2012/31

Relaxations for two-level multi-item lot-sizing problem

Mathieu Van Vyve, Laurence A. Wolsey and Hande Yaman

DISCUSSION PAPER

Center for Operations Research and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

CORE DISCUSSION PAPER
2012/31

Relaxations for two-level multi-item lot-sizing problem

Mathieu VAN VYVE [1], Laurence A. WOLSEY[2]
and Hande YAMAN [3]

August 2012

Abstract

We consider several variants of the two-level lot-sizing problem with one item at the upper level facing dependent demand, and multiple items or clients at the lower level, facing independent demands. We first show that under a natural cost assumption, it is sufficient to optimize over a stock-dominant relaxation. We further study the polyhedral structure of a strong relaxation of this problem involving only initial inventory variables and setup variables. We consider several variants: uncapacitated at both levels with or without start-up costs, uncapacitated at the upper level and constant capacity at the lower level, constant capacity at both levels. We finally demonstrate how the strong formulations described improve our ability to solve instances with up to several dozens of periods and a few hundred products.

Keywords: mixed-integer programming, lot-sizing, extended formulation, multi-level, multi-item.

MSC Classification: 68Q25, 90C11, 90C27, 90C35, 90B05, 90B06

[1] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: Mathieu.vanvyve@uclouvain.be. This author is also member of ECORE, the association between CORE and ECARES.

[2] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: Laurence.wolsey@uclouvain.be.

[3] Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium; Department of Industrial Engineering, Bilkent University, Ankara, Turkey.

This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors. The research of the third author is supported by TUBITAK.

1 Introduction

We study two-level multi-item multi-period planning problems on a finite horizon with time-dependent demand. In this context, multi-level means that there is dependent demand in the system: some goods are consumed by the production of others. We focus on problems with one item at the upper-level facing dependent demand, and multiple items or clients at the lower level, facing independent demands. The two levels can represent different stages of a production process executed at a single location (e.g., making and packing, bulk and end products, component and assembly), but can also represent production and transportation to clients, in which case the problem is known as the one warehouse, multiple retailer (OWMR) problem. One key aspect of the models that we consider is that holding inventory is possible at both levels. We study various polyhedra related to such problems. In particular, we consider the uncapacitated problem, the problem with start-up cost at both levels, and some capacitated variants.

The seminal papers of Wagner and Whitin [29] and Zangwill [30] show how to solve the uncapacitated single-level and multi-level in-series lot-sizing problems in polynomial time. Veinott [27] generalizes the approach to more general product structures leading to non-polynomial-time algorithms. van Hoesel et al. [23] give a polynomial-time algorithm for a two-level problem with constant production capacity at the upper level. Hwang [12] gives polynomial-time algorithms for uncapacitated single-item two-level problems with more general cost structures.

Several important hardness results have been proved. Bitran and Yanasse [7] show that the single-item lot-sizing problem becomes NP-Hard when the production capacity varies over time. Arkin et al. [3] show that the Joint Replenishment Problem (two levels with one item at the upper level without inventory and multiple items at the lower level) is NP-Hard. The one-level multi-item problem with a joint capacity constraint generalizes the problem of optimizing over a single-node flow set and is NP-Hard. Since most realistic problems involve at least one of these three characteristics (varying capacity, divergent product structure, joint capacity) and are therefore NP-Hard, much research in the last 30 years has been devoted to finding (provably) strong reformulations that can then be used in MIP solvers, as opposed to searching for direct optimization algorithms. The present paper follows this line of research of which Pochet and Wolsey [20] provides an in-depth survey.

For single-item lot-sizing, many polyhedral results have been obtained both for the basic uncapacitated model [6, 13] and for extensions including backlogging [14, 16], start-ups [25], constant capacity [18], increasing capacities [21], sales, or a combination of these [28]. These results can be classified into two categories: linear description of the convex hull of solutions in the original variable space, usually of exponential size and accompanied by an efficient separation algorithm on the one hand, and tight extended formulation involving additional variables, usually of polynomial size on the other hand. For the latter, Van Vyve and Wolsey [26] show how to create and manage a trade-off between strength and size of these extended formulations.

Within this line of research Pochet and Wolsey [19] is crucial in terms of motivation. They show that the non-speculative cost assumption, which often is satisfied in practice and has been shown to translate into faster optimization algorithms [2, 10, 24], has an analog in polyhedral combinatorics. Specifically, under this cost assumption, to solve the problem, it suffices to optimize over the stock-dominant of the solution set, without requiring non-negativity of production. The resulting polyhedron has a much simpler polyhedral structure and is a very strong relaxation of the original model.

For multi-item problems, Clark and Scarf [8] introduced the concept of echelon-stock. This later proved to be key in building strong single-item relaxations of multi-level models leading to efficient branch-and-bound algorithms based on Lagrangian relaxation [1] or cutting plane approaches [17]. Less progress has been made on the polyhedral structure of multi-level models beyond such single-item relaxations. The multi-commodity extended reformulation applicable to any single-source fixed-charge network flow problem is known to be very strong, but it is not tight for in-series models, even for two levels and under the non-speculative cost assumption. Melo and Wolsey [15] give a tight $\mathcal{O}(n^3)$ formulation of the uncapacitated two-level in-series model. Zhang et al. [31] give a partial description of the convex hull of solutions in the original variable space for the same model, allowing also for intermediate independent demand.

To the best of our knowledge, no polyhedral work has been done for multi-level lot-sizing models involving start-ups, capacities, or multiple items at the lower level (beyond single-item relaxations based on the echelon-stock concept). The present work partially fills this gap. Following Pochet and Wolsey [19], we consider stock-dominant relaxations of these multi-level problems that we prove are sufficient to solve the problem under specific cost assumptions.

The rest of the paper is organized as follows. In Section 2 we describe the capacitated two-level lot-sizing model *2LS*, its stock-dominant relaxation *2WW* and the closely related two-level discrete lot-sizing problem *2DLS*, whose polyhedral structure we study in order to obtain a good formulation for *2WW*. We prove that solving *2WW* solves *2LS* under a natural cost assumption. Section 3 is devoted to the polyhedral analysis of several variants of *2DLS*. In Section 3.1 we consider the basic uncapacitated *2DLS-(U,U)* model and give a polynomial-size linear programming (LP) extended formulation, together with its projection onto the original variable space. The next sections extend, sometimes partially, these results in several directions. In Section 3.2 we consider the model *2DLS-(U,U)-SC* that includes start-ups and extend the result obtained for *2DLS-(U,U)*. In Sections 3.3 and 3.4 we derive results for the case with constant capacity limits on production of items at the lower level, and at both levels respectively. In Section 4 we demonstrate how these strong formulations improve our ability to solve several variants of two-level planning problems. We also indicate what may be the best modeling options for instances of very large size. We conclude in the last section by discussing some open problems.

2 The two-level multi-item lot sizing problem and its Wagner-Whitin relaxation

Here we present the problem of interest and the non-speculative relaxations that we will study.

Let n be the length of the planning horizon, I be the set of items at the lower level with $m = |I|$ and 0 be the item at the upper level. We define $I0 = I \cup \{0\}$. For integers a and b, we use $[a, b]$ to denote the set of integers $\{a, \ldots, b\}$ from a to b. We denote the demand in period $j \in [1, n]$ for item $i \in I$ by d^i_j and the setup, production, inventory holding costs and the capacity for item $i \in I0$ and period j by f^i_j, p^i_j, $\tilde{h}^i_j$ and Q^i_j, respectively.

We define x^i_j to be the amount of production of item $i \in I0$ in period $j \in [1, n]$, s^i_j to be the amount of item i in the inventory at the end of period $j \in [0, n]$, and y^i_j to be 1 if a setup for item i takes place in period $j \in [1, n]$ and to be 0 otherwise. We can model the two-level multi-item lot-sizing problem (*2LS*) as follows.

$$z^{2LS} = \min \sum_{i \in I0} \left(\tilde{h}^i_0 s^i_0 + \sum_{j=1}^{n} \left(f^i_j y^i_j + p^i_j x^i_j + \tilde{h}^i_j s^i_j \right) \right) \tag{1}$$

$$\text{s.t. } s^0_{j-1} + x^0_j = \sum_{i \in I} x^i_j + s^0_j \qquad j \in [1, n], \tag{2}$$

$$s^i_{j-1} + x^i_j = d^i_j + s^i_j \qquad i \in I, j \in [1, n], \tag{3}$$

$$x^i_j \leq Q^i_j y^i_j \qquad i \in I0, j \in [1, n], \tag{4}$$

$$s^i_j \geq 0 \qquad i \in I0, j \in [0, n], \tag{5}$$

$$y^i_j \in \{0, 1\} \qquad i \in I0, j \in [1, n], \tag{6}$$

$$x^i_j \geq 0 \qquad i \in I0, j \in [1, n]. \tag{7}$$

Constraints (2) and (3) are balance constraints for item 0 and items in set I, respectively. Constraints (4) relate the production and setup variables and impose the capacity restrictions. Constraints (5)-(7) are variable restrictions. The objective function (1) is the sum of the setup, production and inventory holding costs.

In the sequel, we use a_{ut} to denote $\sum_{j=u}^{t} a_j$ for both variables and data, and $a^+ = \max(a, 0)$.

Substituting $x_j^i = d_j^i + s_j^i - s_{j-1}^i$ for $i \in I$ and $x_j^0 = \sum_{i \in I} x_j^i + s_j^0 - s_{j-1}^0 = \sum_{i \in I0} s_j^i + \sum_{i \in I} d_j^i - \sum_{i \in I0} s_{j-1}^i$ for $j \in [1, n]$ in the variable production costs yields

$$\sum_{i \in I0} \sum_{j=1}^{n} p_j^i x_j^i = \sum_{j=1}^{n} \left(p_j^0 \left(\sum_{i \in I0} s_j^i + \sum_{i \in I} d_j^i - \sum_{i \in I0} s_{j-1}^i \right) + \sum_{i \in I} p_j^i (d_j^i + s_j^i - s_{j-1}^i) \right)$$
$$= \sum_{i \in I} \sum_{j=1}^{n} (p_j^0 + p_j^i) d_j^i + \sum_{j=0}^{n} (p_j^0 - p_{j+1}^0) s_j^0 + \sum_{i \in I} \sum_{j=0}^{n} (p_j^0 - p_{j+1}^0 + p_j^i - p_{j+1}^i) s_j^i,$$

where $p_0^i = p_{n+1}^i = 0$ for all $i \in I0$.

For $j \in [0, n]$, let $h_j^0 = p_j^0 - p_{j+1}^0 + \tilde{h}_j^0$ and $h_j^i = p_j^0 - p_{j+1}^0 + p_j^i - p_{j+1}^i + \tilde{h}_j^i$ for $i \in I$. Also, let $K = \sum_{i \in I} \sum_{j=1}^{n} (p_j^0 + p_j^i) d_j^i$.

Let $1 \leq k \leq t \leq n$, $l(i) \in [t, n]$ for $i \in I$, and (x, s, y) be a feasible solution to *2LS*. Summing up (2) for $j \in [k, t]$ and (3) for $j \in [k, l(i)]$ and $i \in I$ gives $\sum_{i \in I0} s_{k-1}^i + \sum_{j=k}^{t} x_j^0 + \sum_{i \in I} \sum_{j=t+1}^{l(i)} x_j^i = \sum_{i \in I} d_{k,l(i)}^i + s_t^0 + \sum_{i \in I} s_{l(i)}^i$. Since $Q_j^0 y_j^0 \geq x_j^0$ for $j \in [k, t]$, $Q_j^i y_j^i \geq x_j^i$ for $j \in [t+1, l(i)]$ and $i \in I$, $s_t^0 \geq 0$, and $s_{l(i)}^i \geq 0$ for $i \in I$, (x, s, y) satisfies

$$\sum_{i \in I0} s_{k-1}^i + \sum_{j=k}^{t} Q_j^0 y_j^0 + \sum_{i \in I} \sum_{j=t+1}^{l(i)} Q_j^i y_j^i \geq \sum_{i \in I} d_{k,l(i)}^i \qquad 1 \leq k \leq t \leq n, l(i) \in [t, n] \text{ for } i \in I. \quad (8)$$

Similarly, the inequality

$$s_{k-1}^i + \sum_{j=k}^{l} Q_j^i y_j^i \geq d_{kl}^i \qquad i \in I, 1 \leq k \leq l \leq n, \quad (9)$$

is satisfied by any feasible solution (x, s, y). Hence the problem *2WW*

$$z^{2WW} = K + \min \sum_{i \in I0} \left(h_0^i s_0^i + \sum_{j=1}^{n} \left(f_j^i y_j^i + h_j^i s_j^i \right) \right)$$
$$\text{s.t. } (5), (6), (8) \text{ and } (9)$$

is a relaxation of $2LS$. We refer to this relaxation as the Wagner-Whitin relaxation. Next we show that if the costs satisfy a certain condition, this relaxation yields the same optimal value as the original problem.

