

UNIVERSITÉ CATHOLIQUE DE LOUVAIN

**Coverings, Factorization systems
and Closure operators in the
category of quandles**

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Introduction

The goal of this thesis is to grasp some categorical properties of the category $\mathbf{Qnd}$ of *quandles* and investigate some well-known concepts coming from category theory in the particular context of the category of quandles. In particular, a complete description of the *coverings*, of the *factorization systems* and of the *closure operators* all related to the adjunction between the category of quandles and its subcategory of *trivial quandles* is provided. This introduction begins with a brief review of the history of quandles before turning to the categorical world where we give a quick introduction to the aforementioned categorical concepts studied in this thesis.

Quandles

In a Euclidean space E , a *reflection* is a mapping from E to itself that is an isometry with an hyperplane as a set of fixed points. A reflection is an *involution*, which means that when applied twice in a row, every point is sent back to its original position, and every geometrical object is returned to its original state. Finding an algebraic structure to capture the properties of reflections actually led to the first signs of the structure of quandle: in 1943, Takasaki [Tak43] defined the notion of *kei*. A kei is defined as a set A with a binary operation $\triangleleft$ satisfying

$$(K1) \quad a \triangleleft a = a$$

$$(K2) \quad (a \triangleleft b) \triangleleft b = a$$

$$(K3) \quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$$

for all $a, b, c \in A$. Writing $a \triangleleft b$ for the reflection of a over b in a space A makes it a kei.

In the 1950's, the idea was rediscovered by Conway and Wraith. In an unpublished correspondence, they discussed a structure they called *wracks* which refers to wrack and ruin of a group after dismissing the multiplicative

operation and only keeping the group conjugation

$$g \triangleleft h = h^{-1} \cdot g \cdot h$$

for all g and h in the multiplicative group $(G, \cdot, 1)$. They remarked that the group conjugation satisfies the axiom (K1) and (K3) and they noticed that the three axioms held when taking for definition

$$g \triangleleft h = h \cdot g^{-1} \cdot h$$

for all elements g and h in a group G .

The early 80's marked the birth of the structure of *quandle*, as defined by Joyce in his thesis [Joy79] and his article [Joy82]. A quandle is a set A equipped with two binary operations $\triangleleft$ and $\triangleleft^{-1}$ satisfying

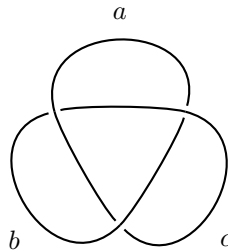
$$(Q1) \quad a \triangleleft a = a$$

$$(Q2) \quad (a \triangleleft b) \triangleleft^{-1} b = a = (a \triangleleft^{-1} b) \triangleleft b$$

$$(Q3) \quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$$

for all a , b and c in A . It is also at that time that Matveev [Mat82] was writing an article about the same algebraic structure which he called *distributive groupoid*. In the terms of Joyce, a kei or a wrack is actually an *involution quandle*, where a quandle is said to be involutive when the two quandle operations $\triangleleft$ and $\triangleleft^{-1}$ coincide: $\triangleleft = \triangleleft^{-1}$.

One of the main motivations behind the study of quandles comes from knot theory. Quandles actually provide a *knot invariant*, i.e. a mathematical object defined on each knot which is the same for "equivalent" knots. Let $A = \{a_1, a_2, \dots, a_n\}$ be the set of arcs (where an arc is a segment from one underpass to the next underpass) of the knot diagram like in the following example:



If the knot is oriented, one can define two operations on the diagram of a knot: for each crossing, we associate the following operation $\triangleleft$:

$$\begin{array}{c} b \\ \uparrow \\ a \text{ --- } \longrightarrow c = a \triangleleft b \\ \downarrow \\ b \end{array}$$

and the other operation $\triangleleft^{-1}$:

$$\begin{array}{c} b \\ \downarrow \\ a \text{ --- } \longrightarrow c = a \triangleleft^{-1} b \\ \uparrow \\ b \end{array}$$

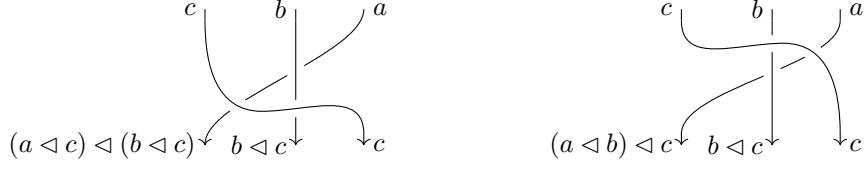
The *knot quandle* associated to a knot is then the quandle generated by the set of arcs $A = \{a_1, a_2, \dots, a_n\}$ and the defining relations are given by the crossing relations of the crossings of the knot [Joy82]. This quandle associated to a knot is actually an invariant since the three identities satisfied by a quandle correspond to the Reidemeister moves [Rei27, AB27]. The first identity $a \triangleleft a = a$ is equivalent to twist or untwist in either direction:

$$\begin{array}{ccc} \begin{array}{c} a \\ | \\ \text{loop} \\ | \\ a \triangleleft a \end{array} & & \begin{array}{c} a \\ | \\ a \end{array} \end{array}$$

The second one, $(a \triangleleft^{-1} b) \triangleleft b = b$ captures the movement of one arc completely over another arc:

$$\begin{array}{ccc} \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{crossing} \\ \diagup \quad \diagdown \\ (a \triangleleft^{-1} b) \triangleleft b \end{array} & & \begin{array}{cc} a & b \\ | & | \\ a & b \end{array} \end{array}$$

Finally, the third identity $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ is the equivalence of moving a crossing completely over or under an arc:



Joyce proved that the knot quandle is a complete invariant up to orientation [Joy82].

