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Contents

Introduction

A new context was introduced in [**25**] providing an efficient way of unifying and comparing results from pointed and non-pointed categorical algebra. The important notion in this approach is that of *ideal* of morphisms in the sense of C. Ehresmann [**16**]: a class $\mathcal{N}$ of morphisms in a category $\mathcal{C}$ is an ideal when, for a composable pair of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$, the composite gf belongs to the class $\mathcal{N}$ whenever either f or g is in $\mathcal{N}$. The concept of a category equipped with an ideal of morphisms has been extensively used by M. Grandis to developed his "categorical foundation of homological and homotopical algebra" (see [**31**] and the references therein). In [**25**] categories $\mathcal{C}$ equipped with an ideal of morphisms $\mathcal{N}$ are called *multi-pointed categories*, and are denoted by the pair $(\mathcal{C}, \mathcal{N})$. This context can be seen as a generalization of the so-called *pointed context*, which is captured by choosing the ideal $\mathcal{N}$ to be the class of zero morphisms of a pointed category; the non-pointed context is captured by choosing $\mathcal{N}$ to be the class of all morphisms of a category, and this situation is refered to as the *total context*. The study of general multi-pointed categories can be seen as a simultaneous treatment of pointed and non-pointed categories. In this context, the notion of *star-relation* plays a central role: a *star* is just a parallel pair of morphisms

$$X \underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}} Y$$

of $\mathcal{C}$ such that δ_1 belongs to $\mathcal{N}$. When studying a general multi-pointed category one usually assumes that the kernels with respect to the class $\mathcal{N}$ exist: this means that given any arrow $f\colon X \to Y$ in $\mathcal{C}$, there is an arrow $k\colon K \to X$ such that the composite fk belongs to $\mathcal{N}$ and the arrow k is universal with this property. In this case, when the category is finitely complete, any arrow in $\mathcal{C}$ turns out to have a *kernel star*. In the pointed context, the notion of kernel star gives the usual one of *kernel* of a morphism, while, in the total context, it becomes the *kernel pair* of a morphism.

This allows one to define a general notion of *star-exact sequence*, extending both the classical notion of short exact sequence (in the pointed context) and the one of exact fork (in the total context). Recall that a short exact sequence in a pointed finitely complete category with zero object 0 is a diagram

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 0$$

where $k = \mathsf{ker}\, f$ and $f = \mathsf{coker}\, k$.

In the total context, short exact sequences are naturally replaced by "exact forks": these are diagrams of the form

$$Eq(f) \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \overset{f}{\twoheadrightarrow} Y$$

in which (r_1, r_2) is the kernel pair of f and $f = \mathsf{coeq}\,(f_1, f_2)$.

These two notions are then unified in the context of star-relations by that of *star-exact sequence* **[23]**, which is a diagram

$$Eq(f)^* \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} X \overset{f}{\twoheadrightarrow} Y$$

where $\kappa = (\kappa_1, \kappa_2)$ is the kernel star of f and f is the coequaliser of (κ_1, κ_2).

Several independent results from pointed and non-pointed algebra can be brought together in multi-pointed categories. For example, some fundamental diagram lemmas, such as the 3×3 *lemma* and the *short five lemma* for star-exact sequences have been investigated by M. Gran, Z. Janelidze and D. Rodelo **[23]** in the realm of *star-regular* categories **[25]**, a common generalization of regular categories (in the total context) and of normal categories **[44]** (in the pointed context). A star-regular category is a regular multi-pointed category $(\mathcal{C}, \mathcal{N})$ in which every morphism has an $\mathcal{N}$-kernel and every regular epimorphism is a coequaliser of a star.

The notion of *ideal* in a variety of universal algebras with a constant 0 has been extended to a categorical context in **[38]**, and is defined as a regular image of a normal monomorphism. A normal variety whose theory has a unique constant 0 is *ideal determined* in the sense of **[33]** (or "has a good theory of ideals" **[56]**) when any ideal is normal, i.e. the 0-class of a congruence. This notion of ideal was extended to the case of pointed categories in **[38]** and it was defined as a monomorphism arising as the regular image of a kernel. The notion of ideal is defined in **[25]** in a star-regular category as an internal relation occurring as the regular image of a kernel star. A star-regular category is said to have a *good theory of ideals* **[25]** when any ideal is a kernel star. In the pointed context, having a good theory of ideals becomes the notion of an ideal determined category **[37]**, i.e. a normal category in which any ideal is a normal monomorphism. In the total context, categories with a good theory of ideals become exact Goursat categories in the sense of **[14]**.

The goal of this thesis is to study two fundamental aspects of star-regular categories.

First of all, we show that star-regular categories are suitable to establish a general version of the Noether isomorphism theorems and of the Zassenhaus Lemma, well known in group theory. The isomorphism theorems are the fundamental diagram lemmas in a category providing some important relationships between subobjects, quotients and homomorphisms. Various versions of these theorems exist in the categories Gp of groups, Rng of rings and R-Mod of modules over a fixed ring R, and in more general contexts. It is then interesting to look at the problem of establishing the isomorphism theorems in a star-regular category. It turns out that this is possible, and the general version of the isomorphism theorems we obtain in this thesis include not only the ones considered in the pointed context by O. Wyler [**57**] and by F. Borceux and D. Bourn [**5**], but also the ones first considered by W. Tholen [**54**] in the total context. These results are then used to establish a general form of the Zassenhaus Lemma in any star-regular category satisfying an additional mild condition which holds true in both the pointed and the total contexts.

The second aspect of star-regular categories we investigate in this thesis concerns *descent theory.* It is well-known that regular epimorphisms are *descent* morphisms in any regular category, but they fail to be effective descent morphisms, in general. This failure never happens when the category is exact [**2**], as observed by G. Janelidze and W. Tholen [**42**]: in an exact category the effective descent morphisms are the same as regular epimorphisms. Accordingly, the regular categories in which regular epimorphisms are effective descent morphisms can be considered as being *almost exact*, in the terminology of G. Janelidze and M. Sobral in [**40**]. In this work we find a sufficient condition on a star-regular category guaranteeing that regular epimorphisms are effective descent morphisms. A star-regular category satisfying this condition is called *semi-effective.* There are many interesting examples of semi-effective star-regular categories, such as any ideal determined category, any almost abelian in the sense of [**53**], and any efficiently regular category in the sense of [**10**].

Structure of the text

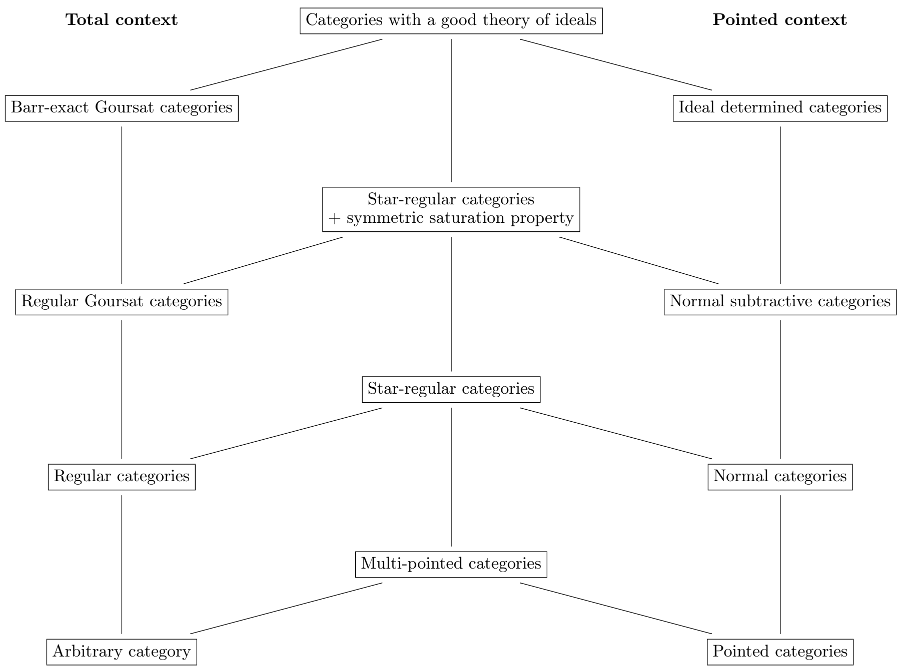
Chapter 1. We give an overview of the categorical structures needed in this thesis, such as *regular*, *Barr-exact*, *normal*, *Goursat*, *ideal determined* categories, and the relevant notions therein: equivalence relations, normal monomorphisms, Bourn-normal monomorphisms, clots and ideals. We then set the context of our work by recalling the definitions of multi-pointed categories, star-regular categories and categories with a good theory of ideals, among other ones. The material presented in this chapter is mainly based on [**25**]. The most interesting new result here is the characterization of monomorphisms in a star-regular category in terms of the triviality of their kernel stars (Lemma 1.1.31): a morphism $f\colon X \to Y$ is a monomorphism if and only if its kernel star $Eq(f)^*$ is Δ_X^*, the kernel star of the identity 1_X. In the pointed context, this gives the well-known characterization of monomorphisms in a normal category as those morphisms $f\colon X \to Y$ such that $\mathsf{Ker}\,(f) \cong 0$.

Chapter 2. In this chapter we unify and generalize the isomorphism theorems and the Zassenhaus Lemma to the context of star-regular categories. To establish the two isomorphism theorems, we define an "asymmetric join" between a kernel star and a monomorphism (see Definition 2.1.1). The *diamond isomorphism theorem* (sometimes called the second isomorphism theorem) is established in a star-regular category. To express the *double quotient isomorphism theorem* (also referred to as the third isomorphism theorem in the literature) and the Zassenhaus Lemma, we need to introduce a mild condition on a star-regular category $\mathcal{C}$, called *property* (C) (Definition 2.1.4), which holds true in both the pointed and the total contexts. Moreover, in a star-regular category, this property (C) guarantees the existence, for a regular epimorphism $f\colon X \to Y$, of a lattice isomorphism between the lattice of kernel stars on X

containing the kernel star $Eq(f)^*$ of f and the lattice of kernel stars on Y (see Section 2.3.1). This result is sometimes referred to as the "fourth isomorphism theorem" in the literature. The connection between the property (C) and a property of regular diamonds introduced in **[24]** is also analysed. The Zassenhaus Lemma is finally established in any star-regular category (with pushouts of regular epimorphisms) satisfying the property (C). The content of the second chapter is essentially based on my preprint **[50]**.

Chapter 3. In the third chapter, we investigate a sufficient condition on a star-regular category guaranteeing that regular epimorphisms are effective descent morphisms. For this, we first identify a useful condition on a reflexive relation (later called $(*\pi_0)$ in **[22]**) which we show to be always satisfied in a star-regular category (Lemma 3.2.2). This observation and the notion of semi-effective star-regular category (Definition 3.2.4) are then used in the proof of the main result, Theorem 3.2.8. We conclude the thesis by examining many examples of semi-effective star-regular categories. The main results of this chapter have been published in the article **[26]** written in collaboration with M. Gran.

We end this Introduction with a small diagram summarizing the different contexts considered so far in the new approach to categorical algebra using star-relations:



CHAPTER 1

Categories with a good theory of ideals

In this chapter, we first recall several notions about regular categories and normal categories. Then we move on to the notion of (regular) multi-pointed categories and of star-regular categories: the concept unifying both notions of regular categories and normal categories and that will play a central role in this thesis. We set almost all the notions and properties in these categories that will be useful in later chapters. Finally, in the last section, before giving the definition of a category with a good theory of ideals, we first recall the notions of Barr-exact Goursat categories and ideal determined categories which are unified by this context of categories with a good theory of ideals. The material presented here in the realm of star-relations is mainly based on the paper of M. Gran, Z. Janelidze and A. Ursini [**25**].

1.1. Star-regular categories

1.1.1. Regular and normal categories.

Recall that a *kernel pair* of a morphism $f\colon X \to Y$ is a pair of parallel morphisms $Eq(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} X$ obtained via the following pullback

$$\begin{tikzcd} Eq(f) \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition 1.1.1. A finitely complete category $\mathcal{C}$ is *regular* [**2**] when

(a) $\mathcal{C}$ has coequalisers of kernel pairs;
(b) regular epimorphisms are stable under pullbacks in $\mathcal{C}$. That is, given a pullback diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "v"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

where f is regular epimorphism, then g is a regular epimorphism.

This definition of regular category is equivalent with the following one:

Definition 1.1.2. A finitely complete category $\mathcal{C}$ is a regular category when

(a) any arrow $f\colon X \to Y$ in $\mathcal{C}$ can be factorised as $f = me$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

with e a regular epimorphism and m a monomorphism;
(b) these factorisations are pullback-stable in $\mathcal{C}$.

Since monomorphisms are always pullback-stable, the property (b) reduces to the stability of regular epimorphisms under pullbacks. In the category Set of sets, and more generally in any variety of universal algebras, an arrow is a regular epimorphism precisely when it is surjective. Accordingly, the factorisation in (a) is the usual one of a function (or a homomorphism) as a surjection $e\colon X \to I$ onto its direct image $I = \mathsf{Im}(f) = \{f(x) \mid x \in X\}$ followed by the inclusion $m\colon \mathsf{Im}(f) \to Y$ of its image in its codomain Y. Furthermore, surjective functions (and surjective homomorphisms) are clearly pullback-stable, so that these categories are all regular. For many purposes the notion of regular epimorphism provides a suitable abstraction of the notion of *quotient* in the context of regular categories.

Let us recall the following definitions about (internal) relations in a category $\mathcal{C}$.

Definition 1.1.3. Let $\mathcal{C}$ be a category with finite limits.

(a) An *(internal) relation* R on an object X is a pair of parallel arrows (r_1, r_2) represented by the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

which is jointly monomorphic, in the sense that for any pair of parallel arrows $\begin{tikzcd} U \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & R \end{tikzcd}$ such that $r_i u = r_i v; i = 1, 2$; one then has $u = v$. Equivalently, this means that the factorisation $(r_1, r_2)\colon R \to X \times X$ is a monomorphism.

(b) The relation $\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$ is *reflexive* if there is a morphism $\delta\colon X \to R$

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \\ X \arrow[u, dashed, "\delta"] \arrow[ur, shift left, "1_X"] \arrow[ur, shift right, "1_X"'] & \end{tikzcd}$$

such that $r_1 \circ \delta = 1_X$ and $r_2 \circ \delta = 1_X$. This is equivalent to saying that the diagonal $(1_X, 1_X)\colon X \to X \times X$ factors through the morphism $(r_1, r_2)\colon R \to X \times X$.

(c) The relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ is *symmetric* if there is a morphism $\sigma\colon R \to R$

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \\ R \arrow[u, dashed, "\sigma"] \arrow[ur, shift left, "r_2"] \arrow[ur, shift right, "r_1"'] & \end{tikzcd}$$

such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$.

(d) Let $(R \times_X R, p_1, p_2)$ denote the pullback of r_1 along r_2:

$$\begin{tikzcd} R \times_X R \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X. \end{tikzcd} \tag{1.1}$$

The relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ is said to be *transitive* if there is a morphism $\tau\colon R \times_X R \to R$ such that $r_1 \circ \tau = r_1 \circ p_1$ and $r_2 \circ \tau = r_2 \circ p_2$.

An *(internal) equivalence relation* is a reflexive, symmetric and transitive relation. An equivalence relation is called *effective* when it is the kernel pair of some morphism. In this case, it is necessarily an internal equivalence relation.

A category $\mathcal{C}$ is *pointed* if $\mathcal{C}$ has an initial object which is also terminal. In this case, this object is called the zero object and is denoted by 0. In such a category, for two objects X and Y in $\mathcal{C}$, the unique morphism $f\colon X \to Y$ which factors through 0 is called a zero morphism and denoted by 0_{XY}, or just by 0. The *cokernel* $Y \xrightarrow{q} \mathrm{Coker} f$ of a morphism $f\colon X \to Y$ is defined by the pushout

$$\begin{tikzcd} X \arrow[r, "\tau_X"] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r, "q"'] & \mathrm{Coker} f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

of f along the terminal arrow $\tau_X\colon X \to 0$. Any cokernel is a regular epimorphism: it is the coequaliser of the parallel pair $X \overset{f}{\underset{0}{\rightrightarrows}} Y$. A morphism which is a cokernel of some morphism is called a normal epimorphism.

Definition 1.1.4. [**44**] A finitely complete pointed category $\mathcal{C}$ is a *normal* category when:

- any arrow $f\colon X \to Y$ in $\mathcal{C}$ has a factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & I \arrow[ur, tail, "i"'] & \end{tikzcd}$$

with q a normal epimorphism and i a monomorphism;

- these factorisations are pullback-stable.

Equivalently, a normal category is a pointed regular category in which every regular epimorphism is a normal epimorphism.

We shall use the following convention: arrows depicted in diagrams (or somewhere else) by

$$\cdot \rightarrowtail \cdot \qquad \cdot \rhd\!\!\longrightarrow \cdot \qquad \cdot \twoheadrightarrow \cdot \qquad \cdot \longrightarrow\!\!\rhd \cdot$$

will represent respectively monomorphism, normal monomorphism, epimorphism and normal epimorphism.

1.1.2. Normal monomorphisms, Bourn-normal monomorphisms, clots.

Definition 1.1.5. The *kernel* $k\colon \mathsf{Ker}\, f \rhd\!\!\longrightarrow X$ of a morphism $f\colon X \to Y$ is defined by the pullback

$$\begin{tikzcd} \mathsf{Ker}\, f \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ 0 \arrow[r, "\alpha_Y"'] & Y \end{tikzcd}$$

of f along the initial arrow $\alpha_Y : 0 \to Y$. Equivalently, one defines the kernel $k\colon \mathsf{Ker}\, f \rhd\!\!\longrightarrow X$ of f as the equaliser of the pair of arrows $X \overset{f}{\underset{0}{\rightrightarrows}} Y$. A morphism which is the kernel of some morphism is called a *normal monomorphism*.

In a variety of universal algebras, a kernel is a normal subalgebra and can be seen as a 0-class of a congruence. Let us recall the notion of normal monomorphism in the sense of D. Bourn [**7**].

Definition 1.1.6. [**7**] Let $\mathcal{C}$ be a finitely complete category. An arrow $f\colon X \to Y$ is said to be *normal to an equivalence relation* $S \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y$ on Y when:

- the inverse image $f^{-1}(S)$ of S along f is the largest equivalence relation $X \times X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$ on X;
- there is a map $\overline{f}\colon X \times X \to S$ making the diagram

$$\begin{tikzcd} X \times X \arrow[r, "\overline{f}"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commute, and moreover the induced arrow $(X \times X, \pi_1, \pi_2) \to (S, s_1, s_2)$ in the category of internal equivalence relations in $\mathcal{C}$ is a discrete fibration (over the equivalence relation S). This means that both the commutative squares involving the first and the second projections are pullbacks.

As shown in [**7**], such an arrow f is necessarily a monomorphism.

Proposition 1.1.7. [**7, 11**] *The following conditions hold in a finitely complete pointed category $\mathcal{C}$.*

(1) *An arrow f is a kernel if and only if f is normal to an effective equivalence relation.*
(2) *Any equivalence relation determines a canonical normal subobject.*

Proof. To prove (1), assume that $f\colon K \to X$ is the kernel of a morphism $g\colon X \to Y$. Take $\mathsf{Eq}(g)$ the kernel pair of g and consider the following diagram

$$\begin{tikzcd} \mathsf{Ker}\, g_1 \arrow[r, "h"] \arrow[d] & \mathsf{Eq}(g) \arrow[r, "g_2"] \arrow[d, "g_1"'] & X \arrow[d, "g"] \\ 0 \arrow[r] & X \arrow[r, "g"'] & Y \end{tikzcd}$$

where h is the kernel of g_1. Both the left-hand square and the right-hand squares are pullbacks, this implies that the outer diagram is a pullback as well. Thus $\mathsf{Ker}\, g_1 \cong K$ and g_2h is the kernel of g. Finally one checks that g_2h is normal to the kernel pair $\mathsf{Eq}(g)$ of g. For the converse, if f is normal to an effective equivalence relation $\mathsf{Eq}(g)$, one uses the Barr-Kock Theorem [**2, 5**] to prove that f is the kernel of g.

(2). First recall that any equivalence relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ on X has the property that $R \times_X R \underset{\tau}{\overset{p_1}{\rightrightarrows}} R$ is the kernel pair of r_1 and $R \times_X R \underset{p_2}{\overset{\tau}{\rightrightarrows}} R$ is the kernel pair of r_2 with p_1, p_2 and τ as in Definition 1.1.3(d). Thus, in the following commutative diagram

$$\begin{tikzcd} \mathsf{Ker}\, r_1 \times \mathsf{Ker}\, r_1 \arrow[r, "\alpha"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & R \times_X R \arrow[r, "p_2"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\tau"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ \mathsf{Ker}\, r_1 \arrow[r, "k"'] \arrow[d] & R \arrow[r, "r_2"'] \arrow[d, "r_1"] & X \\ 0 \arrow[r] & X & \end{tikzcd}$$

both the top right-hand squares and left-hand squares are discrete fibrations, with k the kernel of r_1. This proves that r_2k is normal to the equivalence relation R. □

Definition 1.1.8. [**45, 39**] Let $\mathcal{C}$ be a pointed, finitely complete category. A monomorphism $m\colon M \to A$ is a *clot* if there is a reflexive relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$

on A such that $m = r_2 k$

$$\begin{tikzcd} M \arrow[r, "k"] \arrow[d] \arrow[rr, bend left, "m"] & R \arrow[r, "r_2"] \arrow[d, "r_1"] & A \arrow[d, equal] \\ 0 \arrow[r] & A \arrow[r, equal] & A \end{tikzcd}$$

where $k\colon M \to R$ is the kernel of r_1.

Notice that the first categorical definition of a clot is given in [**38**] in a pointed category with finite coproducts and kernels: that of a monomorphism $m\colon M \to A$ which is *closed under the conjugation action.* In [**39**], the authors prove that this definition is equivalent to Definition 1.1.8 when the category $\mathcal{C}$ is regular. In a variety of universal algebras, this notion of clot existed many years before, and it was introduced by A. Ursini and P. Agliano in [**1**], defined as a 0-class of a semi-congruence i.e., a compatible reflexive relation. A Bourn-normal monomorphism is a 0-class of an equivalence relation and a normal monomorphism is a 0-class of a effective equivalence relation. Then, if we denote by $N(A)$, B-$N(A)$ and $Cl(A)$ the classes of normal monomorphisms, Bourn-normal monomorphisms and clots, on the same object A, respectively, we have the obvious inclusions

$$N(A) \subseteq B\text{-}N(A) \subseteq Cl(A) \tag{1.2}$$

in any pointed category $\mathcal{C}$.

1.1.3. Multi-pointed categories.
In this section, we set the basic notions and results in the framework of star-relations we shall need in the subsequent chapters.

Definition 1.1.9. A multi-pointed category [**25**] is a pair $(\mathcal{C}, \mathcal{N})$ where $\mathcal{C}$ is a category and $\mathcal{N}$ is a class of morphisms of $\mathcal{C}$ that forms an *ideal* in the sense of C. Ehresmann [**16**]: for a composable pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathcal{C}$, the composite fg belongs to $\mathcal{N}$ if either f or g belongs to $\mathcal{N}$.

This notion of category equipped with an ideal of morphisms has been used by M. Grandis as a basis for his categorical foundations of homological and homotopical algebra (see [**31, 32**]). The following two examples of multi-pointed categories will be the guiding ones for our work:

Example 1.1.10.
(1) Any category $\mathcal{C}$ can be seen as a multi-pointed category by choosing the ideal $\mathcal{N}$ to be the class of all morphisms in this category; in this case, one calls it the *total context*.
(2) When $\mathcal{C}$ is a pointed category with a zero object 0, by choosing $\mathcal{N}$ to be the class of zero morphisms in $\mathcal{C}$, $\mathcal{N}$ is then an ideal of morphisms of $\mathcal{C}$, and $(\mathcal{C},\mathcal{N})$ is a multi-pointed category. This context is called *the pointed context*.

Besides the pointed and the total contexts, one has the so-called *proto-pointed context* introduced in [**25**]: this is the situation of a regular category $\mathcal{C}$

in which $\mathcal{N}$ is the class of those morphisms $f\colon X \to Y$ whose regular image is the smallest subobject of Y.

Recall that a finitely complete category is quasi-pointed [**8**] if it has an initial object 0, a terminal object 1, and the unique arrow $0 \to 1$ is a monomorphism. As explained in [**23**] any quasi-pointed category provides an example of proto-pointed context: it suffices to choose for $\mathcal{N}$ the class of morphisms that factor through the initial object.

The category Set of sets and maps between sets is quasi-pointed. Besides this example, for us, an interesting class of non-trivial examples is formed by the categories of internal groupoids over a fixed base object (see for instance [**8**]). Let $\mathcal{C}$ be a finitely complete category and X a fixed object of $\mathcal{C}$. Consider the category $\mathsf{Gpd}_X(\mathcal{C})$ of internal groupoids in $\mathcal{C}$ over the fixed object X of objects. Then the category $\mathsf{Gpd}_X(\mathcal{C})$ is quasi-pointed: indeed, the initial object is the discrete groupoid $\begin{tikzcd} X \arrow[r, shift left=2, "1_X"] \arrow[r, shift right=2, "1_X"'] & X \arrow[l, dashed] \end{tikzcd}$ on X, and the terminal object is the indiscrete groupoid $\begin{tikzcd} X \times X \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, dashed] \end{tikzcd}$ on X, and the canonical arrow φ

$$\begin{tikzcd} X \arrow[rr, dotted, "\varphi"] \arrow[dr, shift left] \arrow[dr, shift right] & & X \times X \arrow[dl, shift left] \arrow[dl, shift right] \\ & X \arrow[ul, dashed] \arrow[ur, dashed] & \end{tikzcd}$$

is a monomorphism.

We shall notice that the notion of quasi-pointed category is stable under "slicing". Recall that for a category $\mathcal{C}$ and a fixed object I of $\mathcal{C}$, the slice category $(\mathcal{C} \downarrow I)$ is defined as follows:

(1) objects are pairs (A, p_A) where A is an object of $\mathcal{C}$ and $p_A : A \to I$ is a morphism of $\mathcal{C}$,
(2) an arrow $f\colon (A, p_A) \to (B, p_B)$ is an arrow $f\colon A \to B$ of $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_A"'] & & B \arrow[dl, "p_B"] \\ & I & \end{tikzcd}$$

and the composition of morphisms in $(\mathcal{C} \downarrow I)$ is given by their composition in $\mathcal{C}$. When $\mathcal{C}$ is a quasi-pointed category with initial object 0 and terminal object 1, then the slice category $(\mathcal{C} \downarrow I)$ is also a quasi-pointed category with initial object $(0, \alpha_I)$ where $\alpha_I\colon 0 \to I$ is the unique morphism from the initial object. The terminal object in $(\mathcal{C} \downarrow I)$ is given by the pair $(I, 1_I)$ and $(0, \alpha_I) \to (I, 1_I)$ is a monomorphism in $(\mathcal{C} \downarrow I)$.

An example of a multi-pointed category in the proto-pointed context which is not a quasi-pointed category is given by the category CRng of commutative unitary rings where the initial object is the ring of integers $\mathbb{Z}$ and the terminal object is the trivial ring.

For a multi-pointed category $(\mathcal{C}, \mathcal{N})$, a *star* in $\mathcal{C}$ is a pair of parallel morphisms

$$\sigma = (\sigma_1, \sigma_2) : \ X \rightrightarrows Y$$

where $\sigma_1 \in \mathcal{N}$. In the pointed context, the notion of a star essentially becomes a morphism of the category (its second component, since the first one is a zero morphism), while in the total context, a star is a pair of parallel morphisms of the category.

The composition of stars and morphisms is possible: it yields a star again and it is defined in a natural way as follows. Given a star $\sigma : Y \rightrightarrows Z$ and two morphisms $f : X \to Y$ and $g : Z \to T$ in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_2"'] & Z \arrow[r, "g"] & T, \end{tikzcd}$$

the composite $g\sigma f = (g\sigma_1 f, g\sigma_2 f)$ is a star.

From now, we shall assume that the category $\mathcal{C}$ is finitely complete.

An $\mathcal{N}$*-kernel* of a morphism $f : X \to Y$ in $\mathcal{C}$ is a morphism $k : K \to X$ in $\mathcal{C}$ such that the composite fk belongs to $\mathcal{N}$ and k is universal with this property i.e., for any other morphism $l : L \to X$ in $\mathcal{C}$ such that $fl \in \mathcal{N}$, there exists a unique morphism $u : L \to K$ such that $ku = l$

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "f"] & Y \\ L \arrow[u, "u"] \arrow[ur, "l"'] & & \end{tikzcd}$$

Such an $\mathcal{N}$-kernel is always a monomorphism. In the pointed context, an $\mathcal{N}$-kernel of a morphism is the usual kernel of this morphism, whereas in the total context, the $\mathcal{N}$-kernel of a morphism $f : X \to Y$ reduces to the identity on X.

A *star-relation* is a star $\sigma = (\sigma_1, \sigma_2) : X \rightrightarrows Y$ which is a jointly monic pair. In the pointed context, this gives the notion of a monomorphism; and in the total context, a star-relation becomes a relation.

A *kernel star* of a morphism $f : X \to Y$ is a star $\sigma : K \rightrightarrows X$ having the property that $f\sigma_1 = f\sigma_2$ and for any other star $\delta : Z \rightrightarrows X$ verifying $f\delta_1 = f\delta_2$, then there exists a unique morphism $u : Z \to K$ such that $\sigma_1 u = \delta_1$ and $\sigma_2 u = \delta_2$

$$\begin{tikzcd} K \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_2"'] & X \arrow[r, "f"] & Y \\ Z \arrow[u, "u"] \arrow[ur, shift left, "\delta_1"] \arrow[ur, shift right, "\delta_2"'] & & \end{tikzcd}$$

A kernel star is always a star-relation. In the pointed context, a kernel star of a morphism gives the notion of kernel, while in the total context, this gives the notion of a kernel pair.

