

REPRESENTATION THEORY IN CLASSICAL AND QUANTUM PHYSICS

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We review the basic notions of group theory, in particular Lie groups and Lie algebras, and of representations of the latter. Then we examine briefly their occurrence in classical physics for the description of invariance properties of physical systems and the concomitant conservation laws resulting from Noether's theorem. In the last section, finally, we give an overview of the applications of group representation theory in quantum physics, with special emphasis on the proper mathematical description of symmetry properties, both in quantum mechanics and in quantum field theory.

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1. Motivation

Before starting this review, there are two basic questions we ought to answer: Why do we need *group theory* in physics ? Why do we need *representations* of groups and algebras ?

As for the first question, it is, of course, closely linked to the notion of *symmetry*. Symmetry, or the lack of it, has always fascinated Man, from the ornaments on Mycenaean jewelry, several centuries B.C., to the most contemporary developments in high energy physics (string theory) or condensed matter physics (quasicrystals). And group theory is the mathematical language for describing symmetries, which really developed in the 19th century.¹ Then, a quick look at the various instances where it appears reveals that the dominant themes are *classification* and *simplification*. Let us give a few examples.

- . *Crystallography*: this was the first occurrence of group theory in physics. Originally at least, the goal was to list all allowed types of crystals (there are 230 of them in 3 dimensions, so that a classification principle was indeed indispensable). The complete classification of crystal into 32 crystal classes was achieved by listing the 32 point groups (finite groups). Combining the latter with lattice translations then leads to the 230 space groups. Their complete listing is due to E. V. Fedorov (1885) and A. Schönflies (1891).
- . *Spectroscopy*: already in the 30s, the use of the rotation group $SO(3)$, under the impetus of Wigner, led to considerable progress in unraveling atomic spectra. This was not easily accepted, however, as is manifested in the term *Gruppenpest* qualifying the group-theoretical approach!
- . Describing *symmetry* or *invariance* properties of physical systems: here dynamics enters, but geometry remains an essential tool (remember, F. Klein defines geometry as the invariance under a certain group of transformations; for instance, Euclidean geometry is the study of properties invariant under Euclidean transformations,

namely, rotations and translations).

- . *Tensorial properties* of physical quantities, a fundamental tool in relativity, are defined in terms of a certain transformation group (rotations, Lorentz).
- . *Elementary particles*: since the 1960s, the proliferation of new particles (which by the way are far from being elementary!) also demanded a systematization, that group theory indeed provided.
- . Classifying *interactions* between particles in terms of their invariance groups: this is now a purely dynamical aspect.
- . Understanding *conservation laws*: by Noether's theorem, invariance of a Lagrangian implies existence of conservation laws (more about this in Section 4.3).

As for the second question, why do we need *representations* of groups and algebras, the answers differ in the classical and in the quantum cases.

- . In *classical (Hamiltonian) physics*, states are functions on phase space and the dynamics is described by the Poisson algebra, defining canonical transformations. Thus generators of symmetry groups obey a Poisson algebra analogous to the Lie algebra $\mathfrak{g}$ of the symmetry group G , i.e., a representation of $\mathfrak{g}$.
- . In *quantum physics*, states are rays in a Hilbert space $\mathfrak{H}$ (from the superposition principle) and observables are linear operators acting on $\mathfrak{H}$. Thus, symmetry operations must be realized by operators acting on $\mathfrak{H}$ and possessing the same properties as the corresponding group G , i.e., a linear representation of G ; the corresponding infinitesimal generators are described by a representation of the Lie algebra $\mathfrak{g}$.

2. Mathematical tools I: Group theory

Since it is important to have precise notions in mind before tackling applications, we shall devote the next two (long) sections to the essential mathematical tools we will need, basic group theory and representations, respectively. Most proofs are omitted, so that this text is not really a crash course in group theory. No prior knowledge is assumed, although the pace might appear fast to the complete debutant. In that case, recourse to a standard textbook may be advised, such as those of Bacry,² Barut–Rączka,³ or Gilmore.⁴

2.1. Basic notions: a quick reminder

Definition 2.1. A *group* is a set G equipped with an internal composition law, called product, $(g, g') \mapsto gg'$, such that

- (i) the product is *associative*: $g_1(g_2g_3) = (g_1g_2)g_3$, for all $g_1, g_2, g_3 \in G$;
- (ii) there exists a (unique) *neutral element* $e \in G$ such that $eg = ge = g$, for all $g \in G$;
- (iii) every element $g \in G$ has an *inverse* $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

The group is *abelian* or *commutative* if $g_1g_2 = g_2g_1$, for all $g_1, g_2 \in G$. Equivalently, all *commutators* are trivial, that is, $g_1g_2g_1^{-1}g_2^{-1} = e$, for all $g_1, g_2 \in G$.

Examples:

- $\mathbb{Z}_2 = \{1, -1\}$
- $V_4 = \{\mathbb{I}_2, R, -\mathbb{I}_2, -R\}$, where $R = \text{diag}(-1, 1)$
- $\mathbb{Z}_4 = \{1, i, -1, -i\}$, cyclic group of order 4 ; this group is different from V_4 (consider the square of each element)!
- $\text{SO}(2) = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, 0 \leq \varphi < 2\pi \right\}$
- $\text{SO}_o(1,1) = \left\{ \begin{pmatrix} \cosh \mu & -\sinh \mu \\ -\sinh \mu & \cosh \mu \end{pmatrix}, \mu \in \mathbb{R} \right\}$
- $\text{SO}(3)$, the rotation group of $\mathbb{R}^3$
- $\text{SO}_o(1,3)$, the (proper) Lorentz group
- $\text{SU}(2)$, the set of all 2×2 unitary matrices U (i.e., $U^\dagger U = \mathbb{I}_2$, implying $UU^\dagger = \mathbb{I}_2$) with $\det U = 1$ (here $U^\dagger$ denotes the hermitian adjoint of U).

It might happen that two groups have exactly the same structure and may be identified. More precisely,

Definition 2.2. An *isomorphism* between two groups G, G' is a bijection $\sigma : G \rightarrow G'$ that preserves the group law:

$$\sigma(g_1g_2) = \sigma(g_1)\sigma(g_2), \text{ for all } g_1, g_2 \in G.$$

It follows easily that the inverse map $\sigma^{-1} : G' \rightarrow G$ is also an isomorphism.

Whenever G, G' are isomorphic, we write $G \simeq G'$. Notice that the symbol $\simeq$ will be used also in the sequel for denoting an isomorphism between any

two mathematical entities of the same nature (Lie groups, Lie algebras, smooth manifolds).

Definition 2.3. A *subgroup* of G is a subset $H \subset G$ which is itself a group for the group law of G . Thus the neutral element e belongs to H and $g, g' \in H$ implies that $gg', g^{-1} \in H$. A subgroup is *nontrivial* if it differs from $\{e\}$ and G itself.

Examples: $\text{SO}(2) \subset \text{SU}(2)$, $\text{SO}(3) \subset \text{SO}_o(1,3)$, $\mathbb{Z}_2 \subset \mathbb{Z}_4$.

Given a subgroup $H \subset G$, we define an equivalence relation on G by

$$g \sim g' \iff g^{-1}g' \in H.$$

The equivalence classes, which are necessarily disjoint or equal, are called *left cosets* and are of the form $gH = \{gh, h \in H\}$. This equivalence relation induces a partition of G into equivalence classes

$$G = \bigcup_{i \in I} g_i H, \quad \text{where } g_i \not\sim g_j \text{ for } i \neq j. \quad (2.1)$$

The index set I in (2.1) may be finite or infinite. Thus every element of G belongs to one and only one coset.

In the same way, the equivalence relation $g \sim g' \iff gg'^{-1} \in H$ defines *right cosets*, of the form $Hg = \{hg, h \in H\}$, and another partition of G , *a priori* different from that in (2.1).

Definition 2.4. The set of all left cosets gH , denoted G/H , is called the (left) *quotient* of G by H . Similarly, the set of all right cosets Hg , denoted $H \backslash G$, is called the (right) *quotient* of G by H .

Without additional structure, both quotients are merely sets, and are not related to each other. Things change, however, when the subgroup H is of a special type, called *invariant*:

Definition 2.5. The subgroup H is *invariant* or *normal* if it is equal to all its conjugates, that is, $gHg^{-1} = H$, for all $g \in G$ or, equivalently, if $gH = Hg$, for all $g \in G$. In that case, we write $H \trianglelefteq G$.

Examples:

- If G is abelian, every subgroup is invariant, for instance, $\mathbb{Z}_2 \trianglelefteq \mathbb{Z}_4$.
- $\text{SO}(2) \trianglelefteq \text{O}(2) = \{g(\varphi), g_-(\varphi), 0 \leq \varphi < 2\pi\}$, where

$$g(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \quad g_-(\varphi) = \begin{pmatrix} -\cos \varphi & -\sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}.$$

Thus, when H is invariant, both partitions of G , into left and right cosets, coincide. But there is more.

Proposition 2.6. *If $H \subseteq G$, then $G/H = \{gH, g \in G\}$ is a group, called the quotient group, with group operation*

$$(g_1H)(g_2H) = (g_1g_2)H. \quad (2.2)$$

The neutral element is $H \equiv eH$ and the inverse of gH is $g^{-1}H$.

The law (2.2) indeed makes sense, for one has:

$$(g_1H)(g_2H) = g_1(Hg_2)H = g_1(g_2H)H = g_1g_2HH = (g_1g_2)H,$$

that is, the l.h.s. is indeed a left coset.

Examples:

- Consider the group $O(2)$ introduced after Definition 2.5: $SO(2) \subseteq O(2)$, $O(2) = SO(2) \cup R SO(2)$ and $O(2)/SO(2) \simeq \{\mathbb{I}, R\} \simeq \mathbb{Z}_2$.
- For any group G , its center $Z(G) = \{z \in G : zg = gz, \text{ for all } g \in G\}$ is an invariant subgroup. A subgroup of $Z(G)$ will hence be called *central*.

Definition 2.7. The *direct product* of two groups G, G' is the set $G \times G' = \{(g, g'), g \in G, g' \in G'\}$, endowed with the product $(g_1, g'_1) \cdot (g_2, g'_2) = (g_1g_2, g'_1g'_2)$. Then $G \simeq \{(g, e'), g \in G\}$ and $G' \simeq \{(e, g'), g' \in G'\}$ are both invariant subgroups of $G \times G'$.

For instance, $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 2.8. (1) The group G is *simple* if it has no nontrivial invariant subgroup.

Examples: $SO(2)$, $SU(2)$.

(2) The group G is *semisimple* if it has no nontrivial invariant abelian subgroup.

Example: $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$ is semisimple, but not simple. On the other hand, $O(2)$ is not semisimple, since $SO(2)$ is an invariant abelian subgroup.

Proposition 2.9. *Every semisimple group G is of the form*

$$G \simeq (G_1 \times G_2 \times \dots \times G_n)/Z, \quad (2.3)$$

where $G_1, G_2, \dots, G_n$ are simple and Z is a discrete central subgroup of the product.

Now we turn to maps from a group into another one, that preserve some of the group structure.

Definition 2.10. A *(group) homomorphism* is a map $\sigma : G \rightarrow G'$ such that

$$\sigma(g_1 g_2) = \sigma(g_1) \sigma(g_2), \text{ for all } g_1, g_2 \in G. \quad (2.4)$$

It follows from (2.4) that $\sigma(g^{-1}) = \sigma(g)^{-1}$, for all $g \in G$, and that $\sigma(e) = e'$. The *kernel* of the homomorphism is the set $\text{Ker } \sigma = \{g \in G : \sigma(g) = e'\}$ and its *range* is the set $\text{Im } \sigma = \{\sigma(g), g \in G\}$.

It is readily seen that $\text{Ker } \sigma$ is an invariant subgroup of G and that $\text{Im } \sigma$ is a subgroup of G' . Of course, if $\sigma : G \rightarrow G'$ is bijective, we recover the Definition 2.2 of an isomorphism.

Actually, the notion of homomorphism is in a sense equivalent to that of invariant subspace, as follows from the next theorem.

Theorem 2.11. (i) If $H \subseteq G$, then $\sigma : g \mapsto gH$ is a (canonical) homomorphism of G onto G/H and $\text{Ker } \sigma = H$.

(ii) Conversely, let $\sigma : G \rightarrow G'$ be a homomorphism. Then $G/\text{Ker } \sigma \simeq \text{Im } \sigma$.

In other words, H is an invariant subgroup of G if and only if it is the kernel of some surjective homomorphism $\sigma : G \rightarrow G'$ such that $G' \simeq G/H$.

Example: Take again the group $O(2)$, with its subgroups $SO(2)$ and $\{\mathbb{I}, R\}$:

- $SO(2) \subseteq O(2)$ and $O(2)/SO(2) \simeq \{\mathbb{I}, R\} \simeq \mathbb{Z}_2$.
- $\{\mathbb{I}, R\} \subset O(2)$, but $\{\mathbb{I}, R\} \not\subseteq SO(2)$.

It follows from Definition 2.7 that $O(2)$ is *not* the direct product of $SO(2)$ and $\{\mathbb{I}, R\}$, otherwise both subgroups would be invariant (it is a semidirect product, see below).

Definition 2.12. Let G be a group. (1) An *automorphism* of G is an isomorphism $\sigma : G \rightarrow G$. The set of all automorphisms of G is a group with respect to the composition of maps. It is denoted by $\text{Aut } G$.

(2) An *inner automorphism* of G is simply the conjugation by an element $h \in G$: $g \mapsto hgh^{-1}$, for all $g \in G$. The set of all inner automorphisms of G is a subgroup of $\text{Aut } G$, denoted by $\text{Int } G$. It follows that two conjugated subgroups of G are always isomorphic: $H' \equiv gHg^{-1} \simeq H$.

(3) An *outer automorphism* of G is an automorphism which is not inner. The outer automorphisms of G also form a group, denoted by $\text{Out } G$, as follows from the next lemma.

Lemma 2.13. *One has*

$$\begin{aligned} \text{Int } G &\subseteq \text{Aut } G \\ \text{Out } G &\simeq \text{Aut } G / \text{Int } G. \end{aligned}$$

Example: consider $\text{SO}(3)$ acting on the additive group $\mathbb{R}^3$; the map $\vec{x} \mapsto R\vec{x}$ is an outer automorphism of $\mathbb{R}^3$ (a precise definition of the notion of action is given in Definition 2.15 below).

The notion of automorphism allows one to build a new group from two smaller ones, in a fashion more general than a direct product. Let H, K be two groups for which there exists a homomorphism $\alpha : K \rightarrow \text{Aut } H$. Then we define a new type of product:

Definition 2.14. The *semidirect product* of H (noted additively) by K (noted multiplicatively), with respect to the homomorphism $\alpha : K \rightarrow \text{Aut } H$, is the group $G = H \rtimes K$ of all pairs $(h, k) \in H \times K$, with composition law

$$(h, k)(h', k') = (h + \alpha(k)(h'), kk'), \text{ for all } (h, k), (h', k') \in H \times K.$$

The neutral element of G is $(0, e_K)$ and the inverse of (h, k) is $(h, k)^{-1} = (-\alpha(k)^{-1}(h), k^{-1})$.

Moreover, it is easy to see that $\{(h, e_K), h \in H\} \simeq H \subseteq G$ and $G/H = (H \rtimes K)/H \simeq K$.

Example: $\text{O}(2) = \text{SO}(2) \rtimes \{\mathbb{I}, R\}$, $\text{SO}(2) \subseteq \text{O}(2)$ and $\text{O}(2)/\text{SO}(2) \simeq \{\mathbb{I}, R\} \simeq \mathbb{Z}_2$.

The most useful type of semidirect product is an *inhomogeneous group*, that is, a group of the form $G = V \rtimes K$, where V is a vector space and $K \subset \text{GL}(V)$, the set of all invertible linear transformations of V , with the composition law

$$(x, \Lambda)(x', \Lambda') = (x + \Lambda x', \Lambda\Lambda'), \text{ for all } (x, \Lambda), (x', \Lambda') \in V \times K$$

Examples:

- The Euclidean group: $\text{E}(n) = \mathbb{R}^n \rtimes \text{SO}(n)$, where $\mathbb{R}^n$ corresponds to translations and $\text{SO}(n)$ to rotations.
- The Poincaré group: $\mathcal{P}(1,3) = \mathbb{R}^4 \rtimes \text{SO}_o(1,3)$.
- The affine group of $\mathbb{R}$: $G_{\text{aff}} = \mathbb{R} \rtimes \mathbb{R}_*^+$, where $\mathbb{R}_*^+$ is the group of dilations and the product law reads as $(b, a)(b', a') = (b + ab', aa')$; this is the group underlying the *1-D wavelet transform*.

- The similitude group of $\mathbb{R}^n$: $\text{SIM}(n) = \mathbb{R}^n \rtimes (\mathbb{R}_*^+ \times \text{SO}(n))$; this is the group underlying the n -D wavelet transform.

Definition 2.15. (1) If G is a group and X is a set, an *action* of G on X is a map $(g, x) \in G \times X \mapsto g[x] \in X$ such that

- (i) $g_1 g_2[x] = g_1[g_2[x]]$, for all $g_1, g_2 \in G$, for all $x \in X$.
- (ii) $e[x] = x$, $e \in G$, for all $x \in X$.

