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Galois theory for reflexive graphs and simplicial objects

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Introduction

From Galois theory to central extensions

The fundamental theorem of classical Galois Theory, as it is commonly stated, gives a bijective order-reversing correspondence between the set of sub-extensions of a Galois extension F of a field K and the set of closed subgroups of its Galois group $\mathrm{Gal}(F/K)$. A slightly more precise way of expressing this correspondence is that the opposite of the category of sub-extensions of F is equivalent to the category of profinite spaces with a continuous action by $\mathrm{Gal}(F/K)$, and in fact this is a restriction of a more general contravariant equivalence between the separable K-algebras split over F and $\mathrm{Gal}(F/K)$-spaces, originally proven by Grothendieck in [74]. Meanwhile, a theory of Galois extensions of commutative rings was developed by M. Auslander and O. Goldman [2]. S. Chase, D. Harrison and A. Rosenberg then showed that the subgroups of the Galois groups could again be linked to sub-extensions of a Galois extension [38], although the correspondence was not quite perfect in cases where the rings involved had more than one nonzero idempotent. To fix this problem, O. Villamayor and D. Zelinsky replaced Galois groups with Galois *groupoids* [128, 129], and A. Magid then developed a Grothendieck-Galois theory for commutative rings, proving an anti-equivalence between the category of locally strongly separable R-algebras and the category of continuous actions of the fundamental groupoid of R [108]. A key point in Magid's approach was the use of the properties of the Pierce spectrum of a ring, which is defined as the spectrum of the Boolean algebra formed by its idempotent elements.

Grothendieck had developed his theory by analogy with the theory of coverings of a locally connected space B, and their equivalence with actions of the fundamental group $\pi_1(B)$. In fact, he introduced an abstract definition of *Galois category*, and proved that any such category was in fact equivalent to the category of actions of its suitably defined fundamental group. This led to the investigation of Galois theory and fundamental groups for toposes [5, 99].

In order to unite these developments, G. Janelidze introduced a notion of normal extension $p\colon E \to B$ in a category $\mathcal{C}$ relatively to a *Galois structure*, which is defined by an adjunction $I \dashv H$, where I is a functor $\mathcal{C} \to \mathcal{X}$, and suitable classes of extensions in $\mathcal{C}$ and $\mathcal{X}$ [80, 81, 82]. In this context, he proved that the category of extensions of B that are split by p is equivalent to a certain full subcategory of the category of actions of the Galois groupoid $\mathrm{Gal}_I(p)$, which is an internal groupoid in $\mathcal{X}$. In the particular case where the left adjoint I above is the Pierce spectrum functor between the opposite category of commutative rings and the category of profinite spaces, one recovers the results of Magid. Many other examples of Galois theories can be seen as particular cases of Janelidze's results [12, 86], including the topos-theoretic cases mentioned above [84, 32].

Other unexpected results, a priori unrelated to Galois Theory, turned out to be special cases as well. For the classical adjunction formed by the abelianization functor from the category of groups to the full subcategory of abelian groups, and using surjective morphisms as extensions, normal extensions turned out to coincide with central extensions, i.e. surjective group homomorphisms $f\colon G \to H$ such that $Ker(f) \subset Z(G)$. More generally, the central extensions for varieties $\mathcal{V}$ of Ω-groups relatively to a subvariety $\mathcal{W}$ of Fröhlich and Lue [62, 104, 63] are precisely the normal extensions induced by Janelidze's Galois theory for the reflector $\mathcal{V} \to \mathcal{W}$, with surjective morphisms as extensions.

Motivated by this case, G. Janelidze and G. M. Kelly used the categorical version of Galois theory to define a general notion of central extension relatively to an *admissible* reflective subcategory [88] $\mathcal{X}$ of a category $\mathcal{C}$. For this, and for the more general purpose of studying the properties of algebraic categories, certain categorical properties play a key role.

Categorical algebra

When moving away from varieties of universal algebra to categories, one needs to replace the classical concepts of image or surjective morphisms. Regular categories [6] were introduced to provide a convenient notion of image factorization of a morphism, which in turn can be used to define composition of internal relations. In such categories, the morphisms with maximal image, called regular or strong epimorphisms, can play the role of surjective morphisms to define extensions. A regular category where furthermore every equivalence relation occurs as the kernel pair of a morphism is called *(Barr)-exact* [6]; this property also plays an important role for Galois theory, as every regular epimorphism in such a category is effective for descent [94].

Another key notion is that of a *Mal'tsev category.* Mal'tsev categories were introduced by A. Carboni, J. Lambek and M. C. Pedicchio [35], and defined by one of the following equivalent properties :

- every reflexive relation is an equivalence relation
- the composition of two equivalence relations on the same object is an equivalence relation
- $R \circ S = S \circ R$ for any two equivalence relations R, S on the same object

(note that the last two properties only make sense in regular categories). They were named by analogy with Mal'tsev varieties [123], themselves named in reference to a theorem of A. I. Mal'tsev [109], who proved that, for varieties of universal algebras, the third property is equivalent to the existence in the theory of a ternary operation p satisfying the identities

$$p(x, y, y) = x \qquad \text{and} \qquad p(x, x, y) = y.$$

The main example of such a variety is the variety **Grp** of groups, where one can take the operation $p(x, y, z) = xy^{-1}z$. Many properties of groups can be extended to Mal'tsev varieties and Mal'tsev categories; notably, J. D. H. Smith developed a notion of commutator of equivalence relations in Mal'tsev varieties [123], which Pedicchio extended to exact Mal'tsev categories with coequalizers [118, 119]. One can then define an *abelian* object X as one for which the commutator $[\nabla_X, \nabla_X]_{SP}$ (where ∇_X is the largest equivalence relation on the object X) is trivial. The subcategory of abelian objects is then a reflective subcategory, with the reflection defined by quotienting out the commutator $[\nabla_X, \nabla_X]_{SP}$, and it is furthermore closed under subobjects and quotients. By analogy with Birkhoff's HSP theorem characterizing subvarieties in universal algebra, a subcategory with these properties is often called a *Birkhoff subcategory.* In an exact Mal'tsev category, such subcategories are admissible in the sense of Galois theory (at least when one takes regular epimorphisms as extensions), and for the subcategory of abelian objects the central extensions in the sense of Galois theory are precisely those for which the commutator $[Eq[f], X]_{SP}$ is trivial [90, 67].

Motivated by the links between centrality in the sense of Galois theory, abstract commutators, and Baer invariants in varieties of Ω-groups, T. Everaert, T. Van der Linden and M. Gran developed the theory of Baer invariants and "1-dimensional" commutator of normal subobjects relative to Birkhoff subcategories of suitable categories [58, 55]. More precisely, they defined a commutator $[X, N]_{\mathcal{X}}$ for any normal subobject N of X, in such a way that if $f \colon X \to Y$ is a regular epimorphism and $K[f]$ denotes its kernel, then $[X, K[f]]_{\mathcal{X}} = 0$ if and only if f is a central extension. Later T. Everaert and T. Van der Linden defined relative commutators $[M, N]_{\mathcal{X}}$ for any two normal subobjects M, N of X [61]. When working with kernels rather than kernel pairs, the notion of *protomodular* category, introduced by D. Bourn [16], is often more convenient than that of Mal'tsev category. For a pointed category, protomodularity is equivalent to the fact that the *Split Short Five Lemma* holds : given a diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, shift left, "p"] \arrow[d, "v"'] & Y \arrow[l, shift left, "s"] \arrow[d, "w"] \\ K' \arrow[r, "k'"'] & X' \arrow[r, shift left, "p'"] & Y' \arrow[l, shift left, "s'"] \end{tikzcd}$$

where k, k' are the kernels of p, p' respectively, $ps = 1_Y$ and $p's' = 1_{Y'}$, and all squares commute, then if u and w are isomorphisms, v is an isomorphism as well.

A category is called *semi-abelian* if it is pointed, protomodular, Barr-exact and has binary coproducts. These categories were introduced by G. Janelidze, L. Márki and W. Tholen [91] as a categorical framework to unify various non-abelian categories sharing similar homological and algebraic properties, similarly to the way abelian categories unify categories of modules. For example, the categories of groups, Lie algebras, non-unital rings, C^*-algebras, and cocommutative Hopf algebras are all semi-abelian. The classical isomorphism theorems hold in semi-abelian categories; one can also investigate notions of commutator and centrality of subobjects [26, 113].

Since semi-abelian categories are regular, there is a notion of image, which can be used to define exact sequences. The classical homological lemmas, such as the Five Lemma, Snake Lemma and 3×3 Lemma, are valid in any semi-abelian category [18, 10]. Thus it is possible to define, for example, Barr-Beck homology [4] with coefficients in a semi-abelian category [59]. In particular, when $\mathcal{C}$ is the category of groups, then the homology with the abelianization as coefficient functor coincides with the classical homology groups $H_n(G, \mathbb{Z})$. In particular, the second homology group $H_2(G)$ is given by the Hopf formula [76]:

$$H_2(G) \cong \frac{K \cap [F, F]}{[K, F]},$$

where F is a free group, K is a normal subgroup of F and $G \cong \frac{F}{K}$. Similarly, the third homology group $H_3(G)$ is given by the formula

$$H_3(G) \cong \frac{[F,F] \cap K_1 \cap K_2}{[K_1 \cap K_2, F] \cdot [K_1, K_2]},$$

where F is free group, K_1, K_2 are normal subgroups of F such that $\frac{F}{K_1}$ and $\frac{F}{K_2}$ are free and $G \cong \frac{F}{K_1 \cdot K_2}$. Such generalized Hopf formulae can be given for all homology groups [29], and similar formulae exist for the homology of groups with coefficients in k-nilpotent groups [46]. In fact, thanks to categorical Galois theory one can develop such formulae for the homology of a semi-abelian monadic category in any Birkhoff subcategory [57]. Alternatively, one can use such formulae to define the homology with coefficients in a Birkhoff subcategory of any semi-abelian category with enough projectives [58, 55, 51, 87].

Crossed modules and Peiffer commutators

A *precrossed module* is defined by two groups X, B, a group homomorphism $\partial\colon X \to B$ and an action of B on X $(b, x) \mapsto b \cdot x$, satisfying the condition $\partial(b \cdot x) = b\partial(x)b^{-1}$. A *crossed module* is then a precrossed module that also satisfies the Peiffer condition $\partial(x) \cdot y = xyx^{-1}$. Crossed modules were initially introduced by Whitehead [130, 131], who used them to study the effect of adding 1-cells on the second homotopy groups of CW complexes. In the following years they found their way to various parts of mathematics : Mac Lane and Whitehead showed that they could be used to model all connected homotopy 2-types [107], while Mac Lane also used them in low-dimensional group cohomology, to give an interpretation of third cohomology groups [105]. Later, they were used as coefficients for non-abelian cohomology by Dedecker and others [44, 45].

An internal reflexive graph in the category of groups is determined by two groups G, B, together with morphisms $c, d\colon G \to B$ with a common section i; and an internal groupoid in the category of groups (also called a *cat*^{1}-group by J.-L. Loday [103]) is a reflexive graph which can be equipped with a (unique) group homomorphism $G \times_B G \to G$ in such a way that the underlying sets and functions form a small groupoid. There is an equivalence between precrossed modules and reflexive graphs of groups, which restricts to an equivalence between crossed modules and internal groupoids [31]; and this equivalence stems from the construction of semi-direct products, and the fact that every split epimorphism of groups or algebras is isomorphic to a semi-direct product. One can define similar notions for other algebraic categories, while preserving these links between actions and split extensions, and between crossed modules and internal groupoids [102, 121]. In fact, actions and semi-direct products can be defined in any semi-abelian category [28, 13]. G. Janelidze used these notions to define internal (pre)crossed modules in semi-abelian categories [85].

For precrossed modules of groups, D. Conduché and G. Ellis defined the *Peiffer commutator* $\langle M, N\rangle$ of two precrossed B-submodules M, N of a precrossed B-module X as the subobject generated by all terms of the form $mnm^{-1}(\partial(m)\cdot n)^{-1}$ and $nmn^{-1}(\partial(n)\cdot m)^{-1}$. They used this commutator to define the homology of a precrossed B-module X via the generalized Hopf formulae

$$H_1(X) = \frac{X}{\langle X, X\rangle} \qquad \text{and} \qquad H_2(X) = \frac{K \cap \langle F, F\rangle}{\langle K, F\rangle},$$

where

$$\{1\} \longrightarrow K \longrightarrow F \longrightarrow X \longrightarrow \{1\}$$

is an exact sequence of precrossed B-modules and F a free precrossed B-module.

Objectives and main results

More recently, the notion of Peiffer commutator was defined in semi-abelian categories by A. S. Cigoli, S. Mantovani and G. Metere [40]. One of the goals of this thesis is to establish a link between this internal Peiffer commutator and centrality for precrossed modules relatively to crossed modules. For this purpose, we study the Galois structure associated with the reflections of reflexive graphs into groupoids in an exact Mal'tsev category with coequalizers, and establish a characterization of central and double central extensions. For Mal'tsev varieties, T. Everaert and M. Gran proved that a surjective morphism

$$\begin{tikzcd} X_1 \arrow[rr, "f_1"] \arrow[dr, shift left=2, "c"] \arrow[dr, shift right=2, "d"'] & & Y_1 \arrow[dl, shift right=2, "d'"'] \arrow[dl, shift left=2, "c'"] \\ & B \arrow[ul, "e"] \arrow[ur, "e'"] & \end{tikzcd}$$

of reflexive graphs over a fixed base B is a central extension with respect to groupoids if and only if its kernel pair satisfies the commutator conditions

$$[Eq[f], Eq[c]]_{SP} = \Delta_X = [Eq[f], Eq[d]]_{SP}.$$

We extend this result to the case where $\mathcal{C}$ is an exact Mal'tsev category with coequalizers. Moreover, we prove that the same commutator conditions characterizes the central extensions of a Galois structure on the category $2\text{-}\mathbf{Eq}(\mathcal{C})$, whose objects are triples (X, R, S) where X is an object of $\mathcal{C}$ and R, S are two equivalence relations on X, and its full subcategory $\mathbf{Conn}(\mathcal{C})$ defined by the triples (X, R, S) where the relations are *connected*, (i.e. centralize each other, in the sense that $[R, S]_{SP} = \Delta_X$). In this framework, we use as extensions the regular epimorphisms $f\colon (X, R, S) \to (X', R', S')$ of pairs of equivalence relations that induce isomorphisms on the quotients $\frac{X}{R} \cong \frac{X'}{R'}$ and $\frac{X}{S} \cong \frac{X'}{S'}$; we call such morphisms *fibrations*.

Theorem 2.6. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers, and $f\colon (X, R, S) \to (X', R', S')$ a fibration in $2\text{-}\mathbf{Eq}(\mathcal{C})$. Then the following conditions are equivalent :*

1. *f is a $\Gamma_{\mathcal{F}}$-central extension;*
2. *f is a $\Gamma_{\mathcal{F}}$-normal extension;*
3. *the following commutator condition holds :*

$$[Eq[f], R \vee S]_{SP} = \Delta_X.$$

We also give a characterization of double central extensions in terms of commutators, inspired by the characterization of double $\mathbf{Ab}(\mathcal{C})$-central extensions in an exact Mal'tsev category with coequalizers [71, 60].

Theorem 2.15. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers. A double extension*

$$\begin{tikzcd}
(X, R, S) \arrow[r, "g"] \arrow[d, "f"'] & (Z, R_Z, S_Z) \arrow[d, "h"] \\
(Y, R_Y, S_Y) \arrow[r, "p"'] & (W, R_W, S_W)
\end{tikzcd}$$

is $\Gamma_{\mathcal{F}}$-central if and only if the commutator conditions

$$[Eq[f], Eq[g]]_{SP} = \Delta_X = [Eq[f] \wedge Eq[g], R \vee S]_{SP}$$

are satisfied.

Using these results, we then give centrality criteria for extensions and double extensions of precrossed modules with respect to crossed modules in semi-abelian categories satisfying the condition of "uniqueness of actions" (UA) introduced in [40] (see page 32).

Theorem 3.12. *Let $\mathcal{C}$ be a semi-abelian category satisfying* (UA)*, and B an object in $\mathcal{C}$. An extension*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\partial_X"'] & & Y \arrow[dl, "\partial_Y"] \\ & B & \end{tikzcd}$$

in $\mathbf{PXMod}_B(\mathcal{C})$ is $\mathbf{XMod}_B(\mathcal{C})$-central with if and only if the internal Peiffer commutator $\langle K[f], X\rangle$ is trivial.

Theorem 3.17. *Let $\mathcal{C}$ be a semi-abelian category satisfying* (UA)*, and B an object in $\mathcal{C}$. A double extension*

$$\begin{tikzcd} (X, \partial_X, \xi_X) \arrow[r, "g"] \arrow[d, "f"'] & (Z, \partial_Z, \xi_Z) \arrow[d, "h"] \\ (Y, \partial_Y, \xi_Y) \arrow[r, "p"'] & (W, \partial_W, \xi_W) \end{tikzcd}$$

in $\mathbf{PXMod}_B(\mathcal{C})$ is a double central extension if and only if

$$\langle K[f] \wedge K[g], X\rangle = 0 = \langle K[f], K[g]\rangle.$$

This allows us to define homology objects and a Stallings-Stammbach exact sequence for internal precrossed modules, thus extending the work of Conduché and Ellis [43].

We also study another Galois structure, based on simplicial objects and groupoids. This is inspired by the work of R. Brown and G. Janelidze [30], who studied a Galois structure on the adjunction between the fundamental groupoid functor and the nerve functor [64], with Kan fibrations as extensions. They showed that Kan complexes are admissible for this adjunction; since regular Mal'tsev categories can be characterized by the fact that every simplicial object satisfies the Kan condition, this suggests that a similar admissible Galois structure might be constructed in a Mal'tsev category $\mathcal{C}$. In fact, we can explicitly construct the reflection of the category $\mathbf{Simp}(\mathcal{C})$ of simplicial objects into the category $\mathbf{Grpd}(\mathcal{C})$ of internal groupoids, by quotienting out the relations $H_n(\mathbb{X})$ defined on each X_n by

$$H_1(\mathbb{X}) = d_1(Eq[d_0] \wedge Eq[d_2]) \quad \text{and} \quad H_n = \bigvee_{0\leq i<j\leq n} (Eq[d_i] \wedge Eq[d_j]).$$

This gives us the following

Theorem 4.10. *Let $\mathcal{C}$ be an exact Mal'tsev category. The inclusion (or nerve) functor* $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$ *admits a left adjoint*

$$\Pi_1 \colon \mathbf{Simp}(\mathcal{C}) \to \mathbf{Grpd}(\mathcal{C}),$$

and this makes $\mathbf{Grpd}(\mathcal{C})$ *a Birkhoff subcategory of* $\mathbf{Simp}(\mathcal{C})$.

Thus $\mathbf{Grpd}(\mathcal{C})$ is in particular an admissible subcategory of $\mathbf{Simp}(\mathcal{C})$. Furthermore, we can characterize the corresponding central extensions by determining the conditions that their kernel pairs have to satisfy :

Theorem 4.15. *A regular epimorphism $f \colon \mathbb{X} \to \mathbb{Y}$ in* $\mathbf{Simp}(\mathcal{C})$ *is a* $\mathbf{Grpd}(\mathcal{C})$*-central extension if and only if*

$$d_1(Eq[f_1] \wedge Eq[d_0] \wedge Eq[d_2]) = \Delta_{X_1}$$

and

$$Eq[f_n] \wedge Eq[d_i] \wedge Eq[d_j] = \Delta_{X_n} \quad \forall 0 \leq i < j \leq n$$

for all $n \geq 2$.

We can also give a more geometric interpretation of these conditions. If we denote $\Lambda^n_k(\mathbb{X})$ the object of (n,k)-horns of $\mathbb{X}$, and $\lambda^n_k(\mathbb{X}) \colon X_n \to \Lambda^n_k(\mathbb{X})$ the canonical morphism, then the conditions above give us

Theorem 4.19. *A regular epimorphism $f \colon \mathbb{X} \to \mathbb{Y}$ in* $\mathbf{Simp}(\mathcal{C})$ *is a* $\mathbf{Grpd}(\mathcal{C})$*-central extension if and only if f is an exact fibration at all dimensions $n \geq 2$ in the sense of Glenn ([65]), i.e. for all $n \geq 2$ and $0 \leq k \leq n$ the square*

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\lambda^n_k"'] & Y_n \arrow[d, "\lambda^n_k"] \\ \Lambda^n_k(\mathbb{X}) \arrow[r, "\Lambda^n_k(f)"'] & \Lambda^n_k(\mathbb{Y}) \end{tikzcd}$$

is a pullback.

We can also draw some parallels between our constructions and the classical construction of the homotopy groupoid of a Kan complex [9]. We also draw parallels between the Smith-Pedicchio commutator $[Eq[d_0], Eq[d_1]]_{SP}$ and the equivalence relation $H_1(\mathbb{X})$, which are both used to provide reflections into a subcategory of groupoids; this connection allows us to give a new result characterizing the internal groupoids in terms of weighted commutators.

Corollary 4.30. *For any reflexive graph in a semi-abelian category $\mathcal{C}$, the weighted commutator $[K[d_0], K[d_1]]_{s_0 \colon X_0 \to X_1}$ coincides with the Ursini-Mantovani commutator $[K[d_0], K[d_1]]_{UM}$.*

Outline of the thesis

In the first chapter we review the main concepts needed in the rest of the thesis, such as regular and exact categories, simplicial objects, internal graphs and groupoids, Mal'tsev categories, semi-abelian categories, and categorical Galois theory.

The second chapter is adapted from the article [50], written jointly with Marino Gran. There we define a Galois structure on the category of pairs of equivalence relations in an exact Mal'tsev category with coequalizers, and characterize central and double central extensions in terms of higher commutator conditions. These results generalize both the ones related to the abelianization functor in exact Mal'tsev categories, and the ones corresponding to the reflection from the category of internal reflexive graphs to the subcategory of internal groupoids.

The third chapter is adapted from the article [41], written with Alan S. Cigoli, Marino Gran and Sandra Mantovani. We describe the construction of the Peiffer commutator and its properties, before proving that, under the condition (UA), central extensions in $\mathbf{PXMod}_B(\mathcal{C})$ can be characterized as those such that the Peiffer commutator $\langle K[f], X\rangle$ is trivial. We then use this characterization and a result from [55] to get a five term exact sequence in homology, where the homology objects in $\mathbf{PXMod}_B(\mathcal{C})$ are expressed in terms of generalized Hopf formulae. When $\mathcal{C}$ is the category of Lie algebras, one obtains an exact sequence in the category of Lie algebra precrossed modules (see Example 3.16). In the last section a characterization of double central extensions relative to the induced adjunctions between the categories of extensions and of central extensions in $\mathbf{PXMod}_B(\mathcal{C})$ will also be established (Theorem 3.17). From this, an explicit Hopf formula describing the Galois group of a weakly universal double central extension will be deduced.

The fourth chapter is adapted from the preprint [49]. We show that the category of internal groupoids in an exact Mal'tsev category is reflective, and in fact a Birkhoff subcategory of the category of simplicial objects. We then characterize the central extensions of the corresponding Galois structure, and show that regular epimorphisms in $\mathbf{Simp}(\mathcal{C})$ admit a relative monotone-light factorization system in the sense of Chikhladze [39]. We also draw some comparisons with Kan complexes. By comparing the reflections of simplicial objects and reflexive graphs into groupoids, we exhibit a connection with weighted commutators.

1 Background

In this chapter, we review the main concepts and notations needed in the rest of the thesis, and give some illustrating examples. We begin with regular and exact categories, including inverse and direct images and the calculus of relations. Then we turn to simplicial objects, internal graphs and groupoids. Next, we consider Mal'tsev categories and double extensions. We then review protomodular and semi-abelian categories, including internal actions and semi-direct products, as well as internal precrossed and crossed modules. Finally, we give an introduction to categorical Galois theory, notably Galois structures and the notion of admissibility, trivial, central and normal extensions, and finally higher (central) extensions.

1.1 Regular and exact categories

Let us recall that a subobject of an object X in a category $\mathcal{C}$ is an equivalence class of monomorphisms with codomain X, with two such morphisms $m\colon M \to X$ and $n\colon N \to X$ considered equivalent whenever there exists an isomorphism $\varphi : M \to N$ such that $n\varphi = m$. More generally, for two subobjects M, N we will write $M \leq N$ when there exists a morphism (not necessarily invertible) φ such that $n\varphi = m$. Note that M and N are then equal as subobjects if and only if $M \leq N$ and $N \leq M$.

If $\mathcal{C}$ is a category with finite limits, an internal relation between two objects X and Y is a subobject R of their product $X \times Y$. This can equivalently be described as a pair of maps $r_1\colon R \to X$ and $r_2\colon R \to Y$ that are *jointly monic*, i.e. such that $r_1u = r_1v$ and $r_2u = r_2v$ imply $u = v$ for all parallel morphisms u, v. If R is a relation between X and Y, we will also denote R^o the relation $(r_2, r_1)\colon R \to Y \times X$.

If $r = (r_1, r_2)\colon R \to X \times X$ is an internal relation, then it is called reflexive (resp. symmetric, transitive) if for every object Y of $\mathcal{C}$, the relation on $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ defined by $\mathrm{Hom}_{\mathcal{C}}(Y, R)$ is reflexive (resp. symmetric, transitive). This can equivalently be formulated by saying that the relation R is

- *reflexive* if there exists $\delta\colon X \to R$ such that $r\delta = (1,1)$.
- *symmetric* if $R^o = R$, i.e. if there exists $\sigma\colon R \to R$ such that $(r_1, r_2)\sigma = (r_2, r_1)$.
- *transitive* if there exists $\tau\colon R \times_X R \to R$ such that $r\tau = (r_1\pi_1, r_2\pi_2)$, where $R \times_X R$ is the pullback of r_2 along r_1.

R is called an internal equivalence relation if it is reflexive, symmetric and transitive.

Example 1.1. For any object X, the *discrete* equivalence relation Δ_X is the monomorphism $\delta_X = (1,1)\colon X \to X \times X$; the *indiscrete* equivalence relation ∇_X is simply the identity $X \times X \to X \times X$. Note that every relation R on X satisfies $R \leq \nabla_X$, and R is reflexive if and only if $R \geq \Delta_X$.

Example 1.2. If $f\colon X \to Y$, then its *kernel pair* $Eq[f]$ is determined by the pullback

$$\begin{tikzcd} Eq[f] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

The kernel pair of f is an equivalence relation. Moreover, f is a monomorphism if and only if $Eq[f] = \Delta_X$.

Alternatively, $Eq[f]$ can be defined as the pullback

$$\begin{tikzcd} Eq[f] \arrow[r] \arrow[d, "{(\pi_1,\pi_2)}"'] & Y \arrow[d, "{\delta_Y=(1,1)}"] \\ X \times X \arrow[r, "f\times f"'] & Y \times Y. \end{tikzcd}$$

If $f\colon X \to Y$ is a morphism of $\mathcal{C}$ and $n\colon N \to Y$ a subobject of N, its inverse image $f^{-1}(N)$ is defined by the pullback

$$\begin{tikzcd} f^{-1}(N) \arrow[r] \arrow[d, "n'"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "n"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

If S is an equivalence relation on Y, then its inverse image $(f \times f)^{-1}(S)$ is an equivalence relation on X, which we will simply denote $f^{-1}(S)$. For example, we have $f^{-1}(\Delta_Y) = Eq[f]$.

Definition 1.3 ([6])**.** A finitely complete category $\mathcal{C}$ is called

- *regular* if every kernel pair has a coequalizer, and regular epimorphisms are stable under pullbacks, i.e. in a pullback square

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

  if p is a regular epimorphism then so is π_1.

- *exact* if it is regular and every equivalence relation in $\mathcal{C}$ is the kernel pair of some morphism.

In a regular category, every morphism $f\colon X \to Y$ has a kernel pair $Eq[f]$, and this kernel pair admits a coequalizer $q\colon X \to I$. Since $f\pi_1 = f\pi_2$ by construction, there exists a unique morphism $m\colon I \to Y$ such that $mq = f$. One can show that this m is a monomorphism. Thus any morphism in a regular category can be factorized as a regular epimorphism followed by a monomorphism. Such a factorization is unique up to unique isomorphism, and stable under pullbacks; given a morphism $f\colon X \to Y$ in $\mathcal{C}$, its image $\mathrm{im}(f)$ is simply the subobject of Y determined by the factorization of f.

Example 1.4. The category **Set** of sets is exact. Regular epimorphisms coincide with surjective functions, and monomorphisms coincide with injective functions. Thus the image of a function $f\colon X \to Y$ is simply the subset $f(X)$ of Y, and its factorization is given by $X \to f(X) \hookrightarrow Y$.

More generally, every category monadic over **Set** is exact [47], and monadic functors into **Set** preserve and reflect the regular epimorphism-monomorphism factorizations; thus for examples all varieties of algebras are exact categories, and the factorizations in such categories are simply the classical factorizations into a surjective homomorphism followed by an injective morphism. Moreover, every quasivariety (i.e. a full subcategory of a variety closed under products and subobjects) is a regular category [120].

Example 1.5. Any abelian category is exact, and the image of a morphism is given by the kernel of its cokernel ; in fact, a category is abelian if and only if it is exact and additive, a fact which is often referenced as the "Tierney equation" (see [3])

$$(\text{Barr-exact}) + (\text{additive}) = (\text{abelian}).$$

Moreover, every quasi-abelian category [122] is regular and coregular.

Proposition 1.6 ([34])**.** *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two morphisms in a regular category $\mathcal{C}$. Then g factors through the regular image of f if and only if there exist an object W of $\mathcal{C}$ with a morphism $h\colon W \to X$ and a regular epimorphism $q\colon W \to Y$ such that $fh = gq$.*

The existence of these factorizations allows one to define the direct image of a subobject M of X along a morphism $f\colon X \to Y$ as the image of the composition $fm\colon M \to Y$. One can also use this to compose relations in the following way : if $R \leq X \times Y$ and $S \leq Y \times Z$ are relations, we form the pullback

$$\begin{tikzcd}
R \times_X S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "s_1"] \\
R \arrow[r, "r_2"'] & Y,
\end{tikzcd}$$

and the composition $SR \leq X \times Z$ is the relation determined by the image of $(r_1\pi_1, s_2\pi_2)\colon X \times Z$. This composition is associative (thanks to the stability of images under pullbacks), and moreover the discrete relation Δ_X is an identity for this composition. Thus for every regular category $\mathcal{C}$ there exists a category $\mathbf{Rel}(\mathcal{C})$ of internal relations, which is in fact a locally posetal 2-category. Note that $(SR)^o = R^o S^o$.

Every $f\colon X \to Y$ can be identified with the relation $(1, f)\colon X \to X \times Y$; one can check that this defines a faithful, injective on objects functor $\mathcal{C} \to \mathbf{Rel}(\mathcal{C})$. Then a relation $r = (r_1, r_2)\colon R \to X \times Y$ is equal to $r_2 r_1^o$, while given two maps $f\colon X \to Z$ and $g\colon Y \to Z$, the composition $g^o f$ is simply the relation from X to Y defined by the pullback $X \times_Z Y$, as a subobject of $X \times Y$. In particular, the kernel pair of $f\colon X \to Y$ coincides with the composition $f^o f$, so that we always have $f^o f \geq \Delta_X$, with equality if and only if f is a monomorphism. On the other hand, ff^o is defined as the image of the map $(f, f)\colon X \to Y \times Y$, and thus $ff^o \leq \Delta_Y$, with equality if and only if f is a regular epimorphism. As a consequence, every morphism f in $\mathcal{C}$ satisfies the identity $f = ff^o f$. If S is a relation on Y, then the inverse image $f^{-1}(S)$ is equal to the composition $f^o S f$, while if R is a relation on X, its direct image $f(R)$ is equal to the composition fRf^o.

The following result plays an important role in the theory of regular categories. It is often referred to as the Barr-Kock theorem ([3, Example 6.10]), although it had been proved earlier by Grothendieck [73, Proposition 4.2]. A proof can also be found in [27, Theorem 2.17].

Proposition 1.7. *Given a commutative diagram*

$$\begin{tikzcd} Eq[f] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\widetilde{u}"'] \arrow[dr, phantom, "(1)"] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "(2)"] & Y \arrow[d, "v"] \\ Eq[f'] \arrow[r, shift left] \arrow[r, shift right] & X' \arrow[r, "f'"'] & Y', \end{tikzcd}$$

if the square (2) *is a pullback, then so are the two squares* (1)*. If f is a regular epimorphism, then the converse is true as well.*

1.2 Internal structures

1.2.1 Simplicial objects

Let Δ denote the category of finite non-empty ordinals and monotone maps. This means that the objects of Δ are the sets $[n] = \{0, \ldots, n\}$ for $n \geq 0$, with functions $f\colon [m] \to [n]$ such that $i \leq j$ implies $f(i) \leq f(j)$ as morphisms. These morphisms are generated by the following two classes of morphisms :

- the *cofaces* $\delta_i\colon [n-1] \to [n]$, $i = 0, \ldots, n$, where

$$\delta_i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

- the *codegeneracies* $\sigma_i = [n+1] \to [n]$, $i = 0, \ldots, n$, where

$$\sigma_i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

In fact every morphism $f\colon [m] \to [n]$ can be written uniquely as a composition $\delta_{i_{n-r}} \cdots \delta_{i_1} \sigma_{j_1} \cdots \sigma_{j_{m-r}}$, with $i_1 \leq \cdots \leq i_{n-r}$ and $j_1 \leq \cdots \leq j_{m-r}$. This decomposition is obtained by considering the factorization of f as a surjective function $[m] \to [r]$ followed by an injective one $[r] \to [n]$. The indices i_l can then be obtained by setting

$$\{i_1, \ldots, i_{n-r}\} = [n] \setminus f([m]),$$

while the indices j_l are determined by setting

$$\{j_1, \ldots, j_{m-r}\} = \{j \in [m] \mid f(j) = f(j+1)\}.$$

For a given category $\mathcal{C}$, the category $\mathbf{Simp}(\mathcal{C})$ of simplicial objects in $\mathcal{C}$ is then defined as the category of functors $[\Delta^{op}, \mathcal{C}]$[64, 48]. Equivalently, an object $\mathbb{X}$ of $\mathbf{Simp}(\mathcal{C})$ is a collection of objects $(X_n)_{n\in\mathbb{N}}$, given by $\mathbb{X}([n]) = X_n$, together with *face maps* $d_i \colon X_n \to X_{n-1}$ for all $n > 0$ and $0 \leq i \leq n$, given by $\mathbb{X}(\delta_i) = d_i$, and *degeneracy maps* $s_i \colon X_n \to X_{n+1}$ for $n \geq 0$ and $0 \leq i \leq n$, given by $\mathbb{X}(\sigma_i) = s_i$. The face and degeneracy maps must then satisfy the following simplicial identities :

$$\begin{array}{rcll}
d_i d_j & = & d_{j-1} d_i & \text{if } 0 \leq i < j \leq n+1 \\
s_i s_j & = & s_{j+1} s_i & \text{if } 0 \leq i \leq j \leq n \\
d_i s_j & = & s_{j-1} d_i & \text{if } 0 \leq i < j \leq n-1 \\
& = & 1 & \text{if } i \in \{j, j+1\} \\
& = & s_j d_{i-1} & \text{if } 0 \leq j < i-1 \leq n-1.
\end{array}$$

When necessary, we will write $d_i^{\mathbb{X}}$ and $s_i^{\mathbb{X}}$ to distinguish the face or degeneracy maps of different simplicial objects. A morphism $f \colon \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Simp}(\mathcal{C})$ is then a collection of morphisms $f_n \colon X_n \to Y_n$ that commute with face and degeneracy maps, in the sense that $d_i^{\mathbb{Y}} f_{n+1} = f_n d_i^{\mathbb{X}}$ and $s_i^{\mathbb{Y}} f_n = f_{n+1} s_i^{\mathbb{X}}$ for all $n \geq 0$ and $0 \leq i \leq n$.

If $\mathbb{X}$ is a simplicial object, we will denote $Dec(\mathbb{X})$ the simplicial object $(X_{n+1})_{n\geq 0}$, whose face and degeneracies are the same as those of $\mathbb{X}$, without the last faces $d_{n+1} \colon X_{n+1} \to X_n$ and last degeneracies $s_n \colon X_n \to X_{n+1}$ for all $n \geq 1$ [78]. The simplicial identities imply that the maps $d_{n+1} \colon X_{n+1} \to X_n$ form a morphism of simplicial objects :

$$\begin{tikzcd}
\cdots & X_3 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_2"'] \arrow[d, "d_3"'] & X_2 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "d_2"] & X_1 \arrow[l, "s_0" description] \arrow[d, "d_1"] & Dec(\mathbb{X}) \arrow[d, "\epsilon_{\mathbb{X}}"] \\
\cdots & X_2 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_2"'] & X_1 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0" description] & \mathbb{X}.
\end{tikzcd}$$

Since all these maps are split (and thus regular) epimorphisms, $\epsilon_{\mathbb{X}}$ is a regular epimorphism in $\mathbf{Simp}(\mathcal{C})$, although it need not be a split epimorphism. Notice that Dec defines an endofunctor of $\mathbf{Simp}(\mathcal{C})$, and ϵ is a natural transformation from Dec to the identity endofunctor. In fact, Dec has a natural comonad structure, with ϵ as the counit.

By definition Δ is a skeleton of, and thus equivalent to, the category of non-empty finite totally ordered sets. In particular, since this category contains the poset $\mathcal{P}_{f,n.e.}(\mathbb{N})$ of non-empty finite subsets of $\mathbb{N}$ (ordered by inclusion) as a subcategory, there is a canonical functor $\Phi\colon \mathcal{P}_{f,n.e.}(\mathbb{N}) \to \Delta$ that maps any set with $n+1$ elements to $\{0,\dots,n\}$ and any inclusion map to an injective morphism in Δ.

For a given simplicial object $\mathbb{X}$, and for every $n \geq 2$, one can consider the restriction of Φ to the poset of proper subsets of $\{0,1,\dots,n\}$; taking the opposite functor and composing with $\mathbb{X}\colon \Delta^{op} \to \mathcal{C}$ gives a diagram in $\mathcal{C}$. The limit of this diagram is the n-th *simplicial kernel* of $\mathbb{X}$, and denoted $K_n(\mathbb{X})$ [125]. In particular, we have maps $\mu_i\colon K_n(\mathbb{X}) \to X_{n-1}$ for $i = 0,\dots,n$, satisfying $d_i\mu_j = d_{j-1}\mu_i$ for all $0 \leq i < j \leq n$, and the maps μ_i are universal with this property. Thus the face maps $d_0,\dots,d_n\colon X_n \to X_{n-1}$ induce a canonical map $\kappa_n\colon X_n \to K_n(\mathbb{X})$. Following [65], we say that $\mathbb{X}$ is *aspherical at dimension* n if κ_n is a regular epimorphism, and aspherical if it is aspherical at dimension n for all $n \geq 1$. In [53] these simplicial objects are called *exact* (at n).

Moreover, for every $n \geq 2$ and $0 \leq k \leq n$, we can also restrict Φ to the poset of proper subsets of $\{0,\dots,n\}$ that contain k, and then compose the opposite functor with $\mathbb{X}$. The limit of this diagram is the object of (n,k)-horns $\Lambda^n_k(\mathbb{X})$, and it is equipped with maps $\nu_i\colon \Lambda^n_k(\mathbb{X}) \to X_{n-1}$ for $0 \leq i \leq n$ and $i \neq k$ that satisfy the identities $d_i\nu_j = d_{j-1}\nu_i$ for all $0 \leq i < j \leq n$ and $i \neq k \neq j$, and are universal with this property. There is then also a canonical morphism $\lambda^n_k\colon X_n \to \Lambda^n_k(\mathbb{X})$ induced by the face maps $d_i\colon X_n \to X_{n-1}$ for $i \neq k$, and $\mathbb{X}$ is said to satisfy the Kan property if all these maps are regular epimorphisms. Moreover, a map $f\colon \mathbb{X} \to \mathbb{Y}$ between simplicial objects is called a *Kan fibration* if for all n and k the canonical morphism θ^n_k in the diagram

$$\begin{tikzcd}
X_n \arrow[dr, "\theta^n_k"] \arrow[drr, bend left, "f_n"] \arrow[ddr, bend right, "\lambda^n_k"'] & & \\
& \Lambda^n_k(\mathbb{X}) \times_{\Lambda^n_k(\mathbb{Y})} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_n \arrow[d, "\lambda^n_k"] \\
& \Lambda^n_k(\mathbb{X}) \arrow[r, "\Lambda^n_k(f)"'] & \Lambda^n_k(\mathbb{Y})
\end{tikzcd} \tag{1.1}$$

is a regular epimorphism.

For every $n \geq 1$, we denote Δ_n the full subcategory of Δ consisting of the ordinals with $n+1$ elements or less, and $\mathbf{Simp}_n(\mathcal{C})$ the category of functors $\Delta^{op}_n \to \mathcal{C}$, whose objects we call n-truncated simplicial objects. The inclusion $\Delta_n \hookrightarrow \Delta$ then induces by precomposition the truncation functor $Tr_n\colon \mathbf{Simp}(\mathcal{C}) \to \mathbf{Simp}_n(\mathcal{C})$. Whenever $\mathcal{C}$ has finite limits (resp. colimits), this truncation functor has a right (resp. left) adjoint, called the *n-coskeleton* (resp. *n-skeleton* Sk_n) [48]; it is determined by the right (resp. left) Kan extension along $\Delta_n \hookrightarrow \Delta$. In particular, the n-coskeleton $Cosk_n$ can be constructed iteratively, by defining X_n as the simplicial kernel of the truncated simplicial object $X_{n-1} \overset{d_0}{\underset{d_{n-1}}{\rightrightarrows}} X_{n-2} \dots$, and the face maps $d_i \colon X_n \to X_{n-1}$ as the canonical projections.

1.2.2 Internal graphs, categories, and groupoids

Definition 1.8. An *internal reflexive graph* in $\mathcal{C}$ is given by two objects X_1, X_0 of $\mathcal{C}$ and morphisms $c, d \colon X_1 \to X_0$ and $e \colon X_0 \to X_1$ such that $ce = 1_{X_0} = de$. A morphism of reflexive graphs $(X_1, X_0, c, d, e) \to (Y_1, Y_0, c', d', e')$ is given by two morphisms $f_1 \colon X_1 \to Y_1$, $f_0 \colon X_0 \to Y_0$ making the following diagrams commute :

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & Y_1 \arrow[d, shift right=2, "d'"'] \arrow[d, shift left=2, "c'"] \\ X_0 \arrow[u, "e" description] \arrow[r, "f_0"'] & Y_0. \arrow[u, "e'" description] \end{tikzcd}$$

In particular, the category $\mathbf{RG}(\mathcal{C})$ of internal reflexive graphs in $\mathcal{C}$ coincides with the category $\mathbf{Simp}_1(\mathcal{C})$ of 1-truncated simplicial objects.

Definition 1.9. 1. An *internal multiplicative graph* is a reflexive graph endowed with a partial multiplication $m \colon X_1 \times_{X_0} X_1 \to X_1$, defined on the pullback

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "c"] \\ X_1 \arrow[r, "d"'] & X_0 \end{tikzcd}$$

that is unital and compatible with the domain and codomain maps [36], in the sense that $m(1, ed) = 1 = m(ec, 1)$ and $cm = c\pi_2$ and $dm = d\pi_1$.

2. An *internal category* is a multiplicative graph whose multiplication is associative, in the sense that $m(1 \times m) = m(m \times 1)$.

3. An *internal groupoid* is an internal category with an additional morphism $\sigma \colon X_1 \to X_1$ satisfying $c\sigma = d$ and $d\sigma = c$, and such that $m(1, \sigma) = 1 = m(\sigma, 1)$.

4. An *internal functor* between two multiplicative graphs, categories or groupoids is a morphism of the underlying reflexive graphs that also preserves the multiplication, in the sense that the following diagram commutes :

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, "f_1 \times_{f_0} f_1"] \arrow[d, "m"'] & Y_1 \times_{Y_0} Y_1 \arrow[d, "m'"] \\
X_1 \arrow[r, "f_1"'] & Y_1.
\end{tikzcd}$$

All these conditions can be summarized by saying that an internal multiplicative graph is an object of $\mathbf{Simp}_2(\mathcal{C})$, such that the square

$$\begin{tikzcd}
X_2 \arrow[r, "d_2"] \arrow[d, "d_0"'] & X_1 \arrow[d, "d_0"] \\
X_1 \arrow[r, "d_1"'] & X_0
\end{tikzcd}$$

is a pullback, and an internal category is an object of $\mathbf{Simp}_3(\mathcal{C})$ such that the square above and the square

$$\begin{tikzcd}
X_3 \arrow[r, "d_3"] \arrow[d, "d_0"'] & X_2 \arrow[d, "d_0"] \\
X_2 \arrow[r, "d_2"'] & X_1
\end{tikzcd}$$

are pullbacks.

