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The polynomial part of the codimension growth of affine PI algebras ☆

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ABSTRACT

Let F be a field of characteristic zero and W an associative affine F-algebra satisfying a polynomial identity (PI). The codimension sequence $\{c_n(W)\}$ associated to W is known to be of the form $\Theta(n^t d^n)$, where d is the well known PI-exponent of W. In this paper we establish an algebraic interpretation of the polynomial part (the constant t) by means of Kemer's theory. In particular, we show that in case W is a *basic* algebra (hence finite dimensional), $t = \frac{q-d}{2} + s$, where q is the number of simple component in $W/J(W)$ and $s+1$ is the nilpotency degree of $J(W)$ (the Jacobson radical of W). Thus proving a conjecture of Giambruno.

1. Introduction

Let W be an affine (i.e. finitely generated) PI algebra over a field F of characteristic 0. The codimension growth of W was studied heavily in the last 40 years or so (see for instance [7,9]). It provides an important tool for measuring the "size" of the T-ideal of identities of W in asymptotic terms. In particular, the exponential part of the asymp-

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totics is key in the classification of varieties of PI algebras. Let us recall briefly some definitions.

Let $F\langle X\rangle$ be the free algebra over a countable set X. A polynomial $f(x_1,\dots,x_n)\in F\langle X\rangle$ in noncommuting indeterminates $x_1,\dots,x_n\in X$, is called a polynomial identity of W if $f(w_1,\dots,w_n)=0$ for any $(w_1,\dots,w_n)\in W^n$. We say W is a PI-algebra if such nonzero polynomial exists. The set of polynomial identities of W, denoted $\mathrm{Id}(W)$, is clearly an ideal of $F\langle X\rangle$. Moreover $\mathrm{Id}(W)$ is a T-ideal, that is, it is invariant under any endomorphism $\phi\in\mathrm{End}_F(F\langle X\rangle)$. It is well known (applying multilinearization) that T-ideals are generated as T-ideal by multilinear elements, i.e. by the elements in $P_n(F)=\mathrm{span}_F\{x_{\sigma(1)}\cdots x_{\sigma(n)}\mid\sigma\in S_n\}$, $n\in\mathbb{N}$. Let $c_n(W)=\dim_F P_n(F)/P_n(F)\cap\mathrm{Id}(W)$ be the nth term of the codimension sequence of W. It is clear that $c_n(W)=c_n(W\otimes_F R)$ where R is a commutative domain and hence, for the calculation of $c_n(W)$ we may assume that *F is an algebraically closed field*.

Regev, in his pioneering article, proved the sequence $\{c_n(W)\}$ is exponentially bounded [12]. Twenty five years later Giambruno and Zaicev showed (as conjectured by Amitsur) that $\lim_{n\to\infty}\sqrt[n]{c_n(W)}$ exists and is an integer. This is the PI-exponent of W, denoted by $\exp(W)$ [5].

Giambruno and Zaicev proved that $\exp(W)$ is determined by the F-dimension of a suitable subalgebra of W in case $W\cong A$ is finite dimensional over F. Let us state their result precisely. Recall that by Wedderburn–Malcev's theorem any finite dimensional algebra A over F can be decomposed into $A=A_{ss}\oplus J(A)$, where A_{ss} is a maximal semisimple subalgebra of A (unique up to isomorphism) and $J(A)$ is the Jacobson radical of A. Moreover, F being algebraically closed, $A_{ss}\cong A_1\times\cdots\times A_q$, where the product is direct and $A_i\cong M_{d_i}(F)$ is the algebra of $d_i\times d_i$-matrices over F. With this notation, Giambruno and Zaicev proved that $\exp(A)=\max\{dim_F(A_{i_1}\times\cdots\times A_{i_r})$: where $i_j\neq i_k$ for $j\neq k$ and $A_{i_1}JA_{i_2}\cdots JA_{i_r}\neq 0\}$.

In order to get their result for arbitrary affine algebras (namely $\lim_{n\to\infty}\sqrt[n]{c_n(W)}$ exists and is an integer) one needs to invoke the fundamental representability theorem of Kemer [10,1].

Theorem 1.1. *For any affine PI algebra W there exists a finite dimensional algebra A such that* $\mathrm{Id}(W)=\mathrm{Id}(A)$*.*

Regev conjectured that the asymptotic behavior of the sequence $\{c_n(W)\}$ is of the form cn^td^n for some constants $c\in\mathbb{R}$, $t\in\frac{1}{2}\mathbb{Z}$ and $d\in\mathbb{Z}_{\geq 0}$ (we write $c_n(W)\simeq cn^td^n$ where $f\simeq g$ means $\lim\frac{f}{g}=1$). A weakened version of Regev's conjecture was confirmed by Berele and Regev in [4], namely that $\{c_n(W)\}$ is asymptotically bounded by the functions

$$c_1n^t(\exp(W))^n\lesssim c_n(W)\lesssim c_2n^t(\exp(W))^n \tag{1.1}$$

for some constants $c_1, c_2 \in \mathbb{R}$ and $t \in \frac{1}{2}\mathbb{Z}$ in case the codimension sequence is eventually nondecreasing (i.e. monotonic nondecreasing for large enough n). Furthermore, they showed that if W is unital (and hence the codimension sequence is nondecreasing) $c_1 = c_2$ and thus confirming in this case Regev's conjecture. Recently, in [8], Giambruno and Zaicev proved that the sequence of codimensions is indeed eventually nondecreasing, showing the asymptotic inequality above holds for any affine PI algebra. We refer to $t = t(W)$ as the polynomial part of the codimension sequence of W.

We emphasize that the proof of Berele and Regev for the existence of the parameter $t(W)$ does not give a formula for its calculation. In this article (following a reduction to basic algebras) we present an interpretation, a la Giambruno and Zaicev, of the polynomial part of the codimension growth for any finite dimensional F-algebra W.

Let A be a finite dimensional algebra and let $\mathrm{Par}(A) = (\dim_F A_{ss}, \mathrm{nildeg}(J(A)) - 1)$ be its *parameter* ($\mathrm{nildeg}(J(A))$ denotes the nilpotency degree of the Jacobson radical). Such an algebra A is said to be basic (or fundamental) if it is not PI equivalent to $B_1 \times \cdots \times B_l$ where $\mathrm{Par}(B_i) < \mathrm{Par}(A)$ for every $1 \leq i \leq l$. By induction one can easily get the following

Lemma 1.2. *Let A be a finite dimensional algebra over F. Then there exist basic algebras $B_1, \ldots, B_l$ such that A is PI-equivalent to $B_1 \times \cdots \times B_l$.*

This can be used to reduce the problem of interpreting the polynomial part (from arbitrary finite dimensional algebras) to basic algebras.

Corollary 1.3. *With the above notation,* $\exp(A) = \max_{1 \leq i \leq l} \exp(B_i)$ *and*

$$t(A) = \max_j \{t(B_j) : \exp(B_j) = \exp(A)\}.$$

Remark 1.4. The definition of a *basic algebra* used in the proof of the representability theorem for affine PI algebras (Theorem 1.1) is different, yet equivalent, to the one we presented above. The decomposition of finite dimensional algebras into the direct product of basic algebras up to PI equivalence (Lemma 1.2 above) using the other definition is a key and nontrivial step in the proof of the Theorem.

For basic algebras Giambruno made the following conjecture.

Conjecture 1.5. *Let A be a basic algebra with Wedderburn–Malcev decomposition $A \cong M_{d_1}(F) \times \cdots \times M_{d_q}(F) \oplus J(A)$. Then*

$$t(A) = \frac{q - d}{2} + s$$

where $d = d_1^2 + \cdots + d_q^2$ and $s + 1$ is the nilpotency degree of $J(A)$.

In case $A = M_d(F)$ this was established by Regev [13]. The conjecture is also known to hold for the algebra of upper-block triangular matrices $UT(d_1, \dots, d_q)$ [6]. This was proved by Giambruno and Zaicev. In their proof they used Lewin's theorem [11] and Berele and Regev's result [3].

Applying Regev's result for matrix algebras, Giambruno's conjecture can be re-stated as follows. Let A be a basic algebra over F and $A \cong A_1 \times \cdots A_q \oplus J(A)$ be its Wedderburn–Malcev decomposition. Then

$$t(A) = t(A_1) + \cdots + t(A_q) + (\mathrm{nildeg}(J(A)) - 1).$$

Our main result appears is Theorem 3.29 where we prove the conjecture for an arbitrary basic algebra.

The paper is organized as follows. In section 2 we study basic algebras. In particular, to any basic algebra A we associate another basic algebra $\mathcal{A}$ which is better understood and yet $\mathrm{Par}(\mathcal{A}) = \mathrm{Par}(A)$. The associated algebra $\mathcal{A}$ will be the main tool in the proof of the upper bound, namely $t(A) \leq \frac{q-d}{2} + s$ (subsection 3.1). Finally in subsection 3.2 we handle the lower bound and achieve the main result.

Notation. Throughout, F will be an algebraically closed field of characteristic 0, W an affine algebra over F, A a finite dimensional algebra over F, A_{ss} a maximal semisimple subalgebra and $J(A)$ its radical. By $\mathrm{nildeg}(J(A))$ we denote the nilpotency degree of $J(A)$, i.e. the number such that $J(A)^l = 0$ but $J(A)^{l-1} \neq 0$. All polynomials are elements of $F\langle X\rangle$, the polynomial ring on countably many noncommuting variables (the elements of X). We denote by S_n the symmetric group on n letters and by $\mathbb{N}$ the set of positive integers.