Proposition 1 *If $h_j^i \geq h_j^0 \geq 0$ for all $i \in I$ and $j \in [0, n]$, then $z^{2LS} = z^{2WW}$.*

Proof. Let (s, y) be an optimal solution to the problem *2WW*. For $i \in I0$, as $h_n^i \geq 0$, there exists an optimal solution to *2WW* with $s_n^i = 0$. For $i \in I$, if there exists $k \in [1, n]$ with $s_{k-1}^i > 0$ and $s_{k-1}^i + \sum_{j=k}^{l} Q_j^i y_j^i > d_{kl}^i$ for all $l \in [k, n]$, then the solution obtained by decreasing s_{k-1}^i and increasing s^0_{k-1} by a small amount does not cost more. If there exists $k \in [1,n]$ with $s^0_{k-1} > 0$ and $\sum_{i\in I0} s^i_{k-1} + \sum_{j=k}^{t} Q^0_j y^0_j + \sum_{i\in I}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j > \sum_{i\in I} d^i_{k,l(i)}$ for all choices of t and $l(i)$ for $i \in I$, then the solution obtained by decreasing s^0_{k-1} by a small amount is feasible and not worse in terms of cost.

Let (s, y) be an optimal solution to *2WW* such that i) for each $i \in I$ and $k \in [1,n]$ with $s^i_{k-1} > 0$, there exists $l \in [k,n]$ with $s^i_{k-1} + \sum_{j=k}^{l} Q^i_j y^i_j = d^i_{kl}$, ii) for each $k \in [1,n]$ with $s^0_{k-1} > 0$, there exist $t \in [k,n]$ and $l(i) \in [t,n]$ for $i \in I$ with $\sum_{i\in I0} s^i_{k-1} + \sum_{j=k}^{t} Q^0_j y^0_j + \sum_{i\in I}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j = \sum_{i\in I} d^i_{k,l(i)}$, and iii) $s^i_n = 0$ for $i \in I0$.

For $i \in I$ and $k \in [1,n]$, let $x^i_k = s^i_k + d^i_k - s^i_{k-1}$. First we show that $x^i_k \geq 0$. If $s^i_{k-1} = 0$, then $x^i_k = s^i_k + d^i_k \geq 0$. If $s^i_{k-1} > 0$, then there exists $l \in [k,n]$ with $s^i_{k-1} = d^i_{kl} - \sum_{j=k}^{l} Q^i_j y^i_j$ and $x^i_k = s^i_k + d^i_k - d^i_{kl} + \sum_{j=k}^{l} Q^i_j y^i_j = s^i_k - d^i_{k+1,l} + \sum_{j=k+1}^{l} Q^i_j y^i_j + Q^i_k y^i_k$. Since $s^i_k + \sum_{j=k+1}^{l} Q^i_j y^i_j \geq d^i_{k+1,l}$ and $Q^i_k y^i_k \geq 0$, we have $x^i_k \geq 0$. Next we show that $x^i_k \leq Q^i_k y^i_k$. If $s^i_k = 0$, then $x^i_k = d^i_k - s^i_{k-1} \leq Q^i_k y^i_k$. If $s^i_k > 0$, then there exists $l \in [k+1,n]$ with $s^i_k = d^i_{k+1,l} - \sum_{j=k+1}^{l} Q^i_j y^i_j$ and $x^i_k = d^i_{k+1,l} - \sum_{j=k+1}^{l} Q^i_j y^i_j + d^i_k - s^i_{k-1} = d^i_{kl} - \sum_{j=k}^{l} Q^i_j y^i_j - s^i_{k-1} + Q^i_k y^i_k$. As $s^i_{k-1} + \sum_{j=k}^{l} Q^i_j y^i_j \geq d^i_{kl}$, we have $x^i_k \leq Q^i_k y^i_k$.

For $k \in [1,n]$, we take $x^0_k = \sum_{i\in I} x^i_k + s^0_k - s^0_{k-1} = \sum_{i\in I0} s^i_k + \sum_{i\in I} d^i_k - \sum_{i\in I0} s^i_{k-1}$. We first show that $x^0_k \geq 0$. f $\sum_{i\in I0} s^i_{k-1} = 0$, then $x^0_k \geq 0$. Otherwise, if there exist $t \in [k,n]$ and $l(i) \in [t,n]$ for $i \in I$ with $\sum_{i\in I0} s^i_{k-1} + \sum_{j=k}^{t} Q^0_j y^0_j + \sum_{i\in I}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j = \sum_{i\in I} d^i_{k,l(i)}$, then

$$
\begin{aligned}
x^0_k =& \sum_{i\in I0} s^i_k + \sum_{i\in I} d^i_k + \sum_{j=k}^{t} Q^0_j y^0_j + \sum_{i\in I}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j - \sum_{i\in I} d^i_{k,l(i)} \\
=& Q^0_k y^0_k + \sum_{i\in I0} s^i_k + \sum_{j=k+1}^{t} Q^0_j y^0_j + \sum_{i\in I}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j - \sum_{i\in I} d^i_{k+1,l(i)} \geq 0.
\end{aligned}
$$

If no such t and $l(i)$ for $i \in I$ exist, then $s^0_{k-1} = 0$. Let $I' = \{i \in I : s^i_{k-1} > 0\}$. For each $i \in I'$, there exists $l(i) \in [k,n]$ with $s^i_{k-1} = d^i_{k,l(i)} - \sum_{j=k}^{l(i)} Q^i_j y^i_j$, and

$$
\begin{aligned}
x^0_k =& \sum_{i\in I0} s^i_k + \sum_{i\in I} d^i_k - \sum_{i\in I'} \left(d^i_{k,l(i)} - \sum_{j=k}^{l(i)} Q^i_j y^i_j \right) \\
=& \sum_{i\in I0\setminus I'} s^i_k + \sum_{i\in I\setminus I'} d^i_k + \sum_{i\in I'} Q^i_k y^i_k + \sum_{i\in I'} \left(s^i_k - d^i_{k+1,l(i)} + \sum_{j=k+1}^{l(i)} Q^i_j y^i_j \right) \geq 0.
\end{aligned}
$$

Now we show that $x^0_k \leq Q^0_k y^0_k$. If $\sum_{i\in I0} s^i_k = 0$, then $x^0_k = \sum_{i\in I} d^i_k - \sum_{i\in I0} s^i_{k-1} \leq Q^0_k y^0_k$ (inequality (8) for $t = k$ and $l(i) = k$ for all $i \in I$). Otherwise, if there exist $t \in [k+1,n]$ and

$l(i) \in [t,n]$ with $\sum_{i \in I0} s_k^i + \sum_{j=k+1}^{t} Q_j^0 y_j^0 + \sum_{i \in I} \sum_{j=t+1}^{l(i)} Q_j^i y_j^i = \sum_{i \in I} d_{k+1,l(i)}^i$, then

$$x_k^0 = \sum_{i \in I} d_{k+1,l(i)}^i - \sum_{j=k+1}^{t} Q_j^0 y_j^0 - \sum_{i \in I} \sum_{j=t+1}^{l(i)} Q_j^i y_j^i + \sum_{i \in I} d_k^i - \sum_{i \in I0} s_{k-1}^i$$
$$= \sum_{i \in I} d_{k,l(i)}^i + Q_k^0 y_k^0 - \sum_{j=k}^{t} Q_j^0 y_j^0 - \sum_{i \in I} \sum_{j=t+1}^{l(i)} Q_j^i y_j^i - \sum_{i \in I0} s_{k-1}^i \leq Q_k^0 y_k^0.$$

If no such t and $l(i)$ for $i \in I$ exist, then $s_k^0 = 0$. Let $I' = \{i \in I : s_k^i > 0\}$. For each $i \in I'$, there exists $l(i) \in [k+1, n]$ with $s_k^i = d_{k+1,l(i)}^i - \sum_{j=k+1}^{l(i)} Q_j^i y_j^i$. In this case,

$$x_k^0 = \sum_{i \in I'} (d_{k+1,l(i)}^i - \sum_{j=k+1}^{l(i)} Q_j^i y_j^i) + \sum_{i \in I} d_k^i - \sum_{i \in I0} s_{k-1}^i$$
$$= Q_k^0 y_k^0 + \sum_{i \in I'} (d_{k,l(i)}^i - \sum_{j=k+1}^{l(i)} Q_j^i y_j^i) + \sum_{i \in I \setminus I'} d_k^i - \sum_{i \in I0} s_{k-1}^i - Q_k^0 y_k^0 \leq Q_k^0 y_k^0,$$

since $\sum_{i \in I'} (d_{k,l(i)}^i - \sum_{j=k+1}^{l(i)} Q_j^i y_j^i) + \sum_{i \in I \setminus I'} d_k^i - \sum_{i \in I0} s_{k-1}^i - Q_k^0 y_k^0 \leq 0$ by inequality (8) with $t = k$ and $l(i) = k$ for all $i \in I \setminus I'$.

Now as $0 \leq x_k^i \leq Q_k^i y_k^i$ for all $i \in I0$ and $k \in [1, n]$, the solution (x, s, y) is feasible for *2LS*. □

Defining X^{2WW} as the set of solutions to (8)–(9) and the associated bound and integrality constraints, and $\bar{X}_k^{2DLS}$ as

$$\sum_{i \in I0} s_{k-1}^i + \sum_{j=k}^{t} Q_j^0 y_j^0 + \sum_{i \in I} \sum_{j=t+1}^{l(i)} Q_j^i y_j^i \geq \sum_{i \in I} d_{k,l(i)}^i \qquad k \leq t \leq n, l(i) \in [t,n] \text{ for } i \in I, \tag{10}$$

$$s_{k-1}^i + \sum_{j=k}^{l} Q_j^i y_j^i \geq d_{kl}^i \qquad i \in I, l \in [k,n], \tag{11}$$

$$s_{k-1} \in \mathbb{R}_+^{m+1}, y \in \{0,1\}^{(m+1)(n-k+1)}, \tag{12}$$

it is easy to see that $X^{2WW} = \bigcap_{k=1}^{n+1} \bar{X}_k^{2DLS}$. Moreover each of the sets $\bar{X}_k^{2DLS}$ is of the same form. It is natural to hope that with a good approximation or an exact formulation for conv$(\bar{X}_k^{2DLS})$, the intersection of these formulations will provide a good approximation of conv(X^{2WW}).