Moreover an internal category is an internal groupoid if and only if any of the squares

$$\begin{tikzcd}
X_2 \arrow[r, "d_1"] \arrow[d, "d_0"'] & X_1 \arrow[d, "d_0"] \\
X_1 \arrow[r, "d_0"'] & X_0
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X_2 \arrow[r, "d_2"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_1"] \\
X_1 \arrow[r, "d_1"'] & X_0
\end{tikzcd}$$

is a pullback. Internal functors are also the same thing as (restricted) simplicial morphisms. Moreover, any internal category can be extended to a simplicial object by simply taking its nerve [48]. We can thus freely consider $\mathbf{Cat}(\mathcal{C})$ and $\mathbf{Grpd}(\mathcal{C})$ as full subcategories of $\mathbf{Simp}(\mathcal{C})$; more precisely, a simplicial object $\mathbb{X}$ is an internal category if and only if the commutative square

$$\begin{tikzcd}
X_n \arrow[r, "d_0"] \arrow[d, "d_n"'] & X_{n-1} \arrow[d, "d_{n-1}"] \\
X_{n-1} \arrow[r, "d_0"'] & X_{n-2}
\end{tikzcd}$$

is a pullback for all $n \geq 2$.

Note that every reflexive relation is in particular a reflexive graph, and moreover a reflexive relation admits a structure of an internal groupoid if and only if it is an equivalence relation.

1.3 Mal'tsev categories

Definition 1.10. A finitely complete category $\mathcal{C}$ is called a *Mal'tsev category* (alternatively Maltsev, Mal'cev or Malcev) if any reflexive relation in $\mathcal{C}$ is an equivalence relation [35].

Example 1.11 (Mal'tsev varieties [123])**.** A variety $\mathcal{V}$ of universal algebras is a Mal'tsev category if and only if its theory contains a ternary operation p such that the identities

$$p(x,y,y) = x \qquad \text{and} \qquad p(x,x,y) = y \tag{1.2}$$

hold. A variety that is a Mal'tsev category is called a Mal'tsev variety.

The main example of such a variety is the variety **Grp** of groups, where the operation $p(x,y,z) = xy^{-1}z$ satisfies the identities (1.2). More generally, any variety whose theory contains a group operation is also a Mal'tsev variety. This means that the varieties of (commutative) (unital) (associative) rings, Lie-algebras, modules (over any ring), Jordan algebras are all Mal'tsev varieties.

Example 1.12. A *naturally Mal'tsev* category is a category $\mathcal{C}$ with finite products and a natural Mal'tsev operation, i.e. a natural transformation $\mu\colon Id \times Id \times Id \Rightarrow Id$ satisfying the identities

$$\mu(1_{Id} \times \Delta) = 1_{Id \times Id} = \mu(\Delta \times 1_{Id}).$$

This is equivalent to the category $\mathcal{C}$ being enriched over the category **AuMal** [95], whose objects are sets equipped with a ternary operation p satisfying the Mal'tsev conditions (1.2) as well as the condition

$$\begin{aligned} p(p(x_1,y_1,z_1),&p(x_2,y_2,z_2),p(x_3,y_3,z_3)) \\ &= p(p(x_1,x_2,x_3),p(y_1,y_2,y_3),p(z_1,z_2,z_3)). \end{aligned}$$

For a pointed category, being naturally Mal'tsev is equivalent to being additive; thus for example every abelian category is a (naturally) Mal'tsev category.

Theorem 1.13 ([95, 17])**.** *If $\mathcal{C}$ is finitely complete, the following conditions are equivalent*

1. *$\mathcal{C}$ is a naturally Mal'tsev category;*
2. *the forgetful functor $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{RG}(\mathcal{C})$ is an equivalence of categories;*
3. *every category $\mathbf{Pt}_B(\mathcal{C})$ is additive.*

Theorem 1.14 ([115, 35, 34])**.** *If $\mathcal{C}$ is a regular category, then the following conditions are equivalent :*

1. *$\mathcal{C}$ is a Mal'tsev category*
2. *the composition of two equivalence relations is an equivalence relation*
3. *$RS = SR$ for any two equivalence relations R, S on the same object*
4. *any reflexive relation is transitive*
5. *any reflexive relation is symmetric*
6. *any relation R between two objects X, Y is difunctional, i.e. it satisfies $RR^oR \leq R$.*

By reflexivity of R and S, we have $R \leq RS$ and $S \leq RS$. Moreover, by transitivity any equivalence relation T bigger than R and S is also bigger than RS. Thus RS is the join of R and S in the poset of equivalence relations on X; this means that this poset is a lattice. This observation also allows one to establish the following result :

Proposition 1.15 ([34])**.** *The lattice of equivalence relations on an object X is modular, i.e. the identity*

$$R \vee (S \wedge T) = (R \vee S) \wedge T$$

holds for all equivalence relations R, S, T such that $R \leq T$.

Moreover, note that if $f \colon X \to Y$ is a regular epimorphism and R is a reflexive relation on X, then

$$f(R) = fRf^o \geq ff^o\Delta_Y,$$

thus $f(R)$ is also a reflexive relation. In a Mal'tsev category, this implies that if R is an equivalence relation, then $f(R)$ is an equivalence relation as well. Moreover, we have

$$\begin{aligned} f(R) \vee f(S) = f(R)f(S) = fRf^ofSf^o = fRSf^off^o \\ = fRSf^o = f(RS) = f(R \vee S), \end{aligned}$$

and thus direct images preserve joins of equivalence relations. Furthermore, we have

$$f^{-1}(f(R)) = f^o f R f^o f = Eq[f] \vee R.$$

Definition 1.16. In any regular category, a commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "p"'] & W \end{tikzcd} \tag{1.3}$$

of regular epimorphisms is called a *double extension* [83] or *regular pushout* [22] if the canonical map $(f, g)\colon X \to Y \times_W Z$ is a regular epimorphism.

Note that the existence of the map (f, g) (and thus the commutativity of the square (1.3) is equivalent to the inequality $gf^o \leq h^o p$ of relations between Y and Z; and the condition that (f, g) is a regular epimorphism is then equivalent to the *equality* $gf^o = h^o p$.

Proposition 1.17. *Let* (1.3) *be a double extension, and*

$$\begin{tikzcd} Z' \arrow[r, "u"] \arrow[d, "q"'] & Z \arrow[d, "h"] \\ W' \arrow[r, "v"'] & W \end{tikzcd}$$

be a commutative square where q is a regular epimorphism. Then the back face of the commutative cube

$$\begin{tikzcd} X' \arrow[rr, "g'"] \arrow[dd, "f'"'] \arrow[dr, "u'"] & & Z' \arrow[dd, "q" near start] \arrow[dr, "u"] & \\ & X \arrow[rr, "g" near start] \arrow[dd, "f" near start] & & Z \arrow[dd, "h"] \\ Y' \arrow[rr, "p'" near end] \arrow[dr, "v'"'] & & W' \arrow[dr, "v"] & \\ & Y \arrow[rr, "p"'] & & W \end{tikzcd}$$

(where the top and bottom faces are pullbacks) is a double extension.

Proof. Taking the pullbacks of p and h, as well as p' and q, yields a commutative cube

$$\begin{tikzcd}
Y' \times_{W'} Z' \arrow[rr, "p''"] \arrow[dr, "v' \times u"] \arrow[dd, "q'"'] & & U \arrow[dr, "u"] \arrow[dd, "q"] & \\
& Y \times_W Z \arrow[rr] \arrow[dd] & & Z \arrow[dd, "h"] \\
Y' \arrow[rr, "p'"] \arrow[dr, "v'"'] & & V \arrow[dr, "v"] & \\
& Y \arrow[rr, "p"'] & & W
\end{tikzcd}$$

where the front, back and bottom faces are pullbacks; then the top face is a pullback as well. This top face is the right-hand square in the diagram

$$\begin{tikzcd}
X' \arrow[r, "{(f',g')}"] \arrow[d, "u'"'] & Y' \times_{W'} Z' \arrow[r, "p''"] \arrow[d, "v' \times u"] & Z' \arrow[d, "u"] \\
X \arrow[r, "{(f,g)}"'] & Y \times_W Z \arrow[r] & Z,
\end{tikzcd}$$

where the whole rectangle is the top face of the original cube, and thus also a pullback; thus the left-hand square is a pullback as well. Since (f,g) is a regular epimorphism, (f',g') is one as well. Moreover q' and p'' are also regular epimorphisms, and thus $f' = q'(f',g')$ and $g' = p''(f',g')$ are also regular epimorphisms. □

In a Mal'tsev category, this yields a useful criterion to characterize double extensions :

Proposition 1.18. *If $\mathcal{C}$ is a regular Mal'tsev category, then a square of regular epimorphisms such as* (1.3) *is a double extension if and only if* $f(Eq[g]) = Eq[p]$.

Proof. If the square (1.3) is a double extension, then its pullback along itself is also a double extension, and in particular the map $Eq[g] \to Eq[p]$ is a regular epimorphism, i.e. $f(Eq[g]) = Eq[p]$.

For the converse, $f(Eq[g]) = Eq[p]$ is equivalent to $fg^ogf^o = p^op$. We then have

$$gf^o = gf^ofg^ogf^o = gf^op^op = g(pf)^op = gg^oh^op = h^op,$$

which proves that the square is a double extension. □

Definition 1.19. In a category $\mathcal{C}$ with finite limits, two maps with the same codomain $f\colon A \to C$ and $g\colon B \to C$ are said to be jointly strongly epimorphic if they do not factor through a proper subobject, i.e. if $m\colon M \to C$ is a monomorphism such that both f and g factor through m, then m is an isomorphism.

For example, the two injections of a coproduct $\iota\colon A \to A+B$ and $\iota_B\colon B \to A+B$ are jointly strongly epimorphic. When f, g are monomorphisms, then this condition means that the corresponding subobjects are comaximal, or in other words, that $A \vee B = C$.

Theorem 1.20 ([19])**.** *Let $\mathcal{C}$ be a regular category. Then the following conditions are equivalent :*

1. *$\mathcal{C}$ is a Mal'tsev category;*
2. *for any two split epimorphisms $p\colon X \to Z$ and $q\colon Y \to Z$ with respective sections s and t, the two maps $(1, tp)$ and $(sq, 1)$ to the pullback*

$$\begin{tikzcd} X \times_Z Y \arrow[r, shift left, "\pi_2"] \arrow[d, shift left, "\pi_1"] & Y \arrow[l, shift left, "(sq,1)"] \arrow[d, shift left, "q"] \\ X \arrow[u, shift left, "(1,tp)"] \arrow[r, shift left, "p"] & Z \arrow[l, shift left, "s"] \arrow[u, shift left, "t"] \end{tikzcd}$$

   are jointly strongly epimorphic;
3. *any square of the form*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, shift left, "f"] & Z \arrow[d, shift left, "h"] \\ Y \arrow[u, shift left, "s"] \arrow[r, "p"'] & W \arrow[u, shift left, "t"] \end{tikzcd} \tag{1.4}$$

   where $pf = hg$, $gs = tp$, $fs = 1_X$, $ht = 1_W$ and g, p are regular epimorphisms is a double extension.

Proof. Assume first that $\mathcal{C}$ is a Mal'tsev category, and let us prove that the second condition holds. Let $m\colon M \to X \times_Z Y$ be a monomorphism such that $(1, tp)$ and $(sq, 1)$ factor through m; we will prove that m is an isomorphism. The pullback $X \times_Z Y$ coincides with the relation $q^o p$, and M can then be interpreted as another relation between X and Y, for which we have $M \leq q^o p$, and m is an isomorphism if and only if $M = q^o p$. Then the existence of a factorization of $(1, tp)$ through m means that $tp \leq M$, while the existence of a factorization of $(sq, 1)$ through m means that $(sq)^o \leq M$. Moreover, $(s,t) = (1,tp)s$ also factorizes through m, hence we have $ts^o \leq M$. Combining these inequalities, we get

$$q^o p = q^o s^o s t^o t p = (sq)^o (ts^o)^o (tp) \leq M M^o M \leq M,$$

(where the last inequality is given by the difunctionality of M as a relation) and thus $q^o p = M$.

For the third condition, note that the morphism induced by f and h between $Eq[g]$ and $Eq[p]$ must be a split, hence regular, epimorphism, so that by Proposition 1.18 the square is a double extension.

Assuming now that either the second or third condition holds, we prove that $\mathcal{C}$ is a Mal'tsev category. Let $(r_0, r_1)\colon R \to X \times X$ be a reflexive relation on an object X, and let us take the simplicial kernel of R :

$$\begin{tikzcd} K_2(R) \arrow[r, shift left=2, "\mu_0"] \arrow[r, "\mu_1" description] \arrow[r, shift right=2, "\mu_2"'] & R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & X. \end{tikzcd}$$

This gives a commutative diagram

$$\begin{tikzcd} K_2(R) \arrow[r, shift left, "\mu_2"] \arrow[d, shift left, "\mu_0"] & R \arrow[l, shift left, "s_1"] \arrow[d, shift left, "r_0"] \\ R \arrow[u, shift left, "s_0"] \arrow[r, shift left, "r_1"] & X \arrow[l, shift left, "\delta"] \arrow[u, shift left, "\delta"] \end{tikzcd}$$

and both the second and third condition imply in this case that the morphism $(\mu_0, \mu_2)\colon K_2(R) \to R \times_X R$ is a regular epimorphism. But by definition of simplicial kernel, the square

$$\begin{tikzcd} K_2(R) \arrow[r, "\mu_1"] \arrow[d, "(\mu_0,\mu_2)"'] & R \arrow[d, "(r_0,r_1)"] \\ R \times_X R \arrow[r, "r_0 \times r_1"'] & X \times X \end{tikzcd}$$

is a pullback, thus (μ_0, μ_2) is also a monomorphism, hence it is an isomorphism. Let us define

$$\tau = \mu_1 (\mu_0, \mu_2)^{-1}\colon R \times_X R \to R.$$

Then we have

$$r_0 \tau = r_0 \mu_1 (\mu_0, \mu_2)^{-1} = r_0 \mu_0 (\mu_0, \mu_2)^{-1} = r_0 \pi_1,$$

and similarly $r_1 \tau = r_1 \pi_2$, which proves that R is transitive.

Thus every reflexive relation is transitive, which implies that $\mathcal{C}$ is a Mal'tsev category. $\square$

The properties given by Theorem 1.20 have some interesting consequences on the properties of internal groupoids and simplicial objects in Mal'tsev categories.

Theorem 1.21 ([36]). *If $\mathcal{C}$ is a regular Mal'tsev category, then every multiplicative reflexive graph is an internal groupoid. Moreover, a reflexive graph admits at most one multiplication, and every morphism of reflexive graphs is a functor.*

Note that in [36] this theorem is proved without the regularity assumption.

Proposition 1.22 ([34, Theorem 5.7]). *Let $\mathcal{C}$ be a regular category. Then the following conditions are equivalent :*

1. *$\mathcal{C}$ is an exact Mal'tsev category;*
2. *given two regular epimorphisms $f\colon X \to Y$ and $g\colon X \to Z$, their pushout exists and is a double extension.*

Proof. Assume $\mathcal{C}$ is an exact Mal'tsev category, and let f and g be given as above. Then the composition $Eq[f]Eq[g]$ is an equivalence relation, which is the join of $Eq[f]$ and $Eq[g]$ in the lattice of equivalence relations on X. By exactness, it is the kernel pair of its coequalizer $q\colon X \to W$; then q must coequalize the two projections of $Eq[f]$, thus there must be a morphism p such that $pf = q$, and similarly there must be some morphism h such that $hg = q$, so that we have a commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "q"] & Z \arrow[d, "h"] \\ Y \arrow[r, "p"'] & W \end{tikzcd} \tag{1.5}$$

Moreover, if $u\colon Y \to T$ and $v\colon Z \to T$ are such that $uf = vg$, then the kernel pair of $uf = vg$ is bigger than those of f and g, and thus also bigger than $Eq[f]Eq[g] = Eq[q]$. Thus there exist a unique morphism $w\colon W \to T$ such that

$$uf = vg = wq = wpf = whg,$$

and thus $u = wp$ and $v = wh$, which proves that (1.5) is a pushout. Moreover, the equality $Eq[f]Eq[g] = Eq[q]$ is equivalent to

$$f^o f g^o g = q^o q = f^o p^o h g,$$

from which we find

$$fg^o = ff^ofg^ogg^o = ff^op^ohgg^o = p^oh,$$

which proves that (1.5) must be a double extension.

For the other implication, let $(r_1, r_2)\colon R \to X \times X$ be a reflexive relation. Then r_1 and r_2 are split epimorphisms, and thus regular epimorphisms. If we consider their pushout

$$\begin{tikzcd}
R \arrow[r, "r_2"] \arrow[d, "r_1"'] & X \arrow[d, "q'"] \\
X \arrow[r, "q"'] & Q,
\end{tikzcd}$$

then we have

$$q = qr_1\delta_R = q'r_2\delta_R = q'.$$

Since the same reasoning shows that any two maps u, v such that $ur_1 = vr_2$ must be equal, the pushout is in fact the coequalizer of r_1 and r_2. Now since the pushout square must also be a double extension, the canonical morphism $(r_1, r_2)\colon R \to X \times_Q X$ is a regular epimorphism; but since R is a relation, it is also a monomorphism, and thus an isomorphism. Thus any reflexive relation is a kernel pair, which proves that $\mathcal{C}$ is an exact Mal'tsev category. $\square$

We can also define a *triple extension* [57] as a commutative cube

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dd, "f"'] \arrow[dr, "\alpha"'] & & Z \arrow[dd, "h", near start] \arrow[dr, "\gamma"] & \\
& X' \arrow[rr, "g'", near start, crossing over] & & Z' \arrow[dd, "h'"] \\
Y \arrow[rr, "p"', near end] \arrow[dr, "\beta"'] & & W \arrow[dr, "\delta"'] & \\
& Y' \arrow[rr, "p'"'] \arrow[from=uu, "f'", near start, crossing over] & & W'
\end{tikzcd}$$

for which all faces, as well as the induced commutative square

$$\begin{tikzcd}
X \arrow[r, "(f,g)"] \arrow[d, "\alpha"'] & Y \times_W Z \arrow[d, "\beta \times_\delta \gamma"] \\
X' \arrow[r, "(f',g')"'] & Y' \times_{W'} Z',
\end{tikzcd}$$

are double extensions. Note that this condition amounts to the identity

$$\alpha(Eq[f] \wedge Eq[g]) = Eq[f'] \wedge Eq[g'].$$

Triple extensions satisfy similar properties as double extensions : in particular, a split cube between double extensions is always a triple extension.

1.4 Semi-abelian categories

1.4.1 Definitions, examples and properties

If $\mathcal{C}$ is a category, the category of points $\mathbf{Pt}(\mathcal{C})$ [16] is the category whose

- objects are pairs of morphisms $p\colon A \to B$ and $s\colon B \to A$ such that $ps = 1_B$
- morphisms $(A, B, p, s) \to (A', B', p', s')$ are pairs of morphisms $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ such that $\beta p = p'\alpha$ and $\alpha s = s'\beta$.

The codomain functor $\mathrm{cod}\colon \mathbf{Pt}(\mathcal{C}) \to \mathcal{C}$, which maps (A, B, p, s) to B and (α, β) to β, is then a bifibration, which is often called the *fibration of points* or fibration of pointed objects. The cartesian morphisms for this fibration are the pairs (α, β) such that the squares $\beta p = p'\alpha$ and $\alpha s = s'\beta$ are pullbacks, and the cocartesian morphisms are those such that these squares are pushouts.

We will denote $\mathbf{Pt}_B(\mathcal{C})$ the fiber of the codomain functor at the object B, i.e. the category whose objects are morphisms $p\colon A \to B$ with a given section $s\colon B \to A$, and morphisms $(A, p, s) \to (A', p', s')$ are just morphisms $\alpha\colon A \to A'$ such that $p'\alpha = \alpha$ and $\alpha s = s'$.

Definition 1.23. A finitely complete category $\mathcal{C}$ is

- *protomodular* if every change-of-basis functor for the fibration of points is conservative [16].
- *sequentiable* if it is quasi-pointed (i.e. $\mathcal{C}$ admits an initial object 0 and the morphism $0 \to 1$ is a monomorphism), regular and protomodular [18].
- *homological* if it is pointed, regular and protomodular [10].
- *semi-abelian* if it is homological, exact, and has binary coproducts [91].

Note that if $\mathcal{C}$ has a zero object 0, then $\mathbf{Pt}_0(\mathcal{C})$ is equivalent to $\mathcal{C}$.

Proposition 1.24. *If $\mathcal{C}$ is pointed, then it is protomodular if and only if the* Split Short Five Lemma *holds : given a diagram*

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, shift left, "p"] \arrow[d, "v"'] & Y \arrow[l, shift left, "s"] \arrow[d, "w"] \\
K' \arrow[r, "k'"'] & X' \arrow[r, shift left, "p'"] & Y' \arrow[l, shift left, "s'"]
\end{tikzcd}
$$

where k, k' are the kernels of p, p' respectively, $ps = 1_Y$ and $p's' = 1_{Y'}$, and all squares commute, then if u and w are isomorphisms, v is an isomorphism as well.

In particular, every additive category is protomodular.

Example 1.25. A variety $\mathcal{V}$ is a protomodular category if and only if its theory contains an $(n+1)$-ary operation m, n binary operations $s_1, \ldots, s_n$ and n constants $e_1, \ldots, e_n$ such that the identities

$$m(s_1(x, y), \ldots, s_n(x, y), y) = x$$

and

$$s_i(x, x) = e_i \quad 0 \leq i \leq n$$

hold [30]. A variety is semi-abelian if and only if its theory has exactly one constant and satisfies the above conditions. It turns out that such varieties were previously studied by Ursini [126, 127], who called them *BIT-speciale* varieties (for *buona teoria degli ideali* or *good theory of ideals*).

In particular, the variety of groups is protomodular, since one can take $n = 1$ and define $m(x, y) = x \cdot y$, $s(x, y) = x \cdot y^{-1}$ and the constant e to be the neutral element of the group, and then check that the axioms are satisfied. More generally, any variety whose theory contains a group operation is protomodular, and it is also pointed if its theory does not contain another constant. Since moreover every variety is exact and cocomplete, the categories of groups, Lie algebras, (non-unital) rings, (non-unital) K-algebras... are all semi-abelian. The categories of unital rings or unital K-algebras are protomodular, but not pointed, and thus not semi-abelian.

Example 1.26. The categories **Grp**(**Top**) and **Grp**(**Top**) of topological groups and topological Hausdorff groups are homological [11, 24], but not exact (and thus not semi-abelian).

The category of **Grp**(**Comp**) of compact Hausdorff groups is semi-abelian, as is the category of profinite groups [11].

More generally, the models of a semi-abelian theory in the categories of topological spaces, Hausdorff spaces or totally disconnected spaces form a homological category, and the models of a semi-abelian theory in the categories of compact Hausdorff or profinite spaces form a semi-abelian category [11].

Example 1.27. The category of C^*-algebra is also semi-abelian [70], and so is the category of cocommutative Hopf algebras over a field [69, 72].

Example 1.28 (Heyting semi-lattices and algebras)**.** A Heyting semi-lattice is a poset which is cartesian closed when seen as a category, and similarly a Heyting algebra is a lattice which is cartesian closed when seen as a category. Thus a Heyting semi-lattice can be described as a set endowed with a constant 1 and binary operations $\wedge$ and $\Rightarrow$ satisfying the identities

$$x \wedge 1 = x = x \wedge x, \qquad x \wedge y = y \wedge x, \qquad (x \wedge y) \wedge z = x \wedge (y \wedge z),$$

$$(x \Rightarrow x) = 1, \qquad x \wedge (x \Rightarrow y) = x \wedge y, \qquad y \wedge (x \Rightarrow y) = y,$$

$$\text{and} \qquad x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z),$$

and a Heyting algebra is a Heyting semi-lattice that is also endowed with a constant 0 and another binary operation $\vee$ satisfying the identities

$$0 \vee x = x = x \vee x, \qquad x \vee y = y \vee x, \qquad (x \vee y) \vee z = x \vee (y \vee z),$$

$$x \wedge (x \vee y) = x \qquad \text{and} \qquad x \vee (x \wedge y) = x.$$

The variety of Heyting semi-lattices is semi-abelian, and the variety of Heyting algebras is protomodular [97]. Indeed, one can define operations

$$s_1(x, y) = x \Rightarrow y, \qquad s_2(x, y) = ((x \Rightarrow y) \Rightarrow y) \Rightarrow x$$

$$\text{and} \qquad m(x, y, z) = (x \Rightarrow z) \wedge y$$

and verify that these operations satisfy the conditions. However, unlike the case of groups, it is not possible to find binary operations m and s satisfying the required conditions for $n = 1$ [97].

Using these facts, one can prove that the dual of any topos is also protomodular, and the dual of the category of pointed sets is semi-abelian [17, 22].

Proposition 1.29. *Let $p\colon A \to B$ be a split epimorphism in a protomodular category $\mathcal{C}$. In the pullback*

$$\begin{tikzcd}
C \times_B A \arrow[r, "f'"] \arrow[d, "p'"'] & A \arrow[d, shift left, "p"] \\
C \arrow[r, "f"'] & B, \arrow[u, shift left, "s"]
\end{tikzcd}$$

the morphisms f' and s are jointly strongly epimorphic.

Proof. Let $m\colon M \to A$ a monomorphism such that f' and s factor as $m\widetilde{f'}$ and $m\widetilde{s}$, respectively. Let $\widetilde{p} = pm$. Then m is a morphism $(p, s, A, B) \to (\widetilde{p}, \widetilde{s}, M, B)$. Furthermore, we have the diagram

$$\begin{tikzcd}
C \times_B A \arrow[rr, "\widetilde{f'}"] \arrow[dd, shift left] \arrow[dr, equal] & & M \arrow[dr, "m"] \arrow[dd, shift left, "\widetilde{p}"] & \\
& C \times_B A \arrow[rr, "f'"'] \arrow[dl, shift left] & & A \arrow[dl, shift left, "p"] \\
C \arrow[uu, shift left] \arrow[rr, "f"'] \arrow[ur, shift left] & & B \arrow[uu, shift left, "\widetilde{s}"] \arrow[ur, shift left, "s"] &
\end{tikzcd}$$

where all the squares are pullbacks; thus $f^*(m)$ is an isomorphism. By protomodularity, m is then an isomorphism. $\square$

Corollary 1.30 ([17])**.** *Every protomodular category is a Mal'tsev category.*

Proof. If the commutative square (b) below is a pullback of split epimorphisms

$$\begin{tikzcd}
Y \arrow[r, "{(sq,1)}"] \arrow[d, shift left, "q"] \arrow[dr, phantom, "(a)"] & X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, shift left, "\pi_1"] \arrow[dr, phantom, "(b)"] & Y \arrow[d, shift left, "q"] \\
Z \arrow[r, "s"'] \arrow[u, shift left, "t"] & X \arrow[r, "p"'] \arrow[u, shift left, "{(1,tp)}"] & Z, \arrow[u, shift left, "t"]
\end{tikzcd}$$

then the commutative square (a) is a pullback as well, since the horizontal sides of the composite rectangle $(a) + (b)$ are identities. Thus $(sq, 1)$ and $(1, tp)$ are jointly strongly epimorphic, which implies that $\mathcal{C}$ is a Mal'tsev category by Theorem 1.20. $\square$

Proposition 1.31. *If $\mathcal{C}$ is a homological category, then every regular epimorphism is the cokernel of its kernel. Thus every morphism factorizes as a coequalizer followed by a monomorphism, and these factorizations are stable under pullback.*

Proof. Every cokernel is a coequalizer, and thus a regular epimorphism. Conversely, let $q\colon X \to Y$ be a regular epimorphism, with kernel $k\colon K \to X$, and consider the diagram

$$\begin{tikzcd} K \times K \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "k'"'] \arrow[dr, phantom, "(a)"] & K \arrow[r] \arrow[d, "k"] \arrow[dr, phantom, "(b)"] & 0 \arrow[d] \\ Eq[q] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & X \arrow[l, "\delta" description] \arrow[r, "q"'] & Y. \end{tikzcd}$$

Since k is the kernel of q, the square (b) is a pullback, and thus so is the square (a), so that k' and δ are jointly strongly epimorphic. Then if $h\colon X \to Z$ is such that $hk = 0$, we have

$$hp_1\delta = h = h\pi_2\delta$$

and

$$hp_1k' = hk\pi_1 = 0 = hk\pi_2 = hp_2k',$$

so that $hp_1 = hp_2$. Since q is the coequalizer of its kernel pair, h factors uniquely through q, which proves that $q = \mathrm{coker}(k)$. □

Corollary 1.32. *In a homological category, $f\colon X \to Y$ is a monomorphism if and only* $\ker(f) = 0$*.*

Another consequence is that if $\mathcal{C}$ is a semi-abelian category, there is a bijection between internal equivalence relations, regular epimorphisms, and normal subobjects (i.e. kernels).

Proposition 1.33. *If $(r_1, r_2)\colon R \to X \times X$ is an equivalence relation, then the kernel of its coequalizer is equal to $r_2 \ker(r_1)$.*

Proof. Since $\mathcal{C}$ is exact, R is the kernel pair of its coequalizer $q\colon X \to \frac{X}{R}$. Then we have a diagram

$$\begin{tikzcd} K[r_1] \arrow[r, "\ker(r_1)"] \arrow[d] & R \arrow[r, "r_2"] \arrow[d, "r_1"'] & X \arrow[d, "q"] \\ 0 \arrow[r] & X \arrow[r, "q"] & \frac{X}{R} \end{tikzcd}$$

where the two squares are pullbacks; thus the outer rectangle is a pullback as well, i.e. $r_2 \ker(r_1) = \ker(q)$. □

Proposition 1.34. *In a semi-abelian category $\mathcal{C}$, a square of regular epimorphisms*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "h"] \\ Y \arrow[r, "p"'] & W \end{tikzcd}$$

is a pushout if and only if $f(K[g]) = K[p]$.

Proof. If $f(K[g]) = K[p]$ and u, v are morphisms such that $uf = vg$, then $u(K[p]) = uf(K[g]) = vg(K[g]) = 0$, thus u factors uniquely as wp. Then we also have $whg = wpf = uf = vg$, and thus $wh = v$. Thus the square is a pushout.

Conversely, if the square is a pushout then it is also a double extension. In the diagram

$$\begin{tikzcd} K[g] \arrow[r, "\ker(g)"] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "(a)"] & X \arrow[r, "g"] \arrow[d, "{(f,g)}"] & Z \arrow[d, equal] \\ K[f] \arrow[r, "{(\ker(p),0)}"'] \arrow[d, equal] & Y \times_W Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "h"] \\ K[f] \arrow[r, "\ker(p)"'] & Y \arrow[r, "p"'] & W, \end{tikzcd}$$

$(\ker(p), 0)$ is the kernel of p_2, and the square (a) is a pullback. Since (f, g) is a regular epimorphism, so is $\widetilde{f}$. □

Definition 1.35. In a homological or sequentiable category $\mathcal{C}$, a sequence of morphisms

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

is called *exact at X_n* if $\ker(f_n) = \mathrm{im}(f_{n-1})$, and *exact* if it is exact at all X_n.

In a semi-abelian category, the classical homological lemmas, such as the Five Lemma, the Snake Lemma and the 3×3 Lemma hold [10]. Moreover the Noether isomorphisms theorems hold as well [10].

1.4.2 Internal actions and semi-direct products

For any object B, the change-of-base functor along $0 \to B$ is given by

$$Ker\colon \mathbf{Pt}_B(\mathcal{C}) \to \mathcal{C}\colon (A, p, s) \mapsto K[p],$$

and its left adjoint is given by

$$B + \_ \colon \mathcal{C} \to \mathbf{Pt}_B(\mathcal{C}) \colon X \mapsto (B + X, [1,0], \iota_B).$$

We will denote $B\flat X \xrightarrow{\kappa_{B,X}} B + X$ the kernel of $[1,0] \colon B + X \to B$. Note that $B\flat X$ is functorial with respect to B and X ; given $g \colon B \to C$ and $f \colon X \to Y$, $g\flat f$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "g\flat f"'] & B + X \arrow[r, shift left, "{[1,0]}"] \arrow[d, "g+f"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, "g"] \\
C\flat Y \arrow[r, "\kappa_{C,Y}"'] & C + Y \arrow[r, shift left, "{[1,0]}"] & C. \arrow[l, shift left, "\iota_C"]
\end{tikzcd}$$

Now $B\flat\_$ is the composition of the two adjoint functors on the object X, and is thus endowed with a monad structure. The unit $\eta_{B,X} \colon X \to B\flat X$ of this monad is the unique morphism such that $\kappa_{B,X}\eta_{B,X} = \iota_X \colon X \to B + X$, while the multiplication μ is the unique morphism making the following diagram commute :

$$\begin{tikzcd}
B\flat(B\flat X) \arrow[r, "\kappa_{B,X}"] \arrow[d, "\mu_{B,X}"'] & B + (B\flat X) \arrow[r, shift left, "{[1,0]}"] \arrow[d, "{[\iota_B, \kappa_{B,X}]}"'] & B \arrow[l, shift left, "\iota_{B\flat X}"] \arrow[d, equal] \\
B\flat X \arrow[r, "\kappa_{C,Y}"'] & B + X \arrow[r, shift left, "{[1,0]}"] & B. \arrow[l, shift left, "\iota_B"]
\end{tikzcd}$$

Definition 1.36. The monad $(B\flat\_, \eta, \mu)$ is called the *monad of internal B-actions*, and its Eilenberg-Moore category is the category of B-actions [13].

An action of B on X is thus a $B\flat\_$-algebra structure on X, i.e. a morphism $\xi \colon B\flat X \to X$ in $\mathcal{C}$ such that the following diagrams commute :

$$\begin{tikzcd}
X \arrow[r, "\eta_{B,X}"] \arrow[dr, "1_X"'] & B\flat X \arrow[d, "\xi"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
B\flat B\flat X \arrow[r, "1\flat\xi"] \arrow[d, "\mu_{B,X}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X.
\end{tikzcd}$$

A morphism of B-action $(X, \xi_X) \to (Y, \xi_Y)$ is then a morphism of $B\flat\_$-actions, i.e. a morphism $f \colon X \to Y$ such that $f\xi_X = \xi_Y(B\flat f)$.

Since semi-abelian categories are finitely cocomplete, the comparison functor $\mathbf{Pt}_B(\mathcal{C}) \to \mathcal{C}^{B\flat -}$ has a left adjoint L [91].

Definition 1.37. Let ξ be an action of B on X. The semi-direct product $B \ltimes_\xi X$ is the domain of the point $L(X, \xi)$ [28].

The left adjoint L is given by the following coequalizer in $\mathbf{Pt}_B(\mathcal{C})$:

$$\begin{tikzcd}
B + B\flat X \arrow[r, shift left, "{[\iota_B,\kappa_{B,X}]}"] \arrow[r, shift right, "1+\xi"'] \arrow[d, shift left, "{[1,0]}"] & B + X \arrow[r, dashed, "q_\xi"] \arrow[d, shift left, "{[1,0]}"] & B \ltimes_\xi X \arrow[d, shift left, "p_B"] \\
B \arrow[u, shift left, "\iota_B"] \arrow[r, equal] & B \arrow[u, shift left, "\iota_B"] \arrow[r, equal] & B. \arrow[u, shift left, "s_B"]
\end{tikzcd}$$

In fact, this coequalizer can be taken in $\mathcal{C}$. Moreover, precomposing the morphisms $[\iota_B, \kappa_{B,X}]$ and $1 + \xi$ with the canonical injection ι_B gives the same result :

$$[\iota_B, \kappa_{B,X}] \circ \iota_B = \iota_B = (1 + \xi) \circ \iota_B.$$

Since the coproduct injections ι_B and $\iota_{B\flat X}$ are jointly epimorphic, a morphism $v\colon B + X \to Y$ coequalizes $[\iota_B, \kappa_{B,X}]$ and $1 + \xi$ if and only if it coequalizes $[\iota_B, \kappa_{B,X}]\iota_{B\flat X} = \kappa_{B,X}$ and $(1 + \xi) \circ \iota_{B\flat X} = \iota_X\xi$, so the semi-direct product can be simply defined as the coequalizer

$$\begin{tikzcd}
B\flat X \arrow[r, shift left, "\kappa_{B,X}"] \arrow[r, shift right, "\iota_X\xi"'] & B + X \arrow[r, dashed, "q_\xi"] & B \ltimes_\xi X.
\end{tikzcd} \tag{1.6}$$

If u, v is any pair of morphisms such that $v\kappa_{B,X} = u\xi$, then

$$u = u\xi\eta_{B,X} = v\kappa_{B,X} = v\iota_X;$$

and thus $v\kappa_{B,X} = v\iota_X\xi$. Thus there is a bijection between morphisms v coequalizing $\kappa_{B,X}$ and $\iota_X\xi$ and pairs of morphisms u, v such that $v\kappa_{B,X} = u\xi$. As a consequence, the semi-direct product can be alternatively defined as the pushout

$$\begin{tikzcd}
B\flat X \arrow[r, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B + X \arrow[d, "{q_\xi = [s_B, j_X]}"] \\
X \arrow[r, "j_X"'] \arrow[ur, phantom, "\ulcorner", very near end] & B \ltimes_\xi X.
\end{tikzcd} \tag{1.7}$$

Example 1.38. In the category of groups, the coproduct $B + X$ is the free product, whose elements are reduced words of elements of B and X. The kernel $B\flat X$ of the morphism $[1_B, 0]\colon B + X \to B$ is then the subgroup of words $b_1x_1 \dots b_nx_nb_{n+1}$ with $x_i \in X$ and $b_i \in B$ such that $b_1b_2 \dots b_{n+1} = e_B$. Such a product can be rewritten inductively as

$$\begin{aligned}
b_1x_1 \dots b_nx_nb_{n+1} &= \left(b_1x_1b_1^{-1}\right)(b_1b_2x_2b_3 \dots b_nx_nb_{n+1}) \\
&= \left(b_1x_1b_1^{-1}\right)\left((b_1b_2)x_2(b_1b_2)^{-1}\right)(b_1b_2b_3x_3 \dots b_nx_nb_{n+1}) \\
&= \left(b_1x_1b_1^{-1}\right) \dots \left(b_n'x_n(b_n')^{-1}\right).
\end{aligned}$$

Thus the group $B\flat X$ is isomorphic to a coproduct of copies of X, indexed by the elements of B; moreover this isomorphism is natural in X. Thus a $B\flat\_$-algebra structure $\xi\colon B\flat X \to X$ is determined by one group homomorphism $\xi_b\colon X \to X$ for every $b \in B$. The unit $\eta_{B,X}\colon X \to B\flat X$ corresponds to the inclusion of elements of X conjugated by the identity of B, thus it can be seen as the canonical inclusion $\iota_{e_B}\colon X \to \coprod_{b\in B} X$. As a consequence, the condition that $\xi\eta_{B,X} = 1_X$ amounts to $\xi_{e_B} = 1_X$. On the other hand, the condition that $\xi(1\flat\xi) = \xi\mu^B_X$ amounts to $\xi_b(\xi_{b'}(x)) = \xi_{bb'}(x)$. Thus a $B\flat\_$-algebra structure on X is equivalent to an action of B on X in the classical sense.

The semi-direct product is then the coequalizer of $\kappa_{B,X}$ and $\iota_X\xi$; the first of these morphisms sends a generator bxb^{-1} to itself in $B+X$, while the other one sends that same generator to $\xi_b(x)$. Thus the semi-direct product must be generated by B and X, with the condition that the conjugation of B on X coincides with ξ.

Proposition 1.39 ([28]). *Let $\mathcal{C}$ be a semi-abelian category. Then the kernel functor $Ker\colon \mathbf{Pt}_B(\mathcal{C}) \to \mathcal{C}$ is monadic; thus we have an equivalence of categories $\mathcal{C}^{B\flat(-)} \simeq \mathbf{Pt}_B(\mathcal{C})$, so that every action $\xi\colon B\flat X \to X$ is induced by a split short exact sequence*

$$0 \longrightarrow X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0,$$

and every such split exact sequence is isomorphic to

$$0 \longrightarrow X \xrightarrow{j_X} B \ltimes_\xi X \underset{s_B}{\overset{p_B}{\rightleftarrows}} B \longrightarrow 0.$$

Let us make the two constructions explicit. Given a split short exact sequence as above, the corresponding action ξ is defined as the unique morphism $B\flat X \to X$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B\flat X \arrow[r, "\kappa_{B+X}"] \arrow[d, "\xi"'] \arrow[dr, phantom, "(a)"] & B+X \arrow[r, shift left, "{[1,0]}"] \arrow[d, "{[s,k]}"'] & B \arrow[l, shift left, "\iota_B"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

commute. Now the commutativity of the square (a) implies the existence of a unique morphism $B \ltimes_\xi X \to A$, which is then an isomorphism.

Proof. (of Proposition 1.39) It is enough, according to Beck's monadicity theorem (see for example [7, Theorem 3.14]), to check that $\mathbf{Pt}_B(\mathcal{C})$ has reflexive coequalizers, and that the kernel functor preserves reflexive coequalizers and is conservative. But reflexive coequalizers are double extensions, and since $\mathbf{Pt}_B(\mathcal{C})$ is semi-abelian, it is in particular an exact Mal'tsev category, thus it has reflexive coequalizers. Moreover, the functor Ker preserves pullback and regular epimorphisms, hence it also preserves double extensions, and thus reflexive coequalizers. Moreover protomodularity is equivalent to the functor Ker being conservative. $\square$

Example 1.40 (Trivial action)**.** For any objects B, X, the trivial action τ is the action of B on X corresponding to the point defined by the split epimorphism $\pi_B\colon B \times X \to B$ and its section $(1,0)\colon B \to B \times X$. Since the kernel of π_B is the canonical morphism $(0,1)\colon X \to B \times X$, τ is determined by the condition that $(0,1)\tau = [(1,0),(0,1)\kappa_{B,X}]$, and thus $\tau = [0,1]\kappa_{B,X}$.

Example 1.41 (Conjugation action)**.** For any object B, the conjugation action $\chi\colon B\flat B \to B$ is the action induced by the point $\begin{tikzcd} B \times B \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "(1,1)"] \end{tikzcd}$.
Again, the kernel of π_1 is the canonical morphism $(0,1)\colon B \to B \times B$, and thus χ must satisfy $(0,1)\chi = [(1,1),(0,1)]\kappa_{B,B}$, so that $\chi = [1,1]\kappa_{B,B}$.

More generally, for any normal subobject $k\colon K \to B$, we have a morphism of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\ker(r_1)"] \arrow[d, "k"'] & R \arrow[r, shift left, "r_1"] \arrow[d, "{(r_1,r_2)}"'] & B \arrow[l, shift left, "\delta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "{(0,1)}"'] & B \times B \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "{(1,1)}"] \arrow[r] & 0
\end{tikzcd}$$

where R is the kernel pair of $B \to B/K$. Since (r_1, r_2) is a monomorphism in the category $\mathbf{Pt}_B(\mathcal{C})$, it induces a monomorphism $k\colon (K, \chi_K) \to (B, \chi)$ in the category of B-actions; thus the action $B\flat K \to K$ induced by the upper split short exact sequence is the restriction of the conjugation of B on itself to K. This action χ_K^B must satisfy

$$k\chi_K^B = \chi_B(1\flat k) = [1,1]\kappa_{B,B}(1\flat k) = [1,1](1+k)\kappa_{B,K} = [1,k]\kappa_{B,K}.$$

Definition 1.42. Let ξ be an action of B on X, and let $f\colon A \to B$ be a morphism. The *pullback action* $f^*\xi$ of A on X is the action corresponding to the point defined by pulling back along f :

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "j'_X"] \arrow[d, equal] & A \ltimes_{f^*\xi} X \arrow[r, shift left, "p_A"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[l, shift left, "s_A"] \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & X \arrow[r, "j_X"'] & B \ltimes_\xi X \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "s_B"] \arrow[r] & 0.
\end{tikzcd}$$

Since this defines a morphism of points, f and 1_X must define a morphism of actions; this means that $f^*\xi$ is equal to the composition

$$A\flat X \xrightarrow{f\flat 1} B\flat X \xrightarrow{\xi} X.$$

We can then define the category $\mathbf{Act}C$ as the category whose objects are triples (X, B, ξ), where X, B are objects of $\mathcal{C}$ and ξ is a B-action, and morphisms $(X, B, \xi) \to (Y, C, \xi')$ are pairs of morphisms $f\colon X \to Y$ and $g\colon B \to C$ that are equivariant, in the sense that the following diagram commutes :

$$\begin{tikzcd} B\flat X \arrow[r, "g\flat f"] \arrow[d, "\xi"'] & C\flat Y \arrow[d, "\xi'"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Note that this is equivalent to f being a morphism $(X, \xi) \to (Y, g^*\xi')$ in the category of B-actions.