Then one says that X is a G -set.

(2) The action of G on X is *transitive* if, for every pair $x, x' \in X$, there exists an element $g \in G$ such that $x = g[x']$; then X is called a (G) -homogeneous set.

(3) The *orbit* of a point $x \in X$ is the set of elements of X to which x can be moved by the elements of G . It is denoted by $\mathcal{O}_x$:

$$\mathcal{O}_x = \{g[x], g \in G\}.$$

(4) The *stabilizer subgroup* of a point $x \in X$ is the set of elements of G that fix x :

$$G_x = \{g \in G, g[x] = x\}.$$

(5) Orbits and stabilizers are related. For each point $x \in X$, one has $G/G_x \simeq \mathcal{O}_x$.

The action is transitive on each orbit and the stabilizer subgroups of any two points of the same orbit are conjugated, hence isomorphic: $y \in \mathcal{O}_x$ means that $y = g[x]$, for some $g \in G$, and then $G_y = gG_xg^{-1}$. If the action is transitive, then X is the unique orbit and thus $X \simeq G/G_x$, for all $x \in G$.

Examples:

- $\text{SO}(3)$ acts transitively on the unit sphere S^2 and the stabilizer of every point of the sphere is isomorphic to $\text{SO}(2)$. Thus one has $S^2 \simeq \text{SO}(3)/\text{SO}(2)$.
- $\text{SO}(3)$ acting on $\mathbb{R}^3$: the orbit of the point $\vec{x} \in \mathbb{R}^3$ is the sphere S_r of radius $r = |\vec{x}|$ and $\mathbb{R}^3 = \bigcup_{r \geq 0} S_r$.

We conclude this section with some crude indications concerning topological groups. For lack of space, we have to drastically simplify the treatment here. Precise statements and definitions may be found in the textbooks.

Definition 2.16. Roughly speaking, a *topological space* is a set X with a topology, the latter being defined by a collection of *open* sets or, equivalently, a collection of *closed* sets.

Then we define the following notions.

- . A map $\alpha : X \rightarrow Y$ from one topological space into another one is *continuous* if $\alpha^{-1}U$ is open in X for every open subset $U \subset Y$.
- . A set X is *compact* if every open covering of X contains a finite open subcovering (a more manageable definition will be given in Section 2.4).
- . A set X is *connected* if X cannot be decomposed into the union of two nonempty disjoint subsets, $X \neq A \cup B$, called *connected components*; thus every set is the disjoint union of connected components.
- . Given x, y in the same connected component, two continuous curves $P_{x \rightarrow y}, Q_{x \rightarrow y}$ from x to y are *homotopic* if they can be deformed continuously into one another. Clearly, homotopy is an equivalence relation. Given $x \in X$, the set of equivalence classes of closed curves $P_{x \rightarrow x}$ is an abelian group with respect to the composition of curves. This group does not depend on the choice of $x \in X$ and is called the (first) *homotopy group* of X .
- . The set X is *simply connected* if every two continuous curves $P_{x \rightarrow y}, Q_{x \rightarrow y}, x, y \in X$, are homotopic or, equivalently, if every closed curve $P_{x \rightarrow x}$ is homotopic to the null curve. In this case, the homotopy group of X is trivial.

Definition 2.17. A *topological group* is a group G which is also a topological space, such that the group multiplication $(g_1, g_2) \mapsto g_1 g_2$ and the inverse operation $g \mapsto g^{-1}$ are continuous maps (here $G \times G$ is viewed as a topological space with the product topology).

A *compact (topological) group* is a topological group which is also a compact space.

Examples:

- . $\text{SO}(2), \text{SO}(3)$ are compact topological groups.
- . $\text{SO}(1,1), \text{SO}(1,3)$ are noncompact topological groups.

Proposition 2.18. Let G be a topological group and G_o the connected component of G containing the identity. Then G_o is a closed invariant subgroup of G and G/G_o is discrete.

Proposition 2.19. *If G is a compact topological group and H is a closed subgroup of G , then H is a compact topological group.*

Examples:

$\mathrm{SO}(2) \subset \mathrm{SO}(3)$, $\mathrm{SO}(2) \subset \mathrm{SU}(2)$.

Note that the result is false if H not closed (for instance, an irrational helix winding on the torus $\mathrm{SO}(2) \times \mathrm{SO}(2)$ is a dense noncompact subgroup).

2.2. Lie groups and Lie algebras

Definition 2.20. A subset M of $\mathbb{R}^n$ is a k -dimensional *smooth* or C^∞ *manifold* if it is locally diffeomorphic to $\mathbb{R}^k$ for some $k \leq n$, i.e., for each point $x \in M$ there exist open sets $U, V \subset \mathbb{R}^n$, $U \ni x$, and a diffeomorphism $h : U \rightarrow V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0, 0, \dots, 0\}) = \{y \in V : y^{k+1} = \dots = y^n = 0\}$$

Given two smooth manifolds M, N , the notion of smooth or C^∞ map $h : M \rightarrow N$ follows immediately from the definition above (a smooth map is also called differentiable or a diffeomorphism)

Examples: the sphere S^n in $\mathbb{R}^{n+1}$, a (one- or two-sheeted) hyperboloid in $\mathbb{R}^{n+1}$, a torus in $\mathbb{R}^3$, the Möbius strip in $\mathbb{R}^3$.

Definition 2.21. A *Lie group* is a group G which is at the same time a smooth manifold such that the map $G \times G \rightarrow G$, $(g, h) \mapsto gh^{-1}$ is (infinitely often) differentiable.

Examples:

- . Abelian Lie groups: finite dimensional vector spaces (with vector addition), S^1 (with multiplication of complex numbers)
- . Matrix groups: $\mathrm{SO}(2)$, $\mathrm{SO}(3)$, $\mathrm{SU}(2)$, $\mathrm{SO}(1,3)$, $\mathrm{GL}(n, \mathbb{R}) \equiv \mathrm{GL}(\mathbb{R}^n)$.

Note that, if G is a Lie group and H a closed subgroup of G , then the quotient G/H is a smooth manifold. For instance, $\mathrm{SO}(3)/\mathrm{SO}(2) \simeq S^2$, the 2-sphere, and the two are isomorphic smooth manifolds.

Lie algebras: intuitive discussion

. The case of $\text{SO}(2)$

Every entry of the matrix $g(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \in \text{SO}(2)$ is an analytic function of φ , thus it may be expanded in a Taylor series:

$$g(\varphi) = \begin{pmatrix} 1 - \frac{1}{2}\varphi^2 + O(\varphi^4) & -\varphi + O(\varphi^3) \\ \varphi + O(\varphi^3) & 1 - \frac{1}{2}\varphi^2 + O(\varphi^4) \end{pmatrix}$$

$$= \mathbb{I}_2 - i\varphi\sigma + O(\varphi^2)M, \text{ where } \sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } M \text{ is a } 2 \times 2 \text{ matrix.}$$

The matrix σ may also be defined as the *infinitesimal generator* of $\text{SO}(2)$:

$$\sigma = i \lim_{\varphi \rightarrow 0} \frac{d}{d\varphi} g(\varphi).$$

Conversely, using $(-i\sigma)^2 = -\mathbb{I}_2$, one gets:

$$g(\varphi) = \exp(-i\sigma\varphi)$$

$$\equiv \mathbb{I}_2 - i\sigma\varphi + \frac{(-i\sigma\varphi)^2}{2!} + \frac{(-i\sigma\varphi)^3}{3!} + \dots$$

Thus σ defines completely the group structure of $\text{SO}(2)$ in a neighborhood of the identity and allows one to reconstruct every group element by *exponentiation*.

Notice that σ *linearizes* the group composition law in a neighborhood of the identity, in the sense that

$$g(\varphi)g(\psi) = (\mathbb{I} - i\sigma\varphi + \dots)(\mathbb{I} - i\sigma\psi + \dots)$$

$$= \mathbb{I} - i\sigma(\varphi + \psi) + \dots$$

. The general case

In general, for studying the group structure of G in a neighborhood of the identity, one considers “sufficiently many” one-parameter subgroups $g_i(\cdot)$:

$$g_i(s)g_i(t) = g_i(s+t), \quad s, t \in \mathbb{R}, \quad i = 1, 2, \dots, p,$$

where p is dimension of G , that is, its dimension as manifold.

As before, one defines Σ_j , the *infinitesimal generator* of j^{th} subgroup g_j , so $g_j(s) = \exp(-i\Sigma_j s)$. Then, the *Lie algebra* $\mathfrak{g}$ of G is the vector space generated by the infinitesimal generators $\Sigma_j, j = 1, \dots, p$.

This definition may be visualized intuitively as follows:

- . in the neighborhood of the identity e , the group G may be seen as a (hyper)surface containing e
- . the one-parameter subgroups $g_j(s)$ are represented by curves on that surface, intersecting at e (only)
- . the generators Σ_j are vectors tangent to these curves at e
- . finally, the Lie algebra $\mathfrak{g}$ is the vector space generated by these tangent vectors, that is, the plane tangent to the surface at e

To give an example, consider the group $SU(2)$, which may be parameterized as

$$SU(2) \ni g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with } a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1$$

($\bar{a}$ is the complex conjugate of a). Writing $a = x_1 + ix_2, b = x_3 + ix_4$, one gets $SU(2) \simeq S^3$, the unit sphere in $\mathbb{R}^4$, with equation:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

The identity element is the point $(1,0,0,0)$, the North Pole, and the Lie algebra $\mathfrak{su}(2)$ is the plane tangent to the sphere S^3 at the North Pole.

Commutation relations

Measuring the non-abelianness of the group G amounts to evaluate the curvature of the surface that models it. Given two one-parameter subgroups $g_k(s) = e^{-is\Sigma_k}, g_\ell(t) = e^{-it\Sigma_\ell}$, their *commutator*

$$g_k(s)g_\ell(t)g_k(s)^{-1}g_\ell(t)^{-1} = \mathbb{I} + st(\Sigma_\ell\Sigma_k - \Sigma_k\Sigma_\ell) + \dots$$

measures the non-abelian character of G . The crucial point, which may be proven using techniques of differential geometry, is that the commutator $[X, Y] \equiv XY - YX$ of two elements of the Lie algebra $\mathfrak{g}$ is again an element of $\mathfrak{g}$, namely,

$$[\Sigma_k, \Sigma_\ell] = i \sum_j C_{k\ell}^j \Sigma_j, \quad (2.5)$$

where the $C_{k\ell}^j$ are called the *structure constants* of $\mathfrak{g}$. Thus the *Lie algebra* of G is the vector space generated by the infinitesimal generators Σ_j , equipped with the bracket $[X, Y] \equiv XY - YX$, an antisymmetric bilinear map of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$. In addition, this bracket satisfies the *Jacobi identity* (which reflects the associativity of G), namely,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad \text{for all } X, Y, Z \in \mathfrak{g}. \quad (2.6)$$

Lie algebra of a Lie group: formal definition

In order to state a precise definition of the Lie algebra, we have to resort to the language of differential geometry. Let us consider the action of the Lie group G on itself by *left translation* : $L_g : G \rightarrow G$, $L_g(h) = gh$. The corresponding differential is a map between tangent bundles:

$$(dL_g)_h : T_h G \rightarrow T_{gh} G.$$

A vector field $\mathcal{X}$ on G is *left-invariant* if $dL_g \mathcal{X} = \mathcal{X}$, i.e., at $h \in G$, $(dL_g)_h \mathcal{X}(h) = \mathcal{X}(gh)$.

The central result is the following.

Theorem 2.22. *The vector space $\mathfrak{g}$ of left-invariant vector fields on a Lie group G is isomorphic to the tangent space at the neutral element: $\mathfrak{g} \simeq T_e G$.*

Examples:

- **SO(3):** the generators are J_1, J_2, J_3 , where J_k generates the rotations around the x_k -axis; they obey the commutation relations

$$[J_1, J_2] = iJ_3, [J_2, J_3] = iJ_1, [J_3, J_1] = iJ_2, \quad \Leftrightarrow \quad [J_k, J_\ell] = i\epsilon_{k\ell m} J_m,$$

where $\epsilon_{k\ell m}$ is the totally antisymmetric unit tensor.

- **SU(2):** the generators $\Sigma_1, \Sigma_2, \Sigma_3$ obey the commutation relations

$$[\Sigma_k, \Sigma_\ell] = i\epsilon_{k\ell m} \Sigma_m.$$

These are identical to those of SO(3)! In other words, the two Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic (but *not* the groups SO(3) and SU(2), as we will see below).

Note that $\Sigma_j = \frac{1}{2}\sigma_j$, the *Pauli matrices*, with commutation relations

$$[\sigma_k, \sigma_\ell] = 2i\epsilon_{k\ell m} \sigma_m,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- **SO_o(1,3),** the Lorentz group: the generators are J_n (rotations), and K_n (boosts), $n = 1, 2, 3$, with commutation relations

$$\begin{aligned} [J_\ell, J_m] &= i\epsilon_{\ell mn} J_n, \\ [J_\ell, K_m] &= i\epsilon_{\ell mn} K_n, \\ [K_\ell, K_m] &= -i\epsilon_{\ell mn} J_n. \end{aligned}$$

More generally, we list the explicit correspondence between some of the so-called classical Lie groups and their Lie algebras, given by the relation $g(t) = e^{-itX}$:

$\cdot \mathbf{SL}(n, \mathbb{R})$ $\det g = 1$	$\cdot \mathfrak{sl}(n)$ $\operatorname{tr} X = 0$
$\cdot \mathbf{SO}(n)$ $g^T g = \mathbb{I}$ $\det g = 1$	$\cdot \mathfrak{so}(n)$ $X^T + X = 0$ (antisymmetric) $\operatorname{tr} X = 0$
$\cdot \mathbf{SU}(n)$ $g^\dagger g = \mathbb{I}$ $\det g = 1$	$\cdot \mathfrak{su}(n)$ $X^\dagger = X$ (hermitian) $\operatorname{tr} X = 0$
$\cdot \mathbf{Sp}(2n)$ $g^\dagger g = \mathbb{I}$ $g^T h g = h$, with $h = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$	$\cdot \mathfrak{sp}(2n)$ $X = \begin{pmatrix} Y & Z \\ Z^T & -Y^T \end{pmatrix}$, with $Y^\dagger = Y$

For deriving the correspondences above, the following identities have been used:

$$\begin{aligned}
 (e^A)^T &= e^{A^T}, & (e^A)^\dagger &= e^{A^\dagger}, \\
 \det e^A &= e^{\operatorname{tr} A}, \\
 e^A e^B &= e^{A+B} \text{ if and only if } AB = BA.
 \end{aligned}$$

Definition 2.23. An *abstract Lie algebra* is a vector space $\mathfrak{g}$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear, antisymmetric bracket $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} : (X, Y) \mapsto [X, Y]$ satisfying the Jacobi identity:

$$[\alpha X + \beta Y, Z] = \alpha[X, Y] + \beta[Y, Z], \text{ for all } X, Y, Z \in \mathfrak{g}, \alpha, \beta \in \mathbb{K},$$

$$[X, Y] = -[Y, X], \text{ for all } X, Y \in \mathfrak{g},$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \text{ for all } X, Y, Z \in \mathfrak{g}.$$

Theorem 2.24. (Ado) Every real or complex finite dimensional Lie algebra is isomorphic to a matrix Lie algebra, where $[X, Y] = XY - YX$.

As in the case of groups, comparing Lie algebras requires a proper notion of homomorphism. The natural definition reads as follows.

Definition 2.25. Let $\mathfrak{g}, \mathfrak{g}'$ be two Lie algebras over $\mathbb{K}$. A map $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$ is a *Lie algebra homomorphism* if:

- (i) φ is linear: $\varphi(\alpha X + \beta Y) = \alpha\varphi(X) + \beta\varphi(Y)$, $X, Y \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{K}$;
- (ii) $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$, $X, Y \in \mathfrak{g}$.

It is easy to see that the *kernel* of φ , namely $\mathfrak{n} = \{X \in \mathfrak{g} : \varphi(X) = 0\}$ is an ideal of $\mathfrak{g}$, that is, $X \in \mathfrak{g}, Y \in \mathfrak{n}$ implies $[X, Y] \in \mathfrak{n}$.

As it is clear from the intuitive discussion above, the link between Lie algebras and Lie groups is the *exponential map*, that we define now in a precise way.

Given a vector field $\mathcal{V}$ on a manifold M , an *integral curve* of $\mathcal{V}$ is a curve $\gamma : (a, b) \rightarrow M$ whose tangent vector $\dot{\gamma}(t)$ at each point t coincides with the value of the vector field there:

$$\dot{\gamma}(t) = \mathcal{V}(\gamma(t)).$$

This notion applies, in particular, to elements of a Lie algebra, since these may be identified with the left-invariant vector fields at the identity, according to Theorem 2.22. Thus the following definition makes sense.

