Tests cases for Lagrangian discretisation of diffusion terms revisited[1]

Eric Deleersnijder, 5 February 2018

A number of transport problems are tackled. In most cases, the focus is on diffusive processes. This is because the aim of the present working note is to list test cases for the stochastic aspects of Lagrangian methods. The concentration of a dissolved quantity, defined as a mass fraction (i.e. a dimensionless variable), is obtained analytically. When an analytical solution cannot be derived, a quantitative diagnosis is relied upon. The Boussinesq approximation is assumed to be valid. Accordingly, until otherwise stated, the density ρ of the fluid under study (a mixture of water and dissolved or particulate constituents) is taken to be a constant.

The position-vector is denoted

$$\mathbf{x} = \underbrace{x\mathbf{e}_x + y\mathbf{e}_y}_{=\mathbf{x}_h} + z\mathbf{e}_z \quad , \tag{1}$$

where x and y are horizontal coordinate, whilst z is the vertical coordinate (pointing upward). The del operator reads

$$\nabla = \underbrace{\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y}}_{=\nabla_h} + \mathbf{e}_z \frac{\partial}{\partial z} \quad , \tag{2}$$

where ∇_h is the horizontal part of the operator.

Advection-diffusion in a channel

Consider an unbounded channel, where x is the along-channel coordinate ($-\infty < x < \infty$). The cross-channel area and the section-averaged velocity are $S(x)$ and $U(x)$, respectively. The volumetric flow rate $S(x)U(x)$ is constant (continuity equation). A mass M of a passive tracer is abruptly released at $t = 0$ and $x = 0$.

The tracer concentration $C(t,x)$ obeys partial differential equation

$$\frac{\partial(SC)}{\partial t} = -\frac{\partial}{\partial x}\left(SCU - S\kappa \frac{\partial C}{\partial x} \right) \quad , \tag{3}$$

which must be solved under the initial condition

$$C(0,x) = \frac{M}{\rho S(0)} \delta(x-0) \tag{4}$$

where constant ρ is the reference density of the fluid (water and dissolved constituents).

At coordinate x, the time-integrated advective and diffusive mass fluxes read

---

[1] *This working note is based essentially on published articles and previous working notes. Most of them are explicitly referred to hereinafter.*

$$\phi_a(x) = \int_0^\infty \rho SCU\, dt = \begin{cases} M\ , & 0 < x \\ \exp\left(-\int_x^0 \frac{U(x')}{\kappa(x')}\, dx'\right) M\ , & x < 0 \end{cases} \tag{5}$$

and

$$\phi_d(x) = -\int_0^\infty \rho S\kappa \frac{\partial C}{\partial x}\, dt = \begin{cases} 0\ , & 0 < x \\ -\exp\left(-\int_x^0 \frac{U(x')}{\kappa(x')}\, dx'\right) M\ , & x < 0 \end{cases} \tag{6}$$

Unsurprisingly, the sum of these integrated mass fluxes is equal to the injected mass downstream of the injection point (x=0) and is zero upstream of it. This is because any tracer particle that, at a given time, moves in the upstream direction (due to diffusion) will eventually be flushed downstream.

If the cross-sectional area is constant, then the velocity is also constant. In this case, the tracer concentration is

$$C(t,x) = \frac{M}{\rho S\sqrt{4\pi\kappa t}} \exp\left[-\frac{(x-Ut)^2}{4\kappa t}\right] \tag{7}$$

References

Deleersnijder E., 2012, An unexpected property of tracer transport in a channel flow, Working note, Université catholique de Louvain, 2 pages, http://hdl.handle.net/2078.1/155261

Rutherford J.C., 1994, *River Mixing*, Wiley, 347 pages

Diffusion in a water column model

Under the hypothesis of horizontal homogeneity, the concentration $C(t,z)$ obeys one-dimensional diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial C}{\partial z}\right) \tag{8}$$

The vertical diffusivity κ is a function of only the vertical coordinate z. The upper and lower boundaries are impermeable

$$\left[\kappa \frac{\partial C}{\partial z}\right]_{z=-h} = 0 = \left[\kappa \frac{\partial C}{\partial z}\right]_{z=0} \tag{9}$$