Using these notions, one can express the universal property of the semi-direct product alternatively as follows :

Proposition 1.43. *Let $\xi\colon B\flat X \to X$ be an action, and let $u\colon B \to D$ and $v\colon X \to D$ be two morphisms. There exists a morphism $\varphi\colon B \ltimes_\xi X \to D$ such that $\varphi s_B = u$ and $\varphi j_X = v$ if and only if u, v form a morphism $(X, B, \xi) \to (D, D, \chi_D)$ in the category $\mathbf{Act}(\mathcal{C})$, i.e. if and only if the diagram*

$$\begin{tikzcd} B\flat X \arrow[r, "u\flat v"] \arrow[d, "\xi"'] & D\flat D \arrow[d, "\chi_D"] \\ X \arrow[r, "v"'] & D \end{tikzcd}$$

commutes.

Proof. Using the pushout (1.7) as definition of the semi-direct product, the morphism φ exists if and only if $[u, v]\kappa_{B,X} = v\xi$. Now we have

$$[u, v]\kappa_{B,X} = [1, 1](u + v)\kappa_{B,X} = [1, 1]\kappa_{D,D}(u\flat v) = \chi_D(u\flat v),$$

which concludes the proof. $\square$

When the morphism φ as above exists, we will denote it $[u, v\rangle$. As a particular case, for a morphism $(f, g)\colon (X, B, \xi) \to (Y, C, \xi')$ in $\mathbf{Act}(\mathcal{C})$, the diagram

$$\begin{tikzcd} B\flat X \arrow[r, "g\flat f"] \arrow[d, "\xi"'] & C\flat Y \arrow[r, "s_C\flat j_Y"] \arrow[d, "\xi'"] & (C \ltimes_{\xi'} Y)\flat(C \ltimes_{\xi'} Y) \arrow[d, "\chi"] \\ X \arrow[r, "f"'] & Y \arrow[r, "j_Y"'] & C \ltimes_{\xi'} Y \end{tikzcd}$$

commutes, so that we have a morphism $g \ltimes f = [s_C g, j_Y f\rangle \colon B \ltimes X \to C \ltimes Y$, which defines a morphism in $\mathbf{Pt}(\mathcal{C})$. This means the semi-direct product defines a functor $\mathbf{Act}(\mathcal{C}) \to \mathcal{C}$. In fact, this functor is part of an equivalence $\mathbf{Pt}(\mathcal{C}) \simeq \mathbf{Act}(\mathcal{C})$, which arises from all the equivalences $\mathbf{Pt}_B(\mathcal{C}) \simeq \mathcal{C}^{B\flat -}$.

1.4.3 Precrossed and crossed modules

Let (X_1, B, c, d, e) be a reflexive graph in a semi-abelian category. In particular, (X_1, B, d, e) is a point in $\mathcal{C}$, and there is a corresponding action ξ of B on $X = Ker(d)$. By Proposition 1.39, we then have $X_1 \cong B \ltimes_\xi X$. Now the additional morphism c induces a morphism $\partial \colon X \to B$ by composition with $k = \ker(d)$. Given such a morphism, by the universal property of the semi-direct product the existence of $c \colon X_1 \to B$ such that $ce = 1$ and $ck = \partial$ is equivalent to $\partial \colon (X, \xi) \to (B, \chi)$ is B-equivariant. This motivates the following definition :

Definition 1.44 ([85])**.** A *precrossed module* in $\mathcal{C}$ is given by two objects X, B of $\mathcal{C}$, an action $\xi \colon B\flat X \to X$ and a morphism $\partial \colon X \to B$ that is B-equivariant (with B acting on itself by conjugation), i.e. such that the following square commutes :

$$\begin{tikzcd} B\flat X \arrow[r, "B\flat\partial"] \arrow[d, "\xi"'] & B\flat B \arrow[d, "\chi"] \\ X \arrow[r, "\partial"'] & B. \end{tikzcd}$$

A precrossed module morphism $(X, B, \partial_X, \xi_X) \to (Y, C\partial_Y, \xi_Y)$ is given by a pair of equivariant morphisms $f \colon X \to Y$ and $g \colon B \to C$ such that $g\partial_X = \partial_Y f$.

We will denote $\mathbf{PXMod}(\mathcal{C})$ the category of precrossed modules in $\mathcal{C}$. For a given object B of $\mathcal{C}$, we also denote $\mathbf{PXMod}_B(\mathcal{C})$ the subcategory of precrossed modules with codomain B, with morphisms $(f \colon X \to Y, 1_B \colon B \to B)$ as morphisms. By construction, we have an equivalence of categories $\mathbf{PXMod}(\mathcal{C}) \simeq \mathbf{RG}(\mathcal{C})$, which restricts to an equivalence $\mathbf{PXMod}_B(\mathcal{C}) \simeq \mathbf{RG}_B(\mathcal{C})$ for all B. The functor

$$\mathbf{RG}(\mathcal{C}) \longrightarrow \mathbf{PXMod}(\mathcal{C})$$

$$(X_1, B, c, d, e) \longmapsto \begin{pmatrix} X = K[d] \\ \partial = c\ker(d) \\ \xi = e^*\chi \end{pmatrix}$$

is often called the *normalization* functor. It can alternatively be described as the pullback of $(c,d)\colon X_1 \to B\times B$ along $(1,0)\colon B \to B \times B$. Indeed, we have the diagram

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "\partial"'] \arrow[dr, phantom, "(a)"] & X_1 \arrow[d, "{(c,d)}"] \arrow[dr, bend left, "c"] & \\
B \arrow[r, "{(1,0)}"] \arrow[d] \arrow[dr, phantom, "(b)"] & B\times B \arrow[r, "p_1"] \arrow[d, "p_2"] & B \\
0 \arrow[r] & B, &
\end{tikzcd}$$

where $(1,0)$ is the kernel of p_2, so that (b) is a pullback. Then since (a) is a pullback if and only if $(a)+(b)$ is one, so the pullback of (c,d) and $(0,1)$ coincides with the kernel of d, and we have again

$$\partial = p_1(1,0)\partial = p_1(c,d)k = ck.$$

Now given a precrossed module (X,B,∂,ξ), the corresponding reflexive graph $(B \ltimes_\xi X, B, [1,\partial\rangle, p_b, s_B)$ is an internal groupoid if and only if it is a multiplicative graph. The multiplication must be defined on the pullback of p_B along $[1,\partial\rangle$, and by definition of the pullback action this coincides with the right-hand square in the diagram

$$\begin{tikzcd}
X \arrow[r, "j'_X"] \arrow[d, equal] & (B\ltimes_\xi X)\ltimes_{[1,\partial]^*\xi} X \arrow[r, shift left, "p_{B\ltimes_\xi X}"] \arrow[d, shift left] \arrow[dr, phantom, very near start, "\lrcorner"] & B\ltimes_\xi X \arrow[l, shift left, "s_{B\ltimes_\xi X}"] \arrow[d, "{[1,\partial]}"] \\
X \arrow[r, "j_X"'] & B\ltimes_\xi X \arrow[u, shift left, "l"] \arrow[r, shift left, "p_B"] & B, \arrow[l, shift left, "s_B"]
\end{tikzcd}$$

and thus the graph is multiplicative if and only if there exists

$$m\colon (B\ltimes_\xi X)\ltimes_{[1,\partial]^*\xi} X \to B\ltimes_\xi X$$

such that $ms_{B\ltimes_\xi X} = 1$ and $ml = 1$. Note that since j_X and s_B are jointly epimorphic, this second identity is equivalent to $mlj_X = j_X$ and $mls_B = s_B$. Now we have

$$mls_B = ms_{B\ltimes_\xi X}s_B = s_B$$

and

$$mlj_X = mj'_X;$$

thus the existence of m is equivalent to the condition that the diagram

$$\begin{tikzcd}
(B\ltimes_\xi X)\flat X \arrow[r, "\kappa_{B\ltimes_\xi X,X}"] \arrow[d, "{[1,\partial]^*\xi}"'] & (B\ltimes_\xi X)+X \arrow[d, "{[1,j_X]}"] \\
X \arrow[r, "j_X"'] & B\ltimes_\xi X
\end{tikzcd} \tag{1.8}$$

commutes. Since by construction $j_X = \ker(p_B)$, we have

$$[1, j_X]\kappa_{B\ltimes_\xi X,X} = j_X\chi.$$

Thus a precrossed module corresponds to an internal groupoid if and only if the action $[1,\partial\rangle^*\xi$ and the conjugation action of $B \ltimes_\xi X$ on X coincide.

Definition 1.45 ([85])**.** A precrossed module is called a *crossed module* if the action $[1,\partial\rangle^*\xi$ and the conjugation action of $B \ltimes_\xi X$ on X coincide.

Remark 1.46. The original definition of internal crossed module given in [85] is slightly different, but equivalent. It follows from the observation that since $q_\xi \flat 1$ is a regular epimorphism, the condition that (1.8) commutes is equivalent to the condition that

$$[1, j_X]\kappa_{B\ltimes_\xi X,X}(q_\xi \flat 1) = j_X([1,\partial]^*\xi)(q_\xi \flat 1). \tag{1.9}$$

The right-hand side in this equation is equal to

$$j_X\xi([1,\partial]\flat 1)(q_\xi \flat 1) = j_X\xi[1,\partial]\flat 1.$$

On the other hand, we have by definition $\kappa_{B\ltimes_\xi X,X}(q_\xi \flat 1) = (q_\xi + 1)\kappa_{B+X,X}$, so that the left-hand side is equal to

$$\begin{aligned}[1, j_X](q_\xi + 1)\kappa_{B+X,X} &= [q_\xi, j_X]\kappa_{B+X,X}\\ &= [s_B, j_X, j_X]\kappa_{B+X,X}\\ &= q_\xi[\iota_B, \iota_X, \iota_X]\kappa_{B+X,X}\end{aligned}$$

Now in the diagram

$$\begin{tikzcd}
(B+X)\flat X \arrow[r, "\kappa_{B+X,X}"] \arrow[d, dotted, "{[\iota_B,\iota_X,\iota_X]^*}"'] \arrow[dr, phantom, "(a)"] & B+X+X \arrow[r, "{[\iota_B,\iota_X,0]}"] \arrow[d, "{[\iota_B,\iota_X,\iota_X]}"] \arrow[dr, phantom, "(b)"] & B+X \arrow[d, "{[1,0]}"] \\
B\flat X \arrow[r, "\kappa_{B,X}"'] \arrow[d, "\xi"'] \arrow[dr, phantom, "(c)"] & B+X \arrow[r, "{[1,0]}"'] \arrow[d, "q_\xi"] & B \\
X \arrow[r, "j_X"'] & B\ltimes_\xi X, &
\end{tikzcd}$$

the square (b) commutes, so that there exists a unique morphism $[\iota_B, \iota_X, \iota_X]^*$ making (a) commute. Moreover (c) commutes by definition of ξ. As a consequence, equation (1.9) is equivalent to

$$j_X\xi[\iota_B, \iota_X, \iota_X]^* = j_X\xi[1,\partial]\flat 1.$$

Thus a precrossed module is a crossed module if and only if the following diagram commutes :

$$\begin{tikzcd}
(B+X)\flat X \arrow[r, "{[1,\partial]\flat 1}"] \arrow[d, "{[\iota_B,\iota_X,\iota_X]^*}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X.
\end{tikzcd} \tag{1.10}$$

Given a precrossed module, the pullback of the action $[1,\partial\rangle^*\xi$ along s_B is simply

$$s_B^*[1,\partial\rangle^*\xi = ([1,\partial\rangle s_B)^*\xi = \xi.$$

The pullback action $s_B^*\chi$ coincides with ξ as well.

On the other hand, pulling back $[1,\partial\rangle^*\xi$ along j_X yields $\partial^*\xi$, whereas pulling back χ along j_X just gives the conjugation of X on itself. Thus every crossed module satisfies the *Peiffer condition* : $\partial^*\xi = \chi$.

In the category of groups, it is well-known [31] that the converse is true as well, and thus a precrossed module is a crossed module if and only if it satisfies the Peiffer condition. This does not hold in every semi-abelian category, but it does hold whenever the category satisfies the "uniqueness of actions" condition :

(UA) Given an extremal epimorphic cospan $A \xrightarrow{f} B \xleftarrow{g} C$ in $\mathcal{C}$, then for any 4-tuple $(\xi_1,\xi_2,\xi_3,\xi_4)$ of actions on a fixed object X making the diagram

$$\begin{tikzcd}
A\flat X \arrow[r, "f\flat 1"] \arrow[dr, "\xi_1"'] & B\flat X \arrow[d, shift right, "\xi_3"'] \arrow[d, shift left, "\xi_4"] & C\flat X \arrow[l, "g\flat 1"'] \arrow[dl, "\xi_2"] \\
& X &
\end{tikzcd} \tag{1.11}$$

commute, we have $\xi_3 = \xi_4$.

This condition was introduced in [40], where the authors show that this property holds in any *action representable* semi-abelian category (see [14]) and in any *category of interest* in the sense of Orzech [117]. In particular, the categories of groups, Lie and Leibniz algebras over a fixed field, rings, associative algebras, Poisson algebras over a commutative ring with unit are all examples of such categories.

Example 1.47. When $\mathcal{C}$ is the category $\mathbf{Lie}_K$ of Lie algebras over a field K, the classical notion of action coincides with the semi-abelian one. Accordingly, a *Lie algebra precrossed module* is given by two Lie algebra homomorphisms $\partial\colon X \to B$ and $\xi\colon B \to \mathrm{Der}(X)$, where $\mathrm{Der}(X)$

is the Lie algebra of derivations of X, such that $\partial(\xi(b)(x)) = [b, \partial(x)]$ for all $x \in X$ and $b \in B$. A *Lie algebra crossed module* [102] is then a precrossed module where the Peiffer identity

$$\xi(\partial(x))(y) = [x, y]$$

holds for all $x, y \in X$.

1.4.4 Commutators and the "Smith is Huq" condition

Definition 1.48. In a semi-abelian category $\mathcal{C}$, two subobjects M, N of an object X are said to commute in the sense of Huq [77] if there exists a morphism $\varphi\colon M \times N \to X$ making the two triangles below commute :

$$\begin{tikzcd} M \arrow[r, "{(1,0)}"] \arrow[dr, "m"'] & M \times N \arrow[d, dashed, "\varphi"] & N \arrow[l, "{(0,1)}"'] \arrow[dl, "n"] \\ & X. & \end{tikzcd}$$

Such a morphism φ, if it exists, is called a *cooperator* for M and N. Two subobjects have at most one cooperator.

Example 1.49. In the category of groups, two subgroups commute in the sense of Huq if and only if they commute in the usual sense. Indeed, if M, N are two subgroups of X, then a cooperator $\varphi\colon M \times N \to X$ must satisfy $\varphi(m, 1) = m$ for all $m \in M$ and $\varphi(1, n) = n$ for all $n \in N$. Then

$$\varphi(m, n) = \varphi((m, 1) \cdot (1, n)) = \varphi(m, 1)\varphi(1, n) = mn,$$

but also

$$\varphi(m, n) = \varphi((1, n) \cdot (m, 1)) = \varphi(1, n)\varphi(m, 1) = nm,$$

thus if a cooperator exists, M and N commute. On the other hand, if M and N commute then $\varphi(m, n) = mn$ defines a homomorphism $M \times N \to X$, which is a cooperator.

Definition 1.50 ([75, 77, 113])**.** Let M, N be two subobjects of an object X.

1. The *Higgins commutator* of M and N is the image of the kernel $M \diamond N$ of $\Sigma\colon M + N \to M \times N$ along the morphism $[m, n]\colon M + N \to X$.

2. The *Huq commutator* of M and N is the kernel of the map q defined by the following pushout :

$$\begin{tikzcd} M+N \arrow[r, "{[m,n]}"] \arrow[d, "\Sigma"'] & X \arrow[d, "q"] \\ M\times N \arrow[r] & Q \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note that this pushout is equivalently the colimit of the diagram of solid morphisms

$$\begin{tikzcd} & M \arrow[dl, "{(1,0)}"'] \arrow[d, dotted] \arrow[dr, "m"] & \\ M\times N \arrow[r, dotted] & Q & X. \arrow[l, dotted, "q"'] \\ & N \arrow[ul, "{(0,1)}"] \arrow[u, dotted] \arrow[ur, "n"'] & \end{tikzcd} \tag{1.12}$$

Thus the subobjects M, N commute if and only if their Higgins commutator $[M,N]_{Hig}$ is trivial; moreover, the Huq commutator is the smallest normal subobject whose cokernel q is such that $q(M)$ and $q(N)$ commute.

Proposition 1.51 ([113])**.** *The Huq commutator is the normal closure of the Higgins commutator.*

Definition 1.52 (Ursini commutator [111])**.** Let M, N be normal subobjects of X, and let R_M, R_N denote the associated equivalence relations. The *Ursini commutator* $[M,N]_U$ is the normal subobject associated to the equivalence relation $[R_M, R_N]_{SP}$.

Proposition 1.53 ([26])**.** *Let R, S be two equivalence relations on an object X, and let M, N be the associated normal subobjects. If R and S are connected, then M, N commute.*

Proof. We have a pair of commutative squares

$$\begin{tikzcd} M \arrow[r, shift left] \arrow[d, "\ker(r_2)"'] & 0 \arrow[l, shift left] \arrow[r, shift right] \arrow[d] & N \arrow[l, shift right] \arrow[d, "\ker(s_1)"] \\ R \arrow[r, shift left, "r_2"] & X \arrow[l, shift left, "\delta_R"] \arrow[r, shift left, "\delta_S"'] & S, \arrow[l, shift left, "s_1"'] \end{tikzcd}$$

which induce a morphism $\ker(r_2)\times\ker(s_1)\colon M\times N \to R\times_X S$. Then if $p\colon R\times_X S$ is a connector, its restriction φ to $M\times N$ is a cooperator :

indeed we have

$$\varphi(1,0) = p(\ker(r_2) \times \ker(s_1))(1,0) = p\sigma_1 \ker(r_2) = r_1 \ker(r_2) = m$$

and similarly $\varphi(0,1) = n$. □

In particular, we have the inequality $[M,N]_{Huq} \leq [M,N]_U$ for all M, N. The converse of this proposition is not necessarily true, so the inequality of commutators can be strict. However, in many semi-abelian categories, the converse does hold. This is known as the "Smith is Huq" condition [114]:

(SH) two equivalence relations whose associated subobjects commute are connected.

Theorem 1.54 ([114])**.** *A semi-abelian category $\mathcal{C}$ satisfies the "Smith is Huq" condition if and only if every precrossed module satisfying the Peiffer condition is a crossed module.*

In particular, the condition (UA) implies (SH).

The construction of the Higgins or Huq commutator can be generalized :

Definition 1.55 (Weighted commutators[68])**.** Let M, N, W be three subobjects of an object X in a semi-abelian category $\mathcal{C}$. Let us consider the diagram

$$\begin{tikzcd}
W+M+N \arrow[dr, "\psi"] \arrow[drr, bend left, "{[\iota_M,0,\iota_N]}"] \arrow[ddr, bend right, "{[\iota_W,\iota_M,0]}"'] & & \\
& (W+M)\times_W (W+N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W+N \arrow[d, "{[1,0]}"] \\
& W+M \arrow[r, "{[1,0]}"'] & W;
\end{tikzcd}$$

then the outer square is a double extension, thus ψ is a regular epimorphism.

- The weighted commutator $[M,N]_W$ is the image of $Ker(\psi)$ along $[w,m,n]\colon W+M+N \to X$
- The normal weighted commutator $[M,N]_W$ is the kernel of the map q defined by the following pushout :

$$\begin{tikzcd}
W+M+N \arrow[r, "{[w,m,n]}"] \arrow[d, "\psi"'] & X \arrow[d, "q"] \\
(W+M)\times_W (W+N) \arrow[r] & Q. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Note that the two definitions coincide whenever $[w, m, n]$ is a regular epimorphism, i.e. whenever $W \vee M \vee N = X$; moreover, if $W = 0$, then we just recover the definition of the Higgins and Huq commutators.

Theorem 1.56 ([111, 68])**.** *Let M, N be two normal subobjects of X, with respective cokernels q_M, q_N. Then $[M, N]_X$ coincides with the Ursini commutator $[M, N]_U$.*

1.5 Extensions and Categorical Galois Theory

1.5.1 Extensions

Definition 1.57. We will call a class $\mathcal{F}$ of morphisms in a category $\mathcal{C}$ a *class of extensions* [57, 51, 53] if :

(E1) any isomorphism is in $\mathcal{F}$;

(E2) the pullback of a morphism in $\mathcal{F}$ along any other morphism exists and is in $\mathcal{F}$;

(E3) $\mathcal{F}$ is closed under composition.

Such a class $\mathcal{F}$ determines a full subcategory of the category of arrows of $\mathcal{C}$, which we will denote $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$. Given an object B of $\mathcal{C}$, we will also denote $\mathbf{Ext}_{\mathcal{F},B}(\mathcal{C})$ the full subcategory of the slice category $\mathcal{C}/B$ consisting of extensions $f\colon C \to B$. Then, thanks to condition (E2), any morphism $p\colon E \to B$ induces a functor $p^*\colon \mathbf{Ext}_{\mathcal{F},B}(\mathcal{C}) \to \mathbf{Ext}_{\mathcal{F},E}(\mathcal{C})$, defined by pulling back along p.

If p is itself an extension, then by (E3) composition with p defines a functor $p_!\colon \mathbf{Ext}_{\mathcal{F},E}(\mathcal{C}) \to \mathbf{Ext}_{\mathcal{F},B}(\mathcal{C})$, which is left adjoint to p^*. The extension p is said to be of *effective $\mathcal{F}$-descent* or simply a *monadic extension* if p^* is monadic.

Example 1.58. If $\mathcal{C}$ is a regular category, then the class $\mathcal{F}$ of all regular epimorphisms is a class of extensions. If moreover $\mathcal{C}$ is exact, then every extension is monadic [124, 94].

1.5.2 Galois structures

We recall some definitions from [88, 89].

Definition 1.59. A *Galois structure* $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ consists of :

- a category $\mathcal{C}$

- a full reflective subcategory $\mathcal{X}$ of $\mathcal{C}$, with inclusion functor $U\colon \mathcal{X} \to \mathcal{C}$ and reflector $I\colon \mathcal{C} \to \mathcal{X}$
- a class $\mathcal{F}$ of extensions for $\mathcal{C}$

such that the reflection preserves extensions :

(G) $UI(\mathcal{F}) \subset \mathcal{F}$.

Since the subcategory $\mathcal{X}$ is closed under limits in $\mathcal{C}$, the restriction of $\mathcal{F}$ to $\mathcal{X}$ is a class of extensions in $\mathcal{X}$, which we will also denote $\mathcal{F}$. We thus have, as above, full subcategories $\mathbf{Ext}_{\mathcal{F}}(\mathcal{X})$ and $\mathbf{Ext}_{\mathcal{F},X}(\mathcal{X})$ (for every object X of $\mathcal{X}$) of the arrow category of $\mathcal{X}$ and of the slice category $\mathcal{X}/X$, respectively.

Furthermore, the condition (G) implies that the reflector I induces for every object B of $\mathcal{C}$ a functor $I_B\colon \mathbf{Ext}_{\mathcal{F},B}(\mathcal{C}) \to \mathbf{Ext}_{\mathcal{F},I(B)}(\mathcal{X})$ which maps $f\colon X \to B$ to $I(f)\colon I(X) \to I(B)$; and every such functor has a right adjoint U_B, defined for any $g\colon Y \to I(B)$ by the pullback

$$\begin{tikzcd} B \times_{UI(B)} U(Y) \arrow[r, "\beta"] \arrow[d, "U_B(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & U(Y) \arrow[d, "U(g)"] \\ B \arrow[r, "\eta_B"'] & UI(B). \end{tikzcd} \tag{1.13}$$

Definition 1.60. Let $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ be a Galois structure. An object B of $\mathcal{C}$ is *admissible* if U_B is fully faithful. Γ is admissible if every object of $\mathcal{C}$ is admissible.

Proposition 1.61 ([88])**.** *If* $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ *is a Galois structure, then the following conditions are equivalent :*

1. Γ *is admissible;*
2. I *preserves all pullbacks of the form* (1.13) *(where* $g \in \mathcal{F}$*);*
3. I *preserves all pullbacks along an extension that lies in* $\mathcal{X}$*.*

Proof. For an object B of $\mathcal{C}$, U_B is fully faithful if and only if the counit of the adjunction $I_B \dashv U_B$ is an isomorphism. For an extension $g\colon X \to I(B)$, the corresponding component of this counit is $\epsilon_X I(\beta)$. Since U is fully faithful, ϵ_X is an isomorphism; so this is invertible if and only if $I(\beta)$ is invertible, and since $I(\eta_B)$ is invertible, this is equivalent to the image of (1.13) being a pullback. Thus the first two conditions are equivalent. Moreover the second is a particular case of the third one. It remains to prove that the second condition implies the third.

Let us consider a pullback square

$$\begin{tikzcd} B \times_{U(Y)} U(X) \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U(X) \arrow[d, "U(g)"] \\ B \arrow[r, "f"'] & U(Y) \end{tikzcd}$$

where the morphism g is in $\mathcal{F}$, and moreover X, Y are in $\mathcal{X}$. Then f can be decomposed as $U(\tilde{f})\eta_B$, so that the pullback square above can be decomposed as

$$\begin{tikzcd} B \times_{U(Y)} U(X) \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "(a)"] & UI(B) \times_{U(Y)} U(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & U(X) \arrow[d, "U(g)"] \\ B \arrow[r, "\eta_B"'] & UI(B) \arrow[r, "U(\tilde{f})"'] & U(Y). \end{tikzcd}$$

Then $UI(B) \times_{U(Y)} U(X) \cong U(I(B) \times_Y X)$, so the pullback square (b) is in the subcategory $\mathcal{X}$ and is thus preserved by the reflector I, and the pullback square (a) is of the form (1.13) and is thus also preserved by I, so that the whole pullback $(a) + (b)$ is preserved by I. $\square$

Given an admissible Galois structure, an extension $f\colon X \to B$ in $\mathbf{Ext}_{\mathcal{F},B}(\mathcal{C})$ is said to be

- *trivial* if it lies in the replete image of U_B, or equivalently if the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UI(X) \arrow[d, "UI(f)"] \\ B \arrow[r, "\eta_B"'] & UI(B) \end{tikzcd}$$

  is a pullback;

- *central*, or alternatively a *covering*, if it is locally trivial, in the sense that there exists a monadic extension $p\colon E \to B$ such that $p^*(f)$ is trivial;

- *normal*, if it is a monadic extension and if $f^*(f)$ is trivial (that is, if the projections π_1, π_2 of the kernel pair of f are trivial).

Proposition 1.62 ([88]). *Trivial and central extensions are stable under pullbacks.*

Proof. Let $f\colon X \to B$ be a trivial extension, and let $h\colon A \to B$ be any morphism in $\mathcal{C}$. Then the two squares

$$\begin{tikzcd}
A \times_B X \arrow[r, "h'"] \arrow[d, "h^*(f)"'] \arrow[dr, phantom, "(a)"] & X \arrow[r, "\eta_X"] \arrow[d, "f"] \arrow[dr, phantom, "(b)"] & UI(X) \arrow[d, "UI(f)"] \\
A \arrow[r, "h"'] & B \arrow[r, "\eta_B"'] & UI(B)
\end{tikzcd}$$

are pullbacks, so the rectangle $(a) + (b)$ is a pullback as well. This outer rectangle is the same as in the diagram

$$\begin{tikzcd}
A \times_B X \arrow[r, "\eta_{A \times_B X}"] \arrow[d, "h^*(f)"'] \arrow[dr, phantom, "(c)"] & UI(A \times_B X) \arrow[r, "UI(h')"] \arrow[d, "UI(h^*(f))"] \arrow[dr, phantom, "(d)"] & UI(X) \arrow[d, "UI(f)"] \\
A \arrow[r, "\eta_A"'] & UI(A) \arrow[r, "UI(h)"'] & UI(B).
\end{tikzcd}$$

The square (d) is equal, up to isomorphisms, to the image of the rectangle $(a) + (b)$ under UI; and by Proposition 1.61 this pullback is preserved by I, and by U since it is a right adjoint. So (d) is a pullback, and thus (c) is a pullback as well, which proves that $h^*(f)$ is trivial.

Let $f\colon X \to B$ be a central extension, split by the monadic extension $p\colon E \to B$, and let $h\colon A \to B$ be any morphism in $\mathcal{C}$. Let us consider the diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, dotted] \arrow[dd, "h''^*(p^*(f))"'] & & E \times_B X \arrow[dr] \arrow[dd, "p^*(f)" near start] & \\
& A \times_B X \arrow[rr, "h'" near start] \arrow[dd, "h^*(f)" near start] & & X \arrow[dd, "f"] \\
A \times_B E \arrow[rr, "h''" near start] \arrow[dr, "q"'] & & E \arrow[dr, "p"] & \\
& A \arrow[rr, "h"'] & & B,
\end{tikzcd}$$

where P is the pullback of h'' along $p^*(f)$. Since $p^*(f)$ is trivial, $h''^*(p^*(f))$ is trivial as well. Since the front face is a pullback, there is a dotted morphism making the whole diagram commute; moreover, since the right, back and front faces are pullbacks, the left face is also a pullback, and thus $h''^*(p^*(f)) = q^*(h^*(f))$. Since the bottom face is a pullback, q is a monadic extension. Thus $h^*(f)$ is a central extension. $\square$

Proposition 1.63 ([88])**.** *If every extension is monadic, then the following conditions are equivalent :*

1. every central extension is normal

2. every central extension that is a split epimorphism is trivial.

Proof. Let us assume that every central extension is normal. Let $f\colon X \to Y$ be a central extension, with a section $s\colon Y \to X$. Since f is normal, $f^*(f)$ is a trivial extension. Then $s^*(f^*(f)) \cong (fs)^*(f) \cong f$ is trivial as well.

Conversely, let us assume that every central extension with a section is trivial. If f is a central extension, then $f^*(f)$ is central as well. But $f^*(f)$ is the first projection π_1 of the kernel pair of f, which has a section $\delta = (1, 1)$; thus it is trivial, which means that f is a normal extension. □

Definition 1.64 (Birkhoff subcategory)**.** A full reflective subcategory $\mathcal{X}$ of a regular category $\mathcal{C}$ is called a *Birkhoff subcategory* if it is

1. closed under subobjects, i.e. if $m\colon A \to B$ is a monomorphism in $\mathcal{C}$ and B is in $\mathcal{X}$ then A is in $\mathcal{X}$
2. closed under quotients, i.e. if $p\colon B \to C$ is a regular epimorphism in $\mathcal{C}$ and B is in $\mathcal{X}$ then C is in $\mathcal{X}$.

Example 1.65. The name "Birkhoff subcategory" comes from the fact that if $\mathcal{V}$ is a variety of universal algebra, then Birkhoff subcategories of $\mathcal{V}$ coincide with subvarieties of $\mathcal{V}$. Indeed, by Birkhoff's HSP theorem the subvarieties are precisely the full subcategories closed under products, subobjects and quotients, and any reflective subcategory is closed under products, thus every Birkhoff subcategory is a subvariety; conversely, every subvariety is a reflective subcategory (see for example [1, Corollary 10.24] or [8, Corollary 10.5.5]).

Example 1.66. If $\mathcal{C}$ is an exact Mal'tsev category and $\mathcal{X}$ is any Birkhoff subcategory, and $\mathcal{F}$ is the class of regular epimorphisms, then the Galois structure $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ is admissible, and moreover every central extension is also normal [88].

When $\mathcal{C}$ is the category of groups and $\mathcal{X}$ the subcategory of abelian groups, the central extensions in this sense are exactly the surjective group homomorphisms whose kernel is included in the center of the domain. More generally, in any exact Mal'tsev category with coequalizers, the central extensions of the Galois structure defined by the subcategory of abelian objects are the extensions such that the Smith-Pedicchio commutator $[Eq[f], \nabla_X]$ is trivial.

Example 1.67. In a finitely complete category, any object X has a corresponding discrete internal groupoid. This defines is a fully faithful functor $D\colon \mathcal{C} \to \mathbf{Grpd}(\mathcal{C})$. If $\mathcal{C}$ is exact, then this functor admits a semi-left-exact left adjoint $\Pi_0\colon \mathbf{Grpd}(\mathcal{C}) \to \mathcal{C}$ [15]. When $\mathcal{C}$ is moreover Mal'tsev, $\mathcal{C}$ is in fact a Birkhoff subcategory of $\mathbf{Grpd}(\mathcal{C})$, and the central extensions of the Galois structure $(\mathbf{Grpd}(\mathcal{C}), \mathcal{C}, \Pi_0, D, \mathcal{F})$ (where $\mathcal{F}$ is the class of regular epimorphisms) are precisely the regular epimorphic discrete fibrations [66].

The following lemma is often useful to prove that an extension is central :

Lemma 1.68. *Let $\mathcal{G}$ be a class of morphisms included in $\mathcal{F}$. Assume $\mathcal{G}$ satisfies the following two conditions :*

1. *for any pullback square*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

   with $f \in \mathcal{F}$, we have $g' \in \mathcal{G}$ if and only if $g \in \mathcal{G}$;

2. *every extension in $\mathcal{X}$ is in $\mathcal{G}$.*

Then every central extension is in $\mathcal{G}$.

Proof. An extension $f\colon X \to B$ is central if and only if there exists a monadic extension $p\colon E \to B$ such that $p^*(f)$ is trivial. This gives a diagram

$$\begin{tikzcd} UI(E \times_B X) \arrow[d, "UI(p^*(f))"'] & E \times_B X \arrow[l, "\eta_{E \times_B X}"'] \arrow[r, "p'"] \arrow[d, "p^*(f)"] & X \arrow[d, "f"] \\ UI(E) & E \arrow[l, "\eta_E"] \arrow[r, "p"'] & B. \end{tikzcd}$$

where both squares are pullbacks. Then we have $UI(p^*(f)) \in \mathcal{G}$, which implies that $p^*(f) \in \mathcal{G}$, and thus $f \in \mathcal{G}$. $\square$

1.5.3 Higher extensions

Given a Galois structure $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$, let us denote $\mathbf{CExt}_\Gamma(\mathcal{C})$ the full subcategory of $\mathbf{Ext}_\mathcal{F}(\mathcal{C})$ defined by the central extensions. It often happens that $\mathbf{CExt}_\Gamma(\mathcal{C})$ is itself reflective in $\mathbf{Ext}_\mathcal{F}(\mathcal{C})$; one can then define a new Galois structure based on this adjunction. To this end it is necessary to define a class of extensions *between extensions*, or double extensions. Since $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$ is by definition a full subcategory of $\mathbf{Arr}(\mathcal{C})$, such a class of double extensions must be a class of morphisms in $\mathbf{Arr}(\mathcal{C})$, and thus a class of objects in $\mathbf{Arr}(\mathbf{Arr}(\mathcal{C}))$, that is, it must be a class of *commutative squares* in $\mathcal{C}$.

Definition 1.69. Given a class $\mathcal{F}$ of morphisms in a category $\mathcal{C}$, we define the class $\mathcal{F}^1$ to be the class of commutative squares in $\mathcal{C}$ such that all morphisms in the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "(f{,}g)"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_W Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "h"] \\
& Y \arrow[r, "p"'] & W
\end{tikzcd} \tag{1.14}$$

are in the class $\mathcal{F}$. Note that thanks to (E2) and (E3), this is equivalent to just (f, g), p and h being in $\mathcal{F}$.

Double extensions form a class of extensions in $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$:

Proposition 1.70. *If $\mathcal{F}$ satisfies* (E1) *to* (E3) *in $\mathcal{C}$, then $\mathcal{F}^1$ satisfies the same three properties in $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$.*

Proof. It is straightforward to see that (E1) still holds for double extensions; moreover, the proof of Proposition 1.17 can be applied to show that $\mathcal{F}^1$ satisfies (E2). For (E3), we need to prove that in a diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "(a)"] & Z \arrow[r, "g'"] \arrow[d, "h"] \arrow[dr, phantom, "(b)"] & U \arrow[d, "q"] \\
Y \arrow[r, "p"'] & W \arrow[r, "p'"'] & V
\end{tikzcd}$$

where the squares (a) and (b) are double extensions, the outer rectangle is a double extension as well. To this end we consider the diagram

$$\begin{tikzcd}
X \arrow[d, "\gamma"] \arrow[dr, bend left, "g"] \arrow[ddd, bend right=60, "f"'] & & \\
Y \times_W Z \arrow[r] \arrow[d, "\beta"] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "\alpha"] \arrow[dd, bend right=40, "h"'] \arrow[dr, bend left, "g'"] & \\
Y \times_V U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W \times_V U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "q"] \\
Y \arrow[r, "p"'] & W \arrow[r, "p'"'] & V;
\end{tikzcd}$$

since the square (b) is a double extension, $\alpha \in \mathcal{F}$, and then $\beta \in \mathcal{F}$ as well since it is a pullback of α. Since the square (a) is a double extension, $\gamma \in \mathcal{F}$, and then $\beta\gamma \in \mathcal{F}$, which proves that the outer rectangle is a double extension. $\square$

This definition can be iterated, to define n-fold extensions for all n. Just as double extensions are objects of $\mathbf{Arr}^2(\mathcal{C})$, n-fold extensions will be objects of the category $\mathbf{Arr}^n(\mathcal{C})$, i.e. n-dimensional commutative hypercubes. Since $\mathbf{Arr}(\mathcal{C})$ can be identified with the category of functors $\{0 \leq 1\} \to \mathcal{C}$, these commutative hypercubes can be identified with functors $\{0 \leq 1\}^n \to \mathcal{C}$, or alternatively with functors $\mathcal{P}(n) \to \mathcal{C}$ (where we interpret n as the set $\{0, \dots, n-1\}$).

Definition 1.71. Given a class $\mathcal{F} = \mathcal{F}^0$ of extensions in $\mathcal{C}$, we define inductively the class $\mathcal{F}^n$ of $(n+1)$-fold extensions for $n \geq 1$ as $\mathcal{F}^n = (\mathcal{F}^{n-1})^1$ in $\mathbf{Ext}^{\mathcal{F}^{n-2}}(\mathcal{C})$, so that $\mathcal{F}^n$ is a class of extensions in $\mathbf{Ext}^{\mathcal{F}^{n-1}}(\mathcal{C})$.

This means that if $X \colon \mathcal{P}(n) \to \mathcal{C}$ is a n-dimensional hypercube, then it is an n-fold extension if all the induced $(n-1)$-dimensional hypercubes in the diagram

$$\begin{tikzcd}
X_{(_,0,0)} \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_{(_,1,0)} \times_{X_{(_,1,1)}} X_{(_,0,1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{(_,0,1)} \arrow[d] \\
& X_{(_,1,0)} \arrow[r] & X_{(_,1,1)}
\end{tikzcd} \tag{1.15}$$

are $(n-1)$-fold extensions.

Restricting the diagram above by fixing some of the $(n-2)$ first indices preserves pullbacks; using this fact we can prove inductively that all m-dimensional faces of an n-fold extension are m-fold extensions themselves. Moreover any composition of such faces is an m-fold extension as well.

The definition of $\mathcal{F}^1$ as a class of commutative squares in $\mathcal{C}$ does not depend on how these squares are identified with objects of $\mathbf{Arr}(\mathbf{Arr}(\mathcal{C}))$, so that a commutative square such as the outer diagram in (1.14) is a double extension whether it is seen as a morphism $(g, p) \colon f \to h$ or as a morphism $(f, h) \colon g \to p$. This is true for n-fold extensions as well; in fact we can give an alternative definition of n-fold extensions that is independent of the way we interpret it as an extension in $\mathbf{Ext}^{\mathcal{F}^{n-1}}(\mathcal{C})$.

Proposition 1.72 ([53]). *A functor $X\colon \{0 \leq 1\}^n \to \mathcal{C}$ representing an n-dimensional hypercube is an n-fold extension if and only if for all $(i_1, \dots, i_n) \in \{0 \leq 1\}^n \setminus \{\vec{1}\}$, the morphism $\alpha_{\vec{i}}\colon X_{\vec{i}} \to \lim_{\vec{i}<\vec{j}} X_{\vec{j}}$ is in $\mathcal{F}$.*

Proof. We proceed by induction on n. For $n = 1, 2$, the definitions are the same. Now let us assume that the equivalence holds for all $m < n$, and let $X\colon \{0 \leq 1\}^n \to \mathcal{C}$ be an n-fold extension. If $\vec{i} \in \{0,1\}^n$, then the set of elements $\vec{j}$ greater than or equal to $\vec{i}$ is isomorphic to $\{0 \leq 1\}^m$, where m is the number of entries of $\vec{i}$ equal to 0. In particular, if $m < n$, then the m-dimensional hypercube X' obtained by restricting X to this set is an m-fold extension, so that $\alpha'_{\vec{i}}\colon X_{\vec{i}} \to \lim_{\vec{i}<\vec{j}} X_{\vec{j}}$ is in $\mathcal{F}$ by the induction hypothesis. Thus it remains to prove that $X_{\vec{0}} \to \lim_{\vec{0}<\vec{j}} X_{\vec{j}}$ is in $\mathcal{F}$. Since X represents an n-fold extension, we have an $(n-1)$-fold extension $X_{(\_,0,0)} \to X_{(\_,1,0)} \times_{X_{(\_,1,1)}} X_{(\_,0,1)}$, which we denote $X'' : \{0 \leq 1\}^{n-1} \to \mathcal{C}$. By the induction hypothesis, the canonical morphism $\alpha''_{\vec{0}}\colon X''_{\vec{0}} \to \lim_{\vec{0}<\vec{j}} X''_{\vec{j}}$ is then in $\mathcal{F}$; but the limit in the codomain coincides with $\lim_{\vec{0}<\vec{j}} X_{\vec{j}}$, so that $\alpha''_{\vec{0}}$ coincides with $\alpha_{\vec{0}}$, which proves that $\alpha_{\vec{0}}$ is in $\mathcal{F}$.

On the other hand, let us assume that X is such that all morphisms $\alpha_{\vec{i}}$ are in $\mathcal{F}$. As above, we can easily deduce that $X_{(\_,0,1)} \to X_{(\_,1,1)}$ and $X_{(\_,1,0)} \to X_{(\_,1,1)}$ are $(n-1)$-fold extensions using the induction hypothesis. So it is enough to prove that the the $(n-1)$-dimensional hypercube X'' defined by the morphism $X_{(\_,0,0)} \to X_{(\_,1,0)} \times_{X_{(\_,1,1)}} X_{(\_,0,1)}$ in $\mathbf{Arr}^{n-2}(\mathcal{C})$ is an $(n-1)$-fold extension as well; by the induction hypothesis, it is enough to check that the corresponding canonical morphisms $\alpha''_{\vec{i}}$ are in $\mathcal{F}$ for all $\vec{i} \in \{0 \leq 1\}^{n-1}$. Since $X_{(\_,1,0)} \times_{X_{(\_,1,1)}} X_{(\_,0,1)}$ is the pullback of two $(n-1)$-fold extensions, it is an $(n-2)$-fold extension, which proves already that $\alpha''_{\vec{i}}$ is in $\mathcal{F}$ for all $\vec{i}$ whose last component is 1. Moreover, if $\vec{i} \in \{0 \leq 1\}^{n-2}$ is non-zero, then restricting the outer square (1.15) to the set of vectors greater than or equal to $(\vec{i}, 0, 0)$ gives an m-fold extension; in particular, we have then an $(m-1)$-fold extension $X_{(\vec{i},0,0)} \to X_{(\vec{i},1,0)} \times_{X_{(\vec{i},1,1)}} X_{(\vec{i},0,1)}$. Since this coincides with the restriction of X'' to the set of vectors greater than or equal to $(\vec{i}, 0)$, we find, again by the induction hypothesis, that all $\alpha''_{(\vec{i},0)}$ are in $\mathcal{F}$. Finally, we see as in the first part of the proof that $\alpha''_{\vec{0}}$ coincides with $\alpha_{\vec{0}}$, and is thus in $\mathcal{F}$, which concludes the proof. □

In order to ensure admissibility of the Galois structure induced by the reflection of $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$ into $\mathbf{CExt}_{\Gamma}(\mathcal{C})$, it is useful to consider the following two additional conditions :

(E4) if $fg \in \mathcal{F}$, then $f \in \mathcal{F}$;

(E5) given any square

$$\begin{tikzcd} X \arrow[r, shift left, "g"] \arrow[d, "f"'] & Z \arrow[l, shift left, "s"] \arrow[d, "h"] \\ Y \arrow[r, shift left, "p"] & W \arrow[l, shift left, "t"] \end{tikzcd}$$

where f and h are in $\mathcal{F}$ and $gs = id_Z$, $pt = id_W$, $hg = pf$ and $fs = th$, the induced morphism $(f, g)\colon X \to Y \times_W Z$ is in $\mathcal{F}$.