Definition 2.26. The *exponential map* $\exp : \mathfrak{g} \rightarrow G$ is the evaluation at $t = 1$ of the integral curve Φ_X of $X \in \mathfrak{g}$ satisfying $\Phi_X(0) = e$:

$$\exp(X) = \Phi_X(1).$$

The exponential $\exp : \mathfrak{g} \rightarrow G$ is a smooth map which has the following properties:

- $\exp(0) = e$
- $\exp(-X) = [\exp(X)]^{-1}$
- $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$
- $\exp$ is a diffeomorphism from an open neighborhood of $0 \in \mathfrak{g}$ onto an open neighborhood of $e \in G$
- All one-parameter subgroups of a Lie group G have the form $\exp(tX)$ for some $X \in \mathfrak{g}$.

Using these tools, Sophus Lie's theory may be summarized as follows.

Theorem 2.27. (Lie)

- (1) Every Lie group G has a unique Lie algebra $\mathfrak{g} \equiv \text{Lie}(G)$ (obtained by derivation).
- (2) Local structure: To every Lie algebra $\mathfrak{g}$ corresponds a unique local Lie group G_{loc} such that $\mathfrak{g} \simeq \text{Lie}(G_{\text{loc}})$.
- (3) Global structure: To every (real) Lie algebra $\mathfrak{g}$ corresponds a unique simply connected Lie group G such that $\mathfrak{g} \simeq \text{Lie}(G)$, obtained by the exponential map $\exp : \mathfrak{g} \rightarrow G$. More precisely, if G is simply connected and D is a discrete invariant (hence central) subgroup of G , then G and G/D have the same Lie algebra and vice versa.

Remarks : (i) In a local Lie group, group operations are defined only in a neighborhood of the identity; (ii) 'Unique' always means 'up to isomorphism'.

Let now G_1 and G_2 be two Lie groups whose Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ are isomorphic. Then, as a consequence of Theorem 2.27, either G_1 and G_2 are globally isomorphic, or G_1 and G_2 are homomorphic images of the same simply connected group $\tilde{G}$, called the *universal covering* group of G_1 and G_2 . In addition, for $j = 1, 2$, G_j is the quotient of $\tilde{G}$ by a discrete central subgroup D_j , $G_j = \tilde{G}/D_j$, and the kernel of the homomorphism $\tilde{G} \rightarrow G_j \simeq \tilde{G}/D_j$ is the homotopy group of G_j .

Thus the general situation may be represented by the following picture:

Examples:

For the classical groups that we are mostly interested in, the general scheme yields the following results.

- . $\text{SU}(n), n \geq 2$ is simply connected;
- . $\text{SO}(n), n \geq 3$, is doubly connected; its universal covering $\widetilde{\text{SO}(n)}$, called $\text{Spin}(n)$, is not a matrix group, for $n > 6$.
In particular:
 $\text{Spin}(3) = \widetilde{\text{SO}(3)} = \text{SU}(2)$, $\text{Spin}(4) = \widetilde{\text{SO}(4)} = \text{SU}(2) \times \text{SU}(2)$.
- . $\text{SO}(4) \simeq (\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2$, corresponding to $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

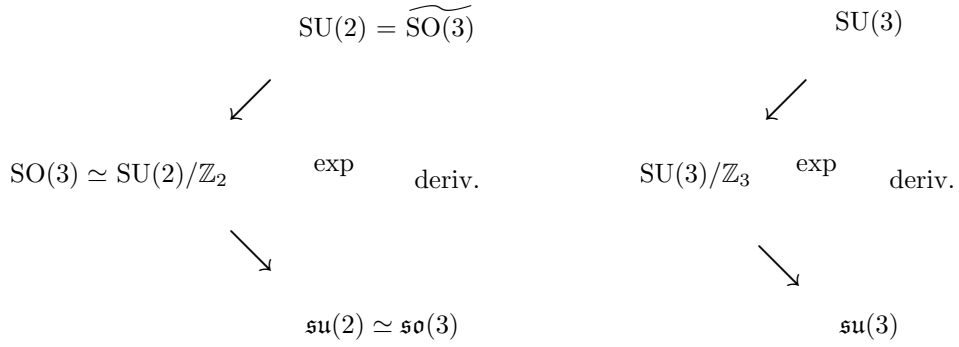
universal covering, simply connected
 $\tilde{G}$

exp

$\tilde{G}/D_1$ $\tilde{G}/D_2$ $\tilde{G}/D_n$

deriv.

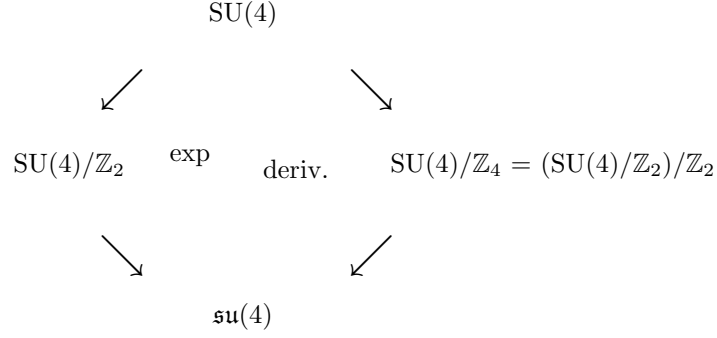
Lie algebra $\mathfrak{g}$



2.3. Simple and semisimple Lie algebras

According to Lie's theorems, there is a systematic parallelism between the properties of Lie groups and those of their Lie algebras. In particular, the following notions correspond to each other.

- . $H \subset G$: Lie subgroup $\iff \mathfrak{h} \subset \mathfrak{g}$: Lie subalgebra
- . $H \in G$: invariant subgroup $\iff \mathfrak{h} \in \mathfrak{g}$: ideal



- *G simple*: no nontrivial invariant subgroup
 $\iff \mathfrak{g}$ *simple* Lie algebra: no nontrivial ideal
- *G semisimple*: no nontrivial abelian invariant subgroup
 $\iff \mathfrak{g}$ *semisimple* Lie algebra: no nontrivial abelian ideal

Thus we get the following equivalence for semisimple Lie groups:

$$G_{\text{ss}} = (G_s^1 \times \cdots \times G_s^n) / D \iff \mathfrak{g}_{\text{ss}} = \mathfrak{g}_s^1 \oplus \cdots \oplus \mathfrak{g}_s^n$$

(“s” $\equiv$ simple, “ss” $\equiv$ semisimple)

The next step is to find a criterion of semisimplicity for a given Lie algebra $\mathfrak{g}$. This is achieved in terms of the so-called *Killing form*, which defines a metric on $\mathfrak{g}$. First we define the *adjoint representation* of $\mathfrak{g}$:

$$X \mapsto \text{ad } X, \text{ ad } X(Y) = [X, Y]. \quad (2.7)$$

Let $\{e_i, i \in I\}$ be a basis of $\mathfrak{g}$ and $X, Y \in \mathfrak{g}$, $X = x^l e_l, Y = y^k e_k$. Then we have

$$(\text{ad } X(Y))^i = [X, Y]^i = C_{lk}^i x^l y^k, \quad i \in I, \text{ i.e., } (\text{ad } X)_k^i = C_{lk}^i x^l,$$

where $\{C_{lk}^i\}$ are called the structure constants of $\mathfrak{g}$.

Definition 2.28. The *Killing form* of $\mathfrak{g}$ is the bilinear symmetric form (scalar product) given by

$$(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y). \quad (2.8)$$

In coordinates:

$$(X, Y) = ((\text{ad } X)_k^i (\text{ad } Y)_i^s) = C_{lk}^i x^l C_{si}^k y^s = g_{ls} x^l y^s,$$

where the second order symmetric tensor $g_{ls} = C_{lk}^i C_{si}^k$ is called the *Cartan metric* of $\mathfrak{g}$.

It is easy to check that the Killing form is invariant under an automorphism ψ of $\mathfrak{g}$: $(\psi(X), \psi(Y)) = (X, Y)$.

Theorem 2.29. (Cartan's criterion) *The Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is regular: $\det g_{ki} \neq 0$.*

Thus the Killing form turns $\mathfrak{g}$ into a metric space (scalar product). A fundamental result of Cartan is the following.

Theorem 2.30. *A semisimple Lie group G is compact if and only if the Killing form of its real Lie algebra $\mathfrak{g}$ is negative definite.*

In that case, the Lie algebra $\mathfrak{g}$ is also said to be *compact* and $\mathfrak{g}$ is a Euclidean space. Take for instance the group $\mathrm{SO}(3)$, which is simple and compact:

$$[X_i, X_j] = \sum_k \varepsilon_{ijk} X_k \Rightarrow g_{im} = \sum_{j,k} \varepsilon_{ijk} \varepsilon_{mkj} = -2\delta_{im}.$$

As we have seen above, every semisimple Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras:

$$\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i, \text{ with } [\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i, \mathfrak{g}_i \text{ simple and } [\mathfrak{g}_i, \mathfrak{g}_j] = 0 \text{ if } i \neq j.$$

Example : $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \Leftrightarrow \mathrm{SO}(4) \simeq [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2$

Thus, in order to classify the *semisimple* Lie algebras, it suffices to classify all the *simple* Lie algebras. This has been done by Cartan in his thesis (1894), another masterpiece of mathematics. We will now sketch this fundamental result.

Cartan's method of classification consists in choosing a standard basis in the Lie algebra and translating its properties in graphical terms, which results in the so-called *root diagram*. Then the latter can be completely classified. To that effect, one considers first a *complex* Lie algebra $\mathfrak{g}$ (i.e., with complex parameters) and one solves the eigenvalue equation

$$\mathrm{ad}A(X) = aX, \quad \text{i.e.,} \quad [A, X] = aX, \quad a \in \mathbb{C}. \quad (2.9)$$

Then the results may be summarized as follows.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $A \in \mathfrak{g}$ an element with a maximal number of distinct eigenvalues. Then:

(1) 0 is the only degenerate eigenvalue; its multiplicity l is called the *rank* of $\mathfrak{g}$.

(2) Choose l independent eigenvectors associated to 0, H_i , $i = 1, 2, \dots, l$:

$$\mathrm{ad}A(H_i) = [A, H_i] = 0.$$

Since $[A, A] = 0$, $A = \sum_i c_i H_i$ and one can choose $A = H_1$. The abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by $\{H_i, i = 1, 2, \dots, l\}$ is called a *Cartan subalgebra*.

(3) Let E_α be an eigenvector of H_1 associated to the (nondegenerate) eigenvalue $\alpha_1 : [H_1, E_\alpha] = \alpha_1 E_\alpha$. Using the Jacobi identity, one gets

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad i = 1, 2, \dots, l.$$

Therefore, all the H_i are diagonalized simultaneously. Denote by $\alpha_1, \alpha_2, \dots, \alpha_l$ the eigenvalues of the common eigenvector E_α . The vector $\alpha \equiv \{\alpha_i\}_{i=1, \dots, l} \in \mathbb{R}^l$ is called a *root vector* and the set of all roots is called the *root diagram* of $\mathfrak{g}$. Thus the root diagram is a set of vectors in $\mathbb{R}^l$, where l is the rank of $\mathfrak{g}$.

In the standard (Cartan) basis $\{H_i, i = 1, 2, \dots, l; E_\alpha\}$ of $\mathfrak{g}$ (complexified), the commutation relations take the form:

$$\begin{aligned} [H_i, H_j] &= 0, \quad i, j = 1, 2, \dots, l, \\ [H_i, E_\alpha] &= \alpha_i E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= \alpha^i H_i, \\ [E_\alpha, E_\beta] &= \begin{cases} N_{\alpha\beta} E_{\alpha+\beta}, & \text{if } (\alpha + \beta) \text{ is a nonzero root,} \\ 0, & \text{if } (\alpha + \beta) \text{ is not a root.} \end{cases} \end{aligned} \quad (2.10)$$

The set of roots has the following properties:

- . $k\alpha$ is a root if and only if $k = \pm 1$.
- . The set of all roots is invariant with respect to the reflection in the hyperplane perpendicular to the pair $\pm\alpha$.
- . All these reflections generate a finite group W , called the *Weyl group*.

Example 1 : $\mathfrak{so}(3)$, with rank 1 and commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = J_3, \quad \text{where } J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2).$$

Root diagram:

$$\begin{array}{ccc} J_- & & J_+ \\ & J_3 & \\ & \bullet & \end{array}$$

Figure 1. The root diagram of $SU(2)$.

Example 2 : $\mathfrak{su}(3) \equiv A_2$, of rank 2

The root diagram consists of six nonzero roots, the tips of which draw a regular hexagon. The Weyl group is isomorphic to the 6-element permutation group S_3 , generated by reflections with respect to the 3 lines orthogonal to roots (dashed lines).¹¹

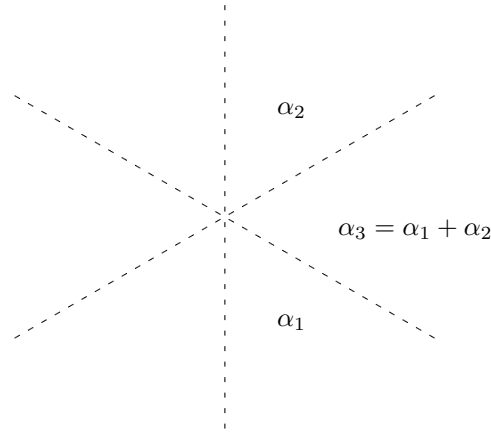


Figure 2. The root diagram of SU(3) (from Ref. 11).

The root diagram has the following further properties:

(i) For any pair of roots α, β :

- the ratio $\frac{2(\alpha\beta)}{(\alpha\alpha)}$ is a positive integer
- $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha$ is a root (reflection of β with respect to the line perpendicular to $\pm\alpha$)

(ii) If $\theta_{\alpha\beta}$ is the angle between the roots α, β , one has

$$\cos^2 \theta_{\alpha\beta} = \frac{(\alpha\beta)^2}{(\alpha\alpha)(\beta\beta)} = \frac{mn}{4}, \quad m, n \in \mathbb{N}^+.$$

As a consequence, only a few values are allowed for the angles $\theta_{\alpha\beta}$ and the ratio of lengths, as follows:

$\theta_{\alpha\beta}$	:	30°	45°	60°	90°	120°	135°	150°	180°
Ratio of lengths :		$\sqrt{3}$	$\sqrt{2}$	1	arbitrary	1	$\sqrt{2}$	$\sqrt{3}$	1

Classification of simple Lie algebras. Dynkin diagrams

One can define an order relation on root systems:

- . $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ is *positive* if the first nonzero component is positive (lexicographic order),
- . $\beta > \alpha$ if $\beta - \alpha > 0$.

A root is said to be *simple* if it is positive and cannot be decomposed into the sum of two positive roots. Then:

- . If $\mathfrak{g}$ has rank l , there exist l linearly independent simple roots (i.e., simple roots are a basis of $\mathbb{R}^l$),
- . If α, β are simple roots, the angle $\theta_{\alpha\beta}$ takes only the values $90^\circ, 120^\circ, 135^\circ, 150^\circ$.

These facts are encoded graphically in terms of the so-called *Dynkin diagrams*. The principles are the following:

- . Each simple root is represented by a small circle.
- . The number of links between two circles is 0,1,2 or 3 whenever the angle between the corresponding roots is $90^\circ, 120^\circ, 135^\circ$ or 150° , respectively.

Example:

Three simple roots r_1, r_2, r_3 with angles $(r_1, r_2) = 120^\circ$, $(r_2, r_3) = 135^\circ$, $(r_1, r_3) = 90^\circ$:

$r_1 \quad r_2 \quad r_3$

- . Closed loops are forbidden:

- . Each circle can support at most three links, so that a connection like the following one is forbidden:

- . Simple roots can have two different lengths only, one uses white disks for the short ones, black disks for the long ones.

The result of Cartan's analysis is that there exists four infinite series of simple complex Lie algebras, corresponding to classical groups, plus five *exceptional* algebras (with no associated classical group). The four infinite series are A_l ($l \geq 1$), B_l ($l \geq 2$), C_l ($l \geq 3$) and D_l ($l \geq 4$), where l denotes the rank of the algebra and the restrictions on l guarantee that all algebras are different. Indeed, there are some isomorphisms for the lower ranks:

$$A_1 \simeq B_1 \simeq C_1, B_2 \simeq C_2, A_3 \simeq D_3, D_2 \simeq A_1 \oplus A_1. \quad (2.11)$$

The five exceptional algebras are denoted G_2, F_4, E_6, E_7 and E_8 . All the simple Lie algebras are listed in Table 2.3, together with their Dynkin diagrams.

The next step is to list the *real forms* of the simple Lie algebras and the corresponding Lie groups. Let $\mathfrak{g}_0$ be a real Lie algebra. Its *complexification* is the complex Lie algebra $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ consisting of all elements of the form $X + iY$, $X, Y \in \mathfrak{g}_0$, the bracket being extended by linearity. Conversely, a real form of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_1$ such that $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_1$. Of course, a complex Lie algebra may have several nonisomorphic real forms. There is one fundamental restriction, however.

