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Local exponential stabilization of nonlinear infinite-dimensional systems

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Abstract— **Local exponential stabilization of nonlinear distributed parameter systems (DPS) by linear state feedbacks is adressed. The results rely on an adapted concept of Fréchet differentiability which is in general easier to deal with. As main contribution, it is first shown how to link the Fréchet differentiability of the nonlinear semigroup generated by the operator dynamics with the Fréchet differentiability of the closed-loop semigroup obtained by injecting a linear state feedback into the dynamics. As a second result, an appropriately stabilizing state feedback for the linearized system around any equilibrium is proved to be locally stabilizing for the nonlinear system, under some boundedness assumption on the control operator. A class of controlled systems satisfying the required assumptions is then identified. The theoretical results are illustrated for the state regulation of a diffusion equation perturbed by a nonlinear term.**

I. INTRODUCTION

Stabilizing an equilibrium of a nonlinear distributed parameter, i.e. infinite-dimensional, system as well as deducing exponential stability of such an equilibrium can be challenging. Some of the existing theories, see e.g. [9], [1], rely on the Fréchet differentiability of the nonlinear semigroup generated by the nonlinear operator dynamics. Fréchet differentiability allows here to link the stability properties of a linear approximation of the nominal system with the stability properties of the latter, locally around an equilibrium. That property can be viewed as an extension of the Lyapunov Indirect's Theorem for finite-dimensional systems, see e.g. [6, Theorem 3.19]. However, checking Fréchet differentiability for nonlinear operators defined on an infinite-dimensional space is difficult or even impossible if these are unbounded. In many cases, the general theory cannot be applied and a case-by-case study has to be performed by working directly on the semigroup instead of its generator. The approach that is proposed here is based on an adapted concept of Fréchet differentiability which takes different spaces and norms into account. This is called the $(Y,X)-$Fréchet differentiability, where X is the (Hilbert) state space and Y is an auxiliary space chosen to handle more easily norm-inequalities when working in infinite dimension (typically L^∞ or Sobolev spaces ($H^p, p \in \mathbb{N}_0$), which are multiplicative algebras). In this new framework sufficient conditions are stated in [7] in order to establish local exponentially stability of an equilibrium of a nonlinear DPS based on an appropriate linearization.

The main contribution here is the extension of the results in [7] to the local stabilization of an equilibrium of a nonlinear DPS by linear feedbacks satisfying particular boundedness conditions. More precisely it is shown that, if the linear control operator is bounded from the input space into the auxiliary space Y and the linear feedback gain operator is bounded from the space X into the input space, then the feedback operator stabilizes locally and exponentially the nonlinear system around the equilibrium, provided that it also stabilizes exponentially the corresponding linearized system on X and Y. A class of LQ-optimal controlled system is identified to fulfill the required assumptions imposed to get local exponential stability of the nonlinear system. The theory is applied to an unstable nonlinear diffusion equation with Neumann boundary conditions.

The paper is organized as follows: the general results for deducing local stability of equilibria of nonlinear DPS are presented in Section II together with the system class, the definitions and the assumptions we are considering. Section III is dedicated to the extension of the results to the stabilization of equilibria while a particular class of systems is identified in Section IV. The general results are applied to an illustrative example that fits the required assumption in Section V. Some conclusions are given in Section VI.

II. BACKGROUND, DEFINITIONS AND PROBLEM STATEMENT

Let us consider a (infinite-dimensional) Banach space X. The systems that are considered here are governed by the following abstract ODE:

$$\dot{x}(t) = Ax(t) + N(x(t)),\ x(0) = x_0, \tag{1}$$

where $A : D(A) \subset X \to X$ and $N : D(N) \subset X \to X$ are linear and nonlinear operators, respectively. Let $x^e \in D(A) \cap D(N)$ be an equilibrium of (1). Obviously, it satisfies the equation $Ax^e + N(x^e) = 0$. It is assumed that the (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ for which $\|T(t)\|_X \leq Me^{\omega t}, M \geq 1, \omega \in \mathbb{R}$, whereas the nonlinear operator N is Lipschitz continuous in the sense that there exists a positive constant l_N such that $\|N(x) - N(y)\| \leq l_N \|x - y\|$, for any $x, y \in D(N)$. Moreover, the closed convex subset $D(N)$ is assumed to be $T(t)-$invariant, i.e. $T(t)D(N) \subset D(N)$, for all $t \geq 0$. In addition, for all $x \in D(N), \lim_{h\to 0} \frac{1}{h} d(x + hN(x); D(N)) = 0$, where the distance function d is defined as $d(x; D(N)) := \inf\{\|x - y\|, y \in D(N)\}$. This setting entails that (1) possesses a unique mild solution on $[0, +\infty[$, for all $x_0 \in D(N)$, see e.g. [10, Theorem 5.1]. By a mild solution

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we mean a solution that satisfies the integral form of (1), i.e., a solution $x(t)$ such that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)N(x(s))ds$$

holds. By defining $\tilde{S}(t)x_0 = x(t)$ for all $t \geq 0$, $(\tilde{S}(t))_{t\geq 0}$ is a nonlinear semigroup on $D(N)$ whose infinitesimal generator is $A+N$. The condition $\lim_{h\to 0} \frac{1}{h} d(x+hN(x); D(N)) = 0$ implies that $D(N)$ is $\tilde{S}(t)-$invariant, see [10, Theorem 5.1]. Let us define $\xi(t) = x(t) - x^e$ and let us write (1) in the variable ξ. It follows that

$$\dot{\xi}(t) = A\xi(t) + N(\xi(t)+x^e) - N(x^e), \xi(0) = x_0 - x^e =: \xi_0. \quad (2)$$

Obviously, 0 is an equilibrium of (2). As a consequence of the fact that $(\tilde{S}(t))_{t\geq 0}$ is the nonlinear semigroup generated by $A+N$, the operator $A+N(\cdot + x^e) - N(x^e)$ is the infinitesimal generator of a nonlinear semigroup $(S(t))_{t\geq 0}$ on $D^e := D(N) - x^e$.

Let us consider the following assumption.

Assumption 2.1: The nonlinear operator N is Gâteaux differentiable at x^e, that is, there exists a linear operator $dN(x^e): X \to X$ such that $\lim_{\varepsilon \to 0} \frac{N(x^e+\varepsilon\xi)-N(x^e)}{\varepsilon} = dN(x^e)\xi$, where x^e and $x^e + \varepsilon\xi \in D(N)$.

Hence, since $D(N)$ is a convex set, any convex combination of x^e and $x^e + \varepsilon\xi$ lies in $D(N)$, that is, for any $a \in [0,1], x^e + \varepsilon(1-a)\xi \in D(N)$. Thus if $x^e + \varepsilon_0\xi \in D(N)$ for some nonnegative ε_0 then $x^e + \varepsilon\xi \in D(N)$ for all nonnegative $\varepsilon < \varepsilon_0$. This assumption allows us to linearize (2) around 0. This yields the linear system

$$\dot{\overline{\xi}}(t) = A\overline{\xi}(t) + dN(x^e)\overline{\xi}(t), \overline{\xi}(0) = \xi_0. \quad (3)$$

By assuming that the linear operator $dN(x^e)$ is bounded on X, the operator $A+dN(x^e)$ is the infinitesimal generator of a C_0-semigroup $(\overline{T}(t))_{t\geq 0}$ of bounded linear operators on X, see e.g. [5, Bounded perturbation theorem].

Let us introduce the following definition, which is an adaptation of the classical Fréchet differentiation with extended and mixed spaces, see [7, Definition 2.1].

