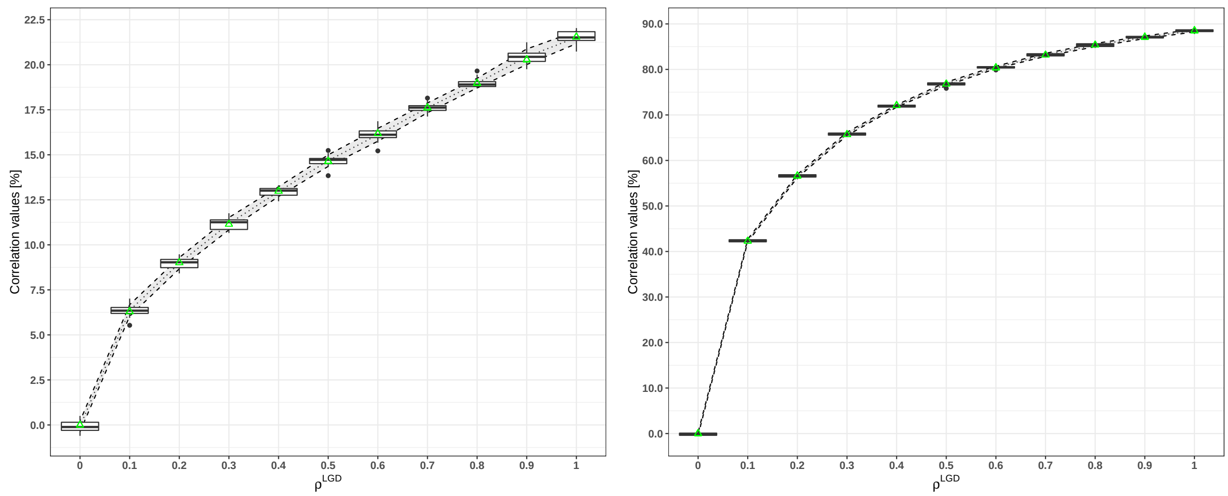
leading to $\rho \approx 19.5\%$.

Following the rating agencies standard practices, the loss given default is modelled by a Beta distribution.[8] We chose $F \sim Beta(2, 3)$, leading the LGDs having an expectation of $\lambda = 40\%$ and a standard deviation of $\sigma = 20\%$. Apart from being the same Beta parameter values as in the seminal work of Pykhtin (2004), these values reflect a good balance between: (i) the fact that under the F-IRB approach, the LGD is fixed at 45%, hence the λ of our LGD random variable should be close to that value, (ii) the latest empirical evidence in terms of recovery rates distributions provided by Gambetti et al. (2019) and García-Céspedes and Moreno (2020). Both authors analyse a wide sample of cross-industry bond RR extracted from the Moody's Analytics Default and Recovery Database (Moody's DRD). The former database spans for the period 1990-2013 December 2013, the latter 1982-2014.

In Figure 1a and Figure 1b, we compute the Monte Carlo estimates of the empirical correlation for various values of ρ^{LGD}. The corresponding estimated mapping functions between the copula parameter ρ^{LGD} and the empirical correlation in (19) coincide with the analytical formula presented in Theorem 1 (displayed as the empty triangles). Interestingly,

[7]The value of p proposed here is considered as standard by banking professionals and meets the prescribed floor of 5 basis points (BCBS, 2017, p. 65, paragraph 68).

[8]Observe that the approaches considered here allow for other choices of distributions. We consider a Beta distribution as it is a standard choice in recovery rates modelling.



(a) Approach 1: $\text{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\hat{\rho}$

(b) Approach 2: $\text{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\tilde{\rho}$

Figure 1: Empirical correlation as a function of the PD-LGD dependence parameter in Approach 1 (a) and Approach 2 (b). The empty triangles (△) depict the values obtained from the analytical formula in Theorem 1. The boxplots are obtained via Monte Carlo simulations. Each boxplot is built from 20 simulated correlation values, each of which being estimated from $m = 50,000$ simulations. The grey area delimited by the dashed lines, represents ± one standard deviation from the estimated correlation mean value. The underlying credit portfolio is fully homogeneous made of $I = 1,500$ loans with $p = 0.925\%$, $\rho = \rho(0.00925) = 19.5\%$ and marginal LGD distributions $Beta(2, 3)$.

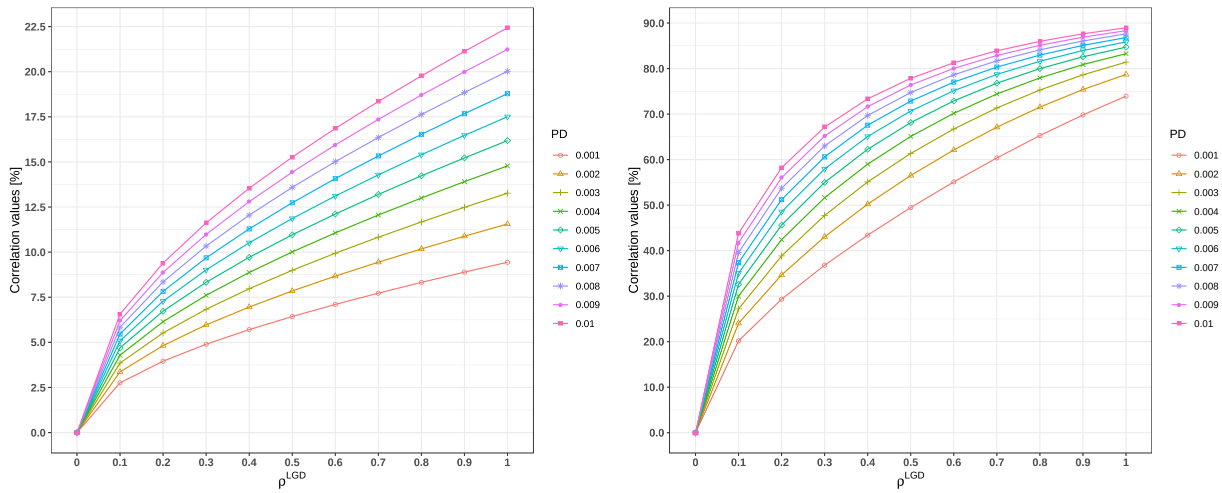
notice the scale difference of the vertical axis. We observe a striking difference between the span of the empirical correlation values that can be captured by both model setups. The extension of Van Damme (2011) presented in Section 3.1 does not seem to be flexible enough to be applied on realistic credit portfolios featuring large values of the $\text{Corr}(\hat{N}, \hat{L}|N > 0)$.

4.2. Empirical correlation sensitivity analysis

Having illustrated the good match between numerical estimates and analytical values of empirical correlations, we conduct a sensitivity analysis using our analytical formula. The purpose is to understand the impact of the model parameters on the mapping function in both approaches and, in particular, to determine the extent to which the difference in the captured correlation spans persists across a set of values for the other risk parameters in the model. Figures 2a and Figure 2b depict the sensitivity of the empirical correlation values to a set of realistic values for the probability of default. The default dependence parameter ρ is adjusted according to (20), while all other inputs are kept constant.

A similar investigation is performed in Figure 3 with respect to the parameters of the marginal LGD distribution. Once again we span across a set of Basel III compliant values for the first two moments.[9] Both Figure 2 and Figure 3 outline that the difference in the captured correlation values is persistent across our sensitivity analysis.

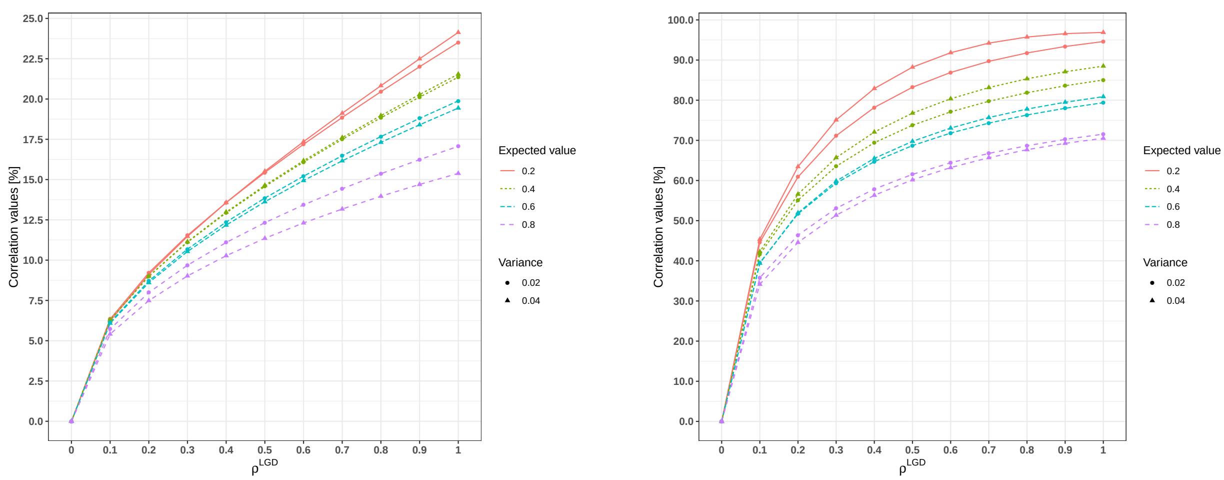
[9]We consider the existence of the 25% floor on the LGD, for unsecured corporate exposures (BCBS, 2017, p. 68, paragraph 85).