Theorem 2.31. *Every semisimple complex Lie algebra has a real form which is compact.*

Starting from this compact real form, one may now obtain a noncompact one, as follows. The tool is the notion of *involutive automorphism* of the compact Lie algebra $\mathfrak{g}$, that is, an automorphism σ of $\mathfrak{g}$ such that $\sigma^2 = I$. Such a map σ has eigenvalues ± 1 and it splits $\mathfrak{g}$ into eigensubspaces: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where the eigenspace $\mathfrak{k}$ corresponds to the eigenvalue $+1$, i.e., it is the set of fixed points of σ . The commutation relations of $\mathfrak{g}$ are the following:

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subseteq \mathfrak{k}, & (\text{thus } \mathfrak{k} \text{ is a subalgebra}) \\ [\mathfrak{k}, \mathfrak{p}] &= \mathfrak{p}, \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq \mathfrak{k}. \end{aligned}$$

Abstract Lie algebra	Real form of corresponding Lie group	Dynkin diagram
$A_l \ (l \geq 1)$	$SU(l+1)$ or $SU(p, q), p+q = l+1$	-----
$B_l \ (l \geq 2)$	$SO(2l+1)$ or $SO(p, q), p+q = 2l+1$	-----
$C_l \ (l \geq 3)$	$Sp(2l)$ or $Sp(p, q), p+q = 2l$	-----
$D_l \ (l \geq 4)$	$SO(2l)$ or $SO(p, q), p+q = 2l$	-----
F_4		
G_2		
E_6		
E_7		
E_8		

Table 2.3. Dynkin diagrams for the simple Lie algebras.

Then the Lie algebra $\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{p}^*$, where $\mathfrak{p}^* = i\mathfrak{p}$, is another real form of $\mathfrak{g}^{\mathbb{C}}$, and it is noncompact (this construction is called *Weyl's unitary trick*). The commutation relations of $\mathfrak{g}^*$ read:

$$\begin{aligned} [\mathfrak{k}, \mathfrak{k}] &\subseteq \mathfrak{k}, \quad (\mathfrak{k} \text{ is still a subalgebra}) \\ [\mathfrak{k}, \mathfrak{p}^*] &= \mathfrak{p}^*, \\ [\mathfrak{p}^*, \mathfrak{p}^*] &\subseteq -\mathfrak{k}. \end{aligned}$$

Thus classifying the real forms amounts to classify the involutive automorphisms, and again the result was obtained by Cartan. For instance, the complex Lie algebra B_2 has three different real forms, namely, $\mathfrak{so}(5)$, which is the compact one, $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$, which are noncompact. All this extends to the corresponding Lie groups. In the example of B_2 , one gets $\mathrm{SO}(5)$, which is compact, $\mathrm{SO}(4,1)$, the de Sitter group, and $\mathrm{SO}(3,2)$, the Anti-de Sitter group, which are both noncompact.

For each nonexceptional Lie algebra, we list in Table 2.3 the real forms of the corresponding classical Lie groups. For each of them, there is one compact form ($\mathrm{SU}(l+1)$, $\mathrm{SO}(2l+1)$, $\mathrm{Sp}(2l)$, $\mathrm{SO}(2l)$) and several noncompact forms (for $l \geq 3$). The isomorphisms of the low-rank Lie algebras (2.11) in turn entail local isomorphisms for the corresponding groups. For instance, $\mathrm{SU}(2) \simeq \mathrm{SO}(3) \simeq \mathrm{Sp}(2)$ or $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ locally (but not always globally).

2.4. Integration on a locally compact group

The main advantage of locally compact groups is that they allow a theory of integration, which will prove crucial in a number of situations. Let G be a locally compact group, in particular a Lie group. Then one defines:

. A *left invariant measure* on G , that is, a measure μ_L on G which satisfies the following relation for any μ_L -integrable function f :

$$\int_G f(g_0 g) d\mu_L(g) = \int_G f(g) d\mu_L(g), \text{ for all } g_0 \in G, \quad (2.12)$$

or, equivalently,

$$d\mu_L(g_0^{-1}g) = d\mu_L(g),$$

or

$$\mu_L(g_0^{-1}E) = \mu_L(E), \text{ for every Borel set } E \text{ of } G.$$

. A right invariant measure μ_R on G :

$$\int_G f(gg_0) d\mu_R(g) = \int_G f(g) d\mu_R(g), \quad (2.13)$$

$$d\mu_R(gg_0^{-1}) = d\mu_R(g),$$

$$\mu_R(Eg_0^{-1}) = \mu_R(E), \text{ for every Borel set } E \text{ of } G.$$

Then the fundamental result of Haar is the following.

Theorem 2.32. *Up to normalization, every locally compact group possesses a unique left invariant measure μ_L and a unique right invariant measure μ_R . These two measures, called Haar measures, are equivalent.*

Remarks:

(1) Two measures are called equivalent if they have the same sets of measure zero.

(2) If μ is a left invariant measure, then $\tilde{\mu}$, image of μ by the homeomorphism $g \mapsto g^{-1}$, is a right invariant measure and vice-versa:

$$\langle f, \tilde{\mu} \rangle = \langle \tilde{f}, \mu \rangle, \quad \tilde{f}(g) = f(g^{-1}).$$

Since $\mu_L \simeq \mu_R$, there exists a continuous function $\Delta : G \rightarrow \mathbb{R}^+$, called the *modular function*, such that:

$$d\mu_L(g) = \Delta(g) d\mu_R(g). \quad (2.14)$$

The modular function has the following properties:

- (1) $\Delta(g) > 0$, for all $g \in G$,
- (2) $\Delta(e) = 1$,
- (3) $\Delta(g_1)\Delta(g_2) = \Delta(g_1g_2)$, for all $g_1, g_2 \in G$.

In other words, Δ is a character of the group G . One has also:

$$\begin{aligned} d\mu_R(g) &= \Delta(g^{-1}) d\mu_L(g) = d\mu_L(g^{-1}), \\ d\mu_L(gg') &= \Delta(g') d\mu_L(g). \end{aligned}$$

The group G is said to be *unimodular* if $\Delta(g) = 1$, for all $g \in G$, i.e., $\mu_L = \mu_R$.

Examples of unimodular groups:

- . Abelian groups
- . Compact groups

- . Simple and semisimple groups
- . Inhomogeneous groups: $E(3)$, $\mathcal{P}(1,3)$, ...
- . Discrete groups.

Examples of nonunimodular groups:

- . The affine group of $\mathbb{R}$: $\{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}, a \neq 0\}$, i.e., $\mathbb{R} \rtimes \mathbb{R}_*$
- . The $ax + b$ subgroup of the affine group ($a > 0$), i.e., $\mathbb{R} \rtimes \mathbb{R}_*^+$
- . The similitude group $\text{SIM}(n) = \mathbb{R}^n \rtimes (\mathbb{R}_*^+ \times \text{SO}(n))$.

The Haar measures provide an easy criterion for compactness of the group: G is compact if and only if $\text{vol } G < \infty$, where

$$\text{vol } G = \int_G d\mu_L(g) = \int_G d\mu_R(g).$$

Examples :

- . $\text{SO}(n), \text{SU}(n)$ are compact
- . $\text{SO}(p, q), \text{SU}(p, q), \mathbb{R}^n, \mathcal{P}(1, 3)$ are noncompact.

Note: a similar discussion may be done for measures on homogeneous spaces $X = G/H$, but there is an essential difference. Indeed, a homogeneous space does not always admit an invariant measure, some (known) criteria have to be satisfied. However, it always admits a *quasi-invariant* measure, i.e., a measure equivalent (but not equal) to its translates.

3. Mathematical tools II: Representations

3.1. Basic notions

Definition 3.1. A *linear representation* of a group G in a vector space V is a homomorphism $T : G \rightarrow \text{GL}(V)$, where $\text{GL}(V)$ denotes the set of all invertible linear operators on V :

$$T(g_1 g_2) = T(g_1) T(g_2), \text{ for all } g_1, g_2 \in G.$$

It follows that $T(g^{-1}) = T(g)^{-1}$ and $T(e) = \mathbb{I}$. The *dimension* of T is defined as the dimension of V .

The most useful case is that where V is a Hilbert space $\mathfrak{H}$ and the operators $T(g)$ are bounded. Then T is a homomorphism of G into $GL(\mathfrak{H})$, the set of bounded operators with bounded inverse.

If G is a Lie group, T is called *strongly continuous* if

$$\|(T(g) - \mathbb{I})\phi\| \rightarrow 0, \text{ when } g \rightarrow e, \text{ for all } \phi \in \mathfrak{H}. \quad (3.1)$$

Definition 3.2. When $\mathfrak{H}$ is a Hilbert space, the representation T is called *unitary* if $T(g)$ is a unitary operator for every $g \in G$, i.e., $T(g^{-1}) = T(g)^{-1} = T(g)^*$:

$$\langle T(g)f | T(g)h \rangle = \langle f | h \rangle, \text{ for all } g \in G, \text{ for all } f, h \in \mathfrak{H}. \quad (3.2)$$

Examples :

$$\cdot G = \text{SO}(2), \mathfrak{H} = L^2(S^1)$$

$$[T(\psi)f](\varphi) = f(\psi + \varphi), \quad f \in L^2(S^1), \psi \in \text{SO}(2), \varphi \in S^1.$$

$$\cdot G = \text{SO}(3), \mathfrak{H} = L^2(S^2),$$

$$[T(g)f](\omega) = f(g^{-1}\omega), \quad f \in L^2(S^2), g \in \text{SO}(3), \omega \in S^2.$$

Both representations are unitary and infinite dimensional.

When G is locally compact, one can define the Hilbert spaces of functions which are square integrable with respect to the left or the right Haar measures. Accordingly, one defines the two *regular* representations:

Left regular representation:

$$[U_L(g_0)f](g) = f(g_0^{-1}g), \quad g_0, g \in G, f \in \mathfrak{H} = L^2(G, d\mu_L). \quad (3.3)$$

Right regular representation:

$$[U_R(g_0)f](g) = f(gg_0), \quad g_0, g \in G, f \in \mathfrak{H} = L^2(G, d\mu_R). \quad (3.4)$$

U_L and U_R are unitary and unitarily equivalent.

Let now H be a closed subgroup of G such that $X = G/H$ possesses a left invariant measure μ . Then one can define the unitary *quasi-regular* representation:

$$[U_{qL}(g)f](x) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}} f(g^{-1}x), \quad f \in \mathfrak{H} = L^2(X, d\mu_L). \quad (3.5)$$

In these expressions, the square root factor is a Radon–Nikodym derivative, which compensates for the non-invariance of the measure.

Examples :

$$\cdot G = \text{SO}(3), H = \text{SO}(2), X = S^2.$$

. $G = \text{SO}_o(1,3)$, $H = \text{SO}(3)$, $X = \text{SO}_o(1,3)/\text{SO}(3) =$ two-sheeted hyperboloid.

Definition 3.3. Two representations, T_1 and T_2 in the Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$, respectively, are *equivalent* if there exists an invertible operator $S : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ such that

$$T_2(g) = ST_1(g)S^{-1}, \text{ for all } g \in G. \quad (3.6)$$

In this case, we note $T_1 \sim T_2$. Two unitary representations T_1, T_2 are *unitarily equivalent* if the operator S in (3.6) is unitary. The set of unitary equivalence classes of unitary irreducible representations of G is called the *dual* of G and is denoted by $\widehat{G}$.

Definition 3.4. If T is finite dimensional, one calls *character* of T the complex-valued function χ_T on G given by the trace of the matrix $T(g)$:

$$\chi_T(g) = \text{Tr } T(g) = \sum_{i=1}^{\dim T} [T(g)]_{ii}.$$

Clearly, equivalent representations have the same character:

$$\chi_{STS^{-1}}(g) = \text{Tr}(ST(g)S^{-1}) = \text{Tr } T(g) = \chi_T(g), \quad g \in G.$$

3.2. Irreducibility of representations

Given a representation T in the Hilbert space $\mathfrak{H}$, a subspace $\mathfrak{H}_1 \subset \mathfrak{H}$ is said to be *invariant* for T if $h \in \mathfrak{H}_1$ implies $T(g)h \in \mathfrak{H}_1$, for all $g \in G$.

Proposition 3.5. Let T be a unitary representation in $\mathfrak{H}$, $\mathfrak{H}_1$ a subspace of $\mathfrak{H}$, and P the orthogonal projection onto $\mathfrak{H}_1$. Then:

- (1) $\mathfrak{H}_1^\perp$ is invariant if and only if $\mathfrak{H}_1$ is invariant.
- (2) $\mathfrak{H}_1$ is invariant if and only if $PT(g) = T(g)P$, for all $g \in G$.

Definition 3.6. The representation T in $\mathfrak{H}$ is called *irreducible* if there exists no nontrivial invariant subspace in $\mathfrak{H}$ (i.e., different from $\{0\}$ or $\mathfrak{H}$). Otherwise, T is called *reducible*.

Let $\mathfrak{H}_1$ be an invariant subspace for T , $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp$. Then one may write

$$T(g) = \begin{bmatrix} T_1(g) & A(g) \\ 0 & T_2(g) \end{bmatrix},$$

where

$$T_1(g) : \mathfrak{H}_1 \rightarrow \mathfrak{H}_1, \quad T_2(g) : \mathfrak{H}_1^\perp \rightarrow \mathfrak{H}_1^\perp, \quad A(g) : \mathfrak{H}_1^\perp \rightarrow \mathfrak{H}_1.$$

One says that T is *completely reducible* if $\mathfrak{H}_1^\perp$ is invariant whenever $\mathfrak{H}_1$ is invariant. In that case, one gets

$$A(g) = 0, \quad \text{and} \quad T(g) = \left[\begin{array}{c|c} T_1(g) & 0 \\ \hline 0 & T_2(g) \end{array} \right].$$

Then one writes $T = T_1 \oplus T_2$, acting in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ (direct sum).

If T is reducible, but not completely reducible, it is called *indecomposable*.

Example: $G = \mathbb{R}$, $\mathfrak{H} = \mathbb{R}^2$, $T(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ is indecomposable:

$\begin{bmatrix} \mathbb{R} \\ 0 \end{bmatrix}$ is invariant, $\begin{bmatrix} 0 \\ \mathbb{R} \end{bmatrix}$ is not.

Theorem 3.7. *Every unitary reducible representation is completely reducible.*

Proof : Let T_1, T_2 be the restriction of T to the subspaces $\mathfrak{H}_1, \mathfrak{H}_2$. Then $\mathfrak{H}_1$ invariant means

$$T(g) = \left[\begin{array}{c|c} T_1(g) & A(g) \\ \hline 0 & T_2(g) \end{array} \right].$$

By the unitarity of $T(g)$, this gives:

$$T(g)^\dagger = \left[\begin{array}{c|c} T_1(g)^\dagger & 0 \\ \hline A(g)^\dagger & T_2(g)^\dagger \end{array} \right] = T(g^{-1}) = \left[\begin{array}{c|c} T_1(g^{-1}) & A(g^{-1}) \\ \hline 0 & T_2(g^{-1}) \end{array} \right].$$

Thus $A = 0$ and T_1, T_2 are unitary. □

If $\dim T < \infty$, the decomposition can be repeated until exhaustion. Thus one gets the following result:

Corollary 3.8. *Every finite dimensional unitary representation is a direct sum of unitary irreducible representations (UIRs).*

3.3. Schur's lemma and its generalizations

Lemma 3.9. (Schur's lemma: classical) *Let T_1, T_2 be two UIRs of G in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and $A : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ an intertwining operator:*

$$AT_1(g) = T_2(g)A, \text{ for all } g \in G.$$

Then, either $A = 0$, or A is invertible and $T_1 \sim T_2$. In the second case, if $\dim T_i < \infty$, A is unique up to a scalar.

Corollary 3.10. *Let T be a finite dimensional unitary representation and $AT(g) = T(g)A$, for all $g \in G$. Then $A = \lambda \mathbb{I}$.*

Given a representation T in the Hilbert space $\mathfrak{H}$, its *commutant* is the set of bounded operators that commute with every $T(g)$, $g \in G$:

$$T' = \{A \in \mathcal{B}(\mathfrak{H}) : AT(g) = T(g)A, \text{ for all } g \in G\}.$$

Corollary 3.11. *Let T be unitary. Then T is irreducible if and only if its commutant T' is trivial, i.e., $T' = \{\lambda \mathbb{I}, \lambda \in \mathbb{C}\}$.*

Corollary 3.12. *If G is abelian, every UIR of G has dimension 1.*

Proof : $T(g)T(g') = T(gg') = T(g'g) = T(g')T(g)$, that is, $T(g) \in T'$. By Corollary 3.11, $T(g) = \lambda(g)\mathbb{I}$ with $|\lambda(g)| = 1$, i.e., $\dim T = 1$. $\square$

Example: $G = \text{SO}(2)$

$$g(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \in \text{SO}(2), \quad g(2\pi) = g(0) = \mathbb{I}.$$

Every UIR of $\text{SO}(2)$ is of the form $T_k(g(\varphi)) = e^{ik\varphi}$, $k \in \mathbb{Z}$. Indeed, one has $T(g(\varphi)) = \lambda(\varphi)\mathbb{I}$, since $\text{SO}(2)$ is abelian. Thus

$$T(g(\varphi_1))T(g(\varphi_2)) = T(g(\varphi_1)g(\varphi_2)) = T(g(\varphi_1 + \varphi_2)).$$

Therefore one gets

$$\lambda(\varphi_1)\lambda(\varphi_2) = \lambda(\varphi_1 + \varphi_2),$$

whose only *continuous* solutions are $\lambda(\varphi) = e^{ik\varphi}$, $k \in \mathbb{R}$. Finally, the condition $T(g(2\pi)) = e^{ik2\pi} = \mathbb{I}$ implies that $k \in \mathbb{Z}$.