The domain of interest is a prism with vertical sides and constant (horizontal) cross-sectional area S. The initial condition is

$$C(0,z) = \frac{M}{\rho S}\delta(z-z_0) \quad . \tag{10}$$

It is convenient to introduce a dimensionless vertical coordinate,

$$\sigma = \frac{z+h}{h} \ , \tag{11}$$

which is zero at the bottom of the water column and equal to unity at the top of it. Then, the diffusivity may be rewritten as follows

$$\kappa(\sigma) = \bar{\kappa}\tilde{\kappa}(\sigma) \ , \tag{12}$$

with

$$\bar{\kappa} = \int_0^1 \kappa(\sigma)\, d\sigma \ , \tag{13}$$

and

$$\int_0^1 \tilde{\kappa}(\sigma)\, d\sigma = 1 \ . \tag{14}$$

Several vertical diffusivity profiles are worth considering. The simplest of them is $\kappa = 1$, i.e. the diffusivity is constant. Then, to account for the presence of the upper and lower boundaries, the parabolic profile $\kappa = 6\sigma(1-\sigma)$ is appropriate; in the vicinity of the seabed, the diffusivity increases as a linear function of the distance to the boundary, which is consistent with the existence of the logarithmic layer. Though the bottom is generally regarded as a solid boundary, the ocean-atmosphere interface is a freely-moving boundary. To take into account the difference in the nature of the lower and upper boundaries, another diffusivity profile may also be worth studying, namely $\kappa = 3\sigma(1-\sigma/2)$.

Table 1. The eigenvalues λ_n and eigenfunctions $\psi_n(\sigma)$ for the diffusivity profiles considered in this study. The order of the mode is identified by the integer index *n*, with *n*=0,1,2,... The symbol P_n represents the *n*-th order Legendre polynomial.

$\kappa(\sigma) = 1$	$\kappa(\sigma) = 6\sigma(1-\sigma)$	$\kappa(\sigma) = 3\sigma(1-\sigma/2)$
$\lambda_n = n^2\pi^2$	$\lambda_n = \frac{3}{2}n(n+1)$	$\lambda_n = 3n(2n+1)$
$\psi_0 = 1$	$\psi_0 = 1$	$\psi_0 = 1$
$\psi_n = \sqrt{2}\cos(n\pi\sigma)$ $(n = 1,2,3,...)$	$\psi_n = \sqrt{2n+1}\, P_n(-1+2\sigma)$ $(n = 1,2,3,...)$	$\psi_n = \sqrt{4n+1}\, P_{2n}(1-\sigma)$ $(n = 1,2,3,...)$

The following eigenvalue problem must be solved:

$$\frac{d}{d\sigma}\left[\tilde{\kappa}(\sigma)\frac{d\psi_n}{d\sigma}\right] = -\lambda_n \psi_n \ , \tag{15}$$

$$\left[\kappa(\sigma)\frac{d\psi_n}{d\sigma}\right]_{\sigma=0} = 0 = \left[\kappa(\sigma)\frac{d\psi_n}{d\sigma}\right]_{\sigma=1} , \tag{16}$$

where $\psi_n(\sigma)$ and λ_n denote the n-th eigenfunction and eigenvalue, respectively (Table 1). The eigenfunctions are orthonormal:

$$\int_0^1 \psi_m \psi_n \, d\sigma = 0 \text{ if } m \neq n . \tag{17}$$

$$\int_0^1 \psi_n^2 \, d\sigma = 1 , \tag{18}$$

with

$$\lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n < \lambda_{n+1} < ... . \tag{19}$$

It is readily seen that for the problem at hand, the following properties are satisfied:

$$\lambda_0 = 0 , \quad \psi_0(1) = 1 , \tag{20}$$

$$\int_0^1 \psi_n \, d\sigma = 0 \text{ if } n \geq 1 . \tag{21}$$

The solution reads

$$C(t,z) = \frac{M}{\rho h S}\left[1 + \sum_{n=1}^{\infty} \exp\left(-\gamma_n t\right) \psi_n(\sigma_0) \psi_n(\sigma)\right] \tag{22}$$

with $\gamma_n = \bar{\kappa}\lambda_n / h^2$ and $\sigma_0 = (z_0 + h)/h$.