2. Basic algebras

2.1. Preliminaries

In this section we present a condition which is equivalent to an algebra being basic. In particular, basic algebras may be viewed as minimal models for a given Kemer index.

As always in this article $A \cong A_1 \times \cdots \times A_q \oplus J(A)$ *with* $A_i \cong M_{d_i}(F)$. *Thus* $d_i^2 = \dim_F A_i$.

Definition 2.1 *(Kemer index).* For any $\nu \in \mathbb{N}$ let

$$\Delta_\nu = \{r \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \mathrm{Id}(A) \text{ alternating in } \nu \text{ disjoint sets of size r}\}.$$

Clearly if $\nu \leq \gamma$, then $\max \Delta_\nu \geq \max \Delta_\gamma$. Let $d = \lim_\nu \max(\Delta_\nu)$.

Next we let

$$S_\nu^d = \{j \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \mathrm{Id}(A) \text{ alternating in } \nu \text{ disjoint sets of size d}$$
$$\text{and alternating in } j \text{ sets of size } d+1\}.$$

Also here $\max S_\nu^d \geq \max S_\gamma^d$ if $\nu \leq \gamma$ and we set $s = \lim_\nu \max S_\nu^d$.

The tuple $\kappa_A = (d, s)$ is called the *Kemer index of A*.

Note that d and s are finite. Indeed, d is finite because $\max \Delta_\nu \leq \dim A$ for any $\nu > 0$ and s is finite due to the definition of d. The alternating sets of size d are called *small sets* and the sets of size $d+1$ are called *big sets*. In other words, there exist nonidentities in an arbitrary number of alternating small sets, but only a finite number of big sets (actually at most s big sets).

Definition 2.2. (Kemer polynomials) Let ν_0 be a number where $\max \Delta_{\nu_0} = d$ and $\max S_{\nu_0}^d = s$. Then a multilinear polynomial f is called a *Kemer polynomial* of A if $f \notin \mathrm{Id}(A)$ has at least ν_0 small sets (cardinality d) and exactly s big sets (cardinality $d+1$).

For a general finite dimensional algebra one has $\kappa_A = (d, s) \leq \mathrm{Par}(A)$ in the lexicographic order [1]. Kemer showed that equality characterizes basic algebras.

Theorem 2.3. *(Kemer, see [10] or ([1], Prop. 7.13)) A finite dimensional algebra A is basic if and only if* $\kappa_A = \mathrm{Par}(A)$*.*

An important consequence of this theorem is that Kemer polynomials do not vanish only for evaluations where all simple components A_i are represented among the substitutions and precisely $J(A) - 1$ variables are substituted by radical elements.

2.2. Examples

2.2.1. Well known examples

The matrix algebra $M_d(F)$ has Kemer index $(d^2, 0)$ and thus is basic. To see this, note that any multilinear polynomial alternating on $d^2 + 1$ variables vanishes on $M_d(F)$. Now recall the nth Capelli polynomial

$$\mathrm{cap}_n(X;Y) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) y_1 x_{\sigma(1)} \cdots y_n x_{\sigma(n)} y_{n+1}.$$

It is well known, [7, Prop. 1.7.1], that $\mathrm{cap}_{d^2+1}(X;Y) \in \mathrm{Id}(M_d(F))$ but $\mathrm{cap}_{d^2}(X;Y) \notin \mathrm{Id}(M_d(F))$ and moreover all diagonal elementary matrices can be realized as a nonzero evaluation. Therefore for any $\mu \in \mathbb{N}$ the polynomial

$$\mathrm{Cap}_{d^2}(X_1,\ldots,X_\mu;Y_1,\ldots,Y_\mu)=\prod_{i=1}^{\mu}\mathrm{cap}_{d^2}(X_i,Y_i)$$

where $X_i=\{x_{i,1},\ldots,x_{i,d^2}\}$ is a Kemer polynomial of $M_d(F)$, proving that $\kappa_{M_d(F)}=(d^2,0)$.

The next natural and important example is the algebra of upper block triangular matrices $UT(d_1,\ldots,d_q)$ for positive integers $d_1,\ldots,d_q$. This is the subalgebra of $M_{d_1+\cdots+d_q}(F)$ consisting of the matrices

$$\begin{pmatrix} M_{d_1}(F) & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & M_{d_q}(F) \end{pmatrix}.$$

This is a basic algebra with Kemer index $(d,q-1)$, where $d=d_1^2+\cdots+d_q^2$. Indeed, it is well known that $UT(d_1,\ldots,d_q)$ has exponent d and hence its Kemer index has the form $\kappa=(d,s)$. Moreover, since the nilpotency degree of $UT(d_1,\ldots,d_q)$ is $q-1$ we have $s\le q-1$. In order to complete the proof we will construct (Kemer) polynomials with arbitrary many small sets of cardinality d and precisely $q-1$ sets of cardinality $d+1$. We start with the construction of polynomials with arbitrary many small sets of cardinality d. For the simple component $M_{d_i}(F)$, $i=1,\ldots,q$, we consider the polynomial $\mathrm{Cap}_{d_i^2}(X_{i,1},\ldots,X_{i,\mu};Y_{i,1},\ldots,Y_{i,\mu})$ and their product bridged by $w_1,\ldots,w_{q-1}$

$$\begin{aligned}&\mathrm{Cap}_{d_1^2}(X_{1,1},\ldots,X_{1,\mu};Y_{1,1},\ldots,Y_{1,\mu})\times w_1\cdot\cdots\cdot w_{q-1}\\ &\quad\times\mathrm{Cap}_{d_q^2}(X_{q,1},\ldots,X_{q,\mu};Y_{q,1},\ldots,Y_{q,\mu}),\end{aligned}$$

which we denote by $\mathrm{Cap}_{d_1^2,\ldots,d_q^2}(X_{i,j};Y_{i,j},i=1,\ldots,q,j=1,\ldots,\mu,W)$ or for short $\mathrm{Cap}_{d_1^2,\ldots,d_q^2}(X_{i,j};Y_{i,j},W)$.

Now for $j=1,\ldots,\mu$ we alternate in the polynomial above the sets $X_{1,j},\ldots,X_{q,j}$ and obtain a polynomial which we denote by $f_{1,\mu}(X_{i,j};Y_{i,j},W)$. Finally we construct $q-1$ big sets by alternating w_j with the set $X_j=X_{1,j}\cup\cdots\cup X_{q,j}$, for $j=1,\ldots,q-1$. The result is a polynomial $f_{2,\mu}$ which alternates on $\mu-(q-1)$ small sets of cardinality d and precisely $q-1$ big sets of cardinality $d+1$. We leave the reader the task to show that $f_{1,\mu}$ and $f_{2,\mu}$ are nonidentities of $UT(d_1,\ldots,d_q)$ by presenting nonzero evaluations (variables of W are evaluated on radical elements). Our construction of $f_{2,\mu}$ shows that κ, the Kemer index of $UT(d_1,\ldots,d_q)$, satisfies $\kappa_{UT(d_1,\ldots,d_q)}\ge(d,q-1)$. On the other hand $\kappa_{UT(d_1,\ldots,d_q)}\le\mathrm{Par}(UT(d_1,\ldots,d_q))=(d,q-1)$ and hence $\kappa=\mathrm{Par}$. This shows $UT(d_1,\ldots,d_q)$ is basic and f_2 is a Kemer polynomial.

2.2.2. The associated basic algebra

As above we let $A\cong A_1\times\cdots\times A_q\oplus J(A)$, $r=\dim_F(J(A))$. For any $u\in\mathbb{N}$ consider the associated algebra

$$\mathcal{A}_u := \frac{A_{ss} * F\{b_1, \ldots, b_r\}}{\langle b_1, \ldots, b_r \rangle^{u+1}_{A_{ss} * F\{b_1, \ldots, b_r\}}}.$$

The case where $u = \mathrm{nildeg}(J(A)) - 1$ is of special interest. We denote $\mathcal{A} = \mathcal{A}_{\mathrm{nildeg}(J(A))-1}$.

Lemma 2.4. *The algebra $\mathcal{A}$ is finite dimensional. Moreover, if the algebra A is basic then $\mathcal{A}$ is also basic.*

Proof. Choose a basis Φ of A_{ss} (e.g. the elementary matrices of the simple components A_j). Consider nonzero monomials in $A_{ss} * F\{b_1, \ldots, b_r\}$. These are words of the form

$$a_{i_1} b_{i_1} a_{i_2} b_{i_2} \cdots a_{i_k} b_{i_k} a_{i_{k+1}}$$

where $k \geq 0$, $a_{i_j} \in \Phi$ and $b_{i_j} \in X$. By definition of $\mathcal{A}$, monomials are zero in $\mathcal{A}$ if $k > u$ and hence their number is finite. This proves the first part of the lemma. For the second part, note that A_{ss} is a maximal semisimple subalgebra which supplements the radical $J(\mathcal{A})$. Moreover, the radical is generated by the variables b_i and its nilpotency degree equals $nildeg(J(A))$. It follows that $\mathrm{Par}(\mathcal{A}) = (d, u)$. But the algebra A is a quotient of $\mathcal{A}$ and hence its Kemer index is at least the Kemer index of A. This implies the Kemer index of $\mathcal{A}$ equals $\mathrm{Par}(\mathcal{A}) = (d, u)$ and the result follows. $\square$

Remark 2.5. It turns out that in the lemma above, the condition on the algebra A being basic is not necessary. Let A_{ss} be a semisimple algebra with q simple components. It was proved by the 3rd named author of this article that the algebra $\mathcal{A}_u$, $u \geq q$ is basic. This will appear in a subsequent paper authored by him.

