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Communications in Computer and Information Science 3020

Information Processing and Management of Uncertainty in Knowledge-Based Systems

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Ideally Exact Categories, Varieties of Universal Algebras and Multi-valued Logics

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Abstract. We study the notions of *coherent* and *ideal* actions in the setting of ideally exact varieties of universal algebras. We prove that, if V is a semi-abelian variety and U is obtained from V by freely adding nullary operations together with suitable identities, then the ideally exact context determined by the associated free–forgetful monadic adjunction with cartesian unit has a *good theory of actions* if and only if a compatibility condition between coherent actions and the added identities is satisfied. Our setting applies in particular to certain categories of interest in algebraic logic, including MV-algebras and product algebras.

Keywords: Ideally exact category · Universal algebra · Internal action · Multi-valued logic · MV-algebra · Product algebra · Hoop

1 Introduction

The notion of *internal action* was developed by D. Bourn and G. Janelidze in [8], and later systematized by F. Borceux, G. Janelidze and G. M. Kelly in [6], with the goal of extending the correspondence between actions and split extensions from the setting of groups to more general semi-abelian categories [24]. In [8], the authors proved that in any semi-abelian category C, every split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} B$$

gives rise to a $B\flat(-)$-algebra (X, ξ) for a suitable monad $B\flat(-)\colon \mathsf{C} \to \mathsf{C}$, where X is given by a kernel $k\colon X \to A$ of p, and $\xi\colon B\flat X \to X$ is a morphism of C which is called *internal action* of B on X.

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When the category is not pointed, the general theory introduced in [8] only allows us to describe split epimorphisms over B in terms of split epimorphisms over another object E, whenever there exists a morphism $E \to B$.

Nevertheless, in various non-pointed algebraic settings, alternative notions of *external* action have long been available—for instance in the context of unital algebras and unital rings [27]. These typically arise by adding suitable axioms to the usual definition of action valid in a pointed—and, hence, semi-abelian—context (see, for instance, [31] where external actions of Orzech categories of interest are studied).

To partially overcome this gap, the authors of [28] defined the notions of *coherent action* and of *ideal action* in the context of *ideally exact categories*, a concept due to G. Janelidze [23] that extends semi-abelian categories and includes important non-pointed examples such as the categories Ring and CRing of (commutative) unital rings. From an algebraic point of view, coherent actions generalise unital actions of rings, where the unit element is required to *act* like a unit, while ideal actions capture the familiar situation of a unital ring acting on an ideal. It was also proved in [28] that every ideal action is coherent, and that the converse statement holds in some relevant ideally exact contexts, which are called *BAT* (see Definition 6). Examples of BAT ideally exact contexts are given by unital non-associative $\mathbb{F}$-algebras or rings, and $\mathsf{Set}^{\mathrm{op}}$, the dual of the category of sets.

The aim of this manuscript is to study a class of ideally exact contexts within the framework of varieties of universal algebras. We characterize the conditions under which an ideally exact context arising from a free–forgetful monadic adjunction with cartesian unit

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"', ""{name=A}] & \mathsf{V}, \arrow[l, bend right, "F"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd} \tag{1}$$

where V is a semi-abelian variety, U is an ideally exact variety obtained from V by freely adding a set of nullary operations together with a certain set of identities, has a *good theory of actions*.

This setting encompasses certain categories of interest in the study of multi-valued logics (see [12] and [19]), including *MV-algebras* and *product algebras*, which form the algebraic semantics for *infinite-valued Łukasiewicz logic* (see [11, 16] and [26]) and *product logic* (see [2, 15] and [22]), respectively.

We end the manuscript with some open questions and future directions, with a focus on the ideally exact contexts determined by the variety of *Gödel algebras* and *BL-algebras* [4], whose associated propositional calculi are, respectively, *Gödel logic* and *basic logic* (see [13] and [21]).