Definition 2.1: [1] Let us consider the nonlinear operator $N: D(A)\cap D(N) \subset X \to X$. Let $(Y, \|\cdot\|_Y)$ be an infinite-dimensional (possibly Banach) space such that $D(A)\cap D^e \subset Y \subseteq X$ and $\|h\|_X \leq \sigma \|h\|_Y$ for all $h \in D(A)\cap D^e$ and some $\sigma > 0$. The operator N is called $(Y,X)-$Fréchet differentiable at x^e if there exists a bounded linear operator $dN(x^e): X \to X$ such that for all $h \in D(A)\cap D^e, N(x^e+h) - N(x^e) = dN(x^e)h + R(x^e,h)$ where $\lim_{\|h\|_Y \to 0} \frac{\|R(x^e,h)\|_X}{\|h\|_X} = 0$, i.e.,

$$\lim_{\|h\|_Y \to 0} \frac{\|N(x^e+h) - N(x^e) - dN(x^e)h\|_X}{\|h\|_X} = 0.$$

Note that convergence in Y implies convergence in X since $\|h\|_X \leq \sigma\|h\|_Y$ for all $h \in D(A)\cap D^e$.

In [7, Sections 2 and 3] it is shown that (Y,X)-Fréchet differentiability and $Y-$Fréchet differentiability of the nonlinear semigroup[2] $(S(t))_{t\geq 0}$ at 0 is sufficient to conclude on the (exponential) stability or instability of the equilibria of that semigroup, under some conditions on the stability of the linearized semigroup, $(\overline{T}(t))_{t\geq 0}$. We recall hereafter the needed assumptions, see [7, Assumptions 2.3, 2.4 and 3.1]. Note also that the Fréchet differentiability of N at x^e is equivalent to the Fréchet differentiability of the operator $N(\cdot + x^e) - N(x^e)$ at 0.

Assumption 2.2: Let us consider $\xi(t)$, the mild solution to (2) for the initial condition ξ_0, where $0 \leq t \leq t_0$. Let us consider a space $(Y, \|\cdot\|_Y)$ that satisfies $D(A)\cap D^e \subset Y \subseteq X$. It is assumed that the nonlinear operator N is $(Y,X)-$Fréchet differentiable at x^e and that ξ is continuously dependent of the initial condition ξ_0 on X and on Y at zero in the sense that the inequalities $\|\xi(t)\|_X \leq \gamma_t \|\xi_0\|_X$ and $\|\xi(t)\|_Y \leq \tilde{\gamma}_t \|\xi_0\|_Y$ hold for some positive $\gamma_t, \tilde{\gamma}_t$ that may depend on time.

Assumption 2.3: The nonlinear abstract Cauchy problem (2) is well-posed on Y. Moreover, it is assumed that the Gâteaux derivative $dN(x^e)$ of N is bounded on Y, hence the linearized dynamics corresponding to (3) is well-posed on Y. The nonlinear C_0-semigroup $(S(t))_{t\geq 0}$ is also assumed to be $Y-$Fréchet differentiable at 0.

Assumption 2.4: For the case where $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on X, it is assumed that $(\overline{T}(t))_{t\geq 0}$ is also exponentially stable on Y, that is $\|\overline{T}(t)\xi_0\|_Y \leq \eta e^{-\theta t}\|\xi_0\|_Y, t \geq 0$, for all $\xi_0 \in Y$, for some $\eta, \theta > 0$.

In [7, Section 2.3], it is shown that $(Y,X)-$Fréchet differentiability of the nonlinear semigroup $(S(t))_{t\geq 0}$ at 0 holds provided that Assumption 2.2 is satisfied, where it is assumed that X is a Hilbert space. We shall present a different proof here where X does not necessarily need to be a Hilbert space. The following lemma is a direct consequence of Assumption 2.2, see [7, Lemma 2.3].

Lemma 2.1: Let us consider $\xi(t)$, the mild solution of the abstract differential equation (2), where $t \in [0,t_0]$ for some positive t_0 and where N is assumed to be in $L^p([0,t_0];X)$ for some $p \geq 1$. Then, under Assumptions 2.1 and 2.2,

$$\lim_{\|\xi_0\|_Y \to 0} \frac{\|N(\xi+x^e) - N(x^e) - dN(x^e)\xi\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X} = 0. \quad (4)$$

We prove in the following Lemma that Assumption 2.2 implies $(Y,X)-$Fréchet differentiability of the nonlinear semigroup $(S(t))_{t\geq 0}$ at 0 with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative, also when X is not necessarily a Hilbert space.

Lemma 2.2: Let us consider a space $(Y, \|\cdot\|_Y)$ satisfying $D(A)\cap D^e \subset Y \subseteq X$ where X is an infinite-dimensional (possibly Banach) space. Under Assumptions 2.1 and 2.2, the nonlinear C_0-semigroup $(S(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0 and its Fréchet derivative is given by the linear C_0-semigroup $(\overline{T}(t))_{t\geq 0}$ whose infinitesimal generator is $A+dN(x^e)$, that corresponds to the Gâteaux derivative of $A+N(\cdot+x^e)-N(x^e)$ at 0.

Proof: Take $\xi_0 \in D(A)\cap D^e$ and $t \in [0,t_0]$. The mild solutions associated to (2) and (3) are given by

$$S(t)\xi_0 = T(t)\xi_0 + \int_0^t T(t-s)[N(\xi(s)+x^e) - N(x^e)]ds$$

[1] Note that when X is Y, we refer to the $Y-$Fréchet differentiability.

[2] Note that, by Fréchet differentiability of the semigroup $(S(t))_{t\geq 0}$, we mean that $S(t)$ is Fréchet differentiable for any $t \geq 0$.

and

$$\overline{T}(t)\xi_0 = T(t)\xi_0 + \int_0^t T(t-s)dN(x^e)\overline{\xi}(s)ds,$$

respectively. Obviously $\|S(t)\xi_0 - \overline{T}(t)\xi_0\|_X =$

$$\begin{aligned}
&= \| \int_0^t T(t-s)[N(\xi(s)+x^e) - N(x^e) - dN(x^e)\overline{\xi}(s)]ds\|_X \\
&= \| \int_0^t T(t-s)[N(\xi(s)+x^e) - N(x^e) - dN(x^e)\xi(s) \\
&\qquad + dN(x^e)\xi(s) - dN(x^e)\overline{\xi}(s)]ds\|_X \\
&\leq M\int_0^t e^{\omega(t-s)}\|N(\xi(s)+x^e) - N(x^e) - dN(x^e)\xi(s)\|_X ds \\
&\qquad + M\int_0^t e^{\omega(t-s)}\|dN(x^e)(\xi(s) - \overline{\xi}(s))\|_X ds.
\end{aligned}$$

By using the boundedness of the Gâteaux derivative of N at x^e, one gets that $\|e^{-\omega t}(S(t)\xi_0 - \overline{T}(t)\xi_0)\|_X \leq$

$$\begin{aligned}
&M\int_0^t e^{-\omega s}\|N(\xi(s)+x^e) - N(x^e) - dN(x^e)\xi(s)\|_X ds \\
&\qquad + M\|dN(x^e)\|_{\mathscr{L}(X)}\int_0^t \|e^{-\omega s}(S(s)\xi_0 - \overline{T}(s)\xi_0)\|_X ds.
\end{aligned}$$

It follows by Gronwall's lemma that

$$\|S(t)\xi_0 - \overline{T}(t)\xi_0\|_X \leq Me^{(\omega+\eta)t}k_0\int_0^t \|R(\xi(s),x^e)\|_X ds \quad (5)$$

where $R(\xi,x^e)$ stands for $N(\xi+x^e) - N(x^e) - dN(x^e)\xi, \eta := M\|dN(x^e)\|_{\mathscr{L}(X)}$ and $k_0 = \max\{1, e^{-\omega t_0}\}$. Consequently, we can derive the following estimate:

$$\|S(t)\xi_0 - \overline{T}(t)\xi_0\|_X \leq Me^{(|\omega|+\eta)t_0}k_0 t_0\|R(\xi,x^e)\|_{L^\infty([0,t_0];X)}.$$

Since (4) holds as a consequence of Assumption 2.2, the nonlinear semigroup $(S(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0 with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative. ■

We end this section by presenting the main result of [7], which consists in deducing exponential (in)stability of the nonlinear semigroup $(S(t))_{t\geq 0}$ on the basis of the linearized semigroup $(\overline{T}(t))_{t\geq 0}$, see [7, Theorem 3.1]. This result is based on the two following definitions, see [7, Section 2].