3. Giambruno conjecture

During this entire section we consider a basic F-algebra A whose Wedderburn–Malcev decomposition is given by

$$A = A_{ss} \oplus J(A),$$

where $A_{ss} = A_1 \times \cdots \times A_q$ is a product of matrix algebras, say $A_i \cong M_{d_i}(F)$, $i = 1, \ldots, q$. Next, denote by $\kappa_A = (d, s)$ the Kemer index of A. In particular, by Theorem 2.3, $d = d_1^2 + \cdots + d_q^2$ and $s = \mathrm{nildeg}(J(A)) - 1$ where $\mathrm{nildeg}(J(A))$ is the nilpotency degree of $J(A)$.

The proof consists of two parts, namely we show $(q-d)/2 + s$ bounds from above and from below $t(A)$. For the upper bound observe that $\mathrm{id}(\mathcal{A}) \subseteq \mathrm{id}(A)$. Also, the algebras A and $\mathcal{A}$ have the same Kemer index and moreover have isomorphic semisimple subalgebras supplementing the corresponding radicals. In particular they have the same exponent. It follows that $t(A) \leq t(\mathcal{A})$ and hence it is sufficient to show $t(\mathcal{A}) \leq (q-d)/2 + s$.

3.1. Upper bound

As remarked above we need to show $t(\mathcal{A}) \leq (q-d)/2+s$ where $\mathcal{A}$ is the basic algebra

$$\mathcal{A} = \frac{A_{ss} * F\{b_1, \ldots, b_{\dim_F J(A)}\}}{\langle b_1, \ldots, b_{\dim_F J(A)}\rangle^{s+1}}.$$

3.1.1. Some reductions

First, we present a preferred basis for $\mathcal{A}$. For $1 \leq l \leq q$ we denote the matrix units of A_l by $e_{j_1,j_2}(A_l)$ and $e_{j_1}(A_l) = e_{j_1,j_1}(A_l)$, $1 \leq j_1, j_2 \leq d_l$. Next, for $1 \leq k, k' \leq q$, $1 \leq i \leq d_k$ and $1 \leq j \leq d_{k'}$ let

$$W_{i,j}(A_k, A_{k'}) = \{e_{i,j_0}(A_k)b_{l_0}e_{i_1,j_1}(A_{k_1})b_{l_1} \cdots e_{i_{s'},j_{s'}}(A_{k_{s'}})b_{l_{s'}}e_{i_{s'+1},j}(A_{k'})|0 \leq s' \leq s\}.$$

Note that in the expression above, the indices j_0 and $i_{s'+1}$ run over the sets $\{1, \ldots, d_k\}$ and $\{1, \ldots, d_{k'}\}$ respectively, the indices i_p, j_p run over the set $\{1, \ldots, d_p\}$, $p = 1, \ldots, s'$, and l_ν, $\nu = 0, \ldots, s'$, runs over the set $\{0, \ldots, dim_F J(A)\}$.

We denote by W the union of the sets $W_{i,j}(A_{k_1}, A_{k_2})$, $1 \leq k_1, k_2 \leq q$, $1 \leq i \leq d_{k_1}$, $1 \leq j \leq d_{k_2}$. Thus, a basis for $\mathcal{A}$ is the set

$$\{e_{j_1,j_2}(A_l) \,|\, 1 \leq l \leq q,\ 1 \leq j_1, j_2 \leq d_l\} \cup W.$$

Since $\mathcal{A}$ is a finite dimensional algebra by Proposition 2.4, we can identify its relatively free algebra $F\langle X_i \mid i \in \mathbb{N}\rangle / \operatorname{Id}(\mathcal{A})$ with a subalgebra of

$$\mathcal{A}_K = \mathcal{A} \otimes_F K,$$

where K is the (commutative) polynomial ring

$$K = F\left[\theta^{(i)}_{j_1,j_2}(A_l), \theta^{(i)}(w) \,|\, i \in \mathbb{N},\ 1 \leq l \leq q,\ 1 \leq j_1, j_2 \leq d_l,\ w \in W\right].$$

Let us make this identification explicit. Write,

$$X_i(A_k) = \sum_{a_1,a_2} \theta^{(i)}_{a_1,a_2}(A_k)e_{a_1,a_2}(A_k).$$

Then the variable X_i of the relatively free algebra of $\mathcal{A}$ is identified with

$$\sum_{k=1}^{q} X_i(A_k) + \sum_{w \in W} \underbrace{\theta^{(i)}(w)w}_{X_i(w)} = \sum_{\Sigma} X_i(\Sigma) \in \mathcal{A}_K,$$

where Σ is a symbol which runs over the set $\mathbf{Symb} = W \cup (\mathbf{SimComp} := \{A_1, \ldots, A_q\})$. As a result of this identification, the spaces $P_n(\mathcal{A})$ are viewed as subspaces of $\mathcal{A}_K$.

We decompose $P_n(\mathcal{A})$ into subspaces as follows. Consider a monomial $X_{\sigma(1)} \cdots X_{\sigma(n)} \in P_n(\mathcal{A})$, where $\sigma \in S_n$. Clearly, by the identification we just described, the corresponding monomial in $\mathcal{A}_K$ is equal to the sum

$$\sum_{\Sigma_1,\ldots,\Sigma_n \in \mathbf{Symb}} X_{\sigma(1)}(\Sigma_1) \cdots X_{\sigma(n)}(\Sigma_n). \tag{3.1}$$

Note that

1. $X_i(A_k)X_j(A_{k'}) = 0$ if $k \neq k'$.
2. If more than s symbols from $\Sigma_1, \ldots, \Sigma_n$ are radical (i.e. from W), then

$$X_{\sigma(1)}(\Sigma_1) \cdots X_{\sigma(n)}(\Sigma_n) = 0.$$

This leads to the following definition.

Definition 3.1. A sequence $\overrightarrow{p} = (p_1, \ldots, p_n)$ of symbols in **Symb** is called a *path* (of length n) if the following two properties are satisfied:

1. If $p_i, p_{i+1} \in \mathbf{SimComp}$, then $p_i = p_{i+1}$.
2. Not more than s symbols (from $p_1, \ldots, p_n$) are in W.

Furthermore, suppose $\overrightarrow{p} = (A_{k_1}, \ldots, A_{k_1}, w_1, A_{k_2}, \ldots, A_{k_2}, w_2, \ldots, w_{s'}, A_{k_{s'+1}}, \ldots, A_{k_{s'+1}})$, then $\mathbf{struc}(\overrightarrow{p})$, the *path structure* of $\overrightarrow{p}$, is defined to be the sequence $(A_{k_1}, w_1, A_{k_2}, w_2, \ldots, w_{s'}, A_{k_{s'+1}})$ (i.e. we record the simple components with no adjacent repetitions and the radical elements). Two paths $\overrightarrow{p_1}, \overrightarrow{p_2}$ (of the same length) are called *equivalent* (denoted by $\overrightarrow{p_1} \sim \overrightarrow{p_2}$) if they have the same path structure (e.g. $(A_{k_1}, A_{k_1}, w_1, A_{k_2})$ and $(A_{k_1}, w_1, A_{k_2}, A_{k_2})$ are equivalent paths whereas $(A_{k_1}, A_{k_1}, w_1, A_{k_2})$ and (A_{k_1}, w_1, A_{k_2}) are not).

The set of all paths of length n is denoted by $\mathbf{Path}_n$ and the set of all equivalence classes of paths of length n is denoted by $\mathbf{Path}_n/\sim$.

Definition 3.2. For a given path $\overrightarrow{p} \in \mathbf{Path}_n$ we denote the number of appearances of a symbol Σ from **Symb** by $\overrightarrow{p}(\Sigma)$.

By definition, the expression in (3.1) can now be rewritten as (indeed, we ignore some vanishing monomials)

$$\begin{aligned}
&\sum_{\overrightarrow{p}=(p_1,\ldots,p_n) \in \mathbf{Path}_n} X_{\sigma(1)}(p_1) \cdots X_{\sigma(n)}(p_n) \\
&= \sum_{[\overrightarrow{p_1}] \in \mathbf{Path}_n/\sim} \left(\sum_{\overrightarrow{p}=(p_1,\ldots,p_n) \in [\overrightarrow{p_1}]} X_{\sigma(1)}(p_1) \cdots X_{\sigma(n)}(p_n) \right).
\end{aligned}$$

Definition 3.3. For a fixed $\overrightarrow{p} \in \mathbf{Path}_n$ denote by $P_{\overrightarrow{p}}(\mathcal{A})$ the F-linear span of all monomials $X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$, where σ varies over S_n. Furthermore, $P_{[\overrightarrow{p_1}]}(\mathcal{A})$ denotes the sum of all $P_{\overrightarrow{p}}(\mathcal{A})$ such that $\overrightarrow{p} \sim \overrightarrow{p_1}$.

Lemma 3.4. *The space $P_n(\mathcal{A})$ is embedded in*

$$\bigoplus_{\overrightarrow{p}\in\mathbf{Path}_n} P_{\overrightarrow{p}}(\mathcal{A}) = \bigoplus_{[\overrightarrow{p}]\in\mathbf{Path}_n/\sim} P_{[\overrightarrow{p}]}(\mathcal{A}).$$

As a result,

$$c_n(A) \leq \sum_{[\overrightarrow{p}]\in\mathbf{Path}_n/\sim} \dim_F P_{[\overrightarrow{p}]}(\mathcal{A}).$$

Moreover, the size of the set $\mathbf{Path}_n/\sim$ is bounded by a constant independent of n.

