Bridge deck analysis through the use of grillage models

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ABSTRACT: The object of the paper is the study of the representativity of the grillage models with which different types of bridge decks are schematized. First, the theoretical principles on which this kind of modelling is based are recalled; the equivalent condition between bi-dimensional continuous elements and corresponding grillage models are imposed through the use of a kinematics and an energetic criterion. Secondly, the same technique is generalized to three-dimensional structures and specialized to the case of cellular decks. For this kind of deck, structural behaviours usually neglected by the current technical approaches, like shear lag, distortion and warping, are considered. The paper presents some methods introducing these effects in a grillage analysis; these methods provide a series of criteria with which it's possible to define the rigidities of the equivalent model. These criteria are applied and compared with finite element solutions. Finally, a series of applications are executed in order to verify the efficiency and the accuracy of this kind of approach.

1 INTRODUCTION

The analysis of the bridge decks through a grillage model is a technique diffused in the second half of the past century, after that some authors, Hrennikoff and Absì particularly, suggest the idea to study the elastic problems modelling the continuous systems through a finite number of elementary frameworks. This type of approach, first applied to the beam and slab decks, spreads widely and its application is extended to the case of much more complex structures, such as cellular decks, skew and curve bridges, and to the case of particular loading conditions, such us the temperature and pre-stress loads.

The fundamental principle lies on the bases of this modelling is clearly expressed in a Hrennikoff's note, referred to the case of bi-dimensional elastic continuous elements but generalizable to the one of three-dimensional structures:

"The basic idea of the method consists in replacing the continuous material of the elastic body under investigation by a framework of bars, arranged according to a definite pattern whose elements are endowed with elastic properties suitable to the type of problem, in analyzing the framework and in spreading the bar stresses over the tributary areas in order to obtained stresses in the original body. The framework so formed is given the same external outline and the boundary restraints, and is subjected to the same loads as the solid body, the loads being all applied at the joints" (Hrennikoff 1941).

Hrennikoff imposes the equivalence between continuous structure and grillage model through a kinematics principle, according to the two models are equivalent if, subjected to the same loading conditions, present equal strains. This technique is taken again by Absì after the spread of the Finite Element Method, and it is re-proposed in a different key; Absì, in fact, supposes continuous structure and grillage model are equivalent if, subjected to the same loading conditions, present equal total potential energy (Absi). The imposition of the equivalence lets to the equations which define the axial, flectional, torsional rigidities of the grillage beams. The two approaches practically let to the same results.

2 DEFORMATIVE MODES OF BRIDGE DECK

2.1 *The problem*

In a bridge deck analysis through the use of a grillage model, the assignment of rigidities to the grillage members is certainly the main phase of this pattern. The expression of the rigidities must be assigned to the beams are given by various manuals for more common types of deck. These estimations of the equivalent rigidities derive from theoretical considerations and experimental observations referred to only "principal" deformation modes, or flections and torsion of deck in longitudinal and transverse directions. As for cellular decks, these principal modes are accompanied by "secondary" deformation modes usually negligent, such as shear lag, distortion and warping. For particular geometric and loading conditions, these effects can become significant and to neglect them can make inaccurate the grillage model. In this paper various techniques considering these effects are proposed.

2.2 *Shear lag*

From the basic assumptions of simple beam theory – where cross section remains plane – the distribution of stress across the top flange of a beam is constant. In broad flange "T" or "I" sections, this is true only for span sections; for end sections or for sections corresponding to points of contra flexure, normal stresses change with a maximum adjacent to the web and reducing to zero at the extremity of the flange.

This effect, usually called shear lag, occurs equally in a cellular deck, ideally composed of a series of adjacent "T" or "I" beams. Since shear lag reduces the effective stiffness of each beam, greater accuracy can be obtained from a grillage analysis if the "effective" section properties arising from shear lag are used in the grillage model. This phenomenon is influenced by loading-restraint conditions and by the type of section used; it's common use to consider an effective flange width established through theoretical valuations and experimental observations (Hambly & Pennels, 1975).