Definition 2.2: The equilibrium 0 of (2) is said to be $(Y,X)-$locally exponentially stable if there exist $\delta,\alpha,\beta > 0$ such that, for all $\xi_0 \in D(A)\cap D^e$ with $\|\xi_0\|_Y < \delta$, there holds $\|\xi(t)\|_X \leq \alpha e^{-\beta t}\|\xi_0\|_X$, for all $t \geq 0$.

Definition 2.3: The equilibrium 0 of (2) is said to be $(Y,X)-$locally stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\xi_0 \in D(A)\cap D^e, \|\xi_0\|_Y < \delta$ implies $\|\xi(t)\|_X < \varepsilon$, for all $t \geq 0$. The equilibrium 0 is $(Y,X)-$(locally) unstable if it is not stable.

Theorem 2.1: Let us consider Assumptions 2.1 to 2.4. If 0 is a globally exponentially stable equilibrium of the linearized model (3), then it is a $(Y,X)-$locally exponentially stable equilibrium of (2). Conversely, if 0 is a $(Y,X)-$unstable equilibrium of (3), it is $(Y,X)-$locally unstable for the nonlinear system (2).

The new concept of Fréchet differentiability allows more easily checkable Fréchet differentiability conditions, yielding the fact that local exponential stability of the equilibrium of (1) holds in a weaker sense, see Theorem 2.1. Based on this new concept, our approach to deduce exponential stability of the equilibrium of (1) can be summarized as in Figure 1. Let $(S(t))_{t\geq 0}$ be the nonlinear semigroup generated by the operator $A + N(\cdot + x^e) - N(x^e)$ on X. The standard concept of Fréchet differentiability is needed for $(S(t))_{t\geq 0}$ on Y, with $(\overline{T}(t))_{t\geq 0}$ as Fréchet derivative, the linear semigroup generated by the Gâteaux derivative $A + dN(x^e)$ of $A + N(\cdot + x^e) - N(x^e)$. After showing that $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on Y in the case where it is exponentially stable on X, the new concept of $(Y,X)-$Fréchet differentiability pops up to make the connection between Y and X, in order to deduce local exponential stability of the equilibrium of (2) on X, see Theorem 2.1. "Local" means here that the $Y-$norm of the initial condition has to be small instead of its $X-$norm. A crucial assumption in our framework is that the nonlinear semigroup $(S(t))_{t\geq 0}$ needs to be continuously dependent on the initial condition at 0, both in the X and Y norms.

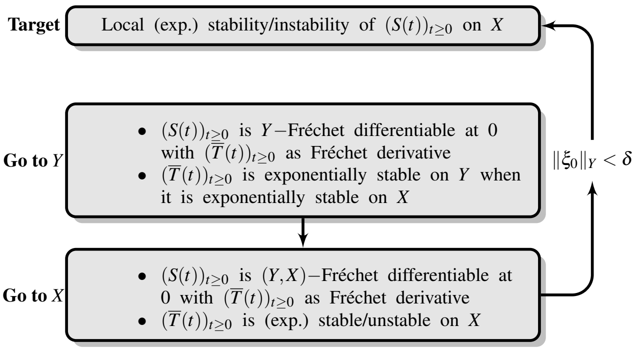

Fig. 1. Illustration of the new theoretical framework.

In the context of this new framework, let us add some control term $u(t)$ to (2) as follows

$$\dot{\xi}_{cl}(t) = A\xi_{cl}(t) + N(\xi_{cl}(t)+x^e) - N(x^e) + Bu(t), \xi_{cl}(0) = \xi_0, \quad (6)$$

where $B : \mathscr{U} \to X$ is a bounded linear operator and where $\mathscr{U}$ is called the control space.

In what follows we shall consider control inputs of the feedback form $u(t) = K\xi_{cl}(t)$ for some operator[3] $K \in \mathscr{L}(X,\mathscr{U})$. The main question will consist in determining a stabilizing state feedback K for the controlled linearized system[4]

$$\dot{\overline{\xi}}_{cl}(t) = A\overline{\xi}_{cl}(t) + dN(x^e)\overline{\xi}_{cl}(t) + BK\overline{\xi}_{cl}(t), \overline{\xi}_{cl}(0) = \xi_0, \quad (7)$$

which still (locally) stabilizes the nonlinear dynamics (6), around the equilibrium 0.

III. LOCAL EXPONENTIAL STABILIZATION OF EQUILIBRIA

In this section, we shall present results that allow to make the gap between the (exp.) stability of the controlled linearized system (7) and the stability of the system (6).

[3] $\mathscr{L}(X,U)$ is the space of linear and bounded operators from X to U.
[4] Which corresponds to the Gâteaux linearization of (6) around 0.

First let us prove that the controlled system (6) with $u(t) = K\xi_{cl}(t)$ is continuously dependent on the initial condition ξ_0 at 0 on X and on Y provided that some boundedness conditions on the operators B and K hold.

Proposition 3.1: Let us consider the nonlinear controlled system (6) where $u(t) = K\xi_{cl}(t)$ for some linear gain operator K. If the operators $B \in \mathscr{L}(\mathscr{U},Y)$ and $K \in \mathscr{L}(X,\mathscr{U})$, then the mild solution $\xi_{cl}(t)$ to (6) depends continuously on the initial condition ξ_0 at 0 on X and Y on any compact interval $[0,t_0]$, where $t_0 > 0$.

Proof: Let us take $\xi_0 \in D(A) \cap D^e$ and $t_0 > 0$. We shall prove the continuous dependence on the initial condition on X. Similar arguments can be used to prove this property on Y. By assumption, the feedback operator $K \in \mathscr{L}(X,\mathscr{U})$. Consequently, for any $\xi_{cl} \in Y, \|K\xi_{cl}\|_{\mathscr{U}} \leq \|K\|_{\mathscr{L}(X,\mathscr{U})}\|\xi_{cl}\|_X \leq \sigma\|K\|_{\mathscr{L}(X,\mathscr{U})}\|\xi_{cl}\|_Y$, which ensures that $K \in \mathscr{L}(Y,\mathscr{U})$ too. A similar argument on the operator B yields the following inequalities: $\|Bu\|_X \leq \sigma\|Bu\|_Y \leq \sigma\|B\|_{\mathscr{L}(\mathscr{U},Y)}\|u\|_{\mathscr{U}}$, where the assumption $B \in \mathscr{L}(\mathscr{U},Y)$ has been used. Hence, $B \in \mathscr{L}(\mathscr{U},X)$. Now consider the mild solution of (6). It is expressed as

$$\xi_{cl}(t) = T(t)\xi_0 + \int_0^t T(t-s)[N(\xi_{cl}(s)+x^e) - N(x^e)]ds + \int_0^t T(t-s)BK\xi_{cl}(s)ds.$$