When $\mathcal{N}$-kernels exists, any pair $\varrho : (\varrho_1, \varrho_2) : R \rightrightarrows X$ has a corresponding star computed as follows: take $k : R^* \to R$ the $\mathcal{N}$-kernel of ϱ_1,

$$\begin{tikzcd}
R^* \arrow[rr, shift left, "\varrho_1 k"] \arrow[rr, shift right, "\varrho_2 k"'] \arrow[dr, "k"'] & & X \\
& R \arrow[ur, shift left, "\varrho_1"] \arrow[ur, shift right, "\varrho_2"'] &
\end{tikzcd}$$

then $(\varrho_1 k, \varrho_2 k) : R^* \rightrightarrows X$ is the star of the pair (ϱ_1, ϱ_2), usually called the star associated with ϱ and denoted by $(\varrho_1 k, \varrho_2 k) = (\varrho_1, \varrho_2)^* = \varrho^*$. Moreover, if ϱ is a relation, then ϱ^* is the largest subrelation of ϱ which is a star. Indeed, let $T \overset{\tau_1}{\underset{\tau_2}{\rightrightarrows}} X$ be another star contained in $R \rightrightarrows X$

$$\begin{tikzcd}
R^* \arrow[r, "k"] & R \arrow[r, shift left, "\varrho_1"] \arrow[r, shift right, "\varrho_2"'] & X \\
& T \arrow[ul, dashed, "m"] \arrow[u, "l"] \arrow[ur, shift left, "\tau_1"] \arrow[ur, shift right, "\tau_2"'] &
\end{tikzcd}$$

i.e. there exists a (mono)morphism l such that $\varrho_i l = \tau_i; i = 1, 2$. We have that $\varrho_1 l = \tau_1 \in \mathcal{N}$, and the universal property of the $\mathcal{N}$-kernel k gives us the induced monomorphism m such that $km = l$.

In particular, if $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$ is the discrete equivalence relation on X, we have $\Delta_X^* = (k_X, k_X) : K \rightrightarrows X$ the star associated with Δ_X, where $k_X : K \to X$ is the $\mathcal{N}$-kernel of 1_X.

In this way, when $\mathcal{C}$ is finitely complete, the kernel star of a morphism $f : X \to Y$ can be built as follows: take the kernel pair $\mathsf{Eq}(f) \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$ of f and $k : \mathsf{Eq}(f)^* \to \mathsf{Eq}(f)$ the $\mathcal{N}$-kernel of f_1

$$\begin{tikzcd}
\mathsf{Eq}(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, "f"] & Y \, , \\
\mathsf{Eq}(f)^* \arrow[u, "k"] \arrow[ur, shift left, "f_1 k"] \arrow[ur, shift right, "f_2 k"'] & &
\end{tikzcd}$$

then $\mathsf{Eq}(f)^* \overset{f_1 k}{\underset{f_2 k}{\rightrightarrows}} X$ is a kernel star of f; it is the largest subrelation of $\mathsf{Eq}(f)$ which is a star-relation. In this sense, we often denote the kernel star of a morphism f by $\mathsf{Eq}(f)^*$. In the pointed context, this gives the notion of a normal monomorphism associated with an effective equivalence relation (a normalisation of an effective equivalence relation, see Proposition 1.1.7(2)).

A *star-pullback* is a commutative square

$$\begin{tikzcd} X \arrow[r, shift left, "\sigma"] \arrow[r, shift right] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, shift left, "\tau"] \arrow[r, shift right] & T \end{tikzcd} \tag{1.3}$$

where $f\sigma = \tau g$ i.e. $f\sigma_1 = \tau_1 g$ and $f\sigma_2 = \tau_2 g$ satisfying the universal property: for any other star $\sigma' : X' \rightrightarrows Y$ and any other morphism $g' : X' \to Z$ such that $f\sigma'_1 = \tau_1 g'$ and $f\sigma'_2 = \tau_2 g'$, there exists a unique morphism $h : X' \to X$ such that $\sigma_1 h = \sigma'_1, \sigma_2 h = \sigma'_2$ and $gh = g'$

$$\begin{tikzcd} X' \arrow[dr, dotted, "h"] \arrow[drr, shift left, bend left, "\sigma'"] \arrow[drr, shift right, bend left] \arrow[ddr, bend right, "g'"'] & & \\ & X \arrow[r, shift left, "\sigma"] \arrow[r, shift right] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ & Z \arrow[r, shift left] \arrow[r, shift right, "\tau"'] & T. \end{tikzcd}$$

As it is the case in a pullback, when such a diagram (1.3) is a star-pullback, the pair (g, σ) is jointly monomorphic. In the pointed context, a star-pullback gives the usual notion of a pullback, while in the total context, it gives the notion of *joint-pullback*. Recall that a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ S \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & Y \end{tikzcd}$$

such that $fr_1 = s_1 g$ and $fr_2 = s_2 g$ is a joint-pullback (see for instance **[9]**) if the following diagram

$$\begin{tikzcd} R \arrow[r, "(r_1, r_2)"] \arrow[d, "g"'] & X \times X \arrow[d, "f \times f"] \\ S \arrow[r, "(s_1, s_2)"'] & Y \times Y \end{tikzcd}$$

is a pullback.

When $\mathcal{C}$ is finitely complete, the presence of $\mathcal{N}$-kernels guarantees the existence of kernel stars and of star-pullbacks. We have explained above why this is the case for kernel stars. Let us now consider the case of star-pullbacks. Given a star $\sigma = (\sigma_1, \sigma_2) : Z \rightrightarrows Y$ and a morphism $f : X \to Y$, the star-pullback of

σ along f can be obtained by first forming the joint-pullback

$$\begin{tikzcd} T \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_2"'] & Y \end{tikzcd}$$

and by then taking the star $(t_1, t_2)^* = (t_1 k, t_2 k) : T^* \rightrightarrows X$ of the pair (t_1, t_2), where $k : T^* \to T$ is the $\mathcal{N}$-kernel of t_1. Then the following diagram

$$\begin{tikzcd} T^* \arrow[r, shift left, "t_1 k"] \arrow[r, shift right, "t_2 k"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\sigma_2"'] & Y \end{tikzcd}$$

is easily checked to be the star-pullback of the star σ along the morphism f.

The following two lemmas are well-known, independently, in the total context and in the pointed context.

LEMMA 1.1.11. *Let $(\mathcal{C}, \mathcal{N})$ be a multi-pointed category with finite limits. The following conditions hold for a star-pullback*

$$\begin{tikzcd} X \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\sigma", yshift=8pt] \arrow[d, "\overline{f}"'] & Y \arrow[d, "f"] \\ Z \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\tau", yshift=8pt] & T \end{tikzcd}$$

in $\mathcal{C}$.

(1) *If τ is a star-relation, then σ is again a star-relation;*
(2) *if τ is a kernel star of a morphism g, then σ is the kernel star of the composite gf.*

LEMMA 1.1.12. **[23, 25]** *Consider a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\kappa'", yshift=8pt] \arrow[d, "e"'] & Y' \arrow[r, "f'"] \arrow[d, "v"] & Z' \arrow[d, "m"] \\ X \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\kappa"', yshift=-8pt] & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

of stars and morphisms.

(1) *Suppose κ is the kernel star of f and m is a monomorphism. Then, the left square is a star-pullback if and only if κ' is the kernel star of f'.*
(2) *Suppose f' is a coequaliser of κ' and e is an epimorphism. Then, the right square is a pushout if and only if f is a coequaliser of κ.*

Star-pullbacks and usual pullbacks compose as well:

Lemma 1.1.13. *In a commutative diagram*

$$\begin{tikzcd} X \arrow[r, Rightarrow, "\sigma"] \arrow[d, "u"'] & Y \arrow[r, "f"] \arrow[d, "v"'] & Z \arrow[d, "w"] \\ X' \arrow[r, Rightarrow, "\sigma'"'] & Y' \arrow[r, "f'"'] & Z' \end{tikzcd}$$

of stars σ and σ' and of morphisms f and f' in a multi-pointed category $\mathcal{C}$, if the right-hand square is a pullback and the outer diagram is a star-pullback, then the left-hand square is a star-pullback.

Proof. Let $\lambda\colon T \rightrightarrows Y$ be a star and $h\colon T \to X'$ a morphism such that $v\lambda = \sigma' h$. This implies that $f'v\lambda = f'\sigma' h$ and we then have $wf\lambda = f'\sigma' h$. The composite $f\lambda$ is a star and since the outer diagram is a star-pullback, there exists a unique morphism $g\colon T \to X$ such that $f\sigma g = f\lambda$ and $ug = h$. It remains to prove that $\sigma g = \lambda$. This follows from the fact that the pair (f, v) is jointly monomorphic. $\square$

In the same way, we have the following

Lemma 1.1.14. *In a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, Rightarrow, "\sigma"] \arrow[d, "v"'] & Z \arrow[d, "w"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, Rightarrow, "\sigma'"'] & Z' \end{tikzcd}$$

of stars and morphisms in a multi-pointed $\mathcal{C}$, where σ and σ' are stars:

1. *if the outer diagram is a star-pullback and the right-hand square is a star-pullback, then the left-hand square is a pullback;*
2. *if the left-hand square is a pullback and the right-hand square is a star-pullback, then the outer diagram is a star-pullback as well.*

As with usual pullbacks we have

Lemma 1.1.15. *Consider a commutative square of stars and morphisms*

$$\begin{tikzcd} \cdot \arrow[r, Rightarrow] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, Rightarrow] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, Rightarrow] & \cdot \end{tikzcd}$$

1. *if both the bottom squares and the top squares are star-pullbacks, then the outer squares are again a star-pullback;*

(2) *if the outer squares and the bottom squares are star-pullbacks, then the top squares is a star-pullback as well.*

Lemma 1.1.16. [**25**] *A subrelation of a star-relation is again a star-relation. This implies in particular that the relational meet of two star-relations is always a star-relation.*

Proof. If $\rho\colon R \rightrightarrows X$ is a star-relation and $\sigma\colon S \rightrightarrows X$ a relation such that $\sigma \leqslant \rho$, then σ factors through ρ i.e., there exists a morphism $u\colon S \to R$ such that $\rho_i u = \sigma_i$, $i = \{1, 2\}$. We then have that $\rho_1 u = \sigma_1 \in \mathcal{N}$ and σ is a star. Moreover, given two star-relations $\rho\colon R \rightrightarrows X$ and $\sigma\colon S \rightrightarrows X$ on X, their meet $\rho \wedge \sigma\colon R \cap S \rightrightarrows X$ is obtained in the pullback

$$\begin{tikzcd}
R \cap S \arrow[r, tail, "a"] \arrow[d, tail, "b"'] & S \arrow[d, tail, "(\sigma_1,\sigma_2)"] \\
R \arrow[r, tail, "(\rho_1,\rho_2)"'] & X \times X
\end{tikzcd}$$

which is the intersection of the two subobjects $(\rho_1, \rho_2)\colon R \to X \times X$ and $(\sigma_1, \sigma_2)\colon S \to X \times X$. Then $\rho \wedge \sigma = (\lambda_1, \lambda_2)\colon R \cap S \rightrightarrows X$ (where $\lambda_1 = \sigma_1 a = \rho_1 b$ and $\lambda_2 = \sigma_2 a = \rho_2 b$) is a subrelation of the star-relation R (and also a subrelation of S); it is then a star-relation. $\square$

Lemma 1.1.17. [**25**] *For any two relations $\rho\colon R \rightrightarrows X$ and $\sigma\colon S \rightrightarrows X$ on the same object X in a multi-pointed category with meets of relations and with $\mathcal{N}$-kernels, one has*

$$(\rho \wedge \sigma)^* = \rho^* \wedge \sigma^*$$

Proof. Since $\rho \wedge \sigma$ is a subrelation of both ρ and σ, then $(\rho \wedge \sigma)^*$ is a subrelation of both ρ^* and σ^*. This implies that $(\rho \wedge \sigma)^*$ is a subrelation of $\rho^* \wedge \sigma^*$. For the converse, the fact that $\rho^* \wedge \sigma^*$ is a subrelation of $\rho \wedge \sigma$ implies that $(\rho^* \wedge \sigma^*)^*$ is a subrelation of $(\rho \wedge \sigma)^*$. But $(\rho^* \wedge \sigma^*)^* = \rho^* \wedge \sigma^*$ since $\rho^* \wedge \sigma^*$ is a star by Lemma 1.1.16. $\square$

1.1.4. Regular multi-pointed categories.

By a regular multi-pointed category [**25**], one means a multi-pointed category $(\mathcal{C}, \mathcal{N})$ where $\mathcal{C}$ is a regular category. If in addition every morphism in $\mathcal{C}$ admits an $\mathcal{N}$-kernel, then $(\mathcal{C}, \mathcal{N})$ will be called a regular multi-pointed category with $\mathcal{N}$-kernels.

Lemma 1.1.18. [**25**] *In a finitely complete multi-pointed category $\mathcal{C}$, for two kernel stars $\kappa : K \rightrightarrows X$ and $\lambda\colon L \rightrightarrows X$ of the morphisms $f : X \to Y$ and $g : X \to Z$, respectively, the meet $\kappa \wedge \lambda\colon K \cap L \rightrightarrows X$ of κ and λ exists and it is the kernel star of the morphism $(f, g) : X \to Y \times Z$.*

Proof. Consider the pullback.

$$\begin{tikzcd}
K \cap L \arrow[r, tail, "u"] \arrow[d, tail, "v"'] & L \arrow[d, tail, "(\lambda_1,\lambda_2)"] \\
K \arrow[r, tail, "(\kappa_1,\kappa_2)"'] & X \times X
\end{tikzcd}$$

The Lemma 1.1.16 gives the existence of the star-relation $(\theta_1, \theta_2) : K \cap L \rightrightarrows X$ with $\theta_1 = \lambda_1 u = \kappa_1 v$ and $\theta_2 = \lambda_2 u = \kappa_2 v$. By looking at the commutative diagram

$$\begin{tikzcd}
 & & & Y \\
K \cap X \arrow[r, shift left, "\theta_1"] \arrow[r, shift right, "\theta_2"'] \arrow[rrru, bend left] \arrow[rrrd, bend right] & X \arrow[r, "(f,g)"] \arrow[rru, bend left, "f"] \arrow[rrd, bend right, "g"'] & Y \times Z \arrow[ru, "\pi_1"'] \arrow[rd, "\pi_2"] & \\
 & & & Z
\end{tikzcd}$$

one sees that $(f, g)\theta_1 = (f, g)\theta_2$. It remains to prove the universal property: for this, let us consider a star $\mu : T \rightrightarrows X$ such that $(f, g)\mu_1 = (f, g)\mu_2$, then one has $f\mu_1 = f\mu_2$ and $g\mu_1 = g\mu_2$. By the universal property of the kernel stars K and L, there exist unique morphisms $a : T \to K$ and $b : T \to L$ such that $\kappa_1 a = \mu_1$, $\kappa_2 a = \mu_2$ and $\lambda_1 b = \mu_1$, $\lambda_2 b = \mu_2$. By the universal property of the pullback above, we have the commutative diagram

$$\begin{tikzcd}
T \arrow[rd, dotted, "\varphi"] \arrow[rrd, bend left, "b"] \arrow[rdd, bend right, "a"'] & & \\
 & K \cap L \arrow[r, "u"] \arrow[d, "v"'] & L \arrow[d, "(\lambda_1,\lambda_2)"] \\
 & K \arrow[r, "(\kappa_1,\kappa_2)"'] & X \times X
\end{tikzcd}$$

with the unique morphism $\varphi : T \to K \cap L$ such that $u\varphi = b$ and $v\varphi = a$. The equality $\theta_i\varphi = \mu_i$ for $i = \{1, 2\}$ follows without difficulty, and the uniqueness of φ satisfying this equality follows from the fact that θ is a star-relation. □

Lemma 1.1.19. **[25]** *Let $(\mathcal{C}, \mathcal{N})$ be a multi-pointed category with $\mathcal{N}$-kernels. Then, for a composable pair of morphisms* $A \xrightarrow{f} B \xrightarrow{g} C$ *such that $gf \in \mathcal{N}$ and f is a strong epimorphism, g also belongs to $\mathcal{N}$.*

Proof. A morphism $g\colon X \to Y$ belongs to $\mathcal{N}$ if and only if 1_X is an $\mathcal{N}$-kernel of g, this is equivalent to the fact that an $\mathcal{N}$-kernel of g is an isomorphism.

So let $k\colon K \to X$ be the $\mathcal{N}$-kernel of g and then consider the following diagram

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "g"] & Y \\ Z \arrow[u, dotted, "u"] \arrow[ur, "f"'] & & \end{tikzcd}$$

where u is the morphism arising from the universal property of k. This implies that k is a strong epimorphism, and then an isomorphism. $\square$

In the pointed context, this lemma says that if $gf = 0$ and f is a strong epimorphism, then $g = 0$. In the total context, this lemma becomes trivial.

Thus, when $(\mathcal{C}, \mathcal{N})$ is a regular multi-pointed category with $\mathcal{N}$-kernels, any star $(\lambda_1, \lambda_2) : X \rightrightarrows Y$ uniquely factorizes (up to isomorphism) as a regular epimorphism followed by a star-relation

$$\begin{tikzcd} X \arrow[rr, shift left, "\lambda_1"] \arrow[rr, shift right, "\lambda_2"'] \arrow[dr, two heads, "e"'] & & Y \\ & Z \arrow[ur, shift left, "\mu_1"] \arrow[ur, shift right, "\mu_2"'] & \end{tikzcd}$$

Indeed, if we consider the usual factorisation in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[rr, "{(\lambda_1,\lambda_2)}"] \arrow[dr, two heads, "e"'] & & Y \times Y \\ & Z \arrow[ur, "{(\mu_1,\mu_2)}"'] & \end{tikzcd}$$

of (λ_1, λ_2) as a regular epimorphism followed by a monomorphism, then $\mu_1 e = \lambda_1 \in \mathcal{N}$, and according to the Lemma 1.1.19, $(\mu_1, \mu_2) : Z \rightrightarrows Y$ is a star-relation.

Definition 1.1.20. **[25]** Let $\mathcal{C}$ be a regular multi-pointed category with $\mathcal{N}$-kernels.

(1) For a morphism $f : X \to Y$ and a star $\lambda : U \rightrightarrows X$, *the image of the star* λ *along the morphism* f is the star-relation $f(\lambda) : f(U) \rightrightarrows Y$ arising from the factorisation of the star $f\lambda$ as a regular epimorphism followed by a star-relation

$$\begin{tikzcd} U \arrow[r, two heads, "f'"] \arrow[d, shift right, "\lambda"'] \arrow[d, shift left] & f(U) \arrow[d, shift right] \arrow[d, shift left, "f(\lambda)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(2) The *star-inverse image* of a star $\tau : T \rightrightarrows Y$ along a morphism $f : X \to Y$ denoted by $f^{-1}(\tau)^* : f^{-1}(T)^* \rightrightarrows X$ is the star obtained in

the following star-pullback

$$\begin{tikzcd} f^{-1}(T)^* \arrow[r] \arrow[d, shift left, "f^{-1}(\tau)^*"'] \arrow[d, shift right] & T \arrow[d, shift left, "\tau"] \arrow[d, shift right] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Equivalently, $f^{-1}(\tau)^* : f^{-1}(T)^* \rightrightarrows X$ is obtained by first considering the usual inverse image of the star τ along f

$$\begin{tikzcd} f^{-1}(T) \arrow[r] \arrow[d, shift left, "f^{-1}(\tau)"'] \arrow[d, shift right] & T \arrow[d, shift left, "\tau"] \arrow[d, shift right] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and by then taking $f^{-1}(T)^* \rightrightarrows X$, which is the star associated with the pair $f^{-1}(T) \rightrightarrows X$. This gives the equality $f^{-1}(\tau)^* = (f^{-1}(\tau))^*$ or, equivalently, $f^{-1}(T)^* = (f^{-1}(T))^*$.

Notation: When f is a monomorphism and σ is a star-relation, for the convenience of our work, we shall often write $f^{-1}(T)^* = (T \cap (X \times X))^*$.

In the pointed context, this gives the intersection of two subobjects

$$\begin{tikzcd} f^{-1}(T) \arrow[r, tail] \arrow[d, tail] & T \arrow[d, tail, "\tau_2"] \\ X \arrow[r, tail, "f"'] & Y \end{tikzcd}$$

and in this case we have the usual notation $f^{-1}(T) = T \cap X$.

In the total context, $f^{-1}(T)$ is in fact the restriction of T to X

$$\begin{tikzcd} f^{-1}(T) \arrow[r, tail] \arrow[d, tail] & T \arrow[d, tail, "(\tau_1,\tau_2)"] \\ X \times X \arrow[r, tail, "f \times f"'] & Y \times Y \end{tikzcd}$$

and we write $f^{-1}(T) = T \cap (X \times X)$.

Proposition 1.1.21. *Let $(\mathcal{C}, \mathcal{N})$ be a regular multi-pointed category and let us denote by $StarRel_{\mathcal{C}}(X)$ and by $StarRel_{\mathcal{C}}(Y)$ the classes of star-relations on objects X and Y in $\mathcal{C}$, respectively. For a morphism $f : X \to Y$, consider the correspondences*

$$\begin{array}{rcl} \phi : StarRel_{\mathcal{C}}(X) & \longrightarrow & StarRel_{\mathcal{C}}(Y) \\ \lambda : U \rightrightarrows X & \mapsto & \phi(\lambda) = f(\lambda) : f(U) \rightrightarrows Y \ \textit{and} \end{array}$$

$$\begin{array}{rcl} \phi' : StarRel_{\mathcal{C}}(Y) & \longrightarrow & StarRel_{\mathcal{C}}(X) \\ \sigma : T \rightrightarrows Y & \mapsto & \phi'(\sigma) = f^{-1}(\sigma)^* : f^{-1}(T)^* \rightrightarrows X. \end{array}$$

Both these correspondences preserve the partial order and satisfy

$$f(\lambda) \leqslant \sigma \Leftrightarrow \lambda \leqslant f^{-1}(\sigma)^*.$$

Proof. Consider the following cube

$$\begin{tikzcd}
U \arrow[rr, "g"] \arrow[dr, dotted, "u"] \arrow[dd, Rightarrow, "\lambda"'] & & T \arrow[dd, Rightarrow, "\tau"] \arrow[dr, dotted, "v"] & \\
& W \arrow[rr, "h"] \arrow[dd, Rightarrow, "\mu"'] & & S \arrow[dd, Rightarrow, "\sigma"] \\
X \arrow[rr, "f"'] \arrow[dr, equal] & & Y \arrow[dr, equal] & \\
& X \arrow[rr, "f"'] & & Y.
\end{tikzcd}$$

If $\mu = f^{-1}(\sigma)^*$ and $\lambda = f^{-1}(\tau)^*$, then both the back squares and the front squares of the cube above are star-pullbacks. Now suppose that $\tau \leqslant \sigma$, this means that there exists an arrow v with such that $\sigma v = \tau$. By the universal property of the front star-pullback, we get the induced arrow u. This proves that $\tau \leqslant \sigma$ implies $f^{-1}(\tau)^* \leqslant f^{-1}(\sigma)^*$. Moreover, if one has $f(\lambda) \leqslant \sigma$, then $\lambda \leqslant f^{-1}(f(\lambda))^* \leqslant f^{-1}(\sigma)^*$. In the same way, if $\tau = f(\lambda)$ and $\sigma = f(\mu)$, then both g and h are regular epimorphisms and if $\lambda \leqslant \mu$ so that the arrow u exists with $\mu u = \lambda$, then the existence of the arrow v follows from the fact that regular epimorphisms are "orthogonal" to monomorphisms, thus $f(\lambda) \leqslant f(\mu)$. Finally, if $\lambda \leqslant f^{-1}(\sigma)^*$ we then have $f(\lambda) \leqslant f(f^{-1}(\sigma)^*) \leqslant \sigma$, this allows one to conclude that $f(\lambda) \leqslant \sigma \Leftrightarrow \lambda \leqslant f^{-1}(\sigma)^*$. □

1.1.5. Star-regular categories.

Definition 1.1.22. **[25]** A regular multi-pointed category $(\mathcal{C}, \mathcal{N})$ with $\mathcal{N}$-kernels is a *star-regular* category if any regular epimorphism in $\mathcal{C}$ is a coequaliser of a star.

In the total context, this additional condition says that a regular epimorphism is a coequaliser of a parallel pair of morphisms and then a star-regular category simply gives the notion of a regular category. In the pointed context, this condition implies that any regular epimorphism is a normal epimorphism, and thus a star-regular category gives in this case the notion of normal category.

Note that in a star-regular category every kernel star has a coequaliser and is the kernel star of this coequaliser.

Example 1.1.23. The category Gp of groups is an example of a normal category. In fact, any semi-abelian category **[36]** is normal. Recall that a semi-abelian category is a pointed, exact and protomodular category with binary coproducts. Thus the categories Ab of abelian groups, R-Mod of modules over a ring R, Lie of Lie algebras are all examples of normal categories.

Many examples of normal categories (not necessarily semi-abelian) arise in universal algebra.

Recall that in a variety of universal algebras, the regular epimorphisms are the surjective homomorphisms. Accordingly, regular epimorphisms correspond to all congruences in the usual way. On the other hand the normal epimorphisms are those surjective homomorphisms which are cokernels: the corresponding congruences are generated by their 0-classes.

Definition 1.1.24. (see e.g. [**38, 1, 33**]) A variety $\mathcal{V}$ of universal algebras with a unique constant 0 is 0-regular when for two congruences θ, θ' on an algebra A, if $[0]_\theta = [0]_{\theta'}$ then $\theta = \theta'$.

The Definition 1.1.24 implies that in a 0-regular variety $\mathcal{V}$ of universal algebras, regular epimorphisms are generated by their 0-class, so regular epimorphisms are normal epimorphisms. In fact, we have that a variety with a unique constant 0 is 0-regular if and only if any regular epimorphism in it is a normal epimorphism [**38**]. The following propositions give a characterization of 0-regular varieties in the language of terms (see for instance [**19, 33**]).

Proposition 1.1.25. [**19, 33**] *A variety $\mathcal{V}$ of universal algebras with a unique constant 0 is 0-regular if and only if there exist n binary terms $d_1, ..., d_n$ and n quaternary terms $q_1, ...q_n$ such that the equations*

- $d_i(x,x) = 0$ *for* $1 \leqslant i \leqslant n$
- $x = q_1(x, y, 0, d_1(x, y))$
- $q_i(x, y, d_i(x, y), 0) = q_{i+1}(x, y, 0, d_{i+1}(x, y))$ *for* $1 \leqslant i < n$
- $q_n(x, y, d_n(x, y), 0) = y$

hold in $\mathcal{V}$.

Corollary 1.1.26. [**19**] *A pointed variety $\mathcal{V}$ with a unique constant 0 is 0-regular if and only if there exist binary terms $d_1, ..., d_n$ such that the equation*

- $d_i(x,x) = 0$ *for* $1 \leqslant i \leqslant n$ *and the implication*
- $d_1(x, y) = d_2(x, y) = ... = d_n(x, y) = 0 \Rightarrow x = y$

hold in $\mathcal{V}$.

Corollary 1.1.26 easily provides examples of 0-regular varieties.

Example 1.1.27.
1. In the case of Gp, take $n = 1$, $0 = 1$ and $d_1(x, y) = xy^{-1}$.
2. For Rng the variety of rings, take $n = 1$ and $d_1(x, y) = x - y$.
3. In the case of Loops the variety of loops, take n=1 and $d_1(x, y) = x\backslash y$ or equivalently $t_1(x, y) = x/y$. Recall that a loop (see for instance [**13**]) is an algebra $(L, /, \cdot, \backslash, 1)$ with three binary operations $/, \cdot, \backslash$ and the identity 1 satisfying the following identities:
   - $x\backslash(x.y) = y, \qquad (x.y)/y = y;$
   - $x.(x\backslash y) = y, \qquad (x/y).y = x;$
   - $x.1 = 1.x = x.$
4. The variety of *implication algebras* is an example of 0-regular variety [**33**] which is not semi-abelian like the three previous examples. Recall that an implication algebra has just one binary operation satisfying the identities:

- $(xy)x = x$;
- $(xy)y = (yx)x$;
- $(xy)z = y(xz)$.

To prove that the variety of implication algebras is 0-regular, take $n = 2$, $d_1(x, y) = xy$ and $d_2(x, y) = yx$.

We return now to the properties of star-regular categories.

LEMMA 1.1.28. **[25]** *Let $\mathcal{C}$ be a star-regular category. Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "\tau"] \arrow[r, shift right] \arrow[d, "g"'] & Y \arrow[r, "p"] \arrow[d, "f"'] & Z \arrow[d, "m"] \\ X' \arrow[r, shift left] \arrow[r, shift right, "\tau'"'] & Y' \arrow[r, "p'"'] & Z' \end{tikzcd}$$

of stars and morphisms, where τ' is the kernel star of p' and p is a regular epimorphism. Any two of the following conditions imply the third one:

(a) *τ is the kernel star of p;*
(b) *the left-hand side square is a star-pullback;*
(c) *m is a monomorphism.*

LEMMA 1.1.29. *Consider a commutative diagram in a star-regular category*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, "f"] \arrow[d, "v"'] & Y \arrow[d, equal] \\ K' \arrow[r, "k'"'] & X' \arrow[r, "f'"'] & M \end{tikzcd}$$

where k and k' are $\mathcal{N}$-kernels of f and f', respectively, then the left-hand square is a pullback.

PROPOSITION 1.1.30. **[25]** *The following conditions are equivalent for a regular multi-pointed category $(\mathcal{C}, \mathcal{N})$ with $\mathcal{N}$-kernels*

(1) *$(\mathcal{C}, \mathcal{N})$ is star-regular;*
(2) *a commutative square*

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & Y \arrow[d, two heads, "h"] \\ Z \arrow[r, two heads, "t"'] & T \end{tikzcd} \tag{1.4}$$

of regular epimorphisms is a pushout if and only if h is the coequaliser of $f(\mathsf{Eq}(g))^* \rightrightarrows Y$, where $\mathsf{Eq}(g)^*$ is the kernel star of g.

