Frequency shift up to the 2-PM approximation

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Abstract

A lot of fundamental tests of gravitational theories rest on highly precise measurements of the travel time and/or the frequency shift of electromagnetic signals propagating through the gravitational field of the Solar System. In practically all of the previous studies, the explicit expressions of such travel times and frequency shifts as predicted by various metric theories of gravity are derived from an integration of the null geodesic differential equations. However, the solution of the geodesic equations requires heavy calculations when one has to take into account the presence of mass multipoles in the field or the tidal effects due to the planetary motions, and the calculations become quite complicated in the post-post-Minkowskian approximation. This difficult task can be avoided using the time transfer function's formalism. We present here our last advances in the formulation of the one-way frequency shift using this formalism up to the post-post-Minkowskian approximation.
In this paper c is the speed of light in a vacuum. The Lorentzian metric of space-time V_4 is denoted by g. We suppose that space-time is covered by some global quasi-Galilean coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$. We employ the vector notation $\boldsymbol{a}$ in order to denote $(a^1, a^2, a^3) = (a^i)$. For any quantity $f(x^\lambda)$, $f_{,\alpha}$ denotes the partial derivative of f with respect to x^α. The indices in parentheses characterize the order of perturbation. They are set up or down, depending on the convenience.

The one-way frequency shift

Consider a clock $\mathcal{O}_\mathcal{A}$ on $\mathcal{A}$ and a clock $\mathcal{O}_\mathcal{B}$ on $\mathcal{B}$ delivering, respectively, the proper frequency $f_\mathcal{A}$ and $f_\mathcal{B}$. Then, suppose that $\mathcal{O}_\mathcal{A}$ is sending photons to $\mathcal{O}_\mathcal{B}$ along null geodesics of the metric (geometric optics approximation).
Then, the one way frequency shift is defined by

$$\left.\frac{\Delta\nu}{\nu}\right|_{\mathrm{A}\to\mathrm{B}}^{\text{one-way}} = \frac{\nu_B}{\nu_A} - 1\,, \tag{1}$$

It is well-known that the ratio ν_B/ν_A can be expressed as [2]

$$\frac{\nu_B}{\nu_A} = \frac{u^\mu_B k^B_\nu}{u^\nu_A k^A_\mu} = \frac{k^B_0}{k^A_0}\frac{u^0_B + u^i_B \hat{k}^B_i}{u^0_A + u^i_A \hat{k}^A_i} = \left(\frac{d\tau}{dt}\right)_A \frac{dt_A}{dt_B}\left(\frac{dt}{d\tau}\right)_B \tag{2}$$

where $u^\mu_{A/B} = (dx^\mu/ds)_{A/B}$ are the four-unit velocity of observers A and B, $\hat{k}_i = \left(\frac{k_i}{k_0}\right)$ and k^A_μ and k^B_μ are the wave vectors (the null tangent vectors) at the point of emission x_A and at the point of reception x_B, respectively. Terms appearing in Eq. (2) can be expressed as

$$\left(\frac{d\tau}{dt}\right)_x = \left[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j\right]^{1/2}_x\,,$$

$$\frac{dt_A}{dt_B} = \frac{k^B_0}{k^A_0}\frac{1+\beta^i_B\hat{k}^B_i}{1+\beta^i_A\hat{k}^A_i}\,, \tag{3}$$

$\beta^i_{A/B} = \frac{1}{c}\frac{dx^i_{A/B}}{dt}$ being the coordinate velocities of observers A and B.

Conclusions

We presented here our last advances in the formulation of the one-way frequency shift using this formalism up to the post-post-Minkowskian approximation. The main result is given by Eq. (9) where the derivatives of Δ_r are given up to 2PM order by Eq. (13-14). Exact formulas up to this order may be required for future space missions to the inner Solar System [4].

References

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Relation between frequency shift and time transfer functions

We put $x_A = (ct_A, \boldsymbol{x}_A)$ the event of emission $\mathcal{A}$ and $x_B = (ct_B, \boldsymbol{x}_B)$ the event of reception $\mathcal{B}$. Moreover, we define $\mathcal{T}_e$ and $\mathcal{T}_r$ as two distinct (coordinate) time transfer functions defined as

$$t_B - t_A = \mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B)\,. \tag{4}$$