This is equivalent to the theory of *Fourier series*! Indeed, an arbitrary function $f \in L^2(S^1)$ may be expanded in a Fourier series:

$$f(\varphi) = \sum_{k=-\infty}^{\infty} c_k e^{ik\varphi}, \quad (3.7)$$

so that

$$L^2(S^1) = \bigoplus_{k=-\infty}^{\infty} \mathfrak{H}_k, \quad \dim \mathfrak{H}_k = 1. \quad (3.8)$$

The regular representation of $\text{SO}(2)$ acts in $L^2(S^1)$ and reads:

$$[U_L(\psi)f](\varphi) = f(\varphi + \psi) = \sum_{k=-\infty}^{\infty} c_k e^{ik(\varphi+\psi)} = \sum_{k=-\infty}^{\infty} c_k T_k(g(\psi)) e^{ik\varphi},$$

i.e., (3.8) corresponds to the decomposition of U_L into 1-dimensional UIRs:

$$U_L = \bigoplus_{k=-\infty}^{\infty} T_k.$$

Lemma 3.13. (Schur's lemma: general) *Let U_1 be a UIR in $\mathfrak{H}_1$, U_2 a unitary representation in $\mathfrak{H}_2$, and $T : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ a bounded operator that intertwines U_1 and U_2 . Then either $T \equiv 0$, or T is a multiple of an isometry, i.e., there exists a constant $\lambda > 0$ such that*

$$\|T\phi\|_{\mathfrak{H}_2}^2 = \lambda \|\phi\|_{\mathfrak{H}_1}^2, \text{ for all } \phi \in \mathfrak{H}_1.$$

Proof: From the hypotheses, we have

$$T^* T U_1(g) = T^* U_2(g) T = U_1(g) T^* T, \text{ for all } g \in G.$$

Then by Schur's classical lemma 3.9, either $T = 0$ or $T^* T = \lambda \mathbb{I}$. $\square$

Lemma 3.14. (Schur's lemma: extended) *Let U be a UIR of G in the Hilbert space $\mathfrak{H}$ and U' a unitary representation in $\mathfrak{H}'$. Let $T : \mathfrak{H} \rightarrow \mathfrak{H}'$ be a closed linear operator with dense, U -invariant domain $D(T)$, that intertwines U and U' .*

Then either $T \equiv 0$, or T is a multiple of an isometry (hence bounded).

3.4. Representations of Lie algebras

Throughout this section, we assume that T is a finite dimensional representation of G .

Consider first the case of a one-parameter group, namely, $\text{SO}(2)$:

$$g(\varphi) = e^{-i\sigma\varphi}, \quad \sigma = \text{infinitesimal generator (see Sec. 2.2).}$$

As before,

$$T(g(\varphi)) = e^{-i\varphi T(\sigma)}, \text{ with } T(\sigma) = \lim_{\varphi \rightarrow 0} \frac{T(g(\varphi)) - \mathbb{I}}{-i\varphi}.$$

Since T is unitary, it follows that $T(\sigma) = T(\sigma)^\dagger$, that is, $T(\sigma)$ is a hermitian matrix.

In the general case, one gets

$$g(x) = e^{-i \sum_j x_j \sigma_j}, x = (x_j), \sigma_j \in \mathfrak{g} = \text{Lie algebra of } G.$$

We could take, for instance, $SU(2)$ or $SO(3)$, as discussed in Section 2.2.

For each one-parameter subgroup $x_j \mapsto g(x_j)$, one has

$$T(g(x_j)) = e^{-ix_j T(\sigma_j)}.$$

The image of the commutator of two such one-parameter subgroups $g(x_1), g(x_2)$ is the operator

$$T(g_1 g_2 g_1^{-1} g_2^{-1}) = T(g_1) T(g_2) T(g_1^{-1}) T(g_2^{-1}),$$

where, for simplicity, we have written $g_j \equiv g(x_j)$, $j = 1, 2$.

To second order, this gives:

$$\begin{aligned} T(g_j) &= T(e^{-ix_j \sigma_j}) = \mathbb{I} - ix_j T(\sigma_j) + \frac{1}{2}(-ix_j T(\sigma_j))^2 + \dots \\ T(g_j^{-1}) &= T(e^{ix_j \sigma_j}) = \mathbb{I} + ix_j T(\sigma_j) + \frac{1}{2}(+ix_j T(\sigma_j))^2 + \dots \end{aligned}$$

Thus:

$$\begin{aligned} T([\sigma_1, \sigma_2]) &= T(\sigma_1) T(\sigma_2) - T(\sigma_2) T(\sigma_1) \\ &= [T(\sigma_1), T(\sigma_2)], \end{aligned}$$

where $[\sigma_1, \sigma_2]$ denotes the Lie bracket in $\mathfrak{g}$ and $[T(\sigma_1), T(\sigma_2)]$ the commutator of the two matrices.

Thus the representation T induces a homomorphism of $\mathfrak{g}$ into the Lie algebra of hermitian matrices, i.e., a linear map $\sigma \mapsto T(\sigma)$ such that

$$T([\sigma_1, \sigma_2]) = [T(\sigma_1), T(\sigma_2)].$$

in other words, a *Lie algebra representation*. In particular, one has $T(0) = 0$.

3.5. Representations of compact groups

Fundamental results

In this section, G denotes a compact topological group with normalized invariant Haar measure dg :

$$\int_G dg = 1 \quad \text{and} \quad \int_G f(g) dg = \int_G f(g^{-1}) dg.$$

T denotes a *strongly continuous* representation of G in the Hilbert space $\mathfrak{H}$, as defined in (3.1). By the principle of uniform boundedness, this implies that T is *bounded*, in the sense that there exists $M > 0$ such that

$$\|T(g)\| \leq M, \text{ for all } g \in G.$$

The main tool for analyzing the UIRs is the integration over G . Their properties may be summarized in the following proposition.

Proposition 3.15. *Let T be a strongly continuous representation of the compact group G in the Hilbert space $\mathfrak{H}$. Then*

- (1) *There exists on $\mathfrak{H}$ a new scalar product, defining a norm equivalent to the initial one, with respect to which T is unitary.*
- (2) *Every UIR of G is finite dimensional.*
- (3) *Every unitary representation of G in a Hilbert space $\mathfrak{H}$ is the direct sum of finite dimensional UIRs.*

Examples : (1) *Regular representation of G , in $\mathfrak{H} = L^2(G, dg)$:*

$$U_L = \bigoplus_{[U_j] \in \widehat{G}} \underbrace{U_j \oplus \cdots \oplus U_j}_{\dim U_j \text{ times}}.$$

Note that *all* elements of $\widehat{G}$ occur in the sum, with a multiplicity equal to their dimension. Thus every UIR of a compact group is a subrepresentation of the regular representation (this is a characteristic property of *square integrable* representations, see Section 3.6 below).

(2) *Quasi-regular representation of G , in $L^2(G/H, d\mu)$, where H is the maximal compact subgroup of G :*

$$U_{qL} = \bigoplus_{[U_j] \in \widehat{G}} U_j.$$

Here too, every UIR occurs in the sum, but only once.

(3) For $\text{SO}(3)$, the UIRs are the well-known representations $U_j \equiv D_j$ of dimension $2j+1$, generated by spherical harmonics. Thus,

$$L^2(S^2) = \bigoplus_{j=0}^{\infty} D_j, \quad L^2(\text{SO}(3)) = \bigoplus_j \underbrace{(D_j \oplus \cdots \oplus D_j)}_{(2j+1) \text{ times}}.$$

Orthogonality relations

Let T_1, T_2 be two UIRs of a compact group G in Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$, respectively. Their matrix elements obey the following *orthogonality relations*:

$$\int_G dg \langle T_1(g)u_1 | v_1 \rangle \overline{\langle T_2(g)u_2 | v_2 \rangle} = \begin{cases} 0, & \text{if } T_1 \not\sim T_2, \\ \frac{1}{d} \langle Vu_1 | u_2 \rangle \overline{\langle Vv_1 | v_2 \rangle}, & \text{if } T_1 \sim T_2, \end{cases}$$

for any $u_1, v_1 \in \mathfrak{H}_1$, $u_2, v_2 \in \mathfrak{H}_2$.

In the second case, T_1 and T_2 are unitarily equivalent: $T_2 = VT_1V^{-1}$, with $V : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ unitary, and $d = \dim T_1 = \dim T_2$.

Equivalently, in terms of matrix elements in suitable orthonormal bases, the orthogonality relations read:

$$\int_G dg (T_1)_{ij}(g) \overline{(T_2)_{kl}(g)} = \begin{cases} 0, & \text{if } T_1 \not\sim T_2, \\ \frac{1}{d} \delta_{ik} \delta_{jl}, & \text{if } T_1 \sim T_2. \end{cases}$$

In fact, these matrix elements are not only orthogonal, but they also constitute a basis of $L^2(G, dg)$, according to the famous Peter–Weyl theorem:

Theorem 3.16. (Peter–Weyl) *Let G be a compact group and $\widehat{G}$ its dual. For each $s \in \widehat{G}$, choose a unique unitary matrix representation $U_s = (U_{ij}^{(s)})$ of dimension d_s . Then the family*

$$\mathcal{B} = \{\sqrt{d_s} U_{ij}^{(s)} \mid s \in \widehat{G}, 1 \leq i, j \leq d_s\}$$

is an orthonormal basis of $L^2(G, dg)$.

For fixed $s \in \widehat{G}$ and $i \in \{1, 2, \dots, d_s\}$, denote by $\mathfrak{H}_i^{(s)}$ the d_s -dimensional subspace of $L^2(G, dg)$ generated by the functions $U_{ij}^{(s)}$, $j = i, \dots, d_s$. Then the spaces $\mathfrak{H}_i^{(s)}$ are invariant under the right regular representation U_R of G and the restriction of U_R to each subspace $\mathfrak{H}_i^{(s)}$ is equivalent to U_s . In other words,

$$L^2(G, dg) = \bigoplus_{s \in \widehat{G}} \bigoplus_{i=1}^{d_s} \mathfrak{H}_i^{(s)}, \quad U_R \sim \bigoplus_{s \in \widehat{G}} d_s U_s,$$

where $d_s U_s$ denotes the direct sum of d_s copies of U_s .

This theorem is essential for the applications. In fact, the matrix elements $U_{ij}^{(s)}(g)$ yield most *special functions*. For instance, $\text{SO}(2)$ yields Fourier analysis; $\text{SO}(3)$ yields spherical harmonics; $\text{E}(2)$ yields Bessel functions. In the case of $\text{SO}(3)$, the matrix elements $U_{ij}^{(s)}(g)$ are the so-called *Wigner functions* $d_{mm'}^j(g)$. In all such cases, the fact that these functions are orthonormal bases simply follows from the Peter–Weyl theorem, whereas addition theorems in general reflect the group law.

Characters of finite dimensional representations

According to Definition 3.4, the character of the finite-dimensional UIR T is the function $\chi_T : G \rightarrow \mathbb{C}$ given by

$$\chi_T(g) = \text{Tr } T(g) = \sum_{i=1}^{\dim T} [T(g)]_{ii}, \quad g \in G.$$

The characters have the following properties:

- $\chi_T(g_o^{-1}gg_o) = \chi_T(g)$ for any $g_o \in G$,
- $\chi_T(g^{-1}) = \overline{\chi_T(g)}$,
- $T_1 \sim T_2$ implies $\chi_1 = \chi_2$, where $\chi_j \equiv \chi_{T_j}$, $j = 1, 2$.
- $\int_G dg \overline{\chi_1(g)} \chi_2(g) = \begin{cases} 0, & \text{if } T_1 \not\sim T_2, \\ 1, & \text{if } T_1 \sim T_2. \end{cases}$

For an arbitrary finite dimensional representation $T = \bigoplus_i m_i T_i$, where $m_i \in \mathbb{N}^+$ is the multiplicity of the UIR T_i in T , one has

$$\chi_T(g) = \sum_{i=1}^n m_i \chi_i(g).$$

Hence,

$$m_i = \int_G dg \overline{\chi_i(g)} \chi_T(g),$$

$$\sum_{i=1}^n m_i^2 = \int_G dg |\chi_T(g)|^2.$$

Therefore, T is irreducible if and only if $\int_G dg |\chi_T(g)|^2 = 1$.

Compact vs. noncompact groups

It is instructive to compare the properties of unitary representations of compact groups with the corresponding ones of noncompact groups.

For a *compact* Lie group G :

- every UIR is finite dimensional;
- every unitary representation T is a (finite or infinite) direct sum of UIRs : $T = \bigoplus_j T_j$.

For a *noncompact* Lie group G :

- If G is connected and semisimple, the only representation of dimension 1 is the trivial one: $T(g) = \mathbb{I}$, for all $g \in G$;
- If G is connected and *simple*, it has no nontrivial finite dimensional UIR:

Examples : $SU(1,1)$, $SO(2,1) \simeq SU(1,1)/\mathbb{Z}_2$, $SO(1,3)$

- If G is connected and *semisimple*, it has no finite dimensional *faithful* unitary representation (a representation is called faithful if its kernel is trivial).

Counterexample (H. Führ): Let $G = O(1, 1) \simeq \mathbb{R} \rtimes \{\pm 1\}$ (this group is not connected!), and let χ be a nontrivial character of $\mathbb{R}$. Then the following representation π_χ of G is unitary, irreducible and of dimension 2:

$$\pi_\chi(x, 1) = \begin{bmatrix} \chi(x) & 0 \\ 0 & \chi(x) \end{bmatrix}, \quad \pi_\chi(x, -1) = \begin{bmatrix} 0 & \chi(x) \\ \chi(x) & 0 \end{bmatrix}, \quad x \in \mathbb{R}$$

Practical analysis of UIRs of compact groups: weight diagrams

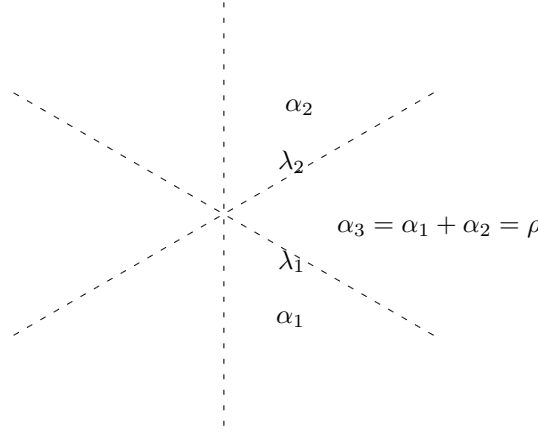


Figure 3. Root diagram and lattice g^c of $SU(3)$ (from Ref. 11).

Let G be a compact Lie group. Following Cartan⁵ and Weyl⁶ (also Hopf⁷ and Stiefel⁸), the following properties hold true:^{9,10,11}

(1) The root diagram (see the discussion in Section 2.3) divides $\mathbb{R}^l$ into *fundamental domains* (also called Weyl chambers), which are permuted by the Weyl group W .

(2) Let $\{\alpha_i\}$ denote the positive roots, $\rho = \frac{1}{2} \sum_{i=1}^m \alpha_i$, D_0 the fundamental domain containing ρ . Then there is a lattice $g^c \subset \mathbb{R}^l$ with basic vectors $\lambda_1, \dots, \lambda_l$, containing the roots, such that:

- . Every vector $v \in g^c$ is of the form $v = \sum_i p_i \lambda_i \equiv (p_1, \dots, p_l) \in \mathbb{Z}^l$.
- . D_0 is defined by $p_i \geq 0$, for all i .
- . The j -th face of D_0 corresponds to $p_j = 0, j = 1, \dots, l$.

(3) In these notations, one has $\rho = \lambda_1 + \dots + \lambda_l = (1, 1, \dots, 1)$

(4) There is a one-to-one correspondence between lattice points λ located strictly inside D_0 and UIRs $D(\lambda)$.

We illustrate these notions on the case of $SU(3)$. Figure 3 shows the root diagram and the lattice g^c . Notice the hexagonal symmetry of both patterns, as results from the invariance under the Weyl group W (see Section 2.3). Figure 4 shows the domain D_0 with the lattice points corresponding to the low-dimensional UIRs, each of them being labeled by its dimension.

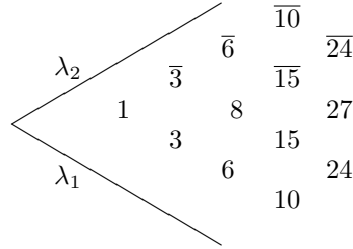


Figure 4. Low-dimensional UIRs of $SU(3)$; each UIR is labeled by its dimension (from Ref. 11).

The *weights* of a representation $D(\lambda)$ are the simultaneous eigenvectors of the commuting generators $\{H_i\}$, and thus vectors in $\mathbb{R}^l$. The *weight diagram* of $D(\lambda)$ is the set of tips of the weight vectors, and it is a set of points in $g^c \subset \mathbb{R}^l$ with integer multiplicity, invariant under the Weyl group W . The number of points of the weight diagram (counting multiplicities) gives the dimension of the corresponding representation $D(\lambda)$.