However, in the next section, we will analyze a slightly different set for the following reason. We remark that X^{2WW} may have extreme points that are not feasible for *2LS*. Because of the cost conditions $h_k^0 \leq h_k^i$ for all $i \in I$ and $k \in [0, n-1]$, these extreme points will not be unique optimal solutions. The same is true for $\bar{X}_k^{2DLS}$. Consider then the set X_k^{2DLS} defined similarly to $\bar{X}_k^{2DLS}$, except that we generate inequalities of the form (10) for all subsets of items $V \subseteq I$

as

$$\sum_{i\in V\cup\{0\}} s^i_{k-1} + \sum_{j=k}^{t} Q^0_j y^0_j + \sum_{i\in V}\sum_{j=t+1}^{l(i)} Q^i_j y^i_j \geq \sum_{i\in V} d^i_{k,l(i)}$$

$$\emptyset \subset V \subseteq I, t\in[k-1,n], l(i)\in[t,n] \text{ for } i\in V. \quad (13)$$

Note that minimizing the objective function $\sum_{i\in I0}(h^i_0 s^i_0 + \sum_{j=1}^n f^i_j y^i_j)$ over X^{2DLS}_1 solves *2LS* when $p^i_j = 0$ for all $j \in [1,n]$, $h^i_0 \geq 0$ and $h^i_j = 0$ for all $j\in[1,n]$ and $i \in I0$. We call this problem the *two-level discrete lot-sizing problem (2DLS)*. In the case of $2DLS$ we do not need the conditions $h^0_0 \leq h^i_0$ for all $i\in I$ to have a valid formulation for *2DLS*, because of the strengthened constraints (13). It is worth noting that this is not true for *2WW*: Proposition 1 does not hold if the assumption that $h^0_k \leq h^i_k$ for all $i \in I$ and $k\in[0,n-1]$ is dropped, even when one replaces constraints (10) by constraints (13) for all k.

3 The two-level discrete lot-sizing problem *2DLS*

In this section, we consider the structure of $X^{2DLS} = X^{2DLS}_1$ when $Q^0 = M$ is large ($M \geq \sum_{i\in I} d^i_{1n}$) except in subsection 3.4. Let e_α denote the α-th unit vector and e_{n+1} the 0-vector in $\mathbb{R}^n$.

Observation 1 *Every extreme point of* $\mathrm{conv}(X^{2DLS})$ *has* $y^0 = e_\alpha$ *for some* $\alpha\in\{1,\dots,n+1\}$.

The following result allows us to largely decompose the problem by item. Let ϕ^i denote the contribution (if any) of item $i\in I$ to the upper level stock s^0_0 .

Proposition 2

$$s^0_0 = \sum_{i\in I}\phi^i, \quad (14)$$

$$\phi^i + s^i_0 + My^0_{1t} + \sum_{j=t+1}^{l} Q^i_j y^i_j \geq d^i_{1l} \qquad i\in I, l\in[1,n], t\in[0,l], \quad (15)$$

$$s^i_0 + \sum_{j=1}^{l} Q^i_j y^i_j \geq d^i_{1l} \qquad i\in I, l\in[1,n], \quad (16)$$

$$s_0\in\mathbb{R}^{m+1}_+, y\in\{0,1\}^{(m+1)n}, \phi\in\mathbb{R}^m_+. \quad (17)$$

is an extended formulation for X^{2DLS}.

Proof: Suppose that (s_0, y, ϕ) satisfies (15)-(17). Let $V\subseteq I$, $k = 1$, $t\in[0,n]$ and $l(i)\in[t,n]$ for $i\in V$. Summing (8) for $l = l(i)$ over $i\in V$ yields $\sum_{i\in V}\phi^i + \sum_{i\in V} s^i_0 + |V|My^0_{1t} +$

$\sum_{i\in V}\sum_{j=t+1}^{l} Q_j^i y_j^i \geq \sum_{i\in V} d_{1l(i)}^i$. As $s_0^0 \geq \sum_{i\in V}\phi^i$, $M \geq \sum_{i\in V} d_{1l(i)}^i$ and y is binary, (s_0, y) satisfies (13). Hence we can conclude that (s_0, y) is in X^{2DLS}.

Let (s_0, y) be an extreme point of conv(X^{2DLS}) with $y^0 = e_\alpha$. Then we know that $s_0^0 = \sum_{i\in I}\max_{l\in[\alpha-1,n]}(d_{1l}^i - \sum_{j=\alpha}^{l} Q_j^i y_j^i - s_0^i)^+$. We can verify that (s_0, y, ϕ) with $\phi^i = \max_{l\in[\alpha-1,n]}(d_{1l}^i - \sum_{j=\alpha}^{l} Q_j^i y_j^i - s_0^i)^+$ for $i \in I$ satisfies (15)-(17). The rest is straightforward. □

3.1 Uncapacitated at both levels *2DLS-(U,U)*

Now we suppose that $Q_j^i = M$ for all $i \in I0$ and $j \in [1, n]$ and we replace the constraints $y^i \in \{0,1\}^n$ by $y^i \in \mathbb{Z}_+^n$ for all $i \in I0$. The constraints (15) now take the form

$$\phi^i + s_0^i + My_{1t}^0 + My_{t+1,l}^i \geq d_{1l}^i.$$

We see that d_l^i is contained in $\phi^i + s_0^i$ if $y_{1t}^0 + y_{t+1,l}^i = 0$ for some $t \in [0, l]$. Taking ζ_l^i to represent $\max_{t\in[0,l]}(1 - y_{1t}^0 - y_{t+1,l}^i)^+$ and δ_l^i to represent $(1 - y_{1l}^i)^+$, one obtains the extended formulation:

$$s_0^0 = \sum_{i\in I}\phi^i, \tag{18}$$

$$\phi^i + s_0^i = \sum_{l=1}^{n} d_l^i \zeta_l^i \qquad i \in I, \tag{19}$$

$$s_0^i = \sum_{l=1}^{n} d_l^i \delta_l^i \qquad i \in I, \tag{20}$$

$$\zeta_l^i \geq \delta_l^i \qquad i \in I, l \in [1, n], \tag{21}$$

$$\zeta_l^i + y_{1t}^0 + y_{t+1,l}^i \geq 1 \qquad i \in I, l \in [1, n], t \in [0, l], \tag{22}$$

$$\delta_l^i + y_{1l}^i \geq 1 \qquad i \in I, l \in [1, n], \tag{23}$$

$$\zeta \in \mathbb{R}_+^{mn}, \delta \in \mathbb{R}_+^{mn}, y \in \mathbb{R}_+^{(m+1)n}, \tag{24}$$

$$\zeta \in \mathbb{Z}^{mn}, \delta \in \mathbb{Z}^{mn}, y \in \mathbb{Z}^{(m+1)n}. \tag{25}$$

Let SC be the set-covering polyhedron described by the constraints (22)-(24) and SC' be $SC \cap (21)$.

Theorem 3 *The polyhedron SC' is integral.*

The proof is in three steps. First we will establish the result for the polyhedron SC when $m = 1$. We then extend this result for all values of m. Finally we show that adding constraints (21) does not create fractional extreme points. Note that the 0-1 constraint matrix associated to SC is neither totally unimodular (TU) nor balanced.

Theorem 4 *The polyhedron SC is integral when $m = 1$.*

Proof: We drop the index i in ζ_l^i and δ_l^i as $m = 1$. For any given non-zero objective function $\min \sum_{u=1}^n h_u^0 \zeta_u + \sum_{u=1}^n h_u^1 \delta_u + \sum_{i=0}^1 \sum_{u=1}^n f_u^i y_u^i$ with bounded optimal value, we determine one inequality among (22)-(24) that is satisfied at equality by all optimal solutions. This proves the result as when the objective function is parallel to a facet, the facet-defining inequality is the only possible answer.

The extreme rays $(y^0, y^1, \zeta, \delta)$ of SC are $(e_j, 0, 0, 0)$, $(0, e_j, 0, 0)$, $(0, 0, e_j, 0)$ and $(0, 0, 0, e_j)$ for $j \in [1, n]$. Hence we need $h^0, h^1, f^0, f^1 \geq 0$ for the problem to be bounded.

If $h^0 = h^1 = 0$, then there exists i, u with $f_u^i > 0$ and all optimal solutions satisfy $y_u^i = 0$. If $f_u^0 < f_{u+1}^0$, then $y_{u+1}^0 = 0$. Therefore, for the remaining cases, we assume that there exists $t \in [0, n]$ such that $f_1^0 \geq f_2^0 \geq \ldots \geq f_t^0 > 0 = f_{t+1}^0 = \ldots = f_n^0$. If $h^0 = 0$ and there exists u with $f_u^0 > 0$, then $y_u^0 = 0$. If $h^0 = f^0 = 0$, then the problem is single-level and the result is known to hold [19].

In the remaining case, there exists l such that $h_l^0 > 0$. Let l be the highest such index. If there exists $k \in [1, l]$ such that $f_k^0 + f_k^1 < h_l^0$ then $\zeta_l = 0$. If $t > l$, then $y_t^0 = 0$. Suppose that $t \leq l$ and $f_k^1 = f_k^0 + f_k^1 \geq h_l^0 > 0$ for $t < k \leq l$. We claim that all optimal solutions satisfy inequality (22) at equality for this choice of t and l. Observe that all variables in the inequality have positive cost, and hence showing the result for all potentially optimal extreme points is sufficient. In every extreme point of SC, $y^0 = e_\alpha$, and $y^1 = e_\beta$ or $y^1 = e_\beta + e_\gamma$ where $\beta \geq \alpha$ and $\gamma < \alpha$. Let $(y^0, y^1, \zeta, \delta)$ be an extreme point of SC. We look at three cases.

1. $\zeta_l = 1$. Then $\beta > l$. If $\alpha \leq t$, then setting $y^0 = e_{t+1}$ improves the cost by $f_\alpha^0 > 0$. If $t + 1 \leq \gamma \leq l$, then setting $y^0 = y^1 = e_\gamma$ and $\zeta_l = 0$ improves the cost by at least $h_l^0 > 0$. Otherwise ($\alpha > t$ and $\gamma < t + 1$ or $\gamma > l$) the inequality (22) is satisfied at equality.

2. $\zeta_l = 0$ and $\alpha \leq t$. If $t + 1 \leq \beta$, then setting $y^0 = e_{t+1}$ improves the cost by $f_\alpha^0 > 0$. Otherwise, the claim holds.

3. $\zeta_l = 0$ and $\alpha \geq t + 1$. If $t + 1 \leq \gamma < \alpha \leq \beta \leq l$, then setting $y^0 = y^1 = e_\gamma$ improves the cost by $f_\beta^1 > 0$. Otherwise, the claim holds.

□

To extend the result to cover multiple items, we first present a somewhat abstract proposition that will then be applied to the set covering problem.

For $k = 1, \ldots, K$, consider the polyhedron P^k

$$Aw^0 + Bw^c \geq \mathbf{1} \quad c = 1, \ldots, k$$
$$w^0 \in R^n_+, \; w^c \in R^{n_1}_+ \quad c = 1, \ldots, k,$$

with $A, B \geq 0$ and $X^k = P^k \cap Z^{N^k}$ with $N^k = n + kn_1$. Suppose that

i. For all k and in every extreme point of $\text{conv}(X^k)$, $\sum_{j=1}^n w^0_j \leq 1$,

ii. for every $(w^0, w^1) \in P^1$ with $\sum_{j=1}^n w^0_j > 1$, there exists a point $(\bar{w}^0, w^1) \in P^1$ such that $\bar{w}^0 \leq w^0$, $\sum_{j=1}^n \bar{w}^0_j = 1$ and $\min(\mathbf{1}, Aw^0) = \min(\mathbf{1}, A\bar{w}^0)$,

iii. P^1 is an integral polyhedron,

iv. $W^\alpha = \{(w^0, w^1) \in R^n_+ \times R^{n_1}_+ : w^0 = e_\alpha, Bw^1 \geq \mathbf{1} - Ae_\alpha\}$ is an integral polyhedron for all $\alpha \in [1, n+1]$.