References

Deleersnijder E., 2007, Analytical solutions to assess Lagrangian models of advective and diffusive transport processes, Working note, Université catholique de Louvain, 24 pages, http://hdl.handle.net/2078.1/155421

Deleersnijder E., 2014, Solutions of a tracer transport problem with a variable vertical eddy diffusivity, Working note, Université catholique de Louvain, 10 pages, http://hdl.handle.net/2078.1/155261

Diffusion in and sinking out of the surface mixed layer

The behaviour of particles exhibiting a negative buoyancy (their density is greater than that of seawater) is investigated in the surface mixed layer. All variables are assumed to be horizontally homogeneous. The settling velocity ($\varpi > 0$) is constant. The vertical diffusivity, κ , depends only on the vertical coordinate, z. The latter is zero at the sea surface and is equal to $-h$ at the bottom of the mixed layer. Under the latter lies the pycnocline in which diffusion is negligible, but not settling, which proceeds with a velocity assumed to be equal to that prevailing in the mixed layer.

The Green's function of the problem, $G(t,z;z_0)$, obeys partial differential equation

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial z}\left(G\varpi + \kappa\frac{\partial G}{\partial z}\right) . \tag{23}$$

The water-air interface is impermeable:

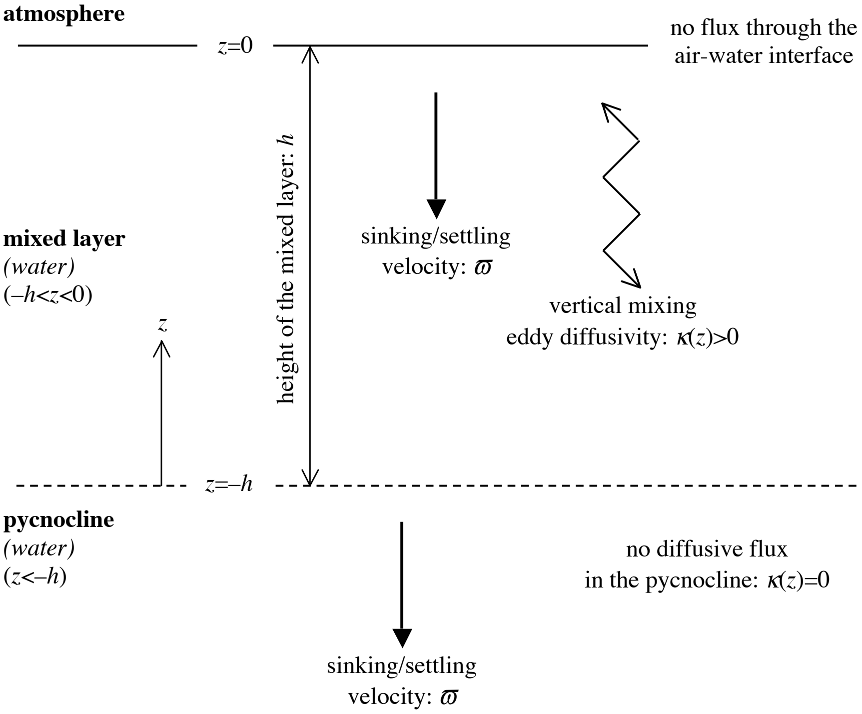
$$\left[G\varpi + \kappa\frac{\partial G}{\partial z}\right]_{z=0} = 0 \quad . \tag{24}$$

At the bottom of the mixed layer, the diffusive flux is zero (diffusion is assumed to be negligible in the pycnocline):

$$\left[\kappa\frac{\partial G}{\partial z}\right]_{z=-h} = 0 \quad . \tag{25}$$

The initial condition reads

$$G(0,z;z_0) = \delta(z-z_0) \quad . \tag{26}$$



Presumably, there is no analytical solution to differential problem (23)-(26). However, it is possible to derive the expression of the residence time in the mixed layer of the particles initially located at $z = z_0$, i.e. the mean time spent in the mixed layer. The residence time reads

$$\theta(z_0) = \frac{\int_0^\infty \int_{-h}^0 G(t,z;z_0)\,dz\,dt}{\int_{-h}^0 G(0,z;z_0)\,dz} = \int_0^\infty \int_{-h}^0 G(t,z;z_0)\,dz\,dt$$

$$= \frac{h+z_0}{\varpi} + \frac{1}{\varpi}\int_{z_0}^0 \exp\left[-\varpi\int_{z_0}^{\xi}\frac{d\zeta}{\kappa(\zeta)}\right]d\xi \tag{27}$$