PROOF. (1) ⇒ (2). Assume that the category $\mathcal{C}$ is star-regular and consider the following diagram

$$\begin{tikzcd}
\mathsf{Eq}(g)^* \arrow[r, two heads, "e"] \arrow[d, shift left] \arrow[d, shift right] & f(\mathsf{Eq}(g)^*) \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[r, two heads, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\
Z \arrow[r, two heads, "t"'] & T.
\end{tikzcd}$$

Then g is a coequaliser of its kernel star $\mathsf{Eq}(g)^*$ and the condition (2) is obtained by applying Lemma 1.1.12(2). For the converse, it remains to prove that any regular epimorphism is a coequaliser of a star. Thus, let $f\colon X \twoheadrightarrow Y$ be a regular epimorphism. By applying (2) to the pushout

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

we get the desired result. □

The following lemma is new, and it provides a characterisation of those arrows that are monomorphisms in terms of their kernel stars.

LEMMA 1.1.31. *In a star-regular category $\mathcal{C}$, the following conditions are equivalent for a morphism $f : X \to Y$:*

(1) *$f : X \to Y$ is a monomorphism;*
(2) *$\mathsf{Eq}(f)^* = \Delta_X^*$;*
(3) *the projections $\pi_1\colon \mathsf{Eq}(f)^* \to X$ and $\pi_2\colon \mathsf{Eq}(f)^* \to X$ are equal.*

PROOF. (1) ⇒ (2) is obvious. For the converse, we first observe that in a star-regular category $\mathcal{C}$ since any regular epimorphism is a coequaliser of its kernel star, the correspondence $\phi\colon$ KernelPairs $\longrightarrow$ KernelStars mapping any kernel pair $\mathsf{Eq}(f)$ to its corresponding star $\mathsf{Eq}(f)^*$ is a bijection. Accordingly, by the injectivity of this correspondence, $\mathsf{Eq}(f)^* = \Delta_X^*$ implies that $\mathsf{Eq}(f) = \Delta_X$, and $f : X \to Y$ is then a monomorphism. (2) ⇒ (3) is clear since the projections of Δ_X^* are equals. (3) ⇒ (2) follows easily from the universal properties of kernel stars $\mathsf{Eq}(f)^*$ and Δ_X^*. □

In the total context, Lemma 1.1.31 says that f is a monomorphism if and only if $\mathsf{Eq}(f) = \Delta_X$; in the pointed context, it says that f is a monomorphism if and only if its kernel $\mathsf{ker}\,(f)$ is trivial: $\mathsf{ker}\,(f) = 0$ (see **[12]** for the case of normal categories).

COROLLARY 1.1.32. *A span $Y \xleftarrow{f} X \xrightarrow{g} Z$ in a star-regular category $\mathcal{C}$ is a relation if and only if $\mathsf{Eq}(f)^* \wedge \mathsf{Eq}(g)^* = \Delta_X^*$.*

Proof. We have

$$\mathsf{Eq}(f)^* \wedge \mathsf{Eq}(g)^* \quad = \quad \mathsf{Eq}(f,g)^* \text{ by Lemma 1.1.18 , with } (f,g) : X \to Y \times Z.$$

Then the span $Y \xleftarrow{f} X \xrightarrow{g} Z$ is a relation if and only if $(f,g) : X \to Y \times Z$ is a monomorphism and $(f,g) : X \to Y \times Z$ is a monomorphism if and only if $\mathsf{Eq}(f,g)^* = \Delta_X^*$ by Lemma 1.1.31. □

Proposition 1.1.33. *In a star-regular category $\mathcal{C}$, consider a commutative diagram*

$$\begin{tikzcd}
E \arrow[dr, "f"] \arrow[drr, bend left, "b"] \arrow[dd, bend right, "a"'] & & \\
& A \arrow[r, "d"] \arrow[d, "c"] & B \\
& C &
\end{tikzcd}$$

where (c,d) is jointly monomorphic. Then

$$\mathsf{Eq}(f)^* = \mathsf{Eq}(a)^* \cap \mathsf{Eq}(b)^*.$$

Proof. Since $cf = a$ and $df = b$, then $\mathsf{Eq}(f)^*$ is obviously a subrelation of both $\mathsf{Eq}(a)^*$ and $\mathsf{Eq}(b)^*$. Then we have the existence of the unique dotted arrow g as in the diagram

$$\begin{tikzcd}
\mathsf{Eq}(f)^* \arrow[dr, dotted, "g"] \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] & & \\
& \mathsf{Eq}(a)^* \cap \mathsf{Eq}(b)^* \arrow[r, "t"] \arrow[d, "w"] & \mathsf{Eq}(b)^* \arrow[d, "(\lambda_1,\lambda_2)"] \\
& \mathsf{Eq}(a)^* \arrow[r, "(\delta_1,\delta_2)"'] & E \times E.
\end{tikzcd}$$

making the two triangles commute. This implies that $\mathsf{Eq}(f)^*$ is a subrelation of $\mathsf{Eq}(a)^* \cap \mathsf{Eq}(b)^*$. For the converse, consider the star $(\theta_1, \theta_2)\colon \mathsf{Eq}(a)^* \cap \mathsf{Eq}(b)^* \rightrightarrows E$ where $\theta_1 = \lambda_1 t = \delta_1 w$ and $\theta_2 = \lambda_2 t = \delta_2 w$. Then the fact that (c,d) is jointly monomorphic gives the equality $f\theta_1 = f\theta_2$ and it follows that $\mathsf{Eq}(a)^* \cap \mathsf{Eq}(b)^* \leqslant \mathsf{Eq}(f)^*$. □

1.2. Categories with a good theory of ideals

1.2.1. Barr-exact Goursat categories and ideal determined categories.

Definition 1.2.1. **[2]** A category $\mathcal{C}$ is called *Barr-exact* when it is regular and every (internal) equivalence relation in $\mathcal{C}$ is a kernel pair.

Any variety of universal algebras is an exact category, as is any elementary topos (see **[4]**, for instance).

Definition 1.2.2. **[14, 15]** A category $\mathcal{C}$ is called a *Goursat* category when

- $\mathcal{C}$ is regular;
- the equivalence relations in $\mathcal{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

The standard reference for Goursat categories is the article by Carboni, Lambek and Pedicchio **[15]**.

Proposition 1.2.3. **[14]** *The following conditions are equivalents for a regular category $\mathcal{C}$.*

(1) *$\mathcal{C}$ is a Goursat category;*
(2) *for any regular epimorphism $f\colon X \twoheadrightarrow Y$ in $\mathcal{C}$ and any equivalence relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ on X*

$$\begin{tikzcd} R \arrow[r, two heads, "f'"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & f(R) \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

the regular image $f(R) \underset{h_2}{\overset{h_1}{\rightrightarrows}} Y$ of R along f is again an equivalence relation.

The following characterization of a *Barr-exact Goursat* category is a consequence of Proposition 1.2.3.

Proposition 1.2.4. *The following conditions are equivalents for a regular category $\mathcal{C}$.*

(1) *$\mathcal{C}$ is an exact Goursat category;*
(2) *for any regular epimorphism $f\colon X \twoheadrightarrow Y$ in $\mathcal{C}$ and any effective equivalence relation $\mathsf{Eq}(g) \underset{g_2}{\overset{g_1}{\rightrightarrows}} X$ on X*

$$\begin{tikzcd} \mathsf{Eq}(g) \arrow[r, two heads, "f'"] \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] & f(\mathsf{Eq}(g)) \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "h_2"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

the regular image $f(\mathsf{Eq}(g)) \underset{h_2}{\overset{h_1}{\rightrightarrows}} Y$ of $\mathsf{Eq}(g)$ along f is again an effective equivalence relation.

It is well known that a variety of universal algebras is n-permutable when it satisfies a suitable Mal'tsev condition (see **[34]**, for instance). In the case of a Mal'tsev variety, the 2-permutability property is characterised by the existence of a ternary operation p called a Mal'tsev term such that $p(x, y, y) = x$

and $p(x,x,y) = y$ [**49**]. The following theorem gives a characterisation of 3-permutable varieties (see [**27**] for a categorical proof of this known result).

THEOREM 1.2.5. *The following conditions are equivalent for a variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is a 3-permutable variety;*
(2) *there exist quaternary terms r and s satisfying the identities*
   - $r(x,y,y,z) = x, \quad s(x,y,y,z) = z;$
   - $r(x,x,y,y) = s(x,x,y,y).$

DEFINITION 1.2.6. [**37**] A pointed finitely complete category $\mathcal{C}$ with finite colimits is an *ideal determined* category if:

- $\mathcal{C}$ is a normal category;
- given a commutative diagram

$$\begin{tikzcd} K \arrow[r, "p'"] \arrow[d, "k"'] & K' \arrow[d, "k'"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

  where p, p' are normal epimorphisms, k a normal monomorphism and k' a monomorphism, then k' is also a normal monomorphism.

The notion of *ideal determined variety* was studied already in the 1970s and 1980s in universal algebra under different names: such varieties were called BIT ("buona teoria degli ideali") in [**55**], and ideal determined in [**33**]. Let us recall the definition usually given in universal algebra:

DEFINITION 1.2.7. [**55, 33**] A variety $\mathcal{V}$ of universal algebras with a unique constant 0 is ideal determined if:

(a) $\mathcal{V}$ is 0-regular;
(b) for any ideal I in $\mathcal{V}$, there exists a congruence θ such that $I = [0]_\theta$.

The following theorem gives a characterization of ideal determined varieties.

THEOREM 1.2.8. [**33**] *The following conditions are equivalent for a pointed variety $\mathcal{V}$ with a unique constant 0:*

(a) *$\mathcal{V}$ is ideal determined;*
(b) *there exists a ternary term $r(x,y,z)$ such that*
   - $r(x,x,y) = y$
   - $r(0,x,x) = 0$

COROLLARY 1.2.9. [**33**] *The following conditions are equivalent for a pointed variety $\mathcal{V}$ with a unique constant 0:*

(1) *$\mathcal{V}$ is ideal determined;*
(2) *there exists a binary term $s(x,y)$ such that*
   - $s(x,x) = 0$
   - $s(0,x) = x$

The condition (2) means that $\mathcal{V}$ is a *subtractive* variety [**56**]. For the proof of above corollary, it suffices to take $s(x,y) = r(0,x,y)$ and conversely $r(x,y,z) = s(s(x,y),z)$.

1.2.2. Categories with a good theory of $\mathcal{I}$.

Let $\mathcal{I}$ be a class of star-relations in $\mathcal{C}$.

Definition 1.2.10. [**25**] A regular multi-pointed category $(\mathcal{C},\mathcal{N})$ with $\mathcal{N}$-kernels is said to have a *good theory of* $\mathcal{I}$ when

1) every star $R \underset{\sigma_2}{\overset{\sigma_1}{\rightrightarrows}} X$ in $\mathcal{I}$ has a coequaliser;
2) $\mathcal{I}$ contains all kernel stars;
3) moreover, the assignment

$$\begin{array}{rcl} \text{KernelPairs} & \longrightarrow & \mathcal{I} \\ \pi & \longmapsto & \pi^* \end{array}$$

is a bijection.

A characterisation of categories with a good theory of $\mathcal{I}$ is then the following

Proposition 1.2.11. [**25**] *A regular multi-pointed category* $(\mathcal{C},\mathcal{N})$ *has a good theory of* $\mathcal{I}$ *if and only if* $(\mathcal{C},\mathcal{N})$ *is star-regular and* $\mathcal{I}$ *coincides with the class of kernel stars.*

Proof. $\Rightarrow$) Assume that $\mathcal{C}$ has a good theory of $\mathcal{I}$ and consider a regular epimorphism $f : A \twoheadrightarrow B$ with $\kappa : \mathsf{Eq}(f)^* \rightrightarrows A$ its kernel star. By 2) in Definition 1.2.10, $\kappa : \mathsf{Eq}(f)^* \rightrightarrows A$ is in $\mathcal{I}$ and by 1), the coequaliser $e : A \to E$ of κ exists. One easily checks that $\mathsf{Eq}(f)^* \cong \mathsf{Eq}(e)^*$. By 3), one has $\mathsf{Eq}(f) = \mathsf{Eq}(e)$ and it follows that f is a coequaliser of $\mathsf{Eq}(f)^*$. Moreover by 3), any star in $\mathcal{I}$ is a kernel star, thus $\mathcal{I}$ coincides with the class of kernel stars.

For the converse, since $\mathcal{C}$ is star-regular, any regular epimorphism in $\mathcal{C}$ is a coequaliser of a star (its kernel star), then all kernel stars have coequalisers and 3) is also satisfied. $\square$

Note that any star-regular category has a good theory of $\mathcal{I}$ with $\mathcal{I}$ the class of kernel stars. The following examples of categories with a good theory of $\mathcal{I}$ were given in [**25**].

Example 1.2.12. (In the total context)
A star-regular category $(\mathcal{C},\mathcal{N})$ has a good theory of $\mathcal{I}$ with $\mathcal{I}$ the class of *all equivalence relations* if and only if $\mathcal{C}$ is a *Barr-exact category.*

Proof. Assume that $\mathcal{C}$ has a good theory of all equivalence relations, then by Proposition 1.2.11, $\mathcal{C}$ is a regular category and the class of equivalence relations coincides with the class of effective equivalence relations: this gives exactly the notion of Barr-exact category.

Conversely, if $\mathcal{C}$ is Barr-exact, then any equivalence relation is effective and then has a coequaliser and 1) in Definition 1.2.10 is satisfied. The conditions 2) and 3) in Definition 1.2.10 obviously hold in $\mathcal{C}$ since any effective equivalence relation is an equivalence relation. $\square$

Example 1.2.13. (In the total context)
If $\mathcal{I}$ is the class of *all reflexive relations*, then a star-regular category $(\mathcal{C}, \mathcal{N})$ has a good theory of $\mathcal{I}$ if and only if $\mathcal{C}$ is a *Barr-exact Mal'tsev category.*

Proof. Recall that a finitely complete category $\mathcal{C}$ is called a Mal'tsev category [**15**] when every reflexive relation in $\mathcal{C}$ is an equivalence relation. It follows that a regular category is a Barr-exact Mal'tsev category if and only if any reflexive relation is an effective equivalence relation. Thus, when $\mathcal{C}$ has a good theory of reflexive relations, by Proposition 1.2.11, $\mathcal{C}$ is a regular category and the class of reflexive relations coincide with the class of effective equivalence relation. Conversely, if $\mathcal{C}$ is a Barr-exact Mal'tsev category, in terms of star-relations it is a star-regular category in which any reflexive relation is an effective equivalence relation. $\square$

Example 1.2.14. (In the pointed context)
Let $(\mathcal{C}, \mathcal{N})$ be a star-regular category and $\mathcal{I}$ the class of *all monomorphisms* in $\mathcal{C}$, then $(\mathcal{C}, \mathcal{N})$ has a good theory of $\mathcal{I}$ if and only if $\mathcal{C}$ is an *abelian* category.

Proof. It is well known that a pointed category with finite products is abelian if and only if any morphism decomposes as a normal epimorphism followed by a normal monomorphism (see [**20**],[**21**] for instance). Now assume that $\mathcal{C}$ has a good theory of monomorphisms and let $f : X \to Y$ be any arrow in $\mathcal{C}$. The star-regularity of $\mathcal{C}$ implies the following factorisation

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

of f as a normal epimorphism p followed by a monomorphism m. Since $\mathcal{C}$ has a good theory of monomorphisms, the class of monomorphisms coincides with the class of normal monomorphisms.

For the converse, any abelian category $\mathcal{C}$ is a normal category and monomorphisms are normal in $\mathcal{C}$. Thus all the conditions $1), 2)$ and $3)$ are obviously satisfied. $\square$

1.2.3. Categories with a good theory of ideals.
The categorical notion of ideal has been recently introduced in [**38**], by extending to categories the original one due to R. Magari and A. Ursini [**48, 55**].

Definition 1.2.15. [**38, 39**] Recall that a momonomphism $m : M \rightarrowtail Y$ is called an *ideal* if there is a commutative square in $\mathcal{C}$

$$\begin{tikzcd} K \arrow[r, "p'"] \arrow[d, tail, "k"'] & M \arrow[d, tail, "m"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

where p and p' are normal epimorphisms and k is a normal monomorphism.

This notion of ideal is extended to the case of star-relations as follows:

DEFINITION 1.2.16. **[25]** In a regular multi-pointed category $(\mathcal{C},\mathcal{N})$, a star-relation $R \overset{*}{\rightrightarrows} Y$ is an *ideal* if we have a commutative diagram of the form

$$\begin{tikzcd} K \arrow[r, twoheadrightarrow, "f'"] \arrow[d, shift right, "\kappa"'] \arrow[d, shift left] & R \arrow[d, shift right] \arrow[d, shift left, "f(\kappa)"] \\ X \arrow[r, twoheadrightarrow, "f"'] & Y \end{tikzcd}$$

where f and f' are regular epimorphisms and $\kappa : K \rightrightarrows X$ is a kernel star.

In particular, this definition of the notion of "ideal" reduces, in the total context, to the following one: a relation $(s_1, s_2)\colon S \rightrightarrows Y$ is an ideal if there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, twoheadrightarrow, "f'"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, twoheadrightarrow, "f"'] & Y \end{tikzcd}$$

where f and f' are regular epimorphisms and $(r_1, r_2) : R \rightrightarrows X$ is a kernel pair.

DEFINITION 1.2.17. **[25]** A regular multi-pointed category $(\mathcal{C},\mathcal{N})$ with kernels is said to have a *good theory of ideals* when $(\mathcal{C},\mathcal{N})$ has a good theory of $\mathcal{I}$ with $\mathcal{I}$ the class of all ideals.

The following Example 1.2.18 and Example 1.2.19 of categories with a good theory of ideals are immediate consequence of Definition 1.2.6. and of Proposition 1.2.4.

EXAMPLE 1.2.18. In the pointed context, a star-regular category $(\mathcal{C},\mathcal{N})$ with finite coproducts has a good theory of $\mathcal{I}$ where $\mathcal{I}$ is the class of *ideals* if and only if $\mathcal{C}$ is an *ideal determined* category.

If we denoted by $Id(A)$ the class of ideals on an object A in a category $\mathcal{C}$, then the inclusions of the diagram (1.2) in Section 1.1.2. can be completed to the following one

$$N(A) \subseteq B\text{-}N(A) \subseteq Cl(A) \subseteq Id(A).$$

By Definition 1.2.6, in an ideal determined category, every ideal is a kernel. We then have the equality $N(A) = Id(A)$; this implies that all the inclusions above become equalities in an ideal determined category. In a pointed Mal'tsev category we have the equality $Cl(A) = B\text{-}N(A)$, and in **[24]**, it is proved that the equality $Cl(A) = Id(A)$ holds in a *subtractive category* **[43]**.

EXAMPLE 1.2.19. In the total context, a star-regular category $(\mathcal{C},\mathcal{N})$ has a good theory of $\mathcal{I}$ with $\mathcal{I}$ the class of *ideals* if and only if $\mathcal{C}$ is a *Barr-exact Goursat* category.

The categories with a good theory of ideals can be characterised as follows

THEOREM 1.2.20. **[25]** *Let $\mathcal{C}$ be a star-regular category with coequalisers of ideals. Then the following are equivalent:*

(a) *$\mathcal{C}$ has a good theory of ideals;*

(b) *a commutative square (1) of regular epimorphisms*

$$\begin{tikzcd} \mathsf{Eq}(e)^* \arrow[r, dotted, "q"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{Eq}(g)^* \arrow[d, shift left] \arrow[d, shift right] \\ X' \arrow[r, two heads, "f"] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "(1)"] & Y' \arrow[d, two heads, "g"] \\ X \arrow[r, two heads, "h"'] & Y \end{tikzcd} \tag{1.5}$$

is a pushout if and only if the regular image of the kernel star $\mathsf{Eq}(e)^*$ of e along f is the kernel star $\mathsf{Eq}(g)^*$ of g, i.e. $f(\mathsf{Eq}(e)^*) = \mathsf{Eq}(g)^*$. (Which is equivalent to saying that the induced arrow $\mathsf{Eq}(e)^* \overset{q}{\dashrightarrow} \mathsf{Eq}(g)^*$ is a regular epimorphism).

PROOF. $(a) \Rightarrow (b)$ Assume that $(\mathcal{C}, \mathcal{N})$ is a category with a good theory of ideals and consider the commutative square of regular epimorphisms (1). Since $\mathcal{C}$ is star-regular then, by Proposition 1.1.30, the square (1) is a pushout if and only if g is the coequaliser of $f(\mathsf{Eq}(e)^*)$. Moreover, by assumption (a), $f(\mathsf{Eq}(e)^*) \rightrightarrows Y'$ is a kernel star so that the square (1) is a pushout if and only if $f(\mathsf{Eq}(e)^*) = \mathsf{Eq}(g)^*$ (since any kernel star has a coequaliser and is the kernel star of this coequaliser). This implies that $\mathsf{Eq}(e)^* \overset{q}{\dashrightarrow} \mathsf{Eq}(g)^*$ is a regular epimorphism.

$(b) \Rightarrow (a)$ For the converse, it suffice to prove that any ideal is a kernel star. For this, let $f : X \twoheadrightarrow Y$ be a regular epimorphism and $\mathsf{Eq}(e)^* \rightrightarrows X$ a kernel star. By definition, any ideal is a star-relation $f(\mathsf{Eq}(e)^*) \rightrightarrows B$ arising from the factorisation of the regular epimorphism f as a regular epimorphims followed by a kernel star as in the top part (3) of the following diagram

$$\begin{tikzcd} \mathsf{Eq}(e)^* \arrow[r, two heads] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "(3)"] & f(\mathsf{Eq}(e)^*) \arrow[d, shift left] \arrow[d, shift right] \\ X' \arrow[r, two heads, "f"] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "(2)"] & Y' \arrow[d, two heads, "g"] \\ X \arrow[r, dashed, "h"'] & Y. \end{tikzcd}$$

Now complete the diagram by the coequaliser e of $\mathsf{Eq}(e)^*$, and g the coequaliser of the ideal $f(\mathsf{Eq}(e)^*) \rightrightarrows B$ which exists by assumption. Then we have the induced arrow h such that $gf = he$. h is then a regular epimorphism since gf is one. By Proposition 1.1.30, (2) is a pushout and by (b), $f(\mathsf{Eq}(e)^*) \rightrightarrows B$ is a kernel star. □

□

In the pointed context, this theorem characterises an ideal determined category by a property of pushouts of regular epimorphisms. Thus we have: a normal category $\mathcal{C}$ is ideal determined if and only if a commutative diagram (1) of normal epimorphisms

$$\begin{tikzcd}
\mathsf{Ker}\, e \arrow[r, dotted, two heads] \arrow[d, "\mathsf{ker}\, e"'] & \mathsf{Ker}\, g \arrow[d, "\mathsf{ker}\, g"] \\
X' \arrow[r, two heads, "h"] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "(1)"] & Y' \arrow[d, two heads, "g"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

is a pushout if and only if the induced arrow between $\mathsf{Ker}\, e$ the kernel of e and $\mathsf{Ker}\, g$ the kernel of g is a normal epimorphism.

In the total context, the above Theorem 1.2.20 gives a characterisation of exact Goursat categories. Let $\mathcal{C}$ be a regular category, then $\mathcal{C}$ is an exact Goursat category if and only if a commutative diagram (1) of regular epimorphisms

$$\begin{tikzcd}
\mathsf{Eq}(e) \arrow[r, dotted, two heads] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{Eq}(g) \arrow[d, shift left] \arrow[d, shift right] \\
X' \arrow[r, two heads, "h"] \arrow[d, two heads, "e"'] \arrow[dr, phantom, "(1)"] & Y' \arrow[d, two heads, "g"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

is a pushout if and only if the regular image of the kernel pair of e along h is the kernel pair of g.

Corollary 1.2.21. *In a category with a good theory of ideals, pushouts of regular epimorphisms always exist.*

Proof. Consider the situation

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & Y \\
Z &
\end{tikzcd}$$

with regular epimorphisms f and g. We are going to construct the pushout of f and g. For this, take $\mathsf{Eq}(g)^*$ the kernel star of g and $f(\mathsf{Eq}(g)^*)$ the regular image of $\mathsf{Eq}(g)^*$ along f and h the coequaliser of $f(\mathsf{Eq}(g)^*)$. We obtain the diagram

$$\begin{tikzcd}
\mathsf{Eq}(g)^* \arrow[r, two heads, "\alpha"] \arrow[d, shift left] \arrow[d, shift right] & f(\mathsf{Eq}(g)^*) \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] \arrow[dr, phantom, "(1)"] & Y \arrow[d, two heads, "h"] \\
Z \arrow[r, two heads, "t"'] & T
\end{tikzcd}$$

with t the induced arrow such that $tg = hf$ which is then a regular epimorphism since all other morphisms are. Since $\mathcal{C}$ has a good theory of ideals, then $f(\mathsf{Eq}(g)^*)$ is a kernel star: it is in fact the kernel star of h. By Theorem 1.2.20, the square (1) is a pushout. $\square$

CHAPTER 2

The Zassenhaus Lemma in star-regular categories

The Noether isomorphism theorems and the Zassenhaus Lemma describe relationships between subobjects, quotients and homomorphisms and can be considered as asserting that certain diagrams have exact rows and exact columns. Various versions of these theorems exist in the categories Gp of groups, Rng of rings and R-Mod of modules over a fixed ring R, and in more general categories. In **[57]** O. Wyler investigated a large class of pointed categories in which the (Noether) isomorphism theorems and the Zassenhaus Lemma hold true. The pointed categories considered by O. Wyler had inverse images, cokernels of kernels, factorisation of any arrow as a normal epimorphism followed by a monomorphism, and normal images of kernels were required to be kernels. These categories are very similar to the ideal determined categories introduced in **[37]** later on.

In varieties of universal algebras, the isomorphism theorems were generalized to the context of algebras and congruences (see **[13]**, for instance). In his doctoral dissertation **[54]**, W. Tholen gave a non-pointed version of the isomorphism theorems and of the Zassenhaus Lemma in a category with pullbacks, pullback-stable regular epimorphisms, coequalisers, regular images, direct images and inverse images of kernel pairs, and with pushouts of regular epimorphisms along arbitrary morphism. To summarize, a non-pointed version of the isomorphism theorems and the Zassenhaus lemma were investigated by W. Tholen in a "regular" category satisfying some suitable conditions. It is then natural to look for a common extension of the Noether isomorphism theorems and of the Zassenhaus Lemma in the framework of star-relations. This is precisely the goal of the present chapter. In the first section, we express the diamond isomorphism theorem in a star-regular category. We then give a condition, called *property* (C) (Definition 2.1.4) in a star-regular category guaranteeing that the double quotient isomorphism theorem holds true. In Section 2.2., we prove the Zassenhaus Lemma in a star-regular category satisfying the property (C) and with pushouts of regular epimorphisms along regular epimorphisms. We end this chapter with a proof of the existence of a lattice isomorphism between the lattice of some kernel stars in a star-regular category (satisfying these conditions) and with some remarks about "regular diamonds". Most of the results of this chapter are contained in my preprint **[50]**.

2.1. Isomorphism theorems

A common generalisation of *short exact sequence* in a pointed finitely complete category with zero object 0 and of *exact forks* in a finitely complete category in the framework of star-relations is the notion of *star-exact sequence* **[23]** which is a diagram

$$K \overset{\kappa}{\rightrightarrows} X \xrightarrow{f} Y$$

where $\kappa = (\kappa_1, \kappa_2)$ is the kernel star of f and f is the coequaliser of (κ_1, κ_2).

When the category is star-regular, this reduces to the assumption that f is a regular epimorphism and κ is a kernel star. In what follows, in such a star-exact sequence diagram, to emphasize the object K instead of the kernel star κ, we shall denote the object Y by X/K.

Definition 2.1.1. Let $\mathcal{C}$ be a regular category. For any regular epimorphism $f : A \to B$ and any monomorphism $m : M \to A$, we define their *asymmetric join*

$$f \mathbin{\vee\!\!\!\vee}_A M = f^{-1}(f(M))$$

as the subobject $f \mathbin{\vee\!\!\!\vee}_A m : f \mathbin{\vee\!\!\!\vee}_A M \to A$ of A in the following pullback

$$\begin{tikzcd}
M \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, tail, bend right, "m"'] & & \\
& f \mathbin{\vee\!\!\!\vee}_A M \arrow[r, two heads] \arrow[d, tail, "f \mathbin{\vee\!\!\!\vee}_A m"] \arrow[dr, phantom, "\lrcorner", very near start] & f(M) \arrow[d, tail, "f(m)"] \\
& A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

In the next proposition and in Proposition 2.1.8, we adopt the terminology used by Mac Lane and Birkhoff in their book *Algebra* **[47]** in the special case of groups.

Proposition 2.1.2. *[**Diamond isomorphism theorem**] Let $\mathcal{C}$ be a star-regular category, $f : A \to B$ a regular epimorphism and $m : M \to A$ a monomorphism. There is an isomorphism*

$$\frac{M}{(\mathsf{Eq}(f)^* \cap (M \times M))^*} \cong \frac{f \mathbin{\vee\!\!\!\vee}_A M}{(\mathsf{Eq}(f)^* \cap ((f \mathbin{\vee\!\!\!\vee}_A M) \times (f \mathbin{\vee\!\!\!\vee}_A M)))^*}$$

of subobjects of B, where $\mathsf{Eq}(f)^* \rightrightarrows A$ is the kernel star of f.