2 Preliminaries

Let C be a category with pullbacks, with initial object I_{C} and terminal object T_{C}. For any object X of C, we denote by ι_X the unique map $I_{\mathsf{C}} \to X$, and by τ_X

the unique map $X \to T_{\mathsf{C}}$. When the category is *pointed*, i.e., when the unique map $I_{\mathsf{C}} \to T_{\mathsf{C}}$ is an isomorphism, we denote the initial/terminal object with 0_{C}.

We recall that the adjunction associated with the unique morphism $I_{\mathsf{C}} \to T_{\mathsf{C}}$ is

$$\begin{tikzcd} \mathsf{C} \cong (\mathsf{C} \downarrow T_{\mathsf{C}}) \arrow[r, "U"', ""{name=A}] & (\mathsf{C} \downarrow I_{\mathsf{C}}), \arrow[l, bend right, "F"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd} \tag{2}$$

where $(\mathsf{C} \downarrow I_{\mathsf{C}})$ and $(\mathsf{C} \downarrow T_{\mathsf{C}})$ denote, respectively, the slice categories over the objects I_{C} and T_{C}, U is the functor defined by pulling back along $I_{\mathsf{C}} \to T_{\mathsf{C}}$ the object $\tau_X : X \to T_{\mathsf{C}}$ of $(\mathsf{C} \downarrow T_{\mathsf{C}})$, and F is its left adjoint.

Remark 1. Let

$$\begin{tikzcd} \mathsf{C} \arrow[r, "U"', ""{name=A}] & \mathsf{D} \arrow[l, bend right, "F"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd} \tag{3}$$

be an adjunction between categories with pullbacks, and assume that D is pointed. Then, the unit of the adjunction $\eta : 1_{\mathsf{D}} \Rightarrow UF$, where $UF = U \circ F$, is cartesian if and only if, for any object X of D, the map $\eta_X : X \to UF(X)$ defines a kernel of $UF(\tau_X) : UF(X) \to UF(0_{\mathsf{D}})$, see [23, Remark 2.6]. Equivalently, the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\tau_X"'] & UF(X) \arrow[d, "UF(\tau_X)"] \\ 0_{\mathsf{D}} \arrow[r, "\eta_0"'] & UF(0_{\mathsf{D}}). \end{tikzcd}$$

This happens, for instance, when $\mathsf{D} = (\mathsf{C} \downarrow I_{\mathsf{C}})$ and 3 coincides with (2).

2.1 Ideally Exact Categories

We now recall the definition of *ideally exact category*, which was introduced by G. Janelidze in [23] as a non-pointed counterpart of semi-abelian categories.

Definition 1. *[23] A category* U *is* ideally exact *if it is Barr exact, Bourn protomodular, has finite coproducts, and the unique morphism* $I_{\mathsf{U}} \to T_{\mathsf{U}}$ *in* U *is a regular epimorphism.*

Examples of ideally exact categories are all semi-abelian categories, the categories of unital and commutative unital rings, every category of unital $\mathbb{F}$-algebras (see [9] and [18]), any cotopos, and all non-trivial protomodular varieties of universal algebras. The latter encompass some categories of particular relevance in algebraic logic, such as the varieties of *BL-algebras* [21], *MV-algebras* [10], *Gödel algebras* and *product algebras* [15].

Since the pullback functor $\mathsf{U} \to (\mathsf{U} \downarrow I_{\mathsf{U}})$ along the regular epimorphism $I_{\mathsf{U}} \to T_{\mathsf{U}}$ is monadic, ideal exactness can be expressed in terms of adjoint pairs.

Theorem 1. *[23] Let* U *be a category with pullbacks. The following conditions are equivalent:*

1. U *is ideally exact.*
2. U *is Barr-exact, has finite coproducts and there exists a monadic functor* $\mathsf{U} \to \mathsf{V}$*, where* V *is a semi-abelian category.*
3. There exists a monadic functor $\mathsf{U} \to \mathsf{V}$*, where* V *is a semi-abelian category, such that the underlying functor of the corresponding monad preserves regular epimorphisms and kernel pairs.*

Remark 2. Let U be an ideally exact category. It follows from Theorem 1 that there exists a monadic adjunction

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{V} \arrow[l, bend right, "F"'] \end{tikzcd}$$

where V is a semi-abelian category. This adjunction is associated with the unique morphism $I_{\mathsf{U}} \to T_{\mathsf{U}}$ (up to an equivalence) if and only if its unit is cartesian.