The *character* of $D(\lambda)$ is expressed in terms of weights:

$$\chi(\lambda) = X(\lambda)/\Delta, \quad (3.9)$$

where

$$X(\lambda) = \sum_{w \in W} \epsilon(w) e^{i(w\lambda, \varphi)}, \quad \lambda \in g^c \cap D_0. \quad (3.10)$$

In this expression, $\Delta = X(\rho)$, since $\chi(\rho) = 1$ (trivial representation), φ^k are the group parameters (corresponding to Cartan subalgebra), and $\epsilon(w) = \pm 1$ is the parity of $w \in W$.

Finally, the *highest weight* of the representation $D(\lambda)$ is $\lambda - \rho$. We present in Figure 5 an example of weight diagram, namely, that of the

representation 15 of $SU(3)$.

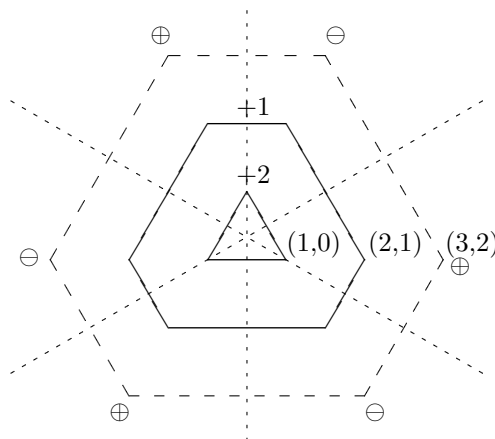


Figure 5. The weight diagram of the representation 15 of $SU(3)$, with highest weight $(2,1)$. The basis vectors are $\lambda_1 = (1,0)$ and $\lambda_2 = (0,1)$ (from Ref. 11).

Altogether, the weight diagram gives a geometric picture of the character. In addition, weight diagrams are very useful for computing and decomposing direct products into UIRs entirely *graphically*:

$$\begin{aligned}\chi(\lambda_1)\chi(\lambda_2) &= \sum_j \chi(\lambda_j) \quad (\text{Clebsch-Gordan decomposition}), \\ \chi(\lambda_1)X(\lambda_2) &= \sum_j X(\lambda_j), \\ \chi(\lambda_1)\lambda_2 &= \sum_j \lambda_j \quad (\text{Speiser's "rubber stamp rule"}).\end{aligned}$$

Thus the weight diagram is a basic tool for classification purposes.

It is interesting to note that most of this machinery extends to infinite dimensional Lie algebras known as *Kac-Moody algebras*.¹¹

3.6. Square integrable representations

Among all representations of noncompact groups, there is a class that enjoys particularly nice properties, closely reminiscent of those of compact groups, namely, the *square integrable* representations. These have a special importance in the theory of coherent states, in particular wavelets.¹²

Let G be a locally compact topological group, with left invariant measure $d\mu(g)$, and U a strongly continuous UIR of G in the Hilbert space $\mathfrak{H}$.

One says that a vector $\eta \in \mathfrak{H}, \eta \neq 0$, is *admissible* for U if

$$I(\eta) = \int_G |\langle U(g)\eta | \eta \rangle|^2 d\mu(g) < \infty, \quad (3.11)$$

or, equivalently,

$$I(\eta, \phi) = \int_G |\langle U(g)\eta | \phi \rangle|^2 d\mu(g) < \infty, \text{ for all } \phi \in \mathfrak{H}. \quad (3.12)$$

The representation U is *square integrable* if it possesses a nonzero admissible vector. It is easy to see that, if a vector $\eta \in \mathfrak{H}$ is admissible, so is the vector $\eta_g = U(g)\eta$, for every $g \in G$. One calls $\{\eta_g, g \in G\}$ the set of *coherent states* associated to U .

The set $\mathcal{A}$ of admissible vectors is a vector subspace of $\mathfrak{H}$, invariant under U . Therefore, since U is irreducible, either $\mathcal{A} = \{0\}$ (in which case U is not square integrable) ; or $\mathcal{A}$ is dense and U is square integrable.

Examples:

- . If G is compact, every UIR is square integrable.
- . For $G = \mathbb{R}$, $\mathfrak{H} = \mathbb{C}$, the representation $U(x) = e^{i\alpha x}, x \in \mathbb{R}$, is not square integrable. Indeed,

$$I(\eta) = \int_{\mathbb{R}} |e^{i\alpha x} \eta|^2 dx = |\eta|^4 \int_{\mathbb{R}} dx = \infty.$$

- . For the *affine group* of $\mathbb{R}$, $G_{\text{aff}} = \{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}, a \neq 0\} = \mathbb{R} \rtimes \mathbb{R}_*$ and $\mathfrak{H} = L^2(\mathbb{R}, dx)$, the representation

$$[U(b, a)f](x) = |a|^{-1/2} f(a^{-1}(x - b))$$

is square integrable (and it is the only one!). A vector $\psi \in L^2(\mathbb{R}, dx)$ is admissible if it satisfies the condition

$$\int_{-\infty}^{\infty} |\widehat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty. \quad (3.13)$$

Square integrable representations have the following characteristic properties:

- (1) Every square integrable representation U is unitarily equivalent to a subrepresentation of the left regular representation U_L (such representations U constitute the *discrete series* of representations of G).

Examples:

- $\text{SO}(1,3)$ (Lorentz), $\text{SO}(1,4)$ (de Sitter), $\mathcal{P}(1,3)$ (Poincaré) have *no* discrete series, hence no square integrable representations.
- $\text{SO}(2,3)$ (Anti-de Sitter), $\text{SO}(2,4)$ (conformal group), $\text{SO}(2,q)$ *do* have a discrete series.

(2) They satisfy orthogonality relations, more precisely the following theorem holds.

Theorem 3.17. (*Duflo–Moore*) *Let U be a square integrable representation of G in $\mathfrak{H}$. Then:*

(1) *There exists a unique positive self-adjoint, invertible operator C on $\mathfrak{H}$, with dense domain $\mathcal{D}(C) = \mathcal{A}$, such that the following orthogonality relations hold:*

$$\int_G \overline{\langle \eta_g | \phi \rangle} \langle \eta'_g | \phi' \rangle dg = \langle C\eta' | C\eta \rangle \langle \phi | \phi' \rangle,$$

for every admissible η, η' , and arbitrary $\phi, \phi' \in \mathfrak{H}$.

(2) *$C = \lambda \mathbb{I}$, $\lambda > 0$, if and only if G is unimodular. In that case, $\mathcal{A} = \mathfrak{H}$, i.e., all vectors are admissible.*

Examples:

- The *Weyl–Heisenberg group* (of quantum harmonic oscillator) is unimodular. Thus every function $f \in L^2(\mathbb{R}, dx)$ is admissible for the Schrödinger representation.
- The *affine group* of the line G_{aff} is not unimodular. Therefore, there exists a nontrivial admissibility condition, namely, (3.13). Thus the Duflo–Moore operator is the operator of multiplication by $|\xi|^{-1/2}$ in Fourier space, which is an unbounded, invertible, self-adjoint operator.
- For the *similitude group* of the plane $\text{SIM}(2) = \mathbb{R}^2 \rtimes (\mathbb{R}_*^+ \times \text{SO}(2))$, exactly the same situation prevails.

In conclusion, we may say that square integrable representations are the natural generalization to noncompact groups of the UIRs of compact groups.

4. Classical physics

4.1. Conservation laws in classical mechanics

For a classical physical system with finitely many degrees of freedom, the general principle of symmetry is that the invariance of the Lagrangian or the Hamiltonian under some symmetry operation implies a *conservation law*. This is expressed in the following terms in the two cases.

(1) In Lagrangian mechanics

The starting point of the theory is the Lagrangian $L(t) = L(q(t), \dot{q}(t))$, a function of (generalized) coordinates $q(t) = \{q_1(t) \dots q_n(t)\}$ and their time derivatives of first order. The basic axiom is the *principle of least action*:

$$\delta \int_{t_1}^{t_2} dt L(q(t), \dot{q}(t)) = 0.$$

This principle leads to the Euler–Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \quad (4.1)$$

From this follows the *symmetry principle*: if $L(t)$ is independent of q_i (ignorable coordinate), then $\frac{\partial L}{\partial \dot{q}_i}$ is a constant.

(2) In Hamiltonian mechanics

Now one starts with conjugate momenta:

$$p(t) = \{p_1(t) \dots p_n(t)\}, \quad p_i = \frac{\partial L}{\partial \dot{q}_i},$$

and considers the Hamiltonian $H(p(t), q(t)) = \sum_i p_i \dot{q}_i - L(q(t), \dot{q}(t))$. One defines the *Poisson brackets*:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

In particular, the canonical Poisson brackets read:

$$\begin{aligned} \{q_i(t), q_j(t)\} &= 0, \\ \{p_i(t), p_j(t)\} &= 0, \\ \{q_i(t), p_j(t)\} &= \delta_{ij}. \end{aligned}$$

As generator of time translations, the Hamiltonian describes the time evolution of observables. Given an observable $A \equiv A(q(t), p(t))$, its time evolution is governed by Hamilton's equation

$$\dot{A} = \{H, A\}. \quad (4.2)$$

As a consequence, $\{H, A\} = 0$ implies that the observable $A(q(t), p(t))$ is conserved.

4.2. *Hamiltonian mechanics: representations with respect to Poisson brackets*

If several conserved observables $A_j(q, p)$ have commutation relations (with respect to Poisson brackets) that close, they constitute a Lie algebra. Thus one obtains a representation of that Lie algebra in terms of Poisson brackets. This applies even in General Relativity, when it is formulated in the Hamiltonian formalism.

Examples:

(1) *Central potential* (2-body problem in center of mass frame):

$$H = \frac{\mathbf{p}^2}{2} + V(r).$$

The components of the angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ are the infinitesimal generators of rotations, with commutation relations

$$\{L_i, L_j\} = \epsilon_{ijk} L_k : \quad \text{Lie algebra } \mathfrak{so}(3).$$

The invariance of the system under rotations is expressed by the relation $\{H, \mathbf{L}\} = 0$. As a consequence, $\mathbf{L}$ is constant and therefore the plane of the orbit is fixed in space. For the Earth, this is the plane of the *ecliptic*, which is indeed fixed, modulo small perturbations due to other bodies, like Jupiter or the Moon.

(2) *Dynamical group for the Coulomb–Kepler potential:*

In the case of the Coulomb–Kepler potential, $V(r) = r^{-1}$, there exists a second invariant, namely, $\mathbf{K} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}$, which satisfies the relations $\{H, \mathbf{K}\} = 0$ and $\mathbf{K} \cdot \mathbf{L} = 0$. However, the commutation relations don't close:

$$\begin{aligned} \{\mathbf{L}, \mathbf{L}\} &= \mathbf{L}, \\ \{\mathbf{L}, \mathbf{K}\} &= \mathbf{K}, \\ \{\mathbf{K}, \mathbf{K}\} &= -2H\mathbf{L}. \end{aligned}$$

Therefore, in the case $H < 0$, one introduces the so-called *Runge–Lenz vector* $\mathbf{A} = \mathbf{K}(-2H)^{-1/2}$. Then,

$$\begin{aligned} \{\mathbf{L}, \mathbf{L}\} &= \mathbf{L}, \\ \{\mathbf{L}, \mathbf{A}\} &= \mathbf{A}, \\ \{\mathbf{A}, \mathbf{A}\} &= \mathbf{L}, \end{aligned}$$

which is the Lie algebra $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Indeed, writing $\mathbf{X}_{\pm} = \frac{1}{2}(\mathbf{L} \pm \mathbf{A})$, one gets

$$\{\mathbf{X}_{\pm}, \mathbf{X}_{\pm}\} = \mathbf{X}_{\pm}, \quad \{\mathbf{X}_{+}, \mathbf{X}_{-}\} = 0.$$

Since the Casimir operators are equal, $\mathbf{X}_{+}^2 = \mathbf{X}_{-}^2$, the corresponding representations of $\text{SO}(4)$ have dimension n^2 , which explains the “accidental” degeneracy of the spectrum of the H-atom.

A similar analysis for $H > 0$ gives $\text{SO}(1,3)$ as symmetry group and $\text{E}(3)$ for $H = 0$.

One can go one step further, by considering additional operators mapping one level to the next one (ladder operators). Altogether, one gets a Lie algebra $\mathfrak{so}(4,1)$ and a *single* UIR of $\text{SO}(4,1)$ yields the full discrete spectrum of the H-atom. This is an example of a *dynamical group*.

It is interesting to note that the same results hold true in quantum mechanics, with the usual commutators of operators. Actually, the $\text{SO}(4)$ symmetry for the bound states was used by Pauli in 1926 for solving the H-atom algebraically, *before* Schrödinger’s quantum mechanics!

4.3. Classical field theory: Symmetries and Noether’s theorem

The most spectacular consequences of invariance properties occur in the case of classical physical systems with infinitely many degrees of freedom, that is, systems that must be described by a (classical) field theory. This covers, for instance, acoustics, fluid dynamics, or classical electromagnetism (Maxwell equations). As in the finite case, one must first identify the various ingredients of the theory.

The canonical variables are (classical) fields: $\varphi_j(x)$, $j = 1, \dots, N$. Notice that, in a relativistic setting, $x = (t, \mathbf{x}) = \{x_{\mu}\}$, $\mu = 0, 1, 2, 3$, and all four variables must be on the same footing.

The Lagrangian is the space integral of a Lagrangian density, $\mathcal{L} \equiv \mathcal{L}(\varphi_i(x), \partial_{\mu}\varphi_i(x))$, which depends only on the fields φ_i and their first derivatives $\partial_{\mu}\varphi_i$:

$$L(t) = \int_{\mathbb{R}^3} d^3\mathbf{x} \mathcal{L}(\varphi_i(x), \partial_{\mu}\varphi_i(x)).$$

The Euler-Lagrange equations are now field equations (the relativistic summation convention is applied):

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi_i)} = 0, \quad i = 1, \dots, N. \quad (4.3)$$

The conjugate fields are defined as

$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i(x)}, \quad i = 1, \dots, N.$$

Notice that, here, time has a privileged role, so that the Hamiltonian formalism is, by definition, not explicitly covariant.

The Hamiltonian is also the integral of a (Hamiltonian) density:

$$H = \int d^3\mathbf{x} \left(\sum_{i=1}^N \pi_i(x) \dot{\varphi}_i(x) - \mathcal{L} \right).$$

As before, the Hamiltonian describes the time evolution of observables. Indeed, given an observable $A \equiv A(\pi_i(x), \varphi_i(x))$, its time evolution is governed by Hamilton's equation (4.2), that we repeat for convenience.

$$\dot{A} = \{H, A\}. \quad (4.4)$$

In the context of classical field theory, the connection between invariance properties and conservation laws is given by the celebrated theorem of Emmy Noether. But before stating the theorem, we have to make more precise the notion of symmetry.

Assume there exists a Lie group G of space-time transformations under which the fields are *covariant*:

$$x \mapsto x' \equiv gx, \quad x \in \mathbb{R}^4, \quad g \in G, \quad (4.5)$$

$$\varphi_i(x) \mapsto \varphi'_i(x') = \sum_{j=1}^N S_i^j(g) \varphi_j(x). \quad (4.6)$$

In (4.6), $S(g) \equiv (S_i^j(g))$ is an N -dimensional representation of G , not necessarily irreducible.

There are several possibilities for such transformations:

(i) *Geometric transformations*, that is, transformations that act both on space-time (thus, $x' \neq x$ in general) and on the components of the fields:

- . translations: $\varphi'_i(x+a) = \varphi_i(x)$
- . Lorentz transformations: $\varphi'_i(\Lambda x) = \sum_j S_i^j(\Lambda) \varphi_j(x)$ (spin)
- . (Anti)-de Sitter transformations, conformal transformations, etc.

(ii) *Internal transformations*, that is, transformations that only mix the fields $\varphi_i, \dots, \varphi_N$, so that $x' = x$. These, in turn, are of two types:

(a) *Global*, i.e., g is independent of x , thus the *same* transformation is applied at all points of space-time:

- abelian, e.g. $G = \text{U}(1) : \varphi'_i(x) = e^{-i\alpha}\varphi_i(x), \quad \alpha \in \mathbb{R},$
- nonabelian, e.g. $G = \text{SU}(2)$ (isospin), $\text{SU}(3)$ (QCD):

$$\varphi'_i(x) = \sum_j S_i^j(g) \varphi_j(x).$$

(b) *Local or gauge transformations*, the same transformations as in (a), but now $g \equiv g(x)$, which means that all points of space-time are transformed independently.

Since G is a Lie group, we may consider an *infinitesimal transformation*

$$x'^\mu = x^\mu + \delta x^\mu \quad \text{with} \quad \delta x^\mu = \omega_r X^{r,\mu},$$

$$S_i^j(g) = \delta_i^j + \omega_r (G^r)_i^j,$$

that is,

$$\varphi'_i(x') = \varphi_i(x) + \delta\varphi_i \quad \text{with} \quad \delta\varphi_i = \omega_r (G^r)_i^j \varphi_j(x).$$

Here $\omega_r, r = 1, 2, \dots, \dim G$, are the infinitesimal parameters of the transformation and G^r are the infinitesimal generators in the representation S (thus each G^r is a $N \times N$ matrix).