The diamond isomorphism theorem may be summarized in the diagram below

$$\begin{tikzcd}
& & f \mathbin{\vee\!\!\!\vee}_A M \arrow[dll, dash] \arrow[dr, dash] & \\
(\mathsf{Eq}(f)^* \cap ((f \mathbin{\vee\!\!\!\vee}_A M) \times (f \mathbin{\vee\!\!\!\vee}_A M)))^* \arrow[drr, dash] & & & M \arrow[dl, dash] \\
& & (\mathsf{Eq}(f)^* \cap (M \times M))^* &
\end{tikzcd}$$

Proof. In order to prove the existence of this isomorphism, we are going to show that both $(\mathsf{Eq}(f)^* \cap ((f \mathbin{\vee\!\!\!\vee}_A M) \times (f \mathbin{\vee\!\!\!\vee}_A M)))^* \rightrightarrows f \mathbin{\vee\!\!\!\vee}_A M$ and $(\mathsf{Eq}(f)^* \cap (M \times M))^* \rightrightarrows M$ are kernel stars with the same quotient. For this, let us set

$$T = (\mathsf{Eq}(f)^* \cap ((f \mathbin{\vee\!\!\!\vee}_A M) \times (f \mathbin{\vee\!\!\!\vee}_A M)))^*$$

and

$$L = (\mathsf{Eq}(f)^* \cap (M \times M))^*$$

and then consider the following diagram

$$\begin{tikzcd}
L \arrow[rr, shift left, "\mu_1"] \arrow[rr, shift right, "\mu_2"'] \arrow[dr, tail] \arrow[ddr, tail, bend right] & & M \arrow[dr, tail] \arrow[ddr, tail, "m"] \arrow[drr, bend left, "f'"] & & \\
& T \arrow[rr, shift left, "\lambda_1"] \arrow[rr, shift right, "\lambda_2"'] \arrow[d, tail, "l"'] & & f \mathbin{\vee\!\!\!\vee}_A M \arrow[r, two heads, "f''"] \arrow[d, tail, "m''"] & f(M) \arrow[d, tail, "m'"] \\
& \mathsf{Eq}(f)^* \arrow[rr, shift left, "\kappa_1"] \arrow[rr, shift right, "\kappa_2"'] & & A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

where $(\kappa_1, \kappa_2) : \ \mathsf{Eq}(f)^* \rightrightarrows A$ is the kernel star of f. One easily checks that $(\lambda_1, \lambda_2) : T \rightrightarrows f \mathbin{\vee\!\!\!\vee}_A M$ is the kernel star of $fm'' = m'f''$ and since m' is a monomorphism, we conclude that (λ_1, λ_2) is the kernel star of the regular epimorphism f''. The star-regularity of $\mathcal{C}$ implies that the middle row of the diagram above is a star-exact sequence, hence $f(M) \cong \dfrac{f \mathbin{\vee\!\!\!\vee}_A M}{T}$ as subobjects of B.

In the same way, $(\mu_1, \mu_2) : L \rightrightarrows M$ is the kernel star of the regular epimorphism f', and this implies that the top row of the diagram above is a star-exact sequence. Thus $f(M) \cong \dfrac{M}{L}$ as subobjects of B. $\square$

In the total context, let $\mathcal{C}$ be a regular category, $f \colon A \to B$ a regular epimorphism and $m \colon M \to A$ a monomorphism. The diamond isomorphism theorem gives us the isomorphism

$$\frac{M}{\mathsf{Eq}(f) \cap (M \times M)} \cong \frac{f \mathbin{\vee\!\!\!\vee}_A M}{\mathsf{Eq}(f) \cap ((f \mathbin{\vee\!\!\!\vee}_A M) \times (f \mathbin{\vee\!\!\!\vee}_A M))}$$

This non-pointed version of the diamond isomorphism theorem was given in **[54]**.

In the pointed context, if $f : A \longrightarrow\!\!\!\triangleright B$ is a normal epimorphism and $m : M \rightarrowtail A$ a monomorphism in a normal category $\mathcal{C}$, then the diamond isomorphism theorem asserts that the isomorphism

$$\frac{M}{K \cap M} \cong \frac{f \mathbin{\vee\!\!\!\vee}_A M}{K \cap (f \mathbin{\vee\!\!\!\vee}_A M)}$$

holds in $\mathcal{C}$, where $K \triangleright\!\!\xrightarrow{k} A$ is the kernel of f. Moreover, in the pointed context, one checks that $f \mathbin{\vee\!\!\!\vee}_A M$ is a subobject of A containing both M and the kernel K of f. Thus, $K \cap (f \vee_A M) = K$. This is summarised in the following diagram

$$\begin{tikzcd}
 & K \cap M \arrow[r, tail] \arrow[d, tail] & M \arrow[r, ->>] \arrow[d, tail] & f(M) \arrow[d, equal] & \\
 & K \arrow[r, tail] \arrow[d, equal] & f \vee_A M \arrow[r, ->>, "f'"] \arrow[d, tail] & f(M) \arrow[d, tail] & \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & A \arrow[r, ->>, "f"'] & B \arrow[r] & 0
\end{tikzcd}$$

In the pointed context, we then find the well-known diamond isomorphism theorem

$$\frac{M}{K \cap M} \cong \frac{f \vee_A M}{K}$$

from group theory, and from module theory (see also [**57**] for a pointed version of this theorem). In the pointed context one can summarize the diamond isomorphism theorem in the diagram

$$\begin{tikzcd}
 & f \vee_A M \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & M \arrow[dl, dash] \\
 & K \cap M &
\end{tikzcd}$$

It is useful to notice that in particular, when the category $\mathcal{C}$ is a *homological category* [**36**], i.e. a pointed, regular, *(Bourn-) protomodular* category, $f \vee_A M = f^{-1}(f(M))$ is exactly the *supremum* of K and M (as subobjects of A). Recall that a pointed category $\mathcal{C}$ is protomodular [**6**] when the *split short five lemma* holds in $\mathcal{C}$. This means that, given a commutative diagram

$$\begin{tikzcd}
K' \arrow[r, tail, "k'"] \arrow[d, "u"'] & X' \arrow[r, shift right, "f'"'] \arrow[d, "v"] & Y' \arrow[l, shift right, "s'"'] \arrow[d, "w"] \\
K \arrow[r, tail, "k"] & X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "s"']
\end{tikzcd}$$

where

- f and f' are split epimorphisms i.e., $f' \circ s' = id_{Y'}, f \circ s = id_Y$;
- k and k' are the kernels of f and f' respectively i.e., $k = \mathsf{ker}\, f$ and $k' = \mathsf{ker}\, f'$;
- $v \circ k' = k \circ u$, $w \circ f' = f \circ v$ and $v \circ s' = s \circ w$.

If u and w are isomorphisms, then v is an isomorphism as well.

When the category $\mathcal{C}$ is regular, the split short five lemma is equivalent to the "short five lemma", where the split epimorphisms f and f' are replaced by regular epimorphisms [**6**].

The following lemma will be useful to prove that $f \mathbin{\vee\!\!\!\vee}_A M = f^{-1}(f(M))$ is the supremum of K and M.

LEMMA 2.1.3. [**8, 6**] *Given a commutative diagram in a homological category* $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, tail, "k'"] \arrow[d, "u"] & X' \arrow[r, two heads, "f'"] \arrow[d, "v"] & Y' \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & X \arrow[r, two heads, "f"'] & Y \arrow[r] & 0
\end{tikzcd}$$

where the horizontal arrows are short exact sequences. Then u is an isomorphism if and only if the right-hand square is a pullback.

In order to prove that $f \mathbin{\vee\!\!\!\vee}_A M = f^{-1}(f(M))$ is the smallest subobject of A containing both K and M, take L to be another subobject of A containing both K and M and consider the following diagram

$$\begin{tikzcd}
& f \mathbin{\vee\!\!\!\vee}_A M \arrow[rr, bend left, two heads] \arrow[dr, dotted] \arrow[ddr, bend left] & M \arrow[l] \arrow[r, two heads] \arrow[d, tail] & f(M) \arrow[d, dashed, "i"] & \\
& K \arrow[u, tail] \arrow[r, tail, "n"] \arrow[d, equal] & L \arrow[r, two heads, "p"] \arrow[d, tail, "l"] & f(L) \arrow[d, tail] & \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & A \arrow[r, two heads, "f"'] & A/K \arrow[r] & 0
\end{tikzcd}$$

where the vertical dotted arrow $i : f(M) \to f(L)$ comes from the fact that regular epimorphisms are orthogonal to monomorphisms. The monomorphism n is a normal monomorphism since $n = \mathsf{ker}\, p$. The protomodularity of $\mathcal{C}$ and Lemma 2.1.3. allow us to assert that the bottom right-hand square is a pullback. This implies the existence of the diagonal dotted arrow as desired.

To prove the double quotient isomorphism theorem and the Zassenhaus Lemma in the framework of star-relations, we need to introduce the following additional condition which holds true in both the pointed and the total contexts.

DEFINITION 2.1.4. A regular multi-pointed category $(\mathcal{C}, \mathcal{N})$ with kernels and with coequalisers of kernel stars is said to satisfy *the property* (C) if the following condition holds in $\mathcal{C}$:

for any kernel star $\mathsf{Eq}(f)^* \rightrightarrows A$ with coequaliser $f : A \to A/\mathsf{Eq}(f)^*$, and for any other kernel star $\mathsf{Eq}(g)^* \rightrightarrows A$ such that $\mathsf{Eq}(f)^* \subseteq \mathsf{Eq}(g)^*$ depicted as

$$\begin{tikzcd} & \mathsf{Eq}(g)^* \arrow[r, two heads] \arrow[d, shift left] \arrow[d, shift right] & f(\mathsf{Eq}(g)^*) \arrow[d, shift left] \arrow[d, shift right] \\ \mathsf{Eq}(f)^* \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, two heads, "f"'] & A/\mathsf{Eq}(f)^* \end{tikzcd}$$

$f(\mathsf{Eq}(g)^*) \rightrightarrows A/\mathsf{Eq}(f)^*$ is a kernel star.

Definition 2.1.5. **[24]** A *diamond* in a regular multi-pointed category $\mathcal{C}$ is a commutative diagram of the form

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "f"] & \\ Z \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & W & \end{tikzcd}$$

Such a diamond is :

(1) a *regular diamond* if all arrows are regular epimorphisms;
(2) *right saturated* if $f(\mathsf{Eq}(e)^*) = \mathsf{Eq}(h)^*$;
(3) *left saturated* if $e(\mathsf{Eq}(f)^*) = \mathsf{Eq}(g)^*$;
(4) *saturated* if it is both left saturated and right saturated.

Since in a star-regular category kernel stars are uniquely determined by the regular epimorphisms, we have this connection between some regular diamonds and the property (C) in the following

Proposition 2.1.6. *The following conditions are equivalent in a star-regular category $\mathcal{C}$.*

(a) *$\mathcal{C}$ satisfies the property (C);*
(b) *any regular diamond of the form*

$$\begin{tikzcd} & X \arrow[dl, two heads, "e"'] \arrow[dr, two heads, "f"] & \\ Z \arrow[dr, two heads, "g"'] & & Y \arrow[dl, equal] \\ & Y & \end{tikzcd} \tag{2.1}$$

is left saturated.

From now, we shall call a regular diamond of the form (2.1) a "right restricted" regular diamond.

Proof. $(a) \Rightarrow (b)$. Consider the following diagram

$$
\begin{tikzcd}
 & \mathsf{Eq}(f)^* \arrow[dl, bend right, "m"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \\
\mathsf{Eq}(g)^* \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "f'"'] & A \arrow[r, two heads, "g"] \arrow[d, "f"] & C \arrow[d, equal] \\
f(\mathsf{Eq}(g)^*) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & B \arrow[r, two heads, "q"'] & C
\end{tikzcd}
\tag{2.2}
$$

where the right-hand square is a right restricted regular diamond. We are going to prove that it is left saturated i.e., $f(\mathsf{Eq}(g)^*) \cong \mathsf{Eq}(q)^*$. For this, we consider the kernel star $\mathsf{Eq}(f)^* \rightrightarrows A$ of f. The fact that $qf = g$ gives us the induced monomorphism m in the diagram above. Then, by the property (C), $f(\mathsf{Eq}(g)^*) \overset{r_1}{\underset{r_2}{\rightrightarrows}} B$ is a kernel star. Moreover, since f' is a regular epimorphism and g is the coequaliser of (g_1, g_2), q is the coequaliser of (r_1, r_2). The fact that in a star-regular category any kernel star is the kernel star of its coequaliser allows us to conclude that $f(\mathsf{Eq}(g)^*) \cong \mathsf{Eq}(q)^*$.

$(b) \Rightarrow (a)$. Conversely, let us first consider the triangle of the diagram (2.2) where $\mathsf{Eq}(f)^*$, $\mathsf{Eq}(g)^*$ are two kernel stars and m is a monomorphism. We complete this triangle with the arrows f and g, the coequalisers of $\mathsf{Eq}(f)^*$ and $\mathsf{Eq}(g)^*$, respectively, and by $(r_1, r_2)f'$, the (regular epimorphism, star-relation) factorisation of the star (fg_1, fg_2). Since $gf_1 = gf_2$, we have the induced arrow q such that $qf = g$ which is then a regular epimorphism since g is a regular epimorphism. We so construct the right-hand square of the diagram (2.2) which is a right restricted regular diamond. Accordingly, by (b), it is left saturated and this implies that $f(\mathsf{Eq}(g)^*)$ is isomorphic to the kernel star $\mathsf{Eq}(q)^*$ of q. $\square$

In both the pointed and the total context, any right restricted regular diamond is left saturated. Indeed, in the following diagram

$$
\begin{tikzcd}
\mathsf{Eq}(g)^* \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "f'"'] & A \arrow[r, two heads, "g"] \arrow[d, "f"] & C \arrow[d, equal] \\
\mathsf{Eq}(q)^* \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & B \arrow[r, two heads, "q"'] & C
\end{tikzcd}
\tag{2.3}
$$

the left-hand square is a star-pullback. Then the left saturating property of such a right restricted regular diamond easily follows from the fact that regular epimorphisms are pullback stable in the pointed context, whereas they are "joint-pullback stable" in the total context. By Proposition 2.1.6., this implies that the property (C) is always satisfied in both the pointed and the total contexts. Also in the quasi-pointed context, the property (C) holds true. To see that, we use the the following lemma (Lemma 1.(1) in **[8]**)

LEMMA 2.1.7. *In any quasi-pointed category $\mathcal{C}$, if we consider the following commutative diagram*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "v"'] & C \arrow[d, "w"] \\
K' \arrow[r, "k'"'] & B \arrow[r, "f'"'] & D
\end{tikzcd}$$

where k' is the kernel of f'. If w is a monomorphism, then k is the kernel of f if and only if the left-hand square of the diagram is a pullback.

According to Lemma 2.1.7 and to diagram 2.3, when $\mathcal{C}$ is a regular quasi-pointed category, the left saturating property of right restricted regular diamonds comes from the stability of regular epimorphisms under pullback.

We may now state the double quotient isomorphism theorem:

PROPOSITION 2.1.8. *[**Double quotient isomorphism theorem**] Let $\mathcal{C}$ be a star-regular category satisfying the property (C). Then for two kernel stars $\mathsf{Eq}(f)^* \rightrightarrows A$, $\mathsf{Eq}(g)^* \rightrightarrows A$ such that $\mathsf{Eq}(f)^* \subseteq \mathsf{Eq}(g)^*$, the isomorphism*

$$A/\mathsf{Eq}(g)^* \cong \frac{A/\mathsf{Eq}(f)^*}{f(\mathsf{Eq}(g)^*)}$$

holds in $\mathcal{C}$.

PROOF. Consider the following diagram

$$\begin{tikzcd}
 & \mathsf{Eq}(f)^* \arrow[dl, bend right, "m"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \\
\mathsf{Eq}(g)^* \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, "f'"'] & A \arrow[r, two heads, "g"] \arrow[d, "f"'] & A/\mathsf{Eq}(g)^* \arrow[d, equal] \\
f(\mathsf{Eq}(g)^*) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A/\mathsf{Eq}(f)^* \arrow[r, two heads, "q"'] & A/\mathsf{Eq}(g)^*
\end{tikzcd}$$

in $\mathcal{C}$ where f and g are coequalisers of their kernel stars $\mathsf{Eq}(f)^* \rightrightarrows A$ and $\mathsf{Eq}(g)^* \rightrightarrows A$, respectively, q the induced arrow such that $qf = g$ (the existence of q follows from the fact that $gf_1 = gf_2$). $(r_1, r_2)f'$ is the factorisation (regular epimorphism, star-relation) of the star (fg_1, fg_2). Assuming that the property (C) is satisfied, it follows that $f(\mathsf{Eq}(g)^*) \overset{r_1}{\underset{r_2}{\rightrightarrows}} A/\mathsf{Eq}(f)^*$ is a kernel star. In fact, by Proposition 2.1.6, $f(\mathsf{Eq}(g)^*) \overset{r_1}{\underset{r_2}{\rightrightarrows}} A/\mathsf{Eq}(f)^*$ is the kernel star of q which is a regular epimorphism since g is a regular epimorphism. Hence the bottom row of the diagram above is a star-exact sequence and this gives the desired isomorphism

$$A/\mathsf{Eq}(g)^* \cong \frac{A/\mathsf{Eq}(f)^*}{f(\mathsf{Eq}(g)^*)}.$$

□

In the total context, let $\mathcal{C}$ be a regular category, and let $\mathsf{Eq}(f) \rightrightarrows A$, $\mathsf{Eq}(g) \rightrightarrows A$ be two kernel pairs such that $\mathsf{Eq}(f) \subseteq \mathsf{Eq}(g)$, then the isomorphism

$$A/\mathsf{Eq}(g) \cong \frac{A/\mathsf{Eq}(f)}{f(\mathsf{Eq}(g))}$$

holds in $\mathcal{C}$, where $f\colon A \twoheadrightarrow A/\mathsf{Eq}(f)$ is the coequaliser of its kernel pair $\mathsf{Eq}(f)$. This non-pointed version of the double quotient isomorphism theorem appears in [**54**].

In the pointed context, when $\mathcal{C}$ is a normal category, then for two normal monomorphisms $K \overset{k}{\rhd\!\!\longrightarrow} A$ and $L \overset{l}{\rhd\!\!\longrightarrow} A$ such that $K \subseteq L$, the double quotient isomorphism theorem can be expressed in this context by the isomorphism

$$A/L \cong \frac{A/K}{f(L)}$$

where $A \overset{f}{\longrightarrow\!\!\rhd} A/K$ is the cokernel of the kernel k. Furthermore, let us consider the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d, tail, "m"'] & K \arrow[d, "k"] & & \\
0 \arrow[r] & L \arrow[r, "l"] \arrow[d, "f'"'] & A \arrow[r, "g"] \arrow[d, "f"'] & A/L \arrow[r] \arrow[d, equal] & 0 \\
 & f(L) \arrow[r, tail, "l'"'] & A/K \arrow[r, "p"'] \arrow[d] & A/L & \\
 & & 0 & &
\end{tikzcd}$$

where g is the cokernel of the kernel l. m is the arrow such that $l \circ m = k$, thus m is a kernel since k is a kernel. p is the induced arrow such that $p \circ f = g$, hence p is a cokernel. By the property (C), we have that $l' = \mathsf{ker}\, p$. Moreover, it is easy to check that m is the kernel of the cokernel f', and since $\mathcal{C}$ is a normal category, this implies that the left column is a short exact sequence and thus

$f(L) \cong L/K$. Then we find the well-known "double quotient" isomorphism theorem

$$A/L \cong \frac{A/K}{K/L}$$

from group theory and module theory and which was generalized in [**57**], and then in [**18**] for the case of normal categories.

2.2. The Zassenhaus Lemma

DEFINITION 2.2.1. Let $(\mathcal{C}, \mathcal{N})$ be a regular multi-pointed category.

1. We define an *asymmetric join* of a regular epimorphism $f : A \to B$, and of a star $\tau : R \rightrightarrows A$ by
$$f \mathbin{\vee\!\!\!\vee}_A R = f^{-1}(f(R))^*$$
which is a star $f \mathbin{\vee\!\!\!\vee}_A \tau : f \mathbin{\vee\!\!\!\vee}_A R \rightrightarrows A$ obtained in the following star-pullback
$$\begin{tikzcd}
R \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, shift right, "\tau"'] \arrow[ddr, shift left] & & \\
& f \mathbin{\vee\!\!\!\vee}_A R \arrow[r] \arrow[d, shift right] \arrow[d, shift left, "f \mathbin{\vee\!\!\!\vee}_A \tau"] \arrow[dr, phantom, "\lrcorner", very near start] & f(R) \arrow[d, shift right] \arrow[d, shift left, "f(\tau)"] \\
& A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$
2. Moreover, if $\mathcal{C}$ has pushouts of regular epimorphisms, then for two kernel stars $\mathsf{Eq}(f)^* \rightrightarrows A$ and $\mathsf{Eq}(g)^* \rightrightarrows A$ on a same object A, we define their *supremum*
$$\mathsf{Eq}(f)^* \vee_A \mathsf{Eq}(g)^* = \mathsf{Eq}(g)^* \vee_A \mathsf{Eq}(f)^*$$
which is the smallest kernel star on A containing both the kernel stars $\mathsf{Eq}(f)^*$ and $\mathsf{Eq}(g)^*$. $\mathsf{Eq}(f)^* \vee_A \mathsf{Eq}(g)^*$ is the kernel star of the arrow $q = g'f = f'g$ defined by the following pushout
$$\begin{tikzcd}
\mathsf{Eq}(q)^* \arrow[dr, shift left] \arrow[dr, shift right] & \mathsf{Eq}(g)^* \arrow[l, dotted] \arrow[d, shift left] \arrow[d, shift right] & \\
\mathsf{Eq}(f)^* \arrow[u, dotted] \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, two heads, "f"] \arrow[d, "g"'] \arrow[dr, "q"] & A/\mathsf{Eq}(f)^* \arrow[d, "g'"] \\
& A/\mathsf{Eq}(g)^* \arrow[r, two heads, "f'"'] & D \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$
where f and g are the coequalisers of $\mathsf{Eq}(f)^*$ and $\mathsf{Eq}(g)^*$, respectively.

In the pointed context, when τ is a star-relation, a monomorphism in this context, Definition 2.2.1.1 becomes the Definition 2.1.1. In general, in a star-regular category, the notions of asymmetric join and of supremum are "totally" different: neither of the two is contained in the other one. But actually, these two notions coincide in a category with a good theory of ideals. The following theorem is given in [**25**] in the context of ideals (which are the same as kernel stars in a category with a good theory of ideals).

THEOREM 2.2.2. *Let $\mathcal{C}$ be a category with a good theory of ideals, then for two kernel stars $\mathsf{Eq}(f)^* \rightrightarrows A$ and $\mathsf{Eq}(g)^* \rightrightarrows A$ of regular epimorphisms $f\colon A \to B$ and $g : A \to C$, respectively, we have that*

$$f \vee\!\!\vee_A \mathsf{Eq}(g)^* = \mathsf{Eq}(f)^* \vee_A \mathsf{Eq}(g)^* = \mathsf{Eq}(g)^* \vee_A \mathsf{Eq}(f)^* = g \vee\!\!\vee_A \mathsf{Eq}(f)^*$$

.

PROOF. Indeed, let us consider the following diagram

$$\begin{tikzcd}
\mathsf{Eq}(g)^* \arrow[dr] \arrow[drr, bend left] \arrow[ddr, Rrightarrow] & & \\
& f^{-1}(f(\mathsf{Eq}(g)^*))^* \arrow[r, two heads] \arrow[d, Rrightarrow] \arrow[dr, phantom, "\lrcorner", very near start] & f(\mathsf{Eq}(g)^*) \arrow[d, Rrightarrow] \\
\mathsf{Eq}(f)^* \arrow[r, Rrightarrow] & A \arrow[r, two heads, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\
& C \arrow[r, two heads, "f'"'] & D
\end{tikzcd}$$

where the bottom square is a pushout of f and q which exist by Corollary 1.2.21. By Theorem 1.2.20, $f(\mathsf{Eq}(g)^*) \rightrightarrows A/\mathsf{Eq}(f)^*$ is the kernel star of q'. Since the top square is a star-pullback, by Lemma 1.1.28, $f^{-1}(f(\mathsf{Eq}(g)^*))^* \rightrightarrows X$ is the kernel star of $q'f$ which is the join $\mathsf{Eq}(f)^* \vee \mathsf{Eq}(g)^*$ of $\mathsf{Eq}(f)^*$ and $\mathsf{Eq}(g)^*$. $\square$

The pointed version of the following lemma appears in [**57**].

LEMMA 2.2.3. *Let $\mathcal{C}$ be a star-regular category satisfying the property (C). Then for two kernel stars $\kappa : \mathsf{Eq}(f)^* \Rrightarrow A$, $\sigma : \mathsf{Eq}(g)^* \Rrightarrow U$ of regular epimorphisms $f\colon A \to B$ and $g : U \to C$, respectively, and a monomorphism $u : U \to A$ such that $(\mathsf{Eq}(f)^* \cap (U \times U))^* \subseteq \mathsf{Eq}(g)^*$, we have that $f \vee\!\!\vee_A \mathsf{Eq}(g)^* \Rrightarrow f \vee\!\!\vee_A U$ is a kernel star and*

$$\frac{f \vee\!\!\vee_A U}{f \vee\!\!\vee_A \mathsf{Eq}(g)^*} \cong \frac{U}{\mathsf{Eq}(g)^*}$$

PROOF. Consider the following diagram

$$
\begin{tikzcd}
 & \mathsf{Eq}(g)^* \arrow[rr, two heads] \arrow[dr] \arrow[d, shift left] \arrow[d, shift right, "\sigma"'] & & f'(\mathsf{Eq}(g)^*) \arrow[dr, equal] \arrow[d, shift left] \arrow[d, shift right] & \\
 & & f \vee_A \mathsf{Eq}(g)^* \arrow[rr, two heads] \arrow[dd, dotted, shift left] \arrow[dd, dotted, shift right] & & f'(\mathsf{Eq}(g)^*) \arrow[dd, shift left] \arrow[dd, shift right] \\
(\mathsf{Eq}(f)^* \cap (U \times U))^* \arrow[uur] \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & U \arrow[rr, two heads, "f'"] \arrow[dr] \arrow[dd, "u"'] & & f(U) \arrow[dr, equal] \arrow[dd] & \\
 & & f \vee_A U \arrow[rr, two heads] \arrow[dd, "\kappa \vee_A u"] & & f(U) \arrow[dd] \\
\mathsf{Eq}(f)^* \arrow[r, shift left, "\kappa"] \arrow[r, shift right] & A \arrow[rr, two heads, "f"] \arrow[dr, equal] & & B \arrow[dr, equal] & \\
 & & A \arrow[rr, two heads, "f"'] & & B
\end{tikzcd}
$$

where the bottom square is obviously a pullback and the bottom front square of the outer cube is a pullback by the Definition 2.1.1. By the uniqueness of the factorisation of a star as a regular epimorphism followed by a star-relation, the isomorphism $f(\mathsf{Eq}(g)^*) \cong f'(\mathsf{Eq}(g)^*)$ holds as stars on B. This implies that the front diagram of the outer cube is a star-pullback by Definition 2.2.1.1. Then, the upper front square is a star-pullback as well (by Lemma 1.1.13). Since $(\mathsf{Eq}(f)^* \cap (U \times U))^* \rightrightarrows U$ is a kernel star of the regular epimorphism f' and $(\mathsf{Eq}(f)^* \cap (U \times U))^* \subseteq \mathsf{Eq}(g)^*$, by property (C) and the double quotient isomorphism theorem

$$
\begin{tikzcd}
 & (\mathsf{Eq}(f)^* \cap (U \times U))^* \arrow[dl, bend right] \arrow[d, shift left] \arrow[d, shift right] & \\
\mathsf{Eq}(g)^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & U \arrow[r, two heads, "g"] \arrow[d, two heads, "f'"'] & C \arrow[d, equal] \\
f'(\mathsf{Eq}(g)^*) \arrow[r, shift left] \arrow[r, shift right] & f(U) \arrow[r, two heads] & C
\end{tikzcd}
$$

we have that $f'(\mathsf{Eq}(g)^*) \rightrightarrows f(U)$ is again a kernel star and

$$
\frac{f(U)}{f'(\mathsf{Eq}(g)^*)} \cong \frac{U}{\mathsf{Eq}(g)^*}.
$$

Thus, since the left-hand square of the commutative diagram below (which is the upper front square of the cube above)

$$
\begin{tikzcd}
f \vee_A \mathsf{Eq}(g)^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & f \vee_A U \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, equal] \\
f'(\mathsf{Eq}(g)^*) \arrow[r, shift left] \arrow[r, shift right] & f(U) \arrow[r, two heads] & C
\end{tikzcd}
$$

is a star-pullback, this implies that $f \Vvee_A \mathsf{Eq}(g)^* \Rrightarrow f \Vvee_A U$ is the kernel star of the diagonal $f \Vvee_A U \twoheadrightarrow f(U) \twoheadrightarrow C$ which is a regular epimorphism as a composite of two regular epimorphisms. Furthermore, by the star-regularity of $\mathcal{C}$, the top row of the last diagram above is a star-exact sequence and we finally have

$$\frac{f \Vvee_A U}{f \Vvee_A \mathsf{Eq}(g)^*} \cong \frac{U}{\mathsf{Eq}(g)^*}$$

. □

Theorem 2.2.4 (**The Zassenhaus Lemma**). *Let $\mathcal{C}$ be a star-regular category with pushouts of regular epimorphisms and satisfying the property (C). Then, for two kernel stars $\kappa : \mathsf{Eq}(f)^* \Rrightarrow U$, $\sigma : \mathsf{Eq}(g)^* \Rrightarrow V$ of regular epimorphisms $f : U \to E$ and $g : V \to F$, respectively, and two monomorphisms $u : U \to A$, $v : V \to A$, we have the isomorphisms*

$$\begin{aligned}
&\frac{f \Vvee_U (U \cap V)}{f \Vvee_U ((\mathsf{Eq}(f)^* \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*)} \\
\cong\ &\frac{U \cap V}{(\mathsf{Eq}(f)^* \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*} \\
\cong\ &\frac{g \Vvee_V (U \cap V)}{g \Vvee_V ((\mathsf{Eq}(f)^* \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*)}
\end{aligned}$$

Proof. First of all, let us construct

$$M = (\mathsf{Eq}(f)^* \cap (V \times V))^* \ \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*.$$

For this, consider the diagram

$$\begin{tikzcd}
M \arrow[drr, shift left] \arrow[drr] \arrow[drr, shift right] & & (\mathsf{Eq}(g)^* \cap (U \times U))^* \arrow[ll, dashed] \arrow[r] \arrow[d, shift left] \arrow[d] \arrow[d, shift right] & \mathsf{Eq}(g)^* \arrow[d, shift left] \arrow[d] \arrow[d, shift right] \\
(\mathsf{Eq}(f)^* \cap (V \times V))^* \arrow[u, dashed] \arrow[rr, shift left] \arrow[rr] \arrow[rr, shift right] \arrow[d] & & U \cap V \arrow[r, tail, "u'"] \arrow[d, tail, "v'"'] & V \arrow[d, tail, "v"] \\
\mathsf{Eq}(f)^* \arrow[rr, shift left] \arrow[rr, "\kappa"] \arrow[rr, shift right] & & U \arrow[r, tail, "u"'] & A
\end{tikzcd}$$

where the bottom right square is a pullback by definition of intersection of subobjects and both the bottom left squares and the top right squares are star-pullbacks by definition of the inverse image of a star-relation. This implies that both $(\mathsf{Eq}(f)^* \cap (V \times V))^* \Rrightarrow U \cap V$ and $(\mathsf{Eq}(g)^* \cap (U \times U))^* \Rrightarrow U \cap V$ are kernel stars of the regular epimorphisms f_1

$$\begin{tikzcd}
(\mathsf{Eq}(f)^* \cap (V \times V))^* \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d] & U \cap V \arrow[r, two heads, "f_1"] \arrow[d, "v'"'] & f(U \cap V) \arrow[d, "v_1"] \\
\mathsf{Eq}(f)^* \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & U \arrow[r, two heads, "f"'] & E
\end{tikzcd}$$

and g_1

$$\begin{tikzcd}
(\mathsf{Eq}(g)^* \cap (U \times U))^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & U \cap V \arrow[r, two heads, "g_1"] \arrow[d, "u'"'] & g(U \cap V) \arrow[d, "u_1"] \\
\mathsf{Eq}(g)^* \arrow[r, shift left] \arrow[r, shift right] & V \arrow[r, two heads, "g"'] & F
\end{tikzcd}$$

respectively (since u_1 and v_1 are monomorphisms). By Definition 2.2.1.2, we have the existence of the kernel star $M \rightrightarrows U \cap V$ which is the supremum of the two kernel stars $(\mathsf{Eq}(f)^* \cap (V \times V))^* \rightrightarrows U \cap V$ and $(\mathsf{Eq}(g)^* \cap (U \times U))^* \rightrightarrows U \cap V$. Moreover, $M \rightrightarrows U \cap V$ is the kernel star of the regular epimorphism q obtained in the following pushout (which exists by assumption):

$$\begin{tikzcd}
U \cap V \arrow[r, two heads, "f_1"] \arrow[d, two heads, "g_1"'] \arrow[dr, two heads, "q"] & f(U \cap V) \arrow[d, two heads, "\overline{g_1}"] \\
g(U \cap V) \arrow[r, two heads, "\overline{f_1}"'] & T
\end{tikzcd} \tag{2.4}$$

Thus, we have $T \cong \dfrac{U \cap V}{M}$.