Note that (E4) implies that split epimorphisms are in $\mathcal{F}$, and (E5) is equivalent to the statement that split epimorphisms in $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$ are in $\mathcal{F}^1$. Note that when $\mathcal{F}$ is the class of regular epimorphism in a regular category, (E5) is equivalent to that category being Mal'tsev. If these properties are satisfied, we can give a useful characterization of double extensions, similar to Proposition 1.18:

Proposition 1.73. *If $\mathcal{F}$ satisfies* (E1) *to* (E5)*, then a square of extensions is a double extension if and only if the morphism $\widetilde{f}\colon Eq[g] \to Eq[p]$ induced by f and h is an extension.*

Proof. Pulling back a double extension along itself gives again a double extension, and this implies that $\widetilde{f}$ is an extension. Conversely, if $\widetilde{f}$ is an extension, then in the commutative cube

$$\begin{tikzcd} Eq[g] \arrow[rr, "\pi_2"] \arrow[dr, "\pi_1"] \arrow[dd, "\widetilde{f}"'] & & X \arrow[dr, "g"] \arrow[dd, "f" near start] & \\ & X \arrow[rr, "g" near start, crossing over] & & Z \arrow[dd, "h"] \\ Eq[p] \arrow[rr, "\psi_2" near start] \arrow[dr, "\psi_1"'] & & Y \arrow[dr, "p"] & \\ & Y \arrow[rr, "p"'] \arrow[from=uu, "f" near start, crossing over] & & W \end{tikzcd}$$

the back face is a double extension by condition (E5) since π_2, ψ_2 have sections which commute with $f, \widetilde{f}$. This means $(\widetilde{f}, \pi_2)\colon Eq[g] \to Eq[p] \times_Y X$ is in $\mathcal{F}$. Moreover, if we take the pullbacks of the front and back faces then g and ψ_1 induce a morphism $\alpha\colon Eq[p] \times_Y X \to Y \times_W Z$, in such a way that the right-hand square in the diagram

$$\begin{tikzcd} Eq[g] \arrow[r, "(\widetilde{f}, \pi_2)"] \arrow[d, "\pi_1"'] & Eq[p] \times_Y X \arrow[r] \arrow[d, "\alpha"] & X \arrow[d, "g"] \\ X \arrow[r, "(f,g)"'] & Y \times_W Z \arrow[r] & Z \end{tikzcd}$$

is a pullback; thus $\alpha \in \mathcal{F}$ since $g \in \mathcal{F}$. Then $\alpha(\widetilde{f}, \pi_2) \in \mathcal{F}$, which implies that $(f, g) \in \mathcal{F}$ as well by (E4). $\square$

These new properties are also preserved by the passage to double extensions :

Proposition 1.74. *If $\mathcal{F}$ satisfies the conditions* (E1) *to* (E5)*, then so does $\mathcal{F}^1$ in $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C})$.*

Proof. For the condition (E4) we need to prove that in the diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "(a)"] & Z \arrow[r, "g'"] \arrow[d, "h"] \arrow[dr, phantom, "(b)"] & U \arrow[d, "q"] \\
Y \arrow[r, "p"'] & W \arrow[r, "p'"'] & V
\end{tikzcd}$$

where $h \in \mathcal{F}$ and the outer rectangle is in $\mathcal{F}^1$, the square (a) is in $\mathcal{F}^1$. Note that f and q are in $\mathcal{F}$, and thanks to the previous proposition, the morphism $Eq[f] \to Eq[h] \to Eq[q]$ is in $\mathcal{F}$; since $\mathcal{F}$ satisfies (E4) this means that $Eq[h] \to Eq[q]$ is in $\mathcal{F}$, and also g' and p' are in $\mathcal{F}$, so that (b) is in $\mathcal{F}^1$.

For the condition (E5), we need to prove that a split epimorphism of double extensions such as

$$\begin{tikzcd}
X \arrow[rr, shift left] \arrow[dr, "g"] \arrow[dd, "f"'] & & X' \arrow[ll, shift left] \arrow[dr, "g'"] \arrow[dd, "f'"' near end] & \\
& Z \arrow[rr, shift left] \arrow[dd, "h"' near start] & & Z' \arrow[ll, shift left] \arrow[dd, "h'"] \\
Y \arrow[rr, shift left] \arrow[dr, "h"'] & & Y' \arrow[ll, shift left] \arrow[dr, "h'"] & \\
& W \arrow[rr, shift left] & & W' \arrow[ll, shift left]
\end{tikzcd}$$

is a triple extension. Note that (E5) implies that all faces are double extensions, and since the notion of triple extension is independent of the way this cube is seen as an extension between double extensions, it suffices to check that the square

$$\begin{tikzcd}
X \arrow[r, "{(f,g)}"] \arrow[d, shift left] & Y \times_W Z \arrow[d, shift left] \\
X' \arrow[r, "{(f',g')}"'] \arrow[u, shift left] & Y' \times_{W'} Z' \arrow[u, shift left]
\end{tikzcd}$$

is a double extension, and this simply follows from (E5). $\square$

Definition 1.75 (Strongly $\mathcal{F}$-Birkhoff subcategory [57]). Given a category $\mathcal{C}$ with a class $\mathcal{F}$ of extensions, a full reflective subcategory $\mathcal{X}$ is a *strongly $\mathcal{F}$-Birkhoff subcategory* if for all $f\colon X \to Y$ in $\mathcal{F}$ the naturality square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UI(X) \arrow[d, "UI(f)"] \\ Y \arrow[r, "\eta_Y"'] & UI(Y) \end{tikzcd}$$

is in $\mathcal{F}^1$, i.e. every component of η and every canonical map (f, η_X) lies in $\mathcal{F}$.

Theorem 1.76 ([52], Theorem 2). *If $\mathcal{F}$ is a class of morphisms satisfying* (E1) *to* (E5) *and such that every $f \in \mathcal{F}$ is an effective $\mathcal{F}$-descent morphism, and $\mathcal{X}$ is a strongly $\mathcal{F}$-Birkhoff subcategory whose reflector I satisfies* (G)*, then*

1. *Every $(f, h) \in \mathcal{F}^1$ is an effective $\mathcal{F}^1$-descent morphism.*
2. *$\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ is an admissible Galois structure;*
3. *$\mathbf{CExt}_\Gamma(\mathcal{C})$ is an $\mathcal{F}^1$-Birkhoff subcategory of $\mathbf{Ext}_\mathcal{F}(\mathcal{C})$.*

Combining this theorem with Proposition 1.74, we find that $\Gamma^1 = (\mathbf{Ext}_\mathcal{F}(\mathcal{C}), \mathbf{CExt}_\Gamma(\mathcal{C}), I^1, U^1, \mathcal{F}^1)$ is again an admissible Galois structure. This allows us to define :

Definition 1.77 (Double central extensions [83]). A *double central extension* is a double extension which is Γ^1-central.

Example 1.78. We've seen before that in any regular Mal'tsev category, regular epimorphisms satisfy the properties (E1) to (E5). When the category is moreover exact, any Birkhoff subcategory is also strongly $\mathcal{F}$-Birkhoff, and moreover, every regular epimorphism is effective for descent. Thus any Birkhoff subcategory $\mathcal{X}$ of $\mathcal{C}$ gives an admissible Galois structure; and moreover all these properties are again satisfied by the induced Galois structure Γ^1.

2 Galois theory for pairs of equivalence relations in exact Mal'tsev categories

This chapter is adapted from the article [50], written jointly with Marino Gran.

We define a Galois structure on the category of pairs of equivalence relations in an exact Mal'tsev category with coequalizers, and characterize central and double central extensions in terms of higher commutator conditions. These results generalize both the ones related to the abelianization functor in exact Mal'tsev categories, and the ones corresponding to the reflection from the category of internal reflexive graphs to the subcategory of internal groupoids.

2.1 The category of pairs, connectors, and commutators

Let $\mathcal{C}$ be a category with finite limits. We define the category $\mathbf{Eq}(\mathcal{C})$ as the category whose objects are pairs (X, R) where R is an equivalence relation on the object X of $\mathcal{C}$, and morphisms $(X, R) \to (X', R')$ are morphisms $f\colon X \to X'$ such that $f(R) \leq R'$. We define similarly the category $2\text{-}\mathbf{Eq}(\mathcal{C})$, whose objects are triples (X, R, S) where R and S are two equivalence relations on an object X, and morphisms $(X, R, S) \to (X', R', S')$ are those morphisms $f\colon X \to X'$ in $\mathcal{C}$ such that $f(R) \leq R'$ and $f(S) \leq S'$. This means that there exist morphisms $f_R\colon R \to R'$ and $f_S\colon S \to S'$ in $\mathcal{C}$ such that the following diagram commutes :

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_R"'] & X \arrow[d, "f"] & S \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "f_S"] \\ R' \arrow[r, shift left] \arrow[r, shift right] & X' & S' \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd} \tag{2.1}$$

One can check that limits in 2 - $\mathbf{Eq}(\mathcal{C})$ are just pointwise limits in $\mathcal{C}$, and if $\mathcal{C}$ is regular Mal'tsev then the same holds for regular epimorphisms; in other words, regular epimorphisms in 2 - $\mathbf{Eq}(\mathcal{C})$ are regular epimorphisms $f\colon X \to X'$ such that $f(R) = R'$ and $f(S) = S'$, i.e. such that f_R and f_S are regular epimorphisms. The categories $\mathbf{Eq}(\mathcal{C})$ and 2 - $\mathbf{Eq}(\mathcal{C})$ are easily seen to be regular Mal'tsev categories whenever $\mathcal{C}$ itself is a regular Mal'tsev category [25]. If $\mathcal{C}$ is exact, then the category $\mathbf{Eq}(\mathcal{C})$, while not exact, shares with $\mathcal{C}$ the property that every regular epimorphism is effective for descent ([93]), and the same is true for 2 - $\mathbf{Eq}(\mathcal{C})$.

Given two equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on the same object X in $\mathcal{C}$, a *double equivalence relation* on R and S is an equivalence relation in $\mathbf{Eq}(\mathcal{C})$, depicted as

$$\begin{tikzcd} C \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X, \end{tikzcd} \tag{2.2}$$

where (C, p_1, p_2) is an equivalence relation on R, (C, q_1, q_2) is an equivalence relation on S, and $r_i p_j = s_j q_i$ for all $i, j \in \{1, 2\}$. A double equivalence relation on R and S is said to be *centralizing* if any of the commutative squares in diagram (2.2) is a pullback (and in this case all of them will be). Two equivalence relations R and S are said to *centralize each other* (or to be *connected*) if they have a centralizing relation. We will write $R \square S$ for the largest double equivalence relation on R and S; it can be constructed as a limit of the diagram

$$\begin{tikzcd} X & R \arrow[l, "r_1"'] \arrow[r, "r_2"] & X \\ S \arrow[u, "s_1"] \arrow[d, "s_2"'] & R \square S \arrow[l, "q_1"] \arrow[r, "q_2"] \arrow[u, "p_1"] \arrow[d, "p_2"] & S \arrow[u, "s_1"'] \arrow[d, "s_2"] \\ X & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & X \end{tikzcd}$$

(as explained in [92]). Note that when R and S are the kernel pairs of morphisms f and g which are part of a commutative square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "h"] \\
Y \arrow[r, "p"'] & W,
\end{tikzcd}$$

then $Eq[f] \,\square\, Eq[g]$, as an equivalence relation on $Eq[f]$ (resp. $Eq[g]$), is the kernel pair of the induced morphism $\widetilde{g}\colon Eq[f] \to Eq[h]$ (resp. $\widetilde{f}\colon Eq[g] \to Eq[p]$).

In a finitely complete category we can denote "generalized elements" of a double relation as $\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)$, where c_{ij} in X are such that $c_{i1} R c_{i2}$ and $c_{1j} S c_{2j}$. With this notation, a double relation C is centralizing if for all a, b, c such that aRb and bSc, there exists a unique d such that $\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right) \in C$.

Let us consider the pullback

$$\begin{tikzcd}
R \times_X S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\
R \arrow[r, "r_2"'] & X.
\end{tikzcd}$$

When $\mathcal{C}$ is a regular Mal'tsev category, then the following conditions are equivalent (see [25, 36, 118, 119]):

1. R and S have a (unique) centralizing relation;
2. there exists a (unique) morphism $\beta\colon R \times_X S \to R \,\square\, S$ such that the diagram

$$\begin{tikzcd}
R \,\square\, S \arrow[dr, shift left, "\alpha"] \arrow[drr, bend left, shift left, "q_2"] \arrow[ddr, bend right, shift right, "p_1"'] & & \\
& R \times_X S \arrow[ul, shift left, "\beta"] \arrow[r, shift left, "\pi_2"] \arrow[d, shift left, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[l, shift left, "\sigma_2"] \arrow[d, shift left, "s_1"] \arrow[ull, bend right, "\delta_Q"'] \\
& R \arrow[u, shift left, "\sigma_1"] \arrow[r, shift left, "r_2"] \arrow[uul, bend left, "\delta_P"] & X \arrow[l, shift left, "\delta_R"] \arrow[u, shift left, "\delta_S"]
\end{tikzcd}$$

commutes, where

$$\alpha\beta = id_{R \times_X S}, \quad \pi_1\sigma_1 = id_R, \quad \pi_2\sigma_2 = id_S,$$

and the morphism δ_R (resp. δ_S, δ_P, δ_Q) is the diagonal of the relation R (resp. S, $P = (R \,\square\, S, p_1, p_2)$, $Q = (R \,\square\, S, q_1, q_2)$), that exists since $R \,\square\, S$ is a double equivalence relation.

3. there exists a (unique) partial Mal'tsev operation $p\colon R \times_X S \to X$, called a *connector* in [26, 25], satisfying $p\sigma_1 = r_1$ and $p\sigma_2 = s_2$, i.e.

$$p(x, y, y) = x \text{ and } p(y, y, z) = z.$$

In addition, the connector p also satisfies the "compatibility axioms"

$$xSp(x, y, z) \quad \text{and} \quad zRp(x, y, z)$$

whenever $xRySz$ and the "associativity axiom"

$$p(p(x, y, z), u, v) = p(x, y, p(z, u, v))$$

whenever $xRySzRuSv$. Thus for every element $\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)$ of $R \,\square\, S$, we have $(p(a, b, c), d) \in R \wedge S$, and the following square is a pullback

$$\begin{tikzcd} R \,\square\, S \arrow[r, "\alpha"] \arrow[d, "q"'] & R \times_X S \arrow[d, "p"] \\ R \wedge S \arrow[r, "p_1"'] & X, \end{tikzcd}$$

where $\alpha\colon R \,\square\, S \to R \times_X S$ associates (a, b, c) with any $\left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right)$ in $R \,\square\, S$.

Let us say a few words about these equivalences. Given a connector $p\colon R \times_X S \to X$, we can form a double centralizing relation

$$\begin{tikzcd} R \times_X S \arrow[r, shift left, "(r_1\pi_1, p)"] \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "(p, s_2\pi_2)"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X, \end{tikzcd}$$

or in other words the generalized elements of this double relation have the form $\left(\begin{smallmatrix} x & y \\ p(x,y,z) & z \end{smallmatrix}\right)$. On the other hand, given a double centralizing relation (2.2), we can define a connector by setting $p = r_1 p_2 = s_2 q_1$, so that $p(x, y, z)$ is the only element such that $\left(\begin{smallmatrix} x & y \\ p(x,y,z) & z \end{smallmatrix}\right) \in C$. Moreover, a map $\beta\colon R \times_X S \to R \square S$ is determined by four maps $\beta_{ij}\colon R \times_X S \to X$, and the condition that $\alpha\beta = 1_{R \times_X S}$ implies that $\beta_{11} = r_1\pi_1$, $\beta_{12} = r_2\pi_1 = s_1\pi_2$ and $\beta_{22} = s_2\pi_2$. Thus a splitting of α is determined by one map $p = \beta_{21}\colon R \times_X S \to X$.

Now the Mal'tsev conditions $p(x, y, y) = y$ and $p(y, y, z) = z$ are equivalent to the condition that $\left(\begin{smallmatrix} x & y \\ x & y \end{smallmatrix}\right) \in C$ and $\left(\begin{smallmatrix} y & y \\ z & z \end{smallmatrix}\right) \in C$, or in other words, that C is a double reflexive relation. The associativity axioms are then equivalent to the symmetry and transitivity of C as a relation on R or S.

Finally, since any connector p must satisfy $p\sigma_1 = r_1$ and $p\sigma_2 = s_2$, the uniqueness just follows from the fact that σ_1, σ_2 are jointly (strongly) epimorphic by Theorem 1.20.

In any exact Mal'tsev category $\mathcal{C}$ with coequalizers, M.C. Pedicchio introduced in [118] a definition of commutator of two equivalence relations R and S on the same object X, which we denote $[R,S]_{SP}$, extending the Smith commutator of congruences (which is defined for varieties of universal algebras [123]). This categorical commutator can be characterized as the smallest equivalence relation on X whose coequalizer q is such that $q(R)$ and $q(S)$ centralize each other. In particular, it has the following properties for all equivalence relations R, S, T on the same object X (see [118, 119]):

1. $[R,S] = [S,R]$;
2. $[R,S] \leq R \wedge S$;
3. if $S \leq T$, then $[R,S] \leq [R,T]$;
4. $[R, S \vee T] = [R,S] \vee [R,T]$;
5. for any regular epimorphism f, $[f(R), f(S)] = f([R,S])$.

In a category with all finite colimits, this commutator can be described in an alternative way, due to Bourn [20]. We first take Z to be the colimit of the diagram of solid arrows

$$\begin{tikzcd} & R \arrow[dl, "\sigma_1"'] \arrow[d, dotted] \arrow[dr, "r_1"] & \\ R \times_X S \arrow[r, dotted, "\beta"] & Z & X. \arrow[l, dotted, "q"'] \\ & S \arrow[ul, "\sigma_2"] \arrow[u, dotted] \arrow[ur, "s_2"'] & \end{tikzcd} \tag{2.3}$$

Then $q\colon X \to Z$ is a regular epimorphism, $q(R)$ and $q(S)$ are connected, and any other regular epimorphism q' with this property factors through q; thus we have $[R,S] = Eq[q]$. Indeed, if $q'\colon X \to X'$ is such that $q'(R)$ and $q'(S)$ centralize each other then by definition q' induce maps $q'_R\colon R \to q'(R)$ and $q'_S\colon S \to q'(S)$. Then it also induces a map $\overline{q'}\colon R \times_X S \to q'(R) \times_{X'} q'(S)$. The connector $p\colon q'(R) \times_{X'} q'(S) \to X'$

then yields a commutative diagram

$$\begin{tikzcd}
 & R \arrow[dl, "\sigma_1"'] \arrow[d] \arrow[dr, "r_1"] & \\
R \times_X S \arrow[r, "p\overline{q'}"] & X' & X. \arrow[l, "q'"'] \\
 & S \arrow[ul, "\sigma_2"] \arrow[u, dotted] \arrow[ur, "s_2"'] &
\end{tikzcd}$$

so that there exists a unique map $\gamma\colon Z \to X'$ such that $\gamma q = q'$.

We must also check that $q(R)$ and $q(S)$ are connected. For this, we note, as above, that q induces morphisms $q_R, q_S, \overline{q}$. q_R and q_S are regular epimorphisms by definition; then by Theorem 1.20, $\overline{q}$ is also a regular epimorphism. As a consequence, to prove the existence of a connector $p\colon q(R) \times_Z q(S) \to Z$, it is enough to prove that β coequalizes the two projections of the kernel pair of $\overline{q}$. If we take the kernel pairs of $q, q_R, q_S, \overline{q}$, then the outer diagram in (2.3) yields a similar diagram

$$\begin{tikzcd}
 & Eq[q_R] \arrow[dl, "\widetilde{\sigma_1}"'] \arrow[dr, "\widetilde{r_1}"] & \\
Eq[\overline{q}] & & Eq[q]. \\
 & Eq[q_S] \arrow[ul, "\widetilde{\sigma_2}"] \arrow[ur, "\widetilde{s_2}"'] &
\end{tikzcd}$$

Here $Eq[\overline{q}]$ can be seen as the pullback of $\widetilde{r_2}$ and $\widetilde{s_1}$, so that by Theorem 1.20, $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ are jointly epimorphic. Thus to check that β coequalizes the two projections of $Eq[\overline{q}]$, it is enough to check that the two compositions coincide after composition with $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$. By construction this amounts to check that $\beta\sigma_1$ and $\beta\sigma_2$ coequalize the projections of $Eq[q_R]$ and $Eq[q_S]$ respectively. But $\beta\sigma_1 = qr_1$ factors through q_R by definition, and similarly $\beta\sigma_2$ factors through $Eq[q_S]$.

Let us denote $\mathbf{Conn}(\mathcal{C})$ the full subcategory of $2\text{-}\mathbf{Eq}(\mathcal{C})$ consisting of triples (X, R, S) where R and S are equivalence relations that admit a connector. It is proved in [26] that when $\mathcal{C}$ is a regular Mal'tsev category, the subcategory $\mathbf{Conn}(\mathcal{C})$ is closed in $2\text{-}\mathbf{Eq}(\mathcal{C})$ under regular quotients and subobjects in $\mathcal{C}$.

By the properties of the Smith-Pedicchio commutator, $\mathbf{Conn}(\mathcal{C})$ is a reflective subcategory of the regular Mal'tsev category $2\text{-}\mathbf{Eq}(\mathcal{C})$, and the (X, R, S)-component of the unit of the corresponding reflection is induced by the canonical quotient $X \to \frac{X}{[R,S]}$ of X by the equivalence relation $[R, S]_{SP}$. $\mathbf{Conn}(\mathcal{C})$ is thus a *Birkhoff subcategory* of $2\text{-}\mathbf{Eq}(\mathcal{C})$.

Convention. For the rest of this chapter, $\mathcal{C}$ will always denote an exact Mal'tsev category with coequalizers.

Proposition 2.1. *If $f\colon (X,R,S) \to (X',R',S')$ is a regular epimorphism in $2\text{-}\mathbf{Eq}(\mathcal{C})$, then the square*

$$\begin{tikzcd}
X \arrow[r, "q_{[R,S]}"] \arrow[d, "f"'] & \frac{X}{[R,S]} \arrow[d, "\bar{f}"] \\
X' \arrow[r, "q_{[R,S]}"'] & \frac{X'}{[R',S']}
\end{tikzcd} \tag{2.4}$$

induces a double extension in $2\text{-}\mathbf{Eq}(\mathcal{C})$. The category $\mathbf{Conn}(\mathcal{C})$ is thus a strongly $\mathcal{E}$-Birkhoff subcategory of $2\text{-}\mathbf{Eq}(\mathcal{C})$.

Proof. Since pullbacks and regular epimorphisms in $2\text{-}\mathbf{Eq}(\mathcal{C})$ are degreewise pullbacks and regular epimorphisms in $\mathcal{C}$, so are double extensions. Let us consider the following commutative diagram :

$$\begin{tikzcd}
 & & R \arrow[rr] \arrow[dr, "f_R"] \arrow[dd, shift left] \arrow[dd, shift right] & & q_{[R,S]}(R) \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & & & R' \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & q_{[R',S']}(R') \arrow[dd, shift left] \arrow[dd, shift right] \\
{[R,S]} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "\tilde{f}"'] & & X \arrow[rr, "q_{[R,S]}"] \arrow[dr, "f"] \arrow[dd, "q_R"'] & & \frac{X}{[R,S]} \arrow[dr, "\bar{f}"] \arrow[dd] & \\
 & {[R',S']} \arrow[rr, shift left] \arrow[rr, shift right] & & X' \arrow[rr, "q_{[R',S']}"] \arrow[dd, "q_{R'}"] & & \frac{X'}{[R',S']} \arrow[dd] \\
 & & \frac{X}{R} \arrow[rr, equal] \arrow[dr] & & \frac{X}{R} \arrow[dr] & \\
 & & & \frac{X'}{R'} \arrow[rr, equal] & & \frac{X'}{R'}.
\end{tikzcd}$$

By the properties of the commutator one has the equalities

$$[R',S'] = [f(R), f(S)] = f([R,S]),$$

thus the arrow $\tilde{f}\colon [R,S] \to [R',S']$ is a regular epimorphism. It follows that the square

$$\begin{tikzcd}
X \arrow[r, "q_{[R,S]}"] \arrow[d, "f"'] & \frac{X}{[R,S]} \arrow[d, "\bar{f}"] \\
X' \arrow[r, "q_{[R',S']}"'] & \frac{X'}{[R',S']}
\end{tikzcd}$$

is a double extension in $\mathcal{C}$. All the sides of the top face are regular epimorphisms; it suffices then to show that the comparison morphism to the pullback is a regular epimorphism. To this end, notice that the bottom face is a pullback, since two opposite sides are isomorphisms. Now by commutativity of limits with limits, the kernel pair of the induced morphism $X' \times_{X'/[R',S']} X/[R,S] \to X/R$ is the pullback $R' \times_{q_{[R',S']}(R')} q_{[R,S]}(R)$. It follows that the comparison morphism is a regular epimorphism, because the induced square of coequalizers

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q_R"'] & X' \times_{X'/[R',S']} X/[R,S] \arrow[d] \\ \frac{X}{R} \arrow[r, equal] & \frac{X}{R} \end{tikzcd}$$

is a double extension. $\square$

A natural example of a class of extensions in the regular category $2\text{-}\mathbf{Eq}(\mathcal{C})$ - i.e. satisfying the axioms (E1), (E2) and (E3)- is provided by the class $\mathcal{E}$ of regular epimorphisms. Since every regular epimorphism in $2\text{-}\mathbf{Eq}(\mathcal{C})$ is an effective descent morphisms and, moreover, $\mathbf{Conn}(\mathcal{C})$ is a strongly $\mathcal{E}$-Birkhoff subcategory of $2\text{-}\mathbf{Eq}(\mathcal{C})$ by Proposition 2.1, we have by Theorem 1.76 an admissible Galois structure $\Gamma_{\mathcal{E}} = (2\text{-}\mathbf{Eq}(\mathcal{C}), \mathbf{Conn}(\mathcal{C}), I, U, \mathcal{E})$.

We will need, however, to restrict our attention to a smaller class of extensions, which we will call *fibrations*; this term was already used in [86] to denote the classes of morphisms occurring in a Galois structure.

Definition 2.2. A *fibration* $f\colon (X,R,S) \to (X',R',S')$ is a regular epimorphism in the category $2\text{-}\mathbf{Eq}(\mathcal{C})$ satisfying one of the following equivalent properties :

- $f^{-1}(R') = R$ and $f^{-1}(S') = S$;
- $Eq[f] \leq R \wedge S$;
- the induced morphisms $\frac{X}{R} \to \frac{X'}{R'}$ and $\frac{X}{S} \to \frac{X'}{S'}$ are isomorphisms.

From now on $\mathcal{F}$ will denote the class of fibrations in $2\text{-}\mathbf{Eq}(\mathcal{C})$. One can check that the class $\mathcal{F}$ satisfies (E1), (E2) and (E3); however, $\mathcal{F}$ does not satisfy (E4), because not every split epimorphism is in $\mathcal{F}$. But the following variant of (E4) holds :

Lemma 2.3. *Consider the following commutative diagram in* $2\text{-}\mathbf{Eq}(\mathcal{C})$ *:*

$$\begin{tikzcd}
(X,R,S) \arrow[rr, "f"] \arrow[dr, "h"'] & & (X',R',S') \arrow[dl, "g"] \\
& (X'',R'',S''). &
\end{tikzcd}$$

If h *is in* $\mathcal{F}$ *and* $f\colon X \to X'$ *is a regular epimorphism in* $\mathcal{C}$*, then both* f *and* g *are in* $\mathcal{F}$*.*

Proof. Taking coequalizers of R, R' and R'' yields a commutative triangle

$$\begin{tikzcd}
\frac{X}{R} \arrow[rr, "\overline{f}"] \arrow[dr, "\overline{h}"'] & & \frac{X'}{R'} \arrow[dl, "\overline{g}"] \\
& \frac{X''}{R''}, &
\end{tikzcd}$$

in $\mathcal{C}$. Now by assumption $\overline{h}$ is an isomorphism, and thus $\overline{f}$ is a monomorphism. Since by construction $\overline{f}$ is a regular epimorphism, it is an isomorphism, and as a consequence so is $\overline{g}$. The same argument applies to the quotients by S, S', S'', hence f and g are in $\mathcal{F}$. □

As a consequence, we have :

Lemma 2.4. *Every fibration* f *induces a monadic pullback functor*

$$f^*\colon \mathbf{Ext}_{\mathcal{F},(X',R',S')}(2\text{-}\mathbf{Eq}(\mathcal{C})) \to \mathbf{Ext}_{\mathcal{F},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C})).$$

Proof. We know already that the pullback functor

$$f^*\colon \mathbf{Ext}_{\mathcal{E},(X',R',S')}(2\text{-}\mathbf{Eq}(\mathcal{C})) \to \mathbf{Ext}_{\mathcal{E},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C})).$$

has a left adjoint $f_!$ defined by composition with f, and that it is monadic. The class $\mathcal{F}$ is stable under pullbacks and composition : accordingly, the functor f^* and its adjoint $f_!$ restrict to an adjunction between $\mathbf{Ext}_{\mathcal{F},(X',R',S')}(2\text{-}\mathbf{Eq}(\mathcal{C}))$ and $\mathbf{Ext}_{\mathcal{F},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C}))$. We need to prove that this restriction is still monadic : using Beck's monadicity theorem [106], we only need to prove that $\mathbf{Ext}_{\mathcal{F},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C}))$ is closed in $\mathbf{Ext}_{\mathcal{E},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C}))$ under coequalizers. Since the domain functor $\mathbf{Ext}_{\mathcal{E},(X,R,S)}(2\text{-}\mathbf{Eq}(\mathcal{C})) \to 2\text{-}\mathbf{Eq}(\mathcal{C})$ creates coequalizers, this follows from the previous lemma. □

Observe that each (X, R, S)-component of the unit of the reflection $2\text{-}\mathbf{Eq}(\mathcal{C}) \to \mathbf{Conn}(\mathcal{C})$ actually belongs to $\mathcal{F}$, and we have shown in the proof of Proposition 2.1 that so is the induced map

$$X \to X' \times_{X'/[R',S']} X/[R, S]$$

whenever $f\colon (X, R, S) \to (X', R', S')$ is a regular epimorphism. Thus $\mathbf{Conn}(\mathcal{C})$ is also a strongly $\mathcal{F}$-Birkhoff subcategory of $2\text{-}\mathbf{Eq}(\mathcal{C})$, and the reflector I preserves morphisms in $\mathcal{F}$. Then by Theorem 1.76, the Galois structure $\Gamma_{\mathcal{F}}$ is admissible.

Moreover, an extension is $\Gamma_{\mathcal{F}}$-trivial if and only if it is $\Gamma_{\mathcal{E}}$-trivial and lies in $\mathcal{F}$; indeed, both conditions are equivalent to the square (2.4) being a pullback. Since in both classes any extension is monadic, the same holds for central and normal extensions. Moreover, the centralization of an extension in $\mathcal{F}$ is again in $\mathcal{F}$, thus the same reasoning works for double central extensions. It follows that double central extensions for $\mathcal{F}$ are exactly the double central extensions for the class of regular epimorphisms which also belong to $\mathcal{F}^1$.

2.2 The centrality criterion

When the category $\mathcal{C}$ is a Mal'tsev variety of universal algebras, T. Everaert and M. Gran proved in [54] that a surjective homomorphism in the category $\mathbf{RG}_B(\mathcal{V})$ of reflexive graphs in a Mal'tsev variety $\mathcal{V}$ is central with respect to the reflection to the subcategory $\mathbf{Grpd}_B(\mathcal{V})$ of groupoids in $\mathcal{V}$ if and only if the *commutator condition*

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_{X_1} \tag{2.5}$$

holds in $\mathcal{C}$. Note that the Smith-Pedicchio commutator distributes over joins, so this condition is equivalent to the two conditions

$$[Eq[f], Eq[c]] = \Delta_{X_1} = [Eq[f], Eq[d]]$$

We now prove a more general result in any exact Mal'tsev category (with coequalizers) by characterizing the central extensions corresponding to the reflection $2\text{-}\mathbf{Eq}(\mathcal{C}) \to \mathbf{Conn}(\mathcal{C})$. For this, we first prove the result for the Galois structure $\Gamma_{\mathcal{E}} = (2\text{-}\mathbf{Eq}(\mathcal{C}), \mathbf{Conn}(\mathcal{C}), I, U, \mathcal{E})$, where $\mathcal{E}$ is the class of regular epimorphisms in $2\text{-}\mathbf{Eq}(\mathcal{C})$.

Lemma 2.5. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers, and $f\colon (X,R,S) \to (X',R',S')$ a regular epimorphism in $2\text{-}\mathbf{Eq}(\mathcal{C})$*

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_R"'] & X \arrow[d, "f"] & S \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "f_S"] \\ R' \arrow[r, shift left] \arrow[r, shift right] & X' & S'. \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

If $[Eq[f], R \vee S] = \Delta_X$, then f is a $\Gamma_{\mathcal{E}}$-normal extension.

Proof. From the definitions above, and because kernel pairs in $2\text{-}\mathbf{Eq}(\mathcal{C})$ are computed levelwise and are maximal double equivalence relations, we need to show that in the diagram

$$\begin{tikzcd} R \,\square\, Eq[f] \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] \arrow[d, shift left] \arrow[d, shift right] & R \arrow[r, "f_R"] \arrow[d, shift left] \arrow[d, shift right] & R' \arrow[d, shift left] \arrow[d, shift right] \\ Eq[f] \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "f"] & X' \\ S \,\square\, Eq[f] \arrow[r, shift left, "\varphi'_1"] \arrow[r, shift right, "\varphi'_2"'] \arrow[u, shift left] \arrow[u, shift right] & S \arrow[r, "f_S"'] \arrow[u, shift left] \arrow[u, shift right] & S' \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

the extension

$$\pi_1\colon (Eq[f], R \,\square\, Eq[f], S \,\square\, Eq[f]) \to (X,R,S)$$

(or, equivalently, π_2) is a $\Gamma_{\mathcal{E}}$-trivial extension in $2\text{-}\mathbf{Eq}(\mathcal{C})$ (with respect to the reflection $2\text{-}\mathbf{Eq}(\mathcal{C}) \to \mathbf{Conn}(\mathcal{C})$).

To this end, we first consider the pushout

$$\begin{tikzcd} X \arrow[r, "q_S"] \arrow[d, "q_R"'] & \frac{X}{S} \arrow[d, "q'"] \\ \frac{X}{R} \arrow[r, "q"'] & \frac{X}{R\vee S}, \end{tikzcd} \tag{2.6}$$

so that $q'q_S = q_{R\vee S} = qq_R$ and moreover

$$q_S(R) = Eq[q'] \qquad q_R(S) = Eq[q].$$

As a consequence, if the condition $[Eq[f], S] = \Delta_X$ holds, then taking the direct image of both sides of this equality by q_R yields

$$[q_R(Eq[f]), Eq[q]] = \Delta_{\frac{X}{R}}.$$

A similar argument shows that

$$[q_S(Eq[f]), Eq[q']] = \Delta_{\frac{X}{S}}.$$

This observation allows us to consider the diagram

$$\begin{tikzcd}
C \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd] & & R \vee S \arrow[dd] \arrow[dr, shift left] \arrow[dr, shift right] & & & \\
& Eq[f] \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dd] & & X \arrow[rr, "f"] \arrow[dr, "q_{R\vee S}"] \arrow[dd, "q_R"] & & X' \arrow[dd, "q_{R'}"] \\
C' \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & & q_R(S) \arrow[dr, shift left] \arrow[dr, shift right] & & \frac{X}{R\vee S} \arrow[dd, equal] & \\
& q_R(Eq[f]) \arrow[rr, shift left, "\psi_1'"] \arrow[rr, shift right, "\psi_2'"'] & & \frac{X}{R} \arrow[rr, "f_0'"] \arrow[dr, "q"] & & \frac{X'}{R'} \\
& & & & \frac{X}{R\vee S} &
\end{tikzcd} \tag{2.7}$$

where C and C' are centralizing relations (so that the upper and lower squares are pullbacks), and the morphism between them is the canonical morphism making the diagram commute. By taking the coequalizers p, p' of the equivalence relations $C \rightrightarrows Eq[f]$ and $C' \rightrightarrows q_R(Eq[f])$ respectively, we obtain the diagram

$$\begin{tikzcd}
R \,\square\, Eq[f] \arrow[rr, shift left, "\varphi_1"] \arrow[rr, shift right, "\varphi_2"'] \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right] & & R \arrow[rr, "f_R"] \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right] & & R' \arrow[dd, shift left] \arrow[dd, shift right] \\
& Eq[\rho] \arrow[rr, shift left, "\widetilde{\varphi}_1"] \arrow[rr, shift right, "\widetilde{\varphi}_2"'] \arrow[dd, shift left] \arrow[dd, shift right] & & \frac{X}{R\vee S} \arrow[dd, shift left] \arrow[dd, shift right] & \\
Eq[f] \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] \arrow[dr, "p"'] \arrow[dd] & & X \arrow[rr, "f"] \arrow[dr, "q_{R\vee S}"] \arrow[dd, "q_R"] & & X' \arrow[dd, "q_{R'}"] \\
& \frac{Eq[f]}{C} \arrow[rr, shift left, "\widetilde{\pi}_1"] \arrow[rr, shift right, "\widetilde{\pi}_2"'] \arrow[dd, "\rho"'] & & \frac{X}{R\vee S} \arrow[dd, equal] & \\
q_R(Eq[f]) \arrow[rr, shift left, "\psi_1"] \arrow[rr, shift right, "\psi_2"'] \arrow[dr, "p'"'] & & \frac{X}{R} \arrow[rr] \arrow[dr, "q"] & & \frac{X'}{R'} \\
& \frac{q_R(Eq[f])}{C'} \arrow[rr, shift left, "\widetilde{\psi}_1"] \arrow[rr, shift right, "\widetilde{\psi}_2"'] & & \frac{X}{R\vee S} &
\end{tikzcd}$$

where all the morphisms in the front face of the lower cube are induced by the universal property of the coequalizers, so that the whole diagram commutes.

Now, by the Barr-Kock theorem (Proposition 1.7), the upper and the lower faces of the lower cube are pullbacks. If we then take the kernel pairs of all the vertical maps, we obtain a pullback of equivalence relations. Proceeding similarly with the roles of R and S reversed, we obtain a pullback

$$\begin{tikzcd}
(Eq[f], R \,\square\, Eq[f], S \,\square\, Eq[f]) \arrow[r, "\pi_1"] \arrow[d, "p"'] & (X, R, S) \arrow[d, "q_{R\vee S}"] \\
\left(\frac{Eq[f]}{C}, Eq[\rho], Eq[\sigma]\right) \arrow[r, "\widetilde{\pi}_1"'] & \left(\frac{X}{R\vee S}, \Delta, \Delta\right)
\end{tikzcd}$$

in $2\text{-}\mathbf{Eq}(\mathcal{C})$; since trivial extensions are pullback stable by Proposition 1.62, to complete the proof all we need to show is that $\widetilde{\pi}_1$ is a $\Gamma_{\mathcal{E}}$-trivial extension.

This will follow immediately if we show that this extension actually lies in $\mathbf{Conn}(\mathcal{C})$. Since the smallest equivalence relation Δ always centralizes itself, it suffices to prove that $Eq[\rho]$ and $Eq[\sigma]$ centralize each other. For this, observe that $\frac{Eq[f]}{C} \overset{\widetilde{\pi}_1}{\underset{\widetilde{\pi}_2}{\rightrightarrows}} \frac{X}{R\vee S}$ is an internal groupoid in $\mathcal{C}$, as a regular quotient of the equivalence relation $Eq[f] \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$, so that $[Eq[\widetilde{\pi}_1], Eq[\widetilde{\pi}_2]] = \Delta_{\frac{Eq[f]}{C}}$. Since $\widetilde{\pi}_1 = \widetilde{\psi}_1 \circ \rho$, we have $Eq[\rho] \leq Eq[\widetilde{\pi}_1]$. A similar argument shows that $Eq[\sigma] \leq Eq[\widetilde{\pi}_2]$, where $\sigma\colon \frac{Eq[f]}{C} \to \frac{q_S(Eq[f])}{C''}$ and C'' is the object part of the centralizing equivalence relation on $q_S(Eq[f])$ and $q_S(R)$. It follows that

$$[Eq[\rho], Eq[\sigma]] \leq [Eq[\widetilde{\pi}_1], Eq[\widetilde{\pi}_2]] = \Delta_{\frac{Eq[f]}{C}},$$

which completes the proof. $\square$

Now, for the Galois structure $\Gamma_{\mathcal{F}}$, we find :

Theorem 2.6. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers, and $f\colon (X, R, S) \to (X', R', S')$ a fibration in $2\text{-}\mathbf{Eq}(\mathcal{C})$. Then the following conditions are equivalent :*

(1) f is a $\Gamma_{\mathcal{F}}$-central extension;
(2) f is a $\Gamma_{\mathcal{F}}$-normal extension;
(3) the commutator condition (2.5) holds :

$$[Eq[f], R \vee S] = \Delta_X.$$

Proof. Any extension which is $\Gamma_{\mathcal{F}}$-central is also $\Gamma_{\mathcal{E}}$-central. It is then also $\Gamma_{\mathcal{E}}$-normal (by Theorem 1.76), and thus $\Gamma_{\mathcal{F}}$-normal. This proves (1) $\Rightarrow$ (2); the converse holds by definition.