(a) Approach 1: $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\hat{\rho}$

(b) Approach 2: $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\tilde{\rho}$

Figure 2: Sensitivity of the empirical correlation in Theorem 1 to a set of realistic Basel III compliant values of p in Approach 1 (a) and Approach 2 (b). The LGD is distributed according to a $Beta(2, 3)$. The remaining pool parameters are set to $I = 1,500$ and $\rho = \rho(p)$.



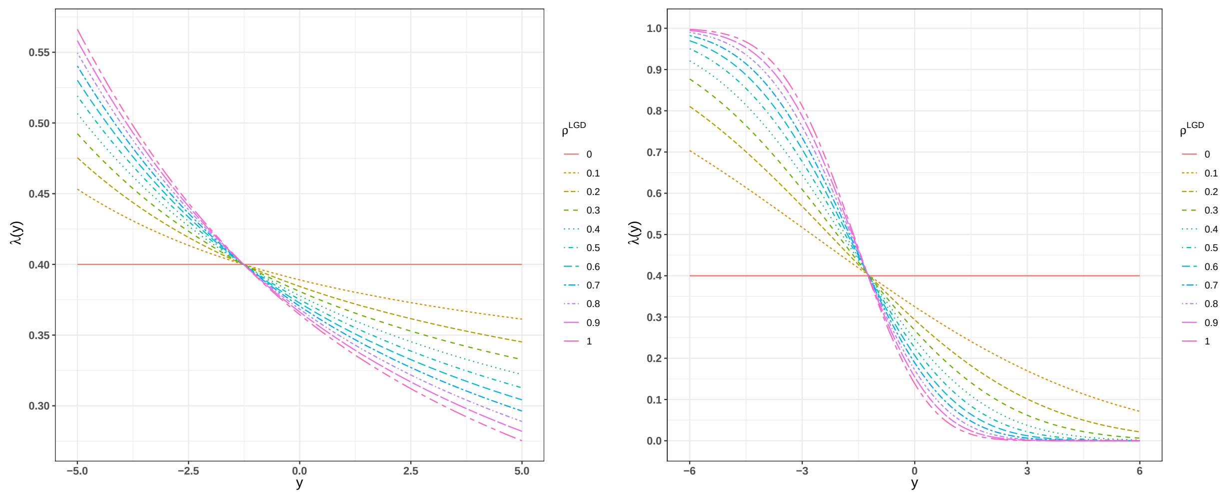
(a) Approach 1: $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\hat{\rho}$

(b) Approach 2: $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ *vs.* $\tilde{\rho}$

Figure 3: Sensitivity of the empirical correlation in Theorem 1 to a set of realistic LGD distributions (whose expectations and variances are given in the legend) in Approach 1 (a) and Approach 2 (b). The remaining pool parameters are set to $I = 1,500$, $p = 0.925\%$ and $\rho = \rho(0.00925) = 19.5\%$.

Compared to the original Basel framework, it is clear from Lemma 1 that the only additional functions that show up in the VaR formula are the conditional expectations $\lambda(y)$. Hence, intuitively, the difference in the correlation bounds across both model setups must stem from the shape of the latter; a flat function being associated to the deterministic LGD case.

Figure 4 shows the conditional LGDs in either approaches. One can see that the image of the function associated with Approach 2 in (16) is wider than that associated with Approach 1 in (13), and this persists across all possible values of ρ^{LGD}. This difference of scales explains why Approach 2 is able to span larger empirical correlation values.



(a) Approach 1: $\lambda(y)$ for various values of $\hat{\rho}$

(b) Approach 2: $\lambda(y)$ for various values of $\tilde{\rho}$

Figure 4: Function $\lambda(y)$ *vs.* y in Approach 1 (a) and Approach 2 (b) across the possible range of values for ρ^{LGD}. The LGDs are distributed according to a $Beta(2,3)$. The remaining pool parameters are set to $p = 0.925\%$ and $\rho = \rho(0.00925)$.

5. Empirical analysis

We now move to the empirical study of the impact of the considered model on VaR figures. More precisely, we would like to compare the VaR computed (i) theoretically using Lemma 1 under the LP and using as λ_i functions the expressions in Lemma 3 and 5 setting $k = 1$ with (ii) those obtained in a Monte Carlo setup. This allows us to validate our approach and to illustrate the quality of the LP approximation as a function of the size of the pool. On the other hand, we want to assess the impact in terms of capital requirements when adopting our new mapping functions for the conditional LGDs. In particular, we would like to compare the UL found in our models when calibrating ρ^{LGD} to actual data using Theorem 1 to that obtained in the Basel framework (10) featuring deterministic downturn LGDs η_i.

5.1. Validating the analytical VaR

The simplest approach consists in estimating our risk figures using Monte Carlo simulation, by sampling the risk factors $Y, Z_1, \ldots, Z_I, \hat{Z}_1, \ldots, \hat{Z}_i, \tilde{Z}_1, \ldots, \tilde{Z}_i$. Under the conditional independence framework detailed in sections 3.1 and 3.2, the $\mathrm{VaR}_\alpha(L)$ is estimated by the corresponding empirical quantile out of m simulations. Denoting by $l_{(1)} \leq l_{(2)} \leq \ldots \leq l_{(m)}$ the order statistics of the m simulated relative portfolio losses $l_1, ..., l_m$, the value-at-risk is approximated by:

$$\mathrm{VaR}_\alpha^{(m)}(L) \coloneqq l_{(i_\alpha)} \qquad \text{for } \; i_\alpha \coloneqq min\{i \in \{1, ..., m\} : i/m \geq \alpha\} \,. \tag{21}$$

This is known to be an asymptotically unbiased and consistent estimator of $\mathrm{VaR}_\alpha(L)$ as discussed in Pitera and Schmidt (2018). More advanced methods, featuring interpolation between empirical quantiles, can be considered as well (see Hyndman and Fan (1996)). As for the EL, we simply use the sample estimator,

$$\mathbb{E}^{(m)}[L] := \frac{1}{m}\sum_{i=1}^{m} l_i \ .$$

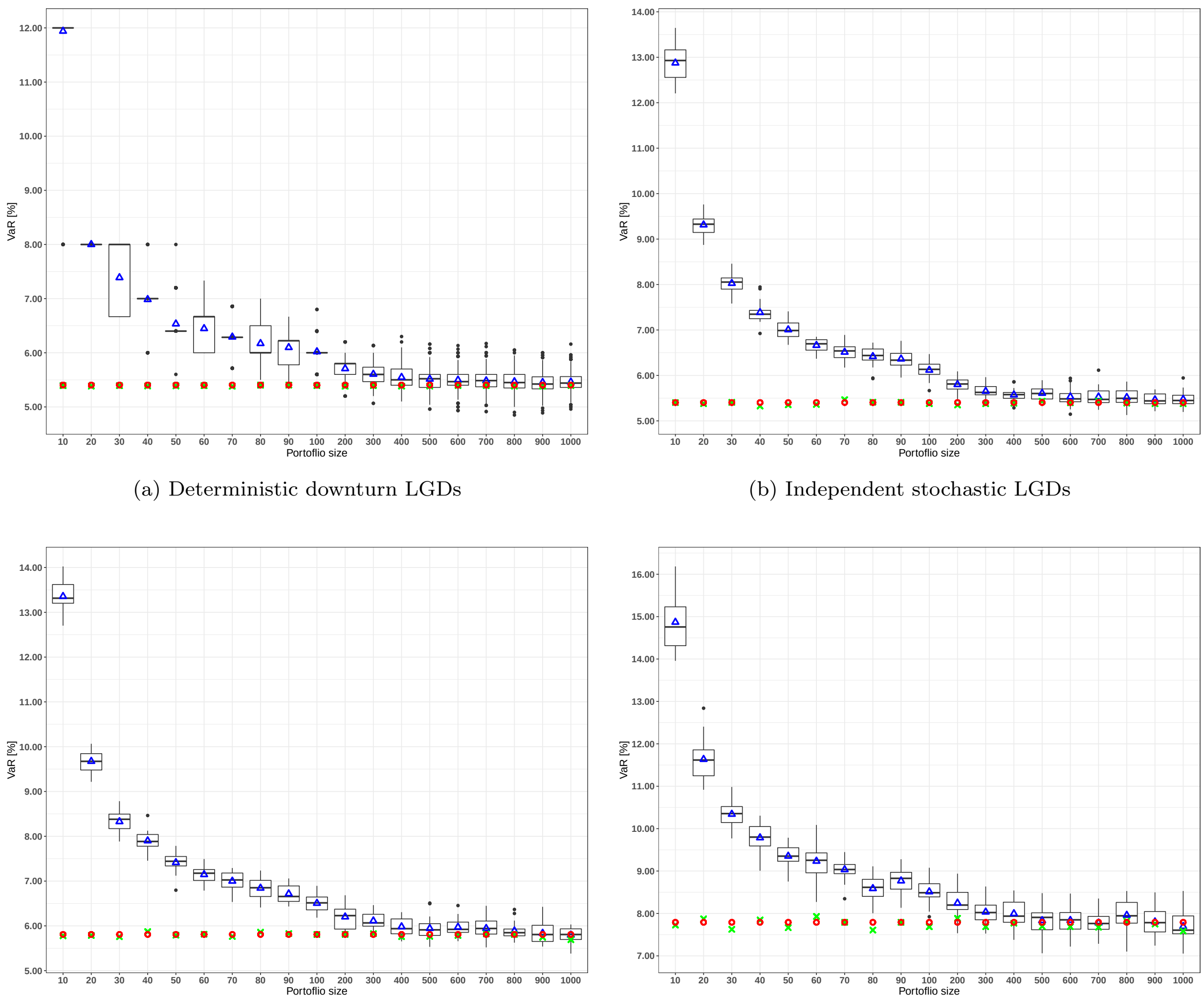
Note that in these expressions, l_i is the weighted sum of the simulated LGDs associated with the obligors for which the default indicators is 1 on the i-th simulation run.