Taking the $X-$norm of both sides yields $\|\xi_{cl}(t)\|_X \leq$

$$\begin{aligned}
&\|T(t)\xi_0\|_X + \int_0^t \|T(t-s)[N(\xi_{cl}(s)+x^e) - N(x^e)]\|_X ds \\
&\qquad + \int_0^t \|T(t-s)BK\xi_{cl}(s)\|_X ds \\
&\leq Me^{\omega t}\|\xi_0\|_X + \int_0^t Me^{\omega(t-s)}\|N(\xi_{cl}(s)+x^e) - N(x^e)\|_X ds \\
&\qquad + \int_0^t Me^{\omega(t-s)}\|BK\xi_{cl}(s)\|_X ds.
\end{aligned}$$

The boundedness of B and K together with the Lipschitz continuity of N imply that

$$\begin{aligned}
\|e^{-\omega t}\xi_{cl}(t)\|_X &\leq M\|\xi_0\|_X + Ml_N\int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds \\
&\quad + M\|B\|_{\mathscr{L}(\mathscr{U},X)}\|K\|_{\mathscr{L}(X,\mathscr{U})}\int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds \\
&= M\|\xi_0\|_X + \tilde{\eta}\int_0^t \|e^{-\omega s}\xi_{cl}(s)\|_X ds,
\end{aligned}$$

where $\tilde{\eta} := Ml_N + M\|B\|_{\mathscr{L}(\mathscr{U},X)}\|K\|_{\mathscr{L}(X,\mathscr{U})}$. Then Gronwall's inequality yields $\|\xi_{cl}(t)\|_X \leq Me^{(\omega+\tilde{\eta})t}\|\xi_0\|_X$, which proves continuous dependence on X. ■

We shall now prove that considering a nonlinear operator N such that $N(\cdot + x^e) - N(x^e)$ is $(Y,X)-$ and $Y-$Fréchet differentiable at 0 entails that the solution of the controlled system (6), $\xi_{cl}(t)$, is $(Y,X)-$ and $Y-$Fréchet differentiable at 0, with the linear semigroup $\overline{T}_{cl}(t)$ generated by the operator $A + dN(x^e) + BK$ as Fréchet derivative.

Proposition 3.2: Under the assumptions that $K \in \mathscr{L}(X,\mathscr{U}), B \in \mathscr{L}(\mathscr{U},Y)$ and that the nonlinear operator $N(\cdot + x^e) - N(x^e)$ is $(Y,X)-$ and $Y-$Fréchet differentiable at 0, the nonlinear semigroup generated by the operator $A + N(\cdot + x^e) - N(x^e) + BK$ is $(Y,X)-$ and $Y-$Fréchet differentiable at 0 with $\overline{T}_{cl}(t)$ the linear semigroup generated by $A + dN(x^e) + BK$ as Fréchet derivative.

Proof: We shall prove the proposition for the $(Y,X)-$Fréchet differentiability. Similar arguments lead to the conclusion for the $Y-$Fréchet differentiability. First note that the assumptions $K \in \mathscr{L}(X,\mathscr{U})$ and $B \in \mathscr{L}(\mathscr{U},Y)$ imply that the solution of (6) depends continuously on the initial condition at 0 on X and on Y (see Proposition 3.1). Hence Assumption 2.2 is satisfied. Consequently the relation

$$\lim_{\|\xi_0\|_Y \to 0} \frac{\|N(\xi_{cl}+x^e) - N(x^e) - dN(x^e)\xi_{cl}\|_{L^\infty([0,t_0];X)}}{\|\xi_0\|_X} = 0 \tag{8}$$

is satisfied according to Lemma 2.1. Then observe that the state trajectories of (6) and (7) are given by $\xi_{cl}(t) =$

$$T(t)\xi_0 + \int_0^t T(t-s)[N(\xi_{cl}(s)+x^e) - N(x^e) + BK\xi_{cl}(s)]ds$$

and

$$\overline{\xi}_{cl}(t) = T(t)\xi_0 + \int_0^t T(t-s)[dN(x^e)\overline{\xi}_{cl}(s) + BK\overline{\xi}_{cl}(s)]ds,$$

respectively. As a consequence, $\|e^{-\omega t}(\xi_{cl}(t) - \overline{\xi}_{cl}(t))\|_X \leq$

$$\begin{aligned}
&M\int_0^t e^{-\omega s}\|N(\xi_{cl}(s)+x^e) - N(x^e) - dN(x^e)\xi_{cl}(s)\|_X ds \\
&\qquad + M\int_0^t e^{-\omega s}\|dN(x^e)(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds \\
&\qquad + M\int_0^t e^{-\omega s}\|BK(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds.
\end{aligned}$$

Using the boundedness of the Gâteaux derivative of N at x^e and the fact that $BK \in \mathscr{L}(X)$, it follows that $\|e^{-\omega t}(\xi_{cl}(t) - \overline{\xi}_{cl}(t))\|_X \leq$

$$\begin{aligned}
&M\int_0^t e^{-\omega s}\|N(\xi_{cl}(s)+x^e) - N(x^e) - dN(x^e)\xi_{cl}(s)\|_X ds \\
&+ \lambda\int_0^t \|e^{-\omega s}(\xi_{cl}(s) - \overline{\xi}_{cl}(s))\|_X ds,
\end{aligned}$$

where $\lambda := M(\|dN(x^e)\|_{\mathscr{L}(X)} + \|BK\|_{\mathscr{L}(X)})$. Hence, by Gronwall's lemma,

$$\|\xi_{cl}(t) - \overline{\xi}_{cl}(t)\|_X \leq Me^{(\omega+\lambda)t}k_0\int_0^t \|R(\xi_{cl}(s),x^e)\|_X ds$$

where $R(\xi_{cl},x^e)$ stands for $N(\xi_{cl}+x^e) - N(x^e) - dN(x^e)\xi_{cl}$ and $k_0 := \max\{1, e^{-\omega t_0}\}$. Since (8) holds, the nonlinear semigroup[5] $(S_{cl}(t))_{t\geq 0}$ is $(Y,X)-$Fréchet differentiable at 0 with $(\overline{T}_{cl}(t))_{t\geq 0}$ as Fréchet derivative. ■

This means that if the state feedback K, designed on the linearized dynamics (7) stabilizes exponentially the latter around 0, it also locally (exp.) stabilizes the nonlinear system (6) around the equilibrium 0 (in the sense of Definition 2.2), provided that the state feedback exponentially stabilizes the linearized dynamics around 0 on Y, see Theorem 2.1 and Assumption 2.4.

[5]We use the notation $\xi_{cl}(t) = S_{cl}(t)\xi_0$ and $\overline{\xi}_{cl}(t) = \overline{T}_{cl}(t)\xi_0$ for any $t \geq 0$ and $\xi_0 \in D(A) \cap D^e$.