Although quandles are mainly used to obtain knot invariants, they are also algebraic structures of interest by themselves. Furthermore, the category of quandles, which is denoted by $\mathbf{Qnd}$, is a variety of universal algebras. The structure of quandles is very rich, and one can for example speak of *connected components* of a given quandle. Taking the connected components $\pi_0(A)$ of a quandle A produces a *trivial quandle*, which is a quandle that satisfies the identity

$$a \triangleleft b = a$$

for all a and b in A . The category of trivial quandles, denoted $\mathbf{Qnd}^*$, is then a subvariety of the variety $\mathbf{Qnd}$ of quandles. The adjunction

$$\begin{array}{ccc}
 & \xrightarrow{\pi_0} & \\
 \mathbf{Qnd} & \perp & \mathbf{Qnd}^* \\
 & \xleftarrow{U} &
 \end{array} \tag{A}$$

between the variety of quandles and its subvariety of trivial quandles is at the core of this thesis.

Categorical Galois theory

More generally, any subvariety $\mathcal{X}$ of a variety $\mathcal{C}$ yields an adjunction

$$\begin{array}{ccc}
 & \xrightarrow{I} & \\
 \mathcal{C} & \perp & \mathcal{X} \\
 & \xleftarrow{H} &
 \end{array} \tag{B}$$

with unit $\eta: 1_{\mathcal{C}} \rightarrow HI$. Such an adjunction together with the class $\mathcal{F}$ of surjective homomorphisms (=regular epimorphisms) in $\mathcal{C}$, forms a *Galois structure* as defined by Janelidze [Jan90]. We denote the Galois structure by

$$\Gamma = \{\mathcal{C}; \mathcal{X}, I, H, \eta, \mathcal{F}\}.$$

The Galois structure is called *admissible* when the left adjoint I preserves the following type of pullbacks:

$$\begin{array}{ccc} E \times_{HI(B)} H(A) & \xrightarrow{p_2} & H(A) \\ p_1 \downarrow & & \downarrow H(f) \text{ with } f \in \mathcal{F} \\ B & \xrightarrow{\eta_B} & HI(B). \end{array}$$

This is the starting point of the categorical Galois theory developed by Janelidze and Kelly [JK94] in this particular context. A typical example of such an admissible Galois structure is given by the adjunction between the variety **Grp** of groups and its subvariety **Ab** of abelian groups:

$$\begin{array}{ccc} & \xrightarrow{ab} & \\ \text{Grp} & \perp & \text{Ab} \\ & \xleftarrow{U} & \end{array} \quad (C)$$

where the functor $ab: \mathbf{Grp} \rightarrow \mathbf{Ab}$ is the classical abelianization functor. More generally, when the variety $\mathcal{C}$ is a *Mal'tsev variety* [Smi76], which means that its algebraic theory is equipped with a ternary operation $p(a, b, c)$ satisfying $p(a, a, b) = b$ and $p(a, b, b) = a$, any subvariety determines an admissible Galois structure. This is for example the case for the adjunction (C) since the variety **Grp** of groups is a Mal'tsev one, where the ternary operation is given by $p(a, b, c) = a \cdot b^{-1} \cdot c$.

The categorical Galois theory developed in this context by Janelidze and Kelly recovers in particular the classical theory of *central extensions* of groups and, more generally, of Ω -groups studied by the Fröhlich school [Frö63, Lue67]. These central extensions are an instance of the *coverings* arising from an admissible Galois structure Γ for a suitable variety $\mathcal{C}$ and subvariety $\mathcal{X}$. When Γ is an admissible Galois structure as above, one calls a surjective homomorphism $f: A \rightarrow B$ a *trivial covering* when the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ f \downarrow & & \downarrow HI(f) \\ B & \xrightarrow{\eta_B} & HI(B) \end{array} \quad (1)$$

is a pullback. A *covering* is a surjective homomorphism $f: A \rightarrow B$ such that one can find a surjective homomorphism $p: E \rightarrow B$ with the property that the