It may be seen that the residence time obeys inequalities

$$\frac{h+z}{\varpi} \le \theta(z_0) \le \frac{h}{\varpi} \quad . \tag{28}$$

If the diffusivity is constant, then (27) simplifies to

$$\kappa = const. \quad \Rightarrow \quad \theta(z_0) = \frac{h+z_0}{\varpi} + \frac{\kappa}{\varpi^2}\left(1-e^{\varpi z/\kappa}\right) \quad . \tag{29}$$

If, on the other hand, the diffusivity exhibits a parabolic profile (and is zero at the top and the bottom of the mixed layer), one has

$$\kappa = 6\bar{\kappa}\frac{|z|}{h}\frac{h+z}{h} \quad \Rightarrow \quad \theta(z_0) = \frac{h+z_0}{\varpi} + \frac{\kappa}{\varpi}\left(\frac{h+z_0}{|z_0|}\right)^{\mu} B_{|z_0|/h}(1+\mu,1-\mu) \tag{30}$$

with $\mu = \varpi h/(6\bar{\kappa})$, where $B_{|z_0|/h}(1+\mu,1-\mu)$ is an incomplete beta function.

References

Deleersnijder E., 2007, Analytical solutions to assess Lagrangian models of advective and diffusive transport processes, Working note, Université catholique de Louvain, 24 pages, http://hdl.handle.net/2078.1/155421

Deleersnijder E., J.-M. Beckers and E.J.M. Delhez, 2006, The residence time of settling particles in the surface mixed layer, *Environmental Fluid Mechanics*, 6, 25-42

Deleersnijder E., J.-M. Beckers and E.J.M. Delhez, 2006, On the behaviour of the residence time at the bottom of the mixed layer, *Environmental Fluid Mechanics,* 6, 541-547

Spivakovskaya D., A.W. Heemink and E. Deleersnijder, 2007, The backward Ito method for the Lagrangian simulation of transport processes with large space variations of the diffusivity, *Ocean Science*, 3, 525-535

One dimensional water column with a pycnocline

Let h represent the height of the water column. The vertical coordinate, increasing upward, is denoted z, with $z=0$ and $z=h$ at its lower and upper boundaries, respectively. A sharp pycnocline is present in the middle of the water column ($z=h/2$). The vertical diffusivity, $\kappa(z)$, is zero at both ends of the domain and at the pycnocline. It is given by

$$\kappa(z) = \begin{cases} \bar{\kappa}\,\dfrac{2(1+a)(1+2a)}{a^2 h^{1+1/a}}(h-z)(2z-h)^{1/a} \ , & 0<z<h/2 \\ \bar{\kappa}\,\dfrac{2(1+a)(1+2a)}{a^2 h^{1+1/a}}z(h-2z)^{1/a} \ , & 0<z<h/2 \end{cases} \tag{31}$$

where constant a is greater than or equal to unity, whilst $\bar{\kappa}$ is the depth mean of the diffusivity, i.e.

$$\bar{\kappa} = \frac{1}{h}\int_0^h \kappa(z)\,dz \quad . \tag{32}$$

The larger the value of a, the larger the gradient of the diffusivity in the vicinity of the pycnocline.

All variables are horizontally homogeneous. There is no diffusive flux through the upper and lower boundaries of the water column. In addition, the diffusive flux crossing the pycnocline is also zero, for the diffusivity is zero at $z = h/2$.

The Green's function of the present diffusion problem, $G(t,z;z_0)$, is the solution of diffusion equation

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial G}{\partial z}\right) \tag{33}$$

under boundary and initial conditions

$$\left[\kappa \frac{\partial G}{\partial z}\right]_{z=0} = 0 = \left[\kappa \frac{\partial G}{\partial z}\right]_{z=h} \quad , \tag{34}$$

$$G(0,z;z_0) = \delta(z - z_0) \quad . \tag{35}$$

There is probably no analytical solution of the problem under consideration. However, it is readily seen that the following limit holds valid

$$\lim_{t\to\infty} G(t,z;z_0) = \frac{2}{h}\left[\Upsilon(h/2 - z_0)\Upsilon(h/2 - z) + \Upsilon(z_0 - h/2)\Upsilon(z - h/2)\right] \quad , \tag{36}$$

where Υ denotes the Heaviside function, i.e. a function whose value is equal to unity (zero) if its argument is positive (negative).