The dependence from the section properties is analysed referring to a unicellular deck of which span, width and height of cell, depth of webs and flanges are changed in turn in order to obtain a great number of cases. The deck is composed of only one span and it's subjected to a loading distribution so that a longitudinal flections arises (Fig. 1).

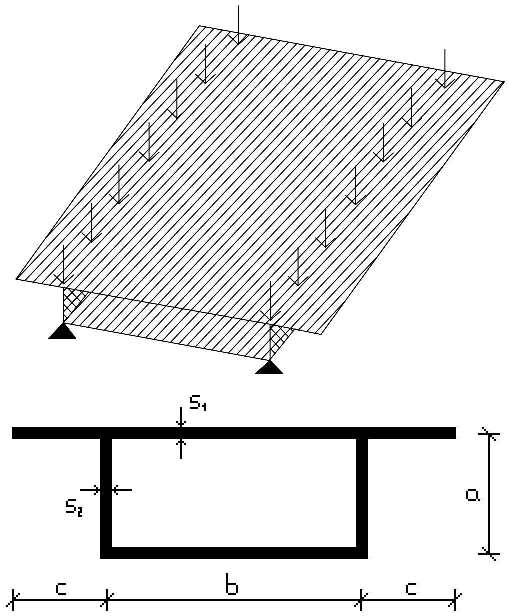

Figure 1. Loading condition and section properties of the reference deck.

$b = 600$ cm $\quad s_1 = 25$ cm
$c = 150$ cm $\quad s_2 = 35$ cm
$h = 150$ cm $\quad l = 3000$ cm
The distributed load is worth 50 kg /cm

The deck is studied through a FE analysis and the value of the vertical displacement in mid-span (corresponding to the intersection of web and flange) is compared to the one deriving from the flections theory for a beam in the same loading-restraint conditions and with the same longitudinal inertia of the whole deck. Drawing the diagram of the ratio $f_{theoretical} / f_{numeric}$ as to the variation in turn of span, height, width and depth of cell, the curve obtained in each case can be considered about linear. It's believed to create a field containing the results of the different analyzed cases, in which the ratio $f_{theoretical} / f_{numerical}$ depends from the value of this ratio obtained in the conditions of l / l_o, h / h_o, w / w_o, t / t_o max and min. The cases corresponding to an increase and diminution of the 50% of these parameters as to the reference one are chosen as extremes of this field. These values are then multiplied by some functions which describe their linear curve in the field. The expression of these functions can be obtained with reference to the shape functions of a truss, of an ISOP4, of a three-dimensional 8 nodes element. The expression of the beam functions in a master field is

$$N_i(\xi) = (1/2)(1+\xi\xi_i) \qquad (1D)$$

$$N_i(\xi,\eta) = (1/4)(1+\xi\xi_i)(1+\eta\eta_i) \qquad (2D)$$

$$N_i(\xi,\eta,\zeta) = (1/8)(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i) \qquad (3D)$$

For an isoparametric finite element ideally lies in a four-dimensional field, the expression is

$$N_i(\xi,\eta,\zeta,\mu) = (1/16)(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)(1+\mu\mu_i)$$

where

ξ, η, ζ, μ — represent coordinate system of a 4D field
$\xi_i, \eta_i, \zeta_i, \mu_i$ — are the nodal coordinates or the extremes of the field

A coordinate transformation from a generic to a master field must be preventively effectuated in order to use these expressions. This transformation, consisting in a translation and a scale change of the field, is shown in Figure 2 for a plane field (offering much more clearly).