Proposition 5 *Under the above conditions, P^k is an integral polyhedron for all $k \geq 1$.*

Proof: First we observe that from (i),

$$X^k = \cup_{\alpha=1}^{n+1} (X^k \cap \{w : w^0 = e_\alpha\}) + Z^{N^k}_+.$$

From (iii)

$$P^1 = \text{conv}(X^1) = \text{conv}(\cup_{\alpha=1}^{n+1} \text{conv}(X^1 \cap \{w : w^0 = e_\alpha\})) + R^{N^1}_+$$

and for $k > 1$ we have

$$\text{conv}(X^k) = \text{conv}(\cup_{\alpha=1}^{n+1} \text{conv}(X^k \cap \{w : w^0 = e_\alpha\})) + R^{N^k}_+ \subseteq P^k.$$

By (iv)

$$\text{conv}(X^1 \cap \{w : w^0 = e_\alpha\}) = \{(w^0, w^1) : w^0 = e_\alpha, Bw^1 \geq \mathbf{1} - Ae_\alpha, w^1 \in R^{n_1}_+\}.$$

Now consider a point $(w^0, w^1) \in P^1$. If $\sum_{j=1}^n w^0_j > 1$, replace w^0 by a vector $\bar{w}^0 \in R^n_+$ with $\bar{w}^0 \leq w^0$, $(\bar{w}^0, w^1) \in P^1$, $\sum_{j=1}^n \bar{w}^0_j = 1$ and $\min(\mathbf{1}, Aw^0) = \min(\mathbf{1}, A\bar{w}^0)$. Otherwise set $\bar{w}^0 = w^0$.

Now from the representation of P^1 as the convex hull of the union of polyhedra, we have that there exist $\lambda \in R^{n+1}_+$ with $\sum_{\alpha=1}^{n+1} \lambda_\alpha = 1$ and points $w^{1,\alpha} \in W^\alpha$ for $\alpha = 1, \ldots, n+1$ such that

$$(\bar{w}^0, w^1) = \sum_{\alpha=1}^{n+1} \lambda_\alpha (e_\alpha, w^{1,\alpha})$$

with $\bar{w}^0_\alpha = \lambda_\alpha$ for $\alpha = 1, \ldots, n$.

Now consider a point $(w^0, w^1, \cdots, w^k) \in P^k$ and select $\bar{w}^0$ as above. For each $c = 1, \ldots, k$, the above argument provides points $w^{c,\alpha}$ and weights λ^c_α such that

$$(\bar{w}^0, w^c) = \sum_{\alpha=1}^{n+1} \lambda^c_\alpha (e_\alpha, w^{c,\alpha}).$$

Note that $\lambda^c_\alpha = \lambda_\alpha = \bar{w}^0_\alpha$ for $\alpha = 1, \ldots, n$, i.e., the weights are identical for each $c = 1, \ldots, k$. Now

$$(w^0, w^1, \cdots, w^k) \geq (\bar{w}^0, w^1, \cdots, w^k) = \sum_{\alpha=1}^{n+1} \lambda_\alpha (e_\alpha, w^{1,\alpha}, \cdots, w^{k,\alpha}).$$

Thus we have shown that $P^k \subseteq \mathrm{conv}\big(\cup_{\alpha=1}^{n+1} \mathrm{conv}(X^k \cap \{w : w^0 = e_\alpha\})\big) + R^{Nk}_+$ and thus P^k =conv(X^k). □

Proof of Theorem 3

We first apply the above to the polyhedron SC and its associated set SC_I of integer points.

To demonstrate that SC is integral, we need to check the four conditions of Proposition 5. Here we have $n_1 = n$ and we take $w^0 = y^0$.

i. Every extreme point of conv(SC_I) is of the form $y^0 = e_\alpha$ for some $\alpha \in \{1, \ldots, n+1\}$.

ii. Given $(y^0, y^1) \in P^1$ with $\sum_{j=1}^n y^0_j > 1$, we select $\bar{w}^0$ as follows: $\bar{w}^0$ is lexicographically maximum subject to $0 \leq \bar{w}^0 \leq y^0$ and $\sum_{j=1}^n \bar{w}^0_j = 1$. It is easily verified that $(\bar{w}^0, y^1) \in P^1$.

iii. (22)-(24) is an integral polyhedron for $m = 1$ by Theorem 4 .

iv. W^α is the polyhedron obtained by setting $y^0_\alpha = 1$. After eliminating dominated constraints one obtains for each fixed $i \in I$:

$$y^0 = e_\alpha, \tag{26}$$

$$\zeta^i_l \geq 1 \qquad l \in [1, \alpha - 1], \tag{27}$$

$$\zeta^i_l + y^i_{\alpha,l} \geq 1 \qquad l \in [\alpha, n], \tag{28}$$

$$\delta^i_l + y^i_{1,l} \geq 1 \qquad l \in [1, n], \tag{29}$$

$$\delta^i, \zeta^i, y^i \in R^n_+. \tag{30}$$

We will prove that the constraint matrix associated to (28)–(29) is TU. A matrix B is TU if and only if each subset J of its columns can be partitioned into two sets $J1$ and $J2$ such that for each row r we have $\sum_{k\in J1} b_{rk} - \sum_{k\in J2} b_{rk} \in \{0, 1, -1\}$ [11]. Given a subset of columns J, we put the column associated with the y^i_j variable with the smallest index j into $J1$, the next one into $J2$, the next into $J1$ and so on. Finally we set ζ^i_l and δ^i_l in the opposite set to y^i_k with k the highest index in J smaller than or equal to l (and $J1$ otherwise). It is easily checked that this partition satisfies the property.

Now the integrality of SC follows from Proposition 5.

It remains to show that adding constraints (21) does not create fractional extreme points. For any $J \subseteq I \times [1, n]$, consider the face of SC' where (21) is tight for $(i, l) \in J$ and not tight for $(i, j) \in \bar{J}$. Since any extreme point of SC' is also an extreme point of such a face for some J, showing that this face is integral for any J implies that SC' is integral.

For $(i, l) \in J$, both (23) (dominated by (22)) and $\delta^i_l \geq 0$ can be dropped from the formulation. Then the face reduces to

$$\zeta^i_l = \delta^i_l \qquad (i, l) \in J, \tag{31}$$

$$\zeta^i_l + y^0_{1t} + y^i_{t+1,l} \geq 1 \qquad i \in I, l \in [1, n], t \in [0, l], \tag{32}$$

$$\delta^i_l + y^i_{1l} \geq 1 \qquad (i, l) \in \bar{J}, \tag{33}$$

$$\zeta \in \mathbb{R}^{mn}_+, \delta \in \mathbb{R}^{|\bar{J}|}_+, y \in \mathbb{R}^{(m+1)n}_+, \tag{34}$$

It is easy to see that (32)–(34) is the projection of SC with δ^i_l for $(i, l) \in J$ being the variables projected out. But this last polyhedron has just been proved to be integral.

End of Proof of Theorem 3

We now return to the two-level discrete lot-sizing problem:

$$\min \left\{ \sum_{i \in I0} \left(h^i_0 s^i_0 + \sum_{j=1}^{n} f^i_j y^i_j \right) |(s_0, y) \in X^{2DLS-(U,U)} \right\}.$$

We have shown that it can be solved as a linear program using the extended formulation

$$\min \left\{ \sum_{i \in I0} \left(h^i_0 s^i_0 + \sum_{j=1}^{n} f^i_j y^i_j \right) |(s_0, \phi, y, \zeta, \delta) \text{ satisfying } (18) - (24) \right\}$$

with $\Theta(mn)$ variables and $\Theta(mn^2)$ constraints.

Observation 2 *Because $\zeta^i_l = \max_{t \in [0,l]}(1 - y^0_{1t} - y^i_{t+1,l})^+$ can be rewritten as $\zeta^i_l = \max(\zeta^i_{l-1} - y^i_l, 1 - y^0_{1l})^+$, a more compact linear program with $\Theta(mn)$ constraints is obtained using the constraints*

$$\zeta^i_0 = 1, \tag{35}$$

$$\zeta^i_l \geq 1 - y^0_{1l} \quad i \in I, l \in [1, n], \tag{36}$$

$$\zeta^i_l \geq \zeta^i_{l-1} - y^i_l \quad i \in I, l \in [1, n] \tag{37}$$

in place of (22).

One can also describe the convex hull in the space of the original (s_0, y) variables. By projection, we obtain

Proposition 6 $\text{conv}(X^{2DLS-(U,U)})$ *is given by:*

$$s_0^0 + \sum_{i \in V} s_0^i \geq \sum_{i \in V} \sum_{u=1}^{l(i)} d_u^i (1 - y_{1t(i,u)}^0 - y_{t(i,u)+1,u}^i)$$

$$V \subseteq I, l(i) \in [1,n], t(i,u) \in \{t(i,u-1), u\}, t(i,0) = 0, \text{ for } u \in [1, l(i)] \text{ and } i \in V, \tag{38}$$

$$s_0^i \geq \sum_{u=1}^{l} d_u^i (1 - y_{1l}^i) \qquad i \in I, l \in [1,n], \tag{39}$$

$$s_0 \in R_+^{m+1}, y \in R_+^{(m+1)n}. \tag{40}$$

Finally observe that the reformulation (35)-(37) of Observation 2 leads to an $\Theta(nm)$ separation algorithm for the inequalities (38). Given $(\bar{s}_0, \bar{y})$, one calculates

$$\bar{\zeta}_l^i = \max\left(1 - \bar{y}_{1l}^0, \bar{\zeta}_{l-1}^i - \bar{y}_l^i\right)^+,$$

$$\bar{\phi}^i = \left(\sum_{u=1}^{n} d_u^i \bar{\zeta}_u^i - \bar{s}_0^i\right)^+$$

and obtains a violated inequality if

$$\bar{s}_0^0 < \sum_{i \in I} \bar{\phi}^i.$$

3.2 Start-up costs *2DLS-(U,U)-SC*

Here we consider the uncapacitated problem with start-ups at both levels. A start-up occurs in the first period of an interval of set-ups. Start-ups often arise at the lower level in make-pack problems. To represent start-ups, we introduce the variables $z_j^i = 1$ if $y_j^i = 1$ and $y_{j-1}^i = 0$, and $z_j^i = 0$ otherwise. Thus we consider the set $X^{2DLS-(U,U)-SC}$ that is the intersection of $X^{2DLS-(U,U)}$ and the additional constraints

$$z_j^i \geq y_j^i - y_{j-1}^i \qquad i \in I0, j \in [1,n], \tag{41}$$

$$z_j^i \leq y_j^i \qquad i \in I0, j \in [1,n], \tag{42}$$

$$z_j^i \in \mathbb{R}_+ \qquad i \in I0, j \in [1,n], \tag{43}$$

$$y_0^i \in \mathbb{Z}_+ \qquad i \in I0. \tag{44}$$

Following a similar proof in three steps, see Appendix, one obtains a result similar to Theorem 3.

Theorem 7 *A tight and compact extended formulation for $X^{2DLS-(U,U)-SC}$ is given by:*

$$s_0^0 = \sum_{i \in I} \phi^i, \tag{45}$$

$$\phi^i + s_0^i = \sum_{l=1}^{n} d_l^i \zeta_l^i \qquad i \in I, \tag{46}$$

$$s_0^i = \sum_{l=1}^{n} d_l^i \delta_l^i \qquad i \in I, \tag{47}$$

$$\zeta_l^i \geq \delta_l^i \qquad i \in I, l \in [1,n], \tag{48}$$

$$\zeta_l^i + y_1^i + z_{2,l}^i \geq 1 \qquad i \in I, l \in [1,n], \tag{49}$$

$$\zeta_l^i + y_1^0 + z_{2t}^0 + y_{t+1}^i + z_{t+2,l}^i \geq 1 \qquad i \in I, t \in [1,l-1], l \in [1,n], \tag{50}$$

$$\zeta_l^i + y_1^0 + z_{2l}^0 \geq 1 \qquad i \in I, l \in [1,n], \tag{51}$$

$$\delta_l^i + y_1^i + z_{2l}^i \geq 1 \qquad i \in I, l \in [1,n], \tag{52}$$

$$z_j^i \geq y_j^i - y_{j-1}^i \qquad i \in I0, j \in [1,n], \tag{53}$$

$$z_j^i \leq y_j^i \qquad i \in I0, j \in [1,n], \tag{54}$$

$$\zeta, \delta \in \mathbb{R}_+^{mn}, y \in \mathbb{R}_+^{(m+1)(n+1)}, z \in \mathbb{R}_+^{(m+1)n}. \tag{55}$$

As above, one can also obtain a formulation with an order of magnitude less constraints, the convex hull in the original (s, y, z) space and a $\Theta(mn)$ separation algorithm.