We consider a fully homogeneous portfolio with the same parameters as before, i.e., $I = 1,500$, $p = 0.925\%$ and $\rho = \rho(0.000925) = 19.5\%$. We set the deterministic downturn LGD to $\eta = 40\%$ in the standard IRB approach. For the stochastic LGD approaches, we model the marginal LGD distributions using a $Beta(2,3)$, starting with a benchmark stochastic LGD setup where the Λ_i are random variables independent from all the others. This is equivalent to both (i) imposing $\rho^{LGD} = 0$, in either one of the models detailed in Section 3.1 or Section 3.2 or (ii) using deterministic downturn LGDs equal to the expectation of F, precisely equal to $\eta = 40\%$ given the choice for the parameters of our $Beta$ distribution. Indeed, it is easy to verify that the analytical values depicted in Figures 5a and 5b (empty circles) are identical on both graphs, equal to 5.4%.

In order to highlight the impact of a PD-LGD correlation on the VaRs, we start by imposing arbitrarily $\rho^{LGD} = 0.2$, leading to $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ close to 9% and 56.5% in Approach 1 and 2, respectively. The results are shown on Figures 5c and 5d. We observe first that the Monte-Carlo estimates of the VaR converge to the corresponding LP approximation, as the number of loans I increases. This suggests that the LP approximation under the standard Basel approach (Figure 5a), the independent stochastic LGD setup (Figure 5a and Figure 5b), and both Approach 1 and Approach 2 (Figure 5c and Figure 5d) are correct. Moreover, one can check that although the confidence intervals are larger in Approach 1 and 2 compared to the independent case, the convergence speed of the VaR to the LP result as the size of the pool increases does not seem to depend on the considered model.[10] In particular, the addition of a new PD-LGD dependence parameter does not entail the quality of the estimation provided by the corresponding LP formula compared to the standard Basel framework.

Figure 5 is helpful in that it shows that the LP formula derived in either models are correct, and the minimum pool size as from which the LP approximation becomes valid (about 600) is not strongly impacted. However, the parameters $\eta = 40\%$ and $\hat{\rho} = \tilde{\rho} = 20\%$ have been chosen for illustration purposes, but in a rather arbitrary way. Therefore, we cannot draw any conclusion about the magnitudes of the differences in the computed VaR figures in real applications. This is, instead, the objective of the next section.

[10] The variability in the width of the boxes in Figure 5a results from the fact that when the LGDs are all set to a same constant, the loss is actually just a multiple of the number of defaults. Hence, the cdf of the loss is discrete and displays jumps at specific values. In this case, the variance of the quantile estimator depends in a complex way of the pool size.



(a) Deterministic downturn LGDs

(b) Independent stochastic LGDs

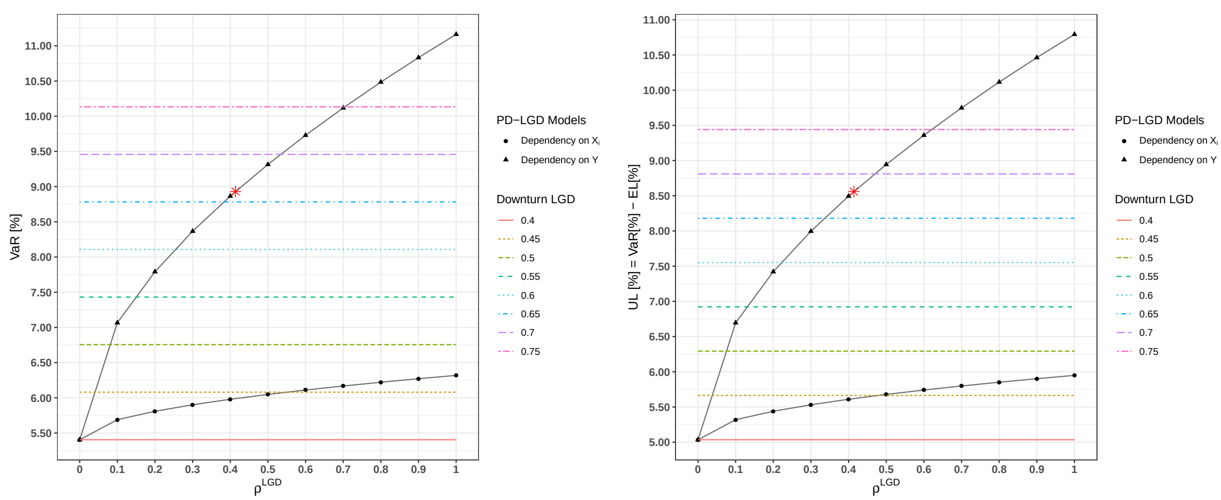
(c) Approach 1 with $\hat{\rho} = 0.2$

(d) Approach 2 with $\tilde{\rho} = 0.2$

Figure 5: Value-at-risk as a function of the portfolio size, comparing different LGD frameworks, i.e., deterministic downturn LGD $\eta = 40\%$ (a) , stochastic LGD framework where $\rho^{LGD} = 0$ (b), Approach 1 with $\rho^{LGD} = \hat{\rho} = 0.2$ (c) and Approach 2 with $\rho^{LGD} = \tilde{\rho} = 0.2$ (d). The markers (○) show the analytical expression obtained under the LP approximation specific to each model, the (x) show the sample mean of the quantiles of $\mathbb{E}[L|Y]$ obtained by sampling the common factor Y, while (△) show the sample mean of the quantiles of L obtained by sampling the common factor Y. The boxplots are obtained via Monte Carlo simulations. A boxplot is built from 20 (1000 for the deterministic downturn LGD setting) simulated VaR values with $\alpha = 99.9\%$, each of which being estimated from $m = 50,000$ simulated relative portfolio losses. The marginal distributions of the stochastic LGDs are $Beta(2, 3)$. The remaining portfolio parameters are $p = 0.925\%$ and $\rho = \rho(0.000925) = 19.5\%$.

5.2. Downturn LGDs vs. *LGD mapping functions*

In order to calibrate the $\hat{\rho}$ and $\tilde{\rho}$ parameters featured in Approach 1 and Approach 2, we exploit the dataset provided in García-Céspedes and Moreno (2020, p. 1870, Table 1). Precisely, we compute the correlation between the DR and the average observed loss given default (simply computed as 1 minus the average RR of all bonds provided in third column of Table 1) for the entire Moody's world bonds portfolio. For the period going from 1892-2014, the observed correlation is 72.81%. Next, for a given portfolio credit risk profile, one can then search for the parameter ρ^{LGD} to insert in the formula in Theorem (1) to reach this level of empirical correlation. Using the same data as in Section 5.1, this amounts to look for the value of ρ^{LGD} leading to 72.81% when considering the curve associated with $(\lambda, \sigma) = (0.4, 0.2)$ in Figure 3. Interestingly, it is clear from the vertical axis of Figure 3a that this value is out of the range of empirical correlations that Approach 1 can achieve; there is no corresponding value for $\hat{\rho}$. Therefore, Figure 3a unveils the inability of Van Damme (2011) standard model to generate a value-at-risk underlying a PD-LGD link strength matching empirical evidence (visible as the $\rho^{LGD} = 1$ point of the curve: Dependency on X_i). By contrast, Figure 3b shows that this is not the case for Approach 2, as an empirical correlation level of 72.81% can be obtained by choosing $\tilde{\rho} \approx 40\%$ (a more precise computation yields 41.40%). For this reason, Approach 2 proves to be more appropriate than Approach 1.

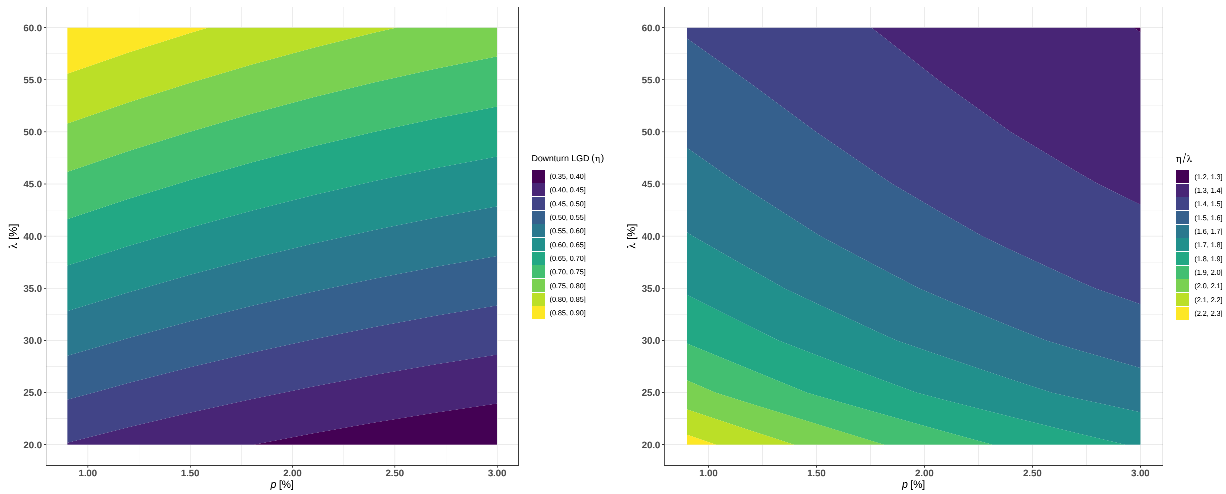


(a) Value-at-risk (VaR) *vs.* ρ^{LGD}

(b) Unexpected loss (UL) *vs.* ρ^{LGD}

Figure 6: Sensitivity of the Basel capital requirements (expressed both in terms of VaR and UL, computed at a 99.9% confidence level) as a function of ρ^{LGD}, for a stochastic LGD $\sim Beta(2,3)$. The highlighted asterisk ($*$) on both graphs depicts the value of ρ^{LGD} one should insert in the PD-LGD dependency model of Approach 2 to match the empirical correlation level $\text{Corr}(\hat{N}, \hat{L}|N > 0)$ recovered from Moody's DRD for the world bond portoflio, for the period 1982–2014. As before, the underlying credit portfolio is fully homogeneous made of $I = 1,500$ loans with $p = 0.925\%$, $\rho = \rho(0.00925) = 19.5\%$ and marginal LGD distributions $Beta(2,3)$.