References

Deleersnijder E., 2011, Test cases for Lagrangian transport models in stratified flows, Working note, Université catholique de Louvain, 11 pages, http://hdl.handle.net/2078.1/194410

Gräwe U., E. Deleersnijder, S.H.A.M. Shah and A.W. Heemink, 2012, *Ocean Dynamics*, 62, 501-514

Multi-dimensional diffusion in an unbounded domain

The domain of interest is $\Re^n$. The concentration $C(t,\mathbf{x})$ is the solution of diffusion equation

$$\frac{\partial C}{\partial t} = \nabla \bullet (\mathbf{K} \bullet \nabla C) \tag{37}$$

under the initial condition

$$C(0,\mathbf{x}) = \frac{M}{\rho}\delta(\mathbf{x} - \mathbf{0}) \quad . \tag{38}$$

where **K** is the diffusivity tensor, which must be positive definite and symmetric. Its components are time- and position-independent.

The solution of diffusion problem (37)-(38) is

$$C(t,\mathbf{x}) = \frac{M}{\rho(4\pi t)^{n/2}\sqrt{\det \mathbf{K}}} \exp\left[-\frac{\mathbf{x} \bullet \mathbf{K}^{-1} \bullet \mathbf{x}}{4t}\right] \quad . \tag{39}$$

The position of the centre of mass of the tracer patch is

$$\mathbf{r}_c(t) = \frac{1}{M} \int_{\Re^n} \rho C(t,\mathbf{x}) \mathbf{x} \, d\mathbf{x} = 0 \quad . \tag{40}$$

The position variance of the tracer distribution increases linearly in time

$$\sigma^2(t) = \frac{1}{M} \int_{\Re^n} \rho C(t,\mathbf{x}) \left| \mathbf{x} - \mathbf{r}_c(t) \right|^2 d\mathbf{x} = 2 \,\mathrm{trace} \mathbf{K} \, t \quad . \tag{41}$$

This is the trademark of harmonic diffusion.

In the interior of the ocean, the diffusivity tensor is usually designed to represent diapycnal/isopycnal mixing. However, in continental seas, coastal regions and estuaries no such phenomena are believe to take place. In general, horizontal-vertical diffusion is to be modelled, yielding the following expression of the diffusivity tensor

$$\mathbf{K} = \kappa_h \mathbf{e}_x \mathbf{e}_x + \kappa_h \mathbf{e}_y \mathbf{e}_y + \kappa_v \mathbf{e}_z \mathbf{e}_z \quad \Rightarrow \quad \begin{cases} \mathrm{trace}\mathbf{K} = 2\kappa_h + \kappa_v \ , \quad \det \mathbf{K} = \kappa_h^2 \kappa_v \\ \mathbf{K}^{-1} = \kappa_h^{-1} \mathbf{e}_x \mathbf{e}_x + \kappa_h^{-1} \mathbf{e}_y \mathbf{e}_y + \kappa_v^{-1} \mathbf{e}_z \mathbf{e}_z \end{cases} \tag{42a}$$

or

$$\mathbf{K} = \begin{pmatrix} \kappa_h & 0 & 0 \\ 0 & \kappa_h & 0 \\ 0 & 0 & \kappa_v \end{pmatrix} \tag{42b}$$

where horizontal diffusivity κ_h and vertical diffusivity κ_v are constant.

The solution of this problem and its properties are a particular case of the general solution (39)-(41):

$$C(t,\mathbf{x}) = \frac{M}{\rho} \frac{\exp\left(-\dfrac{x^2}{4\kappa_h t}\right)}{\sqrt{4\pi\kappa_h t}} \frac{\exp\left(-\dfrac{y^2}{4\kappa_h t}\right)}{\sqrt{4\pi\kappa_h t}} \frac{\exp\left(-\dfrac{z^2}{4\kappa_v t}\right)}{\sqrt{4\pi\kappa_v t}} \quad , \tag{43}$$

$$\mathbf{r}_c(t) = \frac{1}{M} \int_{\Re^3} \rho C(t,\mathbf{x}) \mathbf{x} \, d\mathbf{x} = 0 \quad , \tag{44}$$

$$\sigma^2(t) = \frac{1}{M} \int_{\Re^3} \rho C(t,\mathbf{x}) \left| \mathbf{x} - \mathbf{r}_c(t) \right|^2 d\mathbf{x} = 2(2\kappa_h + \kappa_v)t \quad . \tag{45}$$