Of course the monadic adjunction associated to $I_{\mathsf{U}} \to T_{\mathsf{U}}$ is always there: indeed, one may take $\mathsf{V} = (\mathsf{U} \downarrow I_{\mathsf{U}})$ with U and F defined in the obvious way. However, this construction is not always the most natural choice. In particular, if U is already semi-abelian, it may be preferable to take $\mathsf{V} = \mathsf{U}$.

Example 1. Let $\mathsf{U} = \mathsf{MVAlg}$ be the category of MV-algebras, which has the two-element Boolean algebra $L_2 = \{0, 1\}$ as initial object, and the trivial MV-algebra $\{1\}$ as terminal one. Then a monadic adjunction as in Remark 2 is given by

$$\begin{tikzcd} \mathsf{MVAlg} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{WHoops}, \arrow[l, bend right, "M"'] \end{tikzcd} \tag{4}$$

where WHoops is the semi-abelian category of Wajsberg hoops [3], U is the functor that forgets the bottom element 0[1], and the functor M, which is called the *MV-closure* in [1], maps every Wajsberg hoops $H = (H, \cdot, \to, 1)$ to the MV-algebra

$$M(H) = (H \times \{0, 1\}, \cdot, \to, 0, 1),$$

where $0 := (1, 0)$, $1 := (1, 1)$,

$$(a, i) \cdot (b, j) = \begin{cases} (a \cdot b, 1), & \text{if } i = j = 1, \\ (a \to b, 0), & \text{if } i = 1,\ j = 0, \\ (b \to a, 0), & \text{if } i = 0,\ j = 1, \\ ((a \to (a \cdot b)) \to b, 0), & \text{if } i = j = 0. \end{cases}$$

and

$$(a, i) \to (b, j) = \begin{cases} (a \to b, 1), & \text{if } i = j = 1, \\ (a \cdot b, 0), & \text{if } i = 1,\ j = 0, \\ ((a \to (a \cdot b)) \to b, 1), & \text{if } i = 0,\ j = 1, \\ (b \to a, 1), & \text{if } i = j = 0. \end{cases}$$

[1] It was proved in [14] that the class of MV-algebras is term equivalent to the class of *Wajsberg algebras* [17], i.e., Wajsberg hoops with a bottom element 0 satisfying the identity $0 \to a = 1$.

The unit $\eta\colon 1_{\mathsf{WHoops}} \Rightarrow UM$ of the adjunction $M \dashv U$ is cartesian. In fact, for any Wajsberg hoop H, the homomorphism

$$\eta_H\colon H \to UM(H)\colon x \mapsto (x, 1)$$

defines the kernel of

$$UM(\tau_H)\colon UM(H) \to U(L_2)\colon (x, i) \mapsto i.$$

Thus, by Remark 2, the adjunction (4) is, up to an equivalence, the adjunction associated with the unique morphism $L_2 \to \{1\}$ in MVAlg.

Definition 2. *[28] Let* U *be an ideally exact category, let* V *be a semi-abelian category and let*

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{V} \arrow[l, bend right, "F"'] \end{tikzcd} \tag{5}$$

be a monadic adjunction with cartesian unit. We refer to such a situation as an ideally exact context.

2.2 Coherent and Ideal Actions

In this section, we recall the definitions of *coherent* internal action and *ideal* internal action, which were introduced in [28] in order to generalise different aspects of unital actions of rings and algebras.

Definition 3. *[28] Given an ideally exact context* (5)*, let* B *be an object of* U *and let* X *be an object of* V*. A* relative U-action *of* B *on* X *is an internal action* $\xi\colon U(B)\flat X \to X$ *in* V.