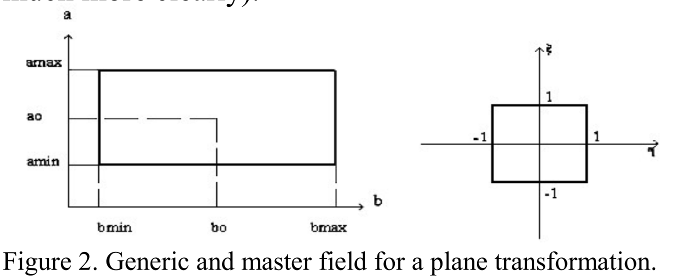

Figure 2. Generic and master field for a plane transformation.

$$\xi = \frac{2(a_j - a_o)}{(a_{max} - a_{min})}$$

for $a = a_o$ $\xi = 0$, $a = a_{max}$ $\xi = 1$, $a = a_{min}$ $\xi = -1$

$$\eta = \frac{2(b_j - b_o)}{(b_{max} - b_{min})}$$

for $b = b_o$ $\eta = 0$, $b = b_{max}$ $\eta = 1$, $b = b_{min}$ $\eta = -1$

For a 4D field there are also the expressions

$$\zeta = \frac{2(c_j - c_o)}{(c_{max} - c_{min})} \qquad \mu = \frac{2(d_j - d_o)}{(d_{max} - d_{min})}$$

For the last, the same properties of the first are valid. The generic coordinates a, b, c and d represent now l / l_o, h / h_o, w / w_o, t / t_o. The values with the "pedice" zero are referred to those cases in which the ratios l_j / l_o, h_j / h_o, w_j / w_o, t_j / t_o, are max and min (where j is referred to the j-th analyzed case).

If $k = f_{theoretical} / f_{numeric}$, since the ratio of the 16 nodal coordinates is known, it's possible to obtain the coefficient for any point of the field using the relationship

$$k = k(\xi, \eta, \zeta, \mu) = \sum_{i=1}^{16} N_i(\xi, \eta, \zeta, \mu) k_i$$

If a grillage analysis is used to study the bridge deck behavior, when it's known the geometric inertia $I_{geometric}$ and the corresponding k coefficient, it's possible to obtain an equivalent correct inertia $I_{equivalent}$ to assign to the longitudinal members of grillage mesh, considering so the shear lag effects. Under linear elastic hypothesis

$$k = f_{theoretical} / f_{numeric} = I_{geometric} / I_{equivalent}$$

and

$$I_{equivalent} = I_{geometric} * k$$

In Tab.1 the k coefficients of the extremes of the corrective field are shown.

l/l_o	h/h_o	w/w_o	t/t_o	k
max	max	max	max	0,9164
max	max	max	min	0,6853
min	max	max	min	0,1553
min	max	max	max	0,5020
min	min	max	max	0,6158
min	min	max	min	0,3850
max	min	max	min	0,8557
max	min	max	max	0,9378
max	min	min	max	0,9894
max	min	min	min	0,9536
min	min	min	min	0,6836
min	min	min	max	0,9125
min	min	min	max	0,7615
min	min	min	min	0,3304
max	max	min	min	0,8571
max	max	min	max	0,9756

Table 1. Corrective coefficients for shear lag.

l span
w width of cell
h height of cell
t ratio between depth of web and flange:
$t = s_{web} / s_{flange}$

2.3 *Distortion*

Distortion of cells occurs when cells have few or no transverse diaphragms or internal bracing, so that a vertical shear force across a cell cause the slabs and webs to flex independently out of plane. The effects of distortion are usually considered in a grillage analysis by giving the transverse grillage members a low shear stiffness, chosen so that when the grillage members and cell are subjected to the same shear force, they experience similar distortion (Hambly, 1991). In this paper the distortional effects are considered assigning a correct inertia to the grillage transverse beams. Likewise shear lag case, the dependence to the section properties is analyzed referring to a unicellular deck of which the span, width and height of cell, depth of webs and flanges are changed in turn in order to obtain a great number of cases. The reference deck, having the same section properties used in shear lag study, is subjected now to a loading distribution so that distortion of cell is caused (Fig. 3).