3.3 Constant capacities for final products *2DLS-(U,CC)*

Here we suppose that $Q_j^0 = M$ and $Q_j^i = Q^i$ for all $j \in [1,n]$ and all $i \in I$. As one again has $y^0 = e_\alpha$ for some $\alpha \in [1, n+1]$ in all extreme points, we define the sets

$$X^\alpha = X^{2DLS-(U,CC)} \cap \{y^0 : y_{1,\alpha-1}^0 = 0, y_\alpha^0 \geq 1\},$$

so the problem decomposes into $n+1$ subproblems

$$X^{2DLS-(U,CC)} = \bigcup_{\alpha=1}^{n+1} X^\alpha.$$

Our goal now is to describe $\text{conv}(X^\alpha)$. Combined with the classical result of Balas [4] this will lead to a description of $\text{conv}(X^{2DLS-(U,CC)})$.

Note that once y^0 is fixed, the set X^α decomposes by item giving $X^\alpha = \bigcap_{i \in I} X^{\alpha,i}$, where

$X^{\alpha,i}$ is the set:

$$y^0_{1,\alpha-1} = 0, \tag{56}$$

$$y^0_\alpha \geq 1, \tag{57}$$

$$\phi^i + s^i_0 \geq d^i_{1,\alpha-1}, \tag{58}$$

$$\phi^i + s^i_0 + Q^i y^i_{\alpha l} \geq d^i_{1l} \quad l \in [\alpha, n], \tag{59}$$

$$s^i_0 + Q^i y^i_{1l} \geq d^i_{1l} \quad l \in [1, n], \tag{60}$$

$$\phi^i, s^i_0 \geq 0, y^i \in \{0,1\}^n. \tag{61}$$

To describe $\mathrm{conv}(X^{\alpha,i})$, we suppose without loss of generality that $Q^i = 1$, and we observe that $X^{\alpha,i}$ can be described as the intersection of two mixing sets, see [9]. Following the standard approach described in [9] to obtain an extended formulation of such sets, we observe that in an extreme point, $\phi^i + s^i_0$ and $s^i_0 \mod 1$ must take either the value 0, or one of the n values $d^i_{1l} \mod 1$. Let $f_1 > f_2 > \cdots > f_{\hat{n}}$ represent these distinct fractional parts in decreasing order, set $f_0 = 1$ and $f_{\hat{n}+1} = 0$, and let $\pi(l)$ be the index in $[1, \hat{n}]$ with $f_{\pi(l)} \equiv d^i_{1l} \mod 1$ for $l \in [1, n]$.

Dropping the superscript i, introducing $\bar{y}_t = y_{1t}$ and noting that $\bar{y}_t - \bar{y}_{\alpha-1} = y_{\alpha t}$, the network dual extended formulation for the two mixing sets gives:

$$\phi + s_0 = \sum_{l=0}^{\hat{n}} (f_l - f_{l+1}) \mu^0_l, \tag{62}$$

$$s_0 = \sum_{l=0}^{\hat{n}} (f_l - f_{l+1}) \mu_l, \tag{63}$$

$$\mu^0_{\pi(\alpha-1)} \geq \lfloor d_{1,\alpha-1} \rfloor + 1, \tag{64}$$

$$\mu^0_{\pi(l)} + \bar{y}_l - \bar{y}_{\alpha-1} \geq \lfloor d_{1l} \rfloor + 1 \qquad l \in [\alpha, n], \tag{65}$$

$$\mu^0_{\hat{n}} - \mu^0_0 = 1, \tag{66}$$

$$\mu^0_l - \mu^0_{l-1} \geq 0 \qquad l \in [1, \hat{n}], \tag{67}$$

$$\mu^0_0 \geq 0, \tag{68}$$

$$\mu_{\pi(l)} + \bar{y}_l \geq \lfloor d_{1l} \rfloor + 1 \qquad l \in [1, n], \tag{69}$$

$$\mu_{\hat{n}} - \mu_0 = 1, \tag{70}$$

$$\mu_l - \mu_{l-1} \geq 0 \qquad l \in [1, \hat{n}], \tag{71}$$

$$\mu_0 \geq 0, \tag{72}$$

$$0 \leq \bar{y}_l - \bar{y}_{l-1} \leq 1 \qquad l \in [1, n], \tag{73}$$

$$\mu^0_l - \mu_l \geq 0 \qquad l \in [0, \hat{n}], \tag{74}$$

$$\bar{y}_0 = 0. \tag{75}$$

Consider now the matrix corresponding to the constraints (64)-(75), and call the associated polyhedron $P^{\alpha,i}$. The constraint matrix is not TU because of (74), but we can show integrality as follows.

We first show that the constraint matrix of (64)-(73) is TU, using again the characterization in [11]. Given a subset J of variables, we put all variables $\bar{y}_l$ for $l \in [\alpha, n]$ in $J1$ and all variables μ_l in $J2$. If $\bar{y}_{\alpha-1}$ is in the set J, then we put $\bar{y}_{\alpha-1}$ and all variables μ_l^0 in $J1$. If $\bar{y}_{\alpha-1}$ is not in the set J, then we put all variables μ_l^0 in $J2$. It is easily checked that this partition satisfies the desired property.

Now, in extreme points of $P^{\alpha,i}$, for each l, either (74) is tight and $\mu_l^0 = \mu_l$ implying that (69) is dominated by (65), so that (69) and therefore (74) can be dropped, or (74) itself can be dropped. In either cases, we have just shown that the resulting system of inequalities is TU. Therefore each extreme point of $P^{\alpha,i}$ is contained in a face that is itself an integral polyhedron and thus $P^{\alpha,i}$ is an integral polyhedron.

We have obtained a description of $\mathrm{conv}(X^\alpha)$:

$$y^0_{1,\alpha-1} = 0, y^0_\alpha \geq 1,$$
$$(\phi^i, s^i, y^i) \in P^{\alpha,i} \; i \in I,$$

which can then be written compactly as the polyhedron

$$F^\alpha(s, y, \phi) \geq g^\alpha.$$

Theorem 8 *An extended formulation for* $\mathrm{conv}(X^{2DLS-(U,CC)})$ *is given by:*

$$s_0^0 = \sum_{i \in I} \phi^i, \tag{76}$$
$$y^i = \sum_{\alpha=1}^{n+1} y^{i,\alpha} \qquad i \in I0, \tag{77}$$
$$\phi^i = \sum_{\alpha=1}^{n+1} \phi^{i,\alpha} \qquad i \in I, \tag{78}$$
$$s_0^i = \sum_{\alpha=1}^{n+1} s^{i,\alpha} \qquad i \in I, \tag{79}$$
$$F^\alpha(s^{\cdot,\alpha}, y^{\cdot,\alpha}, \phi^{\cdot,\alpha}) \geq g^\alpha \omega_\alpha \qquad \alpha \in [1, n+1], \tag{80}$$
$$\sum_{\alpha=1}^{n+1} \omega_\alpha = 1, \tag{81}$$
$$\omega \in \mathbb{R}_+^{n+1}. \tag{82}$$

3.4 Production capacities at both levels

Here we assume that the production capacity is identical at both levels and for all items, i.e., $Q^i = Q$ for all $i \in I0$. Alternatively, one can take $Q = \max_{i \in I0} Q^i$ to build such a relaxation.

Let $X^i = \{(\phi^i, s_0^i, y^0, y^i) \in \mathbb{R}_+^2 \times \{0,1\}^{2n} : \phi^i + s_0^i + Qy_{1t}^0 + Qy_{t+1,l}^i \geq d_{1l}$ for $l \in [1,n], t \in [0,l]\}$. Note that if we set $z_l = \min_{t \in [0,l]}(y_{1t}^0 + y_{t+1,l}^i) \in \mathbb{Z}_+^1$, $s = \phi^i + s_0^i$, and $Y_l^0 = y_{1l}^0$, we obtain a mixing set plus additional constraints:

$$\begin{aligned} s + Qz_l &\geq d_{1l} && l \in [1,n], \\ z_l &\leq Y_l^0 && l \in [1,n], \\ z_l &\leq z_{l-1} + y_l^i && l \in [1,n], \\ s \in \mathbb{R}_+, z \in \mathbb{Z}_+^n,& Y^0 \in \mathbb{Z}_+^n, y^i \in \{0,1\}^n, z_0 = 0. \end{aligned}$$

From [9], we know that an extended formulation of the mixing set $s + Qz_l \geq d_{1l}$ $l \in [1,n], s \in \mathbb{R}_+, z \in \mathbb{Z}_+^n$ is of the form $s = F\mu, A(z,\mu) \geq b$ where A is a network dual matrix and b is integer.

Proposition 9 *The following is a tight and compact extended formulation for X^i.*

$$s = F\mu, \tag{83}$$

$$A(z,\mu) \geq b, \tag{84}$$

$$z_l - Y_l^0 \leq 0 \qquad l \in [1,n], \tag{85}$$

$$z_l - z_{l-1} - y_l^i \leq 0 \qquad l \in [1,n], \tag{86}$$

$$0 \leq Y_l^0 - Y_{l-1}^0 \leq 1 \qquad l \in [1,n], \tag{87}$$

$$s \in \mathbb{R}, z \in \mathbb{R}_+^n, Y^0 \in \mathbb{R}_+^n, y^i \in [0,1]^n. \tag{88}$$

Proof: Consider the matrix associated to constraints (84)-(88). Apart from the columns corresponding to the variables y_l^i each of which appears only once, the remaining matrix is a network dual matrix, and hence TU. It follows that the complete matrix is TU. As the right hand sides and bounds are integer, the extended formulation is integral. □

4 Computational study

4.1 Computational results for the two-level lot-sizing problem with start-up costs

In this section we report the results of our computational experiments for the two-level lot-sizing problem with start-up costs. We performed tests with the original formulation (NF) (2)-(7)

and (41),(42), the multicommodity formulation (MCF), see [22], and our extended formulation (EF) given in Theorem 7 and modified as in Observation 2. We also strengthened the natural formulation (NF-WW) and the multi-commodity formulation (MCF-WW) with (l, S) start-up inequalities [25] based on an echelon-stock reformulation, i.e., we used the inequalities

$$s^i_{k-1} \geq \sum_{t=k}^{l} d^i_t \left(1 - y^i_k - z^i_{k+1,t}\right) \qquad i \in I, k \in [1, n], l \in [k, n],$$

$$\sum_{i \in I0} s^i_{k-1} \geq \sum_{i \in I} \sum_{t=k}^{l} d^i_t \left(1 - y^0_k - z^0_{k+1,t}\right) \qquad k \in [1, n], l \in [k, n],$$

and their disaggregated versions

$$\hat{s}^i_{k-1,l} \geq d^i_l \left(1 - y^i_k - z^i_{k+1,l}\right) \qquad i \in I, k \in [1, n], l \in [k, n],$$

$$\hat{s}^{0i}_{k-1,l} + \hat{s}^i_{k-1,l} \geq d^i_l \left(1 - y^0_k - z^0_{k+1,l}\right) \qquad i \in I, k \in [1, n], l \in [k, n],$$

for NF and MCF respectively, where $\hat{s}^{0i}_{k-1,l}$ and $\hat{s}^i_{k-1,l}$ give the amount of items 0 and i that are in the inventory at the end of period $k-1$ and that are used to satisfy the demand of item i in period l.

We first solve problems with 40 final products and 36 periods. The data is generated as follows. The setup, start-up, and inventory holding costs are constant over time, so we drop the index t. The inventory holding costs for the final products are generated randomly as integers in the interval [0,5] and the cost for item 0 is taken as the minimum of these costs. The demands are generated as integers in the interval [0,50]. For each item $i \in I0$, we generated an integer $\hat{f}^i$ in the interval [10,20]. We use a parameter $\rho \in \{1, 5, 10\}$ to obtain instances with a different ratio of setup and start-up costs between the two levels. We set $q^i = f^i = 100\hat{f}^i$ for $i \in I$ and $q^0 = f^0 = 100\rho\hat{f}^0$.