On the one hand, we have the kernel stars $\mathsf{Eq}(f)^* \rightrightarrows U$, $M \rightrightarrows U \cap V$ of regular epimorphisms $f\colon U \to E$ and $q\colon U \cap V \to T$, respectively, and the monomorphism $v' : U \cap V \to U$ such that $(\mathsf{Eq}(f)^* \cap (V \times V))^* \subseteq M$. By applying Lemma 2.2.3., one obtains the kernel star $f \vee\!\!\vee_U M \rightrightarrows f \vee\!\!\vee_U (U \cap V)$ and the isomorphism

$$\frac{U \cap V}{M} \cong T \cong \frac{f \vee\!\!\vee_U (U \cap V)}{f \vee\!\!\vee_U M}.$$

On the other hand, in the same way, we have the kernel stars $\mathsf{Eq}(g)^* \rightrightarrows V$, $M \rightrightarrows U \cap V$ of regular epimorphisms $g\colon V \to F$ and $q\colon U \cap V \to T$, respectively, and the monomorphism $u' : U \cap V \to V$ such that $(\mathsf{Eq}(g)^* \cap (U \times U))^* \subseteq M$. According to Lemma 2.2.3., $g \vee\!\!\vee_V M \rightrightarrows g \vee\!\!\vee_V (U \cap V)$ is a kernel star and

$$\frac{U \cap V}{M} \cong T \cong \frac{g \vee\!\!\vee_V (U \cap V)}{g \vee\!\!\vee_V M}$$

□

In the total context, the Zassenhaus Lemma holds in a regular category with pushouts of regular epimorphisms. This version of the Zassenhaus Lemma generalises the non-pointed one given by W. Tholen in [**54**] since pushouts of regular epimorphisms along monomorphisms were needed in addition in $\mathcal{C}$ (in this case $\mathcal{C}$ has pushouts of regular epimorphisms along any morphism). The pointed version given by O. Wyler in [**57**], and thus the well-known versions from the categories Gp of groups and R-Mod of modules over a fixed ring R

for instance, are also generalised by Theorem 2.2.4. In order to recover these pointed versions of the Zassenhaus Lemma, we shall need the following lemma:

LEMMA 2.2.5. *Let $\mathcal{C}$ be a star-regular category, $\kappa : \mathsf{Eq}(f)^* \Rrightarrow A$ a kernel star of a regular epimorphism $f\colon A \to B$, $\tau : R \Rrightarrow U$ a star-relation and $u : U \to A$ a monomorphism in $\mathcal{C}$. Then*

(1)
$$f \Vvee_A R \cong f \Vvee_A (f' \Vvee_U R)$$
as stars on A, where $f'\colon U \twoheadrightarrow f(U)$.

(2) *When $\mathcal{C}$ has good theory of ideals and $\tau : R \Rrightarrow U$ is a kernel star, then*
$$f \Vvee_A R \cong f \Vvee_A (\mathsf{Eq}(f')^* \vee_U R).$$
where $\mathsf{Eq}(f')^*$ is the kernel star of f'.

PROOF. (1) Put $D = f \Vvee_A R$, $L = f' \Vvee_U R$ and $D' = f \Vvee_A L$ and then consider the following diagrams

$$\begin{tikzcd}
 & R \arrow[rr, two heads] \arrow[dr] \arrow[d, Rightarrow, "\sigma"'] & & f(R) \arrow[dr, equal] \arrow[d, Rightarrow] & \\
 & & D \arrow[rr, two heads] \arrow[ddl, Rightarrow, "\tau"] & & f(R) \arrow[ddl, Rightarrow] \\
Eq(f')^* \arrow[r, Rightarrow, "\lambda"] \arrow[d] & U \arrow[rr, two heads, "f'"] \arrow[d, tail, "u"'] & & f(U) \arrow[d, tail] & \\
\mathsf{Eq}(f)^* \arrow[r, Rightarrow, "\kappa"] & A \arrow[rr, two heads, "f"'] & & B &
\end{tikzcd}$$

$$\begin{tikzcd}
 & R \arrow[rr, two heads] \arrow[dr] \arrow[dd, Rightarrow, "\tau"'] & & f(R) \arrow[dr, equal] \arrow[dd, Rightarrow] & \\
 & & L \arrow[rr, two heads] \arrow[dl, Rightarrow] \arrow[dd] & & f(R) \arrow[dl, Rightarrow] \arrow[dd] \arrow[dddl, Rightarrow] \\
Eq(f')^* \arrow[r, Rightarrow, "\lambda"] \arrow[dd] & U \arrow[rr, two heads, "f'"] \arrow[dd, tail, "u"'] & & f(U) \arrow[dd, tail] & \\
 & & D' \arrow[rr, two heads] \arrow[dl, Rightarrow] & & f(L) \arrow[dl, Rightarrow] \\
\mathsf{Eq}(f)^* \arrow[r, Rightarrow, "\kappa"] & A \arrow[rr, two heads, "f"'] & & B &
\end{tikzcd}$$

In the top diagram, by the uniqueness of the factorisation (regular epimorphism, star-relation), one finds the isomorphism $f(R) \cong f'(R)$ (as stars on B). In both diagrams, $\lambda : K \Rrightarrow U$ is a kernel star of the regular epimorphism f', and by the Definition 2.2.1.1, we have the existence of the star

(-relation) $L \rightrightarrows U$. In the bottom diagram, we have that $f(L) \cong f'(L)$ (as stars on B). Moreover by Proposition 1.1.21, since the direct image and inverse image of star-relations preserve order we have the isomorphisms $f'(L) = f'(f'^{-1}(f(R))) \cong f(R)$. Then the isomorphisms $f(L) \cong f'(L) \cong f(R) \cong f'(R)$ hold as stars on B. Finally, consider the following cube

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr, dotted] \arrow[dd, shift left] \arrow[dd, shift right] & & f(R) \arrow[dr, equal] \arrow[dd, shift left] \arrow[dd, shift right] & \\
& D' \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & f(R) \arrow[dd, shift left] \arrow[dd, shift right] \\
A \arrow[rr] \arrow[dr, equal] & & B \arrow[dr, equal] & \\
& A \arrow[rr] & & B
\end{tikzcd}$$

where both the back and the front square are star-pullbacks by Definition 2.2.1.1 and the dotted arrow is the one induced by the universal property of this latter star-pullback. Since the bottom square is obviously a pullback, this implies that the top square is a pullback as well by Lemma 1.1.13. It follows that $D \cong D'$, as desired.

(2) When $\mathcal{C}$ has a good theory of ideals and $\tau : R \rightrightarrows U$ is a kernel star, by Theorem 2.2.2 one has

$$f' \mathbb{V}_U R = \mathsf{Eq}(f')^* \vee_U R$$

and the desired isomorphism easily follows. □

Remark 2.2.6. When $\mathcal{C}$ is a category with a good theory of ideals, one has the following isomorphisms:

$$\begin{aligned}
& f \mathbb{V}_U M = f \mathbb{V}_U ((\mathsf{Eq}(f)^* \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*) \\
& \cong f \mathbb{V}_U ((\mathsf{Eq}(f)^* \cap (U \times U) \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (V \times V) \cap (U \times U))^*) \\
& \cong f \mathbb{V}_U ((\mathsf{Eq}(f)^* \cap (U \cap V) \times (U \cap V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \cap V) \times (U \cap V))^*) \\
& \cong f \mathbb{V}_U (\mathsf{Eq}(g)^* \cap (U \cap V) \times (U \cap V))^* \text{ by Lemma 2.2.5 2} \\
& \cong f \mathbb{V}_U (\mathsf{Eq}(g)^* \cap (U \times U))^*
\end{aligned}$$

and

$$\begin{aligned}
& g \mathbb{V}_V M = g \mathbb{V}_V ((\mathsf{Eq}(f)^* \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \times U))^*) \\
& \cong g \mathbb{V}_V ((\mathsf{Eq}(f)^* \cap (U \times U) \cap (V \times V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (V \times V) \cap (U \times U))^*) \\
& \cong g \mathbb{V}_V ((\mathsf{Eq}(f)^* \cap (U \cap V) \times (U \cap V))^* \vee_{U \cap V} (\mathsf{Eq}(g)^* \cap (U \cap V) \times (U \cap V))^*) \\
& \cong g \mathbb{V}_V (\mathsf{Eq}(f)^* \cap (U \cap V) \times (U \cap V))^* \text{ by Lemma 2.2.5 2} \\
& \cong g \mathbb{V}_V (\mathsf{Eq}(f)^* \cap (V \times V))^*
\end{aligned}$$

Then, according to the Lemma 2.2.3 and to Remark 2.2.6, when $\mathcal{C}$ is a category with a good theory of ideals, one has the kernel stars

$$f \mathbb{V}_U (\mathsf{Eq}(g)^* \cap (U \times U))^* \rightrightarrows f \mathbb{V}_U (U \cap V)$$

and

$$g_{\mathbb{V}_V}(\mathsf{Eq}(f)^* \cap (V \times V))^* \Longrightarrow g_{\mathbb{V}_V}(U \cap V)\ .$$

Since such a category with a good theory of ideals obviously satisfies the property (C), and has pushouts of regular epimorphisms 1.2.21, the Zassenhaus Lemma becomes

$$\frac{f_{\mathbb{V}_U}(U \cap V)}{f_{\mathbb{V}_U}(\mathsf{Eq}(g)^* \cap (U \times U))^*} \cong \frac{U \cap V}{M} \cong \frac{g_{\mathbb{V}_V}(U \cap V)}{g_{\mathbb{V}_V}(\mathsf{Eq}(f)^* \cap (V \times V))^*}$$

Corollary 2.2.7. *In the pointed context, let $\mathcal{C}$ be a normal category with pushouts of regular epimorphisms along regular epimorphisms. For two kernels $K \to U$, $L \to V$ with cokernels $f\colon U \to E$, $g\colon V \to F$, respectively, and two monomorphisms $U \longrightarrow A$, $V \longrightarrow A$, we have the isomorphisms*

$$\begin{aligned}\frac{f_{\mathbb{V}_U}(U \cap V)}{f_{\mathbb{V}_U}((K \cap V) \vee_{U \cap V} (L \cap U))} &\cong \frac{U \cap V}{(K \cap V) \vee_{U \cap V} (L \cap U)} \\ &\cong \frac{g_{\mathbb{V}_V}(U \cap V)}{g_{\mathbb{V}_V}((K \cap V) \vee_{U \cap V} (L \cap U))}\end{aligned}$$

When $\mathcal{C}$ is an ideal determined category, the isomorphisms

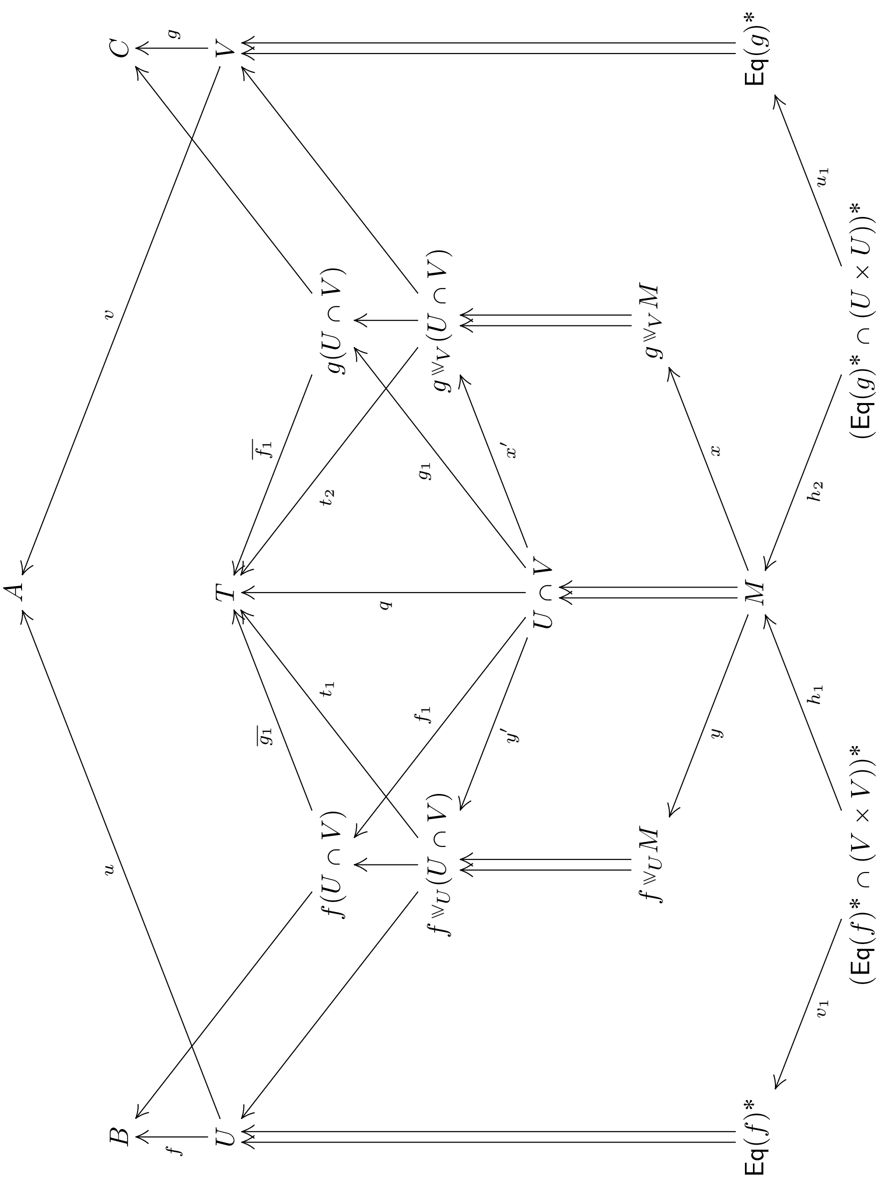
$$f_{\mathbb{V}_U}((K \cap V) \vee_{U \cap V} (L \cap U)) \cong f_{\mathbb{V}_U}(L \cap U) \ \textit{and}$$

$$g_{\mathbb{V}_V}((K \cap V) \vee_{U \cap V} (L \cap U)) \cong g_{\mathbb{V}_V}(K \cap V)$$

hold in $\mathcal{C}$. The Zassenhaus Lemma then takes the following form given in **[57]** *and which is similar to the classical one in group theory*

$$\frac{f_{\mathbb{V}_U}(U \cap V)}{f_{\mathbb{V}_U}(L \cap U)} \cong \frac{U \cap V}{(K \cap V) \vee_{U \cap V} (L \cap U)} \cong \frac{g_{\mathbb{V}_V}(U \cap V)}{g_{\mathbb{V}_V}(K \cap V)}$$

The Zassenhaus Lemma can be summarized in the following diagram



2.3. The lattice of kernel stars

It is known that in a category with a good theory of ideals, the class of ideals (which coincides with the class of kernel stars) on an object X in $\mathcal{C}$ forms a lattice which is modular [**25**]. In a star-regular category $\mathcal{C}$ satisfying the property (C) and with pushouts of regular epimorphisms along regular epimorphisms, for a regular epimorphism $f\colon X \to Y$ in $\mathcal{C}$, the class of kernel stars on X containing $\mathsf{Eq}(f)^*$ is a lattice. In this section we prove that the lattice, which we denote by $\mathsf{KernelStars}_f(\mathsf{X})$, is isomorphic to the lattice $\mathsf{KernelStars}(\mathsf{Y})$ of kernel stars on Y.

LEMMA 2.3.1. *Let $\mathcal{C}$ be a star-regular category satisfying the property (C). Then the following conditions hold for a regular epimorphism $f : X \twoheadrightarrow Y$, and kernel stars $\lambda : K \rightrightarrows X$ such that $\mathsf{Eq}(f)^* \leqslant \lambda$, $\sigma : S \rightrightarrows Y$:*

(1) *$f^{-1}(\sigma)^*$ is a kernel star on X containing $\mathsf{Eq}(f)^*$;*
(2) *$f^{-1}(f(\lambda))^* = \lambda$;*
(3) *$f(f^{-1}(\sigma)^*) = \sigma$.*

PROOF. (1) Consider the diagram

$$\begin{tikzcd}
 & f^{-1}(S)^* \arrow[r, two heads] \arrow[d, shift left, "f^{-1}(\sigma)^*"'] \arrow[d, shift right] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, shift left] \arrow[d, shift right, "\sigma"'] \\
\mathsf{Eq}(f)^* \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

then by Lemma 1.1.15, $f^{-1}(\sigma)^* : f^{-1}(S)^* \rightrightarrows X$ is the kernel star of the composite $X \overset{f}{\twoheadrightarrow} Y \overset{g}{\twoheadrightarrow} Y/S$, then it obviously contains $\mathsf{Eq}(f)^*$ (since $gf\kappa_1 = gf\kappa_2$).

(2) In the following diagram

$$\begin{tikzcd}
 & K \arrow[rr, two heads] \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right, "\lambda"'] & & f(K) \arrow[dd, shift left, "f(\lambda)"] \arrow[dd, shift right] \\
 & & f^{-1}(f(K))^* \arrow[ur, two heads] \arrow[dl, shift left] \arrow[dl, shift right] & \\
\mathsf{Eq}(f)^* \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] \arrow[uur, dotted] & X \arrow[rr, two heads, "f"'] \arrow[d, two heads, "t"] & & Y \arrow[d, two heads, "h"] \\
 & X/K \arrow[rr, two heads, "q"'] & & Y/f(K),
\end{tikzcd}$$

by the property (C), $f(\lambda) : f(K) \rightrightarrows Y$ is a kernel star on Y (the kernel star of h), this implies that $f^{-1}(f(\lambda))^* : f^{-1}(f(K))^* \rightrightarrows X$ is a kernel star on X (the kernel star of hf) and it contains both $\mathsf{Eq}(f)^*$ and K. Moreover, by the double quotient isomorphism theorem, one has the isomorphism

$$X/K \cong Y/f(K),$$

thus q is an isomorphism. This implies that $f^{-1}(f(K))^* = K$, or equivalently with arrows, that $f^{-1}(f(\lambda))^* = \lambda$.

(3) In the following diagram

$$
\begin{tikzcd}
 & & & f(f^{-1}(S)^*) \arrow[dl] \arrow[ddl, bend left, shift left] \arrow[ddl, bend left, shift right] \\
 & f^{-1}(S)^* \arrow[r, two heads] \arrow[urr, bend left, two heads] \arrow[d, shift left] \arrow[d, shift right, "f^{-1}(\sigma)^*"'] & S \arrow[d, shift left, "\sigma"] \arrow[d, shift right] & \\
\mathsf{Eq}(f)^* \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] \arrow[ur, dotted, bend left] & X \arrow[r, two heads, "f"'] \arrow[d, "t"'] & Y \arrow[d, "h"] & \\
 & Y/S \arrow[r, equal] & Y/S, &
\end{tikzcd}
$$

according to (1), $f^{-1}(\sigma)^* : f^{-1}(S)^* \rightrightarrows X$ is a kernel star on X containing $\mathsf{Eq}(f)^*$ (the kernel star of $hf = t$) and by property (C), $f(f^{-1}(\sigma)^*) : f(f^{-1}(S)^*) \rightrightarrows X$ is the kernel star of h, which is $\sigma\colon S \rightrightarrows Y$.

$\square$

Proposition 2.3.2. *Let $\mathcal{C}$ be a star-regular category with pushouts of regular epimorphisms and satisfying the property (C), and let $f : X \twoheadrightarrow Y$ be a regular epimorphism. There is a lattice isomorphism between the lattice $\mathsf{KernelStars}_f\mathsf{(X)}$ of kernel stars on X containing $\mathsf{Eq}(f)^*$ and the lattice $\mathsf{KernelStars(Y)}$ of kernel stars on Y.*

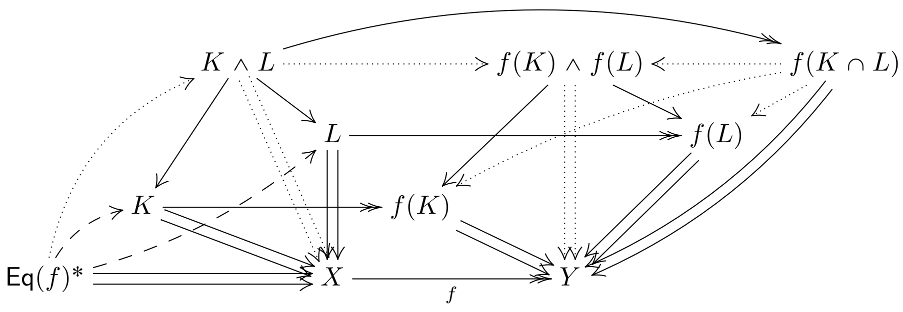
Proof. Consider the map

$$
\begin{array}{rccc}
\Theta : & \mathsf{KernelStars}_f\mathsf{(X)} & \longrightarrow & \mathsf{KernelStars(Y)} \\
 & \kappa : K \rightrightarrows X & \mapsto & \Theta(\kappa) = f(\kappa) : f(K) \rightrightarrows Y
\end{array}
$$

we are going to prove that Θ is a lattice isomorphism.

- Let $\kappa : K \rightrightarrows X$ and $\delta\colon L \rightrightarrows X$ be two kernel stars on X containing $\mathsf{Eq}(f)^*$ such that $\Theta(\kappa) = \Theta(\delta)$. Thus $f(\kappa) = f(\delta)$ by definition of Θ. This implies that $\kappa = f^{-1}(f(\kappa))^* = f^{-1}(f(\delta))^* = \delta$ by Lemma 2.3.1(2). This shows that Θ is injective.
- Let $\tau\colon T \rightrightarrows Y$ be a kernel star on Y. By Lemma 2.3.1.(1) and (2), $f^{-1}(\tau)^*\colon f^{-1}(T)^* \rightrightarrows X$ is a kernel star on X containing $\mathsf{Eq}(f)^*$ and $f(f^{-1}(\tau)^*) = \tau$. This proves that $\Theta(f^{-1}(\tau)^*) = \tau$ and Θ is surjective.

- Let $\kappa : K \rightrightarrows X$ and $\delta : L \rightrightarrows X$ be two kernel stars on X containing $\mathsf{Eq}(f)^*$ and consider the following diagram



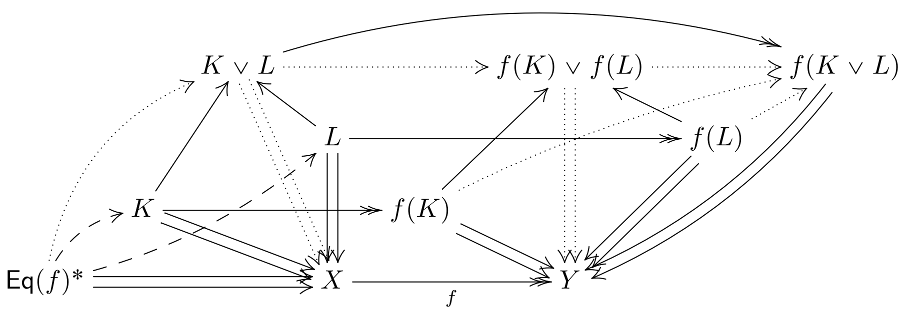
by Lemma 1.1.18, $\kappa \wedge \delta : K \wedge L \rightrightarrows X$ is a kernel star on X and moreover it contains $\mathsf{Eq}(f)^*$. Always by Lemma 1.1.18 and by property (C), $f(\kappa) \wedge f(\delta) : f(K) \wedge f(L) \rightrightarrows Y$ is a kernel star on Y. By the property (C), $f(\kappa \wedge \delta) : f(K \wedge L) \rightrightarrows Y$ is a kernel star on Y, and one sees that $f(\kappa \wedge \delta)$ is contained in both $f(\kappa)$ and $f(\delta)$. Finally, one has

$$\begin{aligned} f^{-1}(f(\kappa) \wedge f(\delta))^* &= (f^{-1}(f(\kappa) \wedge f(\delta)))^* \\ &\cong (f^{-1}(f(\kappa)) \wedge f^{-1}(f(\delta)))^* \\ &\cong f^{-1}(f(\kappa))^* \wedge f^{-1}(f(\delta))^* \text{ by Lemma 1.1.29} \\ &\cong \kappa \wedge \delta \text{ by Lemma 2.3.1} \\ &\cong f^{-1}(f(\kappa \wedge \delta))^* \end{aligned}$$

By Lemma 2.3.1(3), this implies that

$$f(f^{-1}(f(\kappa) \wedge f(\delta))^*) \cong f(\kappa) \wedge f(\delta) \cong f(f^{-1}(f(\kappa \wedge \delta))^*) \cong f(\kappa \wedge \delta).$$

- For $\kappa : K \rightrightarrows X$ and $\delta : L \rightrightarrows X$ two kernel stars on X containing $\mathsf{Eq}(f)^*$, we are going to prove that $f(\kappa \vee \delta) = f(\kappa) \vee f(\delta)$. For this, consider the following diagram



$\kappa \vee \delta : K \vee L \rightrightarrows X$ is a kernel star on X containing $\mathsf{Eq}(f)^*$. Likewise $f(\kappa) \vee f(\delta) : f(K) \vee f(L) \rightrightarrows Y$ is the supremum of the kernel stars $f(K)$ and $f(L)$ and by the universal property of the kernel star $f(K) \vee f(L)$, one can easily checks the existence of the induced arrow $K \vee L \overset{\varphi}{\dashrightarrow} f(K) \vee f(L)$. Moreover, by the property (C),

$f(\kappa \vee \delta) : f(K \vee L) \rightrightarrows Y$ is a kernel on Y and $f(K \vee L)$ contains both $f(K)$ and $f(L)$. Since $f(K) \vee f(L)$ is the smallest kernel star on Y containing both $f(K)$ and $f(L)$, one has the induced arrow $f(K) \vee f(L) \overset{\phi}{\dashrightarrow} f(K \vee L)$. Finally the composite $\phi\varphi$ is a regular epimorphism; this implies that the monomorphism ϕ is a regular epimorphism, thus an isomorphism, and one then has the desired result $f(\kappa \vee \delta) = f(\kappa) \vee f(\delta)$.

□

Corollary 2.3.3. *The following conditions are equivalent in a category $\mathcal{C}$ with a good theory of ideals:*

(1) *$Id_{\mathcal{C}}(X)$ the lattice of kernel stars on X in $\mathcal{C}$ is a distributive lattice;*
(2) *for any regular epimorphism $f : X \twoheadrightarrow Y$ and two kernel stars $\iota : I \rightrightarrows X$, $\lambda \rightrightarrows X$ on X, $f(\iota \cap \lambda) = f(\iota) \cap f(\lambda)$.*

Proof. Recall that the lattice $Id_{\mathcal{C}}(X)$ is a distributive lattice if one of the following two equivalents properties hold

$$\iota \vee (\delta \wedge \tau) = (\iota \vee \sigma) \wedge (\iota \vee \tau), \quad \iota \wedge (\delta \vee \tau) = (\iota \wedge \sigma) \vee (\iota \wedge \tau) \tag{2.5}$$

for kernel stars $\iota : I \rightrightarrows X, \lambda : J \rightrightarrows X$ and $\delta : L \rightrightarrows X$.