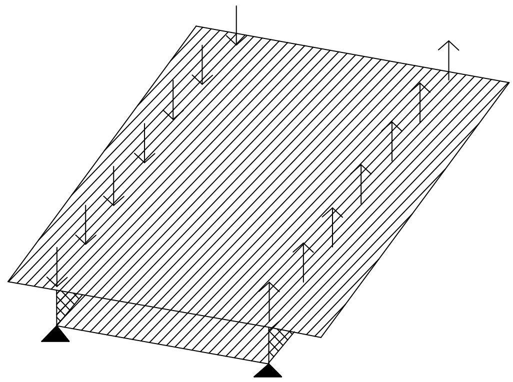

Figure 3. Loading condition for the reference deck.

The deck is studied through a FE analysis and the value of the vertical displacement in mid-span (corresponding to the intersection of web and flange) is compared to the one obtained from a grillage analysis in which distortion is first neglected. The distortional effects are considered by correcting the flectional rigidities of the transverse grillage members; particularly, a series of corrective coefficients, dividing the transverse inertia, are obtained to minimize the error of mid-span vertical displacement.

A corrective coefficient is considered valid when

$$\left|\frac{f - f'}{f}\right| * 100 \leq 1‰$$

where

f vertical displacement of the FE analysis

f' vertical displacement of the grillage analysis

Drawing a diagram of the variations of the corrective coefficients as to the different parameters, the curve obtained can be considered about constant for the parameter l / l_o and about linear for the other parameters. It's so possible to trust distortion depend only from width, height, depth of cell.

Likewise shear lag case, it's believed to create a field containing the corrective coefficients of all the analyzed cases; these coefficients depend from the ones obtained in the conditions h / h_o, w / w_o, t / t_o max and min which are multiplied by some functions describing their linear curve in the field. As for a three-dimensional field, these functions are the beam functions of a 3D 8 nodes element; their expressions is

$$N_i(\xi,\eta,\zeta) = (1/8)(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

where

ξ, η, ζ represent the coordinate system of a 4D field

ξ_i, η_i, ζ_i are the nodal coordinates or the extremes of the field

A coordinate transformation from a generic to a master field must be preventively done in order to use this simple expression. The transformation is similar to the one used in shear lag case. Once the corrective coefficients of the 8 extremes of the field are known, it's possible to obtain the corrective coefficient for any point of the field by using the relationship

$$d = d(\xi,\eta,\zeta) = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta)d_i$$

If a grillage analysis is used, it's so possible to consider the distortional effects correcting the inertia of the grillage transverse members through these coefficients. The equivalent inertia is

$$I_{equivalent} = I_{geometric} / d$$

In Tab.2 the corrective coefficients d for the extremes of the field are given.

h/h_o	w/w_o	t/t_o	d
max	max	max	0,8
max	max	min	1,6
max	min	max	0,95
max	min	max	1,6
min	max	min	0,6
min	max	max	1,2
min	min	min	0,7
min	min	max	1,3

Table 2. Corrective coefficients for distorsion.

w width of cell

h height of cell

t ratio between depth of web and flange:
$t = s_{web} / s_{flange}$

2.4 *Warping*

Warping is an out of plane displacement of point of cross-section. It's composed of two different components, torsional warping displacement, associated to a rigid twist of cross-section, and distortional warping displacement, associated to a distortion of cross section. Both these components give rise to the longitudinal normal stresses when warping is constrained (Maisel & Roll 1974).

In this paper only torsional warping is considered. It is not an immediate operation to introduce the effects of the no uniform torsion in a grillage analysis. A grillage model is avoid of the d.o.f. in warping direction and so is missing a parameter directly linked to warping displacement.