IV. APPLICATION TO A CLASS OF LQ-OPTIMALLY CONTROLLED SYSTEMS

In this section, we consider a class of single-input controlled systems of the form (6) where X is supposed to be a Hilbert space, the operators B and C are given by $Bu = bu$ for all $u \in \mathbb{R}$, for some $b \in Y$ and $C \in \mathscr{L}(X,\mathbb{R}), C\cdot = \langle c, \cdot \rangle_X$, for some $c \in X$. Hence the linearized system (7) reads

$$\dot{\overline{\xi}}_{cl}(t) = A\overline{\xi}_{cl}(t) + dN(x^e)\overline{\xi}_{cl}(t) + bK\overline{\xi}_{cl}(t), \overline{\xi}_{cl}(0) = \xi_0, \quad (9)$$

It is assumed that the pairs $(A + dN(x^e), B)$ and $(C, A + dN(x^e))$ are exponentially stabilizable[6] and exponentially detectable[7], respectively. Moreover, it is assumed that the operator $A + dN(x^e)$ is a Riesz-spectral operator. Hence, the operator $A + dN(x^e)$ admits the spectral decomposition $(A + dN(x^e))\overline{\xi}_{cl} = \sum_{n=0}^{\infty} \lambda_n \langle \overline{\xi}_{cl}, \psi_n \rangle \phi_n$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of eigenvalues of the operator $A + dN(x^e)$, whose spectrum is discrete and consists only of $\{\lambda_n\}_{n\in\mathbb{N}}$. The Riesz basis $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are the corresponding eigenvectors and the eigenvectors of the adjoint of $A + dN(x^e)$. The ϕ_n's and the ψ_n's are biorthonormal, i.e. $\langle \phi_n, \psi_m \rangle = \delta_{nm}$, see e.g [4, Chapter 2].

On the eigenvalues of the operator $A + dN(x^e)$, we make the following additional assumption that there exists $0 < \kappa < \infty$ such that

$$\sup_{m\in\mathbb{N}} \{ \sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{1}{|\lambda_n - \lambda_m|^2} \} = \kappa. \quad (10)$$

This entails that the condition $\inf_{n,m\in\mathbb{N}, n\neq m} |\lambda_n - \lambda_m| = \mu, \mu > 0$, is satisfied. Indeed, assume by contradiction that this does not hold. Let us take any $k \in \mathbb{N}$ such that $\kappa \leq k^2$. Then there exists $\lambda_{n_k}, \lambda_{m_k}$ such that $|\lambda_{n_k} - \lambda_{m_k}| \leq 1/k$, which implies that the sum of the series in (10) exceeds k^2, leading to a contradiction.

For the sake of simplicity, we shall denote $\mathscr{A} := A + dN(x^e)$ in what follows.

Now let us consider the following optimal control problem for system (9). The goal here is to find a control law $u_o \in L^2([0,\infty);\mathbb{R})$ that minimizes the cost functional

$$J(\xi_0, u) = \int_0^{\infty} (|C\overline{\xi}_{cl}(t)|^2 + |u(t)|^2) dt, \quad (11)$$

whose solution is the state feedback $u_o(t) = K_o\overline{\xi}_{cl}(t)$, where $K_o := -B^{\star}Q$ with Q the unique positive self-adjoint solution of the following operator Riccati equation $\mathscr{A}^{\star}Q + Q\mathscr{A} + C^{\star}C - QBB^{\star}Q = 0$ on $D(\mathscr{A})$. For more details about the LQ-optimal control problem for infinite-dimensional systems, see for instance [3], [4] and references therein. Note that the notation * stands for the adjoint operator where we consider $B \in \mathscr{L}(\mathbb{R}, X), C \in \mathscr{L}(X,\mathbb{R})$ and $\mathscr{A} : D(\mathscr{A}) \subset X \to X$.

For the specific framework introduced in this section, the general form of the state feedback $K_o \in \mathscr{L}(X,\mathbb{R})$ is given by $K_o\overline{\xi}_{cl} := \langle k_o, \overline{\xi}_{cl} \rangle$, for any $\overline{\xi}_{cl} \in X$ and some $k_o \in X$.

[6]By definition this means that there exists an operator $K \in \mathscr{L}(X,\mathbb{R})$ such that the operator $A + dN(x^e) + BK$ is the generator of an exponentially stable C_0-semigroup, see [4, Definition 5.2.1].

[7]See [4, Definition 5.2.1].

In order to prove that, once the optimal control law is computed and then plugged in the nonlinear system (6), the corresponding nonlinear semigroup is $(Y,X)-$ and $Y-$Fréchet differentiable, we shall successively check that $K_0 \in \mathscr{L}(X,\mathbb{R})$ and that $B \in \mathscr{L}(\mathbb{R}, Y)$. Obviously, $K_0 \in \mathscr{L}(X,\mathbb{R})$. In addition observe that the linear operator $B \in \mathscr{L}(\mathbb{R}, Y)$ since the vector $b \in Y$. Hence, by considering a nonlinear operator N that satisfies the assumptions of Sections II and III, the nonlinear semigroup generated by the controlled dynamics (6) is $(Y,X)-$ and $Y-$Fréchet differentiable at 0 with the linear semigroup $(\overline{T}_{cl}(t))_{t\geq 0}$ generated by the dynamics of (9), i.e. the operator $\mathscr{A} + b\langle k_o, \cdot \rangle$, as Fréchet derivative.

The last issue consists of analyzing when the state feedback K_o also exponentially stabilizes the linear semigroup generated by $\mathscr{A} + b\langle k_o, \cdot \rangle$ by considering $Y-$norms. Sufficient conditions that solve this question are stated in the next proposition.

Proposition 4.1: Let us consider the linear feedback gain K_o, given by $\langle k_o, \cdot \rangle$, that is obtained as the solution of the optimal control problem (11). Let us denote $\{\lambda_n^o\}_{n\in\mathbb{N}}$ the set of eigenvalues of the operator $\mathscr{A} + b\langle k_o, \cdot \rangle$ with corresponding eigenvectors $\{\phi_n^o\}_{n\in\mathbb{N}}$. The eigenvectors of the adjoint operator of $\mathscr{A} + b\langle k_o, \cdot \rangle$ are denoted by $\{\psi_n^o\}_{n\in\mathbb{N}}$. Under the following assumption:

$$\sum_{n=0}^{\infty} \frac{\|\psi_n^o\|_X \|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|} < \infty, \quad (12)$$

the feedback gain K_o exponentially stabilizes the dynamics (9) around 0 on the space Y.

Proof: According to [12], [13], the closed-loop operator $\mathscr{A} + b\langle k_0, \cdot \rangle$ is a Riesz-spectral operator since so is $\mathscr{A}$ and since (10) holds true. Hence it has a discrete spectrum composed of eigenvalues $\{\lambda_n^o\}_{n\in\mathbb{N}}$ only. Its eigenvectors $\{\phi_n^o\}_{n\in\mathbb{N}}$ together with the eigenvectors of its adjoint, $\{\psi_n^o\}_{n\in\mathbb{N}}$, are biorthonormal Riesz basis of X, such that $\langle \phi_n^o, \psi_m^o \rangle_X = \delta_{nm}$.

Observe that $\sup_{n\in\mathbb{N}} \{\mathfrak{Re}(\lambda_n^o)\} < 0$ since the semigroup generated by $\mathscr{A} + b\langle k_0, \cdot \rangle$ is exponentially stable on X. Take $\overline{\xi}_0 \in D(\mathscr{A})$. It holds that $\|\overline{\xi}_{cl}(t)\|_Y =$

$$\|\sum_{n=0}^{\infty} e^{\lambda_n^o t} \langle \xi_0, \psi_n^o \rangle_X \phi_n^o \|_Y \leq \sigma \sum_{n=0}^{\infty} e^{\mathfrak{Re}(\lambda_n^o) t} \|\psi_n^o\|_X \|\phi_n^o\|_Y \|\xi_0\|_Y.$$