In order to evaluate the impact of the PD-LGD dependency, we proceed with the analysis of the VaR and UL values generated by the IRB Basel approach for various values of the downturn LGD η and compare those to that obtained from Approach 1 and Approach 2 for various values of ρ^{LGD}; this is done in Figure 6. The asterisk ($*$) points the VaR and UL values associated with Approach 2 calibrated as explained above, that is, when $\tilde{\rho} = 41.40\%$. It can be checked that, in order to replicate a similar VaR value in the IRB framework, banks would need to insert a deterministic downturn LGD close to 66.10% for our benchmark credit portfolio. Moreover, as explained in Section 2.3, the standard Basel model used to compute the UL features the downturn LGD η both in the quantile and in the EL parts. Hence, both terms increase with η, explaining that the VaR increases faster than the UL when increasing η. As a consequence, Banks would need to insert a higher donwturn LGD to match the UL than those found from the VaR comparison (here, 68.01% instead of 66.10%). This can be observed by comparing Figure 6a with Figure 6b.



(a) Downturn LGD (η) *vs.* (λ, p)

(b) Conversion factor (η/λ) *vs.* (λ, p)

Figure 7: Contour plots depicting the equivalent downturn LGD (a) and the conversion factor translating expected LGDs into stressed LGD estimates (b) to be used in UL computations under the IRB approach to reflect a PD-LGD link strength corresponding to a $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0) = 72.81\%$ as a function of PD (p) and expected LGD (λ). The variance of the LGD is kept constant to 4%.

In Figure 7, we extend the previous exercise and conduct a sensitivity analysis to provide insights on how to choose stressed LGDs feeding UL computations under the IRB approach in order to obtain capital requirement figures that comply with empirical evidences associated with the PD-LGD relationship. The latter is performed for two key pool parameters, i.e., p and λ. Consistent with what has been done previously, we play with the LGD distribution parameters by varying the first moment while keeping the variance constant to 4%. Figure 7a could serve to define meaningful floors on downturn LGDs, to be inserted under the current IRB approach, as a function of other pool parameters. Additionally, we define a conversion factor η/λ, helping to translate expected LGDs into stressed LGDs estimates, given (i) a credit risk portfolio profile, (ii) a desired PD-LGD link strength, rendered by an observed empirical correlation $\mathrm{Corr}(\hat{N}, \hat{L}|N > 0)$ at the level of the pool; the results are shown in Figure 7b. Going back to the previous example, we conclude from the above exercise that the capital requirements associated with Approach 2 computed assuming an expected LGD of $\lambda = 40\%$ and empirical correlation of 72.81% can be reproduced in the standard Basel IRB formula assuming fixed LGDs of $\eta = 68\%$, suggesting a conversion factor between stressed and expected LGDs of $\eta/\lambda = 1.7$.

6. Conclusion

The 2008 financial crisis revealed the intricacies and vulnerability of the worldwide financial network, stressing the importance of taking appropriate measures in order to enhance its resilience. Strengthening global financial stability is the mission of think tanks, central banks together with national and supra-national regulatory and supervisory instances such as BCBS, who promote better risk management of financial institutions. A cornerstone to achieve this goal is the development of models from which regulatory capital requirement guidelines can be derived. According to the Basel steering committee, three important features that such frameworks must benefit from are *risk-sensitivity, simplicity* and *comparability.*

Our research contributes to this effort by extending the ASRF model underlying the IRB approach governing regulatory capital requirements so as to account for the risk resulting from the PD-LGD dependency. The dependency between default rates and realized LGDs is a well-documented fact supported by strong empirical evidences. Discarding this reality when computing G-SIBs credit risk metrics can lead to severe underestimation of capital requirements and could endanger the global financial architecture. In the Basel IRB approach, this dependency is tackled indirectly: the underlying ASRF model postulates deterministic LGDs but, to offset this unrealistic assumption, banks are required to plug stressed LGD estimates (instead of expected values that should be used under a PD-LGD independence hypothesis). Nevertheless, this procedure is somewhat opaque in the sense that this adjustment is performed via a stressed parameter plugged in a model that does not account for the stochastic link between PDs and LGDs.

To address this drawback, we consider two extensions of the ASRF model that explicitly model the PD-LGD relationship in a consistent manner. Similarly to what is currently done for the treatment of the default events dependency where the default probability is computed conditional upon a systematic risk factor, we derive the associated mapping functions translating unconditional LGDs into conditional LGDs. We contribute to the literature on capital requirements in the following ways. First, in either of the two approaches, the strength of the PD-LGD link is controlled via a single additional parameter compared to the current Basel machinery, while (i) maintaining full flexibility over the marginal distributions of the LGDs and (ii) preserving the appealing regulatory feature of delivering a portfolio-invariant semi-analytical formula for the computation of value-at-risk. By doing so, *risk-sensitivity* is enhanced while *simplicity* is preserved. Second, we connect this new risk parameter to the observed correlation between the default rate and the average observed LGD over a period of time. This provides an interesting route for calibrating our models directly to the data which, precisely, suggest empirically the existence of such a relationship. Generating the PD-LGD dependency via an explicit factor model whose parameters can be connected to observable metrics further enhances both transparency and explainability, and contributes to improve models' *comparability.* Finally, we provide an empirical study that confirms the validity of our formula and illustrate the impact of the PD-LGD correlation parameter on the loss quantiles and capital requirements. As expected, the impact is potentially very substantial. In addition, this study unveils the fact that although similar by design, the second approach is superior to the first in the sense that it is closer to the original Basel spirit, is simpler to tackle, and is able to display more realistic empirical correlation values.

We see two main policy applications of our work. A first route is to enrich the existing regulatory framework with an ASRF model controlling for the stochastic link between PD and LGD, similar to what we propose here. Note that although our study shows a strong impact of the PD-LGD correlation parameter on the unexpected loss, our framework does not necessarily trigger extra capital requirements for G-SIBs compared to the current IRB regulatory framework because the modelling of the PD-LGD relationship would replace the downturn LGD procedure. A second route is to exploit our framework in combination with the current downturn LGD methodology. Differently from the IRB formulas, paragraphs 36.85 to 36.90 of (BCBS, 2019a) merely specify very general guidelines about how downturn LGDs should be computed. In this respect, our approach could help clarifying what downturn LGDs actually are, how they should be determined, or to setup meaningful floors.

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Appendix A. Proof of results

Appendix A.1. Proof of Lemma 1

Proposition 4 in Gordy shows that if $\mathbb{E}[L|Y] = \sum_{i=1}^{I} f_i(Y)$ and that there exists $I_0 < \infty$ such that the $f_i(y)$s are non-decreasing functions locally around $\mathbb{Q}_\alpha[Y]$ for all $I > I_0$, then

$$\mathbb{Q}_\alpha[\mathbb{E}[L|Y]] = \sum_{i=1}^{I} f_i(\mathbb{Q}_\alpha[Y]) \quad \text{for all} \quad I > I_0 \; .$$

Loosely speaking, this expression essentially says that because of the local monotonicity, the worst-case scenario of the conditional expected loss is associated with the worst-case scenario of the systematic factor. A similar development shows that if, instead, the f_i's are non-increasing, the worst-case scenario of the conditional expected loss is associated with the best-case scenario of the systematic factor, i.e., $\mathbb{Q}_\alpha[\mathbb{E}[L|Y]] = \mathbb{E}[L|\, \mathbb{Q}_{1-\alpha}[Y]]$. In our case, $f_i(y) = \lambda_i(y)p_i(y)$ where p_is are decreasing, so that the f_is are decreasing if so are the λ_is. The proof is concluded by noting that Y is a standard normal variable, $\mathbb{Q}_{1-\alpha}[Y] = z_{1-\alpha} = -z_\alpha$.