It's believed to introduce the phenomenon by operating on the terms of the torsion equation

$$EK_{xy}\beta^{IV} - GK_t\beta^{II} = ep$$

where

β torsion

p distributed load across span

e loading eccentricity

GK_t primary torsional rigidity

EK_{xy} secondary torsional rigidity

It's possible to obtain for this equation an approximate solution (Raithel, 1977). It's considered known, for a generic loading condition, the elastic line equation in the form

$$\eta(z) = \eta°f(z)$$

where $\eta°$ is the displacement of a particular section, $z = \zeta$, arbitrary, where $f(z) = 1$.
It's considered β in the form

$$\beta(z) = \beta°f(z)$$

It's supposed that approximately an analogy between $\beta(z)$ e $\eta(z)$ exists. The value $\beta°$ of the torsion is so the only one unknown of the problem. It can be obtained minimizing as to $\beta°$ the functional expression ξ composed of the elastic deformation energy and the loading work.

$$\xi = \int_o^l \left(\frac{EI_x}{2}\eta^{II2} + \frac{EK_{xy}}{2}\beta^{II2} + \frac{GK_t}{2}\beta^{I2} - p\eta - ep\beta \right) dz$$

By making an integral as to z, neglecting the terms in $\eta°$ (constant as to $\beta°$) and imposing the following equalities

$$\int_o^l f^{II2} dz = \Sigma^{II} \qquad \int_o^l f^{I2} dz = \Sigma^{I} \qquad \int_o^l pfdz = \Sigma^{P}$$

It's possible to obtain the value of $\beta°$

$$\xi = \frac{1}{2}\beta^{\circ 2}(EK_{xy}\Sigma^{II} + GK_t\Sigma^{I}) - e\beta^{\circ}\Sigma^{P}$$

$$\frac{\partial \xi}{\partial \beta^{\circ}} = \beta^{\circ}(EK_{xy}\Sigma^{II} + GK_t\Sigma^{I}) - e\Sigma^{P} = 0$$

$$\beta^{\circ} = \frac{e}{EK_{xy}\Sigma^{II} + GK_t\Sigma^{I}}\Sigma^{P}$$

Being equal the external and the internal work

$$\Sigma^{P} = \int_o^l pfdz = \frac{1}{\eta^{\circ}}\int_O^l \eta pdz = \frac{1}{\eta^{\circ}} EI_x \int_o^l \eta^{II2} dz = \eta^{\circ} EI_x \Sigma^{II}$$

Substituting Σ^P expression in $\beta°$ one, is

$$\beta^{\circ} = \eta^{\circ}\frac{eEI_x}{EK_{xy} + GK_t\gamma} \quad \text{where} \quad \gamma = \frac{\Sigma^{I}}{\Sigma^{II}}$$

In a grillage analysis is usually considered only the uniform torsion contribution. To neglect the secondary rigidity associated to the no uniform torsion, let to overestimate the value of $\beta°$.

If both uniform and no uniform torsional contributes are considered, the value of $\beta°$ is

$$\beta^{\circ} = \eta^{\circ}\frac{eEI_x}{EK_{xy} + GK_t\gamma}$$

Call this value β_1 and the corresponding primary rigidity $K_{t\ geometric}$.

If the no uniform rigidity is neglected ($K_{xy} = 0$), the value of $\beta°$ is

$$\beta^{\circ} = \eta^{\circ}\frac{eEI_x}{GK_t\gamma}$$

Call this value β_2 and the corresponding primary rigidity $K_{t\ equivalent}$.

It's clearly $\beta_2 > \beta_1$. If now $K_{t\ geometric}$ is considered known and $K_{t\ equivalent}$ unknown, from the equality $\beta_2 = \beta_1$ it's possible to obtain the value of primary torsional rigidity which includes no uniform torsional contribute. It's so possible to assign to the grillage longitudinal members the torsional constant

$$K_{t\ equivalent} = K_{t\ geometric} * \alpha$$

where α is a corrective coefficient subsequently shown.