For (2) $\Rightarrow$ (3), we recall that if f is normal, then the first projection π_1 of its kernel pair is trivial. The equivalence relation induced on $Eq[f]$ by R and R' is equal to $Eq[f] \square R$, or equivalently $\pi_1^{-1}(R) \wedge \pi_2^{-1}(R)$; and since $R = f^{-1}(R')$ because $f \in \mathcal{F}$, we have

$$\pi_1^{-1}(R) = \pi_1^{-1}(f^{-1}(R')) = (f\pi_1)^{-1}(R') = (f\pi_2)^{-1}(R') = \pi_2^{-1}(R).$$

In a similar way, one can show that the equivalence relation on $Eq[f]$ induced by S and S' is $\pi_1^{-1}(S) = \pi_2^{-1}(S)$, and thus π_1 being trivial means that the square

$$\begin{tikzcd} Eq[f] \arrow[r, "q_{[\pi_2^{-1}(R),\pi_2^{-1}(S)]}"] \arrow[d, "\pi_1"'] & \frac{Eq[f]}{[\pi_2^{-1}(R),\pi_2^{-1}(S)]} \arrow[d, "\overline{\pi_1}"] \\ X \arrow[r, "q_{[R,S]}"'] & \frac{X}{[R,S]} \end{tikzcd}$$

is a pullback; in particular, this implies that

$$Eq[\pi_1] \wedge [\pi_2^{-1}(R), \pi_2^{-1}(S)] = \Delta_{Eq[f]}.$$

By the properties of the Pedicchio commutator, and because $f \in \mathcal{F}$, we then have

$$\begin{aligned} [Eq[\pi_1], \pi_2^{-1}(R)] &\leq [\pi_1^{-1}(S), \pi_2^{-1}(R)] \\ &= [\pi_1^{-1}(f^{-1}(S')), \pi_2^{-1}(R)] \\ &= [(f\pi_1)^{-1}(S'), \pi_2^{-1}(R)] \\ &= [(f\pi_2)^{-1}(S'), \pi_2^{-1}(R)] \\ &= [\pi_2^{-1}(S), \pi_2^{-1}(R)] \\ &= [\pi_2^{-1}(R), \pi_2^{-1}(S)] \end{aligned}$$

and

$$[Eq[\pi_1], \pi_2^{-1}(R)] \leq Eq[\pi_1],$$

hence $[Eq[\pi_1], \pi_2^{-1}(R)] = \Delta_{Eq[f]}$; taking the direct images by π_2 yields

$$[Eq[f], R] = \Delta_X.$$

A similar argument shows that $[Eq[f], S] = \Delta_X$, and then

$$[Eq[f], R \vee S] = [Eq[f], R] \vee [Eq[f], S] = \Delta_X.$$

The result then follows from Lemma 2.5, and the fact that a fibration is $\Gamma_{\mathcal{E}}$-normal if and only if it is $\Gamma_{\mathcal{F}}$-normal. $\square$

Remark 2.7. For a regular epimorphism which does not lie in $\mathcal{F}$, it is not necessarily true that any $\Gamma_{\mathcal{E}}$-central extension must satisfy the commutator condition (2.5) in Theorem 2.6. For example, for any object X in $\mathcal{C}$, we have that (X, Δ_X, ∇_X) is in $\mathbf{Conn}(\mathcal{C})$, since

$$[\Delta_X, \nabla_X] \leq \Delta_X \wedge \nabla_X = \Delta_X.$$

Thus any regular epimorphism $f\colon X \to X'$ gives a $\Gamma_{\mathcal{E}}$-trivial (hence $\Gamma_{\mathcal{E}}$-central) extension in $2\text{-}\mathbf{Eq}(\mathcal{C})$, but in general the condition $[Eq[f], \nabla_X] = \Delta_X$ is not satisfied.

Example 2.8 (Abelian objects)**.** The characterization of central extensions with respect to the category of abelian objects, which was proved by G. Janelidze and G.M. Kelly for the case of Mal'tsev varieties [90] can be seen as a special case of Theorem 2.6.

Indeed, for every object X of $\mathcal{C}$, (X, ∇_X, ∇_X) is an object of $2\text{-}\mathbf{Eq}(\mathcal{C})$; via this identification we can see $\mathcal{C}$ as a subcategory of $2\text{-}\mathbf{Eq}(\mathcal{C})$, which is full since $f(\nabla_X) \leq \nabla_{X'}$ for any morphism $f\colon X \to X'$ in $\mathcal{C}$. It is also easily seen that it is closed under limits and regular quotients. Moreover (X, ∇_X, ∇_X) is in $\mathbf{Conn}(\mathcal{C})$ if and only if $[\nabla_X, \nabla_X] = \Delta_X$ if and only if X is an abelian object of $\mathcal{C}$; and the quotient $X \to \frac{X}{[X,X]}$ serves as X-reflection for both cases. Since the condition $Eq[f] \leq \nabla_X$ is trivially satisfied for any regular epimorphism f of $\mathcal{C}$, an extension $f\colon X \to X'$ is central in $\mathcal{C}$ with respect to $\mathbf{Ab}(\mathcal{C})$ if and only if $f\colon (X, \nabla_X, \nabla_X) \to (X', \nabla_{X'}, \nabla_{X'})$ is a $\Gamma_{\mathcal{F}}$-central extension of $2\text{-}\mathbf{Eq}(\mathcal{C})$ with respect to $\mathbf{Conn}(\mathcal{C})$. Thus in this case, Theorem 2.6 shows that an extension is central if and only if $[Eq[f], \nabla_X] = \Delta_X$.

Example 2.9 (Spans and pregroupoids)**.** Since the category $\mathcal{C}$ is assumed to be exact, the data of two equivalence relations R, S on X is equivalent to the data of two regular epimorphisms with domain X. In other words our category $2\text{-}\mathbf{Eq}(\mathcal{C})$ is equivalent to the category $\mathcal{E}\text{-}\mathbf{Span}(\mathcal{C})$ of spans

$$X_0 \xleftarrow{\alpha} X \xrightarrow{\beta} X_0' \tag{2.8}$$

whose morphisms α and β are regular epimorphisms in $\mathcal{C}$. Through this equivalence the subcategory $\mathbf{Conn}(\mathcal{C})$ corresponds to the subcategory whose objects are spans (2.8) where $[Eq[\alpha], Eq[\beta]] = \Delta_X$ or, equivalently, where there exists a partial Mal'tsev operation $p\colon X \times_{X_0} X \times_{X_0'} X \to X$.

Such a structure on a span can be defined in any category with finite limits. When it also satisfies the associativity axiom $p(p(x, y, z), u, v) = p(x, y, p(z, u, v))$ (which, as we mentioned before, automatically holds in a Mal'tsev category), it forms what has been called a *pregroupoid* by Kock [100, 101] and later a *herdoid* by Johnstone [96]. As noted in Definition 2.2, the condition that a morphism $f\colon \mathbb{X} \to \mathbb{Y}$ belongs to $\mathcal{F}$ is equivalent to the fact that the induced morphisms between the codomains are isomorphisms, so morphisms in $\mathcal{F}$, seen as morphisms in $\mathcal{E}$-$\mathbf{Span}(\mathcal{C})$, must be of the form

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, "f"] & \\ X_0 & & X_0' \\ & Y \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] & \end{tikzcd}$$

with f a regular epimorphism in $\mathcal{C}$. Then our Theorem 2.6 says that an extension of this form in $\mathcal{E}$-$\mathbf{Span}(\mathcal{C})$ is central with respect to the subcategory of pregroupoids if and only if $[Eq[f], Eq[\alpha] \vee Eq[\beta]] = \Delta_X$.

As a special case, one can then consider spans where the two structural morphisms are split epimorphisms, with a given common section. These are the reflexive graphs in $\mathcal{C}$, and a pregroupoid structure on a reflexive graph is in fact equivalent to an internal groupoid structure.

Thus the category $\mathbf{RG}_B(\mathcal{C})$ of reflexive graphs over B, where *all* morphisms are of the form

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & Y_1 \arrow[dl, shift right, "c'"'] \arrow[dl, shift left, "d'"] \\ & B & \end{tikzcd}$$

and also satisfy $fe = e'$, can be seen as a (non full) subcategory of $\mathcal{E}$-$\mathbf{Span}(\mathcal{C})$, and it can be shown that it is closed under pullbacks. Moreover, the condition that B is fixed implies that $Eq[f] \leq Eq(c) \wedge Eq(d)$, so that regular epimorphisms in $\mathbf{RG}_B(\mathcal{C})$ all lie in the class $\mathcal{F}$ of fibrations in $2\text{-}\mathbf{Eq}(\mathcal{C})$. We conclude that an extension is central in $\mathbf{RG}_B(\mathcal{C})$ if and only if it is $\Gamma_{\mathcal{F}}$-central when seen in $2\text{-}\mathbf{Eq}(\mathcal{C})$. Thus, by Theorem 2.6, an extension is central in $\mathbf{RG}_B(\mathcal{C})$ if and only if

$$[Eq[f], Eq[c] \vee Eq[d]] = \Delta_{X_1}.$$

As mentioned previously, this characterization was proved by T. Everaert and M. Gran in [54] in the special case of Mal'tsev varieties, although the proof involved long calculations with elements, which could not be carried out in abstract categories.

Remark 2.10. When the category $\mathcal{C}$ is pointed, so that there is an object 0 in $\mathcal{C}$ that is both initial and terminal, then every object X has a unique structure of reflexive graph above 0, so that $\mathbf{RG}_0(\mathcal{C}) \simeq \mathcal{C}$. Since the kernel pair of the unique morphism $X \to 0$ is the largest equivalence relation ∇_X on X, such a reflexive graph is an internal groupoid if and only if X is an abelian object, which means that $\mathbf{Grpd}_0(\mathcal{C}) \simeq \mathbf{Ab}(\mathcal{C})$; and in this context this is equivalent to the fact that X is an abelian group object. In such a category, we can then obtain the characterization of central extensions in $\mathcal{C}$ with respect to $\mathbf{Ab}(\mathcal{C})$ as the special case of the characterization of central extensions in $\mathbf{RG}_B(\mathcal{C})$ with respect to $\mathbf{Grpd}_B(\mathcal{C})$, where $B = 0$.

This isn't true, however, when the category is not pointed, since in this case defining an internal groupoid above the terminal object 1 requires the choice of an "identity" $1 \to X$, which may not always be possible (for example, in the category of unital rings, the only ring X for which there is a morphism $1 \to X$ is the terminal ring itself). In such a case it is thus necessary to consider spans and pregroupoids or, equivalently, pairs of equivalence relations and connectors, to obtain a context that encompasses both the cases of internal groupoids and of abelian objects. This was our main motivation to consider the Galois structure on $2\text{-}\mathbf{Eq}(\mathcal{C})$ rather than $\mathbf{RG}(\mathcal{C})$.

Example 2.11. The category $\mathbf{Grp}(\mathbf{Comp})$ of compact (Hausdorff) groups is an exact Mal'tsev category which is monadic over the category of sets [110]. As follows from Lemma 6.13 in [56], the Huq commutator of two normal subobjects (which in this context coincide with closed normal subgroups) H and K of a compact group G is simply given by the closure $\overline{[H,K]}$ of the classical group-theoretic commutator $[H,K]$ of H and K. In the category $\mathbf{Grp}(\mathbf{Comp})$ this Huq commutator of normal subobjects is the normal subobject associated with the categorical commutator [118] of the corresponding equivalence relations : this essentially follows from [26], by taking into account the fact that $\mathbf{Grp}(\mathbf{Comp})$ is a strongly protomodular category [21]. From the previous example, it then follows that an extension

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & Y_1 \arrow[dl, shift right, "c'"'] \arrow[dl, shift left, "d'"] \\ & B & \end{tikzcd}$$

in $\mathbf{RG}_B(\mathbf{Grp}(\mathbf{Comp}))$ is *central* relatively to $\mathbf{Grpd}_B(\mathbf{Grp}(\mathbf{Comp}))$ if and only if $\overline{[K[f], K[d] \cdot K[c]]} = \{1\}$, where $K[f]$, $K[d]$ and $K[c]$ denote the kernels of f, d and c in the category $\mathbf{Grp}(\mathbf{Comp})$, respectively, while $K[d] \cdot K[c]$ is the group-theoretic product of these normal closed subgroups, which is a compact (and thus closed) subgroup, since it is the image of the multiplication map

$$\cdot \colon K[d] \times K[c] \to X_1.$$

2.3 Double central extensions

As we explained in section 1.5, the category $\mathbf{CExt}_{\Gamma_{\mathcal{F}}}(2\text{-}\mathbf{Eq}(\mathcal{C}))$ of $\Gamma_{\mathcal{F}}$-central extensions is reflective in $\mathbf{Ext}_{\mathcal{F}}(2\text{-}\mathbf{Eq}(\mathcal{C}))$, and *double central extensions* are double extensions that are central with respect to this induced reflection. We shall show that the double extensions that are central with respect to this induced reflection can be characterized by some natural conditions involving Smith-Pedicchio commutators.

These conditions are a generalization of the ones given by G. Janelidze in [83] characterizing double central extensions in **Grp**, and later extended to Mal'tsev varieties by M. Gran and V. Rossi in [71], and to exact Mal'tsev categories with coequalizers by T. Everaert and T. Van der Linden [60]. In fact, the proofs given below are suitably adapted from the ones appearing in these two papers.

Lemma 2.12. *Consider the following pullback of a double extension (g, p) along a double extension (γ, δ) in $2\text{-}\mathbf{Eq}(\mathcal{C})$ depicted as*

$$\begin{tikzcd}
\mathbb{X}\times_{\mathbb{Z}}\mathbb{U} \arrow[rr, "g'"] \arrow[dr, "\alpha"'] \arrow[dd, "f'"'] & & \mathbb{U} \arrow[dr, "\gamma"] \arrow[dd, "h'" near start] & \\
& \mathbb{X} \arrow[rr, "g" near start] \arrow[dd, "f" near start] & & \mathbb{Z} \arrow[dd, "h"] \\
\mathbb{Y}\times_{\mathbb{W}}\mathbb{V} \arrow[rr, "p'" near end] \arrow[dr, "\beta"'] & & \mathbb{V} \arrow[dr, "\delta"] & \\
& \mathbb{Y} \arrow[rr, "p"'] & & \mathbb{W}.
\end{tikzcd} \tag{2.9}$$

Then we have

$$\alpha([Eq[f'], Eq[g']]) = [Eq[f], Eq[g]]$$

and

$$\alpha([Eq[f'] \wedge Eq[g'], R' \vee S']) = [Eq[f] \wedge Eq[g], R \vee S],$$

where $\mathbb{X} = (X, R, S)$ and $\mathbb{X} \times_{\mathbb{Z}} \mathbb{U} = (X \times_Z U, R', S')$.

Proof. Since morphisms in $\mathcal{F}$ are regular epimorphisms, and the pullbacks and regular epimorphisms in $2\text{-}\mathbf{Eq}(\mathcal{C})$ are computed "levelwise", the cube (2.9) induces a cube in $\mathcal{C}$ that is also a pullback of double extensions.

Since the Pedicchio commutator is preserved by regular images in $\mathcal{C}$, all we have to show is that

$$\alpha(Eq[f']) = Eq[f], \quad \alpha(Eq[g']) = Eq[g], \quad \alpha(R') = R,$$

$$\alpha(S') = S \quad \text{and} \quad \alpha(Eq[f'] \wedge Eq[g']) = Eq[f] \wedge Eq[g].$$

This holds by assumption for R and S; by Proposition 1.18, it suffices to prove in each case that the corresponding coequalizers are part of a double extension in $\mathcal{C}$. This is true for f because the square

$$\begin{tikzcd}
X \times_Z U \arrow[r, "\alpha"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y \times_W U \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

is the pullback of a double extension in $\mathcal{C}$, and is thus a double extension itself, since double extensions are stable under pullbacks in $\mathcal{C}$. It is also true for g because the corresponding square (the top face of (2.9)) is a pullback.

Thus we only have to treat the case of the intersection $Eq[f] \wedge Eq[g]$; its coequalizer is the morphism $X \to Y \times_W Z$ induced by f and g. Accordingly, the corresponding square of coequalizers is the left square in the commutative diagram

$$\begin{tikzcd}
X \times_Z U \arrow[r, "{(f',g')}"] \arrow[d, "\alpha"'] & (Y \times_W V) \times_V U \arrow[r] \arrow[d, "\bar{\alpha}"] & U \arrow[d, "\gamma"] \\
X \arrow[r, "{(f,g)}"'] & Y \times_W Z \arrow[r] & Z.
\end{tikzcd}$$

Now the whole rectangle above is the top face of (2.9), while the right square is the top face of the cube

$$\begin{tikzcd}
(Y \times_W V) \times_V U \arrow[rr] \arrow[dr, "\bar{\alpha}"'] \arrow[dd] & & U \arrow[dr, "\gamma"] \arrow[dd, "h'" near start] & \\
& Y \times_W Z \arrow[rr] \arrow[dd] & & Z \arrow[dd, "h"] \\
Y \times_W V \arrow[rr, "p'" near end] \arrow[dr, "\beta"'] & & V \arrow[dr, "\delta"] & \\
& Y \arrow[rr, "p"'] & & W,
\end{tikzcd}$$

and is then also a pullback, since the front and back faces are pullbacks by construction. $\square$

Proposition 2.13. *Any double $\Gamma_{\mathcal{F}}$-central extension satisfies the commutator conditions*

$$[Eq[f], Eq[g]] = \Delta_X = [Eq[f] \wedge Eq[g], R \vee S]. \qquad (2.10)$$

Proof. By Lemma 1.68, it suffices to check that every double extension between central extensions satisfies the commutator conditions, and that in the diagram (2.9), the front face satisfies the commutator conditions if and only if the back face does.

Suppose then that we have a double extension where f and h are central extensions ; then by Lemma 2.5

$$[Eq[f] \wedge Eq[g], R \vee S] \leq [Eq[f], R \vee S] \leq \Delta_X$$

and

$$[Eq[f], Eq[g]] \leq [Eq[f], R \vee S] \leq \Delta_X,$$

where $Eq[g] \leq R \vee S$ because g is in $\mathcal{F}$.

Suppose now that the back face of (2.9) satisfies the commutator conditions, i.e. that

$$[Eq[f'], Eq[g']] = \Delta_{X'} = [Eq[f'] \wedge Eq[g'], R' \vee S']. \qquad (2.11)$$

Then by Lemma 2.12

$$[Eq[f], Eq[g]] = \alpha([Eq[f'], Eq[g']]) = \Delta_X,$$

and similarly

$$[Eq[f] \wedge Eq[g], R \vee S] = \alpha([Eq[f'] \wedge Eq[g'], R' \vee S']) = \Delta_X.$$

Assume then that the front face of (2.9) satisfies the commutator conditions; then since $\alpha([Eq[f'], Eq[g']]) = \Delta_X$, we have

$$[Eq[f'], Eq[g']] \leq Eq[\alpha];$$

and, moreover,

$$[Eq[f'], Eq[g']] \leq Eq[g']$$

by the monotonicity of the categorical commutator. Since the top face in the cube (2.9) is a pullback, $Eq[g'] \wedge Eq[\alpha] = \Delta_{X \times_Z U}$, and thus

$$[Eq[f'], Eq[g']] = \Delta_{X \times_Z U}.$$

The proof for the other condition is similar. □

We now turn to the proof of the converse implication.

Proposition 2.14. *Any double extension in* $\mathbf{Ext}^2_{\mathcal{F}}(\mathcal{C})$ *satisfying the commutator conditions* (2.10) *is a double* $\Gamma_{\mathcal{F}}$*-central extension.*

Proof. As a consequence of the "denormalized 3×3 lemma" from [19], if we take a double extension and take all the kernel pairs horizontally and vertically in $2\text{-}\mathbf{Eq}(\mathcal{C})$, and then the kernel pairs of the induced morphisms, we obtain a diagram

$$\begin{tikzcd}
Eq[f] \square Eq[g] \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & Eq[g] \arrow[r, "\bar{f}"] \arrow[d, shift left] \arrow[d, shift right] & Eq[p] \arrow[d, shift left] \arrow[d, shift right] \\
Eq[f] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\bar{g}"'] & \mathbb{X} \arrow[r, "f"] \arrow[d, "g"] & \mathbb{Y} \arrow[d, "p"] \\
Eq[h] \arrow[r, shift left] \arrow[r, shift right] & \mathbb{Z} \arrow[r, "h"'] & \mathbb{W}.
\end{tikzcd} \tag{2.12}$$

As explained in section 2.1, the condition that $[Eq[f], Eq[g]] = \Delta_X$ is equivalent to the existence of a section for the comparison morphism $Eq[f] \square Eq[g] \to Eq[f] \times_X Eq[g]$ in $\mathcal{C}$ satisfying certain conditions. Since $\mathcal{F}$ is stable under pullback, all the morphisms involved induce isomorphisms on the quotients by R and S; thus this section in $\mathcal{C}$ induces a similar section at the level of R and S. Thus we have a section in $2\text{-}\mathbf{Eq}(\mathcal{C})$, and thus a connector $p\colon Eq[f] \times_{\mathbb{X}} Eq[g] \to \mathbb{X}$. In particular, we have the following pullback of split epimorphisms in $2\text{-}\mathbf{Eq}(\mathcal{C})$:

$$\begin{tikzcd}
Eq[f] \square Eq[g] \arrow[r, "\pi"] \arrow[d, "x"'] & Eq[f] \times_{\mathbb{X}} Eq[g] \arrow[d, "p"] \\
Eq[f] \wedge Eq[g] \arrow[r, "p_1"'] & \mathbb{X}.
\end{tikzcd}$$

Now the condition $[Eq[f] \wedge Eq[g], R \vee S] = \Delta_X$ exactly means that the morphism $\mathbb{X} \to \mathbb{Y} \times_{\mathbb{W}} \mathbb{Z}$ is a $\Gamma_{\mathcal{F}}$-central (and then $\Gamma_{\mathcal{F}}$-normal) extension (Theorem 2.6). It follows in particular that the first projection $p_1\colon Eq[f] \wedge Eq[g] \to \mathbb{X}$ of its kernel pair is a $\Gamma_{\mathcal{F}}$-trivial extension. As a consequence, the morphism $\pi\colon Eq[f] \square Eq[g] \to Eq[f] \times_{\mathbb{X}} Eq[g]$ is also a $\Gamma_{\mathcal{F}}$-trivial extension.

Now consider the following cube :

$$\begin{tikzcd}
Eq[f] \,\Box\, Eq[g] \arrow[rr, "p_1"] \arrow[dr, "\eta_{Eq[f]\Box Eq[g]}"] \arrow[dd, "q_1"'] & & Eq[f] \arrow[dr, "\eta_{Eq[f]}"] \arrow[dd, "\pi_1"] & \\
& I(Eq[f] \,\Box\, Eq[g]) \arrow[rr, "I(p_1)"] \arrow[dd, "I(q_1)"] & & I(Eq[f]) \arrow[dd, "I(\pi_1)"] \\
Eq[g] \arrow[rr, "\pi_1"'] \arrow[dr, "\eta_{Eq[g]}"'] & & \mathbb{X} \arrow[dr, "\eta_{\mathbb{X}}"] & \\
& I(Eq[g]) \arrow[rr, "I(\pi_1)"'] & & I(\mathbb{X}).
\end{tikzcd}$$

Since **Conn**$(\mathcal{C})$ is a strongly Birkhoff subcategory of 2 - **Eq**$(\mathcal{C})$ and every regular epimorphism is an effective descent morphism, the reflector $I\colon 2\text{-}\mathbf{Eq}(\mathcal{C}) \to \mathbf{Conn}(\mathcal{C})$ preserves pullbacks of split epimorphisms along split epimorphisms (see [52]), so that

$$I(Eq[f] \times_{\mathbb{X}} Eq[g]) = I(Eq[f]) \times_{I(\mathbb{X})} I(Eq[g]).$$

Moreover, since π is a $\Gamma_{\mathcal{F}}$-trivial extension, the induced square to the pullbacks

$$\begin{tikzcd}
Eq[f] \,\Box\, Eq[g] \arrow[r, "\pi"] \arrow[d, "\eta_{Eq[f]\Box Eq[g]}"'] & Eq[f] \times_{\mathbb{X}} Eq[g] \arrow[d, "\eta_{Eq[f]\times_{\mathbb{X}} Eq[g]}"] \\
I(Eq[f] \,\Box\, Eq[g]) \arrow[r, "I(\pi)"'] & I(Eq[f] \times_{\mathbb{X}} Eq[g])
\end{tikzcd}$$

is itself a pullback, and thus the whole cube above is the limit of the diagram formed by its front, right and bottom faces. In particular, the induced square

$$\begin{tikzcd}
Eq[f] \,\Box\, Eq[g] \arrow[r, "p_1"] \arrow[d] & Eq[f] \arrow[d] \\
Eq[g] \times_{I(Eq[g])} I(Eq[f] \,\Box\, Eq[g]) \arrow[r] & \mathbb{X} \times_{I(\mathbb{X})} I(Eq[f])
\end{tikzcd}$$

is also a pullback, and thus the square

$$\begin{tikzcd}
Eq[f] \,\Box\, Eq[g] \arrow[r, "p_1"] \arrow[d, "q_1"'] & Eq[f] \arrow[d, "\pi_1"] \\
Eq[g] \arrow[r, "\pi_1"'] & \mathbb{X}
\end{tikzcd}$$

is a pullback of a double extension between two $\Gamma_{\mathcal{F}}$-trivial (hence $\Gamma_{\mathcal{F}}$-central) extension; it is thus a double $\Gamma_{\mathcal{F}}$-trivial extension. Since it is the kernel pair of the (horizontal) double extension

$$\begin{tikzcd} Eq[f] \arrow[r, "\bar{g}"] \arrow[d, "\pi_1"'] & Eq[h] \arrow[d, "\pi_1"] \\ \mathbb{X} \arrow[r, "g"'] & \mathbb{Z} \end{tikzcd}$$

this latter extension is $\Gamma_{\mathcal{F}}$-normal, and then $\Gamma_{\mathcal{F}}$-central. We can then consider the cube

$$\begin{tikzcd} Eq[f] \arrow[rr, "\bar{g}"] \arrow[dd, "\pi_1"'] \arrow[dr, "\pi_2"] & & Eq[h] \arrow[dd, "\pi_1"] \arrow[dr, "\pi_2"] & \\ & \mathbb{X} \arrow[rr, "g"] \arrow[dd, "f"'] & & \mathbb{Z} \arrow[dd, "h"] \\ \mathbb{X} \arrow[rr, "g"'] \arrow[dr, "f"'] & & \mathbb{Z} \arrow[dr, "h"] & \\ & \mathbb{Y} \arrow[rr, "p"'] & & \mathbb{W}. \end{tikzcd}$$

Since it is a triple extension (as a pullback of double extensions) and the back face is a double $\Gamma_{\mathcal{F}}$-central extension, the front face is a double $\Gamma_{\mathcal{F}}$-central extension, which concludes the proof. ☐

Putting together Proposition 2.13 and Proposition 2.14, we obtain

Theorem 2.15. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers. A double extension*

$$\begin{tikzcd} (X, R, S) \arrow[r, "g"] \arrow[d, "f"'] & (Z, R_Z, S_Z) \arrow[d, "h"] \\ (Y, R_Y, S_Y) \arrow[r, "p"'] & (W, R_W, S_W) \end{tikzcd}$$

is $\Gamma_{\mathcal{F}}$-central if and only if the commutator conditions

$$[Eq[f], Eq[g]] = \Delta_X = [Eq[f] \wedge Eq[g], R \vee S]$$

are satisfied.

We mentioned above that the commutator conditions were a generalization of those given for the double central extensions relative to the subcategory of abelian objects [60]. In fact, that characterization is a special case of ours :

Corollary 2.16. *Let $\mathcal{C}$ be an exact Mal'tsev category with coequalizers. Then a double extension is central with respect to the subcategory of abelian objects in $\mathcal{C}$ if and only if*

$$[Eq[f], Eq[g]] = \Delta_X = [Eq[f] \wedge Eq[g], \nabla_X].$$

Proof. As we have explained for one-dimensional extensions in Example 2.8, the reflection of $\mathcal{C}$ into its subcategory of abelian objects is the special case where $R = \nabla_X = S$. In particular, the results above give us that a double extension is central if and only if

$$[Eq[f] \wedge Eq[g], \nabla_X] = [Eq[f] \wedge Eq[g], \nabla_X \vee \nabla_X] = \Delta_X = [Eq[f], Eq[g]].$$

□

Corollary 2.17. *A double extension*

$$\begin{tikzcd} \mathbb{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "h"] \\ \mathbb{Y} \arrow[r, "p"'] & \mathbb{W} \end{tikzcd} \tag{2.13}$$

in $\mathbf{RG}_B(\mathcal{C})$ is central if and only if

$$[Eq[f] \wedge Eq[g], Eq[c] \vee Eq[d]] = \Delta_{X_1} = [Eq[f], Eq[g]],$$

where $\mathbb{X} = (X_1, B, c, d, e)$.

Proof. As in Example 2.9, any double extension in $\mathbf{RG}_B(\mathcal{C})$ induces a double extension in $2\text{-}\mathbf{Eq}(\mathcal{C})$, and it is $\Gamma_{\mathcal{F}}$-central if and only if the double extension (2.13) is central with respect to the reflection $\mathbf{RG}_B(\mathcal{C}) \to \mathbf{Grpd}_B(\mathcal{C})$. Thus it suffices to take $R = Eq[c]$ and $S = Eq[d]$. □

Example 2.18. Let us consider once again the category $\mathbf{Comp}(\mathbf{Grp})$ of compact (Hausdorff) groups. The meet $K[f] \wedge K[g]$ of two kernels is their set-theoretic intersection, since the intersection of two closed subgroups is a closed (hence compact) subgroup. We then find that a double extension (2.13) is central if and only if

$$\overline{[K[f] \wedge K[g], K[c] \cdot K[d]]} = \{1\} = \overline{[K[f], K[g]]}.$$

3 Galois theory and Peiffer commutators

This chapter is adapted from the article [41], written jointly with Alan S. Cigoli, Marino Gran and Sandra Mantovani (although Example 3.1 is extracted from the article [50], which was the basis of the previous chapter).

In the previous chapter we obtained a characterization of the central and double central extensions in the category $\mathbf{RG}_B(\mathcal{C})$ with respect to the subcategory $\mathbf{Grpd}_B(\mathcal{C})$ for an exact finitely cocomplete Mal'tsev category $\mathcal{C}$. When $\mathcal{C}$ is semi-abelian, these categories are equivalent to the subcategories $\mathbf{PXMod}_B(\mathcal{C})$ and $\mathbf{XMod}_B(\mathcal{C})$, respectively. Thus we can attempt to translate our results to characterize central extensions of the Galois structure induced by the reflection of $\mathbf{PXMod}_B(\mathcal{C})$ into $\mathbf{XMod}_B(\mathcal{C})$.

Example 3.1. Let $f\colon (L, \partial, \xi) \to (L', \partial', \xi')$ be a surjective morphism of precrossed B-modules of Lie algebras. Then under the equivalence $\mathbf{PXMod}_B(\mathbf{Lie}_K) \simeq \mathbf{RG}_B(\mathbf{Lie}_K)$, f becomes the regular epimorphism

$$\begin{tikzcd} B \ltimes L \arrow[rr, "1_B \ltimes f"] \arrow[dr, shift left, "p_B"] \arrow[dr, shift right, "{[1,\partial\rangle}"'] & & B \ltimes L' \arrow[dl, shift right, "{[1,\partial'\rangle}"'] \arrow[dl, shift left, "p_B"] \\ & B & \end{tikzcd} \tag{3.1}$$

(where $1_B \ltimes f$ is the map $(b, l) \mapsto (b, f(l))$ and $[1_B, \partial\rangle$ is the map $(b, l) \mapsto b + \partial(l)$, for any $b \in B$ and any $l \in L$).

Now, by the results of Example 2.9, $1_B \ltimes f$ is a central extension if and only if

$$[Eq[1_B \ltimes f], Eq[p_B] \vee Eq[[1, \partial\rangle]] = \Delta_{B \ltimes L}.$$

In the category of Lie algebras the commutator of equivalence relations is the equivalence relation corresponding to the commutator of the corresponding ideals ([123]), and this condition is then equivalent to

$$[K[1_B \ltimes f], K[p_B] + K[[1, \partial\rangle]] = 0. \tag{3.2}$$

One verifies that the object part of the kernel of $1_B \ltimes f$ is $K[f]$, with inclusion given by the composite $K[f] \to L \to B \ltimes L$, and that the kernel of p_B is simply $j_L \colon L \to B \ltimes L$. Moreover, one easily sees that

$$K[[1, \partial\rangle] = \{(-\partial(l), l) \mid l \in L\}.$$

As a consequence, we find that $[K[1_B \ltimes f], K[\pi_B]]$ is the ideal of $B \ltimes L$ generated by terms of the form

$$[(0, k), (0, l)] = (0, [k, l])$$

for $k \in K[f]$ and $l \in L$, while $[K[1_B \ltimes f], K[[1, \partial\rangle]]$ is generated by terms of the form

$$[(0, k), (-\partial(l), l)] = (0, \xi(\partial(l))(k) + [k, l]).$$

It follows that the commutator described in (3.2) can be seen as an ideal of L, more precisely the subspace generated by terms of the form $[k, l]$ or $\xi(\partial(l))(k)$, for $k \in K[f]$ and $l \in L$. By analogy with the case of precrossed modules of groups [43], we call this subobject the *Peiffer commutator* $\langle K[f], L\rangle$ of $K[f]$ and L.

Let us now consider a double extension

$$\begin{tikzcd} (L_1, \partial_1, \xi_1) \arrow[r, "g"] \arrow[d, "f"'] & (L_2, \partial_2, \xi_2) \arrow[d, "h"] \\ (L_3, \partial_3, \xi_3) \arrow[r, "p"'] & (L_4, \partial_4, \xi_4) \end{tikzcd} \tag{3.3}$$

of precrossed B-modules of Lie algebras. Using again the equivalence between precrossed B-modules and internal reflexive graphs over B in $\mathbf{Lie}_K$, we find that (3.3) is central if and only if

$$\begin{tikzcd} B \ltimes L_1 \arrow[r, "1_B \ltimes g"] \arrow[d, "1_B \ltimes f"'] & B \ltimes L_2 \arrow[d, "1_B \ltimes h"] \\ B \ltimes L_3 \arrow[r, "1_B \ltimes p"'] & B \ltimes L_4 \end{tikzcd}$$

is a double central extension in $\mathbf{RG}_B(\mathbf{Lie}_K)$ (with the graph structures on each object given as in (3.1)), which by Corollary 2.17 is equivalent to

$$[K[1_B \ltimes f] \cap K[1_B \ltimes g], K[p_B] + K[[1, \partial\rangle]] = 0 \tag{3.4}$$

and
$$[K[1_B \ltimes f], K[1_B \ltimes g]] = 0. \tag{3.5}$$
Again, the object parts of the kernels of $1_B \ltimes f$ and $1_B \ltimes g$ are simply $K[f]$ and $K[g]$ respectively, and so the commutator in equation (3.5) is generated by terms of the form
$$[(0, k), (0, k')] = (0, [k, k']),$$
where $k \in K[f]$ and $k' \in K[g]$. Since the commutator in equation (3.4) can be treated exactly as the commutator in (3.2), we find that the double extension (3.3) is central if and only if
$$\langle K[f] \wedge K[g], X\rangle = 0 = [K[f], K[g]],$$
where both sides are ideals of L_1.

A categorical notion of Peiffer commutator was introduced in [40] (see the next section), and under the (SH) condition, the reflection of the precrossed B-module (X, ∂, ξ) associated with the reflexive graph (X_1, d, c, e) into $\mathbf{PXMod}_B(\mathcal{C})$ was shown to be the quotient $\eta_X \colon X \to \frac{X}{\langle X,X\rangle}$ of X by the Peiffer commutator $\langle X, X\rangle$ on X
$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "\partial"'] & & \frac{X}{\langle X,X\rangle} \arrow[dl, "\overline{\partial}"] \\ & B & \end{tikzcd} \tag{3.6}$$
where the B-action $\overline{\xi}$ on $\frac{X}{\langle X,X\rangle}$ is the one induced by the B-action ξ on X.

The correspondence between the Peiffer commutator $\langle X, X\rangle$ on X in the reflection $\mathbf{PXMod}_B(\mathcal{C}) \to \mathbf{XMod}_B(\mathcal{C})$ and the Smith-Pedicchio commutator $[Eq[d], Eq[c]]_{SP}$ in the reflection $\mathbf{RG}_B(\mathcal{C}) \to \mathbf{Grpd}_B(\mathcal{C})$ raises the question of determining whether this is a special case of a more general fact relating centrality conditions coming from categorical Galois theory to this Peiffer commutator (in a context where they are both defined and can then be compared), i.e. whether the results of Example 3.1 are valid for any semi-abelian category $\mathcal{C}$.

In the next section, we will describe the construction of the Peiffer commutator and its properties, before proving that, under the condition (UA), central extensions in $\mathbf{PXMod}_B(\mathcal{C})$ can be characterized as those such that the Peiffer commutator $\langle K[f], X\rangle$ is trivial.

In the third section we shall use this characterization and a result in [55] to get a five term exact sequence in homology (Proposition 3.14), where the homology objects in $\mathbf{PXMod}_B(\mathcal{C})$ are expressed in terms of generalized Hopf formulae. When $\mathcal{C}$ is the category of Lie algebras, one obtains an exact sequence in the category of Lie algebra precrossed modules (see Example 3.16). In the last section a characterization of double central extensions relative to the induced adjunctions between the categories of extensions and of central extensions in $\mathbf{PXMod}_B(\mathcal{C})$ will also be established (Theorem 3.17). From this, an explicit Hopf formula describing the Galois group of a weakly universal double central extension will be deduced (see formula (3.16)).

3.1 The Peiffer commutator

Lemma 3.2. *Let $\partial\colon X \to B$ be a precrossed module in $\mathcal{C}$, and let $g\colon A \to B$ be a morphism in $\mathcal{C}$. Let*

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "\partial'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\partial"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

be the pullback of g and ∂ in $\mathcal{C}$. Then $\partial'\colon P \to A$ can be endowed with a precrossed module structure, such that (g, g') is a precrossed module morphism $(P, A, \partial', \xi') \to (X, B, \partial, \xi)$.

Proof. Let us consider the reflexive graph (X_1, B, c, d, e) associated with the precrossed module $\partial\colon X \to B$. Then taking the pullback of (c, d) along $g \times g$ yields a new reflexive graph, since we have a morphism

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "e'"] \arrow[dd, bend right=60, "{(1,1)}"'] & B \arrow[d, "e"] \arrow[dd, bend left=60, "{(1,1)}"] \\ P_1 \arrow[r, "g_1"] \arrow[d, "{(c',d')}"] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "{(c,d)}"] \\ A \times A \arrow[r, "g\times g"'] & B \times B. \end{tikzcd}$$

Moreover, (g_1, g) defines a morphism of reflexive graphs. Then the normalization functor gives a morphism of precrossed modules, which is obtained as the back face of the cube

$$\begin{tikzcd}
P \arrow[rr, "g'"] \arrow[dr] \arrow[dd, "\partial'"'] & & X \arrow[dr] \arrow[dd, "\partial"] & \\
& P_1 \arrow[rr, "g_1"] \arrow[dd, "{(c',d')}"] & & X_1 \arrow[dd, "{(c,d)}"] \\
A \arrow[rr, "g"] \arrow[dr, "{(1,0)}"'] & & B \arrow[dr, "{(1,0)}"] & \\
& A \times A \arrow[rr, "g\times g"'] & & B \times B,
\end{tikzcd}$$

where the left and right faces are pullbacks; since the front face is a pullback, the back face is indeed a pullback. $\square$

Since the pullback square must be a morphism of precrossed modules, the action $\xi' \colon A\flat P \to P$ is the unique arrow making the following diagram commute :

$$\begin{tikzcd}
A\flat P \arrow[rr, "g\flat g'"] \arrow[dr, "\xi'"] \arrow[dd, "1\flat\partial'"'] & & B\flat X \arrow[d, "\xi"] \\
& P \arrow[r, "g'"] \arrow[d, "\partial'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\partial"] \\
A\flat A \arrow[r, "\chi_A"'] & A \arrow[r, "g"'] & B.
\end{tikzcd}$$

Proposition 3.3. *Let (X, ∂_X, ξ_X) and (Y, ∂_Y, ξ_Y) be two precrossed B-modules. Let us consider the pullback action $\partial_X^*\xi_Y = \xi_Y(\partial_X\flat 1)$ of X on Y, and the corresponding semi-direct product $X \ltimes_{\partial_X^*\xi_Y} Y$; then there is a map $[\partial_X, \partial_Y\rangle \colon X \ltimes_{\partial_X^*\xi_Y} Y \to B$ and an action $\overline{\xi}$ of B on $X \ltimes_{\partial_X^*\xi_Y} Y$, such that $(X \ltimes_{\partial_X^*\xi_Y} Y, [\partial_X, \partial_Y\rangle, \overline{\xi})$ is a precrossed B-module.*

Moreover, the maps $j_Y \colon Y \to X \ltimes_{\partial_X^*\xi_Y} Y$ and $s_X \colon X \to X \ltimes_{\partial_X^*\xi_Y} Y$ are precrossed module morphisms.

Proof. The existence of $[\partial_X, \partial_Y\rangle$ follows from the fact that

$$\chi_B(\partial_X\flat\partial_Y) = \chi_B(1\flat\partial_Y)(\partial_X\flat 1) = \partial_Y\xi_Y(\partial_X\flat 1),$$

since $\partial_Y \colon Y \to B$ is a precrossed module.

We need to define the action $\overline{\xi}$ of B on $X \ltimes Y$. For this, we note that the pullback action $\partial_X^*\xi_Y$ corresponds to a pullback square

$$\begin{tikzcd}
X \ltimes Y \arrow[r, shift left, "p_X"] \arrow[d, "\partial_X \ltimes 1"'] & X \arrow[l, shift left, "s_X"] \arrow[d, "\partial_X"] \\
B \ltimes Y \arrow[r, shift left, "p_B"] & B; \arrow[l, shift left, "s_B"]
\end{tikzcd}$$

then by the previous lemma $\partial_X \ltimes 1\colon X \ltimes Y \to B \ltimes Y$ admits a precrossed module structure, so that the square above is a split epimorphism in $\mathbf{PXMod}(\mathcal{C})$. In particular, there is an action ξ' of $B \ltimes Y$ on $X \ltimes Y$, for which the maps s_B and s_X are equivariant. Let $\overline{\xi} = s_B^*\xi' = \xi'(s_B\flat 1)$; then by definition, $\overline{\xi}$ makes the diagram

$$\begin{tikzcd}
B\flat(X \ltimes Y) \arrow[rr, "1\flat p_X"] \arrow[dr, "\overline{\xi}"] \arrow[dd, "s_B\flat(\partial_X\ltimes 1)"'] & \arrow[d, phantom, "(b)"] & B\flat X \arrow[d, "\xi_X"] \\
\arrow[r, phantom, "(a)"] & X \ltimes Y \arrow[r, "p_X"] \arrow[d, "\partial_X\ltimes 1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\partial"] \\
(B \ltimes Y)\flat(B \ltimes Y) \arrow[r, "\chi_{B\ltimes Y}"'] & B \ltimes Y \arrow[r, "p_B"'] & B
\end{tikzcd}$$

commute. In particular, the commutativity of (a) above and the functoriality of the conjugation action imply the commutativity of the diagram

$$\begin{tikzcd}
B\flat(X \ltimes Y) \arrow[r, "s_B\flat(\partial_X\ltimes 1)"] \arrow[rr, bend left, "1\flat[\partial_X{,}\partial_Y\rangle"] \arrow[d, "\overline{\xi}"'] \arrow[dr, phantom, "(a)"] & (B \ltimes Y)\flat(B \ltimes Y) \arrow[r, "{[1,\partial_Y\rangle\flat[1,\partial_Y\rangle}"'] \arrow[d, "\chi"] & B\flat B \arrow[d, "\chi"] \\
X \ltimes Y \arrow[r, "\partial_X\ltimes 1"'] \arrow[rr, bend right, "{[\partial_X,\partial_Y\rangle}"'] & B \ltimes Y \arrow[r, "{[1,\partial_Y\rangle}"'] & B,
\end{tikzcd}$$

which proves that $\overline{\xi}$ makes $[\partial_X, \partial_Y\rangle\colon X \ltimes Y \to B$ a precrossed module.