Proof. Only the last part requires an explanation. Indeed, the size of $\mathbf{Path}_n/\sim$ is bounded from above by the number of sequences of length at most $2s+1$ whose elements are taken from the finite set **Symb**. So the constant can be taken to be

$$\sum_{t=1}^{2s+1} |\mathbf{Symb}|^t. \quad \square$$

Remark 3.5. By the previous Lemma, it is sufficient to show that $\dim_F P_{[\overrightarrow{p}]}(\mathcal{A})$, $\overrightarrow{p} \in \mathbf{Path}_n$, is bounded from above by $Cn^{\frac{q-d}{2}+s}d^n$, where C is some constant independent of n.

We intend to decompose each $P_{[\overrightarrow{p}]}(\mathcal{A})$ into a (direct) sum of some special subspaces.

Definition 3.6. Let $\overrightarrow{p} = (p_1, \ldots, p_n)$ be a path in $\mathbf{Path}_n$ and let $Z = X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$ be a monomial in $P_{\overrightarrow{p}}(\mathcal{A})$ for some $\sigma \in S_n$. For $1 \leq l \leq q$ we denote by $\mathbf{ind}_l(Z)$ the set of all indices $\sigma(u)$ (here $1 \leq u \leq n$) for which $p_u = A_l$.

Furthermore, we denote by $\mathbf{seq}_{rad}(Z)$ the sequence of indices $(\sigma(i_1), \ldots, \sigma(i_{s'}))$ for which

1. $p_{i_u} \in W$ for every $1 \leq u \leq s'$.
2. $i_1 < \cdots < i_{s'}$.
3. $\{\sigma(i_1), \ldots, \sigma(i_{s'})\} \cup \mathbf{ind}_1(Z) \cup \cdots \cup \mathbf{ind}_q(Z) = \{1, \ldots, n\}$ (that is $\mathbf{seq}_{rad}(Z)$ consists of all the indices whose corresponding variables take values in the radical).

Finally, we set $\overrightarrow{\mathrm{ind}}(Z) = (\mathbf{ind}_1(Z), \ldots, \mathbf{ind}_q(Z); \mathbf{seq}_{rad}(Z))$.

Definition 3.7. Two monomials Z_1 and Z_2 in $P_{[\overrightarrow{p}]}(\mathcal{A})$ are *equivalent* (or $Z_1 \sim Z_2$) if $\overrightarrow{\text{ind}}(Z_1) = \overrightarrow{\text{ind}}(Z_2)$. The set of all equivalence classes corresponding to this relation is denoted by $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$, where $\mathbf{Mon}_{[\overrightarrow{p}]}$ is the set of monomials in $P_{[\overrightarrow{p}]}(\mathcal{A})$. Furthermore, $P_{[Z]}(\mathcal{A}) (\subseteq P_{[\overrightarrow{p}]}(\mathcal{A}))$ denotes the F-span of all monomials in $P_{[\overrightarrow{p}]}(\mathcal{A})$ which are equivalent to Z.

Lemma 3.8. *The following hold:*

1. $P_{[\overrightarrow{p}]}(\mathcal{A})$ *is equal to*

$$\bigoplus_{[Z]\in \mathbf{Mon}_{[\overrightarrow{p}]}/\sim} P_{[Z]}(\mathcal{A}).$$

2. *Denote by* $\mathbf{Mon}_{[\overrightarrow{p}]}(n_1, \dots, n_q)/\sim$ *the subset of* $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$ *consisting of all* $[Z]$ *for which the corresponding path* $\overrightarrow{p}$ *satisfies* $n_1 = |\mathbf{ind}_1(Z)|, \dots, n_q = |\mathbf{ind}_q(Z)|$. *Then,* $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$ *is equal to the (disjoint) union*

$$\bigcup_{n_1+\cdots+n_q=n-s'} \mathbf{Mon}_{[\overrightarrow{p}]}(n_1, \dots, n_q)/\sim,$$

where $s' = |\mathbf{seq}_{rad}(Z)|$ *is the number of symbols from* W *in* $\overrightarrow{p}$.
3. *The size of* $\mathbf{Mon}_{[\overrightarrow{p}]}(n_1, \dots, n_q)/\sim$ *is bounded from above by*

$$n^{s'} \cdot \binom{n-s'}{n_1, \dots, n_q}.$$

Proof. Only the third part requires a proof. There are $s'! \cdot \binom{n}{s'}$ options to choose and order s' indices from the set $\{1, \dots, n\}$, i.e. there are $s'! \cdot \binom{n}{s'}$ ways to choose $\mathbf{seq}_{rad}$ for a fixed $1 \le s' \le s$. From the remaining $n - s'$ indices there are $\binom{n-s'}{n_1,\dots,n_q}$ options to choose n_1 which will correspond to $A_1, \dots, n_q$ which will correspond to A_q. Finally, it is clear that

$$s'! \cdot \binom{n}{s'} \binom{n-s'}{n_1, \dots, n_q} \le n^{s'} \binom{n-s'}{n_1, \dots, n_q}. \quad \square$$

Definition 3.9. For $\overrightarrow{i} = (i_1, \dots, i_l)$ we denote the product $X_{i_1}(A_j) \cdots X_{i_l}(A_j)$ by $\mathbf{X}_{\overrightarrow{\mathbf{i}}}(A_j)$.

Consider monomials in $P_{[Z]}(\mathcal{A}) (\subseteq P_{[\overrightarrow{p}]}(\mathcal{A}))$ of the form

$$\mathbf{X}_{\overrightarrow{\mathbf{i}_1}}(A_{k_1}) X_{\nu_1}(w_1) \mathbf{X}_{\overrightarrow{\mathbf{i}_2}}(A_{k_2}) X_{\nu_2}(w_2) \cdots \mathbf{X}_{\overrightarrow{\mathbf{i}_{s'+1}}}(A_{k_{s'+1}}),$$

namely monomials which satisfy

1. $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \ldots, A_{k_{s'+1}})$
2. $\bigcup_{\alpha: k_\alpha = l} Set_{\overrightarrow{\mathbf{i}_\alpha}} = \mathbf{ind}_l(Z)$, where Set_x consists of all indices appearing in the vector x
3. $\mathbf{seq}_{rad}(Z) = (\nu_1, \ldots, \nu_{s'})$.

Remark 3.10. It is important to stress that there exist other type of monomials, namely monomials where some radical elements are adjacent or monomials which start or end by radical elements. As it will be clear below, the treatment of these monomials is similar to the monomials of the type considered in Definition 3.9.

Consider the spaces $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}) := e_{j_0}(A_{k_1}) P_{[Z]}(\mathcal{A}) e_{j_{s'+1}}(A_{k_{s'+1}}) (\subseteq \mathcal{A}_K)$, where $1 \leq j_0 \leq d_{k_1}$ and $1 \leq j_{s'+1} \leq d_{k_{s'+1}}$ (recall our notation $e_j(B)$, the diagonal matrix $e_{j,j}$ in the matrix algebra B). Note that the simple components A_{k_1} and $A_{k_{s'+1}}$ are determined by the monomial Z if $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}) \neq 0$ and hence we do not record the indices k_1 and $k_{s'+1}$ in the definition of $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$. Since any element f of $P_{[Z]}(\mathcal{A})$ can be written as the sum

$$f = 1(A_{k_1}) \cdot f \cdot 1(A_{k_{s'+1}}) = \sum_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} e_{j_0}(A_{k_1}) \cdot f \cdot e_{j_{s'+1}}(A_{k_{s'+1}}),$$

we obtain an injective map

$$P_{[Z]}(\mathcal{A}) \to \bigoplus_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}).$$

So we have proved

Lemma 3.11. $\dim_F P_{[Z]}(\mathcal{A}) \leq \sum_{j_0, j_{s'+1}} \dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$.

As a result of this observation we will fix also the indices j_0, $j_{s'+1}$ and work in the space $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$. In Lemma 3.12 we describe the asymptotics of $\dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$ and then, subsequently, we will sum up all spaces of that form.

3.1.2. The key lemma and upper bound

In order to be able to carry out manipulations in the vector space $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$ we introduce the following notation:

$$\theta_{a_1, \ldots, a_{l+1}}^{X_1, \ldots, X_l}(A_k) := \theta_{a_1, a_2}^{(X_1)}(A_k) \theta_{a_2, a_3}^{(X_2)}(A_k) \cdots \theta_{a_l, a_{l+1}}^{(X_l)}(A_k)$$

and

$$\theta_{i|j}^{X_1, \ldots, X_l}(A_k) := \sum_{a_2, \ldots, a_l} \theta_{a_1 = i, a_2, \ldots, a_l, a_{l+1} = j}^{X_1, \ldots, X_l}(A_k).$$

Note that we changed slightly the notation we introduced above by replacing $\theta^{(l)}_{a_k,a_r}$ with $\theta^{(X_l)}_{a_k,a_r}$.

Furthermore, if $\overrightarrow{\nu} = (1,\dots,l)$ we simply write

$$\theta^{X_{\overrightarrow{\nu}}}_{i|j}(A_k) = \theta^{X_1,\dots,X_l}_{i|j}(A_k).$$

The next lemma is straightforward (proof is omitted).