Definition 4. *[28] Given an ideally exact context* (5)*, let* B *be an object of* U.

(i) A split epimorphism

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"] \end{tikzcd} \tag{6}$$

in V *is* ideal*, if there exist a split epimorphism*

$$\begin{tikzcd} A' \arrow[r, shift left, "p'"] & B \arrow[l, shift left, "s'"] \end{tikzcd}$$

in U *and an isomorphism* $\sigma\colon U(A') \to A$ *such that the following diagram in* V

$$\begin{tikzcd} U(A') \arrow[rr, "\sigma"] \arrow[dr, shift right, "U(p')"'] & & A \arrow[dl, shift right, "p"'] \\ & U(B) \arrow[ul, shift right, "U(s')"'] \arrow[ur, shift right, "s"'] & \end{tikzcd}$$

is commutative.

(ii) A morphism

$$\begin{tikzcd} A_1 \arrow[rr, "h"] \arrow[dr, shift right, "p_1"'] & & A_2 \arrow[dl, shift left, "p_2"'] \\ & U(B) \arrow[ul, shift right, "s_1"'] \arrow[ur, shift left, "s_2"'] & \end{tikzcd}$$

between ideal split epimorphisms over $U(B)$ is an ideal morphism *if there exists a morphism*

$$\begin{tikzcd} A'_1 \arrow[rr, "h'"] \arrow[dr, shift right, "p'_1"'] & & A'_2 \arrow[dl, shift left, "p'_2"'] \\ & B \arrow[ul, shift right, "s'_1"'] \arrow[ur, shift left, "s'_2"'] & \end{tikzcd}$$

between the corresponding split epimorphisms over B such that $U(h') = h$.

Remark 3. Given an ideal split epimorphism (6), the object $U(B)$ acts on the domain of the *relative U-ideal* $k\colon X \to U(A')$ (see [25]), where (X, k) is a kernel of $U(p')$.

Definition 5. *Let $\xi\colon U(B)\flat X \to X$ be a relative U-action. We say that:*

(i) ξ is a coherent action *if $\xi \circ (U(\iota_B)\flat\, \mathrm{id}_X) = \xi_0$, where $\xi_0\colon UF(0_{\mathsf{C}})\flat X \to X$ is the relative U-action associated with the canonical split epimorphism*

$$UF(X) \underset{UF(\iota_X)}{\overset{UF(\tau_X)}{\rightleftarrows}} UF(0_{\mathsf{V}}).$$

(ii) ξ is an ideal action *if the corresponding split epimorphism*

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$$

is ideal.

(iii) A morphism between ideal actions is ideal *if the corresponding morphism between the corresponding ideal split epimorphisms is ideal.*

In the language of split epimorphisms, we say that

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$$

is a *coherent* split epimorphism if it comes from a coherent relative U-action.

We now recall the following lemma, which allows us to relate coherent actions and split extensions.

Lemma 1. *[28] Let $\xi\colon U(B)\flat X \to X$ be a relative U-action, and let*

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$$

be the split epimorphism associated with ξ. Then, ξ is coherent if and only if there exists a morphism $f\colon UF(X) \to A$ in V such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & UF(X) \arrow[r, shift left, "UF(\tau_X)"] \arrow[d, dashed, "f"'] & UF(0_{\mathsf{V}}) \arrow[l, shift left, "UF(\iota_X)"] \arrow[d, "U(\iota_B)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a morphism of split extensions. Furthermore, the square on the right is a split pullback.

It was proved in [28, Theorem 4.11] that every ideal action is coherent. At the moment, the problem of whether the converse of this statement holds in general remains open. We thus recall the following definition.

Definition 6. *[28] An ideally exact context* (5) *has a* good theory of actions *(or, it is BAT*[2]*, for short) if all coherent actions are ideal, and all morphisms of such actions are ideal.*

In the next section, we characterize a class of BAT ideally exact contexts within the framework of varieties of universal algebras. This class encompasses, among others, some categories of interest in algebraic logic, including the varieties of MV-algebras and product algebras.