$$\alpha = \frac{K_{tequivalent}}{K_{tgeometric}} = \frac{1}{K_{tgeometric}}\eta^{\circ}\frac{eEI_x}{\beta_1 G\gamma}$$

$$\alpha = \frac{1}{K_{tgeometric}}\eta^{\circ}\frac{eEI_x}{G\gamma\left(\frac{\eta^{\circ}eEI_x}{EK_{xy} + GK_{tgeometric}\gamma}\right)}$$

and

$$\alpha = 1 + \frac{EK_{xy}}{\gamma GK_{tgeometric}}$$

So therefore

$\alpha = 1$ if $K_{xy} = 0$
$\alpha > 1$ if $K_{xy} \neq 0$

Once the section properties and the coefficient γ are known, it's possible to calculate the coefficient α correcting torsional rigidities of grillage longitudinal members. As for γ, a good approximation of this coefficient is

$$\gamma = \frac{l^2}{10}$$

where l is the span of deck.

3 AN APPLICATION

It's related the study of a multi-cellular deck in c.a. with a span of 30 metres. The deck is subjected to a vertical load distributed across the outside wall of 50 kg/cm and to a vertical load of 180 ton concentrated in mid-span for the same wall (Fig. 4).

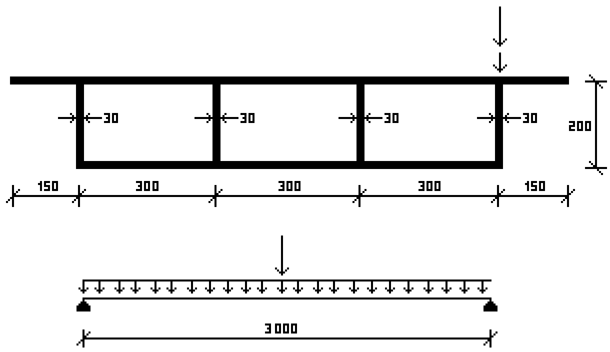

Figure 4. Multi-cellular deck: cross-section and loading condition.

Using the geometrical and loading symmetry, only one half of the deck is studied with a grillage mesh composed of 4 longitudinal members corresponding to the webs of deck and of 10 transverse members with a step of 150 cm (Fig. 5).

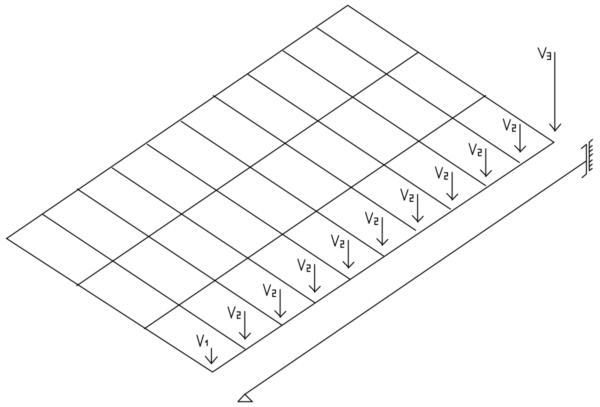

Figure 5. Grillage mesh: geometry and equivalent nodal loads.

In the following tables (Tab.1-4), the equivalent nodal loads, the corrective coefficients, the elementary and correct rigidities are shown. It's also shown a comparison of the longitudinal deformation for the loaded wall obtained with the different analysis (Fig. 6).

Equivalent nodal loads	
V_1	3750
V_2	7500
V_3	93750

Table 1. Equivalent nodal loads.

Beam	Longitudinal		Transverse	
	Internal	External	Internal	External
I	2*E8	1,5*E8	6,7*E5	3,35*E5
C	4,32*E8	2,16*E8	1,3*E6	6,75*E5
A	6*E3	6*E3	4,5*E3	2,25*E5

Table 2. Elementary rigidities.

Corrective coefficients	
k	0,71496
d	1,33611
α	1,067

Table 3. Corrective coefficients.

Beam	Longitudinal		Transverse	
	Internal	External	Internal	External
I	2*E8	1,09*E8	5,05*E5	2,52*E5
C	4,34*E8	2,17*E8	1,3*E6	6,7*E5
A	6*E3	6*E3	4,5*E3	2,25*E5

Table 4. Correct rigidities.