Lemma 3.12. *The following hold.*

1. *For $w_1 \in W_{-,i}(A_-, A_k)$, $w_2 \in W_{j,-}(A_k, A_-)$ we have*

$$w_1 \mathbf{X}_{\overrightarrow{\nu}}(A_k) w_2 = \theta^{X_{\overrightarrow{\nu}}}_{i|j}(A_k) \cdot w_1 e_{i,j}(A_k) w_2.$$

2. *For $w_1 \in W_{j_1,\tilde{j}_1}(A_{k_1}, A_{k_2}), \dots, w_{s'} \in W_{j_{s'},\tilde{j}_{s'}}(A_{k_{s'}}, A_{k_{s'+1}})$ we have*

$$\underbrace{e_{\tilde{j}_0=j_0}(A_{k_1})}_{w_0}\Big(\mathbf{X}_{\overrightarrow{\nu_1}}(A_{k_1}) X_{i_1}(w_1) \mathbf{X}_{\overrightarrow{\nu_2}}(A_{k_2}) X_{i_2}(w_2) \cdots \mathbf{X}_{\overrightarrow{\nu_{s'+1}}}(A_{k_{s'+1}})\Big)$$
$$\times \underbrace{e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})}_{w_{s'+1}}$$

equals to

$$\theta^{(i_1)}(w_1)\cdots\theta^{(i_{s'})}(w_{s'})\left(\prod_{l=1}^{q}\prod_{\alpha:k_\alpha=l}\theta^{X_{\overrightarrow{\nu_\alpha}}}_{\tilde{j}_{\alpha-1}|j_\alpha}(A_l)\right)\cdot \mathbf{w},$$

where

$$\mathbf{w} = w_0 e_{\tilde{j}_0,j_1}(A_{k_1}) w_1 e_{\tilde{j}_1,j_2}(A_{k_2}) \cdots w_{s'} e_{\tilde{j}_{s'},j_{s'+1}}(A_{k_{s'+1}}) w_{s'+1}.$$

Corollary 3.13. *There exists an element $w \in A$ such that for any monomial $Z' \in P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$ (and hence for any element of $P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$)*

$$Z' = f \cdot w$$

where f is a polynomial in $F\big[\theta^{(i)}_{j_1,j_2}(A_l), \theta^{(i)}(w) \,|\, i \in \mathbb{N},\ 1 \le l \le q,\ 1 \le j_1, j_2 \le d_l,$ $w \in W\big]$.

We now turn to the construction of the map which enables to estimate $\dim_F P^{j_0,j_{s'+1}}_{[Z]}$.

For every l, consider the variables $Y_{1,l}, \dots, Y_{v_l-1,l}$, where v_l is the number of appearances of A_l in $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \dots, A_{k_{s'+1}})$.

Let $P_{X_{h_1},\ldots,X_{h_{n_l}};Y_{1,l},\ldots,Y_{v_l-1,l}}$ denote the $(n_l+v_l-1)!$ space of all multilinear polynomials in the prescribed variables, where $\mathbf{ind}_l(Z)=(h_1,\ldots,h_{n_l})$ (i.e. the indices of variables which get values from A_l). Let

$$\begin{aligned}&P_{X_{h_1},\ldots,X_{h_{n_l}};Y_{1,l},\ldots,Y_{v_l-1,l}}(A_l)\\&\quad=P_{X_{h_1},\ldots,X_{h_{n_l}};Y_{1,l},\ldots,Y_{v_l-1,l}}/(P_{X_{h_1},\ldots,X_{h_{n_l}};Y_{1,l},\ldots,Y_{v_l-1,l}}\cap Id(A_l)).\end{aligned}$$

Denote

$$(U_1,\ldots,U_{n_l+v_l+1})=(X_{h_1},\ldots,X_{h_{n_l}},Y_{1,l},\ldots,Y_{v_l-1,l})$$

and let $P_{[\widehat{X_{A_l}}]}(A_l)=P_{[\widehat{X_{A_l}}]}/(P_{[\widehat{X_{A_l}}]}\cap Id(A_l))$ be the subspace of $P_{X_{h_1},\ldots,X_{h_{n_l}};Y_{1,l},\ldots,Y_{v_l-1,l}}(A_l)$ spanned by all monomials $U_{\tau(1)}U_{\tau(2)}\cdots U_{\tau(n_l+v_l-1)}$ where

1. $U_{\tau(1)},U_{\tau(n_l+v_l-1)}\in\{X_{h_1},\ldots,X_{h_{n_l}}\}$
2. If $i<j$ and $U_{\tau(i)},U_{\tau(j)}\in\{Y_{1,l},\ldots,Y_{v_l-1,l}\}$, then there exists k with $i<k<j$ such that $U_{\tau(k)}\in\{X_{h_1},\ldots,X_{h_{n_l}}\}$.
3. The ordering induced by τ on the set $\{Y_{1,l},\ldots,Y_{v_l-1,l}\}$ is precisely the ordering $(Y_{1,l},\ldots,Y_{v_l-1,l})$. That is, for $n_l+1\le i,j\le n_l+v_l-1$ we have $i<j\Longrightarrow\tau(i)<\tau(j)$.

Remark 3.14. We abuse notation and language here by considering monomials as elements of $P_{[\widehat{X_{A_l}}]}(A_l)$

Notation. We denote by $c_{\widehat{X_{A_l}}}(A_l)=dim_F P_{[\widehat{X_{A_l}}]}(A_l)$ and note that $c_{\widehat{X_{A_l}}}(A_l)\le c_{n_l+v_l-1}(A_l)$.

Now, for $l=1,\ldots,q$ let

$$\widehat{X_{A_l}}=X_{\overrightarrow{\mu_{\alpha(1,l)}}}\cdot Y_{1,l}\cdot X_{\overrightarrow{\mu_{\alpha(2,l)}}}\cdot Y_{2,l}\cdots Y_{v_l-1,l}\cdot X_{\overrightarrow{\mu_{\alpha(v_l,l)}}},$$

be a monomial in $P_{[\widehat{X_{A_l}}]}(A_l)$ and let

$$X=(\widehat{X_{A_1}},\ldots,\widehat{X_{A_q}})\in P_{[\widehat{X_{A_1}}]}\times\cdots\times P_{[\widehat{X_{A_q}}]}.$$

Next consider the monomial

$$e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})\in P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}),$$

where

$$X(\mathcal{A})=\mathbf{X}_{\overrightarrow{\nu_{\alpha(t_1,k_1)}}}(A_{k_1})X_{i_1}(w_1)\mathbf{X}_{\overrightarrow{\nu_{\alpha(t_2,k_2)}}}(A_{k_2})X_{i_2}(w_2)\cdots\mathbf{X}_{\overrightarrow{\nu_{\alpha(t_{s'+1},k_{s'+1})}}}(A_{k_{s'+1}})$$

and if $k_{g_1} = k_{g_2} = \ldots = k_{g_{v_l}} = l$ are the indices where the simple component A_l appears in $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \ldots, A_{k_{s'+1}})$, then

$$(\overrightarrow{\nu_{\alpha(t_{g_1},k_{g_1})}}, \ldots, \overrightarrow{\nu_{\alpha(t_{g_{v_l}},k_{g_{v_l}})}}) = (\overrightarrow{\mu_{\alpha(1,l)}}, \ldots, \overrightarrow{\mu_{\alpha(v_l,l)}}).$$

Lemma 3.15. *The following hold.*

1. *The map* $\Psi : P_{[\widehat{X_{A_1}}]} \times \cdots \times P_{[\widehat{X_{A_q}}]} \to P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$

$$X \mapsto e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})$$

is well defined, surjective and multilinear. Hence it determines a surjection

$$P_{\widehat{[X_{A_1}]}}(A_1) \otimes \cdots \otimes P_{\widehat{[X_{A_q}]}}(A_q) \to P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A}).$$

2. $\dim_F P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A}) \leq c_{n_1+s}(A_1) \cdots c_{n_q+s}(A_q) \leq C \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q)$, *where C is a constant which is independent of $n_1, \ldots, n_q$.*

Proof. Suppose f is a linear combination of monomials in $P_{[\widehat{X_{A_l}}]}(A_l)$ which represents the zero element. Clearly, f represents the zero map in $Hom(A_l^{\otimes(n_l+v_l-1)}, A_l)$ and hence, evaluating $Y_{1,l}, \ldots, Y_{v_l-1,l}$ on A_l we obtain the zero map in $Hom(A_l^{\otimes n_l}, A_l)$. In particular we obtain zero if we evaluate

$$Y_{1,l} = e_{j_{g_1},\tilde{j}_{g_2}-1}(A_l), Y_{2,l} = e_{j_{g_2},\tilde{j}_{g_3}-1}(A_l), \ldots, Y_{v_l-1,l} = e_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}-1}(A_l).$$

But in view of the fact that

$$w_{g_1} \in W_{j_{g_1},\tilde{j}_{g_1}}(A_{k_{g_1}}, A_{k_{g_1+1}}), w_{g_2} \in W_{j_{g_2},\tilde{j}_{g_2}}(A_{k_{g_2}}, A_{k_{g_2+1}}), \ldots,$$
$$w_{g_{v_l-1}} \in W_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}}(A_{k_{g_{v_l}-1}}, A_{k_{g_{v_l}}}), w_{g_{v_l-1}} \in W_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}}(A_{k_{g_{v_l}-1}}, A_{k_{g_{v_l}}}),$$

and by Lemma 3.12, we see that up to a scalar the bridge between the ith and the $(i+1)$th appearance of A_l in $X(\mathcal{A})$ is given precisely by the matrix $e_{j_{g_i},\tilde{j}_{g_{i+1}}-1}(A_l)$ and hence $e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}}) = 0$. This shows the map Ψ is well defined. It is clear by construction that Ψ is multilinear and onto.