One of us derived in [1] that the following relations hold for the vector $k^\mu = dx^\mu/d\lambda$ tangent to the null geodesic at point $x(\lambda)$, λ being any affine parameter :

$$\left(\hat{k}_i\right)_A = \left(\frac{k_i}{k_0}\right)_A = c\frac{\partial\mathcal{T}_e}{\partial x^i_A}\left[1 + \frac{\partial\mathcal{T}_e}{\partial t_A}\right]^{-1} = c\frac{\partial\mathcal{T}_r}{\partial x^i_A}\,, \tag{5}$$

$$\left(\hat{k}_i\right)_B = \left(\frac{k_i}{k_0}\right)_B = -c\frac{\partial\mathcal{T}_e}{\partial x^i_B} = -c\frac{\partial\mathcal{T}_r}{\partial x^i_B}\left[1 - \frac{\partial\mathcal{T}_r}{\partial t_B}\right]^{-1}\,, \tag{6}$$

$$\frac{(k_0)_A}{(k_0)_B} = 1 + \frac{\partial\mathcal{T}_e}{\partial t_A} = \left[1 - \frac{\partial\mathcal{T}_r}{\partial t_B}\right]^{-1}\,, \tag{7}$$

It's then straightforward to define the one-way frequency shift (1) as a function of $\mathcal{T}_{e/r}$ and its partial derivatives.

Post-Minkowskian expansion of the frequency shift

In a recent article [3], one of us gives the expression of $\mathcal{T}_{e/r}$ as a formal post-Minkowskian series. In this communication, we focus on $\mathcal{T}_r$, but similar considerations hold for $\mathcal{T}_e$. $\mathcal{T}_r$ can be written in ascending power of G defined as

$$\mathcal{T}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \frac{R_{AB}}{c} + \sum_{n=1}^{\infty}\frac{\Delta_r^{(n)}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B)}{c} \tag{8}$$

where $\Delta^{(n)}_{e/r}$ is of the order $\mathcal{O}(h^n)$ and $R^i_{AB} = x^i_B - x^i_A$, $R_{AB} = |R^i_{AB}|$ and $N^i = \frac{R^i_{AB}}{R_{AB}}$.
Then, we can express the one-way frequency shift (2) as follows (in agreement with expression in [5])

$$\frac{\nu_B}{\nu_A} = \frac{\left[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j\right]^{1/2}_A}{\left[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j\right]^{1/2}_B} \times \frac{1 - N^i\beta^i_B - \beta^i_B\frac{\partial\Delta_r}{\partial x^i_B} - \frac{\partial\Delta_r}{\partial t_B}}{1 - N^i\beta^i_A + \beta^i_A\frac{\partial\Delta_r}{\partial x^i_A}}. \tag{9}$$

The goal of this work is to provide a new way of computing a general form for the derivatives of the time delay function up to the post-post Minkowskian order.
In order to do so, we rewrite $\Delta^{(1)}_r$ and $\Delta^{(2)}_r$ given in [3] as

$$\Delta^{(1)}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \frac{R_{AB}}{2}\int_0^1\left[g^{00}_{(1)} - 2N^i g^{0i}_{(1)} + N^iN^j g^{ij}_{(1)}\right]_{z^\alpha(\lambda)} d\lambda = \int_0^1 m_{(1)}(\lambda)d\lambda\,, \tag{10}$$

$$\Delta^{(2)}_r(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \int_0^1\left[\mathcal{I}_1(\lambda) + \mathcal{I}_2(\lambda) + \mathcal{I}_3(\lambda)\right]d\lambda \tag{11}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= m_{(2)}(\lambda) - \Delta^{(1)}_r(\boldsymbol{z}(\lambda), t_b, \boldsymbol{x}_b)\, m_{(1),0}(\lambda)\quad,\\
\mathcal{I}_2 &= \left[R_{AB}\, g^{0i}_{(1)} - R^k_{AB}\, g^{ik}_{(1)}\right]_{z^\alpha(\lambda)}\frac{\partial\Delta^{(1)}_r}{\partial x^i}(\boldsymbol{z}(\lambda))\quad,\\
\mathcal{I}_3 &= -\frac{R_{AB}}{2}\sum_{j=1}^{3}\left[\frac{\partial\Delta^{(1)}_r}{\partial x^j}(\boldsymbol{z}(\lambda))\right]^2
\end{aligned} \tag{12}$$

and we have defined the null straight line $z^\alpha(\lambda) = (x^0_B - \lambda R_{AB}, x^i_B - \lambda R^i_{AB})$ and

$$m_{(n),\alpha}(\lambda) = \frac{R_{AB}}{2}\left[g^{00}_{(n),\alpha} - 2N^i g^{0i}_{(n),\alpha} + N^iN^j g^{ij}_{(n),\alpha}\right]_{z^\alpha(\lambda)}\quad,\quad \frac{\partial\Delta^{(1)}_r}{\partial x^i}(\boldsymbol{z}(\lambda)) = \int_0^1\left[m_{(1),\alpha}(\lambda\mu)z^\alpha_{A,i}(\lambda\mu) + \tilde{h}^i_{(1)}(\lambda\mu)\right]d\mu$$