It's possible to note how a grillage analysis which considered the effects of shear lag, distortion and warping, gives results which approximate accurately ones obtained from a FE analysis.

Particularly, the error of the vertical displacement in mid-span is reduced from a 13,91% for an "elementary" grillage analysis, to the 1,37% for a "correct" grillage analysis in which secondary deformation modes are included.

Note

The values are expressed in kg-cm. In Figure 6 one half of the deformation is shown; the section 1 corresponds to the bearing, the section 11 corresponds to mid-span.

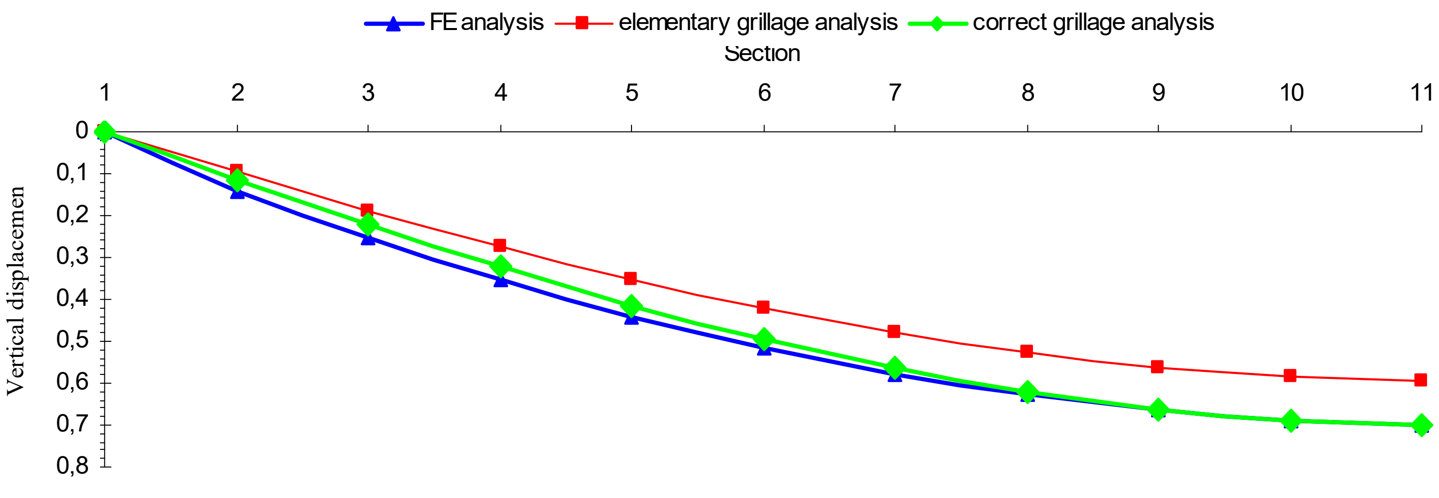

Figure 6. Longitudinal deformation of the loaded wall.

4 CONCLUSIONS

The object of this paper is the study of the representation of the grillage models with which different types of bridge decks can be represented. The purpose of the work is to contribute to this type of approach through the introduction of the effects of shear lag, distortion and warping, usually neglected. The introduction of these effects in a grillage analysis is obtained by applying a series of corrective coefficients to the elementary rigidities of the grillage members. These coefficients are obtained through theoretical considerations and comparisons with other techniques of modelling.

From the effected analysis, it's appeared how the use of these coefficients reinforce the physical equivalence between the real structure and the grillage model; particularly, a correct grillage analysis gives results in terms of stresses and displacements, comparable to the ones obtained from a FE analysis.

The accuracy of the results, the application of this kind of approach also to complex structures, the limited number of d.o.f. and so of dates, the direct use of the results, expressed in terms of generalized stresses (V, M, T), in design procedures, all these factors constitute the main advantages of this modelling.

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