It is stated in [2] that, if

$$\int_0^{\infty} \|\overline{\xi}_{cl}(t)\|_Y^p dt < \infty \quad (13)$$

holds for some $p \in [1,\infty)$ and for any $\xi_0 \in D(\mathscr{A})$, then the state trajectory $\overline{\xi}_{cl}(t)$ is exponentially stable when evaluated with $Y-$norms. The condition (13) in this specific context with $p = 1$ follows from

$$\int_0^{\infty} \|\overline{\xi}_{cl}(t)\|_Y dt \leq \sigma \int_0^{\infty} \sum_{n=0}^{\infty} e^{\mathfrak{Re}(\lambda_n^o) t} \|\psi_n^o\|_X \|\phi_n^o\|_Y \|\xi_0\|_Y dt$$
$$= \sigma \sum_{n=0}^{\infty} \frac{\|\psi_n^o\|_X \|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|} \|\xi_0\|_Y$$

and Assumption (12). ■

V. EXAMPLE: A NONLINEAR DIFFUSION EQUATION

In this section, we study an illustrative example of the theory developed in the previous sections. Consider the following nonlinear controlled partial differential equation (PDE)

$$\begin{cases} \frac{\partial x}{\partial t}(t,z) = \frac{\partial^2 x}{\partial z^2}(t,z) + \sqrt{x^2(t,z)+1} + b(z)u(t), \\ \frac{\partial x}{\partial z}(t,0) = 0 = \frac{\partial x}{\partial z}(t,1), \end{cases} \tag{14}$$

where $t \geq 0$ and $z \in [0,1]$. This system can be interpreted as a (nonlinearly) perturbed heat equation with Neumann boundary conditions. The objective here is to force system (14), i.e. to design the control input $u(t)$ in order to reach exponentially fast a fixed reference profile $r(z)$ that may depend on the spatial variable z. Note that $b(z) = 1_{[0,1]}(z), z \in [0,1]$ whose value is 1 for $z \in [0,1]$ and 0 elsewhere.

Let us fix the reference profile that should be tracked as $r(z) = 1_{[0,1]}(z)$. Obviously, it satisfies the boundary conditions. Writing down the PDE (14) with x replaced by r implies that $1_{[0,1]}(z)\sqrt{2} + 1_{[0,1]}(z)u = 0$, which means that $(r(z) = 1_{[0,1]}(z), u = -\sqrt{2})$ consitutes an equilibrium pair of (14). Now we define the deviation variables $\xi(t,z) = x(t,z) - 1_{[0,1]}(z)$ and $\tilde{u}(t) = u(t) + \sqrt{2}$. Then, system (14) becomes

$$\begin{cases} \frac{\partial \xi}{\partial t}(t,z) = \frac{\partial^2 \xi}{\partial z^2}(t,z) + \sqrt{(\xi(t,z)+1)^2+1} - 1_{[0,1]}(z)\sqrt{2} \\ \qquad\qquad\qquad + b(z)\tilde{u}(t), \\ \frac{\partial \xi}{\partial z}(t,0) = 0 = \frac{\partial \xi}{\partial z}(t,1). \end{cases} \tag{15}$$

Here we consider the Hilbert space of square integrable functions $X = L^2(0,1)$ and the Banach space of continuous functions $Y = C(0,1)$. An abstract representation of (15) is given by

$$\dot{\xi}(t) = A\xi(t) + N(\xi(t)) + b\tilde{u}(t), \xi(0) = \xi_0, \tag{16}$$

where the linear operator $A : D(A) \subset X \to X$ is defined by $A\xi = \frac{d^2\xi}{dz^2}$ for $\xi \in D(A) = \{\xi \in H^2(0,1), \frac{d\xi}{dz}(0) = 0 = \frac{d\xi}{dz}(1)\}$ and the nonlinear operator $N : X \to X$ is expressed as $N(\xi) = \sqrt{(\xi+1)^2+1} - \sqrt{2}$ where $H^2(0,1)$ denotes the Sobolev space of $L^2(0,1)-$functions whose first and second order derivatives are also in $L^2(0,1)$. As the operator A with domain $D(A)$ is the generator of a C_0-semigroup of bounded linear operators and as the nonlinear operator $N : X \to X$ is uniformly Lipschitz continuous with $L = 1$ as least Lipschitz constant, the homogeneous abstract differential equation (16) possesses a unique mild solution on $[0,\infty)$, see [11, Chapter 6, Theorem 1.2]. Moreover, if $\xi_0 \in D(A)$ the mild solution is a classical solution. According to [8, Lecture 4], the linear operator A defined on the domain $\{\xi \in C^2(0,1), \frac{d\xi}{dz}(0) = 0 = \frac{d\xi}{dz}(1)\}$ is also the generator of a contraction C_0-semigroup since it is dissipative and $(\lambda I - A)$ is surjective for positive λ. Moreover, the nonlinear operator N is uniformly Lipschitz continuous viewed as an operator from Y to Y, which allows to conclude on the well-posedness of the nonlinear Cauchy problem both on X and on Y.

Note that the nonlinear operator N is well-defined as an operator from X into X or from Y into Y. Indeed, thanks to the uniformly Lipschitz continuity of N on both spaces and on the fact that f maps the origin into itself, one observes that $\|N(\xi)\|_X \leq \|\xi\|_X < \infty$ and $\|N(\xi)\|_Y \leq \|\xi\|_Y < \infty$.

Concerning the continuous dependence of the solution $\xi(t)$ on the initial condition at 0 on X and on Y, we refer to [10, Lemma 5.1]. Indeed, as the initial conditions are chosen in the domain of the linear operator we actually work we classical solutions, i.e. solutions that are differentiable in time. This implies continuity in time. Hence, continuous dependence is proved.

We shall now focus on the nonlinear operator N in the dynamics and its differentiability properties. Let us consider the following lemma that characterizes the Gâteaux differentiability of N.

Lemma 5.1: The nonlinear operator $N : X \to X$ is Gâteaux differentiable at 0. Its Gâteaux derivative is given by the bounded linear operator $dN(0) : X \to X$ defined as $dN(0)h = \frac{1}{\sqrt{2}}h, h \in X$.

It can also be shown that $dN(0)$ is the Gâteaux derivative of N on Y. Obviously it satisfies $dN(0) \in \mathscr{L}(X)$ and $dN(0) \in \mathscr{L}(Y)$. The operator norm of $dN(0)$ both on X and Y is $\frac{1}{\sqrt{2}}$.

Before starting with the proof of the $(Y,X)-$Fréchet differentiability of the nonlinear operator N, we shall motivate the use of such an adapted concept by proving that the nonlinear operator N introduced in (16) is not Fréchet differentiable on X in the classical sense.

Proposition 5.1: The nonlinear operator N defined in (16) is not Fréchet differentiable on the space X.

Proof: Let us consider the sequence $\{h_n\}_{n\in\mathbb{N}_0}$ of $X-$functions defined by $h_n(z) = \frac{1}{n}1_{[0,1-\frac{1}{n}]}(z) + 1_{[1-\frac{1}{n},1]}(z), z \in [0,1]$. Now we fix $n \in \mathbb{N}_0$. Obviously $\|h_n\|_X = \frac{n^2+n-1}{n^3}$, meaning that the $X-$norm of h_n decreases to 0 as n tends to ∞. Computing the quantity $\frac{\|\sqrt{(h_n+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h_n\|_X^2}{\|h_n\|_X^2}$ gives

$$\frac{n-1}{2(n^2+n-1)}\left(\frac{1}{\sqrt{4n^2+4n+2}+2n+1}\right)^2 + \frac{n^2(19-6\sqrt{10})}{2(n^2+n-1)},$$

which goes to $\frac{19-6\sqrt{10}}{2}$ when n tends to ∞. ■

In order to apply the results of Section III, consider the following proposition.

Proposition 5.2: The nonlinear operator N defined in (16) is $(Y,X)-$Fréchet differentiable by considering $X = L^2(0,1)$ and $Y = C(0,1)$ (equipped with the $L^\infty(0,1)-$norm).