It remains to prove that j_Y and s_X are morphisms in $\mathbf{PXMod}_B(\mathcal{C})$. By definition we have the identities $[\partial_X, \partial_Y\rangle j_Y = \partial_Y$ and $[\partial_X, \partial_Y\rangle s_X = \partial_X$. Moreover, since s_B, s_X are equivariant between ξ_X and ξ', we have

$$s_X\xi_X = \xi'(s_B\flat s_X) = \xi'(s_B\flat 1)(1\flat s_X) = \overline{\xi}(1\flat s_X),$$

thus s_X is a morphism in $\mathbf{PXMod}_B(\mathcal{C})$. To prove that j_Y is B-equivariant, we need to prove that

$$\overline{\xi}(1\flat j_Y) = j_Y\xi_Y;$$

and for this it suffices to check that this holds after precomposition with $\partial_X \ltimes 1$ and π_X, since they are projections of a pullback square. By definition of $\overline{\xi}$ we have

$$\begin{aligned}
(\partial_X \ltimes 1)\overline{\xi}(1\flat j_Y) &= \chi(s_B\flat(\partial_X \ltimes 1))(1\flat j_Y) \\
&= \chi(s_B\flat j_Y) = j_Y\xi_Y \\
&= (\partial_X \ltimes 1)j_Y\xi_Y
\end{aligned}$$

and

$$p_X\overline{\xi}(1\flat j_Y) = \xi_X(1\flat p_X)(1\flat j_Y) = \xi_X(1\flat 0) = 0 = p_X j_Y \xi_Y,$$

which concludes the proof. $\square$

Let us denote $X +_{PX} Y$ the coproduct of X and Y in $\mathbf{PXMod}_B(\mathcal{C})$. Then the two morphisms of precrossed B-modules s_X and j_Y determine a morphism $[s_X, j_Y]_{PX} \colon X +_{PX} Y \to X \ltimes Y$ in $\mathbf{PXMod}_B(\mathcal{C})$. Note that s_X and j_Y are jointly strongly epimorphic in $\mathcal{C}$, and thus also in $\mathbf{PXMod}_B(\mathcal{C})$, so that $[s_X, j_Y]_{PX}$ is a regular epimorphism. In the same way we also have a regular epimorphism

$$[j_X, s_Y]_{PX} \colon X +_{PX} Y \to Y \ltimes X.$$

Definition 3.4 ([40])**.** The *Peiffer product* of two precrossed B-modules (X, ∂_X, ξ_Y) and (Y, ∂_Y, ξ_Y) is defined as the pushout

$$\begin{tikzcd}
X +_{PX} Y \arrow[r, "{[s_X, j_Y]_{PX}}"] \arrow[d, "{[j_X, s_Y]_{PX}}"'] & X \ltimes Y \arrow[d] \\
Y \ltimes X \arrow[r] & X \bowtie Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

in $\mathbf{PXMod}_B(\mathcal{C})$. We denote $\Sigma \colon X +_{PX} Y \to X \bowtie Y$ the diagonal of this pushout, which is a regular epimorphism since $[s_X, j_Y]_{PX}$ and $[j_X, s_Y]_{PX}$ are regular epimorphisms. Σ is determined by two morphisms $l_X \colon X \to X \bowtie Y$ and $l_Y \colon Y \to X \bowtie Y$ in $\mathbf{PXMod}_B(\mathcal{C})$. We also denote $X \boxtimes Y$ the kernel of Σ.

Definition 3.5 ([40])**.** Let M, N be two precrossed B-submodules of X :

$$\begin{tikzcd}
M \arrow[r, tail, "m"] \arrow[dr, "\partial_M"'] & X \arrow[d, dashed, "\partial_A"] & N \arrow[l, tail, "n"'] \arrow[dl, "\partial_N"] \\
& B. &
\end{tikzcd} \tag{3.7}$$

Then their *Peiffer commutator* $\langle M, N \rangle$ is the image of $M \boxtimes N$ through the map $[m, n]_{PX} \colon M +_{PX} N \to X$.

Remark 3.6. When M and N act trivially on each other, the normal closure of their Peiffer commutator coincides with their Huq commutator. In particular, this is the case when both are normal precrossed B-submodules (which implies that ∂_M and ∂_N are zero maps).

We've seen before that the Higgins or Huq commutator illustrates the obstruction to the existence of a cooperator between two normal subobjects. In the same way, we have

Proposition 3.7 ([40, Proposition 3.11])**.** *The Peiffer commutator $\langle M, N\rangle$ of two precrossed B-submodules as in* (3.7) *is trivial if and only if there exists a (unique) morphism φ in $\mathbf{PXMod}_B(\mathcal{C})$ making the diagram*

$$\begin{tikzcd} M \arrow[r, tail, "l_M"] \arrow[dr, "m"'] & M \bowtie N \arrow[d, dashed, "\varphi"] & N \arrow[l, tail, "l_N"'] \arrow[dl, "n"] \\ & X & \end{tikzcd}$$

commute.

Proof. By definition, $\langle M, N\rangle$ is the image of the kernel of the regular epimorphism Σ under $[m, n]_{PX}$, it is trivial if and only if $[m, n]_{PX}$ factors (uniquely) as $\varphi\Sigma$, which is equivalent to $\varphi l_M = m$ and $\varphi l_N = n$. □

Remark 3.8. Notice that the Peiffer commutator of two precrossed B-submodules is not normal in general (although, as the image of a normal precrossed B-submodule, it has the property the underlying map $\langle M, N\rangle \to B$ is trivial). However, it is the case when X is the join of X and Y in $\mathbf{PXMod}_B(\mathcal{C})$ (see Remark 3.9 in [40]). In particular, this happens when considering $\langle X, K\rangle$ for some K normal subobject of X in $\mathbf{PXMod}_B(\mathcal{C})$. Moreover, we have the following lemma:

Lemma 3.9. *For a normal precrossed B-submodule*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[dr, "0"'] & X \arrow[d, "\partial"] \\ & B \end{tikzcd}$$

the inequality $\langle X, K\rangle \leq K$ holds.

Proof. First of all, let us notice that the trivial precrossed B-module map

$$0\colon (K, 0, \xi_K) \to (X, \partial, \xi)$$

exists, and so does $[1, 0]_{\mathsf{PX}} : X +_{\mathsf{PX}} K \to X$. Moreover $X \rtimes_{0^*\xi_X} K \cong X \times K$ (K acts trivially on X). Hence, specializing the pushout (3.4) to our context, by the commutativity of the external square in the diagram

$$\begin{tikzcd} X +_{\mathsf{PX}} K \arrow[r, "{[j_X, i_K]_{\mathsf{PX}}}"] \arrow[d, "{[i_X, j_K]_{\mathsf{PX}}}"'] & X \times K \arrow[d] \arrow[ddr, bend left, "p_1"] & \\ K \rtimes_{\partial^*\xi_K} X \arrow[r] \arrow[drr, bend right, "p_X"'] & X \bowtie K \arrow[dr, dashed, "\tau"] & \\ & & X \end{tikzcd}$$

we get a unique arrow τ such that $\tau \circ \Sigma = [1,0]_{\mathsf{PX}}$. Now, we can proceed as in Section 6 of [112], and consider the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d, tail, "\ker\Sigma"'] & \langle X,K\rangle \arrow[r, tail] \arrow[d, tail] & K \arrow[dl, tail, "k"] \\
X +_{\mathsf{PX}} K \arrow[r, "{[1,k]_{\mathsf{PX}}}"] \arrow[d, "\Sigma"] \arrow[dd, bend right=60, "{[1,0]_{\mathsf{PX}}}"'] \arrow[dr, phantom, "(a)"] & X \arrow[d, "q"] \arrow[dd, bend left=60, "p"] & \\
X \rtimes K \arrow[r] \arrow[d, "\tau"] & \frac{X}{\langle X,K\rangle} \arrow[d, dashed] & \\
X \arrow[r, "p"'] & \frac{X}{K}, &
\end{tikzcd}$$

where p is the cokernel of k. It is easy to check that $p \circ [1,k]_{\mathsf{PX}} = p \circ [1,0]_{\mathsf{PX}}$ by precomposition with the canonical injections. The square (a) is a pushout, since q and Σ are cokernels with a regular epimorphic comparison morphism between the corresponding kernels. By universal property we get that p factors through q and hence $\langle X,K\rangle \leq K$. $\square$

The Peiffer commutator also shares the following two properties with the Huq commutator :

Proposition 3.10 ([40, Proposition 3.13 and Corollary 3.14])**.** *The Peiffer commutator is preserved by regular images: if $q\colon X \to X'$ is a regular epimorphism in $\mathbf{PXMod}_B(\mathcal{C})$ and M and N are precrossed B-submodules of X as in* (3.7)*, then $q(\langle M,N\rangle) = \langle q(M), q(N)\rangle$.*

The Peiffer commutator is monotone: if $M \leq M'$ and $N \leq N'$ are precrossed B-submodules of a given precrossed B-module X, then $\langle M,N\rangle \leq \langle M',N'\rangle$.

As we've mentioned before, one of the most important property of the Peiffer commutator is in relation to the subcategory of crossed B-modules :

Proposition 3.11 ([40, Theorem 3.18])**.** *The quotient $X \to \frac{X}{\langle X,X\rangle}$ gives the reflection of $\mathbf{PXMod}_B(\mathcal{C})$ into the full subcategory of graphs satisfying the Peiffer condition. In particular, whenever $\mathcal{C}$ satisfies the condition* (SH)*, this coincides with the reflection of $\mathbf{PXMod}_B(\mathcal{C})$ into $\mathbf{XMod}_B(\mathcal{C})$.*

Proof. First of all, by Proposition 3.7 the commutator $\langle X,X\rangle$ is trivial if and only if $[1,1]_{PX}\colon X +_{PX} X \to X$ factors through Σ, or equivalently, if it factors through $[j_X, s_X]_{PX}$ and $[s_X, j_X]_{PX}\colon X +_{PX} X \to X \rtimes_{\partial^*\xi} X$ in

$\mathbf{PXMod}_B(\mathcal{C})$. This condition implies the existence of $[1,1\rangle\colon X \ltimes_{\partial^*\xi} X \to X$ in $\mathcal{C}$, which is equivalent to the Peiffer condition. Conversely, if $[1,1\rangle$ exists in $\mathcal{C}$, then using the fact that $[j_X, s_X]_{PX}$ is a regular epimorphism in $\mathcal{C}$, we can prove that $[1,1\rangle$ is in fact a morphism in $\mathbf{PXMod}_B(\mathcal{C})$, so that $[1,1]_{PX}$ factors through Σ, and thus $\langle X, X\rangle = 0$.

Now if (X, ∂_X, ξ_X) is a B-precrossed module, then the pushout η of Σ along $[1,1]_{PX}$ in $\mathbf{PXMod}_B(\mathcal{C})$ coincides with the cokernel of $\langle X, X\rangle$. This is also a pushout in $\mathcal{C}$, and thus the quotient $\frac{X}{\langle X,X\rangle}$ (in $\mathcal{C}$) has a precrossed B-module structure. Moreover,

$$\left\langle \frac{X}{\langle X,X\rangle}, \frac{X}{\langle X,X\rangle} \right\rangle = \langle \eta(X), \eta(X) = \eta(\langle X,X\rangle) = 0,$$

thus (X, ∂_X, ξ_X) satisfies the Peiffer condition.

Finally, if $f\colon (X, \partial_X, \xi_X) \to (Y, \partial_Y, \xi_Y)$ is a morphism of precrossed B-module and $\langle Y, Y\rangle = 0$, then by Proposition 3.10 we have

$$f(\langle X, X\rangle) \leq \langle f(X), f(X)\rangle \leq \langle Y, Y\rangle = 0.$$

Thus $f(\langle X, X\rangle) = 0$, which implies that f factors (uniquely) through $\frac{X}{\langle X,X\rangle}$. $\square$

3.2 Peiffer commutator and centrality

We are now ready to state the main result of this chapter.

Theorem 3.12. *Let $\mathcal{C}$ be a semi-abelian category satisfying* (UA)*, and B an object in $\mathcal{C}$. An extension*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\partial_X"'] & & Y \arrow[dl, "\partial_Y"] \\ & B & \end{tikzcd} \tag{3.8}$$

of precrossed B-modules in $\mathcal{C}$ is $\mathbf{XMod}_B(\mathcal{C})$-central if and only if

$$\langle K[f], X\rangle = 0.$$

Proof. We first prove that if

$$\begin{tikzcd} (X \times_Y Z, \partial_{X\times_Y Z}, \xi_{X\times_Y Z}) \arrow[r, "g'"] \arrow[d, "f'"'] & (X, \partial_X, \xi_X) \arrow[d, "f"] \\ (Z, \partial_Z, \xi_Z) \arrow[r, "g"'] & (Y, \partial_Y, \xi_Y) \end{tikzcd}$$

is a pullback and g a regular epimorphism in the category $\mathbf{PXMod}_B(\mathcal{C})$, then $\langle K[f'], X \times_Y Z\rangle = 0$ if and only if $\langle K[f], X\rangle = 0$. We recall that such a pullback gives in particular a square

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

that is a pullback in $\mathcal{C}$, with g a regular epimorphism in $\mathcal{C}$. This implies that g' is also a regular epimorphism, and that $g'(\ker(f')) = \ker(f)$. Assuming first that $\langle K[f'], X \times_Y Z\rangle = 0$, we then find that

$$\langle K[f], X\rangle = \langle g'(K[f']), g'(X \times_Y Z)\rangle = g'\left(\langle K[f'], X \times_Y Z\rangle\right) = 0$$

because the Peiffer commutator is preserved under regular images by Proposition 3.10. Assuming now that $\langle K[f], X\rangle = 0$, the same reasoning shows that $g'\left(\langle K[f'], X \times_Y Z\rangle\right) = 0$. Moreover

$$f'\left(\langle K[f'], X \times_Y Z\rangle\right) = \langle f'(K[f']), Z\rangle = \langle 0, Z\rangle = 0.$$

Since f' and g' are jointly monic, this implies that $\langle K[f'], X \times_Y Z\rangle = 0$.

Now $\langle K[f], X\rangle \leq \langle X, X\rangle$, so that any extension f between crossed modules must satisfy $\langle K[f], X\rangle = 0$. Then by Lemma 1.68, every central extension satisfies this condition, which proves the "only if" part.

Concerning the "if" part, let us first observe that, for any morphism f in $\mathbf{RG}_B(\mathcal{C})$, the pullback

$$\begin{tikzcd}
K_1 \arrow[r, tail, "k_1"] \arrow[d, "p"'] & X_1 \arrow[d, "f_1"] \\
B \arrow[r, tail, "e'"'] & Y_1
\end{tikzcd} \tag{3.9}$$

determines a kernel (in the sense of quasi-pointed categories) of f_1 in $\mathbf{RG}_B(\mathcal{C})$, described by the following diagram:

$$\begin{tikzcd}
K_1 \arrow[rr, "k_1"] \arrow[dr, shift left=2, "p"] \arrow[dr, shift right=2, "p"'] & & X_1 \arrow[dl, shift right=2, "d"'] \arrow[dl, shift left=2, "c"] \\
& B \arrow[ul, "s"] \arrow[ur, "e"'] &
\end{tikzcd}$$

where s is the unique arrow such that $k_1 s = e$ and $ps = 1_B$.

Taking the kernels in $\mathcal{C}$ of the domain projections of K_1, X_1 and Y_1, and the morphisms between them induced by f_1 and k_1, we get the pullback squares

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[d, tail, "j=\ker(p)"'] & X \arrow[r, "f"] \arrow[d, tail, "h=\ker(d)"] & Y \arrow[d, tail, "h'=\ker(d')"] \\ K_1 \arrow[r, tail, "k_1"'] & X_1 \arrow[r, "f_1"'] & Y_1. \end{tikzcd}$$

It is easy to check that $k = \ker(f)$ and $hk = \ker(f_1)$, so that $K = K[f] = K[f_1]$ is indeed a normal subobject of X_1. The corresponding morphisms of precrossed modules will then look like

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[dr, "0"'] & X \arrow[r, "f"] \arrow[d, "\partial"] & Y \arrow[dl, "\partial'"] \\ & B. & \end{tikzcd}$$

We denote by $\psi\colon B\flat K \to K$ the action of B on K corresponding to the point (p, s), which gives the precrossed module structure on $0\colon K \to B$.

It follows from Proposition 3.7 that the Peiffer commutator $\langle K, X\rangle$ is trivial if and only if there exists an arrow φ making the diagram

$$\begin{tikzcd} K \arrow[r, "l_K"] \arrow[dr, tail, "k"'] & K \bowtie X \arrow[d, dashed, "\varphi"] & X \arrow[l, "l_X"'] \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

commute. By precomposition, this in turn yields the (unique) dashed morphisms making the diagrams

$$\begin{tikzcd} K \arrow[r, "{(1,0)}"] \arrow[dr, tail, "k"'] & K \times X \arrow[d, dashed] & X \arrow[l, "{(0,1)}"'] \arrow[dl, "1_X"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "j_K"] \arrow[dr, tail, "k"'] & K \rtimes_{\partial^*\psi} X \arrow[d, dashed] & X \arrow[l, "s_X"'] \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

commute, where in the left hand diagram we used the isomorphism $K \ltimes_{0^*\xi} X \cong K \times X$. So, in fact, the first diagram tells us that K and X commute in the sense of Huq, i.e. $[K, X] = [K[f_1], K[d]] = 0$. On the other hand, the right hand diagram commutes if and only if the square

$$\begin{tikzcd} X\flat K \arrow[r, "\kappa_{X,K}"] \arrow[d, "\partial^*\psi"'] & X + K \arrow[d, "{[1,k]}"] \\ K \arrow[r, "k"'] & X \end{tikzcd}$$

commutes. If we replace $\partial^*\psi$ by the conjugation action χ_K^X of X on its normal subobject K, we get an analogous commutative diagram. As a consequence $\partial^*\psi = \chi_K^X$, since k is a monomorphism.

Consider now the diagram

$$\begin{tikzcd}
X\flat K \arrow[r, "h\flat 1"] \arrow[dr, "\partial^*\psi"'] & X_1\flat K \arrow[d, shift right, "\chi_K^{X_1}"'] \arrow[d, shift left, "c^*\psi"] & B\flat K \arrow[l, "e\flat 1"'] \arrow[dl, "\psi"] \\
& K, &
\end{tikzcd} \tag{3.10}$$

where $\chi_K^{X_1}$ denotes the conjugation action of X_1 on its normal subobject K. We want to show that both possible choices of the middle vertical arrow make the two triangles commute.

Let us start with the triangles on the left. The equality $\partial^*\psi = c^*\psi \circ (h\flat 1)$ easily follows from the fact that $\partial = c \circ h$, while the equality $\partial^*\psi = \chi_K^{X_1} \circ (h\flat 1)$ holds because $\partial^*\psi = \chi_K^X$, as we proved above, and the commutative diagram

$$\begin{tikzcd}
X\flat K \arrow[rr, "\kappa_{X,K}"] \arrow[dr, "h\flat 1"] \arrow[dd, "\chi_K^X"'] & & X+K \arrow[dr, "h+1"] \arrow[dd, "{[1,k]}"] & \\
& X_1\flat K \arrow[rr, "\kappa_{X_1,K}"] \arrow[dd, "\chi_K^{X_1}"] & & X_1+K \arrow[dd, "{[1,hk]}"] \\
K \arrow[rr, "k"] \arrow[dr, equal] & & X \arrow[dr, "h"] & \\
& K \arrow[rr, "hk"'] & & X_1
\end{tikzcd}$$

shows that $\chi_K^X = \chi_K^{X_1} \circ (h\flat 1) = h^*\chi_K^{X_1}$, since by composing with the monomorphism kh they are equal.

As for the right hand triangles, by definition of pullback action we have $c^*\psi = \psi \circ (c\flat 1)$, hence $c^*\psi \circ (e\flat 1) = \psi \circ (c\flat 1) \circ (e\flat 1) = \psi$. On the other hand, the diagram

$$\begin{tikzcd}
B\flat K \arrow[rr, "\kappa_{B,K}"] \arrow[dr, "e\flat 1"] \arrow[dd, "\psi"'] & & B+K \arrow[dr, "e+1"] \arrow[dd, "{[s,j]}"] & \\
& X_1\flat K \arrow[rr, "\kappa_{X_1,K}"] \arrow[dd, "\chi_K^{X_1}"] & & X_1+K \arrow[dd, "{[1,k_1 j]}"] \\
K \arrow[rr, "j"] \arrow[dr, equal] & & K_1 \arrow[dr, "k_1"] & \\
& K \arrow[rr, "k_1 j"'] & & X_1
\end{tikzcd}$$

shows that $\psi = \chi_K^{X_1} \circ (e \flat 1)$, since by composing with the monomorphism $k_1 j$ they are equal.

By (UA), since the cospan (h, e) is extremal epimorphic by protomodularity (see Proposition 1.29), the above arguments imply that $c^* \psi = \chi_K^{X_1}$.

Finally, if we consider K as a (normal) subobject of $K[c]$:

$$
\begin{tikzcd}
K \arrow[r, tail] \arrow[d, tail, "j=\ker(p)"'] & K[c] \arrow[d, tail, "l=\ker(c)"] \\
K_1 \arrow[r, tail, "k_1"'] & X_1,
\end{tikzcd}
$$

we get, as before, that $\chi_K^{K[c]} = l^* \chi_K^{X_1}$. Hence $\chi_K^{K[c]} = l^* \chi_K^{X_1} = l^* c^* \psi = 0^* \psi$, which means that the conjugation action of $K[c]$ on K is trivial, i.e. $[K, K[c]] = [K[f_1], K[c]] = 0$.

Thanks to Example 2.9, this proves that any extension f of precrossed B-modules as in (3.8) is central with respect to $\mathbf{XMod}_B(\mathcal{C})$ if the Peiffer commutator $\langle K[f], X \rangle$ is trivial. $\square$

The previous characterization of central extensions, together with the properties of the Peiffer commutator, yields the following result.

Corollary 3.13. *If f is an extension in $\mathbf{PXMod}_B(\mathcal{C})$ as in* (3.8)*, then the induced extension*

$$
\begin{tikzcd}
\frac{X}{\langle K[f], X \rangle} \arrow[rr, "\overline{f}"] \arrow[dr, "\overline{\partial}"'] & & Y \arrow[dl, "\partial'"] \\
& B &
\end{tikzcd}
$$

is central and, moreover, any morphism h from f to a central extension g factors uniquely through $\overline{f}$. Accordingly, the category of $\mathbf{XMod}_B(\mathcal{C})$-central extensions in $\mathbf{PXMod}_B(\mathcal{C})$ is a reflective subcategory of the category of extensions in $\mathbf{PXMod}_B(\mathcal{C})$.

Proof. First observe that the extension $\overline{f}$ is central. Indeed, if we write $\eta \colon X \to \frac{X}{\langle K[f], X \rangle}$ for the canonical quotient, then

$$
\left\langle K[\overline{f}], \frac{X}{\langle K[f], X \rangle} \right\rangle = \langle \eta(K[f]), \eta(X) \rangle = \eta \langle K[f], X \rangle = 0,
$$

where we have used the property of preservation of the Peiffer commutator by regular images (Proposition 3.10). Let then $h \colon X \to Z$ be a morphism in $\mathbf{PXMod}_B(\mathcal{C})$ from f to another central extension $g\colon Z \to Y$, so that $gh = f$. Consider the factorization of h in $\mathcal{C}$ as a regular epimorphism q followed by a monomorphism i:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "q"'] & & Z \\ & I \arrow[ur, tail, "i"'] & \end{tikzcd}$$

To show that h factors through η it suffices to prove that q factors through η. First observe that the induced morphism $gi\colon I \to Y$ is a central extension, i.e. $\langle K[gi], I\rangle = 0$ (this follows immediately from Proposition 3.10).

By applying once again Proposition 3.10, this implies that

$$q(\langle K[f], X\rangle) = \langle q(K[f]), q(X)\rangle = \langle K[gi], I\rangle = 0.$$

The last statement is then clear, since we have just proved that η satisfies the universal property of the f-component of the unit of the reflection into the subcategory of $\mathbf{XMod}_B(\mathcal{C})$-central extensions in $\mathbf{PXMod}_B(\mathcal{C})$. $\square$

Since quotienting by $\langle X, K[f]\rangle$ gives the centralization of an extension f, under the hypotheses of Theorem 3.12, the normal precrossed B-submodule $(\langle X, K[f]\rangle, 0, \xi_{\langle X,K[f]\rangle})$ coincides with the relative commutator $[X, K[f]]_{\mathbf{PXMod}_B(\mathcal{C})}$ as defined in [55].

3.3 Hopf formula for the fundamental group and homology

Given a normal extension $f\colon X \to Y$ in $\mathbf{PXMod}_B(\mathcal{C})$, its *Galois groupoid* is defined (see for example [87, 88]) as the reflection of its kernel pair $(Eq[f], p_1, p_2)$ into $\mathbf{XMod}_B(\mathcal{C})$. By analogy with the pointed case, we call the intersection of the kernels of $G(p_1)$ and $G(p_2)$ the *Galois group* of f and denote it by $\mathrm{Gal}(f, 0)$. This is equivalent to the kernel of the normalization of the Galois groupoid, i.e. of the composite $G(p_2)\ker(G(p_1))\colon K[G(p_1)] \to G(X)$. Since f is a normal extension, the square

$$\begin{tikzcd} Eq[f] \arrow[r, "p_1"] \arrow[d, "\eta_{Eq[f]}"'] & X \arrow[d, "\eta_X"] \\ G(Eq[f]) \arrow[r, "G(p_1)"'] & G(X) \end{tikzcd}$$

is a pullback, and thus $\ker(G(p_1))$ is equal to $\eta_{Eq[f]} \ker(p_1)$. We then have

$$G(p_2)\ker(G(p_1)) = G(p_2)\eta_{Eq[f]} \ker(p_1) = \eta_X p_2 \ker(p_1) = \eta_X \ker(f),$$

and thus

$$\mathrm{Gal}(f,0) = K[\eta_X \ker(f)] = K[f] \wedge K[\eta_X] = K[f] \wedge \langle X, X\rangle.$$

Let us assume that the category $\mathbf{PXMod}_B(\mathcal{C})$ has enough (regular) projectives. This is the case, for instance, whenever $\mathcal{C}$ is a semi-abelian variety (see for example [54]). For a given precrossed module (X,∂,ξ), we can then consider a regular epimorphism

$$p\colon (P,\partial_P,\xi_P) \to (X,\partial,\xi)$$

with (P,∂_P,ξ_P) a projective precrossed module, and then its centralization

$$\begin{tikzcd}
(P,\partial_P,\xi_P) \arrow[rr, "q"] \arrow[dr, "p"'] & & \left(\frac{P}{\langle P,K[p]\rangle}, \overline{\partial_P}, \overline{\xi_P}\right) \arrow[dl, "\overline{p}"] \\
& (X,\partial,\xi) &
\end{tikzcd}$$

in $\mathbf{PXMod}_B(\mathcal{C})$. Since (P,∂_P,ξ_P) is projective, thanks to the universal property of the centralization expressed by Corollary 3.13, one can show that $\overline{p}$ is a *weakly universal central extension*: for any other central extension $c\colon (Y,\partial_Y,\xi_Y) \to (X,\partial,\xi)$, there exists a morphism of precrossed modules $t\colon (\frac{P}{\langle P,K[p]\rangle}, \overline{\partial_P}, \overline{\xi_P}) \to (Y,\partial_Y,\xi_Y)$ such that $ct = \overline{p}$. In our context, such a universal central extension is in fact normal, so that we can consider its fundamental groupoid. The Galois groupoid of (X,∂,ξ) can then be defined as the Galois groupoid of $\overline{p}$ since, according to [87], it does not depend on the choice of the weakly universal normal extension of (X,∂,ξ). The fundamental group $\pi_1(X,\partial,\xi)$ is the Galois group $\mathrm{Gal}(\overline{p},0)$. This is given as above by the formula

$$\pi_1(X,\partial,\xi) = \mathrm{Gal}(\overline{p},0) = K[\overline{p}] \wedge \left\langle \frac{P}{\langle P,K[p]\rangle}, \frac{P}{\langle P,K[p]\rangle} \right\rangle.$$

Since the Peiffer commutator is preserved by regular images, we have

$$\left\langle \frac{P}{\langle P,K[p]\rangle}, \frac{P}{\langle P,K[p]\rangle} \right\rangle = \frac{\langle P,P\rangle}{\langle P,K[p]\rangle}.$$

Moreover, since we have a regular epimorphism

$$\overline{p}\colon \frac{P}{\langle P,K[p]\rangle} \to X \cong \frac{P}{K[p]},$$

the Noether isomorphism theorem (see Theorem 2.2 in [55]) gives us

$$K[\overline{p}] = \frac{K[p]}{\langle P, K[p] \rangle}.$$

To sum up, we find that the Galois group of the precrossed B-module (X, ∂, ξ) is given by the Hopf formula

$$\pi_1(X, \partial, \xi) \cong \frac{K[p] \wedge \langle P, P \rangle}{\langle P, K[p] \rangle},$$

which is also the second homology object $H_2(X, \partial, \xi)$ of (X, ∂, ξ) as defined in [55].

Given a short exact sequence

$$0 \longrightarrow (K, \partial_K) \xrightarrow{\ f\ } (X, \partial_X) \xrightarrow{\ g\ } (Y, \partial_Y) \longrightarrow 0 \tag{3.11}$$

and a projective presentation $p\colon (P, \partial_P) \to (X, \partial_X)$ of (X, ∂_X), this also gives a projective presentation $gp\colon (P, \partial_P) \to (Y, \partial_Y)$ of (Y, ∂_Y). It follows that

$$H_2(Y, \partial_Y) \cong \frac{K[gp] \wedge \langle P, P \rangle}{\langle P, K[gp] \rangle},$$

and we then get the following extension of the Stallings-Stammbach theorem for precrossed modules (of groups) given in [43]:

Proposition 3.14. *Any short exact sequence* (3.11) *in* $\mathbf{PXMod}_B(\mathcal{C})$*, with* $p\colon (P, \partial_P) \to (X, \partial_X)$ *a projective presentation of* (X, ∂_X)*, induces a five-term exact sequence*

$$\tfrac{K[p] \wedge \langle P,P \rangle}{\langle P, K[p] \rangle} \longrightarrow \tfrac{K[gp] \wedge \langle P,P \rangle}{\langle P, K[gp] \rangle} \longrightarrow \tfrac{K}{\langle K,X \rangle} \longrightarrow \tfrac{X}{\langle X,X \rangle} \xrightarrow{\overline{g}} \tfrac{Y}{\langle Y,Y \rangle} \tag{3.12}$$

where the morphism $\overline{g}$ *is a regular epimorphism.*

Proof. This follows from Theorem 4.6 in [55] and the remarks above. $\square$

Remark 3.15. Observe that, when $B = 0$, all Peiffer commutators above coincide with Huq commutators (see [40]) and we recover from the above result the internal version of the classical Stalling-Stammbach theorem: a short exact sequence

$$0 \longrightarrow K \xrightarrow{f} X \xrightarrow{g} Y \longrightarrow 0$$

in a semi-abelian category $\mathcal{C}$ yields a five-term exact sequence

$$\tfrac{K[p] \wedge [P,P]}{[P, K[p]]} \longrightarrow \tfrac{K[gp] \wedge [P,P]}{[P, K[gp]]} \longrightarrow \tfrac{K}{[K,X]} \longrightarrow \tfrac{X}{[X,X]} \longrightarrow \tfrac{Y}{[Y,Y]} \longrightarrow 0$$

(see also [55]).

Example 3.16. When $\mathcal{C}$ is the category of Lie algebras, the Peiffer commutator $\langle M, N\rangle$ of two precrossed B-submodules of X is the Lie ideal of X generated by the Peiffer elements

$$[m,n]-\xi(\partial(m))(n) \quad \text{and} \quad [n,m]-\xi(\partial(n))(m)$$

where $m \in M$, $n \in N$. In particular, for a morphism

$$f\colon (X,\partial,\xi) \to (Y,\partial',\xi')$$

in $\mathbf{PXMod}_B(\mathbf{Lie}_K)$, we have $\partial(k) = \partial' f(k) = 0$ for all $k \in K[f]$, so that the Peiffer commutator $\langle K[f], X\rangle$ is generated by the terms $[k,x]$ and $\xi(\partial(x))(k)$. It is thus the same ideal as in Example 3.1, and thus we find the characterization of central extensions given earlier as a special case of Theorem 3.12. Moreover, given a short exact sequence (3.11) in the category of Lie algebra precrossed modules, we obtain an exact sequence of Lie algebra precrossed modules

$$H_2(X,\partial_X) \longrightarrow H_2(Y,\partial_Y) \longrightarrow \tfrac{K}{\langle K,X\rangle} \longrightarrow \tfrac{X}{\langle X,X\rangle} \longrightarrow \tfrac{Y}{\langle Y,Y\rangle} \longrightarrow 0.$$

3.4 Double central extensions and homology

We can also characterize double central extensions, thanks to the results of Corollary 2.17.

Theorem 3.17. *Let $\mathcal{C}$ be a semi-abelian category satisfying* (UA)*, and let*

$$\begin{tikzcd} (X,\partial_X,\xi_X) \arrow[r, "g"] \arrow[d, "f"'] & (Z,\partial_Z,\xi_Z) \arrow[d, "h"] \\ (Y,\partial_Y,\xi_Y) \arrow[r, "p"'] & (W,\partial_W,\xi_W) \end{tikzcd} \tag{3.13}$$

be a double extension in the category $\mathbf{PXMod}_B(\mathcal{C})$. Then (3.13) *is a double central extension if and only if*

$$\langle K[f]\wedge K[g], X\rangle = 0 = \langle K[f], K[g]\rangle.$$

Proof. By Corollary 2.17, the double extension

$$\begin{tikzcd} X_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & Z_1 \arrow[d, "h_1"] \\ Y_1 \arrow[r, "p_1"'] & W_1 \end{tikzcd}$$

of reflexive graphs corresponding to (3.13) is central if and only if it satisfies the conditions

$$[Eq[f_1] \wedge Eq[g_1], Eq[c] \vee Eq[d]]_{SP} = \Delta_{X_1} = [Eq[f_1], Eq[g_1]]_{SP}.$$

The equality on the left may be interpreted as requiring that the comparison map $(f_1, g_1)\colon X_1 \to Y_1 \times_{W_1} Z_1$ is, according to the characterization Theorem 2.6, a central extension. By equivalence, the corresponding morphism $(f, g)\colon X \to Y \times_W Z$ in $\mathbf{PXMod}_B(\mathcal{C})$ is also a central extension, which means, by Theorem 3.12, that

$$\langle K[f] \wedge K[g], X\rangle = 0.$$

Under the (SH) condition, the equality on the right is equivalent to the Huq commutator of $K[f]$ and $K[g]$ being trivial in X_1. But since $K[f]$ and $K[g]$ are subobjects of $X = K[d]$, this is is equivalent to their Huq commutator being trivial in X. This in turn implies that

$$\langle K[f], K[g]\rangle = 0$$

by Remark 3.6, since $K[f]$ and $K[g]$ are normal precrossed B-submodules of X. □

As for the characterization of central extensions, by the previous result, we get a description of the reflection of double extensions into the subcategory of double central extensions.

Proposition 3.18. *The centralization of a double extension as* (3.13) *in* $\mathbf{PXMod}_B(\mathcal{C})$ *is given by*

$$\begin{tikzcd}
\left(\frac{X}{\langle K[f]\wedge K[g], X\rangle \vee \langle K[f], K[g]\rangle}, \overline{\partial}, \overline{\xi}\right) \arrow[r, "\overline{g}"] \arrow[d, "\overline{f}"'] & (Z, \partial_Z, \xi_Z) \arrow[d, "h"] \\
(Y, \partial_Y, \xi_Y) \arrow[r, "p"'] & (W, \partial_W, \xi_W).
\end{tikzcd} \tag{3.14}$$

Proof. Let us first observe that, by Lemma 3.9 and Proposition 3.10:

$$\langle K[f] \wedge K[g], X\rangle \vee \langle K[f], K[g]\rangle \leq K[f] \wedge K[g].$$

Hence, denoting $J = \langle K[f] \wedge K[g], X\rangle \vee \langle K[f], K[g]\rangle$ and $q\colon X \to X/J$, we have a pushout

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"'] & \frac{X}{K[f]\wedge K[g]} \arrow[d, equal] \\
\frac{X}{J} \arrow[r] & \frac{X}{K[f]\wedge K[g]},
\end{tikzcd}$$

and as a consequence, by taking kernels horizontally:

$$q(K[f] \wedge K[g]) = \frac{K[f] \wedge K[g]}{J} = \frac{K[f]}{J} \wedge \frac{K[g]}{J} = q(K[f]) \wedge q(K[g]).$$

We are going to show that the double extension (3.14) is central by means of the characterization given by Theorem 3.17:

$$\begin{aligned}
\left\langle K[\overline{f}] \wedge K[\overline{g}], \frac{X}{J} \right\rangle &= \left\langle \frac{K[f]}{J} \wedge \frac{K[g]}{J}, \frac{X}{J} \right\rangle \\
&= \langle q(K[f]) \wedge q(K[g]), q(X) \rangle \\
&= \langle q(K[f] \wedge K[g]), q(X) \rangle \\
&= q(\langle K[f] \wedge K[g], X \rangle) = 0,
\end{aligned}$$

$$\begin{aligned}
\langle K[\overline{f}], K[\overline{g}] \rangle &= \left\langle \frac{K[f]}{J}, \frac{K[g]}{J} \right\rangle \\
&= \langle q(K[f]), q(K[g]) \rangle = q(\langle K[f], K[g] \rangle) = 0.
\end{aligned}$$

By Theorem 1.76, we know that the category of double central extensions is reflective in the category of double extensions in $\mathbf{PXMod}_B(\mathcal{C})$. Moreover, by a result due to Im and Kelly [79], the reflection must fix all but the "top object" (here X) of the double extension. To prove the universal property, there is then no restriction in considering an arrow φ of the form

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\varphi"] \arrow[dd, "f"'] & & Z \arrow[dr, equal] \arrow[dd] & \\
& X' \arrow[rr, "g'"] \arrow[dd, "f'"] & & Z \arrow[dd] \\
Y \arrow[rr] \arrow[dr, equal] & & W \arrow[dr, equal] & \\
& Y \arrow[rr] & & W
\end{tikzcd}$$

where the front face is a double central extension. Consider the decomposition $\varphi = ip$, where i is a monomorphism and p is a regular epimorphism. Then it induces a diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "p"] \arrow[dd, "f"'] & & Z \arrow[dr, equal] \arrow[dd] & \\
& I \arrow[rr, "g'i"] \arrow[dd, "f'i"] & & Z \arrow[dd] \\
Y \arrow[rr] \arrow[dr, equal] & & W \arrow[dr, equal] & \\
& Y \arrow[rr] & & W
\end{tikzcd}$$

where the front face is a double extension, since p is a regular epimorphism and the back face is a double extension. Moreover, it is a subobject of a double central extension, since i is a monomorphism ; since double central extensions form a strongly $\mathcal{E}^2$-Birkhoff subcategory of the category of double extensions, this inclusion must factor through a triple extension. But then this triple extension is also a (degreewise) monomorphism between double extension, and thus it must be an isomorphism. Thus the front face is a double central extension, and

$$\begin{aligned} p(J) &= p(\langle K[f] \wedge K[g], X\rangle) \vee p(\langle K[f], K[g]\rangle) \\ &= \langle p(K[f] \wedge K[g]), p(X)\rangle \vee \langle p(K[f]), p(K[g])\rangle \\ &= \langle K[f'i] \wedge K[g'i], I\rangle \vee \langle K[f'i], K[g'i]\rangle \\ &= 0, \end{aligned}$$

where the first equality follows from the fact that regular images distribute over joins. So p factors through q, yielding a commutative triangle of double extensions, which shows that q gives indeed the required reflection. $\square$

In particular, if we consider two normal precrossed B-submodules $(H, 0, \xi_H)$ and $(K, 0, \xi_K)$ of a given precrossed module (X, ∂_X, ξ_X), then the join $H \vee K$ in $\mathcal{C}$ is endowed with a precrossed B-module structure, and it is normal in (X, ∂_X, ξ_X) too (see [40]). One can then consider the double extension

$$\begin{array}{ccc} (H \vee K, 0, \xi_{H \vee K}) & \longrightarrow & (\frac{H \vee K}{H}, 0, \xi_{\frac{H \vee K}{H}}) \\ \downarrow & & \downarrow \\ (\frac{H \vee K}{K}, 0, \xi_{\frac{H \vee K}{K}}) & \longrightarrow & (0, 0, \tau), \end{array}$$

and apply Proposition 3.18 to this special case, whose centralization is obtained by quotienting out the object

$$\langle H \wedge K, H \vee K\rangle \vee \langle H, K\rangle = \langle H \wedge K, H\rangle \vee \langle H \wedge K, K\rangle \vee \langle H, K\rangle = \langle H, K\rangle.$$

Let us observe that H and K act trivially on each other by the action induced by B, because their structure maps are zero, whence

$$\langle H, K\rangle = [H, K]_{Huq}$$

by Remark 3.6.

Finally, slightly enlarging the context of [61] to include quasi-pointed categories, we may say that the centralization just described provides a description of the relative commutator of two normal precrossed B-submodules with respect to the reflection of $\mathbf{PXMod}_B(\mathcal{C})$, so that

$$[H,K]_{\mathbf{PXMod}_B(\mathcal{C})} = ([H,K]_{Huq}, 0, \xi_{[H,K]}).$$

3.5 The third homology object

Following the lines of Section 6 in [87] and using the characterization of double central extensions we are now going to establish a Hopf formula for the third homology object in $\mathbf{PXMod}_B(\mathcal{C})$, which specializes in particular to the third integral homology group of a group [29]. To this purpose, we assume again that $\mathbf{PXMod}_B(\mathcal{C})$ has enough regular projectives, and we can first define $\pi_2(X,\partial,\xi)$ as the Galois group of a weakly universal double central extension. To construct such a double extension, we take two projective precrossed B-modules (P,∂_P,ξ_P) and $(P',\partial_{P'},\xi_{P'})$ and regular epimorphisms $p\colon P \to X$ and $p'\colon P' \to X$ in $\mathbf{PXMod}_B(\mathcal{C})$; then we form the pullback $P\times_X P'$ of p and p', and take a projective precrossed B-module (Q,∂_Q,ξ_Q) with a regular epimorphism $Q \to P\times_X P'$. The square

$$\begin{tikzcd} Q \arrow[r, "q"] \arrow[d, "q'"'] & P' \arrow[d, "p'"] \\ P \arrow[r, "p"'] & X \end{tikzcd} \tag{3.15}$$

is then a double extension (in $\mathbf{PXMod}_B(\mathcal{C})$), so that we can see $q \to p$ and $q' \to p'$ as extensions with projective domains in the category $\mathbf{Ext}_{\mathcal{E}}(\mathbf{PXMod}_B(\mathcal{C}))$. As in the one-dimensional case, the centralization

$$\frac{Q}{\langle K[q]\wedge K[q'],Q\rangle \vee \langle K[q],K[q']\rangle} = \begin{tikzcd} \overline{Q} \arrow[r, "\overline{q}"] \arrow[d, "\overline{q'}"'] & P' \arrow[d, "p'"] \\ P \arrow[r, "p"'] & X \end{tikzcd}$$

of this double extension is then a weakly universal double central extension, and we can use it to compute the fundamental group of the extension p' as

$$\begin{aligned} \pi_1(p') &= K[(\overline{q},p)] \wedge K[\eta^1_{\overline{q'}}] \\ &= K[\overline{q}] \wedge \langle K[\overline{q'}], \overline{Q}\rangle \to 0 \\ &= \frac{K[q]\wedge \langle K[q'],Q\rangle}{\langle K[q]\wedge K[q'],Q\rangle \vee \langle K[q],K[q']\rangle} \to 0 \end{aligned}$$

where the second equality is explained by the following (horizontal) pullback in $\mathbf{Ext}_{\mathcal{E}}(\mathbf{PXMod}_B(\mathcal{C}))$:

$$\begin{tikzcd}
K[\overline{q}] \wedge \langle K[\overline{q'}], \overline{Q} \rangle \arrow[rr] \arrow[dr] \arrow[dd] & & \langle K[\overline{q'}], \overline{Q} \rangle \arrow[dr, "\ker(\eta^1_{\overline{q'}})"] \arrow[dd, "K[\eta^1_{\overline{q'}}]" near end] & \\
& K[\overline{q}] \arrow[rr] \arrow[dd, "K[(\overline{q}, p)]"' near start] & & \overline{Q} \arrow[dd, "\overline{q'}"] \\
0 \arrow[rr, equal] \arrow[dr] & & 0 \arrow[dr] & \\
& K[p] \arrow[rr] & & P.
\end{tikzcd}$$

Analogously

$$\pi_1(p) = \frac{K[q'] \wedge \langle K[q], Q \rangle}{\langle K[q] \wedge K[q'], Q \rangle \vee \langle K[q], K[q'] \rangle} \to 0.$$

Since $q \colon Q \to P'$ is also a regular epimorphism with projective domain, the fundamental group of P' can be calculated as

$$\pi_1(P') = \frac{K[q] \wedge \langle Q, Q \rangle}{\langle K[q], Q \rangle},$$

but since P' is projective, this fundamental group must be trivial, which implies that $K[q] \wedge \langle Q, Q \rangle = \langle K[q], Q \rangle$. By analogy, we must also have that $K[q'] \wedge \langle Q, Q \rangle = \langle K[q'], Q \rangle$, and as a consequence, we obtain

$$\pi_1(p) = \frac{K[q'] \wedge K[q] \wedge \langle Q, Q \rangle}{\langle K[q] \wedge K[q'], Q \rangle \vee \langle K[q], K[q'] \rangle} \to 0 = \pi_1(p').$$

Since this must be true for any p and p', and $\pi_1(p)$ and $\pi_1(p')$ only depend on p and p' respectively, $\pi_1(p)$ only depends on its codomain (X, ∂, ξ); thus we can define $\pi_2(X, \partial, \xi)$ as the domain of $\pi_1(p)$, and this gives us the Hopf formula

$$H_3(X, \partial) = \pi_2(X, \partial) = \frac{K[q'] \wedge K[q] \wedge \langle Q, Q \rangle}{\langle K[q] \wedge K[q'], Q \rangle \vee \langle K[q], K[q'] \rangle}, \tag{3.16}$$

which is independent of the chosen double extension (3.15).