References

Deleersnijder E., 2012, Homogenisation of a passive tracer concentration in an isolated domain, Working note, Université catholique de Louvain, 7 pages, http://hdl.handle.net/2078.1/155297

Deleersnijder E., 2013, Test cases for isopycnal (and diapycnal) diffusion in the ocean, Working note, Université catholique de Louvain, 18 pages, http://hdl.handle.net/2078.1/155331

Spivakovskaya D., A.W. Heemink and E. Deleersnijder, 2007, Lagrangian modelling of multi-dimensional advection-diffusion with space varying diffusivities: theory and idealized tests cases, *Ocean Dynamics*, 57, 189-203

Multi-dimensional diffusion in semi-infinite domain: revisiting the previous problem

The previous problem is revisited. It is henceforth assumed that the domain of interest is semi-infinite:

$$-\infty < x,y < +\infty \quad , \quad 0 < z < +\infty \tag{46}$$

The $z = 0$ plane is impermeable:

$$\left[\kappa_v \frac{\partial C}{\partial z} \right]_{z=0} = 0 \quad , \tag{47}$$

and the initial condition now reads

$$C(0,\mathbf{x}) = \frac{M}{\rho} \delta(x-0)\delta(y-0)\delta(z-z_0) \quad , \tag{48}$$

with $z_0 > 0$. Nothing else is modified.

The solution is obtained by the method of mirror images:

$$C(t,\mathbf{x}) = \frac{M}{\rho} \frac{\exp\left(-\dfrac{x^2}{4\kappa_h t}\right)}{\sqrt{4\pi\kappa_h t}} \frac{\exp\left(-\dfrac{y^2}{4\kappa_h t}\right)}{\sqrt{4\pi\kappa_h t}} \left[\frac{\exp\left(-\dfrac{(z-z_0)^2}{4\kappa_v t}\right)}{\sqrt{4\pi\kappa_v t}} + \frac{\exp\left(-\dfrac{(z+z_0)^2}{4\kappa_v t}\right)}{\sqrt{4\pi\kappa_v t}} \right] \tag{49}$$

This analytical solution may allow one to assess the implementation of impermeability conditions in Lagrangian models, which is much less straightforward than in Eulerian models.

Depth-integrated modelling

The domain of interest is two-dimensional (horizontal) and unbounded ($\mathbf{x}_h \in \Re^2$). The depth-integrated concentration $C(t,\mathbf{x}_h)$ of a passive tracer obeys equation

$$\frac{\partial (hC)}{\partial t} = \nabla_h \bullet (h\kappa \nabla_h C) \quad , \tag{50}$$

where

$$h(\mathbf{x}_h) = h_0 \, e^{\mathbf{k} \cdot \mathbf{x}_h} \tag{51}$$

is the water column height, whilst vector $\mathbf{k}$ is a constant vector whose physical dimension is length^{-1}. The initial condition is

$$C(0,\mathbf{x}) = \frac{M}{\rho h_0} \delta(x-0)\delta(y-0) \quad . \tag{52}$$

Let $\mathbf{v}$ represent the following “velocity”:

$$\mathbf{v} = -\frac{\nabla_h (h\kappa)}{h} = -\kappa \mathbf{k} \quad . \tag{53}$$

Then, the solution reads

$$C(t,\mathbf{x}_h) = \frac{M}{4\pi\rho h_0 \kappa t} \exp\left[-\frac{|\mathbf{x}_h - \mathbf{v}t|^2}{4\kappa t} \right] \quad . \tag{54}$$

The point where the concentration is maximum is located at $\mathbf{v}t = -\kappa \mathbf{k} t$. It moves towards the shallower part of the domain as opposed to the centre of mass, which moves in the opposite direction,

$$\mathbf{r}_c(t) \equiv \frac{1}{M} \int\limits_{\Re^2} \rho\, h(\mathbf{x}_h) C(t,\mathbf{x}_h) \mathbf{x}_h \, d\mathbf{x}_h = \frac{1}{4\pi\kappa t} \int\limits_{\Re^2} \exp\left[-\frac{\left|\mathbf{x}_h + \mathbf{v}t\right|^2}{4\kappa t} \right] \mathbf{x}_h \, d\mathbf{x}_h = \kappa \mathbf{k} t \quad . \qquad (55)$$