All experiments are carried out using Xpress-IVE version 1.22.04 on a Dell notebook with 2.20 GHz Intel core i7-2720QM processor and 8 GB RAM. The time limit is 600 seconds. For each ρ value, we solve three instances and report the average results. We report the number of instances solved to optimality, the gap of the LP relaxation (LP-gap, computed using the best upper bound), the gap at termination (f-gap, computed using the upper and lower bounds at termination), the number of nodes explored, and the solution time in seconds. The results are presented in Table 1.

We observe that NF and MCF have huge duality gaps and adding the (l, S) start-up inequalities results in a considerable improvement. MCF-WW and EF have very similar duality gaps, but, more instances are solved to optimality with EF and the final gaps for those that are not solved are smaller. The results of this first experiment suggest that we may be able to compute

Table 1: Results for the two-level lot-sizing problem with start-up costs

$n.m.\rho$	formulation	solved	LP-gap	f-gap	nodes	time
	NF	0	72.6	21.7	38980.3	600
	NF-WW	0	3.0	1.4	474.3	600
36.40.1	MCF	0	22.9	35.4	316.3	600
	MCF-WW	0	0.2	8.2	90.3	600
	EF	2	0.1	0.1	38.0	277.9
	NF	0	72.6	23.1	38114.3	600
	NF-WW	0	4.9	4.3	90.0	600
36.40.5	MCF	0	23.5	45.8	173.0	600
	MCF-WW	1	0.3	13.8	4.0	408.9
	EF	2	0.3	0.2	20.0	344.8
	NF	0	72.0	22.7	33595.0	600
	NF-WW	0	4.9	4.8	106.0	600
36.40.10	MCF	0	23.2	54.9	112.3	600
	MCF-WW	1	0.1	0.1	11.7	432.4
	EF	1	0.1	0.1	30.7	446.7

good bounds for larger instances using NF-WW, MCF-WW and EF. This is what we test in our second experiment.

In Table 2, we present results for four instances with $m = 40$ items final products and up to 60 periods and also for four instances with $n = 36$ periods and up to 200 final products. Here we set $\rho = 10$. We report the individual results rather than the averages. For each instance and formulation, we report the best lower and upper bounds (all in millions) and the gap on termination (BLB, BIP, and f-gap, respectively) when the time limit is set to 600 seconds and 1800 seconds respectively. If an instance is solved to optimality, we report the solution time in parentheses in the column f-gap. We highlight the best lower and upper bounds in bold and present the gap between these best bounds in column "b-gap". We observe that the solver usually finds good solutions with NF-WW, however the lower bounds are significantly worse than those of the other two formulations. With MCF-WW, upper bounds are of poor quality and letting the solver run for half an hour only leads to an improvement for the instances with 48 periods and 40 products. Using EF, one can obtain good solutions with a less than 1% gap in ten minutes when $n = 48$, however the results are not good for $n = 60$. In one of the instances with 60 periods, letting the solver run for another twenty minutes resulted in a significant improvement with EF. If the number of periods is not large, EF remains the most efficient formulation for our instances with larger values of m. Overall, except for one instance, we could obtain solutions with a less than 2% gap in half an hour using EF.

4.2 Computational results for the two-level lot-sizing problem with constant capacities for final products

Now we present computational results for the capacitated lot-sizing problem where $Q^0 = M$ and $Q^i = Q$ for all $i \in I$. Here, we compare again the natural formulation (NF), the multicommodity formulation (MCF), and our extended formulation (EF) (76)-(82). We also test NF and MCF with an approximation of the constant capacity Wagner-Whitin extended formulation [19, 26]. We refer to the resulting formulations as NF-WW and MCF-WW.

In Table 3, we report the results for the discrete lot-sizing problem (only the initial stock variables and setup variables have nonzero costs). Here we consider instances with 40 final products and 60 periods and take the costs for the initial stocks to be equal to 1. The setup cost at level 0 in period t is obtained by multiplying ρ by an integer generated randomly in the interval $[50, 50 + 20(n - t)]$ and for the other items, f_t^i is randomly generated in the interval [50,70]. The demands are generated as integers in the interval [0,50] and the capacity is taken to be 100. The time limit is 180 seconds. For each ρ value, we report the averages for three instances. All instances are solved to optimality with formulations NF-WW and EF within the time limit and none is solved with the other three formulations. The final gaps can be as large as 20%. In most cases, NF-WW proves optimality sooner than EF.

The results for the two-level lot-sizing problem are given in Table 4. Here we take $n = 18$ and $m = 20$. The data is generated in the same way as for the instances with start-ups except that we set $f^i = 200\hat{f}^i$ for $i \in I$ and $f^0 = 200\rho\hat{f}^0$. We take the capacity to be equal to 100. In this experiment, the time limit is set to 600 seconds. We report the average results for three instances for each ρ value. Here, it is clear that NF and MCF have large duality gaps and cannot obtain optimal solutions within the time limit. However, when strengthened, these formulations outperform EF in terms of computation time.

Due to its large size, EF takes longer to solve for larger instances. In our final experiment, we use NF-WW and MCF-WW to see the quality of bounds that one can obtain as n and m increase. The results are given in Table 5. Here the results are given for individual instances. Except for the instances solved to optimality, the best lower bound is obtained using MCF-WW and the best upper bound using NF-WW. We see that the lower bounds obtained by MCF-WW in half an hour are very close to those obtained after ten minutes. However, for several instances, there was a significant improvement in the upper bounds obtained with NF-WW after half an

Table 2: Results for the two-level lot-sizing problem with start-up costs - larger instances

		600 seconds				1800 seconds			
$n.m.\rho$	formulation	BLB	BIP	f-gap	b-gap	BLB	BIP	f-gap	b-gap
	NF-WW	1.32556	1.42863	7.2		1.32672	1.39296	4.8	
48.40.10	MCF-WW	**1.38409**	2.45373	43.6	0.8	1.38415	**1.38828**	0.3	0.2
	EF	1.38399	**1.39513**	0.8		**1.38601**	**1.38828**	0.2	
	NF-WW	1.33228	1.41290	5.7		1.33311	1.41290	5.6	
48.40.10	MCF-WW	1.37867	2.16587	36.3	0.0	1.37876	1.38554	0.5	0.0
	EF	**1.37919**	**1.37919**	(528)		**1.37919**	**1.37919**	(528)	
	NF-WW	1.63304	**1.78248**	8.4		1.63424	**1.77430**	7.9	
60.40.10	MCF-WW	**1.70761**	3.49178	51.1	4.2	**1.70762**	3.41090	49.9	3.8
	EF	1.70755	1.89162	9.7		1.70755	1.80135	5.2	
	NF-WW	1.64781	1.94792	15.4		1.64967	1.76339	6.4	
60.40.10	MCF-WW	**1.71224**	3.14452	45.5	9.0	**1.71224**	3.14452	45.5	1.2
	EF	1.71220	**1.88121**	9.0		1.71220	**1.73387**	1.2	
	NF-WW	2.38869	2.52650	5.5		2.39063	2.52650	5.4	
36.100.10	MCF-WW	2.47379	4.18509	40.9	0.7	2.47381	4.18509	40.9	0.6
	EF	**2.47398**	**2.49237**	0.7		**2.47441**	**2.49040**	0.6	
	NF-WW	2.30817	2.45065	5.8		2.31267	2.44202	5.3	
36.100.10	MCF-WW	2.38724	3.82288	37.6	1.1	2.38724	3.82288	37.6	1.0
	EF	**2.38863**	**2.41537**	1.1		**2.38863**	**2.41243**	1.0	
	NF-WW	4.46134	4.88696	8.7		4.46355	4.70116	5.1	
36.200.10	MCF-WW	4.61008	7.27327	36.6	4.7	4.61009	7.27327	36.6	0.8
	EF	**4.61471**	**4.84130**	4.7		**4.61471**	**4.65059**	0.8	
	NF-WW	4.45811	4.72269	5.6		4.46117	**4.66521**	4.4	
36.200.10	MCF-WW	4.57967	6.68247	31.5	2.0	4.57969	6.68247	31.5	1.7
	EF	**4.58782**	**4.67975**	2.0		**4.58782**	4.67975	2.0	

Table 3: Results for the discrete two-level lot-sizing problem with constant capacities for final products

$n.m.\rho$	formulation	solved	LP-gap	f-gap	nodes	time
	NF	0	3.0	1.7	35819.3	180
	NF-WW	3	0.7	0	3.0	75.5
60.40.1	MCF	0	1.6	1.7	222.3	180
	MCF-WW	0	0.2	5.6	0.3	180
	EF	3	0	0	1.0	105.6
	NF	0	5.3	1.6	37442.0	180
	NF-WW	3	1.3	0	7.0	110.6
60.40.5	MCF	0	1.5	1.6	8.0	180
	MCF-WW	0	0.7	14.5	0	180
	EF	3	0	0	1.0	117.4
	NF	0	6.4	1.8	39027.7	180
	NF-WW	3	1.4	0	5.0	120.9
60.40.10	MCF	0	1.5	2.0	669.0	180
	MCF-WW	0	0.6	21.8	0	180
	EF	3	0	0	1.0	112.3

Table 4: Results for the two-level lot-sizing problem with constant capacities for final products

$n.m.\rho$	formulation	solved	LP-gap	f-gap	nodes	time
	NF	0	19.2	4.2	74299.3	600
	NF-WW	3	3.1	0	443.7	14.4
18.20.1	MCF	0	5.5	7.5	37522.3	600
	MCF-WW	3	0.3	0	132.3	36.4
	EF	3	0	0	1.7	247.2
	NF	0	19.7	4.6	75682.0	600
	NF-WW	3	4.5	0.0	93.7	10.4
18.20.5	MCF	0	4.9	6.1	42839.3	600.0
	MCF-WW	3	0.4	0	179.0	46.2
	EF	3	0	0	1.0	94.3
	NF	0	19.1	4.4	67517.3	600
	NF-WW	3	5.0	0	46.3	8.0
18.20.10	MCF	0	4.3	5.2	44697.7	600.0
	MCF-WW	3	0.4	0	265.7	34.3
	EF	3	0	0	1.0	77.8

hour. Overall, we obtain good solutions with small duality gaps even for problems with 24 periods and 200 final products in half an hour using NF-WW and compute good lower bounds in ten minutes using MCF-WW.

5 Conclusions

In this paper, we have proposed exact and approximate extended formulations for two-level multi-item discrete lot-sizing problems and reported some computational results on using these reformulations to solve one-producer multiple item lot-sizing, or equivalently one-warehouse multiple-retailer problems. We have proposed an exact extended formulation for the uncapacitated problem and modified it to handle start-up costs. In our computational experiments, we have observed that the extended formulation for the problem with start-up costs outperforms the existing formulations. We note that this formulation can be extended easily to problems with more levels and to problems with demand at intermediate levels.