3 Ideally Exact Varieties of Algebras

Let U (resp. V) be a non-trivial variety of universal algebras with signature Σ_{U} (resp. Σ_{V}) defined by a set Z_{U} (resp. Z_{V}) of identities. Moreover, we require V to be semi-abelian with a unique constant 0, $\Sigma_{\mathsf{V}} \subseteq \Sigma_{\mathsf{U}}$, $Z_{\mathsf{V}} \subseteq Z_{\mathsf{U}}$ and $\Sigma_{\mathsf{U}} \setminus \Sigma_{\mathsf{V}}$ to be a set of nullary operations, that can be interpreted as constants of the variety. We denote by $\Gamma := I_{\mathsf{U}} \setminus 0_{\mathsf{V}} = I_{\mathsf{U}} \setminus T_{\mathsf{U}}$ the set of constants in U which do not belong to V, and by $U\colon \mathsf{U} \to \mathsf{V}$ the functor which forgets all the constants in Γ as well as all the identities in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, i.e., all the identities involving such constants. Furthermore, for any object A of U and for any $c \in \Gamma$, we denote by c^A the interpretation of the constant c in A.

Remark 4. The forgetful functor U is a faithful conservative right adjoint to the functor $F\colon \mathsf{V} \to \mathsf{U}$ which freely adds the constants of Γ. Thus, we have the free-forgetful adjunction

$$\begin{tikzcd}
\mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathsf{V}, \arrow[l, bend right, "F"']
\end{tikzcd} \tag{7}$$

which is the adjunction associated with the unique morphism $I_{\mathsf{U}} \to T_{\mathsf{U}}$ if and only if the unit $\eta\colon 1_{\mathsf{V}} \Rightarrow UF$ is cartesian.

[2] The acronym BAT is inspired by the notion of BIT-variety, where BIT stands for **B**uona (good, in Italian) **I**deal **T**heory, introduced by A. Ursini in [33]. Analogously, BAT stands for **B**uona **A**ction **T**heory.

From now on, we suppose that the unit of the free-forgetful adjunction (7) is cartesian.

We recall from [7] that, if U is a protomodular variety, then there exists n 0-ary terms $c_1, \ldots, c_n$, n binary terms $\alpha_1, \ldots, \alpha_n$, and an $(n+1)$-ary term θ such that

$$\alpha_i(x, x) = c_i \tag{8}$$

and

$$\theta(\alpha_1(x, y), \alpha_2(x, y), \ldots, \alpha_n(x, y), y) = x. \tag{9}$$

We further recall that a protomodular variety U is called *classically ideal determined* [32] when it satisfies the conditions (8) and (9), with all the 0-ary terms c_i equal to the unique constant 0 of V.

Remark 5. [23] Since it is a right adjoint, U reflects protomodularity. In particular, U is a classically ideal determined and protomodular variety, and therefore an ideally exact category.

Proposition 1. *Let U be a classically ideal determined variety, and let s, t be terms of Σ_{U}. Then, the identity $s = t$ holds in U if and only if $\alpha_i(s, t) = 0$, for any $i = 1, \ldots, n$.*

Proof. Suppose that $s = t$. Then,

$$\alpha_i(s, t) = \alpha_i(t, t) = 0$$

for any $i = 1, \ldots, n$. Conversely, if $\alpha_i(s, t) = 0$ for any $i = 1, \ldots, n$, then

$$s = \theta(\alpha_1(s, t), \ldots, \alpha_n(s, t), t) = \theta(0, \ldots, 0, t) = t.$$

It follows that any identity $s = t$ in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, where

$$s = s(a_1, \ldots, a_{p_s}, c_1, \ldots, c_{q_s}), \quad t = t(b_1, \ldots, b_{p_t}, d_1, \ldots, d_{q_t})$$

and $c_1, \ldots, c_{q_s}, d_1, \ldots d_{q_t} \in \Gamma$, may be written as

$$\omega(a_1, \ldots, a_p, c_1, \ldots, c_q) = 0,$$

where ω is a term and $c_1, \ldots, c_q \in \Gamma$.