Appendix A.2. Proof that the CDF of Λ_i in (11) is F

Because Λ_i is only defined conditional upon default, it is enough to check that $U_i := 1 - \frac{\Phi(X_i)}{p_i}$ is uniformly distributed conditional upon $B_i = 1$. To see this, observe first that

$$B_i = 1 \quad \Leftrightarrow \quad X_i \le c_i \quad \Leftrightarrow \quad \Phi(X_i) \le p_i$$

which, in turns, implies that

$$\mathbb{P}(U_i \le 0 | B_i = 1) = 0 \quad \text{and} \quad \mathbb{P}(U_i \le 1 | B_i = 1) = 1 \; .$$

Now, for any $x \in [0, 1]$,

$$\begin{aligned}
\mathbb{P}(U_i \le x | B_i = 1) &= \mathbb{P}(U_i \le x | X_i \le c_i) \\
&= \frac{\mathbb{P}\left(\frac{\Phi(X_i)}{p_i} \ge 1 - x \cap X_i \le c_i\right)}{\Phi(c_i)} \\
&= \frac{\mathbb{P}\left(\Phi^{-1}(p_i \times (1-x)) \le X_i \le c_i\right)}{p_i} \\
&= \frac{p_i - p_i \times (1-x)}{p_i} = x \; .
\end{aligned}$$

Appendix A.3. Proof of Lemma 2

Notice that the distribution of Λ_i remains equal to F_i for all $\hat{\rho}_i \in [0,1]$. This is clear for $\hat{\rho} = 1$, and it is obvious for $\hat{\rho} = 0$ since then

$$\mathbb{P}(L_i \leq l | B_i = 1) = \mathbb{P}(F_i^{-1}(\Phi(\hat{Z}_i)) \leq l) = \mathbb{P}(\hat{Z}_i \leq \Phi^{-1}(F_i(l))) = F_i(l). \qquad \text{(A.1)}$$

For the other values of $\hat{\rho}_i$ we have

$$\begin{aligned}
\mathbb{P}(L_i \leq l | B_i = 1) &= \mathbb{P}(\Lambda_i \leq l | B_i = 1) \\
&= \mathbb{P}\left(\Lambda_i \leq l \cap \hat{X}_i \leq c_i\right)/p_i \\
&= \mathbb{P}\left(F_i^{-1}\left(\Phi\left(\sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(\hat{X}_i)}{p_i}\right) + \sqrt{1-\hat{\rho}_i}\hat{Z}_i\right)\right) \leq l \cap \hat{X}_i \leq c_i\right)/p_i \\
&= \mathbb{P}\left(\sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(\hat{X}_i)}{p_i}\right) + \sqrt{1-\hat{\rho}_i}\hat{Z}_i \leq \Phi^{-1}(F_i(l)) \cap \hat{X}_i \leq c_i\right)/p_i \\
&= \mathbb{P}\left(\hat{Z}_i \leq \frac{\Phi^{-1}(F_i(l)) - \sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(\hat{X}_i)}{p_i}\right)}{\sqrt{1-\hat{\rho}_i}}) \cap \hat{X}_i \leq c_i\right)/p_i \\
&= \frac{1}{p_i}\int_{-\infty}^{c_i} \mathbb{P}\left(\hat{Z}_i \leq \frac{\Phi^{-1}(F_i(l)) - \sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(x)}{p_i}\right)}{\sqrt{1-\hat{\rho}_i}}\right)\phi(x)dx \\
&= \frac{1}{p_i}\int_{-\infty}^{\Phi^{-1}(p_i)} \Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{\Phi(x)}{p_i}\right)}{\sqrt{1-\hat{\rho}_i}}\right)\phi(x)dx\ .
\end{aligned}$$

Now, let us apply two changes of variables: $u = \Phi(x)$ ($du = \phi(x)dx$) and $v = \Phi^{-1}(1 - u/p_i)$ ($dv = \frac{-1}{p_i\phi(v)}du$), leading to write

$$\begin{aligned}
\mathbb{P}(L_i \leq l | B_i = 1) &= \frac{1}{p_i}\int_0^{p_i} \Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\hat{\rho}_i}\Phi^{-1}\left(1 - \frac{u}{p_i}\right)}{\sqrt{1-\hat{\rho}_i}}\right)du \\
&= -\frac{1}{p_i}\int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(F_i(l)) - \sqrt{\hat{\rho}_i}v}{\sqrt{1-\hat{\rho}_i}}\right)(-p_i\phi(v))dv
\end{aligned}$$

Defining

$$f(Y,p,\rho) := \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}Y}{\sqrt{1-\rho}}\right) \quad \text{with} \quad \rho \in [0,1)$$

and cancelling the -p_i's, the above integral takes the form $\mathbb{E}[f(Y, F_i(l), \hat{\rho}_i)]$. The proof is concluded by noting that $\mathbb{E}[p_i(Y)] = \mathbb{E}[f(Y, p_i, \rho_i)] = p_i$.

Appendix A.4. Proof of Lemma 3

The expectation can be computed by fixing Y in (12) (which is hidden behind X_i) and imposing further $\{B_i = 1\}$. For Y given, the default event becomes

$$\begin{aligned}\{B_i = 1\} &= \{X_i \leq \Phi^{-1}(p_i)\}\\ &= \left\{Z_i \leq (\Phi^{-1}(p_i) - \sqrt{\rho_i}Y)/\sqrt{1-\rho_i}\right\}\\ &= \{Z_i \leq c_i(Y)\}\,.\end{aligned}$$

Hence,

$$\mathbb{E}[\Lambda_i^k | B_i = 1, Y] = \mathbb{E}\left[\Lambda_i^k | Z_i \leq c_i(Y), Y\right].$$

The proof is concluded by computing the expectation of $\Lambda_i = F_i^{-1}(\Phi(\hat{X}_i))$ over the set $\{Z_i \leq c_i(Y)\}$ assuming Y is fixed, where $\hat{X}_i$ is expressed as a function of the i.i.d. variables $Y, Z_i, \hat{Z}_i$ given in (13) and rescaling by the probability of the conditionning event given Y, $\mathbb{P}(Z_i \leq c_i(Y)) = p_i(Y)$.

Appendix A.5. Proof of Lemma 4

The cdf of $\tilde{X}_i$ conditional upon $\{B_i = 1\}$ evaluated at x is

$$\begin{aligned}F_{\tilde{X}_i|B_i=1}(x) &= \mathbb{P}(\tilde{X}_i \leq x | B_i = 1)\\ &= \mathbb{P}\left(\tilde{X}_i \leq x | X_i \leq c_i\right)\\ &= \mathbb{P}\left(\tilde{X}_i \leq x \cap X_i \leq c_i\right)/p_i,\\ &= \mathbb{P}\left(-\sqrt{\tilde{\rho}_i}Y + \sqrt{1-\tilde{\rho}_i}\tilde{Z}_i \leq x \cap \sqrt{\rho_i}Y + \sqrt{1-\rho_i}Z_i \leq c_i\right)/p_i\\ &= \mathbb{E}\left[\mathbb{P}\left(-\sqrt{\tilde{\rho}_i}Y + \sqrt{1-\tilde{\rho}_i}\tilde{Z}_i \leq x \cap \sqrt{\rho_i}Y + \sqrt{1-\rho_i}Z_i \leq c_i | Y\right)\right]/p_i\\ &= \mathbb{E}\left[\mathbb{P}\left(\tilde{Z}_i \leq \frac{x + \sqrt{\tilde{\rho}_i}Y}{\sqrt{1-\tilde{\rho}_i}}\middle| Y\right) \times \mathbb{P}\left(Z_i \leq \frac{c_i - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\middle| Y\right)\right]/p_i\\ &= \mathbb{E}\left[\Phi\left(\frac{x + \sqrt{\tilde{\rho}_i}Y}{\sqrt{1-\tilde{\rho}_i}}\right) \times \Phi\left(\frac{c_i - \sqrt{\rho_i}Y}{\sqrt{1-\rho_i}}\right)\right]/p_i\ ,\end{aligned}$$

where the numerator takes the form $\mathbb{E}\left[\Phi\left(a_1 + b_1 Y\right) \times \Phi\left(a_2 + b_2 Y\right)\right]$ with

$$a_1 := x/\sqrt{1-\tilde{\rho}_i}, \quad b_1 := \sqrt{\tilde{\rho}_i}/\sqrt{1-\tilde{\rho}_i},$$

$$a_2 := c_i/\sqrt{1-\rho_i}, \quad b_2 := -\sqrt{\rho_i}/\sqrt{1-\rho_i}.$$

The proof is concluded by noting that for $X \sim \mathcal{N}(0,1)$ and any real values a_1, a_2, b_1, b_2,

$$\mathbb{E}\left[\Phi\left(a_1 + b_1 X\right)\Phi\left(a_2 + b_2 X\right)\right] = \Phi_2\left(\frac{a_1}{\sqrt{1+b_1^2}}, \frac{a_2}{\sqrt{1+b_2^2}}, \frac{b_1 b_2}{\sqrt{1+b_1^2}\sqrt{1+b_2^2}}\right)$$

and replacing the a_i's and b_i's coefficients by the expressions given above.

Appendix A.6. Proof of Lemma 5

Recall from the definition of the LGD that $\Lambda_i = F_i^{-1} \circ H_i(-\sqrt{\tilde{\rho}_i}Y + \sqrt{1-\tilde{\rho}_i}\tilde{Z}_i)$ is defined on the set $\{B_i = 1\}$. Because, the condition $\{B_i = 1\}$ is a set that features the pair (Y, Z_i) and that $\tilde{Z}_i$ is independent from (Y, Z_i),

$$\mathbb{E}[\Lambda_i^k | B_i = 1, Y] = \mathbb{E}\left[\left(F_i^{-1} \circ H_i(-\sqrt{\tilde{\rho}_i}Y + \sqrt{1-\tilde{\rho}_i}\tilde{Z}_i) \right)^k \middle| Y \right] .$$

From the independence between $\tilde{Z}_i$ and (Y, Z_i), the conditional expectation is recovered by integrating the latter with respect to the standard normal density.