Proof: Let us take $h \in D(A)$. We have that

$$\begin{aligned} &\|\sqrt{(h+1)^2+1} - \sqrt{2} - \frac{1}{\sqrt{2}}h\|_X \\ &= \|\frac{((h+1)^2-1)\sqrt{2} - h(\sqrt{(h+1)^2+1}+\sqrt{2})}{\sqrt{2}(\sqrt{(h+1)^2+1}+\sqrt{2})}\|_X \\ &\leq \|\sqrt{2}h^2 + \sqrt{2}h - h\sqrt{(h+1)^2+1}\|_X \\ &\leq \sqrt{2}\|h\|_Y\|h\|_X + \|h\|_X\|\frac{2-(h+1)^2-1}{\sqrt{2}+\sqrt{(h+1)^2+1}}\|_Y \\ &\leq \sqrt{2}\|h\|_Y\|h\|_X + \|h\|_X\|h\|_Y(2+\|h\|_Y) \end{aligned}$$

Consequently, the following estimate

$$\frac{\|\sqrt{(h+1)^2+1}-\sqrt{2}-\frac{1}{\sqrt{2}h}\|_X}{\|h\|_X} \leq \|h\|_Y(\sqrt{2}+2+\|h\|_Y)$$

holds true. Taking the limit for $\|h\|_Y$ going to 0 yields the conclusion. ■

Note that similar arguments lead to the $Y-$Fréchet differentiability of the nonlinear operator N at 0 (hint: use the fact that Y is a multiplicative algebra).

The Gâteaux linearization of system (15) around 0 reads as follows:

$$\begin{cases} \frac{\partial\overline{\xi}}{\partial t}(t,z) = \frac{\partial^2\overline{\xi}}{\partial z^2}(t,z) + \frac{1}{\sqrt{2}}\overline{\xi}(t,z) + 1_{[0,1]}(z)\tilde{u}(t), \\ \frac{\partial\overline{\xi}}{\partial z}(t,0) = 0 = \frac{\partial\overline{\xi}}{\partial z}(t,1). \end{cases} \tag{17}$$

The previous system may be written as the following abstract Cauchy problem $\dot{\overline{\xi}}(t) = \mathscr{A}\overline{\xi}(t) + b\tilde{u}(t), \overline{\xi}(0) = \xi_0$, where the (unbounded) linear operator $\mathscr{A}$ is given by $\mathscr{A}\cdot = \frac{d^2\cdot}{dz^2} + \frac{1}{\sqrt{2}}\cdot$ on $D(\mathscr{A}) = D(A)$. The operator $\mathscr{A}$ generates a C_0-semigroup of linear operators both on X and on Y. Moreover, the latter admits the Riesz-spectral decomposition $\mathscr{A}\cdot = \sum_{n=0}^{\infty}\lambda_n\langle\cdot,\phi_n\rangle\phi_n$, where $\{\lambda_n\}_{n\in\mathbb{N}} = \{-n^2\pi^2 + \frac{1}{\sqrt{2}}\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}} = \{1_{[0,1]}(z)\} \cup \{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}_0}$. Since $\sup_{n\in\mathbb{N}}\{\lambda_n\}_{n\in\mathbb{N}} = \frac{1}{\sqrt{2}}$, the system (17) is obviously unstable. We shall consider the optimal control problem for (17) in which we seek $\tilde{u} \in L^2([0,\infty);\mathbb{R})$ that minimizes the cost functional

$$J(\tilde{u},\xi_0,\infty) = \int_0^\infty \left(|\langle c,\overline{\xi}(t)\rangle_X|^2 + \tilde{u}^2(t)\right)dt, \tag{18}$$

where $c(\cdot) = 1_{[0,1]}(\cdot)$. First we check that the pairs $(\mathscr{A},b)$ and $(c,\mathscr{A})$ are exponentially stabilizable and detectable, respectively (this ensures that the optimal control problem is well-posed), that is, $\langle b(\cdot),\phi_n(\cdot)\rangle_X \neq 0$ and $\langle c(\cdot),\phi_n(\cdot)\rangle_X \neq 0$ for all n such that $\lambda_n \geq 0$. The only eigenvalue that is positive here is $\lambda_0 = \frac{1}{\sqrt{2}}$ and the corresponding eigenfunction is given by $\phi_0 = 1_{[0,1]}(z)$. It holds that $\langle 1_{[0,1]}(\cdot), 1_{[0,1]}(\cdot)\rangle_X = 1$. The exponential detectability of the pair $(c,\mathscr{A})$ is also ensured by using the same argument.

We shall now isolate the unstable part of the system. Due to the parabolicity of the PDE (17), the solution $\overline{\xi}(t)$ admits the following series expansion $\overline{\xi}(t,z) = \sum_{n=1}^{\infty}\alpha_n(t)\phi_n(z)$, where $\{\alpha_n\}_{n\in\mathbb{N}}$ are time dependent functions. Plugging the expansion of $\overline{\xi}(t,z)$ into (17) and then projecting the equation on the basis of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ yields the following dynamical system for the α_n's

$$\dot{\alpha}_0(t) = \frac{1}{\sqrt{2}}\alpha_0(t) + \tilde{u}(t), \dot{\alpha}_n(t) = (-n^2\pi^2 + \frac{1}{\sqrt{2}})\alpha_n(t), n \geq 1. \tag{19}$$

In what follows we shall focus on the unstable part of (19), i.e. we shall design $\tilde{u}$ such that $\alpha_0(t)$ converges exponentially fast to 0. For this we rewrite the cost functional (18) with the spectral decomposition of $\overline{\xi}$, i.e.

$$J(\tilde{u},\xi_0,\infty) = \int_0^\infty \left(y^2(t) + \tilde{u}^2(t)\right)dt, \tag{20}$$

where $y(t) = \alpha_0(t)$. The optimal feedback that stabilizes $\alpha_0(t)$ and that minimizes the cost functional (20) is obtained by solving the following Riccati equation $\sqrt{2}Q + 1 - Q^2 = 0$, whose solutions are obviously given by $Q_1 = \frac{\sqrt{2}+\sqrt{6}}{2}$ and $Q_2 = \frac{\sqrt{2}-\sqrt{6}}{2}$. Since the solution has to be positive, we shall keep $Q = \frac{\sqrt{2}+\sqrt{6}}{2}$ as unique solution. The optimal gain k is obtained as $k = -Q$. Then the optimal feedback $\tilde{u}(t) = k\alpha_0(t)$ is known to exponentially stabilize (19) and to minimize the cost functional (20). Another way to see the optimal feedback (as an operator acting on the whole state $\overline{\xi}$) is

$$\tilde{u}(t) = k\langle 1_{[0,1]}(\cdot),\overline{\xi}(t,\cdot)\rangle_X = k\int_0^1 \overline{\xi}(t,z)dz. \tag{21}$$

The linearized system (17) together with the optimal control $\tilde{u}$ (21) yields the following closed-loop system

$$\begin{cases} \frac{\partial\overline{\xi}}{\partial t}(t,z) = \frac{\partial^2\overline{\xi}}{\partial z^2}(t,z) + \frac{1}{\sqrt{2}}\overline{\xi}(t,z) + k1_{[0,1]}(z)\int_0^1 \overline{\xi}(t,z)dz, \\ \frac{\partial\overline{\xi}}{\partial z}(t,0) = 0 = \frac{\partial\overline{\xi}}{\partial z}(t,1). \end{cases} \tag{22}$$