Let us prove the second part. Applying part (1) and the inequalities $c_{\widehat{X_{A_l}}}(A_l) \leq c_{n_l+v_l-1}(A_l)$ we have

$$\dim_F P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A}) \leq c_{\widehat{X_{A_1}}}(A_1) \cdots c_{\widehat{X_{A_q}}}(A_q) \leq c_{n_l+v_l-1}(A_l) \cdots c_{n_l+v_l-1}(A_l).$$

Furthermore, since the number of Y's is bounded by s, and the sequence $c_n(A_l)$ is an eventually nondecreasing function in n [8], we obtain

$$\dim_F P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}) \le c_{n_1+s}(A_1)\cdots c_{n_q+s}(A_q).$$

The last inequality in the lemma follows from the fact that $c_n(B) \simeq \Theta(n^t d^n)$ for PI algebras B (as proved by Berele and Regev [4] for unital algebras and later by Giambruno and Zaicev [8] for arbitrary algebras). Indeed, we have that

$$\lim_{n\to\infty} \frac{c_{n+s}(A_l)}{c_n(A_l)} = K_2$$

for some constant $K_2 \in \mathbb{R}$ and the result follows. This completes the proof of the lemma. □

As mentioned in Remark 3.10 other type of monomials Z should be considered. The proofs of the statements that correspond to Lemmas 3.11–3.15 are similar and left to the reader.

Theorem 3.16 *(Upper bound). There is a constant C such that*

$$c_n(A) \le C \cdot n^{\frac{q-d}{2}+s} d^n.$$

Proof. By part (5) of Lemma 3.12 it follows that

$$\dim_F P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}) \le C_1 \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q),$$

where $n_1,\dots,n_q$ are determined by the path corresponding to $[Z]$. Combining this with Lemma 3.11 it gives

$$\dim_F P_{[Z]}(\mathcal{A}) \le C_2 \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q).$$

By Lemma 3.8 it follows that

$$\sum_{[Z]\in \mathbf{Mon}_{[\vec{p}]}(n_1,\dots,n_q)/\sim} \dim_F P_{[Z]}(\mathcal{A}) \le C_3 \cdot n^{s'} \binom{n-s'}{n_1,\dots,n_q} \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q),$$

where $s' = n - n_1 - \cdots - n_q$. Thus,

$$\dim_F P_{[\vec{p}]}(\mathcal{A}) \le C_4 \cdot n^{s'} \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\dots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q).$$

By Lemma 3.4 we obtain

$$c_n(A) \le C_5 \cdot \sum_{s'=0}^{s} \left(n^{s'} \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\dots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q) \right).$$

Next, by [13],

$$\sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q)$$
$$\leq C_6 \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} n_1^{\frac{1-d_1^2}{2}} d_1^{2n_1} \cdots n_q^{\frac{1-d_q^2}{2}} d_q^{2n_q},$$

for some constant C_6.

This, by a theorem of Regev and Beckner (which we recall below for the reader convenience), is asymptotically equal to

$$C_7 \cdot (n-s')^{\frac{q-d}{2}} d^n.$$

All in all we have

$$c_n(A) \leq C_8 \cdot \sum_{s'=0}^{s} \left(n^{s'} \cdot (n-s')^{\frac{q-d}{2}} d^n \right) \leq C \cdot n^{\frac{q-d}{2}+s} d^n$$

as desired. □

Theorem 3.17. *(Regev and Beckner [2]) Let $r_1,\ldots,r_q,k_1,\ldots,k_q \in \mathbb{R}$ such that $0 < k_1,\ldots,k_q$. Then,*

$$\sum_{n_1+\cdots+n_q=n} \binom{n}{n_1,\ldots,n_q} k_1^{n_1}\cdots k_q^{n_q} n_1^{r_1}\cdots n_q^{r_q} \sim \left(\left(\frac{k_1}{k}\right)^{r_1} \cdots \left(\frac{k_q}{k}\right)^{r_q} \right) \cdot n^r k^n,$$

where $k = k_1+\cdots+k_q$ and $r = r_1+\cdots+r_q$.

3.2. Lower bound

Let A be a basic algebra over a field F. Write $A = (A_{ss} = A_1 \times \cdots \times A_q) \oplus J$ and denote by d_i^2 the dimension of A_i, $i = 1,\ldots,q$. Furthermore, $d = d_1^2+\cdots+d_q^2$ and s denotes the nilpotency degree of J minus 1.

Convention 3.18. *In the sequel, as we may by linearity, all evaluations we consider of multilinear polynomials are from $A_1 \cup \cdots \cup A_q \cup J$.*

Since A is basic, by Theorem 2.3, it possesses a multilinear polynomial

$$f_0 = f_0\left(z_1,\ldots,z_q; B := B_1 \cup \cdots \cup B_s; E\right),$$

where

1. $|B_1| = \cdots = |B_s| = d + 1$.
2. f_0 alternates on each set B_i, $i = 1, \ldots, s$. Therefore, any nonzero evaluation of the variables of f_0 takes exactly one variable of every B_i to a radical element and the remaining variables (including the z's and the ones from E) to a semisimple element.
3. There is a nonzero evaluation of the variables of f_0 such that $\tilde{z}_i = 1(A_i)$ for $i = 1, \ldots, q$.

Till the end of this section we fix a nonzero evaluation, denoted by $\tilde{f}_0 = f_0(\tilde{z}_1, \ldots, \tilde{z}_q; \tilde{B}; \tilde{E})$, which satisfies (3). Moreover, for $i = 1, \ldots, s$, we denote by $w_i \in B_i$ the variable such that $\tilde{w}_i \in J$.

Remark 3.19. In the sequel we'll consider partial evaluations of multilinear polynomials. The properties of these evaluations rely on the existence of the evaluation $\tilde{f}_0$ of f_0.

Consider the multilinear polynomial

$$f_1 = f_1(z_1, \ldots, z_q; B; Y; E) = f_0(y_{1,1}y_{1,2}z_1y_{1,3}y_{1.4}, \ldots, y_{q,1}y_{q,2}z_qy_{q,3}y_{q,4}; B; E),$$

where $Y = \{y_{i,j} | i = 1, \ldots, q, j = 1, \ldots, 4\}$. We shall abuse notation by writing $f_1(z_1, \ldots, z_q)$ (omitting E, Y and B). Furthermore, we denote $B \cup Y \cup E$ by BYE. Note that since f_0 is a nonidentity of A, f_1 is a nonidentity as well.

Remark 3.20. In what follows, roughly speaking, we shall replace the variables $z_1, \ldots, z_q$, with multilinear polynomials $g_1, \ldots, g_q$. The polynomials $g_1, \ldots, g_q$ will be elements in $F\langle x_1, \ldots, x_n\rangle$, $n \in \mathbb{N}$, where different polynomials depend on disjoint sets of variables. This will give rise to a multilinear polynomial in the variables $\{x_1, \ldots, x_n\} \cup BYE$.

For any $n \in \mathbb{N}$ let $X_n = \{x_1, \ldots, x_n\}$, and let $X_n \cup BYE$ be the corresponding set of variables. Let $P_{X_n;BYE}$ be the F-space of all multilinear polynomials on $X_n \cup BYE$. The symmetric group S_n acts on $P_{X_n;BYE}$ by

$$\sigma \cdot f(x_1, \ldots, x_n; BYE) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}; BYE)$$

and it is well known that this action induces an S_n-module structure on the space

$$P_{X_n;BYE}(A) = \frac{P_{X_n;BYE}}{P_{X_n;BYE} \cap id(A)}.$$

Now, consider a partition $\mathbf{p}$ of the set X_n into q subsets, denoted by $X[A_1], \ldots, X[A_q]$, where each $X[A_i]$ is nonempty (we are interested in $n \to \infty$, so we may assume that $n \geq q$).

Consider the symmetric groups $S_{X[A_1]}, \ldots, S_{X[A_q]}$ and their product $S_{\boldsymbol{p}} = S_{X[A_1]} \times \cdots \times S_{X[A_q]} < S_n$. Clearly, by restriction, we obtain $S_{\boldsymbol{p}}$-modules structures on $P_{X_n;BYE}$ and on $P_{X_n;BYE}(A)$.

By the well known embedding of the relatively free algebra of A in $U_A = A \otimes_F F(\theta_{i,j} : i \in \mathbb{N},\ j = 1, \dots, \dim_F A)$ we may view the space $P_{X_n;BYE}(A)$ as a subspace of U_A. This allows us to consider partial evaluation which is a key idea in this section.

Definition 3.21. For $\tilde{\mathbf{j}}, \mathbf{j} \in \{1, \dots, d_1\} \times \cdots \times \{1, \dots, d_q\}$, let $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A) \subseteq U_A$ be the space obtained from $P_{X_{\mathbf{n}};BYE}(A)$ by performing the following evaluations on some of the variables:

1. $y_{i,2} \to \bar{y}_{i,2} = e_{\tilde{j}_i}(A_i)$ and $y_{i,3} \to \bar{y}_{i,3} = e_{j_i}(A_i)$ for $i = 1, \dots, q$.
2. For $i = 1, \dots, s$, and for any $w \in B_i \setminus \{w_i\}$ we replace w by its value $\tilde{w}$. Note that $\tilde{w}$ is necessary a semisimple element.

For $g \in P_{X_n;BYE}(A)$ we denote by $\bar{g}$ its image in $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$.

Notation. In order to simplify our notation for elements in $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ we shall write $\overline{f(x_1, \dots, x_n)}$ if the variables BYE do not play a role in our variable manipulations. In Lemma 3.27 we'll need to manipulate the variables of X_n and B and so we'll write $\overline{f(x_1, \dots, x_n; w_1, \dots, w_s)}$.