Remark 6. Let $\omega(a_1, \ldots, a_p, c_1, \ldots, c_q) = 0$ be an identity in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$ and consider a split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B).$$

Then $\omega(a_1, \ldots, a_p, s(c_1^B), \ldots, s(c_q^B)) \in \operatorname{Ker} p$, for any $a_1, \ldots, a_p \in A$. Indeed, one has

$$\begin{aligned} p(\omega(a_1, \ldots, a_p, s(c_1^B), \ldots, s(c_q^B))) &= \omega(p(a_1), \ldots, p(a_p), p(s(c_1^B)), \ldots, p(s(c_q^B))) \\ &= \omega(p(a_1), \ldots, p(a_p), c_1^B, \ldots, c_q^B) = 0^B. \end{aligned}$$

We now determine the conditions under which the converse of [28, Theorem 4.11] holds in the ideally exact context (7); namely, we seek to identify when a coherent relative U-action is in fact ideal.

Theorem 2. *Consider the ideally exact context (7), and let*

$$\xi \colon U(B)\flat X \to X$$

be a coherent relative U-action with associated split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B). \tag{10}$$

Then, ξ is an ideal action if and only if

$$\omega_A = \omega(\omega_A, \dots, \omega_A, s(c_1^B), \dots, s(c_q^B)), \tag{11}$$

for every identity $\omega = 0$ in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, and for every $a_1, \dots, a_p \in A$, where ω_A denotes the element $\omega(a_1, \dots, a_p, s(c_1^B), \dots, s(c_q^B)) \in A$.

Proof. If ξ is an ideal action, then its associated split epimorphism is of the form

$$U(A) \underset{U(s)}{\overset{U(p)}{\rightleftarrows}} U(B).$$

Hence, for any identity $\omega = 0$ in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, one has

$$\omega_A = \omega(a_1, \dots, a_p, s(c_1^B), \dots, s(c_q^B)) = 0^A$$

for any $a_1, \dots, a_p \in A$, since $s(c_i^B) = c_i^A$. Thus, (11) holds.

Conversely, suppose that (11) holds. By Lemma 1, there exists a morphism $f \colon UF(X) \to A$ in V such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & UF(X) \arrow[r, shift left, "UF(\tau_X)"] \arrow[d, dashed, "f"'] & UF(0_{\mathsf{V}}) \arrow[l, shift left, "UF(\iota_X)"] \arrow[d, "U(\iota_B)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"]
\end{tikzcd}$$

is commutative, where $X = \operatorname{Ker} p = \{a \in A \mid p(a) = 0^B\}$ and k denote the canonical inclusion.

In order to prove that ξ is an ideal action, we need to check that $\omega_A = 0^A$, for any identity $\omega = 0$ in $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, and for any $a_1, \dots, a_p \in A$.

By Remark 6, one has that $\omega_A \in X$. Hence

$$\begin{aligned}
0^A &= f(0^{UF(X)}) = f(\omega(\eta_X(\omega_A), \dots, \eta_X(\omega_A), c_1^{UF(X)}, \dots, c_q^{UF(X)})) \\
&= f(\omega(\eta_X(\omega_A), \dots, \eta_X(\omega_A), UF(\iota_X)(c_1^{UF(0_{\mathsf{V}})}), \dots, UF(\iota_X)(c_q^{UF(0_{\mathsf{V}})})) \\
&= \omega(f(\eta_X(\omega_A)), \dots, f(\eta_X(\omega_A)), f(UF(\iota_X)(c_1^{UF(0_{\mathsf{V}})})), \dots, f(UF(\iota_X)(c_q^{UF(0_{\mathsf{V}})}))) \\
&= \omega(k(\omega_A), \dots, k(\omega_A), s(U(\iota_B)(c_1^{UF(0_{\mathsf{V}})})), \dots, s(U(\iota_B)(c_q^{UF(0_{\mathsf{V}})}))) \\
&= \omega(\omega_A, \dots, \omega_A, s(c_1^B), \dots, s(c_q^B)) \\
&= \omega(a_1, \dots, a_p, s(c_1^B), \dots, s(c_q^B)) = \omega_A.
\end{aligned}$$