(1) $\Rightarrow$ (2). Let $f : X \to Y$ be a regular epimorphism with kernel star $\kappa : \mathsf{Eq}(f)^* \rightrightarrows X$, and $\iota : K \rightrightarrows X, \delta : J \rightrightarrows X$ be two kernel stars. By Proposition 2.3.2., it suffices to prove that $f^{-1}(f(\iota \wedge \delta))^* = f^{-1}(f(\iota) \wedge f(\delta))^*$

$$\begin{aligned}
f^{-1}(f(\iota \wedge \delta))^* &\cong \kappa \vee (\iota \wedge \delta) \text{ by Theorem 2.2.2} \\
&\cong (\kappa \vee \iota) \wedge (\kappa \vee \delta) \text{ by assumption} \\
&\cong f^{-1}(f(\iota))^* \wedge f^{-1}(f(\delta))^* \\
&\cong (f^{-1}(f(\iota)) \wedge f^{-1}(f(\delta)))^* \text{ by Lemma 1.1.29} \\
&\cong f^{-1}(f(\iota) \wedge f(\delta))^*
\end{aligned}$$

Conversely, let us consider three kernel stars $\iota : K \rightrightarrows X, \delta : J \rightrightarrows X$ and $\tau : L \rightrightarrows X$ and $g : X \twoheadrightarrow Y$ the coequaliser of ι.

$$\begin{aligned}
(\iota \vee \delta) \wedge (\iota \vee \tau) &\cong g^{-1}(g(\delta))^* \wedge g^{-1}(g(\tau))^* \\
&\cong (g^{-1}(g(\delta)) \wedge g^{-1}(g(\tau)))^* \\
&\cong (g^{-1}(g(\delta) \wedge g(\tau)))^* \\
&\cong (g^{-1}(g(\delta \wedge \tau)))^* \text{ by assumption} \\
&\cong g^{-1}(g(\delta \wedge \tau))^* \\
&\cong \iota \vee (\delta \wedge \tau)
\end{aligned}$$

□

2.4. Some remarks on regular diamonds

Under the assumption that any right restricted regular diamond (2.1) is left saturated, the following lemma gives a characterization of star-regular categories.

Lemma 2.4.1. *In a regular multi-pointed category $(\mathcal{C}, \mathcal{N})$ with $\mathcal{N}$-kernels, with coequalisers of kernels stars and verifying the property that any right restricted regular diamond (2.1) is left saturated, the following are equivalent:*

(1) *$(\mathcal{C}, \mathcal{N})$ is a star-regular category;*
(2) *an arrow $m : U \to V$ is a monomorphism if and only if*

$$\mathsf{Eq}(m)^* = \Delta_U^* = \mathsf{Eq}(1_U)^*$$

.

Proof. (1) $\Rightarrow$ (2). See Lemma 1.1.31
(2) $\Rightarrow$ (1). It suffices to prove that any regular epimorphism is a coequaliser of a star. Let $f : A \twoheadrightarrow B$ be a regular epimorphism and consider the following commutative diagram

$$\begin{tikzcd}
\mathsf{Eq}(f)^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\tau"'] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "q"] & B \arrow[d, equal] \\
\Delta^*_{A/_{\mathsf{Eq}(f)^*}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha"'] & A/_{\mathsf{Eq}(f)^*} \arrow[r, "m"'] \arrow[d, equal] & B \arrow[d, equal] \\
\mathsf{Eq}(m)^* \arrow[r, shift left] \arrow[r, shift right] & A/_{\mathsf{Eq}(f)^*} \arrow[r, "m"'] & B
\end{tikzcd}$$

where q is a coequaliser of the kernel star $\mathsf{Eq}(f)^* \rightrightarrows A$ of f, m is the induced arrow such that $mq = f$. The arrow m is then a regular epimorphism and it remains to prove that m is a monomorphism. Since we assume that any right restricted regular diamond (2.1) is left saturated in $\mathcal{C}$, the composite $\alpha\tau$ is a regular epimorphism. This implies that α is a regular epimorphism as well, thus an isomorphism. It follows that $\mathsf{Eq}(m)^* = \Delta^*_{A/_{\mathsf{Eq}(f)^*}}$ and, by assumption, m is a monomorphism. $\square$

Corollary 2.4.2. *In the pointed context, for pointed regular category with cokernels of kernels, the following conditions are equivalent:*

(1) *$\mathcal{C}$ is a normal category;*
(2) *a morphism $m \colon A \to B$ is a monomorphism if and only if* $\mathsf{Ker}\, m = 0$

Definition 2.4.3. [**24**] A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be *saturating* if a diamond of the form

$$\begin{tikzcd}
& X \arrow[dl, equal] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, equal] \\
& Y &
\end{tikzcd}$$

is right saturated. That is the induced arrow $\Delta_X^* \overset{\alpha}{\dashrightarrow} \Delta_Y^*$ from the kernel star Δ_X^* of 1_X to the kernel star Δ_Y^* of 1_Y is a regular epimorphism, which is equivalent to saying that the induced arrow $K_X \overset{\beta}{\dashrightarrow} K_Y$ is a regular epimorphism, where K_X and K_Y denote the $\mathcal{N}$-kernel of 1_X and 1_Y, respectively.

In the total context, saturating morphisms are regular epimorphisms, while in the pointed context all morphisms are saturating. The following lemma gives a connection between some regular diamonds and saturating regular epimorphisms.

Lemma 2.4.4. *In a regular multi-pointed category $\mathcal{C}$, the following are equivalent:*

(a) *any regular epimorphism is saturating;*
(b) *any right restricted regular diamond (2.1) is right saturated.*

Proof. $(a) \Rightarrow (b)$. Consider the following diagram

$$\begin{tikzcd}
\Delta_X^* \arrow[r, "\theta"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{Eq}(e)^* \arrow[r, "f'"] \arrow[d, shift left] \arrow[d, shift right] & \Delta_Y^* \arrow[d, shift left] \arrow[d, shift right] \\
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, two heads, "f"] \arrow[d, "e"] & Y \arrow[d, equal] \\
X \arrow[r, two heads, "e"'] & Z \arrow[r, two heads, "g"'] & Y
\end{tikzcd}$$

where the bottom right square is a right restricted regular diamond. By assumption, since any regular epimorphism is saturating, the composite $f'\theta$ is a regular epimorphism. This implies that f' is a regular epimorphism, as desired.
$(b) \Rightarrow (a)$. Conversely, given a regular epimorphism $f : X \twoheadrightarrow Y$, the right saturation property obviously implies that f is saturating. □

Lemma 2.4.5. [**24**] *Let $\mathcal{C}$ be a regular multi-pointed category with $\mathcal{N}$-kernels. Then the following conditions are equivalent for every regular epimorphism $f\colon X \to Y$:*

(a) *f is saturating;*
(b) *for any relation $\sigma\colon R \rightrightarrows Y$ one has $f(f^{-1}(\sigma)^*) = \sigma^*$.*

Definition 2.4.6. [**24**] A multi-pointed category $\mathcal{C}$ is said to have the *symmetric saturation property* if the following equivalent conditions hold:

(a) any left saturated regular diamond is right saturated;
(b) any right saturated regular diamond is left saturated;
(c) left/right saturated regular diamonds are the same as the saturated ones.

This notion of right/left saturated diamonds leads to a characterization of star-regular categories and categories with a good theory of ideals.

Theorem 2.4.7. [**24**] *For a regular multi-pointed category $\mathcal{C}$ with $\mathcal{N}$-kernels, the following conditions are equivalent*

(1) *$\mathcal{C}$ is a star-regular category;*
(2) • *$\mathcal{C}$ admits coequalisers of kernel stars;*
  • *every left saturated regular diamond in $\mathcal{C}$ is a pushout.*

Proof. Assume that $\mathcal{C}$ is a star-regular category. To prove the first point, let us consider a kernel star $\mathsf{Eq}(f)^*$ of a morphism $f\colon X \to Y$. If me is the factorisation (regular epimorphism, monomorphism) of f, then $\mathsf{Eq}(f)^* = \mathsf{Eq}(e)^*$. The arrow e is a regular epimorphism and $\mathcal{C}$ is star-regular, thus e is the coequaliser of its kernel star. To prove the second point, consider a regular diamond

$$\begin{tikzcd} & X \arrow[dl, two heads, "e"'] \arrow[dr, two heads, "f"] & \\ Z \arrow[dr, two heads, "g"'] & & Y \arrow[dl, two heads, "h"] \\ & W & \end{tikzcd}$$

which is left saturated. Then by assumption, $e(\mathsf{Eq}(f)^*)$ is the kernel star of the regular epimorphism g and by star-regularity of $\mathcal{C}$, g is the coequaliser of $e(\mathsf{Eq}(f)^*)$. So, by Proposition 1.1.30, the above left saturated diamond is a pushout. For the converse, let us assume (2). It remains to prove that any regular epimorphism is a coequaliser of a star (its kernel star). For this, consider a regular epimorphism $f\colon X \twoheadrightarrow Y$ and $g\colon X \twoheadrightarrow Z$ the coequaliser of $\mathsf{Eq}(f)^*$ the kernel star of f; this implies that $\mathsf{Eq}(f)$ is the kernel star of g. Now, consider the diamond

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, two heads, "g"] & \\ X \arrow[dr, two heads, "f"'] & & Z \arrow[dl, two heads, "h"] \\ & Y & \end{tikzcd}$$

where h is the induced arrow coming from the universal property of the coequaliser g and then is a regular epimorphism since f and g are. This diamond is regular and left saturated, then by assumption it is a pushout. Then h is an isomorphism and f is the coequaliser of $\mathsf{Eq}(f)^*$. □

Theorem 2.4.8. **[24]** *The following conditions are equivalent for a star-regular category $\mathcal{C}$ with coequalisers of ideals:*

(1) *$\mathcal{C}$ has a good theory of ideals;*
(2) • *$\mathcal{C}$ admits coequalisers of ideals;*
  • *pushouts of regular epimorphisms in $\mathcal{C}$ are the same as left saturated regular diamonds.*
(3) • *$\mathcal{C}$ admits coequalisers of ideals;*
  • *pushouts of regular epimorphisms in $\mathcal{C}$ are the same as right saturated regular diamonds.*

This theorem also proves that any category with a good theory of ideals has the symmetric saturation property. It was proved in **[24]** that when the symmetry saturation property holds in a regular category, in the pointed context this characterises a normal subtractive category, whereas in the total context, it characterizes a regular Goursat category. When the symmetric saturation property holds for every right restricted regular diamond, one can say that the multi-pointed category $\mathcal{C}$ has the *right restricted symmetric saturation property.*

Lemma 2.4.9. *Let $(\mathcal{C}, \mathcal{N})$ be a star-regular category with saturating regular epimorphisms. Then the following conditions are equivalent:*

(a) *$\mathcal{C}$ satisfies the property (C);*
(b) *$\mathcal{C}$ has the right restricted symmetric saturation property.*

Proof. $(a) \Rightarrow (b)$. If every regular epimorphism in $\mathcal{C}$ is saturated, then by Lemma 2.4.4, any right restricted regular diamond is right saturated. If we assume (a), by Lemma 2.1.6 one has that any right restricted regular diamond is left saturated. Then $\mathcal{C}$ has the right restricted symmetric saturation property. The converse is straightforward. □

Recall that in a regular category, when such a commutative square of regular epimorphisms

$$\begin{tikzcd} X \arrow[r, two heads, "h"] \arrow[d, two heads, "e"'] & Z \arrow[d, two heads, "g"] \\ Y \arrow[r, two heads, "f"'] & T \end{tikzcd} \tag{2.6}$$

is pullback, it is always a pushout.

Definition 2.4.10. **[9]** In a regular category, a commutative square of regular epimorphisms (2.6) is said to be a *regular pushout* (or alternatively a *double extension* in **[29]**, **[35]**) when the comparison arrow

$$\begin{tikzcd} X \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, two heads, "h"] \arrow[ddr, bend right, two heads, "e"'] & & \\ & Y \times_T Z \arrow[r, two heads, "f'"] \arrow[d, two heads, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, two heads, "g"] \\ & Y \arrow[r, two heads, "f"'] & T \end{tikzcd}$$

$\alpha \colon X \to Y \times_T Z$ to the pullback of g along f is a regular epimorphism.

Such commutative squares are always pushouts. It was shown in **[14]** that exact Mal'tsev categories can be characterised through regular pushouts. That is, a regular category is an exact Mal'tsev category if and only if any diagram of the form (2.6) is a regular pushout.

Observation: Any right restricted regular diamond is a regular pushout.

Indeed, first notice that any right restricted regular diamond is always a pushout. Next, in a diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, two heads, "h"] \arrow[ddr, bend right, two heads, "e"'] & & \\
& Y \arrow[r, two heads, "f'"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & Z \arrow[d, equal] \\
& Y \arrow[r, two heads, "f"'] & Z
\end{tikzcd}$$

where the outer diagram is a right restricted regular diamond, the comparison arrow α is obviously a regular epimorphism. This notion of regular pushout leads M. Gran and D. Rodelo to introduce the following

Definition 2.4.11. **[28]** A commutative square (2.6) is a *star-regular pushout* when the comparison dotted arrow $N_e \overset{\varphi}{\dashrightarrow} N_{\pi_1}$

$$\begin{tikzcd}
N_e \arrow[dr, dotted, two heads] \arrow[dd] & & \\
& N_{\pi_1} \arrow[dd] & \\
X \arrow[rr, two heads, "h"] \arrow[dr, "\alpha"] \arrow[dd, two heads, "e"'] & & Z \arrow[dd, two heads, "g"] \\
& Y \times_T Z \arrow[ur, two heads, "\pi_0"] \arrow[dl, two heads, "\pi_1"] & \\
Y \arrow[rr, two heads, "f"'] & & T
\end{tikzcd}$$

from the $\mathcal{N}$-kernel N_e of e to the $\mathcal{N}$-kernel N_{π_1} of π_1 is a regular epimorphism.

In the total context, since $\mathcal{N}$-kernels are isomorphisms, this gives exactly the fact that $X \xrightarrow{\alpha} Y \times_T Z$ is a regular epimorphism, and then (2.6) is a regular pushout. In the pointed context, since $\mathcal{N}$-kernels are the same as kernels, a star-regular pushout becomes the pointed version of the right saturation property for any commutative diagram (2.6): that is the induced arrow $\mathsf{Ker}\, e \overset{\varphi}{\dashrightarrow} \mathsf{Ker}\, \pi_1$ is a regular epimorphism.

Lemma 2.4.12. *Any right restricted regular diamond is a star-regular pushout.*

Proof. Consider the following diagrams

$$\begin{tikzcd}
& X \arrow[dl, two heads, "e"'] \arrow[dr, two heads, "g"] & \\
Y \arrow[dr, two heads, "f"'] & & Z \arrow[dl, equal] \\
& Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
N_e \arrow[r, dotted, "\varphi"] \arrow[d, tail] & N_{1_Y} \arrow[d, tail] & \\
X \arrow[r, "\alpha"] \arrow[d, two heads, "e"'] & Y \arrow[r, two heads, "h"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & Z \arrow[d, equal] \\
Y \arrow[r, equal] & Y \arrow[r, two heads, "f"'] & Z
\end{tikzcd}$$

where the left diagram is a right restricted regular diamond, which is a regular pushout; this implies that α is a regular epimorphism. Moreover, by Lemma 1.1.29, the upper diagram is a pullback. Thus φ is a regular epimorphism, as desired. □

In the following, we shall prove that any star-regular pushout is a right saturated regular diamond. For this we need in addition the following condition in a star-regular category introduced in [**23**]. This condition will also be used in Chapter 3, and holds true in both the pointed and the total context and also in any quasi-pointed category.

Definition 2.4.13. [**23**] An object X is said to be $\mathcal{N}$*-trivial* when $1_X \in \mathcal{N}$; equivalently, X is $\mathcal{N}$-trivial when any morphism whose domain or codomain is X belongs to $\mathcal{N}$. Because of the presence of $\mathcal{N}$-kernels, we have that if a composite fg belongs to $\mathcal{N}$ and g is a regular epimorphism (Lemma 1.1.19), then also f belongs to $\mathcal{N}$. This easily implies that $\mathcal{N}$-trivial objects are closed under quotients.

Definition 2.4.14. [**23**] A multi-pointed category $(\mathcal{C}, \mathcal{N})$ has *enough trivial objects* when $\mathcal{N}$ is a closed ideal [**31**] i.e., any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object, and moreover, the class of $\mathcal{N}$-trivial objects, besides being closed under quotients, is also closed under subobjects and squares. A class of objects is closed under square means that for any $\mathcal{N}$-trivial object X, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial.

There are several equivalent conditions defining when a category has enough trivial objects (see Proposition 3.5 in [**25**]). For our work, the following will be the most suitable one.

Proposition 2.4.15. *The following conditions are equivalent in a regular multi-pointed category* $(\mathcal{C}; \mathcal{N})$ *with* $\mathcal{N}$*-kernels*

(1) $\mathcal{C}$ *has enough trivial objects;*
(2) *for any relation* $(r_1; r_2) : R \rightrightarrows X$ *in* $\mathcal{C}$ *and any arrow* $f : K \to R$ *such that* $r_1 f, r_2 f \in \mathcal{N}$*, one then has that* $f \in \mathcal{N}$*.*

Proof. (1) ⇒ (2). Consider the diagram

$$
\begin{tikzcd}
K \arrow[r, "f"] \arrow[dr, "e"'] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \\
& T \arrow[u, dashed, "t"] \arrow[ur, shift left, "m_1"] \arrow[ur, shift right, "m_2"'] &
\end{tikzcd}
$$

where $(m_1, m_2)e$ is the factorisation of the star $(r_1 f, r_2 f)$ as a regular epimorphism followed by a star-relation and t the induced arrow making the two triangles commute. Since $m_1 e$ and $m_2 e$ belong to $\mathcal{N}$ and e is a regular epimorphism, then by Lemma 1.1.19 one has that m_1 and m_2 belong to $\mathcal{N}$. Then by assumption, m_1 and m_2 factor through the trivial object L_X the $\mathcal{N}$-kernel of the identity 1_X on X depicted in

$$\begin{tikzcd} T \arrow[rr, shift left, "m_1"] \arrow[rr, shift right, "m_2"'] \arrow[dr, dotted, shift left, "s_1"] \arrow[dr, dotted, shift right, "s_2"'] & & X \arrow[r, "1_X"] & X \\ & L_X. \arrow[ur, "l_X"'] & & \end{tikzcd}$$

This gives the monomorphism $T \rightarrowtail L_X \times L_X$ to the trivial object $L_X \times L_X$ since trivial objects are closed under subobjects. Accordingly, T is a $\mathcal{N}$-trivial object and consequently, f belongs to $\mathcal{N}$. For the converse, let $g\colon X \to Y$ be a morphism in $\mathcal{N}$, and let me be a factorisation of g

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, two heads, "e"'] & & Y \\ & M \arrow[ur, tail, "m"'] & \end{tikzcd}$$

as a regular epimorphism e followed by a monomorphism m. Then me belongs to $\mathcal{N}$ and e is a regular epimorphism, this implies that the monomorphism m is in $\mathcal{N}$. By (2), one then has that $1_M\colon M \to M$ is in $\mathcal{N}$ i.e., M is a $\mathcal{N}$-trivial object. It is easy to see that under the assumption (2), the class of $\mathcal{N}$-trivial objects is closed under subobjects, and under squares; for this latter case use the projections $(\pi_1, \pi_2)\colon X \times X \rightrightarrows X$, for a $\mathcal{N}$-trivial object X. □

Remark 2.4.16. In particular, Proposition 2.4.15 says that when a category $(\mathcal{C}, \mathcal{N})$ has enough trivial objects, for a monomorphism $f\colon X \to Y$ and for any arrow $g\colon Z \to X$, such that the composite fg belongs to $\mathcal{N}$, then g also belongs to $\mathcal{N}$.

Thanks to this proposition, it is clear that both in the total context and in the pointed context $\mathcal{C}$ has enough trivial objects. In the proto-pointed context, this is not true in general. In the case where $\mathcal{C}$ is a quasi-pointed category, it was shown in [**23**] that $\mathcal{C}$ has enough trivial objects. But in the category Rng of unitary rings, there are not enough trivial objects. Indeed, if we consider Rng the category of unitary rings, the ring $\mathbb{Z}$ of integers is a $\mathcal{N}$-trivial object but not $\mathbb{Z} \times \mathbb{Z}$. The following diagram

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "1_{\mathbb{Z}\times\mathbb{Z}}"] & \mathbb{Z} \times \mathbb{Z} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbb{Z} \end{tikzcd}$$

shows that $\mathbb{Z} \times \mathbb{Z}$ is not a $\mathcal{N}$-trivial object.

Remark 2.4.17. Let $\kappa\colon \mathsf{Eq}(f)^* \rightrightarrows X$ be a kernel star of a morphism $f\colon X \to Y$ and $k\colon K \to X$ the $\mathcal{N}$-kernel of f in a regular multi-pointed category $\mathcal{C}$ with $\mathcal{N}$-kernels. Since $f\kappa_1 = f\kappa_2 \in \mathcal{N}$, then by the universal property of the $\mathcal{N}$-kernel k, there exist two morphisms $\lambda_1, \lambda_2\colon \mathsf{Eq}(f)^* \to K$ making the following triangle

$$\begin{tikzcd}
\mathsf{Eq}(f)^* \arrow[rr, shift left, "\kappa_1"] \arrow[rr, shift right, "\kappa_2"'] \arrow[dr, dotted, shift left, "\lambda_1"] \arrow[dr, dotted, shift right, "\lambda_2"'] & & X \arrow[r, "f"] & Y \\
& K \arrow[ur, "k"'] & &
\end{tikzcd}$$

commute. If moreover $\mathcal{C}$ has enough trivial objects, then $\lambda\colon \mathsf{Eq}(f)^* \rightrightarrows K$ is a star (-relation).

Lemma 2.4.18. *In a star-regular category with saturating regular epimorphisms and with enough trivial objects, a star-regular pushout is a right saturated regular diamond.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathsf{Eq}(e)^* \arrow[rr, "\phi"] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\sigma"] & & \mathsf{Eq}(\pi_1)^* \arrow[rr, "\varphi"] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\lambda"] & & \mathsf{Eq}(g)^* \arrow[dd, shift left] \arrow[dd, shift right] \\
& N_e \arrow[rr, "\theta"] \arrow[dl, tail] & & N_{\pi_1} \arrow[dl, tail] & \\
X \arrow[rr, "\alpha"] \arrow[dd, two heads, "e"'] & & Y \times_T Z \arrow[rr, two heads, "\pi_0"] \arrow[dd, two heads, "\pi_1"'] \arrow[dd, phantom, very near start, "\lrcorner"] & & Z \arrow[dd, two heads, "g"] \\
& & & & \\
Y \arrow[rr, equal] & & Y \arrow[rr, two heads, "f"'] & & T
\end{tikzcd}$$

where the outer bottom square is a commutative square of regular epimorphisms (2.6) with $h = \pi_0\alpha$. Assuming that this diagram is a star-regular pushout, θ is then a regular epimorphism. Under the assumption that every regular epimorphism is saturating in $\mathcal{C}$, applying Lemma 2.4.5 to conclude that φ is a regular epimorphism. It remains to prove that ϕ is also a regular epimorphism. By the Remark 2.4.17, one has the existence of the star-relations λ and σ. Moreover the upper left-hand square with vertical arrows is a star-pullback by Lemma 1.1.28 and the middle diagram is a pullback by Lemma 1.1.29; this implies that the upper left-hand diagram with diagonal arrows is a star-pullback by Lemma 1.1.13. Finally, applying Lemma 2.4.5 to obtain that ϕ is a regular epimorphism. $\square$

CHAPTER 3

Effective descent morphisms in star-regular categories

The regular epimorphisms are known to be *descent* morphisms in any regular category, but they fail to be *effective descent* morphisms, in general [**42**]. This is due to the fact that not all equivalence relations in a regular category $\mathcal{C}$ are *effective*, namely they do not necessarily occur as the kernel pair of a morphism in $\mathcal{C}$. The recent work on descent theory (see [**41**], for instance, and the references therein) has made it clear that the right notion of "good quotient" in a general category is precisely provided by the notion of effective descent morphism. This is obviously the case when the regular category is *exact* [**2**], since all equivalence relations are effective. However, this is also the case for many (not necessarily exact) regular categories: G. Janelidze and M. Sobral call such categories *almost exact* [**40**]. The present chapter deals with the problem of finding a simple condition guaranteeing that regular epimorphisms and effective descent morphisms coincide in a star-regular category. In the first section, we briefly recall the definition of effective descent morphism in two equivalent way: the "elementary descent" theory which is the descent via discrete fibrations and the "monadic descent" theory, the descent theory via the monads and their algebras. Then, we give the condition on a star-regular category guaranteeing that regular epimorphisms are effective descent morphisms. In the last section, we analyse several further examples of categories satisfying this condition i.e., categories in which regular epimorphims coincide with effective descent morphisms; some of which have been studied in [**17**] by T. Everaert using a different approach. All results in this chapter are joint work with Marino Gran and also appear in our paper [**26**].

3.1. Descent theory

3.1.1. Descent via discrete fibrations.

Let $\mathcal{C}$ be a category with pullbacks.

Definition 3.1.1. A discrete fibration of equivalence relations in $\mathcal{C}$ is an internal functor $(f_0, f_1)\colon S \to R$ as in the commutative diagram

$$\begin{tikzcd} S \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "f_1"'] & D \arrow[d, "f_0"] \\ R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \end{tikzcd} \tag{3.1}$$

having the property that the square involving the second projections is a pullback. This implies that also the square involving the first projections is a pullback.

An arrow φ between discrete fibrations (f_0, f_1) and (f'_0, f'_1) is a quadruple $(\varphi_0, \varphi'_0, \varphi_1, \varphi'_1)$ making the following diagram

$$\begin{tikzcd}
S \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dr, "\varphi'_1"'] \arrow[dd, "f_1"'] & & D \arrow[dr, "\varphi'_0"] \arrow[dd, "f_0"] & \\
& S' \arrow[rr, shift left, "\pi'_1"] \arrow[rr, shift right, "\pi'_2"'] \arrow[dd, "f'_1"'] & & D' \arrow[dd, "f'_0"] \\
R \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] \arrow[dr, "\varphi_1"'] & & E \arrow[dr, "\varphi_0"] & \\
& R' \arrow[rr, shift left, "p'_1"] \arrow[rr, shift right, "p'_2"'] & & E'
\end{tikzcd} \tag{3.2}$$

commute. Composition of discrete fibrations is defined in the natural way, by using the composition of morphisms in $\mathcal{C}$. Thus the class of discrete fibrations of equivalence relations in a category $\mathcal{C}$ forms a category.

Let $p\colon E \to B$ be a morphism in $\mathcal{C}$ and $(p_1, p_2)\colon \mathsf{Eq}(p) \rightrightarrows E$ be the kernel pair of p. We write $\mathsf{DiscFib}(\mathsf{Eq}(p))$ for the category of discrete fibrations of equivalence relations over $\mathsf{Eq}(p)$ and morphisms between them. Let

$$K^p : (\mathcal{C} \downarrow B) \to \mathsf{DiscFib}(\mathsf{Eq}(p))$$

be the functor sending an object (A, α_A) in the comma category $(\mathcal{C} \downarrow B)$ to the discrete fibration (C) in the commutative diagram

$$\begin{tikzcd}
\mathsf{Eq}(q) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\overline{\sigma}"'] \arrow[dr, phantom, "(C)"] & E \times_B A \arrow[r, "q"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "(D)"] & A \arrow[d, "\alpha_A"] \\
\mathsf{Eq}(p) \arrow[r, shift left] \arrow[r, shift right] & E \arrow[r, "p"'] & B
\end{tikzcd} \tag{3.3}$$

where (D) is the pullback of p and α_A, $\mathsf{Eq}(q)$ is the kernel pair of q, and $\overline{\sigma}$ the induced arrow making the two left squares commutative. For a morphism $f\colon (A, \alpha_A) \to (A', \alpha'_A)$ in $(\mathcal{C} \downarrow B)$, $K^p(f) = (\varphi, \overline{\varphi})$ is defined as in the diagram

$$\begin{tikzcd}
\mathsf{Eq}(q) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "\overline{\varphi}"] \arrow[dd, "\overline{\sigma}"'] & & E \times_B A \arrow[rr, "q"] \arrow[dr, "\varphi"] \arrow[dd, "\sigma"] & & A \arrow[dr, "f"] \arrow[dd, "\alpha_A"] & \\
& \mathsf{Eq}(q') \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, "\overline{\lambda}"] & & E \times_B A' \arrow[rr, "q'"'] \arrow[dl, "\lambda"] & & A' \arrow[dl, "\alpha_{A'}"] \\
\mathsf{Eq}(p) \arrow[rr, shift left] \arrow[rr, shift right] & & E \arrow[rr, "p"'] & & B &
\end{tikzcd} \tag{3.4}$$

where φ and $\overline{\varphi}$ are induced arrows making all the diagrams commute.

DEFINITION 3.1.2. [**42, 41**]

A morphism $p : E \to B$ in $\mathcal{C}$ is said to be:

(1) a *descent morphism* if the functor $K^p : (\mathcal{C} \downarrow B) \to \mathsf{DiscFib}(\mathsf{Eq}(p))$ is full and faithful;
(2) an *effective descent morphism* if the functor K^p is a category equivalence.

The following theorem gives a characterization of effective descent morphisms in a finitely complete regular category. (see [**41, 40, 17**], for instance).

THEOREM 3.1.3. *Let $\mathbb{C}$ be a regular category. Then:*

(1) *$p\colon E \to B$ in $\mathbb{C}$ is a descent morphism if and only if it is a regular epimorphism;*
(2) *a regular epimorphism $p : E \to B$ in $\mathbb{C}$ is an effective descent morphism if and only if, for any discrete fibration of equivalence relations*

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "f_1"'] & D \arrow[d, "f_0"] \\ \mathsf{Eq}(p) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \end{tikzcd} \tag{3.5}$$

over the kernel pair $\mathsf{Eq}(p)$ of p, the equivalence relation R is effective.

For the morphism $p\colon E \to B$, let

$$U^p\colon \mathsf{DiscFib}(\mathsf{Eq}(p)) \to (\mathcal{C} \downarrow E)$$

be the forgetful functor sending any discrete fibration of the form (3.5) of equivalence relations over $\mathsf{Eq}(p)$ to the object (D, f_0) in $(\mathcal{C} \downarrow E)$. For a morphism $(\varphi, \overline{\varphi})$ in $\mathsf{DiscFib}(\mathsf{Eq}(p))$ as in diagram (3.4), define $U^p(\varphi, \overline{\varphi}) = \varphi$ which is a morphism in $(\mathcal{C} \downarrow E)$. Let

$$p^*\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)$$

be the induced "change-of-base" functor sending an object (A, α_A) in $(\mathcal{C} \downarrow B)$ to $\sigma\colon E \times_B A \to E$ in $(\mathcal{C} \downarrow B)$ obtained by pulling back

$$\begin{tikzcd} E \times_B A \arrow[r, "p'"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\alpha_A"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

along p. This functor sends a morphism $f\colon (A, \alpha_A) \to (A', \alpha'_A)$ in $(\mathcal{C} \downarrow B)$ to the morphism $p^*(f) = \varphi$ in $(\mathcal{C} \downarrow E)$ as in diagram (3.4).