4 Fundamental groupoids of simplicial objects

This chapter is adapted from the preprint [49].

The category of groupoids can be identified, thanks to the nerve functor, with a full subcategory of the category of simplicial sets. Furthermore, the nerve functor admits a left adjoint, the fundamental groupoid functor **Simp** $\to$ **Grpd** [64]. In [30], R. Brown and G. Janelidze defined a Galois structure based on this adjunction, using Kan fibrations as extensions between simplicial sets. They showed that Kan complexes are admissible for this Galois structure, and called the corresponding central extensions *second order coverings*.

By a theorem of Moore [116], the underlying simplicial set of a simplicial group is always a Kan complex. In fact, any simplicial object in a regular Mal'tsev category satisfies the Kan condition, and in fact this property characterizes Mal'tsev categories, as proved by A. Carboni, G. M. Kelly and M. C. Pedicchio in [34] (and conjectured by Barr in [3]). Furthermore, this also implies that regular epimorphisms coincide with Kan fibrations [59].

This suggests that the subcategory of internal groupoids in any exact Mal'tsev category might be an admissible reflective subcategory of the category of simplicial objects. In fact, we can show that it is in fact a Birkhoff subcategory; furthermore, the central extensions coincide with exact fibrations in the sense of [65].

The chapter is organized as follows : first we include a proof of the fact that the Kan property characterizes Mal'tsev categories. We then construct the reflection of the category of simplicial objects into the subcategory of internal groupoids. Next, we characterize the central extensions for the Galois structure. In the next section we compare our construction with the homotopy relations for the simplices in a Kan complex, which are used to define its homotopy groupoid. Then we prove that the Galois structure admits a relative monotone-light factorization system. We end the chapter with a discussion of reflexive graphs, seen as truncated simplicial objects, and a connection with commutators.

4.1 The Kan condition in regular Mal'tsev categories

Theorem 4.1 ([34, 59, 53])**.** *A regular category $\mathcal{C}$ is a Mal'tsev category if and only if every simplicial object in $\mathcal{C}$ is a Kan complex. When this is the case, then every regular epimorphism in* $\mathbf{Simp}(\mathcal{C})$ *is a Kan fibration.*

Proof. Assume first that $\mathcal{C}$ is a Mal'tsev category. If $\mathbb{X}$ is a simplicial object in $\mathcal{C}$, then for all $n \geq 2$ and $0 \leq k \leq n$, the restriction of $\mathbb{X}$ to the poset of proper subsets of $\{0, \dots, n\}$ containing k forms an n-dimensional hypercube. Then by Proposition 1.72, to prove that $\mathbb{X}$ is a Kan complex it suffices to prove that every such hypercube is an n-fold extension. Moreover, a simplicial morphism $f \colon \mathbb{X} \to \mathbb{Y}$ induces a morphism of n-dimensional hypercubes, and thus an $(n+1)$-dimensional hypercube, and to prove that f is actually a Kan fibration it suffices to prove that this hypercube is an $(n+1)$-fold extension.

We will prove by induction on n that all such hypercubes can in fact be seen as split epimorphisms of $(n-1)$ or n-fold extensions. For $n = 2$, the simplicial identities, together with Theorem 1.20, imply that the squares

$$\begin{tikzcd} X_2 \arrow[r, "d_2"] \arrow[d, shift left, "d_1"] & X_1 \arrow[d, shift left, "d_1"] \\ X_1 \arrow[r, "d_1"] \arrow[u, shift left, "s_0"] & X_0 \arrow[u, shift left, "s_0"] \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "d_2"] \arrow[d, shift left, "d_0"] & X_1 \arrow[d, shift left, "d_0"] \\ X_1 \arrow[r, "d_1"] \arrow[u, shift left, "s_0"] & X_0 \arrow[u, shift left, "s_0"] \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, shift left, "d_1"] \arrow[d, "d_0"'] & X_1 \arrow[l, shift left, "s_1"] \arrow[d, "d_0"] \\ X_1 \arrow[r, shift left, "d_0"] & X_0 \arrow[l, shift left, "s_1"] \end{tikzcd}$$

are all double extensions. Now if $f \colon \mathbb{X} \to \mathbb{Y}$ is a regular epimorphism in $\mathbf{Simp}(\mathcal{C})$, then first of all the induced commutative cube

$$\begin{tikzcd} X_2 \arrow[rr, "d_2"] \arrow[dr, "f_2"] \arrow[dd, shift left, "d_1"] & & X_1 \arrow[dr, "f_1"] \arrow[dd, shift left, "d_1"] & \\ & Y_2 \arrow[rr, "d_2"] \arrow[dd, shift left, "d_1"] & & Y_1 \arrow[dd, shift left, "d_1"] \\ X_1 \arrow[rr, "d_1"] \arrow[dr, "f_1"'] \arrow[uu, shift left, "s_0"] & & X_0 \arrow[dr, "f_0"] \arrow[uu, shift left, "s_0"] & \\ & Y_1 \arrow[rr, "d_1"'] \arrow[uu, shift left, "s_0"] & & Y_0 \arrow[uu, shift left, "s_0"] \end{tikzcd}$$

is a (vertical) split epimorphism, and its domain and codomain are double extensions, since d_2 and d_1 are split by s_1 and s_0 respectively, and the components of f commute with these sections. The cases where $k = 1, 2$ can be treated in the same way.

Let us now assume that for all $k \neq n$ and all regular epimorphism $f \colon \mathbb{X} \to \mathbb{Y}$, the corresponding $(n+1)$-dimensional hypercube is a split epimorphism of $(n+1)$-fold extension. In particular, we can apply this to $f = \epsilon \colon Dec(\mathbb{X}) \to \mathbb{X}$, so that the $(n+1)$-dimensional hypercube obtained as above when choosing $k < n+1$ is a split epimorphism of n-fold extensions. Moreover, f will commute with the corresponding section, so this becomes a split epimorphism between the $(n+1)$-fold extensions determined by f and $Dec(f)$. The case where $k = n+1$ is just symmetric to the case where $k = 0$.

Let us now assume that every simplicial object is a Kan complex. If R is a reflexive relation on an object X, then by taking iterated simplicial kernels we can extend the corresponding reflexive graph to a simplicial object, which is then a Kan complex. Then we can proceed as in the proof of Theorem 1.20 to prove that R is a transitive relation, and since this applies to every R and X, this means $\mathcal{C}$ is a Mal'tsev category. $\square$

Convention. From now on, $\mathcal{C}$ will denote a regular Mal'tsev category. Note that $\mathbf{Simp}(\mathcal{C})$, being a functor category, is then also regular Mal'tsev.

For a given simplicial object $(X_n)_{n\geq 0}$ with face maps $d_i \colon X_n \to X_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$, we will denote D_i the kernel pair of d_i.

If $f \colon \mathbb{X} \to \mathbb{Y}$ is a map in $\mathbf{Simp}(\mathcal{C})$, we will denote F_n the kernel pair of the corresponding map $f_n \colon X_n \to Y_n$, for all $n \geq 0$. Similarly, for maps $g \colon \mathbb{Z} \to \mathbb{W}$ and $f' \colon \mathbb{X}' \to \mathbb{Y}'$ in $\mathbf{Simp}(\mathcal{C})$, we will denote the corresponding kernel pairs G_n and F'_n (for $n \geq 0$), respectively.

Corollary 4.2. *As a consequence of Theorem 4.1, for all $n \geq 2$ and $0 \leq i < j \leq n$, the commutative square*

$$\begin{tikzcd}
X_n \arrow[r, "d_j"] \arrow[d, "d_i"'] & X_{n-1} \arrow[d, "d_i"] \\
X_{n-1} \arrow[r, "d_{j-1}"'] & X_{n-2}
\end{tikzcd}$$

is a double extension, and for all $n \geq 3$ and all $0 \leq i < j < k \leq n$, the *commutative cube*

$$\begin{tikzcd}
X_n \arrow[rr, "d_j"] \arrow[dr, "d_k"'] \arrow[dd, "d_i"'] & & X_{n-1} \arrow[dr, "d_{k-1}"] \arrow[dd, "d_i"] & \\
& X_{n-1} \arrow[rr, "d_j"] \arrow[dd, "d_i"] & & X_{n-2} \arrow[dd, "d_i"] \\
X_{n-1} \arrow[rr, "d_{j-1}"] \arrow[dr, "d_{k-1}"'] & & X_{n-2} \arrow[dr, "d_{k-2}"'] & \\
& X_{n-2} \arrow[rr, "d_{j-1}"'] & & X_{n-3}
\end{tikzcd}$$

is a triple extension.

Moreover, if $f\colon \mathbb{X} \to \mathbb{Y}$ *is a regular epimorphism in* $\mathbf{Simp}(\mathcal{C})$*, then for all* $n \geq 2$ *and* $0 \leq i < j \leq n$*, the commutative cube*

$$\begin{tikzcd}
X_n \arrow[rr, "d_j"] \arrow[dr, "f_n"'] \arrow[dd, "d_i"'] & & X_{n-1} \arrow[dr, "f_{n-1}"] \arrow[dd, "d_i"] & \\
& Y_n \arrow[rr, "d_j"] \arrow[dd, "d_i"] & & Y_{n-1} \arrow[dd, "d_i"] \\
X_{n-1} \arrow[rr, "d_{j-1}"] \arrow[dr, "f_{n-1}"'] & & X_{n-2} \arrow[dr, "f_{n-2}"] & \\
& Y_{n-1} \arrow[rr, "d_{j-1}"'] & & Y_{n-2}
\end{tikzcd}$$

is a triple extension.

Thus we have

Corollary 4.3. *For every simplicial object* $\mathbb{X}$ *in* $\mathcal{C}$*, and for all* $0 \leq i < j \leq n$ *we have*

$$d_i(D_j) = D_{j-1} \quad \text{and} \quad d_j(D_i) = D_i,$$

and for all $0 \leq i < j < k \leq n$ *we also have*

$$\begin{array}{rcl}
d_k(D_i \wedge D_j) & = & D_i \wedge D_j \\
d_j(D_i \wedge D_k) & = & D_i \wedge D_{k-1} \\
d_i(D_j \wedge D_k) & = & D_{j-1} \wedge D_{k-1}.
\end{array}$$

Furthermore, for all regular epimorphism $f\colon \mathbb{X} \to \mathbb{Y}$ *in* $\mathbf{Simp}(\mathcal{C})$ *and all* $n \geq 2$ *we have*

$$\begin{array}{rcl}
f_n(D_i^{\mathbb{X}} \wedge D_j^{\mathbb{X}}) & = & D_i^{\mathbb{Y}} \wedge D_j^{\mathbb{Y}} \\
d_i^{\mathbb{X}}(F_n \wedge D_j^{\mathbb{X}}) & = & F_{n-1} \wedge D_{j-1}^{\mathbb{X}} \\
d_j^{\mathbb{X}}(F_n \wedge D_i^{\mathbb{X}}) & = & F_{n-1} \wedge D_i^{\mathbb{X}}.
\end{array}$$

4.2 The reflection of simplicial objects into groupoids

Lemma 4.4. *For any simplicial object $\mathbb{X}$, the following equivalence relations in X_1 are all equal :*

$$d_0(D_1 \wedge D_2) = d_1(D_0 \wedge D_2) = d_2(D_0 \wedge D_1).$$

Proof. Since $D_1 \wedge D_2 = d_2(D_1 \wedge D_3)$, we have

$$d_0(D_1 \wedge D_2) = d_0(d_2(D_1 \wedge D_3)) = d_1(d_0(D_1 \wedge D_3)) = d_1(D_0 \wedge D_2).$$

The proof for the other identity is similar. □

Definition 4.5. Let $\mathbb{X}$ be a simplicial object in $\mathcal{C}$. We will denote $H_1(\mathbb{X})$ the equivalence relation $d_1^{\mathbb{X}}(D_0^{\mathbb{X}} \wedge D_2^{\mathbb{X}})$ on the object X_1.

Proposition 4.6. *Let $\mathbb{X}$ be a simplicial object in $\mathcal{C}$. Then for all $n \geq 2$ the following conditions are equivalent :*

1. $D_i \wedge D_j = \Delta_{X_n}$ *for all* $0 \leq i < j \leq n$*;*
2. $D_0 \wedge D_n = \Delta_{X_n}$*;*
3. *there exist* $0 \leq i < j \leq n$ *such that* $D_i \wedge D_j = \Delta_{X_n}$*.*

Moreover, $\mathbb{X}$ is an internal groupoid if and only if it satisfies these conditions for all $n \geq 2$.

Proof. It is clear that the first condition implies the second and that the second implies the third; we prove that the third implies the first by induction. We first consider the case where $n = 2$; for this case it is enough to prove that $D_i \wedge D_k = \Delta_{X_2}$ for all $k \notin \{i, j\}$. We have $d_i(D_i \wedge D_k) = \Delta_{X_1}$ and

$$d_j(D_i \wedge D_k) = d_k(D_i \wedge D_j) = \Delta_{X_1}$$

by Lemma 4.4, and thus $D_i \wedge D_k \leq D_i \wedge D_j = \Delta_{X_2}$.

Assuming now that the condition holds for n, we prove that it holds for $n+1$. Assume that $D_i \wedge D_j = \Delta_{X_{n+1}}$; then taking images by d_k (for $k \notin \{i, j\}$) on both sides shows that $D_{i'} \wedge D_{j'} = \Delta_{X_n}$ for some i', j', and thus, by the induction hypothesis, for all i', j'. In particular, for any $0 \leq r < s \leq n+1$, we have for some r', s'

$$d_i(D_r \wedge D_s) \leq D_{r'} \wedge D_{s'} = \Delta_{X_n},$$

so that $D_r \wedge D_s \leq D_i$; and similarly $D_r \wedge D_s \leq D_j$, so that $D_r \wedge D_s = \Delta_{X_{n+1}}$.

Now $\mathbb{X}$ is an internal groupoid if and only if the squares

$$\begin{tikzcd} X_n \arrow[r, "d_0"] \arrow[d, "d_n"'] & X_{n-1} \arrow[d, "d_{n-1}"] \\ X_{n-1} \arrow[r, "d_0"'] & X_{n-2} \end{tikzcd}$$

are all pullbacks. Since we know already that they are all double extensions, this is equivalent to the fact that the pair d_0, d_n is jointly monic, and this is equivalent to the second condition. Thus any internal category always satisfies the second condition, and conversely any simplicial object satisfying the first one is an internal category where the square

$$\begin{tikzcd} X_2 \arrow[r, "d_0"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_0"'] & X_0 \end{tikzcd}$$

is a pullback. This condition is equivalent to the internal category being a groupoid. □

Note that in the above proof we only needed to know that $\mathbb{X}$ was an internal category to prove that it satisfied the conditions; so this gives us a new proof of the fact that any internal category in a regular Mal'tsev category is an internal groupoid.

Corollary 4.7. *The subcategory* **Grpd**$(\mathcal{C})$ *of* **Simp**$(\mathcal{C})$ *is closed under quotients and subobjects.*

Proof. All the intersections that characterize internal groupoids in the previous proposition are preserved by regular epimorphisms of simplicial objects, which shows that groupoids are closed under quotients. Moreover they are also closed under subobjects ; indeed, if $m \colon \mathbb{X} \to \mathbb{Y}$ is a monomorphism in **Simp**$(\mathcal{C})$ with $\mathbb{Y}$ a groupoid, then for any $0 \leq i < j \leq n$, the cube induced by the identity $d_i d_j = d_{j-1} d_i$ and m yields a commutative square

$$\begin{tikzcd} X_n \arrow[r, "m_n"] \arrow[d, "(d_i^{\mathbb{X}}, d_j^{\mathbb{X}})"'] & Y_n \arrow[d, "(d_i^{\mathbb{Y}}, d_j^{\mathbb{Y}})"] \\ X_{n-1} \times_{X_{n-2}} X_{n-1} \arrow[r, "\widetilde{m}"'] & Y_{n-1} \times_{Y_{n-2}} Y_{n-1} \end{tikzcd}$$

where the horizontal sides are monomorphisms and the right-hand vertical side is an isomorphism, and thus the left-hand vertical side is a monomorphism. Since it is also a regular epimorphism by Corollary 4.2, this means $(d_i^{\mathbb{X}}, d_j^{\mathbb{X}})$ is an isomorphism, hence $\mathbb{X}$ is an internal groupoid. $\square$

Remark 4.8. In fact the previous corollary also characterizes Mal'tsev categories among the regular (or even finitely complete) ones : indeed a reflexive relation $R \hookrightarrow X \times X$ is just a subobject of the reflexive graph $(X \times X, X, \pi_1, \pi_2, \delta_X)$, and by taking iterated simplicial kernels, one can extend this to a monomorphism in $\mathbf{Simp}(\mathcal{C})$, whose codomain is just the nerve of the indiscrete equivalence relation/groupoid on X. Thus every reflexive relation is a subobject of a groupoid, and a relation is a groupoid if and only if it is an equivalence relation. Accordingly :

Corollary 4.9. *A category $\mathcal{C}$ is a Mal'tsev category if and only if $\mathbf{Grpd}(\mathcal{C})$ is closed under subobjects in $\mathbf{Simp}(\mathcal{C})$.*

Convention. We now assume that the category $\mathcal{C}$ is also exact. Note that this implies that $\mathbf{Simp}(\mathcal{C})$ is also exact.

In this setting, we have

Theorem 4.10. *The inclusion $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$ admits a left adjoint $\Pi_1 \colon \mathbf{Simp}(\mathcal{C}) \to \mathbf{Grpd}(\mathcal{C})$, which makes $\mathbf{Grpd}(\mathcal{C})$ a Birkhoff subcategory of $\mathbf{Simp}(\mathcal{C})$.*

Proof. Note first that since by definition $H_1(\mathbb{X}) \leq D_0 \wedge D_1$, d_0 and $d_1 \colon X_1 \to X_0$ both factor through the coequalizer η_1 of $H_1(\mathbb{X})$, and their factorizations have a common section $\eta_1 s_0$, which we will also denote $\overline{s_0}$, so that we get a morphism of reflexive graphs

$$\begin{tikzcd}
X_1 \arrow[rr, "\eta_1"] \arrow[dr, shift left, "d_1"] \arrow[dr, shift right, "d_0"'] & & \frac{X_1}{H_1(\mathbb{X})} \arrow[dl, shift right, "\overline{d_0}"'] \arrow[dl, shift left, "\overline{d_1}"] \\
& X_0. &
\end{tikzcd}$$

Let us then form the pullback

$$\begin{tikzcd}
\frac{X_1}{H_1(\mathbb{X})} \times_{X_0} \frac{X_1}{H_1(\mathbb{X})} \arrow[r, "\overline{d_2}"] \arrow[d, "\overline{d_0}"'] & \frac{X_1}{H_1(\mathbb{X})} \arrow[d, "\overline{d_0}"] \\
\frac{X_1}{H_1(\mathbb{X})} \arrow[r, "\overline{d_1}"'] & X_0
\end{tikzcd}$$

By Proposition 1.17, $\eta_1 \times_1 \eta_1 \colon X_1 \times_{X_0} X_1 \to \frac{X_1}{H_1(\mathbb{X})} \times_{X_0} \frac{X_1}{H_1(\mathbb{X})}$ is a regular epimorphism, and as a consequence so is

$$(\eta_1 d_0, \eta_1 d_2) = (\eta_1 \times_1 \eta_1) \circ (d_0, d_2) \colon X_2 \to \frac{X_1}{H_1(\mathbb{X})} \times_{X_0} \frac{X_1}{H_1(\mathbb{X})},$$

which we will denote η_2. We also define $H_2(\mathbb{X}) = Eq[\eta_2]$. Now we need to show that $\eta_1 d_1 \colon X_2 \to \frac{X_1}{H_1(\mathbb{X})}$ factorizes through η_2; for this it is enough to show that $\eta_1 d_1(H_2(\mathbb{X})) = \Delta_{X_2}$, which is equivalent to $d_1(H_2(\mathbb{X})) \leq H_1(\mathbb{X})$. Since $\overline{d_0}$ and $\overline{d_2}$ are jointly monic by construction, we find that

$$\begin{aligned} H_2(\mathbb{X}) &= Eq[\eta_2] = Eq[\overline{d_0}\eta_2] \wedge Eq[\overline{d_2}\eta_2] \\ &= Eq[\eta_1 d_0] \wedge Eq[\eta_1 d_2] \\ &= d_0^{-1}(H_1(\mathbb{X})) \wedge d_2^{-1}(H_1(\mathbb{X})) \\ &= d_0^{-1}(d_0(D_1 \wedge D_2)) \wedge d_2^{-1}(d_2(D_0 \wedge D_1)) \\ &= (D_0 \vee (D_1 \wedge D_2)) \wedge (D_2 \vee (D_0 \wedge D_1)). \end{aligned}$$

Using the modularity of the lattice of equivalence relations on X_2 twice (Proposition 1.15), one sees that this is equal to

$$((D_0 \vee (D_1 \wedge D_2)) \wedge D_2) \vee (D_0 \wedge D_1) = (D_0 \wedge D_2) \vee (D_1 \wedge D_2) \vee (D_0 \wedge D_1).$$

From this last expression, we get that $d_1(H_2(\mathbb{X})) = d_1(D_0 \wedge D_2) = H_1(\mathbb{X})$. This proves the existence of $\overline{d_1}$ such that $\overline{d_1}\eta_2 = \eta_1 d_1$. Let us also denote $\overline{s_0}$ the unique map $\frac{X_1}{H_1(\mathbb{X})} \to \frac{X_1}{H_1(\mathbb{X})} \times_{X_0} \frac{X_1}{H_1(\mathbb{X})}$ such that $\overline{d_0}\,\overline{s_0} = 1_{\frac{X_1}{H_1(\mathbb{X})}}$ and $\overline{d_2}\,\overline{s_0} = \overline{s_0}\,\overline{d_1}$, and $\overline{s_1}$ the unique map $\frac{X_1}{H_1(\mathbb{X})} \to \frac{X_1}{H_1(\mathbb{X})} \times_{X_0} \frac{X_1}{H_1(\mathbb{X})}$ such that $\overline{d_2}\,\overline{s_1} = 1_{\frac{X_1}{H_1(\mathbb{X})}}$ and $\overline{d_0}\,\overline{s_1} = \overline{s_0}\,\overline{d_0}$. Using the fact that η_1 and η_2 are (regular) epimorphisms, one can now easily prove that all the simplicial identities are satisfied. This endows the quotient graph with the structure of a multiplicative graph, which is then automatically a groupoid by Theorem 1.21. We denote $\Pi_1(\mathbb{X})$ this internal groupoid, $\eta_{\mathbb{X}} \colon \mathbb{X} \to \Pi_1(\mathbb{X})$ the morphism of simplicial objects induced by 1_{X_0}, η_1 and η_2. We can show that η_n is a regular epimorphism for all n, by iterating the argument showing that η_2 is a regular epimorphism.

The only thing that remains to be checked is that this construction is universal. For this we must prove that for every morphism $f \colon \mathbb{X} \to \mathbb{Y}$ to a groupoid $\mathbb{Y}$, there exists a factorization of f_n through $\eta_n \colon X_n \to \Pi_1(\mathbb{X})_n$ for all n (note that such a factorization is unique, as every η_n is a regular epimorphism). The case $n = 0$ is trivial as η_0 is the identity. For $n = 1$, it is enough to prove that $Eq[f_1] \geq H_1(\mathbb{X})$, or equivalently

$f_1(H_1(\mathbb{X})) = \Delta_{Y_1}$. Now

$$\begin{aligned} f_1(H_1(\mathbb{X})) &= f_1 d_1^{\mathbb{X}}(D_0^{\mathbb{X}} \wedge D_2^{\mathbb{X}}) = d_1^{\mathbb{Y}} f_2(D_0^{\mathbb{X}} \wedge D_2^{\mathbb{X}}) \\ &\leq d_1^{\mathbb{Y}}(D_0^{\mathbb{Y}} \wedge D_2^{\mathbb{Y}}) = H_1(\mathbb{Y}) = \Delta_{Y_1}, \end{aligned}$$

where the last equality is due to the fact that $\mathbb{Y}$ is a groupoid. This shows that the truncation of f to a morphism (f_1, f_0) of reflexive graph factors through the groupoid $X_1/H_1(\mathbb{X})$, with a factorization $(g_1, g_0 = f_0)$; applying the nerve functor allows us to extend this factorization to higher levels, resulting in morphisms $g_n\colon \Pi_1(\mathbb{X})_n \to Y_n$. Then the factorizations $f_n = g_n\eta_n$ for $n \geq 2$ can be obtained from the universal property of the pullbacks defining each $\overline{X}_n$ and Y_n. Then since each η_n is a regular epimorphism, the morphisms g_n define a morphism of simplicial objects. □

For the next section, it will be useful to have a formula for the quotients defining each component $\eta_n\colon X_n \to \Pi(\mathbb{X})_n$.

Proposition 4.11. *For all $n \geq 2$, the kernel pair $H_n(\mathbb{X})$ of η_n is given by*

$$H_n(\mathbb{X}) = \bigvee_{0 \leq i < j \leq n} (D_i \wedge D_j).$$

Proof. We prove the result by induction on n. The case $n = 2$ was done in the proof of Theorem 4.10. Now let us assume that it holds for n; since by construction the square

$$\begin{tikzcd} \Pi_1(\mathbb{X})_{n+1} \arrow[r, "d_0"] \arrow[d, "d_{n+1}"'] & \Pi_1(\mathbb{X})_n \arrow[d, "d_n"] \\ \Pi_1(\mathbb{X})_n \arrow[r, "d_0"'] & \Pi_1(\mathbb{X})_{n-1} \end{tikzcd}$$

is a pullback, so that the two maps d_0, d_{n+1} are jointly monic, we have for $n+1$

$$\begin{aligned} H_{n+1}(\mathbb{X}) &= Eq[\eta_{n+1}] = Eq[d_0\eta_{n+1}] \wedge Eq[d_{n+1}\eta_{n+1}] \\ &= Eq[\eta_n d_0] \wedge Eq[\eta_n d_{n+1}] = d_0^{-1}(H_n(\mathbb{X})) \wedge d_{n+1}^{-1}(H_n(\mathbb{X})) \end{aligned}$$

Moreover, using Corollary 4.3 and the fact that the images preserve joins, we find that

$$d_0\left(\bigvee_{0 < i < j \leq n+1} (D_i \wedge D_j)\right) = \bigvee_{0 \leq i < j \leq n} (D_i \wedge D_j),$$

which by the induction hypothesis is equal to $H_n(\mathbb{X})$. In the same way we see that

$$d_{n+1}\left(\bigvee_{0\leq i<j<n+1}(D_i \wedge D_j)\right) = H_n(\mathbb{X}).$$

Combining all these identities, we obtain

$$H_{n+1}(\mathbb{X}) = \left(D_0 \vee \bigvee_{0<i<j\leq n+1}(D_i \wedge D_j)\right) \wedge \left(D_{n+1} \vee \bigvee_{0\leq i<j<n+1}(D_i \wedge D_j)\right).$$

From there we already see that

$$H_{n+1}(\mathbb{X}) \geq \bigvee_{0\leq i<j\leq n+1}(D_i \wedge D_j).$$

For the converse inequality, first note that

$$\bigvee_{0<i<j\leq n+1}(D_i \wedge D_j) \leq \left(D_{n+1} \vee \left(\bigvee_{0\leq i<j<n+1}(D_i \wedge D_j)\right)\right),$$

and thus, since the lattice of equivalence relations of X_{n+1} is modular, we have

$$H_{n+1}(\mathbb{X}) = \bigvee_{0<i<j\leq n+1}(D_i \wedge D_j) \vee \left(D_0 \wedge \left(D_{n+1} \vee \bigvee_{0\leq i<j<n+1}(D_i \wedge D_j)\right)\right).$$

Now to conclude the proof it is enough to prove that

$$D_0 \wedge \left(D_m \vee \bigvee_{0\leq i<j<m}(D_i \wedge D_j)\right) = \bigvee_{0<j\leq m}(D_0 \wedge D_j) \tag{4.1}$$

for all $m \geq 1$, which we will do by induction. The case where $m = 1$ is trivial, so let us now assume that (4.1) holds for some m. Then we have

$$\begin{aligned}
&d_m\left(D_0 \wedge \left(D_{m+1} \vee \bigvee_{0\leq i<j<m+1}(D_i \wedge D_j)\right)\right)\\
&\quad\leq d_m(D_0) \wedge \left(d_m(D_{m+1}) \vee \bigvee_{0\leq i<j<m+1} d_m(D_i \wedge D_j)\right)\\
&\quad= D_0 \wedge \left(D_m \vee \bigvee_{0\leq i<j<m}(D_i \wedge D_j)\right)
\end{aligned}$$

and by the induction hypothesis this is equal to

$$\bigvee_{0<j\leq m}(D_0 \wedge D_j) = d_m\left(\bigvee_{0<j\leq m+1}(D_0 \wedge D_j)\right).$$

As a consequence we have

$$D_0 \wedge \left(D_{m+1} \vee \bigvee_{0\leq i<j<m+1}(D_i \wedge D_j)\right) \leq D_m \vee \bigvee_{0<j\leq m+1}(D_0 \wedge D_j).$$

It follows that the left-hand side must be equal to

$$D_0 \wedge \left(D_{m+1} \vee \bigvee_{0\leq i<j<m+1}(D_i \wedge D_j)\right) \wedge \left(D_m \vee \bigvee_{0<j\leq m+1}(D_0 \wedge D_j)\right).$$

Now since $\bigvee_{0<j\leq m+1}(D_0 \wedge D_j) \leq D_0$, using again the modularity law, we find that

$$D_0 \wedge \left(D_m \vee \bigvee_{0<j\leq m+1}(D_0 \wedge D_j)\right) = \bigvee_{0<j\leq m+1}(D_0 \wedge D_j),$$

and this is smaller than $\left(D_{m+1} \vee \bigvee_{0\leq i<j<m+1}(D_i \wedge D_j)\right)$, which concludes the proof. $\square$

Remark 4.12. If the category $\mathcal{C}$ is not only exact Mal'tsev but also arithmetical ([119]), then the category $\mathbf{Grpd}(\mathcal{C})$ coincides with the category of equivalence relations, which is thus a Birkhoff subcategory of $\mathbf{Simp}(\mathcal{C})$. Note that in that case, $H_1(\mathbb{X}) = d_0(D_1 \wedge D_2) = D_0 \wedge D_1$, since direct images preserve intersections of equivalence relations (by Theorem 5.2 of [23]). Accordingly our reflection becomes a reflection of $\mathbf{Simp}(\mathcal{C})$ into $\mathbf{Eq}(\mathcal{C})$.

Since every groupoid is a quotient of an equivalence relation, $\mathbf{Eq}(\mathcal{C})$ is closed under quotients in $\mathbf{Simp}(\mathcal{C})$ if and only if $\mathbf{Eq}(\mathcal{C}) = \mathbf{Grpd}(\mathcal{C})$.

Corollary 4.13. *An exact Mal'tsev category is arithmetical if and only if $\mathbf{Eq}(\mathcal{C})$ is a Birkhoff subcategory of $\mathbf{Simp}(\mathcal{C})$.*

Remark 4.14. Note that, by contrast with the Smith-Pedicchio commutator, whose quotient gives a left adjoint of the forgetful/inclusion functor $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{RG}(\mathcal{C})$, we don't need to assume the existence of any colimits to define $H_1(\mathbb{X})$.

4.3 Characterization of central extensions

Being a Birkhoff subcategory of the exact Mal'tsev category $\mathbf{Simp}(\mathcal{C})$, $\mathbf{Grpd}(\mathcal{C})$ is admissible in the sense of categorical Galois theory, when $\mathcal{F}$ is the class of all regular epimorphisms. In this section we will characterize the central extensions with respect to this reflection.

First, we note that Proposition 4.2 of [88] implies, in our case, that trivial extensions $f\colon \mathbb{X} \to \mathbb{Y}$ are characterized by the property that $F_n \wedge H_n(\mathbb{X}) = \Delta_{X_n}$ for all $n \geq 0$, that is :

$$F_n \wedge \left(\bigvee_{0 \leq i < j \leq n} D_i \wedge D_j \right) = \Delta_{X_n}$$

for $n \geq 2$ and

$$F_1 \wedge d_0(D_1 \wedge D_2) = \Delta_{X_1}.$$

Our characterization of central extensions is then obtained simply by "distributing" the intersection with F_n appearing in these equations with the join or image. In other words we have

Theorem 4.15. *A regular epimorphism $f\colon \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Simp}(\mathcal{C})$ is a central extension with respect to the Galois structure induced by the reflection of $\Pi_1\colon \mathbf{Simp}(\mathcal{C}) \to \mathbf{Grpd}(\mathcal{C})$ if and only if*

$$d_1(F_1 \wedge D_0 \wedge D_2) = \Delta_{X_1} \tag{4.2}$$

and

$$F_n \wedge D_i \wedge D_j = \Delta_{X_n} \quad \forall 0 \leq i < j \leq n \tag{4.3}$$

for all $n \geq 2$.

To prove this we will need a couple of lemmas.

Lemma 4.16. *Let*

$$\begin{tikzcd} \mathbb{P} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathbb{X} \arrow[d, "f"] \\ \mathbb{Z} \arrow[r, "g"'] & \mathbb{Y} \end{tikzcd}$$

be a pullback square of regular epimorphisms in $\mathbf{Simp}(\mathcal{C})$, and let $n \geq 2$ and $0 \leq i < j \leq n$. Let us denote d'_i the face maps of the simplicial object $\mathbb{P}$, and D'_i their kernel pairs. Then

$$F_n \wedge D_i \wedge D_j = \Delta_{X_n} \Leftrightarrow F'_n \wedge D'_i \wedge D'_j = \Delta_{P_n}.$$

Proof. Since pullbacks in $\mathbf{Simp}(\mathcal{C})$ are computed "levelwise" in $\mathcal{C}$, for all n the square

$$\begin{tikzcd} P_n \arrow[r, "g'_n"] \arrow[d, "f'_n"'] & X_n \arrow[d, "f_n"] \\ Z_n \arrow[r, "g"'] & Y_n \end{tikzcd}$$

is a pullback. Since moreover limits commute with limits, in the cube

$$\begin{tikzcd} P_n \arrow[rr, "g'_n"] \arrow[dr, "f'_n"] \arrow[dd, "(d'_i, d'_j)"'] & & X_n \arrow[dr, "f_n"] \arrow[dd, "(d_i, d_j)" near end] & \\ & Z_n \arrow[rr, "g_n" near start] \arrow[dd] & & F \arrow[dd] \\ P_{n-1} \times_{P_{n-2}} P_{n-1} \arrow[rr] \arrow[dr] & & X_{n-1} \times_{X_{n-2}} X_{n-1} \arrow[dr] & \\ & Z_{n-1} \times_{Z_{n-2}} Z_{n-1} \arrow[rr] & & Y_{n-1} \times_{Y_{n-2}} Y_{n-1} \end{tikzcd}$$

the top and bottom faces are pullbacks; one can then show that the square

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, "g'_n"'] & (P_{n-1} \times_{P_{n-2}} P_{n-1}) \times_{Z_{n-1} \times_{Z_{n-2}} Z_{n-1}} Z_n \arrow[d] \\ X_n \arrow[r] & (X_{n-1} \times_{X_{n-2}} X_{n-1}) \times_{Y_{n-1} \times_{Y_{n-2}} Y_{n-1}} Y_n \end{tikzcd}$$

is a pullback, which implies that

$$g'_n(Eq[f'_n] \wedge D'_i \wedge D'_j) = Eq[f_n] \wedge D_i \wedge D_j. \tag{4.4}$$

In particular, $F'_n \wedge D'_i \wedge D'_j = \Delta_{P_n}$ implies that $F_n \wedge D_i \wedge D_j = \Delta_{X_n}$.

For the converse, the equation (4.4) shows already that if $F_n \wedge D_i \wedge D_j = \Delta_{X_n}$, then $F'_n \wedge D'_i \wedge D'_j \leq G'_n$. Since it is also smaller than F'_n, and since f'_n and g'_n are jointly monic by construction, we have $F'_n \wedge D'_i \wedge D'_j = \Delta_{P_n}$. □

Lemma 4.17. *Let $f\colon X \to Y$ be a split epimorphism, with section $s\colon Y \to X$, and let A, B be two equivalence relations on X, with respective coequalizers q_A, q_B. Assume that we have a diagram*

$$\begin{tikzcd} X/A \arrow[d, shift left] & X \arrow[l, "q_A"'] \arrow[r, "q_B"] \arrow[d, shift left, "f"] & X/B \arrow[d, shift left] \\ Y/A' \arrow[u, shift left] & Y \arrow[l, "q_{A'}"] \arrow[r, "q_{B'}"'] \arrow[u, shift left, "s"] & Y/B' \arrow[u, shift left] \end{tikzcd} \tag{4.5}$$

where the vertical downward arrows are split epimorphisms, and the upward and downward squares commute. Then the following conditions are equivalent :

1. $Eq[f] \wedge (A \vee B) = \Delta_X$
2. $Eq[f] \wedge A = \Delta_X = Eq[f] \wedge B$.

Proof. First of all, we have the inequality

$$(Eq[f] \wedge A) \vee (Eq[f] \wedge B) \leq Eq[f] \wedge (A \vee B),$$

which immediately proves that the first condition implies the second.

For the converse, we can complete the diagram (4.5) by taking the pushouts of the top and bottom spans. This yields a cube

$$\begin{tikzcd}
X \arrow[rr, "q_B"] \arrow[dr, shift left, "f"] \arrow[dd, "q_A"'] & & X/B \arrow[dr, shift left] \arrow[dd] & \\
& Y \arrow[ul, shift left, "s"] \arrow[rr, "q_{B'}"] \arrow[dd, "q_{A'}"] & & Y/B' \arrow[ul, shift left] \arrow[dd] \\
X/A \arrow[rr] \arrow[dr, shift left] & & X/(A \vee B) \arrow[dr, shift left] & \\
& Y/A' \arrow[ul, shift left] \arrow[rr] & & Y/(A' \vee B') \arrow[ul, shift left]
\end{tikzcd} \tag{4.6}$$

which is a split epimorphism between double extensions, hence a triple extension. In particular, the square

$$\begin{tikzcd}
X \arrow[r, "q_B"] \arrow[d, "{(q_A, f)}"'] & X/B \arrow[d, "\gamma"] \\
X/A \times_{Y/A'} Y \arrow[r] & X/(A \vee B) \times_{Y/(A' \vee B')} Y/B',
\end{tikzcd}$$

where γ is the morphism induced by the commutativity of the right face of the cube (4.6), is a double extension. Assume now that $Eq[f] \wedge A = \Delta_X = Eq[f] \wedge B$. The first equality implies that (q_A, f) is a monomorphism, hence an isomorphism; then so is γ in the diagram above, and thus the left and right faces of the cube (4.6) are pullbacks. Similarly, the second equality $B \wedge Eq[f]$ implies that the top face is a pullback as well, and then so is the square

$$\begin{tikzcd}
X \arrow[r, "q_{A \vee B}"] \arrow[d, "f"'] & X/(A \vee B) \arrow[d] \\
Y \arrow[r] & Y/(A' \vee B'),
\end{tikzcd}$$

since it is the composite of the top and right faces of (4.6). This implies that $Eq[f] \wedge (A \vee B) = \Delta_X$. □

Proof of Theorem 4.15. Let us consider the diagram

$$\begin{tikzcd}
\Pi_1(\mathbb{X} \times_{\mathbb{Y}} \mathbb{X}) \arrow[d, "\Pi_1(\pi_1)"'] & \mathbb{X} \times_{\mathbb{Y}} \mathbb{X} \arrow[l, "\eta_{\mathbb{X}\times_{\mathbb{Y}}\mathbb{X}}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{X} \arrow[d, "f"] \\
\Pi_1(\mathbb{X}) & \mathbb{X} \arrow[l, "\eta_{\mathbb{X}}"] \arrow[r, "f"'] & \mathbb{Y}.
\end{tikzcd}$$

Now assume first that f is a central extension, so that the left-hand square is a pullback. Since by construction $\Pi_1(\mathbb{X} \times_{\mathbb{Y}} \mathbb{X})$ is an internal groupoid, (4.3) holds for $\Pi_1(\pi_1)$, and then by Lemma 4.16 it also holds for π_1 and thus for f.

Assuming now that (4.3) holds, then again by Lemma 4.16 it also holds with $\pi_1 \colon \mathbb{X} \times_{\mathbb{Y}} \mathbb{X} \to \mathbb{X}$, so that

$$Eq[(\pi_1)_n] \wedge D'_i \wedge D'_j = \Delta_{X_n \times_{Y_n} X_n} \quad \forall 0 \leq i < j \leq n.$$

But π_1 is a split epimorphism in the category of simplicial objects of $\mathcal{C}$. Thus in particular, for all $0 \leq i < j \leq n$, $(\pi_1)_n$ and $D'_i \wedge D'_j$ satisfy the assumptions of Lemma 4.17, and thus we have

$$Eq[(\pi_1)_n] \wedge H_n(\mathbb{X} \times_{\mathbb{Y}} \mathbb{X}) = Eq[(\pi_1)_n] \wedge \left(\bigvee_{0 \leq i < j \leq n} D'_i \wedge D'_j \right) = \Delta_{X_n \times_{Y_n} X_n}.$$

This implies that the left-hand square is a pullback; thus π_1 is a trivial extension, and f is a central extension. □

In fact, we can characterize central extensions with a slightly different set of conditions, which in turn translate in a more geometric criterion.

Proposition 4.18. *A regular epimorphism $f \colon \mathbb{X} \to \mathbb{Y}$ in $\mathbf{Simp}(\mathcal{C})$ is a central extension with respect to the Galois structure induced by the reflection of $\Pi_1 \colon \mathbf{Simp}(\mathcal{C}) \to \mathbf{Grpd}(\mathcal{C})$ if and only if*

$$F_n \wedge \bigwedge_{i \neq k} D_i = \Delta_{X_n} \quad \forall 0 \leq k \leq n \tag{4.7}$$

for all $n \geq 2$.

Proof. By Theorem 4.15 it suffices to prove that (4.7) holding for all $n \geq 2$ is equivalent to (4.3) holding for all $n \geq 2$. It is easily seen that (4.3) implies (4.7) for all $n \geq 2$, and that the two are equivalent for $n = 2$.

To conclude the proof, it then suffices to prove inductively that (4.3) holds for all n, assuming (4.7) holds for all n.

Let us thus assume that (4.3) holds for n. Given $i < j$, if k is distinct from i and j, then for some i' and j' we have

$$d_k(F_{n+1} \wedge D_i \wedge D_j) = F_n \wedge D_{i'} \wedge D_{j'} = \Delta_{X_n},$$

and thus $F_{n+1} \wedge D_i \wedge D_j \leq D_k$. Since we also have $F_{n+1} \wedge D_i \wedge D_j \leq F_{n+1}, D_i, D_j$, we find that

$$F_{n+1} \wedge D_i \wedge D_j \leq F_{n+1} \wedge \bigwedge D_i = \Delta_{X_{n+1}}.$$

$\square$

Theorem 4.19. *An extension $f\colon \mathbb{X} \to \mathbb{Y}$ in* **Simp**$(\mathcal{C})$ *is central with respect to the reflection of* **Simp**$(\mathcal{C})$ *into* **Grpd**$(\mathcal{C})$ *if and only if f is an exact fibration at all dimensions $n \geq 2$ in the sense of Glenn ([65]), i.e. all squares*

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \lambda^n_k}\downarrow & & \downarrow{\scriptstyle \lambda^n_k} \\ \Lambda^n_k(\mathbb{X}) & \xrightarrow[\Lambda^n_k(f)]{} & \Lambda^n_k(\mathbb{Y}) \end{array}$$

are pullbacks.