Observe that $\dim_F P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A) \leq \dim_F P_{X_n;BYE}(A)$. Therefore, for the lower bound, it will be sufficient to bound from below one of the spaces $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$.

The elements of $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$, as the elements of $P_{X_n;BYE}(A)$, will be referred as *multilinear polynomials.*

As for the elements of $P_{X_{\mathbf{n}};BYE}(A)$, we perform the partial evaluations 1 and 2 in Definition 3.21 also on the polynomial

$$f_1 = f_1(z_1, \dots, z_q; B; Y; E) = f_0(y_{1,1}y_{1,2}z_1y_{1,3}y_{1.4}, \dots, y_{q,1}y_{q,2}z_qy_{q,3}y_{q,4}; B; E),$$

and denote the obtained polynomial by $\overline{f_1}$.

Lemma 3.22. *For any nonzero evaluation of $\overline{f_1}$ the following hold.*

1. *The variables $\{z_1, \dots, z_q\} \cup Y \cup E$ are all evaluated by semisimple elements.*
2. *Each $w_i \in B_i$, $i = 1, \dots, s$, is evaluated by a radical element (note the other elements of B_i were already replaced by semisimple elements).*
3. *For every $i = 1, \dots, q$, the variable z_i is evaluated by an element of A_i.*

Moreover, any nonzero evaluation $\bar{g}_i$ of a polynomial g_i by elements of A_i, $i = 1, \dots, q$, such that $e_{j_i}\bar{g}_i e_{\tilde{j}_i} \neq 0$ may be extended to a nonzero evaluation of $\overline{\mathbf{f_1}(g_1, \dots, g_q)}$.

Proof. Parts (1) and (2) are clear. Part (3) now follows, since $\bar{y}_{i,2}$ (and $\bar{y}_{i,3}$) is an element from A_i and z_i must be a semisimple element (by (1)).

We turn to the last part of the Lemma. Since $\bar{y}_{i,2}\bar{g}_i\bar{y}_{i,3} \neq 0$, we can find suitable evaluations of $y_{i,1}$ and $y_{i,4}$ so that the expression $y_{i,1}\bar{y}_{i,2}\bar{g}_i\bar{y}_{i,3}y_{i,4}$ is evaluated by any element of A_i. We are done since there is a nonzero evaluation of f_0 were each z_i (as variable of f_0) is evaluated by an element of A_i. $\square$

Lemma 3.23. *Let V and W be finite dimensional vector spaces and let U be a subspace of $Hom_F(V, W)$. Fix a basis $w_1, \ldots, w_{\dim_F W}$ of W. Then there is an element ψ in the dual basis of W such that*

$$\dim_F(\psi \circ U) \geq \frac{\dim_F U}{\dim_F W}$$

where $\psi \circ U$ is the space of all elements $\psi \circ T$ where $T \in U$.

Proof. It is clear that the map $\Psi : Hom_F(V, W) \to V^* \oplus \cdots \oplus V^*$ ($\dim_F W$ times) given by

$$\Psi(T) = (\psi_1 \circ T, \ldots, \psi_{\dim_F W} \circ T),$$

is injective (in fact, it is an isomorphism), where $\psi_1, \ldots, \psi_{\dim_F W}$ is any basis of W^* (in our case it is the dual basis of $(w_1, \ldots, w_{\dim_F W})$). As a result,

$$\dim_F U \leq \sum_{i=1}^{\dim_F W} \dim_F \psi_i \circ U.$$

So there is some i_0 such that

$$\dim_F U \leq \dim_F W \cdot \dim_F \psi_{i_0} \circ U.$$

So $\psi = \psi_{i_0}$ is the required dual basis element. $\square$

Next, consider the spaces $T_{X[A_i]}(A_i) = Hom_F(A_i^{\otimes n_i}, A_i)$ where $i = 1, \ldots, q$ (n_i is the number of elements of $X[A_i]$ in the partition of n). Furthermore, we may view the space $P_{X[A_i]}(A_i)$ as a subspace of $T_{X[A_i]}(A_i)$ via the embedding $g \to \eta(g)$ which is determined by

$$\eta(g)(a_1 \otimes \cdots \otimes a_{n_i}) = g(a_1, \ldots, a_{n_i}), (a_1, \ldots, a_{n_i}) \in A_i^{\otimes n_i}.$$

We apply the above lemma in the following setup: $V = A_i^{\otimes n_i}$, $W = A_i$ and $U = P_{X[A_i]}(A_i) \subseteq T_{X[A_i]}(A_i)$. We use the basis of elementary matrices as a basis of W.

Now, for $i = 1, \ldots, q$, we denote by $\psi_{\tilde{j}_i, j_i}$ the element in the dual basis of $W = A_i$ as given by Lemma 3.23. Note that $\psi_{\tilde{j}_i, j_i} : A_i \to F$ is the map assigning to each matrix of A_i its $e_{\tilde{j}_i, j_i}$ coefficient. For the given $\mathbf{j} = (j_1, \ldots, j_q)$ and $\tilde{\mathbf{j}} = (\tilde{j}_1, \ldots, \tilde{j}_q)$ we simplify our notation and denote the space $P^{\tilde{\mathbf{j}}, \mathbf{j}}_{X_n;BYE}(A)$ by $\bar{P}_{X_n;BYE}(A)$.

Theorem 3.24. *The mapping*

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \cdots \otimes P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A)$$

given by

$$\phi_{\mathbf{p}}(g_1 \otimes \cdots \otimes g_q) = \overline{f_1(g_1, \ldots, g_q)},$$

is well defined.

Moreover, if we denote by $M(\mathbf{p})$ the image of $\phi_{\mathbf{p}}$, then

$$\dim M(\mathbf{p}) \geq \frac{1}{d_1^2 \cdots d_q^2} \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q),$$

where $n_1 = |X[A_1]|, \ldots, n_q = |X[A_q]|$.

Proof. It is convenient to introduce the following auxiliary spaces which we denote by

$$\bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A).$$

As for the construction of $\bar{P}_{X_n;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ above, the space $\bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A)$, $i = 1, \ldots, q$, is obtained from the space

$$P_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A)$$

by performing evaluations 1 and 2 of Definition 3.21.

Note that

$$\overline{f_1|_{z_i \to g_i}} \in \bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A).$$

Furthermore the map $\phi_i : P_{X[A_i]}(A_i) \to \bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A)$ given by $\phi_i(g_i) = \overline{f_1|_{z_i \to g_i}}$ is well defined, since in any nonzero evaluation of $\overline{f_1}$ all variables of $X[A_i]$ must be evaluated by elements of A_i (see Lemma 3.22), so for $g_i \in Id(A_i)$ there is no nonzero evaluation of $\overline{f_1|_{z_i \to g_i}}$. In other words,

$$g_i \in id(A_i) \Longrightarrow \phi_i(g_i) = 0.$$

The same argument shows that the map

$$\phi'_{\mathbf{p}} : P_{X[A_1]}(A_1) \times \cdots \times P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A)$$

given by

$$\phi'_{\mathbf{p}}(g_1, \ldots, g_q) = \overline{f_1(g_1, \ldots, g_q)}$$

is well defined. Since $\phi'_{\mathbf{p}}$ is multilinear, it induces the map

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \cdots \otimes P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A).$$

Suppose $\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,1)}, \ldots, \psi_{\tilde{j}_i,j_i} \circ g_{(A_i,t_i)} \in \left(A_i^{\otimes n_i}\right)^*$ is a basis of $\psi_{\tilde{j}_i,j_i} \circ P_{X[A_i]}(A_i)$, $i = 1, \ldots, q$.

Applying the lemma it is clear now that in order to complete the proof it is enough to prove that $\phi_{\mathbf{p}}$ is injective when restricted to the subspace T spanned by $g_{(A_1,\alpha_1)} \otimes \cdots \otimes g_{(A_q,\alpha_q)}$, where $\alpha_1 = 1, \ldots, t_1; \ldots; \alpha_q = 1, \ldots, t_q$.

Suppose there are scalars $c_{\alpha_1,\ldots,\alpha_q} \in F$ such that

$$\begin{aligned} 0 &= \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \phi_{\mathbf{p}}(g_{(A_1,\alpha_1)} \otimes \cdots \otimes g_{(A_q,\alpha_q)}) \\ &= \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \overline{f_1\left(g_{(A_1,\alpha_1)}, \ldots, g_{(A_q,\alpha_q)}\right)}. \end{aligned}$$

For $i = 1, \ldots, q$, let $\mathbf{a}_1^{(A_i)}, \ldots, \mathbf{a}_{t_i}^{(A_i)} \in A_i^{\otimes n_i}$ be a dual basis of $\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,1)}, \ldots, \psi_{\tilde{j}_i,j_i} \circ g_{(A_i,t_i)} \in \left(A_i^{\otimes n_i}\right)^*$. Write $\mathbf{a}_1^{(A_1)} = \sum_l a_{1,l} \otimes \cdots \otimes a_{n_1,l}$. Recall the action of $g_{(A_1,\alpha_1)}$ on $\mathbf{a}_1^{(A_i)}$ is determined linearly via the action on $a_{1,l} \otimes \cdots \otimes a_{n_1,l}$ and the latter is determined via the substitutions $x_1 \to a_1(l), \ldots, x_{n_1} \to a_{n_1}(l)$, where (without loss of generality) $X[A_1] = \{x_1, \ldots, x_{n_1}\}$. We remind the reader that $\overline{f_1\left(g_{(A_1,\alpha_1)}, \ldots, g_{(A_q,\alpha_q)}\right)} \in \bar{P}_{X_n;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ and $\tilde{\mathbf{j}}$, $\mathbf{j}$ were chosen in Lemma 3.23. Hence we have,

$$\begin{aligned} 0 &= \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \overline{f_1\left(g_{(A_1,\alpha_1)}(\mathbf{a}_1^{(A_1)}), g_{(A_2,\alpha_2)}, \ldots, g_{(A_q,\alpha_q)}\right)} \\ &= \sum_{\alpha_1=1,\alpha_2,\ldots,\alpha_q} c_{\alpha_1=1,\alpha_2,\ldots,\alpha_q} \overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1), g_{(A_2,\alpha_2)}, \ldots, g_{(A_q,\alpha_q)}\right)}. \end{aligned}$$

By considering $\mathbf{a}_1^{(A_2)}, \ldots, \mathbf{a}_1^{(A_q)}$ and applying the same argument on the last expression we conclude that

$$c_{1,\ldots,1} \cdot \overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1), \ldots, e_{\tilde{j}_q,j_q}(A_q)\right)} = 0.$$

By Lemma 3.22, it follows that $\overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1), \ldots, e_{\tilde{j}_q,j_q}(A_q)\right)} \neq 0$, hence $c_{1,\ldots,1} = 0$.