We have also proposed an exact extended formulation for the problem with constant capacities for final products and no capacity constraints at the upper level. Here the behavior of the formulations appears to be different. Even though the LP relaxation of the extended formulation

Table 5: Results for the two-level lot-sizing problem with constant capacities for final products - larger instances

		600 seconds				1800 seconds			
n.m.ρ	formulation	BLB	BIP	f-gap	b-gap	BLB	BIP	f-gap	b-gap
36.40.10	NF-WW	1.52054	**1.57272**	3.3	1.4	1.55001	**1.56881**	1.2	1.1
	MCF-WW	**1.55142**	1.66952	7.1		**1.55184**	1.66952	7	
36.40.10	NF-WW	1.55135	**1.59773**	2.9	1.2	1.57268	**1.59713**	1.5	1.2
	MCF-WW	**1.57807**	1.92693	18.1		**1.57847**	1.92693	18.1	
48.40.10	NF-WW	1.99234	**2.13121**	6.5	2.9	1.99371	**2.13121**	6.5	2.9
	MCF-WW	**2.07001**	2.81594	26.5		**2.07009**	2.81594	26.5	
48.40.10	NF-WW	2.04227	**2.14571**	4.8	2.9	2.04997	**2.13676**	4.1	2.5
	MCF-WW	**2.08359**	2.89916	28.1		**2.08359**	2.89916	28.1	
18.200.10	NF-WW	3.47911	**3.53825**	1.7	0.4	**3.53703**	**3.53703**	(1251)	0.0
	MCF-WW	**3.52526**	3.87502	9.0		3.52652	3.53747	0.3	
18.200.10	NF-WW	3.46157	**3.51112**	1.4	0.4	**3.50997**	**3.50997**	(1588)	0.0
	MCF-WW	**3.49872**	3.52542	0.8		**3.50997**	**3.50997**	(1394)	
24.200.10	NF-WW	4.57146	**4.72855**	3.3	1.1	4.60086	**4.71983**	2.5	0.9
	MCF-WW	**4.67881**	5.48446	14.7		**4.67881**	4.74974	1.5	
24.200.10	NF-WW	4.59760	**4.99806**	8.0	6.4	4.62476	**4.70941**	1.8	0.7
	MCF-WW	**4.67798**	5.39522	13.3		**4.67798**	4.74297	1.4	

has a duality gap smaller than those of the existing formulations, it is impractical due to its large size. One interesting extension of the current work may be to study the projection of this large formulation onto the space of the original variables and devise a branch-and-cut algorithm. For the more general problem in which capacity constraints are also introduced at the upper level, we have only provided an extended formulation for a relaxation. Testing the performance of this extended formulation in practice and finding an exact extended formulation for this version of the problem remain for further investigation.

Finally, we conjecture that the following is an exact extended formulation for the two-level discrete lot-sizing problem with a single final product and backlogging:

$$
\begin{aligned}
& s^0 + s^1 = \sum_{l=1}^{n} d_l \zeta_l, \\
& s^1 = \sum_{l=1}^{n} d_l \delta_l, \\
& r_l = \sum_{j=1}^{l} d_j \sigma_{jl} \quad l \in [1, n], \\
& \zeta_j + \sigma_{jl} \geq 1 - y^0_{1t} - y^1_{t+1,l} \quad l \in [1, n], t \in [0, l], j \in [1, l], \\
& \delta_j + \sigma_{jl} \geq 1 - y^1_{1l} \quad l \in [1, n], j \in [1, l], \\
& \zeta, \delta, y^0, y^1 \in \mathbb{R}^n_+, \sigma \in \mathbb{R}_+^{\frac{n(n-1)}{2}},
\end{aligned}
$$

where r_l is the amount backlogged at the end of period l. The approach in Proposition 5 can then be used to extend this formulation to multiple final products.

References

[1] P. Afentakis and B. Gavish. Optimal lot-sizing algorithms for complex product structures. *Operations Research*, 34:237–249, 1986.

[2] A. Aggarwal and J. Park. Improved algorithms for economic lot-size problems. *Operations Research*, 41:549–571, 1993.

[3] E. Arkin, D. Joneja, and R. Roundy. Computational complexity of uncapacitated multi-echelon production planning problems. *Operations Research Letters*, 8:61–66, 1989.

[4] E. Balas. On the convex hull of the union of certain polyhedra. *Operations Research Letters* 7(6):279–283, 1988.

[5] I. Barany, J. Edmonds, and L.A. Wolsey. Packing and covering a tree by subtrees. *Combinatorica*, 6:245–257, 1986.

[6] I. Barany, T.J. Van Roy, and L.A. Wolsey. Uncapacitated lot sizing: The convex hull of solutions. *Mathematical Programming*, 22:32–43, 1984.

[7] G.R. Bitran and H.H. Yanasse. Computational complexity of the capacitated lot size problem. *Management Science*, 28:1174–1186, 1982.

[8] A.J. Clark and H. Scarf. Optimal policies for multi-echelon inventory problems. *Management Science*, 6:475–490, 1960.

[9] M. Conforti, M.Di Summa, F. Eisenbrand, and L.A. Wolsey. Network formulations of mixed-integer programs. *Mathematics of Operations Research*, 34:194–209, 2009.

[10] A. Federgrün and M. Tzur. A simple forward algorithm to solve general dynamic lot-size models with n periods in $O(n\, log\, n)$ or $O(n)$ time. *Management Science*, 37:909–925, 1991.

[11] A. Ghouila-Houri. Caracterisation des matrices totalement unimodulaires. *C.R. Academy of Sciences of Paris*, 254:1192–1194, 1962.

[12] H.-C.Hwang. Economic lot-sizing for integrated production and transportation. *Operations Research* 58(2):428–444, 2010.

[13] J. Krarup and O. Bilde. Plant location, set covering and economic lot sizes: An $O(mn)$ algorithm for structured problems. In L. Collatz et al., editor, *Optimierung bei Graphentheoretischen und Ganzzahligen Probleme*, pages 155–180. Birkhauser Verlag, Basel, 1977.

[14] S. Küçükyavuz and Y. Pochet. Uncapacitated lot-sizing with backlogging: the convex hull. *Mathematical Programming*, 118:151–175, 2009.

[15] R. Melo and L.A. Wolsey. Uncapacitated two-level lot-sizing. *Operations Research Letters*, 38:241–245, 2010.

[16] Y. Pochet and L.A. Wolsey. Lot-size models with backlogging: Strong formulations and cutting planes. *Mathematical Programming*, 40:317–335, 1988.

[17] Y. Pochet and L.A. Wolsey. Solving multi-item lot-sizing problems using strong cutting planes. *Management Science* 37(1): 53–67, 1991.

[18] Y. Pochet and L.A. Wolsey. Lot-sizing with constant batches: Formulation and valid inequalities. *Mathematics of Operations Research*, 18:767–785, 1993.

[19] Y. Pochet and L.A. Wolsey. Polyhedra for lot-sizing with Wagner–Whitin costs. *Mathematical Programming*, 67:297–324, 1994.

[20] Y. Pochet and L.A. Wolsey. *Production planning by mixed integer programming*. Springer, 2006.

[21] Y. Pochet and L.A. Wolsey. Single item lot-sizing with non-decreasing capacities. *Mathematical Programming*, 121:123–143, 2010.

[22] R.L. Rardin and U. Choe. Tighter relaxations of fixed charge network flow problems. Technical Report report J-79-18, School of Industrial and Systems Engineering, Georgia Institute of Technology, 1979.

[23] C.P.M. van Hoesel, H.E. Romijn, D. Romero Morales, and A. Wagelmans. Integrated lot-sizing in serial supply chains with production capacities. *Management Science*, 51:1706–1719, 2005.

[24] C.P.M. van Hoesel, A. Wagelmans, and B. Moerman. Using geometric techniques to improve dynamic programming algorithms for the economic lot-sizing problem and extensions. *European Journal of Operational Research*, 75:312–331, 1994.

[25] C.P.M. van Hoesel, A. Wagelmans, and L.A. Wolsey. Polyhedral characterization of the economic lot-sizing problem with start-up costs. *SIAM Journal of Discrete Mathematics*, 7:141–151, 1994.

[26] M. Van Vyve and L.A. Wolsey. Approximate extended formulations. *Mathematical Programming B*, 105:501–522, 2006.

[27] A.F. Veinott. Minimum concave cost solution of Leontief substitution systems of multifacility inventory systems. *Operations Research*, 17:262–291, 1969.

[28] B. Verweij and L.A. Wolsey. Uncapacitated lot-sizing with buying, sales and backlogging. *Optimization Methods and Software*, 19:427–436, 2004.

[29] H.M. Wagner and T.M. Whitin. Dynamic version of the economic lot size model. *Management Science*, 5:89–96, 1958.

[30] W.I. Zangwill. A backlogging model and a multi-echelon model of a dynamic economic lot size production system – A network approach. *Management Science*, 15:506–526, 1969.

[31] M. Zhang, S. Küçükyavuz, and H. Yaman. A polyhedral study of multi-echelon lot sizing with intermediate demands. *Operations Research*, to appear.

A Appendix: Proof of Theorem 7

Similar to Theorem 3, the proof is in three steps. In the first step, we show that the polyhedron SCC defined by (49)–(55) is integral for $m = 1$. Then we extend the result to $m > 1$ and finally prove that adding constraints (48) does not destroy integrality.

Let SSC_I denote the set of integral solutions in SSC and consider the case $m = 1$. As we did in the proof of Theorem 4, for a given non-zero objective function $\min \sum_{u=1}^{n} h_u^0 \zeta_u + \sum_{u=1}^{n} h_u^1 \delta_u + \sum_{i=0}^{1} \sum_{u=0}^{n} f_u^i y_u^i + \sum_{i=0}^{1} \sum_{u=1}^{n} q_u^i z_u^i$ with bounded optimal value, we determine one inequality among (49)-(55) that is satisfied at equality by all optimal solutions.

We use the following observation. Let $\alpha_1 \in [0, n+1]$, $\alpha_2 \in [\alpha_1, n+1]$, $\beta_2 \in [\alpha_1, n+1]$, $\beta_1 \in [0, \beta_2]$ with $\beta_1 = n+1$ if $\beta_2 = n+1$, $\gamma_2 \in [1, \alpha_1 - 1] \cup \{\beta_2\}$, $\gamma_1 \in [0, \gamma_2]$ if $\gamma_2 \leq \alpha_1 - 1$ and $\gamma_1 = \beta_1$ if $\gamma_2 = \beta_2$. The y and z vectors in the extreme points of $conv(SSC_I)$ are of the following form: $y_u^0 = 1$ for $u \in [\alpha_1, \alpha_2]$, $z_{\alpha_1}^0 = 1$, $y_u^1 = 1$ for $u \in [\gamma_1, \gamma_2] \cup [\beta_1, \beta_2]$, $z_{\gamma_1}^1 = z_{\beta_1}^1 = 1$, the other entries of y and z vectors are zero. In the sequel, we use the values $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2$ to represent the corresponding extreme points.

a. Let $q_0^0 = q_0^1 = 0$. We need $h^0, h^1 \geq 0$, $q_t^0 + \sum_{u=t}^{t+k} f_u^0 \geq 0$ and $q_t^1 + \sum_{u=t}^{t+k} f_u^1 \geq 0$ for all $t \in [0, n]$, $k \in [0, n-t]$ for the problem to be bounded.

b. For $i = 0, 1$, if there exists $u \in [1, n]$ with $f_u^i < 0$, then $y_u^i = z_u^i + y_{u-1}^i$.

c. For $i = 0, 1$, if there exists $u \in [1, n]$ with $q_u^i < 0$, then $z_u^i = y_u^i$.

d. For $i = 0, 1$, if $f_0^i < q_1^i$, then $z_1^i = 0$. If $f_0^i > q_1^i$, then $y_0^i = 0$. So $f_0^i = q_1^i$. In the remaining, we study the case where $h^0, h^1, f^0, f^1, q^0, q^1 \geq 0$.

e. If $h^0 = f^0 = q^0 = 0$, then the problem is single-level and the result is known to hold [19].

f. Suppose that $h^0 = 0$. If $f_0^0 > 0$, then $y_0^0 = 0$. If there exists $u \in [1, n]$ with $f_u^0 + q_u^0 > 0$, then $z_u^0 = 0$. In the remaining, we assume that there exists l such that $h_l^0 > 0$. Let l be the highest such index.

g. If $f_u^0 + q_u^0 < f_{u+1}^0 + q_{u+1}^0$ for some $u \in [1, n-1]$, then $z_{u+1}^0 = 0$. Let $t \in [1, n]$ be the largest index with $f_t^0 + q_t^0 > 0$. If no such t exists, then let $t = 0$. If $t > l$, then $z_t^0 = 0$. So we assume that $t \leq l$.

h. If there exist $k \in [1, l]$, $m_1 \in [0, k]$ and $m_2 \in [k, n]$ such that $q_k^0 + f_k^0 + q_{m_1}^1 + \sum_{u=m_1}^{m_2} f_u^1 < h_l^0$, then $\zeta_l = 0$. Therefore, as $q_k^0 + f_k^0 = 0$ for $k > t$, we assume that $q_{m_1}^1 + \sum_{u=m_1}^{m_2} f_u^1 \geq h_l^0 > 0$ for all $m_2 \in [t+1, n]$ and $m_1 \in [0, m_2]$.

i. If $f^1_{t+1} > 0$, we show that with t, l chosen in this way, the inequality (49) if $t = 0$, (50) if $t \in [1, l-1]$ or (51) if $t = l$ is satisfied at equality by all optimal solutions.