Proof. The Kan condition implies that these squares are all double extensions, so that all $\theta^n_k\colon X_n \to \Lambda^n_k(\mathbb{X}) \times_{\Lambda^n_k(\mathbb{Y})} Y_n$ are regular epimorphisms. Since $F_n \wedge \bigwedge_{i \neq k} D_i$ coincides with the kernel pair of θ^n_k, the conditions (4.7) are equivalent to all θ^n_k being isomorphisms. $\square$

4.4 Comparison with simplicial sets

As noted before, the left adjoint to the nerve functor between groupoids and simplicial sets is the *fundamental groupoid* functor [64]. For a simplicial set $\mathbb{X}$ which satisfies the Kan condition, also called a *quasigroupoid*, this left adjoint can alternatively be described as the *homotopy groupoid* (see [9, 98]). One defines the homotopy relation on X_1 by saying that two elements (or 1-simplices) $f, g \in X_1$ are homotopic if and only if there exists $\alpha \in X_2$ such that $d_0(\alpha) = f$, $d_1(\alpha) = g$ and $d_2(\alpha) = s_0 d_1(f) = s_0 d_1(g)$. This is always a reflexive relation (since for a given f one can take $\alpha = s_0 f$), and using the Kan condition one can then prove that this is actually an equivalence relation. The homotopy groupoid is then the groupoid whose objects are just the elements of

X_0, arrows are homotopy classes of 1-simplices, identities defined by the classes of degenerate 1-simplices, and composition defined by the existence of fillers for $(2, 1)$-horns (with two sided inverses defined by the existence of fillers for the outer horns).

This relation can be interpreted in any regular category as follows : first take the pullback

$$\begin{tikzcd} X_2 \times_{X_1} X_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_0 \arrow[d, "s_0"] \\ X_2 \arrow[r, "d_2"'] & X_1, \end{tikzcd} \tag{4.8}$$

and then factorize the map $(d_0, d_1)\pi_1 \colon X_0 \times_{X_1} X_2 \to X_1 \times X_1$ as a regular epimorphism $q \colon P \to R$ followed by a monomorphism $r = (\rho_1, \rho_2) \colon R \to X_1 \times X_1$, so that R is a relation on X_1. As in the case of sets, this is a reflexive relation; indeed, the simplicial identities imply that

$$(\rho_1, \rho_2)(q(d_1, s_0)) = (d_0, d_1)\pi_2(d_1, s_0) = (d_0, d_1)s_0 = (1_{X_1}, 1_{X_1}).$$

This relation is in fact equal to $d_0(D_1 \wedge D_2)$ whenever $\mathbb{X}$ satisfies the Kan condition, as we shall now see. In fact it will be helpful to prove a slightly more general result:

Lemma 4.20. *Given any regular epimorphism $f \colon \mathbb{X} \to \mathbb{Y}$ between two simplicial objects in a regular category $\mathcal{C}$, if we take the pullback*

$$\begin{tikzcd} X_1 \times_{Y_1} Y_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y_0 \arrow[d, "s_0^{\mathbb{Y}}"] \\ X_1 \arrow[r, "f_1"'] & Y_1, \end{tikzcd}$$

then $d_0(D_1 \wedge F_1)$ is equal to the regular image of $(d_0, d_1)\pi_1 \colon X_1 \times_{Y_1} Y_0 \to X_0 \times X_0$.

The case where f is the morphism $\epsilon_{\mathbb{X}} \colon Dec(\mathbb{X}) \to \mathbb{X}$ gives the desired identity.

Proof. Consider the diagram

$$\begin{tikzcd}[column sep=small]
Y_1 \times_{Y_2} X_2 \arrow[rr, "\pi_1'"] \arrow[dr, "\pi_2'"'] \arrow[dd, dotted, "d_0^{\mathbb{Y}} \times_{d_0^{\mathbb{Y}}} d_0^{\mathbb{X}}"'] & & X_2 \arrow[rr, "(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})"] \arrow[dr, "f_2"] \arrow[dd, "d_0^{\mathbb{X}}" near end] & & X_1 \times X_1 \arrow[dd, "d_0^{\mathbb{X}} \times d_0^{\mathbb{X}}"] \\
& Y_1 \arrow[rr, "s_1^{\mathbb{Y}}" near start] \arrow[dd, "d_0^{\mathbb{Y}}" near start] & & Y_2 \arrow[dd, "d_0^{\mathbb{Y}}" near start] & \\
Y_0 \times_{Y_1} X_1 \arrow[rr, "\pi_1" near end] \arrow[dr, "\pi_2"'] & & X_1 \arrow[rr, "(d_0^{\mathbb{X}}, d_1^{\mathbb{X}})" near start] \arrow[dr, "f_1"] & & X_0 \times X_0 \\
& Y_0 \arrow[rr, "s_0^{\mathbb{Y}}"'] & & Y_1 &
\end{tikzcd} \tag{4.9}$$

where the top and bottom faces of the cube are pullbacks. Since all the vertical arrows are split by a degeneracy map s_0, and the horizontal maps commute with these sections, the dotted arrow is a split epimorphism as well. In particular, the image factorization of $(d_0^{\mathbb{X}}, d_1^{\mathbb{X}})\pi_1$ is the same as that of $(d_0^{\mathbb{X}}, d_1^{\mathbb{X}})\pi_1(d_0 \times d_0) = (d_0^{\mathbb{X}} \times d_0^{\mathbb{X}})(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})\pi_1'$. If we prove that the image of $(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})\pi_1'$ in $X_1 \times X_1$ is the equivalence relation $D_1 \wedge F_1$, then it would follow that the image of $(d_0^{\mathbb{X}} \times d_0^{\mathbb{X}})(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})\pi_1'$ is $d_0(D_1 \wedge F_1)$, which would concludes the proof.

Since we have a decomposition of f_2 given by the diagram

$$\begin{tikzcd}
X_2 \arrow[dr, "\theta_0^2"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\lambda_0^2(\mathbb{X})"'] & & \\
& \Lambda_0^2(\mathbb{X}) \times_{\Lambda_0^2(\mathbb{Y})} Y_2 \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "\lrcorner" very near start] & Y_2 \arrow[d, "\lambda_0^2(\mathbb{Y})"] \\
& \Lambda_0^2(\mathbb{X}) \arrow[r, "\Lambda_0^2(f)"'] & \Lambda_0^2(\mathbb{Y}),
\end{tikzcd}$$

we can rewrite the top pullback in (4.9) as the upper rectangle in the following diagram :

$$\begin{tikzcd}
X_2 \times_{Y_2} Y_1 \arrow[r, "q"] \arrow[d, "\pi_1'"'] \arrow[dr, phantom, "\lrcorner" very near start] & P \arrow[r] \arrow[d, "m"] \arrow[dr, phantom, "\lrcorner" very near start] & Y_1 \arrow[d, "s_1^{\mathbb{Y}}"] \\
X_2 \arrow[r, "\theta_0^2"] \arrow[d, "\lambda_0^2(\mathbb{X})"'] & \Lambda_0^2(\mathbb{X}) \times_{\Lambda_0^2(\mathbb{Y})} Y_2 \arrow[r, "\varphi_2"'] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & Y_2 \arrow[d, "\lambda_0^2(\mathbb{Y})"] \\
\Lambda_0^2(\mathbb{X}) \arrow[r, equal] & \Lambda_0^2(\mathbb{X}) \arrow[r, "\Lambda_0^2(f)"'] & \Lambda_0^2(\mathbb{Y})
\end{tikzcd}$$

Now since $\Delta_{Y_1} = (d_1^{\mathbb{Y}}, d_2^{\mathbb{Y}})s_1^{\mathbb{Y}} \colon Y_1 \to Y_1 \times Y_1$, the composition $\lambda_0^2(\mathbb{Y})s_1$ is a monomorphism, and thus so is $\varphi_1 m$. By the Kan condition, $\lambda_0^2(\mathbb{X})$ is the regular epimorphism in the factorization of $(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})\colon X_2 \to X_1 \times X_1$, so that P is the image of the map $(d_1^{\mathbb{X}}, d_2^{\mathbb{X}})\pi_1'$. On the other hand, the right-hand rectangle above coincides with the left-hand square in the rectangle

$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail, "\varphi_1 m"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "\Delta"] \\
\Lambda_0^2(\mathbb{X}) \arrow[r, "\Lambda_0^2(f)"'] & \Lambda_0^2(\mathbb{Y}) \arrow[r, tail] & Y_1 \times Y_1.
\end{tikzcd}$$

Since the two squares are pullbacks, the whole rectangle is one as well. But this is the same as the outer rectangle in

$$\begin{tikzcd}
P \arrow[r] \arrow[d, tail, "\varphi_1 m"'] \arrow[dr, phantom, "\lrcorner", very near start] & F_1 \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "\Delta"] \\
\Lambda_0^2(\mathbb{X}) \arrow[r] & X_1 \times X_1 \arrow[r, "f_1 \times f_1"'] & Y_1 \times Y_1,
\end{tikzcd}$$

where the two squares are again pullbacks. Thus P coincides with the intersection $D_1 \wedge F_1$, which concludes the proof. □

Remark 4.21. If one sees a Kan complex as a quasigroupoid or ∞-groupoid, then the left adjoint to the nerve or inclusion functor $\mathbf{Grpd} \to \mathbf{Kan}$ is in a sense a “strictification”, which turns quasigroupoids into actual groupoids.

The equivalence relation $d_0(D_1 \wedge D_2 \wedge F_2)$ which appears in our characterization of central extensions admits an alternative construction, similar to that of $H_1(\mathbb{X})$. More precisely, if we take now L to be the limit of the lower part of the diagram

$$\begin{tikzcd}
 & & & L \arrow[dlll, dotted, "\rho_1"'] \arrow[d, dotted, "\rho_2"] \arrow[drrr, dotted, "\rho_3"] & & & \\
X_0 \arrow[dr, "s_0"'] & & & X_2 \arrow[dll, "d_2"] \arrow[dr, "f_2"'] & & & Y_1 \arrow[dll, "s_0"] \\
 & X_1 & & & Y_2 & &
\end{tikzcd} \tag{4.10}$$

(with the dotted arrows forming the limit cone) then $d_0(D_1 \wedge D_2 \wedge F_2)$ is equal to the image of the map $(d_0, d_1)\rho_2\colon L \to X_1 \times X_1$.

Indeed, the limit in diagram (4.10) can also be obtained as the pullback

$$\begin{tikzcd} L \arrow[r, "{(\rho_1,\rho_3)}"] \arrow[d, "\rho_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times Y_1 \arrow[d, "s_0\times s_0"] \\ X_2 \arrow[r, "{(d_2,f_2)}"'] & X_1 \times Y_2. \end{tikzcd}$$

Now the image of the map (d_2, f_2) is the pullback $X_1 \times_{Y_1} Y_2$ of f_1 along d_2. Moreover, we have

$$\begin{aligned} f_0\rho_1 &= d_0 s_0 f_0 \rho_1 = d_0 f_1 s_0 \rho_1 = d_0 f_1 d_2 \rho_2 = d_0 d_2 f_2 \rho_2 \\ &= d_0 d_2 s_0 \rho_3 = d_1 d_0 s_0 \rho_3 = d_1 \rho_3, \end{aligned}$$

so that (ρ_1, ρ_3) factors through $X_0 \times_{Y_0} X_1$. Thus the pullback square above factorizes as a rectangle

$$\begin{tikzcd} L \arrow[r, "{(\rho_1,\rho_3)}"] \arrow[d, "\rho_2"'] & X_0 \times_{Y_0} Y_1 \arrow[r] \arrow[d, "s_0\times_{s_0} s_0"] & X_0 \times Y_1 \arrow[d, "s_0\times s_0"] \\ X_2 \arrow[r, "{(d_2,f_2)}"'] & X_1 \times_{Y_1} Y_2 \arrow[r] & X_1 \times Y_2, \end{tikzcd}$$

and one can easily show that the right-hand square is a pullback, and as a consequence so is the left-hand side square. But this square is exactly the pullback that appears if we apply Lemma 4.20 to the induced map $(\epsilon_{\mathbb{X}}, Dec(f))\colon Dec(\mathbb{X}) \to \mathbb{X} \times_{\mathbb{Y}} Dec(\mathbb{Y})$, which is a regular epimorphism between simplicial objects, because the square

$$\begin{tikzcd} Dec(\mathbb{X}) \arrow[r, "Dec(f)"] \arrow[d, "\epsilon_{\mathbb{X}}"'] & Dec(\mathbb{Y}) \arrow[d, "\epsilon_{\mathbb{Y}}"] \\ \mathbb{X} \arrow[r, "f"'] & \mathbb{Y} \end{tikzcd}$$

is a double extension in $\mathbf{Simp}(\mathcal{C})$; indeed the square

$$\begin{tikzcd} X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d^{\mathbb{X}}_{n+1}"'] & Dec(\mathbb{Y}) \arrow[d, "d^{\mathbb{Y}}_{n+1}"] \\ X_n \arrow[r, "f_n"'] & Y_n \end{tikzcd}$$

is a double extension in $\mathcal{C}$ for all $n \in \mathbb{N}$. Thus Lemma 4.20 implies that the two constructions are equal.

4.5 Relative monotone-light factorization

If $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ is a Galois structure where $\mathcal{F}$ is the class of all morphisms in $\mathcal{C}$, admissibility is equivalent to the the reflector I being semi-left-exact in the sense of [37]. In that case the class $\mathcal{M}$ of trivial extensions is a full reflective subcategory of $\mathbf{Ext}_{\mathcal{F}}(\mathcal{C}) = \mathbf{Arr}(\mathcal{C})$, and the reflection of a morphism $f\colon X \to Y$ of $\mathcal{C}$ into this class is given by the morphism f'' in the diagram

$$\begin{tikzcd}
X \arrow[dr, "f'"] \arrow[drr, bend left, "\eta_X"] \arrow[ddr, bend right, "f"'] & & \\
& B \times_{UI(B)} UI(X) \arrow[r] \arrow[d, "f''"] \arrow[dr, phantom, "\lrcorner"] & UI(X) \arrow[d, "UI(f)"] \\
& B \arrow[r, "\eta_B"'] & UI(B).
\end{tikzcd}$$

Here f' is the unit $f \to f'' = U_B I_B(f)$, and thus $I(f) = I_B(f')$ is an isomorphism. Moreover in that case the classes $\mathcal{E}$ of morphisms inverted by I and the class $\mathcal{M}$ are orthogonal to one another, and thus the two classes form a factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$ [37]. The trivial extensions are then stable under pullbacks, but the class $\mathcal{E}$ does not have this property in general. In order to obtain a stable factorization system, one can localize $\mathcal{M}$ and stabilize $\mathcal{E}$, as explained in [33]; this means that we replace $\mathcal{E}$ by the class $\mathcal{E}'$ of maps for which every pullback along a monadic extension is in $\mathcal{E}$, and $\mathcal{M}$ by the class $\mathcal{M}^*$ of maps f that are locally in $\mathcal{M}$, in the sense that there exists a monadic extension p such that $p^*(f) \in \mathcal{M}$. In the context of Galois Theory these are precisely the central extensions. The two classes $\mathcal{E}'$ and $\mathcal{M}^*$ are orthogonal, but in general they do not form a factorization system. When this is the case, the resulting factorization system is called the *monotone-light factorization system* $(\mathcal{E}', \mathcal{M}^*)$ associated with Γ.

In the case where $\mathcal{F}$ is no longer the class of all morphisms in $\mathcal{C}$, it need not be true that every morphism admits an $(\mathcal{E}, \mathcal{M})$-factorization. Nevertheless, this is still true for extensions; it is then natural to extend the notion of factorization system to the case where only *some* morphisms have a factorization. This was done by Chikhladze in [39] :

Definition 4.22. If $\mathcal{C}$ is a category and $\mathcal{F}$ a class of morphisms of $\mathcal{C}$ containing the identities and closed under composition and pullbacks, a *relative factorization system* for $\mathcal{F}$ consists of two classes $\mathcal{E}$ and $\mathcal{M}$ of maps such that

1. $\mathcal{E}$ and $\mathcal{M}$ contain identities and are closed under composition with isomorphisms
2. $\mathcal{E}$ and $\mathcal{M}$ are orthogonal to one another
3. $\mathcal{M} \subset \mathcal{F}$
4. every morphism f in $\mathcal{F}$ can be written as me for some $m \in \mathcal{M}$ and $e \in E$.

Then any admissible Galois structure $\Gamma = (\mathcal{C}, \mathcal{X}, I, U, \mathcal{F})$ yields a relative factorization system for $\mathcal{F}$ with $\mathcal{E}$ and $\mathcal{M}$ consisting of the maps inverted by I and the trivial extensions, respectively. When moreover this factorization system can be stabilized as described above, then the stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ is called a *relative monotone-light factorization system* for $\mathcal{F}$.

In order to prove that our Galois structure admits a relative monotone-light factorization system, we use the following criterion, due to Carboni, Janelidze, Kelly and Paré in the absolute case and to Chikhladze in the relative case :

Proposition 4.23 ([33, 39]). *Let $(\mathcal{C}, \mathcal{X}, I, U\mathcal{F})$ be an admissible Galois structure. The class $\mathcal{F}$ admits monotone-light factorization if for each object B of $\mathcal{C}$ there is an effective $\mathcal{F}$-descent morphism $p\colon C \to B$ where C is a stabilizing object, i.e. an object such that if $h = me$ is the $(\mathcal{E}, \mathcal{M})$-factorization of any morphism $h\colon X \to C$, then any pullback of e along a map in $\mathcal{F}$ is still in $\mathcal{E}$.*

Example 4.24.
1. In [39], Chikhladze showed that the Galois structure induced by the nerve functor between groupoids and Kan complexes and the class of Kan fibrations as extensions admits a relative monotone-light factorization system; in fact this example was his motivation for defining relative factorization systems.
2. When $\mathcal{C}$ is an exact Mal'tsev category, the Galois structure of Example 1.67 admits a relative monotone-light factorization system [42]. In this case the functors stably inverted by Π_0 are precisely the final functors, so that this relative factorization system coincides with the restriction of the comprehensive factorization [15] to regular epimorphisms.

We will prove that, in our case, the shifting $Dec(\mathbb{X})$ of a simplicial object $\mathbb{X}$ is always stabilizing. For this it suffices to prove that aspherical objects are stabilizing since we have :

Proposition 4.25 ([53], Proposition 3.9)**.** *Any simplicial object that is contractible and also satisfies the Kan condition is aspherical.*

As a consequence, if $\mathbb{X}$ satisfies the Kan condition, then its shifting $Dec(\mathbb{X})$ is aspherical.

Lemma 4.26. *If $\mathbb{Y}$ is aspherical, then $\mathbb{Y}$ is stabilizing : given any morphism $f\colon \mathbb{X} \to \mathbb{Y}$, the induced map $(f, \eta_{\mathbb{X}})\colon \mathbb{X} \to \mathbb{Y} \times_{\Pi_1(\mathbb{Y})} \Pi_1(\mathbb{X})$ is stably in $\mathcal{E}$.*

Proof. To simplify the diagrams, we denote $\mathbb{P} = \mathbb{Y} \times_{\Pi_1(\mathbb{Y})} \Pi_1(\mathbb{X})$. Let us consider a pullback square

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "g'"] \arrow[d, "h"'] & \mathbb{X} \arrow[d, "{(f,\eta_{\mathbb{X}})}"] \\ \mathbb{Z} \arrow[r, "g"'] & \mathbb{P} \end{tikzcd} \tag{4.11}$$

with g a regular epimorphism in $\mathbf{Simp}(\mathcal{C})$.

We need to prove that $\Pi_1(h)\colon \Pi_1(\mathbb{Q}) \to \Pi_1(\mathbb{Z})$ is invertible. Since it is a map between internal groupoids, it is enough to prove that $\Pi_1(h)_0$ and $\Pi_1(h)_1$ are invertible. Note that the functor Π_1 leaves the objects X_0 unchanged, and thus (f_0, η_0) is an isomorphism, and thus so are h_0 and $\Pi_1(h)_0$. So we only need to prove that $\Pi_1(h)_1$ is an isomorphism.

Since $\mathbf{Grpd}(\mathcal{C})$ is a Birkhoff subcategory of $\mathbf{Simp}(\mathcal{C})$ and h is a regular epimorphism, the square

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\eta_{\mathbb{Q}}"] \arrow[d, "h"'] & \Pi_1(\mathbb{Q}) \arrow[d, "\Pi_1(h)"] \\ \mathbb{Z} \arrow[r, "\eta_{\mathbb{Q}}"'] & \Pi_1(\mathbb{Z}) \end{tikzcd}$$

is a double extension in $\mathbf{Simp}(\mathcal{C})$, and thus the square

$$\begin{tikzcd} Q_1 \arrow[r, "(\eta_{\mathbb{Q}})_1"] \arrow[d, "h_1"'] & \frac{Q_1}{H_1(\mathbb{Q})} \arrow[d, "\overline{h_1}"] \\ Z_1 \arrow[r, "(\eta_{\mathbb{Z}})_1"'] & \frac{Z_1}{H_1(\mathbb{Z})} \end{tikzcd} \tag{4.12}$$

is a (regular) pushout in $\mathcal{C}$. This already proves that $\overline{h_1} = \Pi_1(h)_1$ is a regular epi. Now if there exists a map $t\colon Z_1 \to Q_1/H_1(\mathbb{Q})$ such that $th_1 = (\eta_{\mathbb{Q}})_1$, then using the universal property of the pushout (4.12) we can construct a retraction for $\overline{h_1}$, which proves that it is an isomorphism. So we are left to prove that such a map t exists ; since h_1 is a regular epimorphism, it is enough to prove that $Eq[h_1] \leq H_1(\mathbb{Q})$.

To prove this, we denote $\psi_1, \psi_2 \colon Eq[h_1] \to Q_1$ the two projections of the kernel pair. Then the commutativity of (4.11) (or rather, the corresponding commutative square involving h_1 in $\mathcal{C}$) implies that

$$g_1'(Eq[h_1]) \leq Eq[(f_1, (\eta_{\mathbb{X}})_1] = F_1 \wedge H_1(\mathbb{X}).$$

Using the fact that $\mathbb{Y}$ is aspherical, we can show that in fact this is equal to $= d_0(D_1 \wedge D_2 \wedge F_2)$ (see the following lemma).

As a consequence, we know that there must exist an arrow $\alpha \colon A \to L$ and a regular epimorphism $p \colon A \to Eq[h_1]$ such that $(d_0, d_1)\rho_2\alpha = (g_1' \times g_1')(\psi_1, \psi_2)p$, where L and ρ_2 are defined by the limiting cone (4.10).

We now prove that $(f_2, \eta_2)\rho_2\alpha$ factors through a degeneracy of $\mathbb{P}$. More precisely, we prove that

$$(f_2, \eta_2)\rho_2\alpha = s_0^{\mathbb{P}} d_0^{\mathbb{P}} (f_2, \eta_2)\rho_2\alpha. \tag{4.13}$$

Since the degeneracy map $s_0^{\mathbb{P}}$ is induced by those of $\Pi_1(\mathbb{X})$ and $\mathbb{Y}$, it is enough to prove that $f_2\rho_2\alpha$ and $\eta_2\rho_2\alpha$ factorize in the same manner through $s_0^{\mathbb{Y}}$ and $s_0^{\Pi_1(\mathbb{X})}$ respectively.

By construction we must have

$$s_0^{\mathbb{Y}} d_0^{\mathbb{Y}} f_2\rho_2\alpha = s_0^{\mathbb{Y}} d_0^{\mathbb{Y}} s_0^{\mathbb{Y}} \rho_3\alpha = s_0^{\mathbb{Y}} \rho_3\alpha = f_2\rho_2\alpha.$$

On the other hand we have

$$d_0^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha = d_0^{\Pi_1(\mathbb{X})} s_0^{\Pi_1(\mathbb{X})} d_0^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha$$

by the simplicial identities, and

$$\begin{aligned} d_1^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha &= \eta_1 d_1^{\mathbb{X}} \rho_2\alpha = \eta_1 g_1' \psi_2 p = g_1 h_1 \psi_2 p = g_1 h_1 \psi_1 p \\ &= \eta_1 g_1' \psi_1 p = \eta_1 d_0^{\mathbb{X}} \rho_2\alpha = d_0^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha \\ &= d_1^{\Pi_1(\mathbb{X})} s_0^{\Pi_1(\mathbb{X})} d_0^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha. \end{aligned}$$

By construction the two maps $d_0^{\Pi_1(\mathbb{X})}, d_1^{\Pi_1(\mathbb{X})} \colon \frac{X_2}{H_2(\mathbb{X})} \to \frac{X_1}{H_1(\mathbb{X})}$ are jointly monic, and thus these equalities imply that

$$\eta_2\rho_2\alpha = s_0^{\Pi_1(\mathbb{X})} d_0^{\Pi_1(\mathbb{X})} \eta_2\rho_2\alpha,$$

and this in turn implies that (4.13) hold. From this we find that

$$\begin{aligned}(f_2,\eta_2)\rho_2\alpha &= s_0^{\mathbb{P}} d_0^{\mathbb{P}}(f_2,\eta_2)\rho_2\alpha\\ &= s_0^{\mathbb{P}}(f_1,\eta_1)d_0^{\mathbb{X}}\rho_2\alpha\\ &= s_0^{\mathbb{P}}(f_1,\eta_1)g_1'\psi_1 p\\ &= s_0^{\mathbb{P}} g_1 h_1\psi_1 p\\ &= g_2 s_0^{\mathbb{Z}} h_1\psi_1 p.\end{aligned}$$

Since Q_2 is the pullback of g_2 along (f_2,η_2), there exists a unique map $\alpha'\colon A\to Q_2$ such that $h_2\alpha' = s_0^{\mathbb{Z}}h_1\psi_1 p$ and $g_2'\alpha'=\rho_2\alpha$. From this, we find that

$$g_1' d_0^{\mathbb{Q}}\alpha' = d_0^{\mathbb{X}} g_2'\alpha' = d_0^{\mathbb{X}}\rho_2\alpha = g_1'\psi_1 p$$

and

$$h_1 d_0^{\mathbb{Q}}\alpha' = d_0^{\mathbb{Z}} h_2\alpha' = d_0^{\mathbb{Z}} s_0^{\mathbb{Z}} h_1\psi_1 p = h_1\psi_1 p,$$

and since g_1' and h_1 are jointly monic, we have $d_0^{\mathbb{Q}}\alpha'=\psi_1 p$, and similarly $d_1^{\mathbb{Q}}\alpha'=\psi_2 p$.

Now we prove that $d_2^{\mathbb{Q}}\alpha' = s_0^{\mathbb{Q}}d_0^{\mathbb{Q}}d_2^{\mathbb{Q}}\alpha'$; from the definition of Q_1 it suffices to check that the identity holds after composition with h_1 and g_1'. We have

$$\begin{aligned}h_1 s_0^{\mathbb{Q}} d_0^{\mathbb{Q}} d_2^{\mathbb{Q}}\alpha' &= d_2^{\mathbb{Z}} h_2 s_0^{\mathbb{Q}} d_0^{\mathbb{Q}}\alpha' = d_2^{\mathbb{Z}} s_0^{\mathbb{Z}} d_0^{\mathbb{Z}} h_2\alpha' = s_0^{\mathbb{Z}} d_1^{\mathbb{Z}} d_0^{\mathbb{Z}} s_0^{\mathbb{Z}} h_1\psi_1 p\\ &= s_0^{\mathbb{Z}} d_1^{\mathbb{Z}} h_1\psi_1 p = d_2^{\mathbb{Z}} s_0^{\mathbb{Z}} h_1\psi_1 p = d_2^{\mathbb{Z}} h_2\alpha' = h_1\alpha'\end{aligned}$$

and

$$\begin{aligned}g_1' s_0^{\mathbb{Q}} d_0^{\mathbb{Q}} d_2^{\mathbb{Q}}\alpha' &= s_0^{\mathbb{X}} d_0^{\mathbb{X}} d_2^{\mathbb{X}} g_2'\alpha' = s_0^{\mathbb{X}} d_0^{\mathbb{X}} d_2^{\mathbb{X}}\rho_2\alpha = s_0^{\mathbb{X}} d_0^{\mathbb{X}} s_0^{\mathbb{X}}\rho_1\alpha\\ &= s_0^{\mathbb{X}}\rho_1\alpha = d_2^{\mathbb{X}}\rho_2\alpha = d_2^{\mathbb{X}} g_2'\alpha' = g_1' d_2^{\mathbb{Q}}\alpha'\end{aligned}$$

Thus α' factorizes through the pullback of s_0^Q along d_2^Q, and thus $(\psi_1,\psi_2)p = (d_0^{\mathbb{Q}}, d_1^{\mathbb{Q}})\alpha'$ factorizes through the inclusion of $H_1(\mathbb{Q})$ in $Q_1\times Q_1$, which concludes the proof. $\square$

Lemma 4.27. *If $\mathbb{Y}$ is a simplicial object aspherical at dimension 3, and $f\colon\mathbb{X}\to\mathbb{Y}$ is a regular epimorphism in* $\mathbf{Simp}(\mathcal{C})$*, then*

$$d_0(D_1\wedge D_2)\wedge F_1 = d_0(D_1\wedge D_2\wedge F_2).$$

Note that this equality means that an extension whose codomain is aspherical is trivial if and only if it is central.

Proof. The inequality

$$d_0(D_1 \wedge D_2 \wedge F_2) \leq d_0(D_1 \wedge D_2) \wedge F_1$$

always holds. To prove the converse, we consider the inclusion $\varphi = (\varphi_1, \varphi_2)$ of the equivalence relation $d_0(D_1 \wedge D_2) \wedge F_1$ into $X_1 \times X_1$. Since this relation is smaller than $d_0(D_1 \wedge D_2)$, by the characterization given in Proposition 1.6 and the alternative construction of $d_0(D_1 \wedge D_2)$ given in section 4.4, there must exist a regular epimorphism $p\colon Z \to d_0(D_1 \wedge D_2) \wedge F_1$ and a morphism $\alpha = (\alpha_1, \alpha_2)\colon Z \to X_2 \times_{X_1} X_0$ such that $d_0\alpha_1 = \varphi_1 p$ and $d_1\alpha_1 = \varphi_2 p$; and since moreover it is smaller than F_1 we have $f_1 d_0 \alpha_1 = f_1 d_1 \alpha_1$, which can be rewritten $d_0 f_2 \alpha_1 = d_1 f_2 \alpha_1$.

Now consider the maps

$$\begin{aligned} y_0 = y_1 &= s_1 d_0 f_2 \alpha_1 = s_1 d_1 f_2 \alpha_1 \\ y_2 &= s_0 d_1 f_2 \alpha_1 \\ y_3 &= f_2 \alpha_1. \end{aligned}$$

One can check that the identity $d_i y_j = d_{j-1} y_i$ holds for all $0 \leq i < j \leq 3$, so that these maps determine a map y from Z to the third simplicial kernel $K_3(\mathbb{Y})$, and we can consider the pullback

$$\begin{tikzcd} Z' \arrow[r, "p'"] \arrow[d, "\alpha'"'] & Z \arrow[d, "y"] \\ Y_3 \arrow[r, "\kappa_3"'] & K_3(\mathbb{Y}). \end{tikzcd}$$

$\mathbb{Y}$ being aspherical at dimension 3 means that κ_3 is a regular epimorphism, and as a consequence so is p'. Consider now the maps

$$x_0 = s_1 d_0 \alpha_1 p'$$

$$x_1 = s_1 d_1 \alpha_1 p'$$

$$x_3 = \alpha_1 p',$$

from Z' to X_2. One can check that the identity $d_i x_j = d_{j-1} x_i$ holds for all $i < j$ and $i \neq 2 \neq j$, thus they determine a map $x\colon Z' \to \Lambda^3_2(\mathbb{X})$; and moreover we have

$$d_i \alpha' = \mu_i \kappa_3 \alpha' = \mu_i y p' = y_i p' = f_2 x_i$$

for $i = 0, 1, 3$, which implies that $\lambda^3_2 \alpha' = \Lambda^3_2(f) x$. Thus x and α' induce a map $Z' \to \Lambda^3_2(\mathbb{X}) \times_{\Lambda^3_2(\mathbb{Y})} Y_3$. Consider then the pullback

$$\begin{array}{ccc} Z'' & \xrightarrow{p''} & Z' \\ {\scriptstyle \alpha''}\downarrow & & \downarrow{\scriptstyle (x,\alpha')} \\ X_3 & \xrightarrow[\theta_2^3]{} & \Lambda_2^3(\mathbb{X}) \times_{\Lambda_2^3(\mathbb{Y})} Y_3. \end{array}$$

Since θ_2^3 is a regular epimorphism, so is p'', and by construction we have $d_i\alpha'' = x_i p''$ for $i = 0, 1, 3$ and $f_3\alpha'' = \alpha' p''$. Now the map $d_2\alpha'' \colon Z'' \to X_2$ is such that

$$f_2 d_2 \alpha'' = d_2 f_3 \alpha'' = d_2 \alpha' p'' = y_2 p' p'' = s_0 d_1 f_2 \alpha_1 p' p''$$

and

$$d_2 d_2 \alpha'' = d_2 d_3 \alpha'' = d_2 x_3 p'' = d_2 \alpha_1 p' p'' = s_0 \alpha_2 p' p'',$$

hence there exists a unique map $\beta \colon Z'' \to L$ (where L is defined by (4.10)) such that $\rho_1\beta = \alpha_2 p' p''$, $\rho_2\beta = d_2\alpha''$ and $\rho_3\beta = f_1 d_1 \alpha_1 p' p''$. Now we can check that

$$\begin{aligned} (d_0, d_1)\rho_2\beta &= (d_0, d_1) d_2 \alpha'' = (d_1 d_0, d_1 d_1)\alpha'' = (d_1 x_0, d_1 x_1) p'' \\ &= (d_1 s_1 d_0, d_1 s_1 d_1)\alpha_1 p' p'' = (d_0, d_1)\alpha_1 p' p'' \\ &= (\varphi_1, \varphi_2) p p' p''. \end{aligned}$$

This proves that $d_0(D_1 \wedge D_2) \wedge F_1 \leq d_0(D_1 \wedge D_2 \wedge F_2)$. □

As a consequence, we then have

Theorem 4.28. *The Galois structure* $(\mathbf{Simp}(\mathcal{C}), \mathbf{Grpd}(\mathcal{C}), \Pi_1, U, \mathcal{F})$ *admits a relative monotone-light factorization system* $(\mathcal{E}', \mathcal{M}^*)$*, where* $\mathcal{E}'$ *is the class of maps stably inverted by* Π_1 *and* $\mathcal{M}^*$ *is the class of central extensions of this Galois structure.*

4.6 Truncated simplicial objects and weighted commutators

For all $n \geq 2$, we can define a nerve functor $\mathbf{Grpd}(\mathcal{C}) \hookrightarrow \mathbf{Simp}_n(\mathcal{C})$; this amounts to compose the usual nerver functor $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$ with the truncation functor $\mathbf{Simp}(\mathcal{C}) \to \mathbf{Simp}_n(\mathcal{C})$. The characterization of groupoids in truncated simplicial objects is then identical. Moreover, the construction of the equivalence relations $H_n(\mathbb{X})$ does not depend on the objects X_m for $m > n$. Thus $\mathbf{Grpd}(\mathcal{C})$ can also be seen as a Birkhoff subcategory of $\mathbf{Simp}_n(\mathcal{C})$, with the reflection defined in the same way, in the sense that the reflectors commute with the truncation functor

$\mathbf{Simp}(\mathcal{C}) \to \mathbf{Simp}_n(\mathcal{C})$. The characterization of central extensions also extends in the same way. Note that for $n = 2$, truncated simplicial sets coincide with internal *precategories* in the sense of [82].

The forgetful functor $\mathbf{Grpd}(\mathcal{C}) \hookrightarrow \mathbf{Simp}_1(\mathcal{C}) = \mathbf{RG}(\mathcal{C})$ also coincides with the composition of the nerve functor with the truncation functor $\mathbf{Simp}(\mathcal{C}) \to \mathbf{RG}(\mathcal{C})$, and it is also fully faithful by Theorem 1.21. On the other hand, this time the reflection does not commute with the truncation, as the construction of $H_1(\mathbb{X})$ depends on X_2 and the face maps $X_2 \to X_1$. We know that the reflection $\mathbf{RG}(\mathcal{C}) \to \mathbf{Grpd}(\mathcal{C})$ is obtained by taking the quotient of X_1 by the Smith-Pedicchio commutator $[D_0, D_1]_{SP}$, and this commutator is preserved by regular images, and is always smaller than the intersection of D_0 and D_1; as a consequence, we always have the inequalities

$$[D_0, D_1]_{SP} \leq H_1(\mathbb{X}) = d_2(D_0 \wedge D_1) \leq D_0 \wedge D_1. \tag{4.14}$$

It turns out that this reflection can also be obtained by applying our results, at least when the category $\mathcal{C}$ is finitely cocomplete.

Indeed, in that case the truncation functor $\mathbf{Simp}(\mathcal{C}) \to \mathbf{RG}(\mathcal{C})$ is right adjoint to the 1-skeleton functor Sk_1 [48]. Since the inclusion $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{RG}(\mathcal{C})$ is the composition of the nerve functor and the truncation Tr_1, the functor $\Pi_1 Sk_1$ must be a left adjoint to this inclusion. Thus our work can be used to give an alternative description of the Smith-Pedicchio commutator as the equivalence relation $H_1(Sk_1(X_1, X_0, d_0, d_1, s_0))$.

Let us make this construction explicit. The object X_2 that appears in the construction of $Sk_1(X_1, X_0, d_0, d_1, s_0)$ is the pushout $X_1 +_{X_0} X_1$ of $s_0 \colon X_0 \to X_1$ along itself, with s_0 and s_1 the two canonical maps $X_1 \to X_1 +_{X_0} X_1$. In order to satisfy the simplicial identities $d_0 s_0 = 1$ and $d_0 s_1 = s_0 d_0$, we must define $d_0 = [1, s_0 d_0] \colon X_1 +_{X_0} X_1 \to X_1$; similarly, we must have $d_1 = [1, 1]$ and $d_2 = [s_0 d_1, 1]$.

In the case where $\mathcal{C}$ is not only exact Mal'tsev and cocomplete but also semi-abelian, the correspondence between equivalence relations and normal subobjects allows us to easily translate the results of the previous sections in terms of normal subobjects, by replacing every kernel pair by the kernel of the corresponding morphism.

In the case where $X_0 = 0$ is the zero object in $\mathcal{C}$, X_2 is simply the coproduct $X_1 + X_1$, and the face maps are just the maps $[1, 0], [1, 1], [0, 1]$. Then our construction of $d_1(K[d_0] \wedge K[d_2])$ is nothing but the Higgins commutator $[X_1, X_1]_H$. More generally, $d_1(D_0 \wedge D_2)$ coincides with a weighted commutator :

Theorem 4.29. *When $\mathcal{C}$ is a semi-abelian category, the (normal) subobject $d_1(K[d_0] \wedge K[d_2])$, where $d_0 = [1, s_0d_0]$ and $d_2 = [s_0d_1, 1]$, coincides with the weighted commutator $[K[d_0], K[d_1]]_{s_0\colon X_0 \to X_1}$.*

Proof. Let us denote $k_0\colon K[d_0] \to X_1$ the kernel of $d_0\colon X_1 \to X_0$, and similarly k_1 for the kernel of d_1.

Consider the following commutative diagram :

$$\begin{tikzcd}
X_0 + K[d_0] \arrow[r, shift right, "{[1,0]}"'] \arrow[d, "{[s_0,k_0]}"'] & X_0 \arrow[l, shift right, "\iota_{X_0}"'] \arrow[r, shift left, "\iota_{X_0}"] \arrow[d, equal] & X_0 + K[d_1] \arrow[l, shift left, "{[1,0]}"] \arrow[d, "{[s_0,k_1]}"] \\
X_1 \arrow[r, shift right, "d_0"'] & X_0 \arrow[l, shift right, "s_0"'] \arrow[r, shift left, "s_0"] & X_1. \arrow[l, shift left, "d_1"]
\end{tikzcd}$$

The vertical maps are regular epimorphisms by Proposition 1.29, and thus the induced map

$$\gamma\colon X_0 + K[d_0] + K[d_1] \to X_1 +_{X_0} X_1$$

between the pushouts of the upper and lower spans (i.e. the cokernel pairs of ι_1 and s_0) is also a regular epimorphism. This gives a commutative cube

$$\begin{tikzcd}
X_0 + K_0 + K_1 \arrow[rr, "{[\iota_1,0,\iota_2]}"] \arrow[dr, "{[\iota_1,\iota_2,0]}"] \arrow[dd, "\gamma"'] & & X_0 + K[d_1] \arrow[dr, "{[1,0]}"] \arrow[dd, "{[s_0,k_1]}"' near start] & \\
& X_0 + K[d_0] \arrow[rr, crossing over, "{[1,0]}" near start] & & X_0 \arrow[dd, equal] \\
X_1 +_{X_0} X_1 \arrow[rr, "{[1,s_0d_0]}"' near start] \arrow[dr, "{[s_0d_1,1]}"'] & & X_1 \arrow[dr, "d_1"] & \\
& X_1 \arrow[rr, "d_0"'] \arrow[from=uu, crossing over, "{[s_0,k_0]}" near start] & & X_0
\end{tikzcd}$$

where every edge is a regular epimorphism. In fact this cube is a triple extension, as it can be seen as a split morphism between (vertical) double extensions. As a consequence, we have

$$\gamma(K[\psi]) = K[(d_0, d_2)] = K[d_0] \wedge K[d_2].$$

Now we also have

$$d_1\gamma = [1, 1] \circ ([s_0, k_0] + [s_0, k_1]) = [s_0, k_0, k_1],$$

and thus the weighted commutator $[K[d_0], K[d_1]]_{X_0}$, which is by definition the image of $K[\psi]$ under $[s_0, k_0, k_1]$, coincides with $d_1(K[d_0] \wedge K[d_2])$, which completes the proof. $\square$

Corollary 4.30. *A reflexive graph in a semi-abelian category $\mathcal{C}$ is an internal groupoid if and only if $[K[d_0], K[d_1]]_{s_0\colon X_0 \to X_1} = 0$.*

In particular, this weighted commutator $[K[d_0], K[d_1]]_{s_0\colon X_0 \to X_1}$ is equal to the Ursini commutator $[K[d_0], K[d_1]]_{Urs}$.

We have shown that when taking the 1-skeleton of a reflexive graph, the first inequality of (4.14) is an equality, so that $H_1(\mathbb{X})$ is as small as possible. In the same way, when taking the 1-coskeleton of a reflexive graph, the induced equivalence relation $H_1(\mathbb{X})$ turns out to be equal to $D_0 \wedge D_1$, so that this time $H_1(\mathbb{X})$ is as big as possible. In fact we can prove something a bit more general :

Proposition 4.31. *If $\mathbb{X}$ is a simplicial object aspherical at dimension 2, i.e. if $\kappa_2\colon X_2 \to K_2(\mathbb{X})$ is a regular epimorphism, then $d_0(D_1 \wedge D_2) = D_0 \wedge D_1$.*

Proof. Consider the following diagram, where all the squares are pullbacks :

$$\begin{tikzcd}
X_2 \times_{X_1} X_0 \arrow[r, "q"] \arrow[d, "\pi_1"'] & K_2(\mathbb{X}) \times_{X_1} X_0 \arrow[r] \arrow[d] & X_0 \arrow[d, "s_0"] \\
X_2 \arrow[r, "\kappa_2"] \arrow[dr, "(d_0,d_1)"'] & K_2(\mathbb{X}) \arrow[r, "\nu_2"] \arrow[d, "(\nu_0,\nu_1)"'] & X_1 \arrow[d, "(d_0,d_1)"] \\
& D_0 \arrow[r, "d_1 \times d_1"'] & X_0 \times X_0.
\end{tikzcd}$$

By definition $\nu_2\kappa_2 = d_2$, and thus the upper rectangle is the pullback of d_2 along s_0, i.e. it is the same pullback as in (4.8), and thus $d_0(D_1 \wedge D_2)$ is the image of the composition $(d_0, d_1)\pi_1$. Since the upper left square is a pullback and κ_2 is a regular epimorphism by hypothesis, q is a regular epimorphism, and thus the image of this map $(d_0, d_1)\pi_1$ is the image of the middle vertical map. Moreover the right-hand rectangle is the pullback of $d_1 \times d_1$ along Δ_{X_0}, and thus this middle vertical map is a monomorphism, and the corresponding subobject of D_0 coincides with $D_0 \wedge D_1$, which concludes the proof. □

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