Let (A, α_A) be a morphism in $(\mathcal{C} \downarrow B)$, one has that

$$\begin{aligned} U^p K^p(A, \alpha_A) &= U^p((C)) \text{ where } (C) \text{ is the discrete fibration in diagram (3.3)} \\ &= \sigma \\ &= p^*(A, \alpha_A), \end{aligned}$$

and

$$\begin{aligned} U^pK^p(f\colon (A,\alpha_A)\to(A',\alpha_{A'})) &= U^p(\varphi,\overline{\varphi}) \text{ where } (\varphi,\overline{\varphi}) \text{ is as in diagram (3.4)} \\ &= \varphi \\ &= p^*(f\colon (A,\alpha_A)\to(A',\alpha_{A'})). \end{aligned}$$

Thus, for a morphism $p\colon E\to B$, we get the commutative diagram

$$\begin{tikzcd} (\mathcal{C}\downarrow B) \arrow[rr, "K^p"] \arrow[dr, "p^*"'] & & \mathsf{DiscFib}(\mathsf{Eq}(p)) \arrow[dl, "U^p"] \\ & (\mathcal{C}\downarrow E) & \end{tikzcd} \tag{3.6}$$

3.1.2. Monadic descent theory.

In this section we are going to briefly describe the equivalence between descent via discrete fibrations and "monadic descent theory". For any arrow $p\colon E\to B$, the functor $p^*\colon (\mathcal{C}\downarrow B)\to(\mathcal{C}\downarrow E)$ has a left adjoint

$$\Sigma_p\colon (\mathcal{C}\downarrow E)\to(\mathcal{C}\downarrow B)$$

given by composition with p: for an object (I,α_I) in $(\mathcal{C}\downarrow E)$, one sets $\Sigma_p(I,\alpha_I)=(I,p\alpha_I)$ which is an object in $(\mathcal{C}\downarrow B)$; Σ_p is defined on arrows in an obvious way. Let us recall that any adjunction gives rise to a *monad*, and this fact is important for the description of monadic descent theory. In order to explain the equivalence between descent via discrete fibrations and "monadic descent theory", we need to recall the following notions. The reader may consult **[3, 4, 41]** for instance, for more information on monads and the monadic description of effective descent morphisms.

Definition 3.1.4. (1) A *monad* on a category $\mathcal{C}$ is a triple $T=(T,\eta,\mu)$ with $T\colon\mathcal{C}\to\mathcal{C}$ an endofunctor, $\eta\colon 1_X\to T$ and $\mu\colon T^2\to T$ two natural transformations making the diagram

$$\begin{tikzcd} T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"'] & T \arrow[l, "\eta_T"'] \arrow[d, "T\eta"] \arrow[dl, "1_T"] \\ T^2 \arrow[r, "\mu"'] & T & T^2 \arrow[l, "\mu"] \end{tikzcd}$$

commute.

(2) • Let $T=(T,\eta,\mu)$ be a monad in $\mathcal{C}$. An *algebra* on this monad, called a T-algebra, is a pair (C,ξ) with $C\in\mathcal{C}$ and an arrow $\xi\colon T(C)\to C$ making the following diagram

$$\begin{tikzcd} T^2(C) \arrow[r, "T\xi"] \arrow[d, "\mu_C"'] & T(C) \arrow[d, "\xi"'] & C \arrow[l, "\eta_C"'] \arrow[d, equal] \\ T(C) \arrow[r, "\xi"'] & C \arrow[r, equal] & C \end{tikzcd}$$

commute.

- An arrow $f\colon (C,\xi) \to (C',\xi')$ of T-algebras is an arrow $f\colon C \to C'$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} T(C) \arrow[r, "Tf"] \arrow[d, "\xi"'] & T(C') \arrow[d, "\xi'"] \\ C \arrow[r, "f"'] & C' \end{tikzcd}$$

commutes.
- T-algebras and their morphisms form a category called the *Eilenberg-Moore* category of the monad T, denoted by $\mathcal{C}^T$.

Since any adjunction gives rise to a monad, let us denote by $T^p = p^* \circ \Sigma_p$ the monad on $(\mathcal{C} \downarrow E)$ corresponding to the adjoint functors $\Sigma_p \dashv p^*$. Write $(\mathcal{C} \downarrow E)^{T^p}$ for the corresponding Eilenberg-Moore category, and $U^{T^p}\colon (\mathcal{C} \downarrow E)^{T^p} \to (\mathcal{C} \downarrow E)$ for the forgetful functor sending any algebra $((A,\alpha_A),\xi)$ to the object (A,α_A) of $(\mathcal{C} \downarrow E)$. Finally write $K^{T^p}\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)^{T^p}$ for the comparison functor sending an object (D,β_D) of $(\mathcal{C} \downarrow B)$ to the T^P-algebra $(p^*(D,\beta_D),\xi')$ and we obtain the following commutative triangle

$$\begin{tikzcd} (\mathcal{C} \downarrow B) \arrow[rr, "K^{T^p}"] \arrow[dr, "p^*"'] & & (\mathcal{C} \downarrow E)^{T^p} \arrow[dl, "U^{T^p}"] \\ & (\mathcal{C} \downarrow E) & \end{tikzcd} \tag{3.7}$$

It was shown in [**41**] that the triangles (3.6) and (3.7) are canonically equivalent, establishing a categorical equivalence between the categories $(\mathcal{C} \downarrow E)^{T^p}$ and $\mathsf{DiscFib}(\mathsf{Eq}(p))$.

To sum up, we have the functor $p^*\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)$ with left adjoint $\Sigma_p\colon (\mathcal{C} \downarrow E) \to (\mathcal{C} \downarrow B)$ and a monad $T^p = p^* \circ \Sigma_p$ on $(\mathcal{C} \downarrow E)$. When the arrow $p\colon E \to B$ is an effective descent morphism, then via the categorical equivalence $(\mathcal{C} \downarrow E)^{T^p} \simeq \mathsf{DiscFib}(\mathsf{Eq}(p))$, the category equivalence of the Definition 3.1.2 corresponds to $K^{T^p}\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)^{T^p}$: this gives exactly the fact that the functor p^* is *monadic*. Accordingly, the following "monadic definition" of effective descent morphism is equivalent to the "elementary" one given in Definition 3.1.2.

Definition 3.1.5. A morphism $p\colon E \to B$ in a category $\mathcal{C}$ with pullbacks is an effective descent morphism if the functor $p^*\colon (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E)$ is monadic.

3.2. Semi-effective star regular categories

In this section, we give a sufficient condition on a star-regular category guaranteeing that regular epimorphisms are effective descent morphisms.

Proposition 3.2.1. *Let $\mathcal{C}$ be a star-regular category. If the following diagram*

$$\begin{tikzcd} A \arrow[r, "(\sigma_1,\sigma_2)"] \arrow[d, "g"'] & B\times B \arrow[d, "f\times f"] \\ C \arrow[r, "(\kappa_1,\kappa_2)"'] & D\times D \end{tikzcd}$$

is a pullback with $\sigma_1, \kappa_1 \in \mathcal{N}$ then the following commutative diagram

$$\begin{tikzcd} A \arrow[r, shift left, "\sigma"] \arrow[r, shift right] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, shift left] \arrow[r, shift right, "\kappa"'] & D \end{tikzcd}$$

of stars and morphisms is a star-pullback. The converse is true when f is a monomorphism and $\mathcal{C}$ has enough trivial objects.

Proof. Let $\lambda : E \rightrightarrows B$ be a star and $h : E \to C$ a morphism such that $f\lambda_i = \kappa_i h; i = 1, 2$. Then $(\kappa_1, \kappa_2)h = (f \times f)(\lambda_1, \lambda_2)$ and by the assumption, there exist a unique factorisation $\varphi : E \to A$ such that $(\sigma_1, \sigma_2)\varphi = (\lambda_1, \lambda_2)$ and $g\varphi = h$. For the converse, let $u\colon U \to B \times B$ and $v\colon U \to C$ be two morphisms such that $(f \times f)u = (\kappa_1, \kappa_2)v$; one then has that $f\pi_1 u = \kappa_1 v \in \mathcal{N}$ and $f\pi_2 u = \kappa_2 v$ with $(\pi_1, \pi_2)\colon B \times B \rightrightarrows B$ the product projections. Assuming that $\mathcal{C}$ has enough trivial objects and f is a monomorphism, one can assert that $\pi_1 u$ belongs to $\mathcal{N}$ and $(\pi_1 u, \pi_1 v)\colon U \rightrightarrows B$ is then a star. By assumption, there exists a unique morphism $w\colon U \to A$ verifying $\sigma_1 w = \pi_1 u$, $\sigma_2 w = \pi_2 u$ and $gw = v$ and the fact that the pair (π_1, π_2) is jointly monomorphic allows us to conclude that the first diagram is a pullback. $\square$

The following condition was introduced in [**26**] in the case of internal reflexive relations and was generalized by M. Gran and Z. Janelidze [**22**] in a suitable sense to study regular completions of star-regular categories. Here we generalize the one given in [**26**] but we adopt the notation used in [**22**].

The condition $(*\pi_0)$ *:* Consider a parallel pair of morphisms

$$\begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \end{tikzcd}$$

in a multi-pointed category $\mathcal{C}$ and $(f_1k, f_2k)\colon K \rightrightarrows Y$ the star of the pair (f_1, f_2) where $k\colon K \to X$ is the $\mathcal{N}$-kernel of f_1. Then for any morphism $g\colon Y \to Z$, one has

$$gf_1 = gf_2 \quad \Leftrightarrow \quad gf_1k = gf_2k.$$

Notice that when either the pair (f_1k, f_2k) or the pair (f_1, f_2) has a coequaliser q, then the condition $(*\pi_0)$ is equivalent to the fact that q is the coequaliser of both the pairs (f_1k, f_2k) and (f_1, f_2).

Lemma 3.2.2. *In a star-regular category $\mathcal{C}$, any reflexive relation satisfies the condition $(*\pi_0)$.*

Proof. Consider the situation

$$\begin{tikzcd} R^* \arrow[r, "k"] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] & Y. \end{tikzcd}$$

where (r_1, r_2) is a reflexive relation, k the $\mathcal{N}$-kernel of r_1 and q is any arrow. We must prove that

$$qr_1 = qr_2 \quad \Leftrightarrow \quad qr_1k = qr_2k.$$

One implication is obvious. Assume now that $qr_1k = qr_2k$. Since r_1 is a split epimorphism in a star regular category, r_1 is the coequaliser of its kernel star $(\mu_1, \mu_2) : \mathsf{Eq}(r_1)^* \rightrightarrows R$, and since $r_1\mu_1 = r_1\mu_2 \in \mathcal{N}$, there are unique morphisms θ_i such that $k\theta_i = \mu_i$ for $i \in \{1, 2\}$, as in the diagram

$$\begin{tikzcd} & R^* \arrow[dr, "k"] & & \\ \mathsf{Eq}(r_1)^* \arrow[ur, shift left, "\theta_1"] \arrow[ur, shift right, "\theta_2"'] \arrow[rr, shift left, "\mu_1"] \arrow[rr, shift right, "\mu_2"'] & & R \arrow[r, "r_1"'] & X. \end{tikzcd}$$

Consequently, we have:

$$\begin{aligned} (qr_2)\mu_1 &= qr_2k\theta_1 \\ &= qr_1k\theta_1 \text{ (by assumption)} \\ &= qr_1\mu_1 \\ &= qr_1\mu_2 \text{ (since } r_1 \text{ coequalises } (\mu_1, \mu_2)) \\ &= qr_1k\theta_2 \\ &= qr_2k\theta_2 \\ &= (qr_2)\mu_2. \end{aligned}$$

Since r_1 is the coequaliser of (μ_1, μ_2), it follows that there exists a unique $t : X \to Y$ such that $tr_1 = qr_2$. Since the relation R is reflexive, there is an arrow $\delta \colon X \to R$ such that $r_1\delta = 1_X = r_2\delta$, and this implies that $t = tr_1\delta = qr_2\delta = q$, as desired. $\square$

Actually, it was shown in **[22]** that this result gives a new characterization of star-regular categories. We then have

Proposition 3.2.3. *A regular multi-pointed category with $\mathcal{N}$-kernels is a star-regular category if and only if any reflexive relation satisfies condition* $(*\pi_0)$.

Proof. $\Rightarrow$) Lemma 3.2.2.
$\Leftarrow$) It remains to prove that every regular epimorphism is a coequaliser of its kernel star. Let $e \colon X \to Y$ be a regular epimorphism and consider the diagram

$$\begin{tikzcd} \mathsf{Eq}(e)^* \arrow[r, "k"] & \mathsf{Eq}(e) \arrow[r, shift left, "e_1"] \arrow[r, shift right, "e_2"'] & X \arrow[r, "e"] \arrow[dr, "t"'] & Y \arrow[d, dotted] \\ & & & T. \end{tikzcd}$$

where (e_1, e_2) is the kernel pair of e, $k\colon \mathsf{Eq}(e)^* \to \mathsf{Eq}(e)$ is the $\mathcal{N}$-kernel of e_1 and t is any arrow such that $te_1k = te_2k$. We need to prove the existence of the dotted arrow. By condition $(*\pi_0)$, one has that $te_1 = te_2$ and since e is the coequaliser of its kernel pair in the regular category $\mathcal{C}$, one has the existence of the dotted arrow, as desired. □

Definition 3.2.4. A star-regular category $\mathcal{C}$ is said to be *semi-effective* when any equivalence relation $R \rightrightarrows X$ in $\mathcal{C}$ has the following property: if the star R^* associated with R is a subobject of a kernel star $\mathsf{Eq}(f)^*$

$$\begin{tikzcd}
R^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "i"'] & & X \\
& \mathsf{Eq}(f)^*, \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

with i a split monomorphism in $\mathcal{C}$, then R^* is itself a kernel star.

Let us then observe that, in the pointed context, the star $(r_1k, r_2k)\colon R^* \rightrightarrows X$ associated with an equivalence relation $(r_1, r_2)\colon R \rightrightarrows X$ on X is the star whose first component is the zero arrow, and the second one a *normal subobject* in the sense of D. Bourn (see Section 1.1.2).

Remark 3.2.5. If the category $\mathcal{C}$ has coequalisers of equivalence relations, then $\mathcal{C}$ is a semi-effective star-regular category if and only if for any equivalence relation $(r_1, r_2) : R \rightrightarrows X$ on an object X and q the coequaliser of the equivalence relation R

$$\begin{tikzcd}
R^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "i"'] & & X \arrow[r, "q", two heads] & X/R \\
& \mathsf{Eq}(q)^*, \arrow[ur, shift left] \arrow[ur, shift right] & &
\end{tikzcd}$$

with i a split monomorphism in $\mathcal{C}$, the star R^* associated with R is the kernel star of q, i.e $R^* \cong \mathsf{Eq}(q)^*$.

Proof. This observation easily follows from Lemma 3.2.2. and from Definition 3.2.4. □

Examples 3.2.6.
- In the total context, it is obvious that any Barr-exact category is semi-effective star-regular. Recall that a regular category is said to be *efficiently regular* [**10**] if whenever we have the situation

$$\begin{tikzcd}
R \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "j"'] & & X \\
& \mathsf{Eq}(f) \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

where R is an equivalence relation and $\mathsf{Eq}(f)$ is an effective equivalence relation on the same object X and j a regular monomorphism (an equaliser of a parallel pair of morphisms), then the equivalence relation R is itself effective. Since any split monomorphism is a regular monomorphism, any efficiently regular category is a semi-effective star-regular category. Examples of efficiently regular categories are provided by the category of topological groups and, more generally, by any category of topological models of a Mal'tsev algebraic theory [**46**].

- In the pointed context, among the examples of semi-effective star-regular categories there is also any category with a good theory of ideals, and any *almost abelian category* in the sense of [**53**]. These examples, and many other ones, will be examined in the last section.

Lemma 3.2.7. *Let $\mathbb{C}$ be a semi-effective star-regular category. Then, given a discrete fibration of equivalence relations*

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "\varphi_1"'] & D \arrow[d, "\varphi_0"] \\ \mathsf{Eq}(p) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & E \end{tikzcd} \tag{1}$$

over the kernel pair of p, the coequaliser of π_1 and π_2 exists.

Proof. Let us build the following diagram

$$\begin{tikzcd} & \mathsf{Eq}(t)^* \arrow[rr, "k''"] \arrow[dl, dotted, shift left, "g'"] & & \mathsf{Eq}(t) \arrow[dl, shift left, "g"] \arrow[dr, shift left, "t_1"] \arrow[dr, shift right, "t_2"'] \arrow[ddl, "u"] & & \\ R^* \arrow[ur, dotted, shift left, "j'"] \arrow[rr, "k'"] \arrow[dd, "\varphi_2"'] & & R \arrow[ur, shift left, "j"] \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dd, "\varphi_1"'] \arrow[ddrr, phantom, "(1)"] & & D \arrow[dd, "\varphi_0"'] \arrow[ddr, "t"] & \\ & & & & & \\ \mathsf{Eq}(p)^* \arrow[rr, "k"'] & & \mathsf{Eq}(p) \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & E \arrow[r, "p"'] & B \end{tikzcd}$$

where:

- $t = p\varphi_0$;
- the morphisms k, k' and k'' are the $\mathcal{N}$-kernels of p_1, π_1 and t_1, respectively;
- u is the unique arrow such that $\varphi_0 t_l = p_l u$, for $l \in \{1; 2\}$;
- g is the unique arrow such that $\pi_1 g = t_1$ and $\varphi_1 g = u$;
- j is the unique morphism such that $t_l j = \pi_l$, for $l \in \{1; 2\}$.

Since (φ_1, π_1) is jointly monomorphic and $uj = \varphi_1$, we have the equality $gj = 1_R$. Moreover, one has $\pi_1 g k'' = t_1 k'' \in \mathcal{N}$ and $t_1 j k' = \pi_1 k' \in \mathcal{N}$, hence by the universal property of $\mathcal{N}$-kernels k' and k'', there exist unique morphisms g' and j' such that $k'g' = gk''$ and $k''j' = jk'$. The fact that k' is a monomorphism allows us to assert that j' is a split monomorphism. By assumption $R^* \rightrightarrows D$ is then a kernel star, since $\mathsf{Eq}(t)^* \rightrightarrows D$ is a kernel star. By the star-regularity of $\mathcal{C}$, this implies that the coequaliser $q\colon D \to D/R^*$ of $\pi_1 k'$ and $\pi_2 k'$ exists, and q is also the coequaliser of π_1 and π_2 by Lemma 3.2.2. □

THEOREM 3.2.8. *Let $\mathcal{C}$ be a semi-effective star-regular category with enough trivial objects. Then the following conditions are equivalent for an arrow $p : E \to B$:*

(a) *p is an effective descent morphism;*
(b) *p is a descent morphism;*
(c) *p is a regular epimorphism.*

PROOF. The implication $(a) \Rightarrow (b)$ is trivial, whereas (b) is equivalent to (c) in any finitely complete regular category (see Theorem 3.1.3).

We are now going to prove that $(c) \Rightarrow (a)$. Assume that $p : E \to B$ is a regular epimorphism, and consider the commutative diagram (1) as in Lemma 3.2.7. We would like to prove that the equivalence relation $R \rightrightarrows D$ is effective.

Let us then build the commutative diagram

$$\begin{tikzcd}
& \mathsf{Eq}(q)^* \arrow[rr, "k''"] \arrow[dl, dotted, shift left, "f'"] & & \mathsf{Eq}(q) \arrow[dl, shift left, "f"] \arrow[dr, shift left, "q_1"] \arrow[dr, shift right, "q_2"'] \arrow[ddl, "\psi"] & & \\
R^* \arrow[rr, "k'"] \arrow[ur, dotted, shift left, "i'"] \arrow[dd, "\varphi_2"'] & & R \arrow[ur, shift left, "i"] \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dd, "\varphi_1"'] \arrow[ddrr, phantom, "(1)"] & & D \arrow[r, "q"] \arrow[dd, "\varphi_0"'] & D/R \arrow[dd, "\varphi"'] \\
& & & & & \\
\mathsf{Eq}(p)^* \arrow[rr, "k"'] & & \mathsf{Eq}(p) \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] & & E \arrow[r, "p"'] & B
\end{tikzcd}$$

where:

- q is the coequaliser of π_1 and π_2 (which exists by Lemma 3.2.7);
- i is the unique arrow such that $q_h i = \pi_h$, for $h \in \{1; 2\}$;
- k, k' and k'' are the $\mathcal{N}$-kernels of p_1, π_1 and q_1, respectively;
- ψ is the unique arrow such that $\varphi_0 q_h = p_h \psi$, for $h \in \{1; 2\}$;
- f is the unique arrow such that $\pi_1 f = q_1$ and $\varphi_1 f = \psi$.

A similar argument to the one used in Lemma 3.2.7 (to show that $gj = 1_R$) implies that $fi = 1_R$. We also know that the induced split monomorphism $i'\colon R^* \to \mathsf{Eq}(q)^*$ is an isomorphism, thanks to Lemma 3.2.7 and to Remark 3.2.5.

It will suffice to show that the split epimorphism $f\colon \mathsf{Eq}(q) \to R$ is a monomorphism, and this will imply that $i\colon R \to \mathsf{Eq}(q)$ is an isomorphism, as desired.

For this, consider the following commutative cube

$$
\begin{tikzcd}
\Delta_R^* \arrow[rr] \arrow[dr, dotted, "\alpha"] \arrow[dd] & & \mathsf{Eq}(\varphi_1)^* \arrow[dr] \arrow[dd] & \\
& \mathsf{Eq}(f)^* \arrow[rr] \arrow[dd] & & \mathsf{Eq}(\psi)^* \arrow[dd] \\
\mathsf{Eq}(\pi_1)^* \arrow[rr] \arrow[dr, "i''"'] & & R \times R \arrow[dr, "i \times i"] & \\
& \mathsf{Eq}(q_1)^* \arrow[rr] & & \mathsf{Eq}(q) \times \mathsf{Eq}(q)
\end{tikzcd}
$$

where the back square is a pullback by Corollary 1.1.32, the front square is a pullback by Proposition 1.1.33, and the dotted arrow α is induced by the universal property of this latter pullback. By Lemma 1.1.28, the commutative diagram

$$
\begin{tikzcd}
\mathsf{Eq}(\pi_1)^* \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "i''"'] & R \arrow[d, "i"] \\
\mathsf{Eq}(q_1)^* \arrow[r, shift left] \arrow[r, shift right] & \mathsf{Eq}(q)
\end{tikzcd}
$$

is a star-pullback, and by Proposition 3.2.1 the bottom square of the cube above is a pullback. The same arguments can be applied to conclude that the right square is a pullback as well, so that the left square is a pullback. The arrow i'' is an isomorphism: indeed, in the following diagram

$$
\begin{tikzcd}
\mathsf{Eq}(\pi_1)^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\mu"'] \arrow[dd, "i''"'] & & R \arrow[r, "\pi_1"] \arrow[dd, "i"] & D \arrow[dd, equal] \\
& R^* \arrow[ur, "k'"] \arrow[dd, "i'"] & & \\
\mathsf{Eq}(q_1)^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, shift left] \arrow[dr, shift right, "\lambda"'] & & \mathsf{Eq}(q) \arrow[r, "q"'] & D \\
& \mathsf{Eq}(q)^* \arrow[ur, "k''"] & &
\end{tikzcd}
$$

the outer square is the star-pullback above and the middle square is a pullback by Lemma 1.1.29. The assumption that $\mathbb{C}$ has enough trivial objects allows us to conclude that both μ and λ are stars (see Remark 2.4.17), and by applying Lemma 1.1.13, one has that the left-hand square is a star pullback. Since isomorphisms are stable under star-pullback and i' is an isomorphism, it follows that i'' is an isomorphism as well. Accordingly, the dotted arrow α is an isomorphism as well. This implies that the projections $f_1 \colon \mathsf{Eq}(f)^* \to \mathsf{Eq}(q)$ and $f_2 \colon \mathsf{Eq}(f)^* \to \mathsf{Eq}(q)$ of the kernel star of f are equal, as one can see from the commutativity of the following diagram

$$\begin{tikzcd}
\Delta^*_R \arrow[r, shift left, "k_R"] \arrow[r, shift right, "k_R"'] \arrow[d, "\alpha"'] & R \arrow[d, "i"] \\
\mathsf{Eq}(f)^* \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & \mathsf{Eq}(q)
\end{tikzcd}$$

where k_R is the $\mathcal{N}$-kernel of 1_R. From Lemma 1.1.31 we conclude that f is a monomorphism, as desired. $\square$

3.3. Examples

3.3.1. Categories with a good theory of ideals.

In any category $\mathbb{C}$ with a good theory of ideals, it is possible to show that any star $(r_1k, r_2k)\colon R^* \rightrightarrows X$ associated with an equivalence relation $(r_1, r_2)\colon R \rightrightarrows X$ on X is necessarily a kernel star. Indeed, in the diagram

$$\begin{tikzcd}
 & & \mathsf{Eq}(q)^* \arrow[ddd, shift left, bend left] \arrow[ddd, shift right, bend left] \\
\mathsf{Eq}(r_1)^* \arrow[r, "\varphi"] \arrow[d, "k"'] \arrow[urr, bend left, "e"] & R^* \arrow[ur, "\phi"'] \arrow[d, "l"] & \\
\mathsf{Eq}(r_1) \arrow[r, "\sigma"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \\
R \arrow[r, "r_2"] \arrow[d, "r_1"'] & X \arrow[d, "q"] & \\
X \arrow[r, "q"'] & X/R. &
\end{tikzcd}$$

the middle square with $(r_2, \sigma)\colon \mathsf{Eq}(r_1) \to R$ determines a canonical discrete fibration associated with the equivalence relation R (see Definition 1.1.7 (2)), the arrows k and l are $\mathcal{N}$-kernels of p_1 and r_1, respectively. Since r_1 and r_2 are regular epimorphisms, the bottom pushout exists in $\mathcal{C}$ by Corollary 1.2.21. Thanks to the characterisation of the categories with a good theory of ideals by a property of pushouts (Theorem 1.2.20), it follows that the induced arrow e is a regular epimorphism. This implies that the canonical monomorphism ϕ is also a regular epimorphism, thus an isomorphism.

Accordingly: *any category $\mathbb{C}$ with a good theory of ideals is semi-effective star-regular*. By Theorem 3.2.8, when $\mathbb{C}$ also has enough trivial objects, any regular epimorphism in $\mathcal{C}$ is then an effective descent morphism.

In the pointed context, this asserts that any ideal determined category $\mathcal{C}$ is a semi-effective star-regular category, thus any regular epimorphism in $\mathcal{C}$ is an effective descent morphism. This particular case of ideal determined categories has been considered by T. Everaert **[17]**, who arrived at the conclusion that regular epimorphisms therein are effective for descent by using a completely different approach. Observe that Theorem 3.2.8 applies to any star-regular category with enough trivial objects for which the class of "stars of equivalence relations" coincides with the class of "kernels stars". This latter condition is weaker than the one asserting that the class of "ideals" coincides with the class of "kernel stars".

3.3.2. Regular epimorphisms in a category with a good theory of ideals.

When $\mathbb{C}$ is a category, we denote by $\mathsf{RegEpi}(\mathcal{C})$ the category of regular epimorphisms in $\mathcal{C}$: an object in $\mathsf{RegEpi}(\mathcal{C})$ is a regular epi $a : A_1 \twoheadrightarrow A_0$ in $\mathcal{C}$, and a morphism $f : a \to b$ in $\mathsf{RegEpi}(\mathcal{C})$ is a pair (f_0, f_1) of morphisms in $\mathcal{C}$ with $f_0 : A_0 \to B_0$ and $f_1 : A_1 \to B_1$

$$\begin{tikzcd} A_1 \arrow[r, two heads, "a"] \arrow[d, "f_1"'] & A_0 \arrow[d, "f_0"] \\ B_1 \arrow[r, two heads, "b"'] & B_0 \end{tikzcd}$$

such that $f_0 a = b f_1$.

In the following, we shall also assume that $\mathcal{N}$-trivial objects are closed under binary products (see [28]). When the category has enough trivial objects, this assumption is equivalent to the following one:

for a relation $(r_1, r_2)\colon R \rightarrowtail X \times Y$ from X to Y such that $r_1 f$, $r_2 f$ belong to $\mathcal{N}$, where $f\colon T \to R$ is a morphism, then f belongs to $\mathcal{N}$.

In both the pointed context and the total contexts, $\mathcal{N}$-trivial objects are closed under binary products.

Lemma 3.3.1. *Let $\mathcal{C}$ be a category with a good theory of ideals, with enough trivial objects, and such that $\mathcal{N}$-trivial objects are closed under binary products. Then $\mathsf{RegEpi}(\mathcal{C})$ is a star-regular category with enough trivial objects and such that $\mathcal{N}$-trivial objects are closed under binary products.*

Proof. First of all, we are going to give a description of pullbacks, kernel pairs and regular epimorphisms in $\mathsf{RegEpi}(\mathcal{C})$.

- A pullback diagram

$$\begin{tikzcd} a \arrow[r, "f'"] \arrow[d, "g'"'] & c \arrow[d, "g"] \\ b \arrow[r, "f"'] & d \end{tikzcd}$$

in $\mathsf{RegEpi}(\mathcal{C})$ is given by the following cube in $\mathcal{C}$

$$\begin{tikzcd}
A_1 \arrow[rr, "f_1'"] \arrow[dr, two heads, "a"] \arrow[dd, "g_1'"'] & & C_1 \arrow[dr, two heads, "c"] \arrow[dd, "g_1" near end] & \\
& A_0 \arrow[rr, "u" near start] \arrow[dd, "v" near start] & & C_0 \arrow[dd, "g_0"] \\
B_1 \arrow[rr, "f_1" near end] \arrow[dr, two heads, "b"'] & & D_1 \arrow[dr, two heads, "d"] & \\
& B_0 \arrow[rr, "f_0"'] & & D_0
\end{tikzcd}$$

where the back square of the cube is a pullback in $\mathcal{C}$ and the pair (u, v) is jointly monomorphic. The regular epimorphism a is obtained by taking the regular epimorphism part of the (regular epimorphism, monomorphism) factorisation of the induced arrow $A_1 \to B_0 \times_{D_0} C_0$.