It is clear that the same argument will work for every $\alpha_1, \ldots, \alpha_q$, thus every $c_{\alpha_1,\ldots,\alpha_q} = 0$. $\square$

Next we study the connection between the different $M(\mathbf{p})$.

Lemma 3.25. *Fix some* $\mathbf{p}_0 = (X[A_1], \ldots, X[A_q])$ *and denote* $\overrightarrow{n} = (n_1, \ldots, n_q)$ *where* n_i *is the number of elements in* $X[A_i]$*. Let* $\mathcal{T} = \{e = \tau_1, \tau_2, \ldots, \tau_l\}$ *be a transversal of* $S_{\mathbf{p}_0}$ *in* S_n*. Then, the sum of vector spaces*

$$M_{\mathbf{tot}}(\overrightarrow{n}) := M(\tau_1 \mathbf{p}_0) + \cdots + M(\tau_l \mathbf{p}_0),$$

is direct.

Note that $M_{\mathbf{tot}}(\overrightarrow{n}) = \overline{FS_{\mathbf{p}_0} \cdot f_1(x_1, \ldots, x_{n_1}, \ldots, x_{n_1+\cdots+n_{q-1}+1}, \ldots, x_{n_q})}$.

Proof. Suppose

$$0 = \sum_{k=1}^{l} \alpha_k h_k,$$

where $0 \neq h_k \in M(\tau_k \mathbf{p}_0)$ and $\alpha_k \in F$. Here, $l = \frac{n!}{n_1! \cdots n_q!} = \binom{n}{n_1, \ldots, n_q}$.

Since

$$h_1 = \overline{\sum_{\sigma \in S_{p_0}} \beta_\sigma f_1(x_{\sigma(1)} \cdots x_{\sigma(n_1)}, \ldots, x_{\sigma(n_1+\cdots+n_{q-1}+1)} \cdots x_{\sigma(n)})} \neq 0$$

we obtain by Lemma 3.22 a nonzero evaluation $\xi : F\{X_n, BYE\} \to A$ which sends the variables of $X[A_i]$ to elements of A_i (for $i = 1, \ldots, q$). We claim this evaluation sends each h_k ($k \neq 1$) to zero. Indeed, at least one of the sets $\tau_k X[A_1], \ldots, \tau_k X[A_q]$ must have an element $x \in \tau_{i_0} X[A_{i_0}]$ which is sent to some A_i, $i \neq i_0$ and hence

$$h_k = \overline{\sum_{\sigma \in S_{p_0}} \beta_{\tau_k \sigma} f_1(x_{\tau_k \sigma(1)} \cdots x_{\tau_k \sigma(n_1)}, \ldots, x_{\tau_k \sigma(n_1+\cdots+n_{q-1}+1)} \cdots x_{\tau_k \sigma(n)})}$$

is sent to zero. It follows that $0 = \alpha_1 \cdot \xi(h_1)$ and hence $\alpha_1 = 0$.

Repeating this argument for $h_2, \ldots, h_l$ yields $\alpha_2 = \alpha_3 = \cdots = \alpha_l = 0$. □

Write

$$M_{\mathbf{tot}}(n) := \sum_{n_1+\cdots+n_q=n} M_{\mathbf{tot}}\left(\overrightarrow{n} = (n_1, \ldots, n_q)\right).$$

Note that in view of our notation above for $M_{\mathbf{tot}}(\overrightarrow{n})$ we have

$$M_{\mathbf{tot}}(n) = \overline{FS_n \cdot f_1(x_1, \ldots, x_{n_1}, \ldots, x_{n_1+\cdots+n_{q-1}+1}, \ldots, x_{n_q})}.$$

Using a similar argument as in the previous lemma we obtain the following (proof is omitted).

Lemma 3.26. *For every n, the above sum in $M_{\mathbf{tot}}(n)$ is direct.*

Consider now the group $S_{X_n \cup \{w_1, \ldots, w_s\}}$ (see the beginning of the section for the definition of $w_1, \ldots, w_s$) and let $\{e = \sigma_1, \sigma_2, \ldots, \sigma_u\}$ be a transversal of S_{X_n} inside the aforementioned group. Note that $u = \frac{(n+s)!}{n!} = \Theta(n^s)$.

Lemma 3.27. *The sum*

$$M_{\mathbf{total}}(n) = \sum_{k=1}^{u} \sigma_k M_{\mathbf{tot}}(n)$$

is direct, where

$$\sigma_k M_{\mathbf{tot}}(n) = \overline{\sigma_k F S_n \cdot f_1(x_1, \ldots, x_{n_1}, \ldots, x_{n_1+\cdots+n_{q-1}+1}, \ldots, x_{n_q}; w_1, \ldots, w_s)}.$$

Proof. The idea used in the proof of Lemma 3.25 works also here. Suppose

$$0 = \sum_{k=1}^{u} \alpha_k h_k,$$

where $0 \neq h_k \in \sigma_k M_{\mathbf{tot}}(n)$ and $\alpha_k \in F$. Since $h_1 \neq 0$, we obtain by Lemma 3.22, a nonzero evaluation $\xi : F\{X_n, BYE\} \to A$ which sends the variables in X_n to elements of A_{ss} and $w_1, \ldots, w_q$ to elements of J. This evaluation sends each h_k $(k \neq 1)$ to zero. Indeed, $\sigma_k B_1, \ldots, \sigma_k B_s$ are alternating sets of cardinality $d+1$ in h_k. However, there is some i_0 for which $\sigma_k B_{i_0}$ does not contain any of the variables $w_1, \ldots, w_s$. Hence $\xi(\sigma_k B_{i_0}) \subseteq A_{ss}$. As a result, $\xi(h_k) = 0$. Thus, $0 = \alpha_1 \xi(h_1)$ and so $\alpha_1 = 0$.

Repeating this argument to $h_2, \ldots, h_l$ yields $\alpha_2 = \cdots = \alpha_l = 0$. $\square$

Now we show that asymptotically $\dim_F M_{\mathbf{tot}}(n)$ has the desired lower bound. Recall that $c_{n_i}(A_i) \sim K_i n^{\frac{1-d_i^2}{2}} d_i^{2n}$ for some constant $K_i \in \mathbb{R}$ [13]. As a result of Lemma 3.25, Lemma 3.26 and the above lemma we get (the inequality $\lesssim$ is asymptotically)

$$\sum_{n_1+\ldots+n_q=n} \frac{K_1 \cdots K_q}{d_1^2 \cdots d_q^2} \cdot \binom{n}{n_1, \ldots, n_q} \cdot n_1^{\frac{1-d_1^2}{2}} \cdots n_q^{\frac{1-d_q^2}{2}} \cdot d_1^{2n_1} \cdots d_q^{2n_q} \cdot n^s$$
$$\lesssim \dim_F M_{\mathbf{total}}(n).$$

Finally we apply Theorem 3.17 and obtain

$$\dim_F M_{\mathbf{total}}(n) \geq C n^{\frac{q-d}{2}+s} d^n$$

(here C is some nonnegative constant).

Corollary 3.28 *(Lower bound). Let $A = A_1 \times \cdots \times A_q \oplus J(A)$ be a basic algebra of Kemer index $\kappa_A = (d, s)$. Then*

$$c_n(A) \geq C n^{\frac{q-d}{2}+s} d^n,$$

for some constant $C \in \mathbb{R}$.

Proof. It is enough to show that

$$\dim_F c_{n+\gamma}(A) \geq \dim_F M_{\mathbf{total}}(n),$$

for some γ independent of n. Indeed,

$$M_{\mathbf{total}}(n) \subseteq \overline{P_{X_n;BYE}(A)}$$

and $\overline{P_{X_n;BYE}(A)}$ is a projection of $P_{X_n;BYE}(A)$. Hence, the statement stands for $\gamma = |BYE|$. $\square$

So combined with Theorem 3.16 we finally get a positive answer on Giambruno's conjecture.

Theorem 3.29. *Let $A = A_1 \times \cdots \times A_q \oplus J(A)$ be a basic algebra of Kemer index $\kappa_A = (d, s)$. Then*

$$c_n(A) = \Theta(n^{\frac{q-d}{2}+s} d^n).$$

In the special case where the algebra A has a unit we have

$$c_n(A) \simeq C n^{\frac{q-d}{2}+s} d^n,$$

for some constant $0 < C \in \mathbb{R}$.

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