Therefore, A is an object of U, $c^A = s(c^B)$ for any constant $c \in \Gamma$, and (10) is a split epimorphism in the ideally exact variety U since

$$p(c^A) = p(s(c^B)) = (p \circ s)(c^B) = c^B.$$

Thus, the internal action ξ is ideal.

Remark 7. We observe that the condition (11) is not too restrictive, since, for instance, it holds in any variety of hoops (see [5] and [29]). Consider the ideally exact context

$$\begin{tikzcd} \mathsf{MVAlg} \arrow[r, "U"', ""{name=U}] & \mathsf{WHoops} \arrow[l, bend right, "M"', ""{name=M}] \arrow[from=M, to=U, phantom, "\perp"] \end{tikzcd},$$

where we recall that MV-algebras may be described in terms of Wajsberg algebras. In this case, 1 is the unique constant of the semi-abelian variety WHoops, $\Gamma = L_2 \setminus \{1\} = \{0\}$, and the unique identity which involves the constant 0 is $\omega(a, 0) = 1$, where $\omega(a, 0) = 0 \to a$. Thus, for any split epimorphism (10), one has

$$\begin{aligned} \omega(\omega(a, s(0^B)), s(0^B)) &= s(0^B) \to (s(0^B) \to a) \\ &= (s(0^B) \cdot s(0^B)) \to a \\ &= s(0^B \cdot 0^B) \to a \\ &= s(0^B) \to a = \omega(a, s(0^B)), \end{aligned}$$

for any $a \in A$, since the identity $a \to (b \to c) = (a \cdot b) \to c$ holds in any hoop.

We can now state the following characterization, whose proof is a direct consequence of Theorem 2.

Theorem 3. *Consider the ideally exact context* (7) *and suppose that* (11) *holds in* V, *for any coherent split epimorphism* (10) *and for any identity* $\omega = 0$ *in* $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$. *Then, a relative* U*-action*

$$\xi\colon U(B) \flat X \to X$$

with associated split epimorphism

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"] \end{tikzcd}$$

is coherent if and only if A *is an object of* U *with* $c^A = s(c^B)$ *for any* $c \in \Gamma$, *and* $\omega(a_1, \ldots, a_p, s(c_1^B), \ldots, s(c_q^B)) = 0_A$ *for any identity* $\omega = 0$ *in* $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$, *and for any* $a_1, \ldots, a_p \in A$.

Furthermore, we can conclude with the following.

Theorem 4. *The ideally exact context*

$$\begin{tikzcd} \mathsf{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=1.5ex, "\perp"] & \mathsf{V} \arrow[l, bend right, "F"'] \end{tikzcd}$$

is BAT if and only if condition (11) *holds for any coherent split epimorphism and for any identity* $\omega = 0$ *in* $Z_{\mathsf{U}} \setminus Z_{\mathsf{V}}$.

Proof. By Theorem 2, we know that any coherent action is ideal if and only (11) holds. It remains to show that every morphism between ideal split epimorphisms is ideal.

Let B be an object of U and consider a morphism

$$\begin{tikzcd} A_1 \arrow[rr, "h"] \arrow[dr, shift right, "p_1"'] & & A_2 \arrow[dl, shift left, "p_2"'] \\ & U(B) \arrow[ul, shift right, "s_1"'] \arrow[ur, shift left, "s_2"'] & \end{tikzcd}$$

of ideal split epimorphisms over $U(B)$. It follows from Theorem 3 that A_i is an algebra of U with $c^{A_i} = s_i(c^B)$, for any $i = 1, 2$ and for any $c \in \Gamma$, and

$$h(c^{A_1}) = h(s_1(c^B)) = s_2(c^B) = c^{A_2}.$$

Thus, h is a morphism in the ideally exact variety U, i.e., it is an ideal morphism.