Appendix B. Proof that $\lambda_i(y)$ in Approach 1 is decreasing

Setting $k = 1$ in (13), we have $\lambda_i(Y) = I(Y)/p_i(Y)$ where

$$I(y) := \int_{-\infty}^{c_i(y)} f(y, \hat{z})\phi(\hat{z})d\hat{z} \tag{B.1}$$

and

$$f(y, \hat{z}) = \int_{-\infty}^{\infty} F_i^{-1} \left(\Phi \left(\sqrt{\hat{\rho}_i}\Phi^{-1} \left(1 - \frac{\Phi(\sqrt{\rho_i}y + \sqrt{1-\rho_i}z)}{p_i} \right) + \sqrt{1-\hat{\rho}_i}\hat{z} \right) \right) \phi(z)dz.$$

Hence,

$$\lambda_i'(y) = \frac{d}{dy}\lambda_i(y) = \frac{I'(y)p_i(y) - I(y)p_i'(y)}{(p_i(y))^2} .$$

We compute the derivative of I using Leibniz rule,

$$I'(y) = \underbrace{f(y, c_i(y)) \times \phi(c_i(y)) \times c_i'(y)}_{:=I_1(y)} + \underbrace{\int_{-\infty}^{c_i(y)} \phi(\hat{z}) \frac{\partial}{\partial y} f(y, \hat{z})d\hat{z},}_{:=I_2(y)} \tag{B.2}$$

where

$$c_i(y) = (\Phi^{-1}(p_i) - \sqrt{\rho_i}y)/\sqrt{1-\rho_i}, \quad \text{and} \quad c_i'(y) := -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}.$$

To show that λ_i is decreasing, we must prove that

$$I_1(y)p_i(y) + I_2(y)p_i(y) - I(y)p_i'(y) \leq 0 .$$

Let us first show that $I_2(y)p_i(y)$ is negative, which amounts to prove that $I_2(y) \leq 0$ for all y. To that end, we rewrite f using $G_i := F_i^{-1}$, $\Psi := \Phi^{-1}$ and $g(y, z) := \sqrt{\rho_i}y + \sqrt{1-\rho_i}z$ and interchange integration with respect to z and differentiation with respect to y,

$$\frac{\partial}{\partial y} f(y, \hat{z}) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} G_i \left(\Phi \left(\sqrt{\hat{\rho}_i}\Psi \left(1 - \frac{\Phi(g(y, z))}{p_i} \right) + \sqrt{1-\hat{\rho}_i}\hat{z} \right) \right) \phi(z)dz . \tag{B.3}$$

Using the chain rule for differentiation, we find

$$\begin{aligned}\frac{\partial}{\partial y}G_i(\cdot) = G_i'&\left(\Phi\left(\sqrt{\hat\rho_i}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\hat\rho_i}\hat z\right)\right)\\ \times\quad &\Phi'\left(\sqrt{\hat\rho_i}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\hat\rho_i}\hat z\right)\\ \times\quad &\sqrt{\hat\rho_i}\Psi'\left(1-\frac{\Phi(g(y,z))}{p_i}\right)\times\frac{-\Phi'(g(y,z))}{p_i}\times g'(y,z)\ .\end{aligned}$$

Moreover, $\Phi'(x)=\phi(x)$, $g'(x,z)=\sqrt{\rho_i}x$ and, from the expression of the derivative of an inverse, $G_i'(x)=1/F_i'(F_i^{-1}(x))$ and $\Psi'(x)=1/\phi(\Phi^{-1}(x))$. Rearranging the terms, one gets

$$\begin{aligned}\frac{\partial}{\partial y}G_i(\cdot) = -\frac{\phi(g(y,z))}{p_i}&\times\frac{\sqrt{\rho_i\hat\rho_i}}{\phi\left(\Phi^{-1}\left(1-\frac{\Phi(g(y,z))}{p_i}\right)\right)}\\ \times\quad &\phi\left(\sqrt{\hat\rho_i}\Psi\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\hat\rho_i}\hat z\right)\\ \times\quad &\frac{1}{F_i'\left(F_i^{-1}\left(\Phi\left(\sqrt{\hat\rho_i}\Phi^{-1}\left(1-\frac{\Phi(g(y,z))}{p_i}\right)+\sqrt{1-\hat\rho_i}\hat z\right)\right)\right)}.\end{aligned}$$

Because $F_i^{'}$ and ϕ are probability densities and $p_i>0$, we conclude that the derivative in (B.3) is negative, and so is the function appearing inside the integral I_2 in (B.2). Consequently, the integral $I_2(y)$ itself is negative for all y.

It remains to show that

$$I_1(y)p_i(y)-I(y)p_i'(y)\le 0\ .$$

Plugging $I_1(y)$ in this expression and noting that $p_i(y)=\Phi(c_i(y))$, the above inequality becomes

$$I(y)\phi(c_i(y))c_i'(y)\ge f(y,c_i(y))\phi(c_i(y))c_i'(y)p_i(y)$$

or, dividing both sides by the negative quantity $c_i'(y)\phi(c_i(y))$,

$$I(y)\le f(y,c_i(y))p_i(y)\ .$$

To see that this equality indeed holds, it suffices to notice that $f(y,\hat z)$ is increasing in $\hat z$. This implies that $f(y,\hat z)\le f(y,c_i(y))$ for all $\hat z\le c_i(y)$, leading to

$$I(y)=\int_{-\infty}^{c_i(y)}f(y,\hat z)\phi(\hat z)d\hat z\le f(y,c_i(y))\int_{-\infty}^{c_i(y)}\phi(\hat z)d\hat z=f(y,c_i(y))p_i(y)\ .$$

Appendix C. Proof of Theorem 1

Density of Y given $N>0$

To start with, let us compute the conditional density of Y given $N>0$ for further reference. Observe first that

$$\mathbb{P}(Y\le y|N>0)=\frac{\mathbb{P}(Y\le y\cap N>0)}{\mathbb{P}(N>0)}\ . \tag{C.1}$$

Using the fact that $N|Y \sim Bin(I, p(Y))$, $\mathbb{P}(N > 0|Y) = 1 - \mathbb{P}(N = 0|Y) = 1 - \prod_{i=1}^{N} q_i(Y)$, and the numerator and the denominator in the right-hand side above read

$$\int_{x=-\infty}^{y} (1 - q^I(x))\phi(x)dx = \Phi(y) - \int_{x=-\infty}^{y} q(x)^I \phi(x)dx$$

and

$$\mathbb{P}(N > 0) = \mathbb{E}[1 - \mathbb{P}(N = 0|Y)] = 1 - \int_{-\infty}^{\infty} q(x)^I \phi(x)dx \ . \tag{C.2}$$

Plugging those expressions in eq. (C.1) and differentiating with respect to y yields the conditional density $f_{Y|N>0}$ given in the lemma.

Covariance

We compute the analytical expression of each of the three terms in the conditional covariance expression

$$\mathrm{Cov}(\hat{N}, \hat{L}|N > 0) = \mathbb{E}[\hat{N}\hat{L}|N > 0] - \mathbb{E}[\hat{N}|N > 0]\,\mathbb{E}[\hat{L}|N > 0] \ . \tag{C.3}$$

Using the tower law, the first expectation in (C.3) can be written as

$$\mathbb{E}[\hat{N}\hat{L}|N > 0] = \frac{1}{I}\,\mathbb{E}\left[\sum_{i=1}^{I} L_i \middle| N > 0\right] = \frac{1}{I}\sum_{i=1}^{I} \mathbb{E}\left[\mathbb{E}\left[L_i|N > 0, Y\right]|N > 0\right] \ .$$

The inner expectation is

$$\begin{aligned}
\mathbb{E}\left[L_i|N > 0, Y\right] &= \mathbb{E}[L_i|B_i = 1, N > 0, Y]\mathbb{P}(B_i = 1|N > 0, Y) \\
&\quad + \underbrace{\mathbb{E}[L_i|B_i = 0, N > 0, Y]}_{=0}\mathbb{P}(B_i = 0|N > 0, Y) \\
&= \mathbb{E}[\Lambda_i|B_i = 1, N > 0, Y] \times \mathbb{P}(B_i = 1|N > 0, Y) \ .
\end{aligned}$$

Because $\{B_i = 1 \cap N > 0\} = \{B_i = 1 \cap N \geq 0\} = \{B_i = 1\}$, we have

$$\mathbb{E}\left[\Lambda_i|B_i = 1, N > 0, Y\right] = \mathbb{E}\left[\Lambda_i|B_i = 1, Y\right] = \lambda_i(Y)$$

for all $i \in \{1, ..., I\}$ where the last equality results from the homogeneity of the pool. On the other hand, Bayes rule yields

$$\mathbb{P}(B_i = 1|N > 0, Y) = \frac{\mathbb{P}(B_i = 1 \cap N > 0|Y)}{\mathbb{P}(N > 0|Y)} = \frac{\mathbb{P}(B_i = 1|Y)}{\mathbb{P}(N > 0|Y)} = \frac{p_i(Y)}{1 - \prod_{j=1}^{I} q_j(Y)} \ , \tag{C.4}$$

where the last equality comes from the conditional independence of the default indicators:

$$\mathbb{P}(N > 0|Y) = 1 - \mathbb{P}(N = 0|Y) = 1 - \prod_{j=1}^{I} \mathbb{P}(B_j = 0|Y) = 1 - \prod_{j=1}^{I} q_j(Y) \ .$$

Using the pool homogeneity ($\lambda_i = \lambda$, $p_i = p$ and $q_j = q$), one finally gets

$$\mathbb{E}[\hat{N}\hat{L}|N>0] = \mathbb{E}\left[\left.\frac{\lambda(Y)p(Y)}{1-q(Y)^I}\right| N>0\right] = \frac{\mathbb{E}[\lambda(Y)p(Y)]}{\mathbb{P}(N>0)}$$

because

$$\int_{-\infty}^{+\infty} \frac{\lambda(y)p(y)}{1-q(y)^I} f_{Y|N>0}(y)dy = \int_{-\infty}^{+\infty} \frac{\lambda(y)p(y)}{1-q(y)^I} \frac{(1-q(y)^I)\phi(y)}{\mathbb{P}(N>0)}dy = \frac{\int_{-\infty}^{+\infty} \lambda(y)p(y)\phi(y)dy}{\mathbb{P}(N>0)} .$$