Note that the cost assumptions imply that all rays with non-zero contribution in this inequality have positive cost.

Let $(\zeta, \delta, y^0, y^1, z^0, z^1)$ be an extreme point optimal solution. Suppose that the inequality (49), (50), or (51) corresponding to the above choice of t and l is not tight.

(a) Case $\zeta_l = 1$.

If $y^1_{t+1} + \sum_{u=t+2}^{l} z^1_u \geq 1$, then $t+1 \leq \beta_2$ and $\sum_{u=1}^{t} y^0_u = 0$. In this case, setting $\alpha_1, \alpha_2 \leftarrow t+1$ and $\zeta_l = 0$ decreases the cost by $h^0_l > 0$.

If $y^1_{t+1} + \sum_{u=t+2}^{l} z^1_u = 0$, then $y^0_1 + \sum_{u=2}^{t} z^0_u = 1$ and $\beta_1 > l$. Now, setting $\alpha_1, \alpha_2 \leftarrow t+1$ yields a better solution since $f^0_u + q^0_u > 0$ for all $u \in [1, t]$ and $f^0_0 = q^0_1$.

(b) Case $\zeta_l = 0$.

If $y^0_1 + \sum_{u=2}^{t} z^0_u = 0$, then $y^1_{t+1} + \sum_{u=t+2}^{l} z^1_u \geq 2$. Hence $t+1 \leq \gamma_2$ and $\beta_1 \geq t+2$. Now setting $\alpha_1, \alpha_2 \leftarrow \gamma_2$, $\beta_1 \leftarrow \gamma_1$ and $\beta_2 \leftarrow \gamma_2$ decreases the cost by $q_{\beta_1} + \sum_{u=\beta_1}^{\beta_2} f^1_u$, which is positive since $\beta_1 \in [t+2, l]$.

If $y^0_1 + \sum_{u=2}^{t} z^0_u = 1$, then $y^1_{t+1} + \sum_{u=t+2}^{l} z^1_u \geq 1$. If $\beta_1 \geq t+1$, then setting $\alpha_1, \alpha_2 \leftarrow t+1$ gives a better solution. If $\beta_1 \leq t$, then $y^1_{t+1} = 1$ and $\sum_{u=t+1}^{\beta_2} f^1_u > 0$, so it is better to set $\beta_2 \leftarrow t$.

j. If $h^0_t > 0$ (and $f^1_{t+1} = 0$, but this is not necessary here), then inequality $\zeta_t \geq 1 - y^0_1 - \sum_{u=2}^{t} z^0_u$ (i.e., of type (51)) is satisfied at equality. Indeed, if not, then $\zeta_t = 1$, $y^0_1 + \sum_{u=2}^{t} z^0_u = 1$, and $\beta_1 \geq t+1$. Then setting $\alpha_1, \alpha_2 \leftarrow t+1$ gives a better solution.

k. If $f^1_{t+1} = 0$ and $h^0_t = 0$, then $z^0_t = 0$, or equivalently $\alpha_1 \neq t$, in any optimal solution. Indeed, if $\alpha_1 = t$ and $\beta_2 > t$, then setting $\alpha_1, \alpha_2 \leftarrow t+1$ gives a better solution. If $\alpha_1 = t$ and $\beta_2 = t$, then setting $y^1_{t+1} = 1$ at zero cost (and therefore $\beta_2 = t+1$) and $\alpha_1, \alpha_2 \leftarrow t+1$ gives a better solution.

Thus we can conclude that all optimal solutions lie on a face defined by one of the inequalities (49)-(55). This proves that SSC is integral when $m = 1$.

To prove that the result is true for $m > 1$, we need a variant of Proposition 5. First, we observe that in an extreme point of $conv(SSC_I)$, we have $y^0_0 + \sum_{j=1}^{n} z^0_j \leq 1$ and that given any $(\zeta, \delta, y^0, y^1, z^0, z^1) \in SSC$, the solution $(\zeta, \delta, \bar{y}^0, y^1, \bar{z}^0, z^1)$ is also in SSC where $\bar{y}^0_0 = \min\{y^0_0, 1\}$, $\bar{y}^0_1 = \min\{z^0_1 + y^0_0, 1\}$, $\bar{z}^0_j = \min\{(1 - y^0_0 - z^0_{1j-1})^+, z^0_j\}$ for $j = 1, \ldots, n$, and $\bar{y}^0_j = \bar{z}^0_j$ for $j = 2, \ldots, n$. Now, we can use similar arguments to those of Proposition 5 to obtain the result.

Finally we need to show that adding constraints (48) does not destroy integrality. As in the proof of Theorem 3, the key argument is that when such an inequality is tight, constraint (52) is dominated.

Recent titles

CORE Discussion Papers

2011/62. Florian MAYNERIS. A new perspective on the firm size-growth relationship: shape of profits, investment and heterogeneous credit constraints.
2011/63. Florian MAYNERIS and Sandra PONCET. Entry on difficult export markets by Chinese domestic firms: the role of foreign export spillovers.
2011/64. Florian MAYNERIS and Sandra PONCET. French firms at the conquest of Asian markets: the role of export spillovers.
2011/65. Jean J. GABSZEWICZ and Ornella TAROLA. Migration, wage differentials and fiscal competition.
2011/66. Robin BOADWAY and Pierre PESTIEAU. Indirect taxes for redistribution: Should necessity goods be favored?
2011/67. Hylke VANDENBUSSCHE, Francesco DI COMITE, Laura ROVEGNO and Christian VIEGELAHN. Moving up the quality ladder? EU-China trade dynamics in clothing.
2011/68. Mathieu LEFEBVRE, Pierre PESTIEAU and Grégory PONTHIERE. Measuring poverty without the mortality paradox.
2011/69. Per J. AGRELL and Adel HATAMI-MARBINI. Frontier-based performance analysis models for supply chain management; state of the art and research directions.
2011/70. Olivier DEVOLDER. Stochastic first order methods in smooth convex optimization.
2011/71. Jens L. HOUGAARD, Juan D. MORENO-TERNERO and Lars P. ØSTERDAL. A unifying framework for the problem of adjudicating conflicting claims.
2011/72. Per J. AGRELL and Peter BOGETOFT. Smart-grid investments, regulation and organization.
2012/1. Per J. AGRELL and Axel GAUTIER. Rethinking regulatory capture.
2012/2. Yu. NESTEROV. Subgradient methods for huge-scale optimization problems.
2012/3. Jeroen K. ROMBOUTS, Lars STENTOFT and Francesco VIOLANTE. The value of multivariate model sophistication: An application to pricing Dow Jones Industrial Average options.
2012/4. Aitor CALO-BLANCO. Responsibility, freedom, and forgiveness in health care.
2012/5. Pierre PESTIEAU and Grégory PONTHIERE. The public economics of increasing longevity.
2012/6. Thierry BRECHET and Guy MEUNIER. Are clean technology and environmental quality conflicting policy goals?
2012/7. Jens L. HOUGAARD, Juan D. MORENO-TERNERO and Lars P. ØSTERDAL. A new axiomatic approach to the evaluation of population health.
2012/8. Kirill BORISSOV, Thierry BRECHET and Stéphane LAMBRECHT. Environmental maintenance in a dynamic model with heterogenous agents.
2012/9. Ken-Ichi SHIMOMURA and Jacques-François THISSE. Competition among the big and the small.
2012/10. Pierre PESTIEAU and Grégory PONTHIERE. Optimal lifecycle fertility in a Barro-Becker economy.
2012/11. Catherine KRIER, Michel MOUCHART and Abderrahim OULHAJ. Neural modelling of ranking data with an application to stated preference data.
2012/12. Matthew O. JACKSON and Dunia LOPEZ-PINTADO. Diffusion and contagion in networks with heterogeneous agents and homophily.
2012/13. Claude D'ASPREMONT, Rodolphe DOS SANTOS FERREIRA and Jacques THEPOT. Hawks and doves in segmented markets: A formal approach to competitive aggressiveness.
2012/14. Claude D'ASPREMONT and Rodolphe DOS SANTOS FERREIRA. Household behavior and individual autonomy: An extended Lindahl mechanism.
2012/15. Dirk VAN DE GAER, Joost VANDENBOSSCHE and José Luis FIGUEROA. Children's health opportunities and project evaluation: Mexico's *Oportunidades* program.
2012/16. Giacomo VALLETTA. Health, fairness and taxation.
2012/17. Chiara CANTA and Pierre PESTIEAU. Long term care insurance and family norms.
2012/18. David DE LA CROIX and Fabio MARIANI. From polygyny to serial monogamy: a unified theory of marriage institutions.

Recent titles

CORE Discussion Papers - continued

2012/19. Carl GAIGNE, Stéphane RIOU and Jacques-François THISSE. Are compact cities environmentally friendly?
2012/20. Jean-François CARPANTIER and Besik SAMKHARADZE. The asymmetric commodity inventory effect on the optimal hedge ratio.
2012/21. Concetta MENDOLICCHIO, Dimitri PAOLINI and Tito PIETRA. Asymmetric information and overeducation.
2012/22. Tom TRUYTS. Stochastic signaling: Information substitutes and complements.
2012/23. Pierre DEHEZ and Samuel FEREY. How to share joint liability: A cooperative game approach.
2012/24. Pilar GARCIA-GOMEZ, Erik SCHOKKAERT, Tom VAN OURTI and Teresa BAGO D'UVA. Inequity in the face of death.
2012/25. Christian HAEDO and Michel MOUCHART. A stochastic independence approach for different measures of concentration and specialization.
2012/26. Xavier RAMOS and Dirk VAN DE GAER. Empirical approaches to inequality of opportunity: principles, measures, and evidence.
2012/27. Jacques H. DRÈZE and Erik SCHOKKAERT. Arrow's theorem of the deductible : moral hazard and stop-loss in health insurance.
2012/28 Luc BAUWENS and Giuseppe STORTI. Computationally efficient inference procedures for vast dimensional realized covariance models.
2012/29. Pierre DEHEZ. Incomplete-markets economies: The seminar work of Diamond, Drèze and Radner.
2012/30 Helmuth CREMER, Pierre PESTIEAU and Grégory PONTHIÈRE. The economics of long-term care: a survey.
2012/31 Mathieu VAN VYVE, Laurence A. WOLSEY and Hande YAMAN. Relaxations for two-level multi-item lot-sizing problem.

Books

G. DURANTON, Ph. MARTIN, Th. MAYER and F. MAYNERIS (2010), *The economics of clusters – Lessons from the French experience*. Oxford University Press.
J. HINDRIKS and I. VAN DE CLOOT (2011), *Notre pension en heritage*. Itinera Institute.
M. FLEURBAEY and F. MANIQUET (2011), *A theory of fairness and social welfare*. Cambridge University Press.
V. GINSBURGH and S. WEBER (2011), *How many languages make sense? The economics of linguistic diversity*. Princeton University Press.
I. THOMAS, D. VANNESTE and X. QUERRIAU (2011), *Atlas de Belgique – Tome 4 Habitat*. Academia Press.
W. GAERTNER and E. SCHOKKAERT (2012), *Empirical social choice*. Cambridge University Press.
L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of volatility models and their applications*. Wiley.
J-C. PRAGER and J. THISSE (2012), *Economic geography and the unequal development of regions*. Routledge.
M. FLEURBAEY and F. MANIQUET (2012), *Equality of opportunity: the economics of responsibility*. World Scientific.
J. HINDRIKS (2012), *Gestion publique*. De Boeck.

CORE Lecture Series

R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.
A. SHAPIRO (2010), Stochastic programming: modeling and theory.