The derivatives of the first PM order of the delay function can be then easily calculated

$$\frac{\partial\Delta^{(1)}_r}{\partial x^i_{A/B}}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \int_0^1\left[m_{(1),\alpha}(\lambda)z^\alpha_{A/B,i}(\lambda) \pm \tilde{h}^i_{(1)}(\lambda)\right]d\lambda\quad,\quad \frac{\partial\Delta^{(1)}_r}{\partial t_B}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \int_0^1\left[m_{(1),0}(\lambda)\; c\right]d\lambda. \tag{13}$$

The same for the 2PM order, even if resulting in more complex formulas

$$\frac{\partial\Delta^{(2)}_r}{\partial x^i_{A/B}}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \int_0^1\left[\frac{\partial\mathcal{I}_1}{\partial x^i_{A/B}} + \frac{\partial\mathcal{I}_2}{\partial x^i_{A/B}} + \frac{\partial\mathcal{I}_3}{\partial x^i_{A/B}}\right]d\lambda\quad,\quad \frac{\partial\Delta^{(2)}_r}{\partial t_B}(\boldsymbol{x}_A, t_B, \boldsymbol{x}_B) = \int_0^1\left[\frac{\partial\mathcal{I}_1}{\partial t_B} + \frac{\partial\mathcal{I}_2}{\partial t_B} + \frac{\partial\mathcal{I}_3}{\partial t_B}\right]d\lambda \tag{14}$$

where the derivatives can be expressed as follows

$$\begin{aligned}
\frac{\partial\mathcal{I}_1}{\partial x^i_{A/B}} &= m_{(2),\alpha}z^\alpha_{A/B,i} \pm \tilde{h}^i_{(2)} - \Delta^{(1)}_r(z(\lambda))\left[m_{(1),0\alpha}z^\alpha_{A/B,i} \pm\ \tilde{h}^i_{(1),0}\right] - m_{(1),0}\frac{\partial\Delta^{(1)}_r}{\partial x^i_{A/B}}(\boldsymbol{z}(\lambda))\\
\frac{\partial\mathcal{I}_2}{\partial x^i_{A/B}} &= \left[\mp N^i_{AB}g^{0j}_{(1)} \pm g^{ij}_{(1)} + (R_{AB}g^{0j}_{(1),\alpha} - g^{jk}_{(1),\alpha}R^k_{AB})z^\alpha_{A/B,i}\right]\frac{\partial\Delta^{(1)}_r}{\partial x^j}(\boldsymbol{z}(\lambda)) + \left[R_{AB}g^{0j}_{(1)} - R^k_{AB}g^{jk}_{(1)}\right]\frac{\partial^2\Delta^{(1)}_r}{\partial x^i_{A/B}\partial x^j}(\boldsymbol{z}(\lambda))\\
\frac{\partial\mathcal{I}_3}{\partial x^i_{A/B}} &= \pm\frac{N^i_{AB}}{2}\sum_{j=1}^{3}\left(\frac{\partial\Delta^{(1)}_r}{\partial x^j}(\boldsymbol{z}(\lambda))\right)^2 - R_{AB}\sum_{j=1}^{3}\left[\frac{\partial\Delta^{(1)}_r}{\partial x^j}(\boldsymbol{z}(\lambda))\frac{\partial^2\Delta^{(1)}_r}{\partial x^i_{A/B}\partial x^j}(z(\lambda))\right],
\end{aligned} \tag{15}$$

all quantities being taken at (λ) and where we define

$$\frac{\partial^2\Delta^{(1)}_r}{\partial x^i_{A/B}\partial x^j}(\boldsymbol{z}(\lambda)) = \int_0^1\left[\ m_{(1),\alpha\beta}z^\alpha_{A,j}z^\beta_{A/B,i} \pm \tilde{h}^i_{(1),\alpha}z^\alpha_{A,j} + m_{(1),\alpha}z^\alpha_{AA/AB,ji} + \tilde{h}^j_{(1),\alpha}z^\alpha_{A/B,i} \pm \bar{h}^{ji}_{(1)}\ \right]_{\lambda\mu}d\mu,$$

$$\tilde{h}^i_{(n)}(g^{\mu\nu}, N^i_{AB}) = \frac{\partial m_{(i),j}}{\partial x^k_A}\Big|_{z^\alpha=const}\quad,\quad \bar{h}^{ik}_{(n)}(g^{\mu\nu}, N^i_{AB}) = \frac{\partial\tilde{h}^i_{(n)}}{\partial x^k_A}\Big|_{z^\alpha=const}\quad.$$

Similar expressions can be written for $\partial\mathcal{I}_i/\partial t_B$.