Example 2. As mentioned above, the free-forgetful adjunction

$$\begin{tikzcd} \mathsf{MVAlg} \arrow[r, "U"'] \arrow[r, phantom, shift left=1.5ex, "\perp"] & \mathsf{WHoops} \arrow[l, bend right, "M"'] \end{tikzcd}$$

has cartesian unit. Moreover, the condition (11) is satisfied by any split epimorphism (10) over $U(B)$. Hence, this ideally exact context has a good theory of actions (see [28, Theorem 6.31]).

Example 3. In a similar way, let PHoops be the semi-abelian variety of product hoops [30], let PAlg be the ideally exact variety of product algebras and consider the free-forgetful adjunction

$$\begin{tikzcd} \mathsf{PAlg} \arrow[r, "U"'] \arrow[r, phantom, shift left=1.5ex, "\perp"] & \mathsf{PHoops}, \arrow[l, bend right, "K"'] \end{tikzcd}$$

where U denotes the forgetful functor and K is its left adjoint (we refer the reader to [20], where an explicit description of the functor K is available). One may check that $K \dashv U$ has cartesian unit and the condition (11) is satisfied by any split epimorphism (10) over $U(B)$. Thus, the ideally exact context determined by the variety of product algebras is BAT (see [28, Theorem 6.40]).

Example 4. Let Rng be the category of rings and let Ring be the subcategory of unital rings, which has $\mathbb{Z}$ as initial object and the zero ring $\{0\}$ as terminal one. Thus, a monadic adjunction with cartesian unit associated with the unique morphism $\mathbb{Z} \to \{0\}$ in Ring is

$$\begin{tikzcd} \mathsf{Ring} \arrow[r, "\perp", "U"'] & \mathsf{Rng}, \arrow[l, bend right, "F"'] \end{tikzcd} \tag{12}$$

where U is the forgetful functor and F maps every ring A to the semidirect product $\mathbb{Z} \ltimes A$. In this case, 0 is the unique constant of the semi-abelian variety Rng, $\Gamma = \mathbb{Z} \setminus \{0\}$, and the identities in $Z_{\mathsf{Ring}} \setminus Z_{\mathsf{Rng}}$ are $\omega_1(a, 1) = \omega_2(a, 1) = 0$, where $\omega_1(a, 1) = a - a \cdot 1$ and $\omega_2(a, 1) = a - 1 \cdot a$. One may check that, for any split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B),$$

one has

$$\omega_i(\omega_i(a, s(1^B)), s(1^B)) = \omega_i(a, s(1^B)),$$

for any $a \in A$ and for any $i = 1, 2$. Therefore, the ideally exact context (12) is BAT.

4 Conclusions and Future Directions

Examples 2, 3 and 4 show that Theorem 2 allows us to recover a class of BAT ideally exact contexts already known in the literature.

As a direction for future research, one may attempt to extend this analysis to the varieties of Gödel algebras and BL-algebras. However, in these cases no explicit description is currently available for the left adjoints of the forgetful functors

$$U \colon \mathsf{GAlg} \to \mathsf{GHoops} \qquad \text{and} \qquad U \colon \mathsf{BLAlg} \to \mathsf{BHoops},$$

where GHoops and BHoops denote the semi-abelian varieties of Gödel hoops and basic hoops, respectively.

As a consequence, it is not known whether the units of the adjunctions

$$\begin{tikzcd} \mathsf{GAlg} \arrow[r, "\perp", "U"'] & \mathsf{GHoops} \arrow[l, bend right, "G"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathsf{BLAlg} \arrow[r, "\perp", "U"'] & \mathsf{BHoops} \arrow[l, bend right, "B"'] \end{tikzcd}$$

are cartesian. Consequently, it remains an open problem whether the ideally exact contexts arising from Gödel algebras and BL-algebras admit a good theory of actions.

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