The second expectation in (C.3) can be decomposed using the same trick as before,

$$\mathbb{E}[\hat{N}|N>0] = \frac{1}{I}\,\mathbb{E}[N|N>0] = \frac{1}{I}\,\mathbb{E}[\mathbb{E}[N|N>0,Y]|N>0] .$$

Using $\mathbb{E}\left[B_i \,|N>0,Y\right] = \mathbb{P}(B_i = 1|N>0,Y)$ given in (C.4), the inner expectation can be computed as follows

$$\begin{aligned}
\mathbb{E}[N|N>0,Y] &= \mathbb{E}\left[\left.\sum_{i=1}^{I} B_i \right| N>0,Y\right] \\
&= \sum_{i=1}^{I} \mathbb{P}(B_i = 1\,|N>0,Y\,) \\
&= \frac{\sum_{i=1}^{I} p_i(Y)}{1-\prod_{j=1}^{I} q_j(Y)} \\
&= \frac{I \times p(Y)}{1-q(Y)^I} \,,
\end{aligned}$$

where the last equality results from the homogeneity of the pool. Hence,

$$\mathbb{E}[\hat{N}|N>0] = \mathbb{E}\left[\left.\frac{p(Y)}{1-q(Y)^I}\right| N>0\right] . \tag{C.5}$$

Using $f_{Y|N>0}$, the conditional expectation of the default rate collapses to

$$\mathbb{E}[\hat{N}|N>0] = \int_{-\infty}^{\infty} \frac{p(y)}{1-q(y)^I} \times \frac{(1-q(y)^I)\phi(y)}{\mathbb{P}(N>0)}dy = \frac{p}{\mathbb{P}(N>0)} . \tag{C.6}$$

The last expectation in (C.3) is

$$\mathbb{E}[\hat{L}|N>0] = \mathbb{E}\left[\left.\frac{\sum_{i=1}^{I} L_i}{N}\right| N>0\right] = \sum_{i=1}^{I} \mathbb{E}\left[\left.\mathbb{E}\left[\left.\frac{L_i}{N}\right| N>0,Y\right]\right| N>0\right] .$$

To evaluate the inner expectation, we first partition the event $N>0$ as

$$\{N>0\} = \left\{B_i = 0 \,\cap\, \sum_{j\neq i} B_j > 0\right\} \quad\cup\quad \left\{B_i = 1 \,\cap\, \sum_{j\neq i} B_j \geq 0\right\}.$$

Note that $L_i = 0$ on the first event, and that the condition $\sum_{j\neq i} B_j \geq 0$ showing up in the second event is trivial, this leads to

$$\begin{aligned}
\mathbb{E}\left[\frac{L_i}{N}\middle| N>0, Y\right] &= \mathbb{E}\left[\frac{\Lambda_i}{1+\sum_{j\neq i} B_j}\middle| B_i = 1, Y\right] \times \mathbb{P}(B_i = 1 | N > 0, Y) \\
&= \mathbb{E}\left[\Lambda_i | B_i = 1, Y\right] \times \mathbb{E}\left[\frac{1}{1+\sum_{j\neq i} B_j}\middle| Y\right] \times \mathbb{P}(B_i = 1 | N > 0, Y) \\
&= \lambda_i(Y) \times \mathbb{E}\left[\frac{1}{1+W(Y)}\right] \times \frac{p_i(Y)}{1-\prod_{j=1}^{I} q_j(Y)} \ .
\end{aligned}$$

The second equality above results from $\Lambda_i \perp B_{j\neq i}$ conditionally on Y, which is valid under both Approach 1 and Approach 2. Under the homogeneous pool assumption, we further have that $W(Y) := \sum_{j\neq i} B_j | Y$ is distributed according to a Binomial distribution with parameters $(I-1, p(Y))$. From Chao and Strawderman (1972), it is known that if $W \sim Bin(n,p)$, then

$$\mathbb{E}\left[\frac{1}{1+W}\right] = \frac{1-q^{n+1}}{(n+1)p} \ , \quad q := 1-p,$$

Hence,

$$\mathbb{E}\left[\frac{L_i}{N}\middle| N>0, Y\right] = \lambda(Y) \times \frac{1-q(Y)^I}{Ip(Y)} \times \frac{p(Y)}{1-q(Y)^I} = \lambda(Y)/I$$

so that one simply gets

$$\mathbb{E}[\hat{L}|N>0] = \mathbb{E}[\lambda(Y)|N>0] = \int_{-\infty}^{+\infty} \lambda(y) \frac{1-q^I(y)}{\mathbb{P}(N>0)} \phi(y) dy = \frac{\mathbb{E}[\lambda(Y)(1-q^I(Y))]}{\mathbb{P}(N>0)} \ . \tag{C.7}$$

Combining the three expectations yields the claim.

Variances

We compute the analytical expression of the conditional variances of $\hat{L}$ and $\hat{N}$ which, combined to (C.3) lead to their correlation:

$$\mathbb{V}[\hat{N}|N>0] = \mathbb{E}[\hat{N}^2|N>0] - \Big(\mathbb{E}[\hat{N}|N>0]\Big)^2$$

$$\mathbb{V}[\hat{L}|N>0] = \mathbb{E}[\hat{L}^2|N>0] - \Big(\mathbb{E}[\hat{L}|N>0]\Big)^2 .$$

Regarding the variance of $\hat{N}$, the second expectation is known from (C.5). The first one can be recovered from the tower law, noting that

$$\mathbb{E}[\hat{N}^2|N>0] = \frac{1}{I^2}\,\mathbb{E}[\mathbb{E}[N^2|N>0,Y]|N>0]$$

The inner expectation is

$$\begin{aligned}
\mathbb{E}[N^2|N>0,Y] &= \mathbb{E}\left[\left(\sum_i B_i\right)^2 \middle| N>0, Y\right] \\
&= \mathbb{E}\left[\sum_i B_i^2 + \sum_{i,j\neq i} B_i B_j \middle| N>0, Y\right] \\
&= \sum_i \mathbb{E}\left[B_i^2 \middle| N>0, Y\right] + \sum_i \sum_{j\neq i} \mathbb{E}\left[B_i B_j \middle| N>0, Y\right] .
\end{aligned}$$

The expectation in the first sum is just $\mathbb{P}(B_i = 1|N>0, Y)$ because $B_i^2 = B_i$ for a Bernoulli random variable. The expectation in the second sum is

$$\mathbb{P}(B_i = 1 \cap B_j = 1|N>0,Y) = \frac{\mathbb{P}(B_i = 1|Y) \times \mathbb{P}(B_j = 1|Y)}{\mathbb{P}(N>0|Y)} = \mathbb{P}(B_i = 1|N>0,Y)p_j(Y)$$

Using (C.4) for an homogeneous pool,

$$\mathbb{E}[N^2|N>0,Y] = \sum_i \mathbb{P}(B_i = 1|N>0,Y)\left(1 + \sum_{j\neq i} p_j(Y)\right) = \frac{Ip(Y)\,(1+(I-1)p(Y))}{1-q(Y)^I}$$

One thus gets

$$\mathbb{E}[\hat{N}^2|N>0] = \frac{1}{I}\,\mathbb{E}\left[\frac{p(Y)}{1-q(Y)^I}\middle| N>0\right] + \frac{I-1}{I}\,\mathbb{E}\left[\frac{p(Y)^2}{1-q(Y)^I}\middle| N>0\right]$$

Combining (C.5) and (C.6), the first expectation is $p/\mathbb{P}(N>0)$. The second is

$$\int \frac{p(y)^2}{1-q(y)^I} f_{Y|N>0}(y)dy = \int_{-\infty}^{\infty} \frac{p^2(y)}{1-q(y)^I} \times \frac{(1-q(y)^I)\phi(y)}{1-\int_{-\infty}^{\infty} q^I(x)\phi(x)dx} dy = \frac{\mathbb{E}[p_i(Y)^2]}{\mathbb{P}(N>0)}$$

On the other hand, we have that for $X \sim \mathcal{N}(0,1)$,

$$\mathbb{E}[\Phi^2(\mu+\sigma X)] = \Phi_2\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\mu}{\sqrt{1+\sigma^2}}; \frac{\sigma^2}{1+\sigma^2}\right) .$$

Applying this result to $\mu = \frac{\Phi^{-1}(p)}{\sqrt{1-\rho}}$ and $\sigma = \frac{-\sqrt{\rho}}{\sqrt{1-\rho}}$ leads to

$$\mathbb{E}[p(Y)^2] = \Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \rho)$$

so that the considered variance then becomes

$$\mathbb{V}[\hat{N}|N>0] = \frac{p}{I\mathbb{P}(N>0)}\left(1 - \frac{Ip}{\mathbb{P}(N>0)} + (I-1)\frac{\Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \rho)}{p}\right)$$

Let us now proceed to the computation of the variance of $\hat{L}$. The second expectation is known from (C.7). The first one can be decomposed using the same trick as before,

$$\mathbb{E}[\hat{L}^2|N>0] = \mathbb{E}[\mathbb{E}[\hat{L}^2|N>0,Y]|N>0] .$$

The inner expectation is

$$\begin{aligned}
\mathbb{E}[\hat{L}^2|N>0,Y] &= \mathbb{E}\left[\left.\frac{\sum_i L_i^2 + \sum_{i,j\neq i} L_i L_j}{N^2}\right| N>0, Y\right] \\
&= \sum_i \mathbb{E}\left[\left.\frac{L_i^2}{N^2}\right| N>0,Y\right] + \sum_i \sum_{j\neq i} \mathbb{E}\left[\left.\frac{L_i L_j}{N^2}\right| N>0,Y\right] . \qquad \text{(C.8)}
\end{aligned}$$