The corresponding abstract representation is given as $\dot{\overline{\xi}}(t) = \mathfrak{A}\overline{\xi}(t), \overline{\xi}(0) = \xi_0$ where the unbounded linear and self-adjoint operator $\mathfrak{A}\cdot = \frac{d^2\cdot}{dz^2} + \frac{1}{\sqrt{2}}\cdot + k1_{[0,1]}(z)\int_0^1 \cdot dz$ on $D(\mathfrak{A}) = D(A)$. Note that the operator $\mathfrak{A}$ satisfies $\mathfrak{A} = \mathscr{A} + k1_{[0,1]}(z)\langle 1_{[0,1]}(z),\cdot\rangle_X$. Since $\mathscr{A}$ is a Riesz-spectral operator, the operator $\mathfrak{A}$ is still a Riesz-spectral operator provided that assumption (10) is satisfied for the operator $\mathscr{A}$, see e.g. [12]. Observe that $\sup_{m\in\mathbb{N}}\{\sum_{\substack{n=0\\n\neq m}}^{\infty}\frac{1}{|\lambda_n-\lambda_m|^2}\} < \infty$. Indeed, for $m \in \mathbb{N}$,

$$\sum_{\substack{n=0\\n\neq m}}^{\infty}\frac{1}{|\lambda_n-\lambda_m|^2} = \frac{1}{\pi^4}\sum_{\substack{n=0\\n\neq m}}^{\infty}\frac{1}{(m-n)^2(m+n)^2} \leq \frac{1}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6\pi^2}.$$

Then we conclude that the closed-loop operator $\mathfrak{A}$ is a Riesz-spectral operator of the form $\mathfrak{A}\cdot = \sum_{n=0}^{\infty}\lambda_n^o\langle\cdot,\phi_n^o\rangle_X\phi_n^o$. Due to previous considerations in the resolution of the Riccati equation and the stabilization of (17), the closed-loop eigenvalues $\{\lambda_n^o\}_{n\in\mathbb{N}}$ and eigenfunctions $\{\phi_n^o\}_{n\in\mathbb{N}}$ are given by $\{\lambda_n^o\}_{n\in\mathbb{N}} = \{-\frac{\sqrt{6}}{2}\} \cup \{-n^2\pi^2 + \frac{1}{\sqrt{2}}\}_{n\in\mathbb{N}_0}$ and $\{\phi_n^o\}_{n\in\mathbb{N}} = \{1_{[0,1]}(z)\} \cup \{\sqrt{2}\cos(n\pi z)\}_{n\in\mathbb{N}}$. Consequently the closed-loop state trajectory solution of the system (22) is given by the series expansion $\overline{\xi}(t,z) = \sum_{n=0}^{\infty}e^{\lambda_n^o t}\langle\xi_0(\cdot),\phi_n^o(\cdot)\rangle_X\phi_n^o(z) =: \overline{T}(t)\xi_0$, where ξ_0 is the initial condition given to the system and $\left(\overline{T}(t)\right)_{t\geq 0}$ denotes the C_0-semigroup whose $\mathfrak{A}$ is the infinitesimal generator. By construction, $(\overline{T}(t))_{t\geq 0}$ is exponentially stable on X. Let us prove that it remains exponentially stable when evaluated with $Y-$norms. According to [2], [11], it is exponentially stable on Y if the convergence condition (13) holds (and depends possibly on $\|\xi_0\|_Y$) for some $1 \leq p < \infty$. According to Proposition 4.1, (13) with $p = 1$ is finite provided that the series (12) converges. Observe that $\|\phi_n^o\|_X = 1$ and $\|\phi_n^o\|_Y = \sqrt{2}$. Consequently,

$$\sum_{n=0}^{\infty}\frac{\|\phi_n^o\|_X\|\phi_n^o\|_Y}{|\mathfrak{Re}(\lambda_n^o)|} = \frac{\sqrt{2}}{|-\frac{\sqrt{6}}{2}|} + \sum_{n=1}^{\infty}\frac{\sqrt{2}}{|-n^2\pi^2 + \frac{1}{\sqrt{2}}|},$$

hence the series (12) is convergent. This proves the exponential stability of $(\overline{T}(t))_{t\geq 0}$ on the space Y.

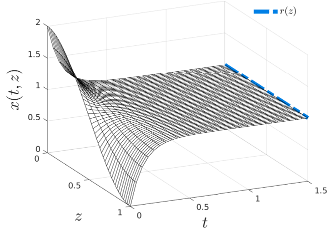

Fig. 2. Closed-loop nonlinear state trajectory $x(t,z)$.

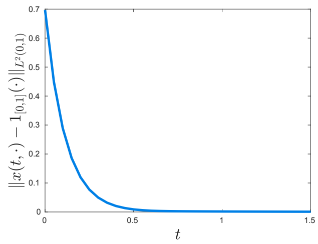

Fig. 3. X-norm of the closed-loop nonlinear state trajectory $x(t,z) - 1_{[0,1]}(z)$.

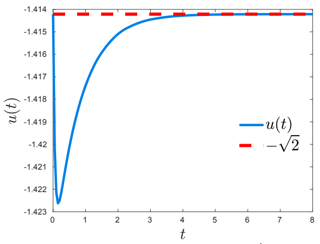

Fig. 4. Control input $u(t) = \tilde{u}(t) - \sqrt{2} = k\int_0^1 (x(t,z) - 1_{[0,1]}(z))dz - \sqrt{2}$.

Let us consider the following concluding theorem that makes the link between the stability of the linearized system (22) with the tracking of the reference profile $r(z)$ for the nonlinear system (14).

Theorem 5.1: The optimal control law $u(t) = \tilde{u}(t) - \sqrt{2}$ stabilizes locally and exponentially the nonlinear system (14) around the reference profile $r(z) = 1_{[0,1]}(z)$, where $\tilde{u}(t)$ is given by the feedback law (21).

Proof: See above in this Section and Section IV. ■

We end up this Section with some numerical simulations in order to show the efficiency of the proposed method. We take $\xi_0(z) = 4z^3 - 6z^2 + 2$ as initial condition. The state trajectory of the nonlinear system (14) with the optimal control $u(t) = \tilde{u}(t) - \sqrt{2}$ is depicted with its $L^2(0,1)-$norm in Figures 2 and 3, respectively. Exponential stability is highlighted according to Theorem 5.1. In figure 4 the control input $u(t) = \tilde{u}(t) - \sqrt{2}$ is shown. The asymptotic value $-\sqrt{2}$ is explained by the expression of $u(t)$. Indeed $|u(t) + \sqrt{2}| = |k\int_0^1 \xi(t,z)dz| \leq |k| \cdot \|\xi(t,\cdot)\|_{L^1(0,1)} \leq |k| \cdot \|\xi(t,\cdot)\|_X$, which tends to 0 exponentially fast as t goes to ∞.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper it is shown how to locally stabilize an infinite-dimensional nonlinear system around any of its equilibria. The method that is proposed relies on the stabilization of the linearized dynamics by linear feedback on two different spaces. Then a new concept of Fréchet differentiability has been introduced to conclude that, under boundedness assumptions for the control operator, the linear feedback gain locally stabilizes the nonlinear system around its equilibrium. The main result is based on the continuous dependence of the nonlinear semigroup at 0, on the two introduced spaces and also on the adapted Fréchet differentiability conditions that the nonlinear operator has to satisfy. A class of LQ-optimally controlled systems has also been described, that fulfills the required assumptions. We illustrated the theory by an example for which numerical simulations have been performed.

Further research could be the identification of other classes of systems to which the new framework developed here can be applied. As another perspective, the consideration of other types of control inputs than state feedbacks interacting with the concepts of Fréchet differentiability should be also interesting to investigate.

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