- A kernel pair $r \overset{u}{\underset{v}{\rightrightarrows}} a$ in $\mathsf{RegEpi}(\mathcal{C})$ is the same as a commutative square

$$\begin{tikzcd}
R_1 \arrow[r, shift left, "u_1"] \arrow[r, shift right, "v_1"'] \arrow[d, two heads, "e"'] & A_1 \arrow[d, two heads, "a"] \\
R_0 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "v_0"'] & A_0
\end{tikzcd}$$

in $\mathsf{RegEpi}(\mathcal{C})$ such that $R_1 \overset{u_1}{\underset{v_1}{\rightrightarrows}} A_1$ is a kernel pair in $\mathcal{C}$ and the pair (u_0, v_0) is jointly monic.

- A regular epimorphism $(f_0, f_1) : a \twoheadrightarrow b$ in $\mathsf{RegEpi}(\mathcal{C})$ is given by a commutative diagram of regular epimorphisms in $\mathcal{C}$

$$\begin{tikzcd}
A_1 \arrow[r, two heads, "a"] \arrow[d, two heads, "f_1"'] & A_0 \arrow[d, two heads, "f_0"] \\
B_1 \arrow[r, two heads, "b"'] & B_0
\end{tikzcd}$$

determining a pushout.

$\mathsf{RegEpi}(\mathcal{C})$ is finitely complete and has coequalisers of kernel pairs since $\mathcal{C}$ has pushouts of regular epimorphisms (see Corollary 1.2.21). For the regularity of $\mathsf{RegEpi}(\mathcal{C})$ it remains to show that regular epimorphisms are stable under pullback in $\mathsf{RegEpi}(\mathcal{C})$; for this, assume that the following diagram

$$\begin{tikzcd}
a \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & c \arrow[d, "g"] \\
b \arrow[r, two heads, "f"'] & d
\end{tikzcd} \tag{3.8}$$

is a pullback in $\mathsf{RegEpi}(\mathcal{C})$ and f a regular epimorphism in $\mathsf{RegEpi}(\mathcal{C})$. Now consider the following diagram in $\mathcal{C}$

$$\begin{tikzcd}
\mathsf{Eq}(a)^* \arrow[rrr, "k_2"] \arrow[dr, shift left] \arrow[dr, shift right, "\kappa"'] \arrow[ddd, "h_1"'] & & & \mathsf{Eq}(b)^* \arrow[dr, shift left, "\mu"] \arrow[dr, shift right] \arrow[ddd, "h_0"'] & & \\
& A_1 \arrow[rrr, "f_1'"] \arrow[dr, "a"] \arrow[ddd, "g_1'"'] & & & B_1 \arrow[dr, "b"] \arrow[ddd, "g_1"] & \\
& & A_0 \arrow[rrr, "f_0'"] \arrow[ddd, "g_0'"'] & & & B_0 \arrow[ddd, "g_0"] \\
\mathsf{Eq}(c)^* \arrow[rrr, two heads, "k_1"] \arrow[dr, shift left] \arrow[dr, shift right, "\lambda"'] & & & \mathsf{Eq}(d)^* \arrow[dr, shift left, "\sigma"] \arrow[dr, shift right] & & \\
& C_1 \arrow[rrr, two heads, "f_1"'] \arrow[dr, "c"'] & & & D_1 \arrow[dr, "d"] & \\
& & C_0 \arrow[rrr, two heads, "f_0"'] & & & D_0
\end{tikzcd}$$

where κ, μ, λ and σ are the kernel stars of a, b, c and d, respectively. The bottom square of the first cube is a pushout of regular epimorphisms in $\mathcal{C}$ because f is a regular epimorphism in $\mathsf{RegEpi}(\mathcal{C})$. Then k_1 is a regular epimorphism by Theorem 1.2.20. Since the square 3.8 is a pullback in $\mathsf{RegEpi}(\mathcal{C})$, this implies that the middle square of the diagram is a pullback in $\mathcal{C}$ and the pair (f_0', g_0') is jointly monomorphic. By the pullback stability of regular epimorphisms in $\mathcal{C}$, f_1' is a regular epimorphism, and then f_0' is a regular epimorphism as well. It remains to prove that the upper square of the first cube is a pushout. For this, we are going to prove that the back face of the outer cube is a pullback. Let us consider the diagram

$$\begin{tikzcd}
U \arrow[dr, dotted, "\varphi"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& \mathsf{Eq}(a)^* \arrow[r, "k_2"] \arrow[d, "h_1"'] & \mathsf{Eq}(b)^* \arrow[d, "h_0"] \\
& \mathsf{Eq}(c)^* \arrow[r, "k_1"'] & \mathsf{Eq}(d)^*
\end{tikzcd}$$

where u and v are any arrows verifying $h_0 u = k_1 v$. We must prove the existence of a unique arrow φ making the two triangles commute. One has $\sigma_i h_0 u = \sigma_i k_1 v, i = 1, 2$; this implies that $g_1 \mu_i u = f_1 \lambda_i v, i = 1, 2$. The universal property of the middle pullback gives two induced morphisms $\delta_i : U \to A_1$ such that $f_1' \delta_i = \mu_i u$ and $g_1' \delta_i = \lambda_i v$. Thus $f_1' \delta_1 = \mu_1 u \in \mathcal{N}$ and $g_1' \delta_1 = \lambda_1 v \in \mathcal{N}$; since the pair (f_1', g_1') is jointly monomorphic, $\mathcal{C}$ has enough trivial objects and $\mathcal{N}$-trivial objects are closed under binary products, this implies that $\delta = (\delta_1, \delta_2) : U \rightrightarrows A_1$ is a star in $\mathcal{C}$. Furthermore, in the diagram

$$\begin{tikzcd}
\mathsf{Eq}(a)^* \arrow[r, shift left, "\kappa_1"] \arrow[r, shift right, "\kappa_2"'] & A_1 \arrow[r, "a"] & A_0 \\
U \arrow[u, dotted, "w"] \arrow[ur, shift left, "\delta_1"] \arrow[ur, shift right, "\delta_2"'] & &
\end{tikzcd}$$

the fact that the pair (f_0', g_0') is jointly monomorphic gives that $a\delta_1 = a\delta_2$ and by the universal property of the kernel star κ of a, there exists a unique factorisation $w : U \to \mathsf{Eq}(a)^*$ such that $\kappa_i w = \delta_i$, $i = 1, 2$. The fact that the pairs (μ_1, μ_2) and (λ_1, λ_2) are jointly monomorphic gives that $k_2 w = u$ and $h_1 w = v$. Take $w = \varphi$, and the uniqueness of φ follows from the fact that both (f_1', g_1') and (κ_1, κ_2) are jointly monomorphic pairs. Thus the back square of the outer cube is a pullback, and k_2 is a regular epimorphism, as desired. By Theorem 1.2.20, this implies that the top square of the front cube is a pushout.

Accordingly, $\mathsf{RegEpi}(\mathcal{C})$ is a regular category. Let us now denote by $\mathcal{M}$ the class of morphisms $f = (f_0, f_1)$ in $\mathsf{RegEpi}(\mathcal{C})$ defined by $f = (f_0, f_1) \in \mathcal{M}$ if and only if $f_1 \in \mathcal{N}$. This implies that also $f_0 \in \mathcal{N}$ by Lemma 1.1.19. This class $\mathcal{M}$ is an ideal of morphisms in $\mathsf{RegEpi}(\mathcal{C})$ (since $\mathcal{N}$ is an ideal in $\mathcal{C}$), so that $(\mathsf{RegEpi}(\mathcal{C}), \mathcal{M})$ is a regular multi-pointed category. An $\mathcal{M}$-kernel $k : a \to b$ of a morphism $f : b \to c$ in $\mathsf{RegEpi}(\mathcal{C})$ is the same as the left-hand square of the commutative square

$$\begin{tikzcd}
A_1 \arrow[r, "k_1"] \arrow[d, two heads, "a"'] & B_1 \arrow[r, "f_1"] \arrow[d, two heads, "b"] & C_1 \arrow[d, two heads, "c"] \\
A_0 \arrow[r, "k_0"'] & B_0 \arrow[r, "f_0"'] & C_0
\end{tikzcd}$$

where k_1 is the $\mathcal{N}$-kernel of f_1 and k_0 is a monomorphism. Since any morphism in $(\mathcal{C}, \mathcal{N})$ has an $\mathcal{N}$-kernel by assumption, then any morphism in $\mathsf{RegEpi}(\mathcal{C})$ has an $\mathcal{M}$-kernel. A kernel star of a morphism $f = (f_0, f_1) : a \to b$ in $\mathsf{RegEpi}(\mathcal{C})$ is a diagram

$$\begin{tikzcd}
K_1 \arrow[r, shift left, "k_1"] \arrow[r, shift right, "l_1"'] \arrow[d, two heads, "p"'] & A_1 \arrow[d, two heads, "a"] \\
K_0 \arrow[r, shift left, "k_0"] \arrow[r, shift right, "l_0"'] & A_0
\end{tikzcd}$$

where $K_1 \overset{k_1}{\underset{l_1}{\rightrightarrows}} A_1$ is the kernel star of f_1 in $\mathcal{C}$ while $K_0 \overset{k_0}{\underset{l_0}{\rightrightarrows}} A_0$ is a star-relation. To see that any regular epimorphism in $\mathsf{RegEpi}(\mathcal{C})$ is the coequaliser of its kernel star, consider a regular epimorphism $(e_0, e_1) : a \twoheadrightarrow b$ in $\mathsf{RegEpi}(\mathcal{C})$ and the following diagram in $\mathsf{RegEpi}(\mathcal{C})$

$$\begin{tikzcd}
\mathsf{Eq}(e_1)^* \arrow[r, shift left, "k_1"] \arrow[r, shift right, "l_1"'] \arrow[d, two heads, "p"'] & A_1 \arrow[r, two heads, "e_1"] \arrow[d, two heads, "a"] & B_1 \arrow[d, two heads, "b"] \\
R_0^* \arrow[r, shift left, "k_0"] \arrow[r, shift right, "l_0"'] & A_0 \arrow[r, two heads, "e_0"'] & B_0
\end{tikzcd}$$

where (l_1, k_1) is the kernel star of e_1 and $(l_0, k_0)p$ is the factorisation (regular epimorphism, star-relation) of the star (al_1, ak_1). $((l_0, k_0), (l_1, k_1))$ is then the kernel star of $e = (e_0, e_1)$ in $\mathsf{RegEpi}(\mathcal{C})$. Since e_1 is the coequaliser of its kernel star in $\mathcal{C}$ and e_0 is the coequaliser of (l_0, k_0) (because p is a regular epimorphism), one has that $e = (e_0, e_1)$ is the coequaliser of $((l_0, k_0), (l_1, k_1))$ in $\mathsf{RegEpi}(\mathcal{C})$. Conclusion: $\mathsf{RegEpi}(\mathcal{C})$ is a star-regular category. If $\mathcal{C}$ has enough trivial objects and $\mathcal{N}$-trivial objects are closed under binary products, it is easy to verify that $\mathsf{RegEpi}(\mathcal{C})$ has enough trivial objects and $\mathcal{N}$-trivial objects are closed under binary products as well. □

REMARK 3.3.2. It is not true, in general, that $\mathsf{RegEpi}(\mathcal{C})$ has a good theory of ideals when $\mathcal{C}$ is a category with a good theory of ideals (and enough trivial objects). For instance, in the total context, even when $\mathcal{C}$ is abelian, the category $\mathsf{RegEpi}(\mathcal{C})$ is regular but not *exact* Goursat (= "with a good theory of ideals", in the total context).

We need the following lemma to prove that regular epimorphisms are effective descent morphisms in $\mathsf{RegEpi}(\mathcal{C})$.

LEMMA 3.3.3. **[17]** *For a regular category $\mathcal{C}$ with pushouts of regular epimorphisms along regular epimorphisms, the following conditions are equivalent:*

(1) *every regular epimorphism in* $\mathsf{RegEpi}(\mathcal{C})$ *is an effective descent morphism;*
(2) $\mathsf{RegEpi}(\mathcal{C})$ *is a regular category.*

From Lemma 3.3.1, and Lemma 3.3.3, we get:

COROLLARY 3.3.4. *Let $\mathcal{C}$ be a category with a good theory of ideals, with enough trivial objects and such that $\mathcal{N}$-trivial objects are closed under binary products. Then regular epimorphisms are effective descent morphisms in* $\mathsf{RegEpi}(\mathcal{C})$.

Corollary 3.3.4 can be iterated more generally for the categories $\mathsf{RegEpi}(\mathcal{C})^n$ of n-fold regular epimorphisms in $\mathcal{C}$.

PROPOSITION 3.3.5. *Let $\mathcal{C}$ be a category with a good theory of ideals, with enough trivial objects and such that $\mathcal{N}$-trivial objects are closed under binary products. Then, for $n \geqslant 1$,* $\mathsf{RegEpi}(\mathcal{C})^n$ *is star-regular with enough trivial objects.*

It was shown in [**17**] that when $\mathcal{C}$ is a regular category with pushouts of regular epimorphisms along regular epimorphisms, then for $n \geqslant 1$, the category $\mathsf{RegEpi}(\mathcal{C})^n$ is a regular category. If in addition $(\mathcal{C},\mathcal{N})$ is a star-regular category, to prove the star-regularity of $\mathsf{RegEpi}(\mathcal{C})^n$, $n \geqslant 1$, one can use the same arguments as above to prove that $\mathsf{RegEpi}(\mathcal{C})$ is a star-regular category. In the total context, Corollary 3.3.4 says that when $\mathcal{C}$ is a Barr-exact Goursat category, then any regular epimorphism is an effective descent morphism in $\mathsf{RegEpi}(\mathcal{C})$. This result was obtained in [**40**] and in [**17**]. In the pointed context, Corollary 3.3.4 says that when $\mathcal{C}$ is an ideal determined category, then any regular epimorphism is an effective descent morphisms in $\mathsf{RegEpi}(\mathcal{C})$ (see also [**17**]).

3.3.3. Monomorphisms in a category with a good theory of ideals. Let us denote by $\mathsf{Mono}(\mathcal{C})$ the category of monomorphisms in a category $\mathcal{C}$. The category $\mathsf{Mono}(\mathcal{C})$ is defined as follows:

(1) objects are monomorphisms $m : M_1 \rightarrowtail M_0$ of $\mathcal{C}$;
(2) an arrow $\lambda : m \to m'$ in $\mathsf{Mono}(\mathcal{C})$ is a pair (λ_0, λ_1) of morphisms of $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} M_1 \arrow[r, "m"] \arrow[d, "\lambda_1"'] & M_0 \arrow[d, "\lambda_0"] \\ M_1' \arrow[r, "m'"'] & M_0' \end{tikzcd}$$

(3) composition of morphisms in $\mathsf{Mono}(\mathcal{C})$ is obtained by the composition of morphisms in $\mathcal{C}$.

Lemma 3.3.6. *Let $(\mathcal{C},\mathcal{N})$ be a semi-effective star-regular category with enough trivial objects. Then $\mathsf{Mono}(\mathcal{C})$ is a semi-effective star-regular category with enough trivial objects.*

Proof. It is well-known that when $\mathcal{C}$ is a regular category, then $\mathsf{Mono}(\mathcal{C})$ is also a regular category and limits are obtained pointwise. Assume moreover that $(\mathcal{C},\mathcal{N})$ is a multi-pointed category and let us denote by $\mathcal{M}$ the class of morphisms $k = (k_0, k_1)$ of $\mathsf{Mono}(\mathcal{C})$ such that k_0 and k_1 are in $\mathcal{N}$. $(\mathsf{Mono}(\mathcal{C}), \mathcal{M})$ is a then a regular multi-pointed category. An $\mathcal{M}$-kernel of a morphism $(\lambda_0, \lambda_1)\colon m \to m'$ in $\mathsf{Mono}(\mathcal{C})$ is a morphism $k = (k_0, k_1)\colon t \to m$ in $\mathsf{Mono}(\mathcal{C})$

$$\begin{tikzcd} T_1 \arrow[r, "k_1"] \arrow[d, "t"'] & M_1 \arrow[r, "\lambda_1"] \arrow[d, "m"] & M_1' \arrow[d, "m'"] \\ T_0 \arrow[r, "k_0"'] & M_0 \arrow[r, "\lambda_0"'] & M_0' \end{tikzcd}$$

such that k_0 and k_1 are $\mathcal{N}$-kernels of λ_0 and λ_1, respectively. Thus when any morphism in $\mathcal{C}$ has an $\mathcal{N}$-kernel, it follows that every morphism in $\mathsf{Mono}(\mathcal{C})$ has an $\mathcal{M}$-kernel. Finally, a kernel star of a morphism $(\lambda_0, \lambda_1)\colon m \to m'$ in

$\mathsf{Mono}(\mathcal{C})$ is the same as a diagram

$$\begin{tikzcd} K_1 \arrow[r, shift left, "n_1"] \arrow[r, shift right, "l_1"'] \arrow[d, tail, "k"'] & M_1 \arrow[d, tail, "m"] \\ K_0 \arrow[r, shift left, "n_0"] \arrow[r, shift right, "l_0"'] & M_0 \end{tikzcd}$$

such that (n_1, l_1) and (n_0, l_0) are the respective kernel stars of λ_1, λ_0 in$\mathcal{C}$. When (e_0, e_1) is a regular epimorphism in $\mathsf{Mono}(\mathcal{C})$, then e_0 and e_1 are regular epimorphisms in $\mathcal{C}$ and whenever any regular epimorphism is a coequaliser of its kernel star in $\mathcal{C}$, this implies also the same for regular epimorphisms in $\mathsf{Mono}(\mathcal{C})$. It is not difficult to check that when $\mathcal{C}$ is a semi-effective star-regular category, then $\mathsf{Mono}(\mathcal{C})$ also is a semi-effective star-regular category. □

As a consequence, under the assumptions of the Lemma 3.3.6, the regular epimorphisms are effective descent morphisms in $\mathsf{Mono}(\mathcal{C})$.

3.3.4. Birkhoff subcategories.

By a *regular-epireflective subcategory* $\mathcal{D}$ of a regular category $\mathcal{C}$, we mean a full replete reflective subcategory

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

with the property that the A-component $\eta_A : A \to UF(A)$ of the unit of the adjunction is a regular epimorphism for each $A \in \mathcal{C}$. It is well known that the last requirement is equivalent to the fact that $\mathcal{D}$ is closed in $\mathcal{C}$ under subobjects. $\mathcal{D}$ is a *Birkhoff subcategory* of $\mathcal{C}$ if, moreover, $\mathcal{D}$ is closed in $\mathcal{C}$ under regular quotients.

LEMMA 3.3.7. *Let $\mathcal{D}$ be a Birkhoff subcategory of a semi-effective star-regular category $(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$. Then $\mathcal{D}$ is a semi-effective star-regular category as well.*

PROOF. It is well known that if $\mathcal{C}$ is regular, then $\mathcal{D}$ is regular as well. This essentially follows from the fact that the regular epi-mono factorisation in $\mathcal{C}$ of an arrow in $\mathcal{D}$ is still the regular epi-mono factorisation of this arrow in $\mathcal{D}$. Since $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, we can choose the ideal of morphisms of $\mathcal{D}$ induced by the ideal of morphisms in $\mathcal{C}$, so that, for any X,Y in $\mathcal{D}$, $\mathcal{N}_{\mathcal{D}}(X, Y) = \mathcal{N}_{\mathcal{C}}(X, Y) = \mathcal{N}(X, Y)$. The fact that $\mathcal{D}$ is closed in $\mathcal{C}$ under subobjects implies that the $\mathcal{N}$-kernel of an arrow in $\mathcal{D}$ is computed in the same way in the categories $\mathcal{C}$ and $\mathcal{D}$. The category $(\mathcal{D}, \mathcal{N})$ is a regular multi-pointed category. Since it is a regular-epireflective subcategory of $\mathcal{C}$, any regular epimorphism in $\mathcal{D}$ is a regular epimorphism in $\mathcal{C}$, and $(\mathcal{D}, \mathcal{N})$ is a star-regular category.

Consider then the following diagram in $\mathcal{D}$

$$\begin{tikzcd} R^* \arrow[r, "k"] \arrow[d, shift left, "s"] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "q"] & E \\ \mathsf{Eq}(q)^* \arrow[r, "l"'] \arrow[u, shift left, "p"] & \mathsf{Eq}(q) \arrow[ur, shift left, "q_1"] \arrow[ur, shift right, "q_2"'] & & \end{tikzcd}$$

where $R \rightrightarrows A$ is an equivalence relation in $\mathcal{D}$ and $\mathsf{Eq}(q)^* \rightrightarrows A$ is a kernel star with coequaliser q (in $\mathcal{D}$), k and l are the $\mathcal{N}$-kernels of r_1 and q_1, respectively, and s is a split monomorphism with $ps = 1_{R^*}$, and $q_i ls = r_i k$ for $i \in \{1, 2\}$. If we look at this diagram in $\mathcal{C}$, then q is still the coequaliser of (q_1, q_2) in $\mathcal{C}$, since $\mathcal{D}$ is stable in $\mathcal{C}$ under quotients. The category $\mathcal{C}$ is semi-effective star-regular, so that $(r_1k, r_2k) \colon R^* \rightrightarrows A$ is a kernel star in $\mathcal{C}$ of its coequaliser $q' \colon A \to E'$ in $\mathcal{C}$. Moreover, $\mathcal{D}$ is closed in $\mathcal{C}$ under quotients, so that E' lies in $\mathcal{D}$, and $(r_1k, r_2k) \colon R^* \rightrightarrows A$ is a kernel star in $\mathcal{D}$. $\square$

The property of being a category with a good theory of ideals is stable under Birkhoff subcategories. More precisely, one has the following

PROPOSITION 3.3.8. *Let $\mathcal{D}$ be a regular-epireflective subcategory of a category $\mathcal{C}$ with a good theory of ideals. Then the following conditions are equivalent:*

(a) *$\mathcal{D}$ is a Birkhoff subcategory of $\mathcal{C}$;*
(b) *$\mathcal{D}$ is a category with a good theory of ideals;*
(c) *for any span of regular epimorphisms $C \xleftarrow{g} A \xrightarrow{f} B$ in $\mathcal{D}$ their pushout (P, f', g') in $\mathcal{C}$ is also their pushout in $\mathcal{D}$.*

PROOF. $(a) \Rightarrow (b)$. By Lemma 3.3.7, we know that the category $\mathcal{D}$ is star-regular. Consider then the diagram

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, shift right, "\lambda"'] \arrow[d, shift left] & I \arrow[d, shift left, "\mu"] \arrow[d, shift right] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where f is a regular epimorphism in $\mathcal{D}$, λ a kernel star in $\mathcal{D}$ and μg the factorisation (regular epi)-(monic-star) of $f\lambda$ in $\mathcal{C}$. One clearly has that $I \in \mathcal{D}$. By assumption, μ is then the kernel star of its coequaliser $q \colon B \to Q$ in $\mathcal{C}$. The category $\mathcal{D}$ is closed in $\mathcal{C}$ under quotients, and this implies that $q \colon B \to Q$ is also the coequaliser of μ in $\mathcal{D}$. Accordingly, μ is a kernel star in $\mathcal{D}$.
$(b) \Rightarrow (c)$. Let $f \colon A \to B$ and $g \colon A \to C$ be regular epimorphisms lying in $\mathcal{D}$. In the category $\mathcal{C}$ with a good theory of ideals the pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\ C \arrow[r, "f'"'] & P' \end{tikzcd} \tag{3.9}$$

always exists (see Corollary 1.2.21). The assumption that $\mathcal{D}$ is a category with a good theory of ideals implies that the pushout (P'', f'', g'') of f and g exists also in $\mathcal{D}$, and g'' is the coequaliser in $\mathcal{D}$ of its kernel star $f(\mathsf{Eq}(g)^*) \rightrightarrows B$. The fact that both g' and g'' are regular epimorphisms in $\mathcal{C}$ with the same kernel star implies that the canonical comparison $\eta\colon P' \to P''$ such that $\eta g' = g''$ $\eta\colon P' \to P''$ is an isomorphism.
$(c) \Rightarrow (a)$. Let $f : A \to B$ be a regular epimorphism in $\mathcal{C}$, with $A \in \mathcal{D}$, and consider the kernel pair $(f_1, f_2)\colon \mathsf{Eq}(f) \rightrightarrows A$ of f, which lies in $\mathcal{D}$, since $\mathcal{D}$ is closed in $\mathcal{C}$ under subobjects and binary products. The projections $f_1\colon \mathsf{Eq}(f) \to A$ and $f_2\colon \mathsf{Eq}(f) \to A$ are regular epimorphisms in $\mathcal{D}$, so that (B, f, f) is their pushout in $\mathcal{D}$ by the assumption. This shows that $B \in \mathcal{D}$. ☐

Observe that Proposition 3.3.8 is useful to find examples of semi-effective star-regular categories which are not categories with a good theory of ideals.

This is the case, for instance, for the category $\mathsf{Ab}_{t.f.}$ of torsion-free abelian groups. Indeed, $\mathsf{Ab}_{t.f.}$ is obviously a normal category, and it is also semi-effective star-regular (see Example 3.3.5); however, by Proposition 3.3.8, it does not have a good theory of ideals. This is due to the fact that, although $\mathsf{Ab}_{t.f.}$ is a regular-epireflective subcategory of the category Ab of abelian groups, it is not stable in Ab under quotients.

3.3.5. Almost abelian categories.

Another class of examples to which Theorem 3.2.8 applies is provided by the so-called "almost abelian categories" in the sense of W. Rump [**53**], also called "Raikov semi-abelian" [**51**] in the literature. An almost abelian category can be defined as an additive category with kernels and cokernels with the property that normal epimorphisms are pullback-stable and normal monomorphisms are pushout-stable. As explained in [**52**], G. Janelidze has observed that a category $\mathcal{C}$ is almost abelian if and only if it is both homological (in the sense of F. Borceux and D. Bourn [**5**]) and co-homological. It is well known that, in an almost abelian category, any arrow $f\colon A \to B$ has a canonical factorisation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathsf{coker}(\mathsf{ker}(f))"'] & B \\ \mathsf{Coim}(f) \arrow[r, "\overline{f}"'] & \mathsf{Im}(f) \arrow[u, "\mathsf{ker}(\mathsf{coker}(f))"'] \end{tikzcd} \tag{3.10}$$

with $\overline{f}\colon \mathsf{Coim}(f) \to \mathsf{Im}(f)$ a bimorphism, i.e. an arrow which is at the same time monic and epic. Any almost abelian category $\mathcal{C}$ is normal (it is even homological), and we are now going to prove that it is also semi-effective star-regular (with respect to the class $\mathcal{N}$ of zero arrows). For this, consider a commutative diagram

$$\begin{tikzcd} R^* \arrow[rr, "n"] \arrow[dr, "i"'] & & X \\ & \mathsf{Im}(n) \arrow[ur, "k"'] & \end{tikzcd}$$

where $n\colon R^* \to X$ is the (Bourn-)normal monomorphism yielding the 0-class of an equivalence relation R, $k = \mathsf{ker}(\mathsf{coker}(n))$ is the kernel of the cokernel $\mathsf{coker}(n)$ of n, and i is a split monomorphism. Then, by factorising $n = k\overline{n}\mathsf{coker}(\mathsf{ker}(n))$ as in diagram (3.10), we see that the arrow $i = \mathsf{coker}(\mathsf{ker}(n))\overline{n}$ is an epimorphism, as a composite of two epimorphisms. It follows that i is an isomorphism, and by Remark 3.2.5, $\mathcal{C}$ is a semi-effective star-regular category, as desired.

We observe that, more generally, any normal category $\mathcal{C}$ such that any arrow in $\mathcal{C}$ has a factorisation as an epimorphism followed by a normal monomorphism is semi-effective star-regular.

It is explained in [**53**] that any *torsion-free* subcategory of an abelian category $\mathcal{C}$ is necessarily almost abelian, as is any *torsion* subcategory of $\mathcal{C}$. Further examples of almost abelian categories are given, for instance, by the categories of real (or complex) *normed vector spaces*, *Banach spaces* (with bounded linear maps as morphisms), and also by the category of *locally compact abelian groups*. By Theorem 3.2.8 the regular epimorphisms are then effective descent morphisms in all these categories.

3.3.6. Categories of topological Mal'tsev algebras.

Consider $\mathbb{T}$ a Mal'tsev theory, i.e. an algebraic theory containing a ternary term $p(x,y,z)$ satisfying the identities $p(x,x,y) = y$ and $p(x,y,y) = x$. The category $\mathbb{T}(\mathsf{Top})$ of topological models of such a theory (i.e. models in the category Top of topological spaces) is called a category of *topological Mal'tsev algebras*. The category $\mathbb{T}(\mathsf{Top})$ is a regular category, as shown in [**46**]. We now prove that $\mathbb{T}(\mathsf{Top})$ is semi-effective star-regular (thinking of $(\mathbb{T}(\mathsf{Top}), \mathcal{N})$ as a star-regular category with $\mathcal{N}$ the ideal of all morphisms). When $(R, \tau_R) \rightrightarrows (X, \tau_X)$ is an equivalence relation in $\mathbb{T}(\mathsf{Top})$, and we consider a commutative diagram

$$\begin{tikzcd}
(R,\tau_R) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "i"'] & & (X,\tau_X) \\
& (\mathsf{Eq}(f),\tau) \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

with the property that i is a split monomorphism, then the topology of (R, τ_R) is the one induced by the topology of the product $(X \times X, \tau_{X\times X})$. Accordingly, the equivalence relation (R, τ_R) is the kernel pair of its coequaliser. Hence, the category $\mathbb{T}(\mathsf{Top})$ is semi-effective star-regular and, by Theorem 3.2.8, every regular epimorphism is an effective descent morphism in $\mathbb{T}(\mathsf{Top})$ (this result is known, see [**30**] for instance, although the proof presented here is different).

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