Unsurprisingly, the position-variance increases as a linear function of time

$$\begin{aligned} \sigma^2(t) &\equiv \frac{1}{M} \int\limits_{\Re^2} \rho\, h(\mathbf{x}_h) C(t,\mathbf{x}_h) \left|\mathbf{x}_h - \mathbf{r}_c(t)\right|^2 d\mathbf{x}_h \\ &= \frac{1}{4\pi\kappa t} \int\limits_{\Re^2} \exp\left[-\frac{\left|\mathbf{x}_h + \mathbf{v}t\right|^2}{4\kappa t} \right] \left|\mathbf{x}_h - \mathbf{r}_c(t)\right|^2 d\mathbf{x}_h = 4\kappa t \end{aligned} \qquad (56)$$

The vertical inventory is

$$h(\mathbf{x}_h) C(t,\mathbf{x}_h) = \frac{M}{4\pi\rho\kappa t} \exp\left[-\frac{\left|\mathbf{x}_h + \mathbf{v}t\right|^2}{4\kappa t} \right] \quad . \qquad (57)$$

The point where the vertical inventory is maximum is the centre of mass of the tracer distribution, which is quite natural.

References

Deleersnijder E., 2015, A depth-integrated diffusion problem in a depth-varying, unbounded domain for assessing Lagrangian schemes, Working note, Université catholique de Louvain, 7 pages, http://hdl.handle.net/2078.1/160980

Spagnol S., E. Wolanski, E. Deleersnijder, R. Brinkman, F. McAllister, B. Cushman-Roisin and E. Hanert, 2002, An error frequently made in the evaluation of advective transport in two-dimensional Lagrangian models of advection-diffusion in coral reef waters, *Marine Ecology Progress Series*, 235, 299-302

Appendix: integrals of use for dealing with advection-diffusion problems

$$\int\limits_0^\infty e^{-a\varsigma^2} \, d\varsigma = \sqrt{\frac{\pi}{4a}} \ , \quad \int\limits_0^\infty \varsigma^2 \, e^{-a\varsigma^2} \, d\varsigma = \sqrt{\frac{\pi}{16a^3}} \ , \quad a > 0$$

$$\int\limits_0^\infty \varsigma^{-3/2} \exp\left(-\frac{a}{\varsigma} - b\varsigma \right) d\varsigma = \sqrt{\frac{\pi}{a}} \exp\left(-2\sqrt{ab} \right) \ , \quad a,b > 0$$

$$\int\limits_0^\infty \varsigma^{-1/2} \exp\left(-\frac{a}{\varsigma} - b\varsigma \right) d\varsigma = \sqrt{\frac{\pi}{b}} \exp\left(-2\sqrt{ab} \right) \ , \quad a,b > 0$$

$$\int_0^\infty \varsigma^{1/2} \exp\left(-\frac{a}{\varsigma} - b\varsigma\right) d\varsigma = \sqrt{\frac{\pi}{4b^3}} \left(1 + 2\sqrt{ab}\right) \exp\left(-2\sqrt{ab}\right) , \quad a,b > 0$$

$$\int_0^\infty \varsigma^{3/2} \exp\left(-\frac{a}{\varsigma} - b\varsigma\right) d\varsigma = \sqrt{\frac{\pi}{16b^5}} \left(3 + 6\sqrt{ab} + 4ab\right) \exp\left(-2\sqrt{ab}\right) , \quad a,b > 0$$

$$\int_0^\infty \operatorname{erfc}\left(a\sqrt{\varsigma} + \frac{b}{\sqrt{\varsigma}} \right) d\varsigma = \frac{e^{-4ab}}{2a^2} , \quad a,b > 0$$

$$\int_0^\infty \varsigma \operatorname{erfc}\left(a\sqrt{\varsigma} + \frac{b}{\sqrt{\varsigma}} \right) d\varsigma = \frac{3 + 4ab}{8a^4} e^{-4ab} , \quad a,b > 0$$

$\delta(\varsigma) = 0$ for $\varsigma \neq 0$

$$\int_{-\infty}^{+\infty} \delta(\varsigma)\, d\varsigma = 1$$

$$\int_{-\infty}^{+\infty} \psi(\zeta) \delta(\zeta - \varsigma)\, d\zeta = \psi(\varsigma)$$