Let us evaluate the expecation in the first sum of (C.8) by partitioning the event $N>0$ as

$$\{N>0\} = \left\{B_i = 0 \,\cap\, \sum_{k\neq i} B_k > 0\right\} \quad \cup \quad \left\{B_i = 1 \,\cap\, \sum_{k\neq i} B_k \geq 0\right\}.$$

note that $L_i = 0$ on the first event, and that the condition $\sum_{k\neq i} B_k \geq 0$ showing up in the second event is trivial. Hence,

$$\begin{aligned}
\mathbb{E}\left[\left.\frac{L_i^2}{N^2}\right| N>0,Y\right] &= \mathbb{E}\left[\left.\frac{\Lambda_i^2}{N^2}\right| B_i=1,Y\right] \times \mathbb{P}(B_i=1|N>0,Y) \\
&= \mathbb{E}\left[\left.\frac{\Lambda_i^2}{(1+\sum_{k\neq i} B_k)^2}\right| B_i=1,Y\right] \times \mathbb{P}(B_i=1|N>0,Y) \\
&= \underbrace{\mathbb{E}[\Lambda_i^2|B_i=1,Y]}_{\gamma_i(Y)} \times \mathbb{E}\left[\left.\frac{1}{(1+\sum_{k\neq i} B_k)^2}\right| Y\right] \times \mathbb{P}(B_i=1|N>0,Y) .
\end{aligned}$$

The above result is valid under both Approach 1 and Approach 2, as conditionally on Y, we have that $\Lambda_i \perp B_{j\neq i}$. Moreover, if $W \sim Bin(n,p)$ with probability mass function $P(i,n,p) := \mathbb{P}(W=i)$, we have

$$g(n,p,k) := \mathbb{E}\left[\frac{1}{(k+W)^2}\right] = \sum_{i=0}^{n} \frac{P(i,n,p)}{(k+i)^2} ,$$

which admits the integral representation (see, e.g., Chao and Strawderman (1972))

$$g(n,p,k) = \int_0^1 t^{-1} \int_0^t u^{k-1}((1-p)+pu)^n du dt \; .$$

Under the homogeneous pool assumption, $\sum_{k\neq i} B_k | Y \sim Bin(I-1, p(Y))$. Hence,

$$G_1(Y) := \mathbb{E}\left[\left. \frac{1}{(1+\sum_{k\neq i} B_k)^2} \right| Y \right] = g(I-1, p(Y), 1) \; .$$

Applying (C.4) to the homogeneous pool case, we finally have that

$$\sum_i \mathbb{E}\left[\left. \frac{L_i^2}{N^2} \right| N>0, Y \right] = \gamma(Y) G_1(Y) \times \frac{I p(Y)}{1-q(Y)^I} \; ,$$

where $\gamma_i(Y) = \gamma(Y), \forall i \in \{1, ..., I\}$.

To evaluate the expectation in the second sum of (C.8), we further partition the event $N>0$ as

$$\begin{aligned} \{N>0\} = &\left\{ B_i = 0 \,\cap\, B_j = 0 \,\cap\, \sum_{s\neq i,j} B_s > 0 \right\} \;\cup\; \left\{ B_i = 1 \,\cap\, B_j = 0 \,\cap\, \sum_{s\neq i,j} B_s \geq 0 \right\} \\ &\cup\; \left\{ B_i = 0 \,\cap\, B_j = 1 \,\cap\, \sum_{s\neq i,j} B_s \geq 0 \right\} \;\cup\; \left\{ B_i = 1 \,\cap\, B_j = 1 \,\cap\, \sum_{s\neq i,j} B_s \geq 0 \right\} . \end{aligned}$$

Note that $L_i L_j = 0$ on all the events except the last one, and that the condition $\sum_{s\neq i,j} B_s \geq 0$ is trivial. For the second term, imposing $j \neq i$ leads to

$$\begin{aligned} \mathbb{E}\left[\left. \frac{L_i L_j}{N^2} \right| N>0, Y \right] &= \mathbb{E}\left[\left. \frac{\Lambda_i \Lambda_j}{(2+\sum_{s\neq i,j} B_s)^2} \right| B_i = 1, B_j = 1, Y \right] \\ &\qquad \times \mathbb{P}(B_i = 1, B_j = 1 | N>0, Y) \\ &= \mathbb{E}\left[\Lambda_i | B_i = 1, Y \right] \times \mathbb{E}\left[\Lambda_j | B_j = 1, Y \right] \\ &\qquad \times \mathbb{E}\left[\left. \frac{1}{(2+\sum_{s\neq i,j} B_s)^2} \right| Y \right] \times \mathbb{P}(B_i = 1, B_j = 1 | N>0, Y) \\ &= \lambda_i(Y) \lambda_j(Y) \times \mathbb{E}\left[\left. \frac{1}{(2+\sum_{s\neq i,j} B_s)^2} \right| Y \right] \times \frac{p_i(Y) \times p_j(Y)}{1-\prod_{s=1}^I q_s(Y)} \end{aligned}$$

The above result is valid under both Approach 1 and Approach 2, as conditionally on Y, we have that $\Lambda_i \perp B_{j\neq i}$. Under the homogeneous pool assumption, we further have that $\sum_{s\neq k,j} B_s | Y \sim Bin(I-2, p(Y))$. Hence,

$$G_2(Y) := \mathbb{E}\left[\left. \frac{1}{(2+\sum_{s\neq i,j} B_s)^2} \right| Y \right] = g(I-2, p(Y), 2) \; .$$

Due to the homogeneity of the pool we have that

$$\sum_i \sum_{j \neq i} \mathbb{E}\left[\frac{L_i L_j}{N^2} \middle| N > 0, Y \right] = \frac{I p(Y)}{1 - q(Y)^I} \times (I-1)\lambda(Y)^2 G_2(Y) p(Y)$$

One finally gets,

$$\mathbb{E}[\hat{L}^2 | N > 0, Y] = \frac{I p(Y)}{1 - q(Y)^I} \big(\gamma(Y) G_1(Y) + (I-1)\lambda(Y)^2 G_2(Y) p(Y) \big)$$

Using $f_{Y|N>0}$, the conditional expectation of the default rate leads to

$$\mathbb{E}[\hat{L}^2 | N > 0] = \frac{I}{\mathbb{P}(N>0)} \left(\mathbb{E}\left[\gamma(Y) G_1(Y) p(Y)\right] + (I-1)\,\mathbb{E}\left[\lambda(Y)^2 G_2(Y) p(Y)^2\right] \right) \ . \quad \text{(C.9)}$$

It is easy to see that the integral form of g satisfies the recursion

$$g(n,p,2) = \frac{1 - (1-p)^{n+2} - p(n+2) g(n+1,p,1)}{p^2 (n+2)(n+1)} \ .$$

Hence, we conclude that

$$G_2(Y) = g(I-2, p(Y), 2) = \frac{1 - q(Y)^I - I p(Y) G_1(Y)}{p(Y)^2 I (I-1)} \ .$$

Plugging this expression in the second expectation of the RHS of (C.9) yields

$$\begin{aligned}(I-1)\,\mathbb{E}\left[\lambda(Y)^2 G_2(Y) p(Y)^2\right] &= \mathbb{E}\left[\lambda(Y)^2 \frac{1 - q(Y)^I - I p(Y) G_1(Y)}{I}\right] \\ &= \frac{1}{I}\Big(\mathbb{E}\left[\lambda(Y)^2 (1 - q(Y)^I)\right] \Big) - \mathbb{E}\left[\lambda(Y)^2 p(Y) G_1(Y)\right]\end{aligned}$$

Hence, noting that

$$\gamma_i(Y) - \lambda_i(Y)^2 = \mathbb{E}[\Lambda_i^2 | B_i = 1, Y] - (\mathbb{E}[\Lambda_i | B_i = 1, Y])^2 = \mathbb{V}[\Lambda_i | B_i = 1, Y] =: \nu(Y) \ ,$$

$$\begin{aligned}\mathbb{V}[\hat{L} | N > 0] &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[(\gamma(Y) - \lambda(Y)^2) G_1(Y) p(Y)\right] + \frac{\mathbb{E}\left[\lambda(Y)^2 (1 - q(Y)^I)\right]}{\mathbb{P}(N>0)} \\ &\quad - (\mathbb{E}[\lambda(Y) | N > 0])^2 \\ &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[\nu(Y) G_1(Y) p(Y)\right] + \mathbb{E}\left[\lambda(Y)^2 | N > 0\right] - (\mathbb{E}[\lambda(Y) | N > 0])^2 \\ &= \frac{I}{\mathbb{P}(N>0)} \times \mathbb{E}\left[\nu(Y) G_1(Y) p(Y)\right] + \mathbb{V}\left[\lambda(Y) | N > 0\right]\end{aligned}$$

The proof is concluded by noting that $I G_1(Y) p(Y) = \Psi(Y)$, which can be obtained by inserting the integral representation of g in $G_1(y) = g(I-1, p(y), 1)$:

$$G_1(y) = \int_0^1 t^{-1} \int_0^t (q(y) + p(y) u)^{I-1} du dt = \frac{1}{p(y) I} \int_0^1 t^{-1} \left((q(y) + p(y) t)^I - q(y)^I \right) dt \ .$$

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