We introduce now the following terminology:

- A *current* is a four-vector $J_\mu(x) \equiv J_\mu(\varphi_i, \partial_\nu \varphi_i)$.
- The current J_μ is *conserved* if $\partial^\mu J_\mu(x) = 0$.
- The *charge* associated to the current J_μ is the quantity

$$Q(t) = \int d^3\mathbf{x} J_0(\mathbf{x}, t).$$

Theorem 4.1. (Noether) Consider a classical field theory with Lagrangian $\mathcal{L}(x) \equiv \mathcal{L}(\varphi_i, \partial_\mu \varphi_i), i = 1, \dots, N$ and a R -dimensional Lie group G of space-time transformations under which the fields are covariant:

$$x \mapsto x' = gx,$$

$$\varphi_i(x) \mapsto \varphi'_i(x') = S_i^j(g) \varphi_j(x).$$

Assume that the fields satisfy the Euler–Lagrange equations (4.3) and that the action integral is invariant under G :

$$\int_{\Omega'} d^4x' \mathcal{L}'(x') = \int_{\Omega} d^4x \mathcal{L}(x),$$

for any region $\Omega \subset \mathbb{R}^4$ and its transform Ω' under the map $x \mapsto x'$.

Then there exist R conserved currents $J_\mu^r(x)$:

$$\partial^\mu J_\mu^r(x) = 0, \quad r = 1, 2, \dots, R = \dim G.$$

In addition, if the Lagrangian vanishes fast enough for $|\mathbf{x}| \rightarrow \infty$, the corresponding charges $Q^r(t)$ are constant:

$$\frac{d}{dt}Q^r(t) = 0, \quad r = 1, 2, \dots, R.$$

The proof of this theorem is essentially computational, using a variational approach (as with the Euler–Lagrange equations). Instead of giving it, we will discuss a series of concrete examples, which are of crucial importance in physics.

(1) *Translations*

An infinitesimal translation reads $x'^\mu = x^\mu + a^\mu$, with $|a^\mu| \ll 1$. Thus $r \equiv \mu$ is a Lorentz index and

$$\delta x^\mu = a^\mu = \omega_\nu X^{\nu\mu}, \text{ so that } X^{r,\mu} \equiv X^{\nu\mu} = g^{\nu\mu}.$$

Each field transforms separately, i.e., $\varphi'_i(x+a) = \varphi_i(x)$. Thus

$$S_i^j = \delta_i^j \text{ (trivial representation) and } (G^r)_i^j \equiv 0.$$

The conserved current becomes

$$J^{r,\mu} \equiv T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial^\nu \varphi_i$$

and the conserved charges read

$$P^\nu = \int d^3\mathbf{x} T^{0\nu} = \int d^3\mathbf{x} (-g^{0\nu} \mathcal{L} + \pi_i \partial^\nu \varphi_i),$$

$$P^0 = \int d^3\mathbf{x} (-\mathcal{L} + \pi_i \dot{\varphi}_i) = \int d^3\mathbf{x} \mathcal{H}(x).$$

Thus P^0 coincides with the Hamiltonian, and represents the total energy of the system. By covariance, P^k represents the total momentum and $T^{\mu\nu}$ is the energy–momentum tensor.

(2) *Lorentz transformations*

An infinitesimal Lorentz transformation is simply an infinitesimal rotation in Minkowski space, hence

$$x'^\mu = x^\mu + \epsilon_\nu^\mu x^\nu, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}, \text{ with } |\epsilon_\nu^\mu| \ll 1,$$

$$\delta x^\mu = \epsilon_\alpha^\mu x^\alpha = \epsilon_{\alpha\beta} g^{\beta\mu} x^\alpha = \frac{1}{2} \epsilon_{\alpha\beta} (g^{\beta\mu} x^\alpha - g^{\alpha\mu} x^\beta).$$

Thus

$$\omega_r \equiv \frac{1}{2}\epsilon_{\alpha\beta}, \quad r = 01, 02, 03, 12, 13, 23,$$

$$X^{\alpha\beta,\mu} = x^\alpha g^{\beta\mu} - x^\beta g^{\alpha\mu} = -X^{\beta\alpha,\mu}.$$

The field transformation law is given by *group theory*: the (proper) Lorentz group has irreducible finite dimensional representations (*nonunitary* for $s \neq 0$) of dimension $(2s+1)$, with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, corresponding to scalar, spinor, vector, tensor fields, etc.

The infinitesimal generators are

$$(G_{(s)}^r)_i^j \equiv (G_{(s)}^{\alpha\beta})_i^j = -(G_{(s)}^{\beta\alpha})_i^j, \\ \text{with } \alpha, \beta = 0, 1, 2, 3, \quad i, j = 1, 2, \dots, 2s+1.$$

The conserved currents become

$$J^{\alpha\beta,\mu} = (x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} (G_{(s)}^{\alpha\beta})_i^j \varphi_j, \\ = L^{\mu\alpha\beta} + S^{\mu\alpha\beta}.$$

The two terms represent orbital angular momentum and spin, respectively. It is important to notice that they are *not* conserved separately, only the total angular momentum is!

(3) Global gauge transformations: $U(1)$

In the abelian case, each field transforms separately, simply by a phase factor,

$$\varphi_i(x) \mapsto \varphi'_i(x) = e^{-i\alpha q_i} \varphi_i(x),$$

where $\alpha \in \mathbb{R}$ is the transformation parameter and the number $q_i \in \mathbb{R}$ specifies the behavior of the field φ_i . Since this is a purely internal transformation, we have $X^{r,\mu} \equiv 0$. The parameters may thus be identified, for an infinitesimal transformation:

$$\delta\varphi_i = -i\alpha q_i \varphi_i, \text{ so that } \omega_r \equiv \alpha, \text{ with } |\alpha| \ll 1,$$

$$(G^r)_i^j = -iq_i \delta_i^j.$$

As for the conserved current and the charge, we get

$$J^\mu(x) = i \sum_{j=1}^N q_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j)} \varphi_j,$$

$$Q = i \sum_{j=1}^N q_j \int d^3 \mathbf{x} \pi_j(x) \varphi_j(x).$$

(4) *General (global) internal transformations*

In this case, which covers, for instance, SU(2) (isospin), SU(3) (color), SU(3)×SU(2)×U(1)(Standard Model), the situation is exactly the same as in the case of U(1):

$$\varphi'_i(x) = \sum_{j=1}^N S_i^j(g) \varphi_j(x)$$

$$\simeq \varphi_i(x) + \omega_r \sum_{j=1}^N (G^r)_i^j \varphi_j(x), \text{ for } |\omega_r| \ll 1.$$

Since this is a purely internal transformation, we have again $X^{r,\mu} \equiv 0$. The conserved current and the charges read as before:

$$J^{r,\mu} = - \sum_{i,j} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} (G^r)_i^j \varphi_j,$$

$$Q^r = - \int d^3 \mathbf{x} \sum_{ij} \pi_i (G^r)_i^j \varphi_j.$$

Remark: in the case of purely internal transformations, abelian or not, the invariance condition is simply

$$\mathcal{L}(\varphi'_i, \partial^\mu \varphi'_i) = \mathcal{L}(\varphi_i, \partial^\mu \varphi_i).$$

Such a condition, which is in general easy to verify, is the primary constraint in the derivation of classical field theory models. This is one of the strengths of the Lagrangian formalism. By extension, as we shall see in the next section, the same situation prevails in quantum field theory. This is the way in which successive models describing the interactions between elementary particles have been set up along the years, with the so-called Standard Model as ultimate example, so far at least.

(5) *Local internal transformations*

An infinitesimal U(1) gauge transformation

$$\varphi_j(x) \mapsto e^{iq_j\theta} \varphi_j(x) \simeq (1 + iq_j\theta) \varphi_j(x), \text{ for } |\theta| \ll 1,$$

yields for the variation of the action

$$\delta A = \int_{\Omega} d^4x \partial_{\mu} \left(i\theta \sum_j q_j \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi_j)} \varphi_j \right).$$

Now two situations may arise:

- (i) For a *global transformation*, θ is constant and may be taken out of the integral, so that everything is as before.
- (ii) For a *local transformation* $\theta = \theta(x)$, the argument may not work. For instance, a spinor field requires coupling to an electromagnetic field (Weyl). However, there is no additional conservation law, but the coupling is uniquely determined (covariant derivative, leading to minimal coupling).

In the case of a nonabelian local internal transformations, the result is the same, but a proper derivation requires the language of differential geometry (concepts as a connection one-form and a curvature two-form are needed), but we shall refrain to pursue this any longer, for lack of space.

5. Quantum physics

5.1. *Symmetries in Quantum Mechanics*

We turn now to Quantum Mechanics. The following assumptions are standard:

- (i) *Pure states* are represented by normalized rays (one-dimensional subspaces) in a Hilbert space $\mathfrak{H}$:

$$\widehat{\psi} = \{\psi e^{i\alpha}, \|\psi\|^2 = 1, 0 \leq \alpha < 2\pi\}$$

We denote by $\widehat{\mathfrak{H}}$ the corresponding projective Hilbert space, that is, the set of all one-dimensional subspaces of $\mathfrak{H}$. Note that $\widehat{\mathfrak{H}}$ is not a vector space!

- (ii) The transition *amplitude* between two states $\widehat{\psi}, \widehat{\phi} \in \widehat{\mathfrak{H}}$ is given by

$$\langle \widehat{\psi}, \widehat{\phi} \rangle = |\langle \psi | \phi \rangle|$$

Clearly, the choice of particular vectors $\psi \in \widehat{\psi}, \phi \in \widehat{\phi}$ is arbitrary.

- (iii) The transition *probability* between these two states is given by

$$P(\widehat{\phi} \rightarrow \widehat{\psi}) = \langle \widehat{\psi}, \widehat{\phi} \rangle^2.$$

- (iv) *Observables* are represented by self-adjoint operators acting on $\mathfrak{H}$.

The motivation for these assumptions is twofold. First the superposition principle imposes a vector space and the notion of transition amplitude requires a scalar product space, thus one needs a prehilbert space. Then mathematical convenience suggests that this state space be complete, i.e., be a Hilbert space.

In this set-up, a symmetry is defined as follows.

Definition 5.1. A *symmetry* is an automorphism of $\widehat{\mathfrak{H}}$, that is, a bijective map between states, $\tau : \widehat{\mathfrak{H}} \rightarrow \widehat{\mathfrak{H}}$ that preserves transition amplitudes:

$$\langle \tau\widehat{\psi}, \tau\widehat{\phi} \rangle = \langle \widehat{\psi}, \widehat{\phi} \rangle, \text{ for all } \widehat{\psi}, \widehat{\phi} \in \widehat{\mathfrak{H}}.$$

The question now is how to translate this notion into operations in $\mathfrak{H}$. Since $\widehat{\mathfrak{H}}$ is the quotient of $\mathfrak{H}$ by an equivalence relation, this is a nontrivial lifting problem, which was first solved by Wigner in a famous paper.¹³

Theorem 5.2. (Wigner) *Every symmetry in $\widehat{\mathfrak{H}}$ is induced by an operator in $\mathfrak{H}$, which is unitary or antiunitary. This operator is unique up to a phase.*

It is useful at this point to introduce some more notations, following Simms.¹⁴ We will denote by $\text{Aut } \widehat{\mathfrak{H}}$ the set of all automorphisms of $\widehat{\mathfrak{H}}$, by $\mathcal{U}(\mathfrak{H})$ the set of all unitary operators on $\mathfrak{H}$, and by $\widetilde{\mathcal{U}}(\mathfrak{H})$ the set of all unitary and all antiunitary operators on $\mathfrak{H}$.

Define a map $\pi : \widetilde{\mathcal{U}}(\mathfrak{H}) \rightarrow \text{Aut } \widehat{\mathfrak{H}}$ by $\pi(A)\widehat{\psi} = \widehat{A\psi}$. Denote by $\mathcal{U}(\widehat{\mathfrak{H}}) = \pi(\mathcal{U}(\mathfrak{H}))$ the image of unitary operators of $\mathfrak{H}$ and by $\text{U}(1) = \{e^{i\lambda}\mathbb{I}\} \subset \mathcal{U}(\mathfrak{H})$ the phase operators on $\mathfrak{H}$. All these objects are groups. Then Wigner's theorem may be rewritten as

$$\text{Aut } \widehat{\mathfrak{H}} \simeq \widetilde{\mathcal{U}}(\mathfrak{H})/\text{U}(1), \quad \mathcal{U}(\widehat{\mathfrak{H}}) \simeq \mathcal{U}(\mathfrak{H})/\text{U}(1).$$

Let us now turn to the case of a *symmetry group*. Suppose that the system possesses a Lie group G of symmetries, that is, to every $g \in G$ one associates a symmetry τ_g that represents g . How does that fit into the present language?

First of all, by Wigner's theorem 5.2, every τ_g is represented by an operator $U(g)$, unitary or antiunitary. Next, in a neighborhood of the identity, every g is a square, $g = g'g'$, hence $U(g)$ must be unitary. This is

true for all $g \in G$ if G is connected, otherwise for all $g \in G_0$, where G_0 is the connected component of the identity of G .

In order to proceed, we introduce new notions of representations. A *projective representation* of G in $\hat{\mathfrak{H}}$ is a homomorphism $T : G \rightarrow \mathcal{U}(\hat{\mathfrak{H}})$. Similarly, a *projective representation* of G in $\mathfrak{H}$ (or *representation up to a phase*), is a map $U_\omega : G \rightarrow \mathcal{U}(\mathfrak{H})$ such that

$$U_\omega(g_1)U_\omega(g_2) = \omega(g_1, g_2) U_\omega(g_1g_2) \quad \text{with} \quad |\omega(g_1, g_2)| = 1.$$

Next we have to introduce some notion of *continuity* of a representation. The correct way is to define topologies on $\mathcal{U}(\mathfrak{H})$ and $\mathcal{U}(\hat{\mathfrak{H}})$ that turn them into topological groups, but this is technically rather difficult. Instead, we will restrict ourselves to a simple-minded approach, namely, we will require, for physical reasons, that the matrix elements $\langle \hat{\psi}, \tau_g \hat{\phi} \rangle$ be continuous functions of g in a neighborhood of the identity. The main result is the following theorem, that summarizes the discussion given by Wigner and Bargmann.

Theorem 5.3. (Wigner–Bargmann) *If the symmetry group G is continuous in the sense that the matrix elements $\langle \hat{\psi}, \tau_g \hat{\phi} \rangle$ are continuous functions of g in a neighborhood of the identity, then the arbitrary phases of the operators $U(g)$ may be chosen in such a way that, in a certain neighborhood of the identity $\mathcal{V}(e)$, the set $\{U(g), g \in \mathcal{V}(e)\}$ is continuous in the strong topology of $\mathfrak{H}$. If G is connected and simply connected, then $\mathcal{V}(e) = G$. Otherwise, the choice stems from a strongly continuous projective unitary representation of the connected component of the universal covering of G .*

Notice that, in general, one needs projective representations! However, if G is connected and simply connected, one can get rid of the projective character of U .¹⁶

Theorem 5.4. (Voisin) *Let G be a connected and simply connected group and let U_ω be any projective unitary representation of G , with factor*

$$\omega(g_1, g_2) = e^{i\xi(g_1, g_2)}.$$

Then, U_ω may be deduced from a genuine unitary representation U of a group G_ω , which is a central extension of G by $\mathbb{R}$, namely, $G_\omega = \{(\theta, g), \theta \in \mathbb{R}, g \in G\}$, with multiplication law

$$(\theta_1, g_1)(\theta_2, g_2) = (\theta_1 + \theta_2 + \xi(g_1, g_2), g_1g_2).$$

The link between the two representations is given by the relation

$$U_\omega(g) = e^{-i\theta} U(\theta, g).$$

Here the extension is called central since $\{(\theta, e)\} \subset Z(G)$.

Examples

(1) Many groups of physical interest do not have nontrivial factors θ , for instance:

- . abelian, connected and simply connected groups, e.g. $\mathbb{R}^n$;
- . abelian, connected and compact groups;
- . semisimple groups (because the Killing form is nondegenerate);
- . In particular, $\mathrm{SO}(p, q)$, which is semisimple;
- . $\mathrm{ISO}(p, q)$, the inhomogeneous pseudo-orthogonal group, provided that $p + q > 2$; the result is false for $\mathrm{ISO}(1, 1)$, which has a one parameter family of nontrivial factors.

(2) *Poincaré group*: the factor $\omega(g_1, g_2)$ may be reduced to ± 1 . This is the fundamental result of Wigner,¹³ in his pioneering paper (the first paper treating infinite dimensional representations).

(3) *Galilei group*: there is a one-parameter family of nonequivalent factors $\xi(g_2, g_1)$, indexed by a parameter m , which is interpreted as the *mass* of the particle described by the representation. Thus, in *nonrelativistic* quantum mechanics, mass is *absolutely conserved* (“superselection rule”). This is the fundamental result of Bargmann.¹⁵

A proper discussion of all this requires tools from cohomology theory, which goes far beyond the present course.

As we have just seen, symmetries are represented by (possibly projective) strongly continuous unitary representations of Lie groups. What about observables? It turns out that most of them are described by elements of the corresponding Lie algebra. Consider a continuous *one-parameter group* of symmetries, for instance time translations (time evolution). This means a family of unitary operators $U(t), t \in \mathbb{R}$, such that

- . $U(t_1)U(t_2) = U(t_1 + t_2)$, for all $t_1, t_2 \in \mathbb{R}$,
- . $U(0) = \mathbb{I}$, which implies $U(t)^* = U(t)^{-1} = U(-t)$,
- . $U(t)$ is strongly continuous at $t = 0$.

By *Stone's theorem*, there exists a self-adjoint operator H , the infinitesimal generator (here the Hamiltonian), such that $U(t) = e^{-iHt}$, and defined as

$$H = \text{s-lim}_{t \rightarrow 0} \left[\frac{U(t) - \mathbb{I}}{-it} \right] \quad (\text{s-lim} = \text{strong limit})$$

on the domain $D(H) = \{\phi \in \mathfrak{H} \text{ such that the limit exists } \}$. In general, this operator is unbounded, so that some care must be exercised !

More generally, for a given unitary representation $U(g)$, the generators of the corresponding Lie algebra are represented by unbounded self-adjoint operators, which represent *observables*, for instance:

- . Time translations $\Rightarrow$ Hamiltonian, total energy
- . Space translations $\Rightarrow$ total momentum
- . Lorentz transformations $\Rightarrow$ total angular momentum
- . Gauge transformations $\Rightarrow$ charges

All these are automatically self-adjoint, but there are, of course, domain problems (which are mostly ignored in the physics literature). These problems lead naturally to the notions of analytic vectors or C^∞ vectors, Gårding domain, etc. Thus we are back to the link between Lie groups and their Lie algebras, this time in terms of operators on Hilbert space.

5.2. Symmetries in Quantum Field Theory

We have seen in Section 4.3 that symmetries in classical field theory are described by representations of Lie groups, while Noether's theorem leads to the conservation laws associated to these symmetries. How to translate this in quantum field theory?

A possible solution is to consider matrix elements of field operators and treat them as classical field variables (*correspondence principle*):

$$F_j^{\alpha\beta}(x) = \langle \Phi_\alpha | \varphi_j(x) | \Phi_\beta \rangle, \quad (5.1)$$

where Φ_α, Φ_β are two state vectors. Consider for instance a Poincaré transformation

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu, \quad (\Lambda, a) \in \mathcal{P}(1, 3).$$

The transformation law of a classical field reads

$${}^{cl}\varphi_i(x) \mapsto {}^{cl}\varphi'_i(\Lambda x + a) = S_i^j(\Lambda) {}^{cl}\varphi_j(x).$$

Apply the correspondence principle:

$$F_i^{\alpha\beta}(x) \mapsto F'^{\alpha\beta}_i(\Lambda x + a) = S_i^j(\Lambda) F_j^{\alpha\beta}(x),$$

where

$$F'^{\alpha\beta}_i(x') \equiv \langle \Phi'_\alpha | \varphi_i(x') | \Phi'_\beta \rangle$$

and $\Phi'_\alpha, \Phi'_\beta$ are the state vectors representing the states α, β in the new frame $\{x'\}$.

By the Wigner–Bargmann theorem 5.3, $\Phi'_\alpha = U(\Lambda, a)\Phi_\alpha$ and $\Phi'_\beta = U(\Lambda, a)\Phi_\beta$, where $U(\Lambda, a)$ is a continuous unitary representation of $\mathcal{P}(1, 3)$. Hence,

$$\begin{aligned} F_i^{\prime\alpha\beta}(x') &\equiv \langle \Phi'_\alpha | \varphi_i(\Lambda x + a) \Phi'_\beta \rangle \\ &= \langle \Phi_\alpha | U^{-1}(\Lambda, a) \varphi_i(\Lambda x + a) U(\Lambda, a) \Phi_\beta \rangle \\ &= S_i^j(\Lambda) \langle \Phi_\alpha | \varphi_j(x) \Phi_\beta \rangle. \end{aligned}$$

Assuming that the states Φ_α, Φ_β run over a dense subset of $\mathfrak{H}$, we may conclude that

$$U^{-1}(\Lambda, a) \varphi_i(\Lambda x + a) U(\Lambda, a) = S_i^j(\Lambda) \varphi_j(x), \quad (5.2)$$

or, equivalently,

$$U(\Lambda, a) \varphi_j(x) U^{-1}(\Lambda, a) = (S^{-1}(\Lambda))_j^i \varphi_i(\Lambda x + a). \quad (5.3)$$

The same analysis can be made for an arbitrary Lie group of symmetries: The classical transformation law

$${}^{cl}\Phi \mapsto {}^{cl}\Phi'_i(gx) = S_i^j(g) {}^{cl}\Phi_j(x), \quad g \in G$$

becomes in its quantum translation

$$U(g) \varphi_j(x) U^{-1}(g) = (S^{-1}(g))_j^i \varphi_i(gx), \quad g \in G,$$

where $U(g)$ is a unitary (projective) representation of G .

For an infinitesimal transformation $x \mapsto gx \simeq x + \omega_r X^r$, $|\omega_r| \ll 1$, we get

$$\begin{aligned} U(g) &= e^{i\omega_r K^r} \simeq \mathbb{I} + i\omega_r K^r + \mathcal{O}(\omega_r^2), \\ S(g) &= e^{\omega_r G^r} \simeq \mathbb{I} + \omega_r G^r + \mathcal{O}(\omega_r^2), \end{aligned}$$

where K^r, G^r are the infinitesimal generators in the representations U, S , respectively. This gives

$$(\mathbb{I} + i\omega_r K^r) \varphi_j(x) (\mathbb{I} - i\omega_r K^r) = (\mathbb{I} - \omega_r G^r)_j^i \varphi_i(x + \omega_r X^r). \quad (5.4)$$

On the other hand

$$\begin{aligned} \varphi_i(x^\mu + \omega_r X^{r,\mu}) &\simeq \varphi_i(x^\mu) + \frac{\partial \varphi_i}{\partial x^\nu} \omega_r X^{r,\nu} \\ &= (\mathbb{I} + \omega_r X^{r,\nu} \partial_\nu) \varphi_i(x^\mu). \end{aligned}$$

To first order in ω_r , this gives

$$i[K^r, \varphi_j(x)] = X^{r,\mu} \partial_\mu \varphi_j(x) - (G^r)_j^i \varphi_i(x).$$

In the case of Poincaré transformations, we thus obtain the *consistency relations*

$$\begin{aligned} i [P^\mu, \varphi_j(x)] &= \partial^\mu \varphi_j(x), \\ i [J^{\alpha\beta}, \varphi_j(x)] &= (x^\alpha \partial^\beta - x^\beta \partial^\alpha) \varphi_j(x) - (G^{\alpha\beta})_j^i \varphi_i(x). \end{aligned}$$

These relations have to be checked explicitly for guaranteeing invariance of the *quantum* theory. Indeed, the quantum theory *cannot be derived* from the classical theory, the correspondence principle is only a guide!

As a matter of fact, in the axiomatic formulation of QFT, the transformation properties of the field operators are part of their very *definition* (tensorial character):

$$U(g) \varphi_j(x) U^{-1}(g) = (S^{-1}(g))_j^i \varphi_i(gx), \quad g \in G. \quad (5.5)$$

5.3. Coherent states

Among the many applications of group representations in quantum physics, we point out one that has enjoyed a considerable success (including the Nobel Prize in physics 2005 to R.J. Glauber), namely, *coherent states*. These were originally introduced by Schrödinger in 1926 in the context of the classical limit of quantum mechanics, but then they were forgotten. They were reconsidered in the 60s for the purpose of modelling the coherent light emitted by a laser, but it was soon recognized, by Perelomov¹⁷ and Gilmore,⁴ independently, that this was essentially an application of group representation theory. A comprehensive discussion of the subject may be found in the recent monograph of Ali *et al.*¹² Here we sketch only the general construction.

Let G be a locally compact topological group, with left invariant measure $d\mu(g)$, and let U be a square integrable UIR of G in a Hilbert space $\mathfrak{H}$. Choose a nonzero admissible vector $\eta \in \mathfrak{H}$ and define

$$c(\eta) = \frac{1}{\|\eta\|^2} \int_G |\langle U(g)\eta | \eta \rangle|^2 d\mu(g).$$

Then the corresponding family of *coherent states* (CS) is the set

$$\mathcal{S} = \{\eta_g = U(g)\eta, \quad g \in G\}.$$

The essential properties of the CS family $\mathcal{S}$ are neatly summarized in the following theorem

Theorem 5.5. *Let G be a locally compact topological group, U a square integrable UIR of G in the Hilbert space $\mathfrak{H}$ and $\eta \in \mathfrak{H}$ a nonzero admissible vector. Define the (CS) map $W_\eta : \mathfrak{H} \rightarrow L^2(G, d\mu(g))$ by*

$$(W_\eta \phi)(g) = \frac{1}{\sqrt{c(\eta)}} \langle \eta_g | \phi \rangle, \quad \phi \in \mathfrak{H}. \quad (5.6)$$

Then:

(1) W_η is an isometry, that is, $W_\eta^* W_\eta = \mathbb{I}$, and $\mathcal{S}$ defines a resolution of the identity:

$$\frac{1}{c(\eta)} \int_G |\eta_g\rangle \langle \eta_g| d\mu(g) = \mathbb{I}.$$

This implies that $\mathcal{S}$ is total in $\mathfrak{H}$.

(2) *The range $\mathfrak{H}_\eta$ of W_η is a closed subspace of $L^2(G, d\mu(g))$ and the corresponding orthogonal projection $P_\eta = W_\eta W_\eta^*$ is an integral operator, with kernel*

$$K_\eta(g, g') = \frac{1}{c(\eta)} \langle \eta_g | \eta_{g'} \rangle.$$

Thus $\mathfrak{H}_\eta$ is a reproducing kernel Hilbert space, that is, $\Phi \in L^2(G, d\mu(g))$ belongs to $\mathfrak{H}_\eta$ if and only if it satisfies the reproduction property

$$\Phi(g) = \int_G K_\eta(g, g') \Phi(g') d\mu(g').$$

(3) W_η intertwines the representation U and the left regular representation U_L of G in $L^2(G, d\mu(g))$:

$$W_\eta U(g) = U_L(g) W_\eta, \quad \text{for all } g \in G.$$

(4) *The CS map W_η may be inverted on its range by the adjoint map W_η^* , so that one gets a reconstruction formula for every vector $\phi \in \mathfrak{H}$:*

$$\phi = W_\eta^* \Phi = \frac{1}{\sqrt{c(\eta)}} \int_G \Phi(g) \eta_g d\mu(g), \quad \Phi \in \mathfrak{H}_\eta.$$

The map W_η is usually called the *CS transform* associated to the group G . Property (3) expresses its *covariance* under the operations of G . It also means that U belongs to the discrete series of representations of G . As for Property (4), it means that every vector (signal) ϕ may be expressed as a (continuous) linear superposition of CSs. Thus the latter play the role of a continuous basis in $\mathfrak{H}$.

The two most conspicuous examples of CS transforms are the Gabor and the wavelet transforms, based on the Weyl–Heisenberg and the similitude (or affine) groups, respectively.

(1) *The Weyl–Heisenberg group*

$$G_{\text{WH}} = \mathbb{R}^2 \rtimes S^1 = \{(s, q, p) : s \in S^1, (q, p) \in \mathbb{R}^2\}.$$

The center of G_{WH} is $Z = S^1 = \{(s, 0, 0)\}$ and the quotient G_{WH}/Z is isomorphic to $\mathbb{R}^2$.

The UIRs of G_{WH} have a very simple form, thanks to *von Neumann's uniqueness theorem*. Indeed, the latter states that, for fixed $\lambda \neq 0$, all the UIRs $U^\lambda(s, q, p)$ of G_{WH} are unitarily equivalent and have the form

$$U^\lambda(s, q, p) = e^{i\lambda s} D^\lambda(q, p),$$

where $D^\lambda(q, p)$ is a displacement operator. In the Schrödinger representation (for simplicity, we put $\lambda = 1$),

$$(D(q, p)f)(x) = e^{ipq/2} e^{ipx} f(x - q), \quad x \in \mathbb{R}.$$

Moreover, each D^λ is *square integrable* modulo Z :

$$\int_{G_{\text{WH}}/Z} |\langle D^\lambda(q, p)\phi | \phi \rangle|^2 dq dp < \infty, \quad \text{for all } \phi \in \mathfrak{H}^\lambda \equiv L^2(\mathbb{R}^2).$$

Then, since the group G_{WH} is unimodular, every function $f \in L^2(\mathbb{R}, dx)$ is admissible.

The CS family so obtained are the *canonical coherent states*, the original ones introduced by Schrödinger in 1926 and rediscovered in the 60s by Glauber, Klauder and Sudarshan.¹⁹ The associated CS transform is variously called the *Gabor transform*, the *Windowed Fourier transform*, or the *Short Time Fourier transform*.

(2) *The affine group of $\mathbb{R}$*

$$G_{\text{aff}} = \{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}, a \neq 0\} = \mathbb{R} \rtimes \mathbb{R}_*.$$

The affine group, which is *not* unimodular, has only one square integrable representation (up to unitary equivalence, of course), acting in $L^2(\mathbb{R}, dx)$, namely,

$$[U(b, a)f](x) = |a|^{-1/2} f(a^{-1}(x - b)).$$

Then $\psi \in L^2(\mathbb{R}, dx)$ admissible if and only if

$$\int_{-\infty}^{\infty} |\widehat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty.$$

The associated CS transform is the *one-dimensional wavelet transform*.

(3) *The similitude group of the plane*

$$\text{SIM}(2) = \mathbb{R}^2 \rtimes (\mathbb{R}_*^+ \times \text{SO}(2)).$$

The situation is exactly the same as in one dimension. First, one notices that the natural operations that may be applied to a signal $s \in L^2(\mathbb{R}^2, d^2x)$ are obtained by combining three elementary transformations:

$$s_{b,a,\theta}(x) \equiv [U(b, a, \theta)s](x) = a^{-1}s(a^{-1}r_{-\theta}(x - b)),$$

where $b \in \mathbb{R}^2$ is a translation parameter, $a > 0$ is a dilation parameter and r_θ is a 2×2 rotation (orthogonal) matrix, $\theta \in [0, 2\pi)$. Then one checks that U is indeed a UIR of $\text{SIM}(2)$, acting in $L^2(\mathbb{R}^2, d^2x)$ and, again, it is the only one. Moreover U is square integrable and a vector $\psi \in L^2(\mathbb{R}^2, d^2x)$ is admissible, and called a *wavelet*, if it satisfies the condition

$$\int |\widehat{\psi}(k)|^2 \frac{d^2k}{|k|^2} < \infty.$$

The associated CS transform is the *two-dimensional wavelet transform*, which is studied thoroughly in our recent monograph.²⁰

5.4. Applications in quantum physics

There are plenty of applications of groups and group representations in quantum physics. Here are some of them (already mentioned in Section 1). A detailed description may be found in our article in the Encyclopedia of Physics.¹

Relativity: in order to completely define a physical system, one must choose a relativity group G^{rel} , that is, the group that leaves invariant the chosen class of equivalent reference frames. The standard examples are the Euclidean group (Euclidean geometry), the Galilei group (nonrelativistic mechanics), the Lorentz or the Poincaré group (nonrelativistic mechanics and electromagnetism). This choice then determines the tensorial properties of quantities, via the representations of the group G^{rel} . Although these considerations apply to classical physics, they are essential in quantum physics as well, as we have seen in Sections 5.1 and 5.2. For instance, the Poincaré group plays a central role in quantum field theory, in particular in the axiomatic approach.

In studying *atoms and molecules*, the rotation group $\text{SO}(3)$ simplifies enormously the spectroscopic data (e.g. selection rules); here again, this answers classification purposes. As we have seen in Section 4.2, the concept

of symmetry may be extended further, for instance, to approximate symmetries, accidental symmetries (H-atom), or dynamical groups. Furthermore, the interaction of matter (mostly atoms) with light is a hot topic now that powerful lasers are available. There, of course, coherent states and their relatives, the so-called squeezed states, play a prominent role. As we have seen in Section 5.3, this is another field of application of group theory.

As for *solid state* physics, it relies in an essential way on the crystallographic properties of solids, which are derived from group theory, as we have seen in Section 1.

Elementary particles: since the early 60s, the whole world of particle physics is dominated by group theory. One may distinguish several successive stages:

- . The original quark model is entirely based on the representations of $SU(3)$.
- . For understanding *dynamical properties*, additional symmetries are postulated: chiral symmetry $SU(2) \times SU(2)$, or $SU(3) \times SU(3)$, current algebra with a local $SU(3) \times SU(3)$ Lie algebra.
- . Going over to *gauge symmetries*, the same scheme repeats itself: QCD is based on $SU(3)$, the electroweak interactions are based on $SU(2) \times U(1)$, the Standard Model is based on $SU(3) \times SU(2) \times U(1)$.
- . The next step may be *supersymmetry*, which mixes bosons and fermions. But this leads also to new “super” mathematics: Lie superalgebras, supermanifolds, supergroups, etc.

In conclusion, one may say without exaggeration “... Except for calculus and linear algebra, no mathematical technique has been so successful”.¹

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