first projection $p_1: E \times_B A \rightarrow E$ of the pullback of f along p

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{p} & B \end{array}$$

is a trivial covering. The covering $f: A \rightarrow B$ is called *normal* when it is possible to choose $p = f$. When considering the adjunction (C), the corresponding coverings are precisely the central extensions of groups, i.e. the surjective group homomorphisms such that the kernel is in the center of their domain. Most non-trivial examples of the categorical theory of coverings in the context of varieties essentially come from a Galois structure where $\mathcal{C}$ is a Mal'tsev variety (see [Xar13] for some other examples in a non Mal'tsev context). By showing the admissibility of the adjunction (A), we add to the theory an interesting non-trivial example where the variety $\mathcal{C}$ is not a Mal'tsev (or even a Goursat) variety. This result still uses some permutability property of some congruences called *orbit congruences* by Bunch, Lofgren, Rapp and Yetter [BLRY10]. We also show that the *algebraic quandle coverings* defined by Eisermann [Eis03], are precisely the coverings arising from the adjunction (A). An algebraic quandle covering $f: A \rightarrow B$ is a surjective quandle homomorphism such that, when $f(a) = f(a')$ for some a and a' in A , it follows that

$$c \triangleleft a = c \triangleleft a'$$

for all $c \in A$. The proof of this fact is the main result of the paper [Eve14a].

Factorization systems

A well-known example of *factorization system* in any variety is the factorization of any homomorphism $f: A \rightarrow B$ into a surjective homomorphism $e: A \rightarrow I$ followed by an injective homomorphism $m: I \rightarrow B$. To construct such a factorization, one considers the kernel pair $\text{Eq}(f)$ of f and then produces the quotient $e: A \rightarrow A/\text{Eq}(f)$, which leads to the factorization of $f = m \circ e$ where $m: A/\text{Eq}(f) \rightarrow B$. When the adjunction (B) is *semi-left-exact* [CHK85], there is an induced *factorization system* $(\mathcal{E}, \mathcal{M})$ of all morphisms where the class $\mathcal{E}$ is the class of morphisms inverted by the left adjoint I and the class $\mathcal{M}$ is the class of morphisms $f: A \rightarrow B$ such that the square (1) is a pullback (see [CHK85, CJKP97] for more details). We give some details about this construction here below. From any morphism $f: A \rightarrow B$, we consider its

naturality square (1) and we take the pullback of $HI(f)$ along η_B .

$$\begin{array}{ccccc}
 A & & & & \\
 \searrow w & & \eta_A & \searrow & \\
 & P & \xrightarrow{p_2} & HI(A) & \\
 \searrow f & \downarrow p_1 & & \downarrow HI(f) & \\
 & B & \xrightarrow{\eta_B} & HI(B) &
 \end{array}$$

The factorization of f in the semi-left-exact case is thus given by the composite $p_1 \circ w$. When working with categorical Galois theory, the important morphisms are the ones belonging to the class $\mathcal{F}$ considered in the Galois structure $\Gamma = \{\mathcal{C}, \mathcal{X}, I, H, \eta, \mathcal{F}\}$ related to an adjunction (B). These are thus the morphisms for which factorization systems are really interesting. The adjunction (A) is not a semi-left-exact adjunction (see Remark 3.1.6). However, we prove that there is a similar factorization system $(\mathcal{E}, \mathcal{M})$ for the morphisms in the class $\mathcal{F}$ of surjective quandle homomorphism (for which we use the definition of factorization system for a class of morphism due to Chikhladze [Chi04]). The class $\mathcal{E}$ is shown to be the class of surjective homomorphisms inverted by the left adjoint π_0 while $\mathcal{M}$ consists of the class of trivial coverings. Janelidze and Kelly showed in [JK97] that coverings over an object B of any variety $\mathcal{C}$ are reflective in the category of surjective homomorphisms with codomain B . By means of the factorization system for surjective homomorphisms arising from the adjunction (A), we give a constructive proof of this result in the case of quandles. The existence of another factorization system $(\mathcal{E}', \mathcal{M}')$ for the class $\mathcal{F}$ that we derive from a recent article by Bunch, Lofgren, Rapp and Yetter [BLRY10] is also demonstrated. These two factorization systems can be compared to each other, and the relationship between them is given by the inclusions

$$\mathcal{E} \subset \mathcal{E}' \text{ and } \mathcal{M} \supset \mathcal{M}'.$$

Factorization systems arising from an adjunction (B) are called *reflective* and are those for which $g \in \mathcal{E}$ whenever $f \circ g \in \mathcal{E}$ and $f \in \mathcal{E}$. We show that $(\mathcal{E}', \mathcal{M}')$ does not satisfy this property, whereas $(\mathcal{E}, \mathcal{M})$ satisfies it.

Closure operators

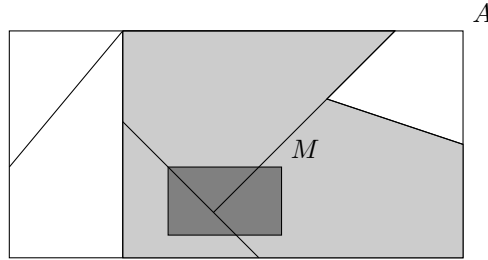
There are natural ways to obtain *closure operators* from an adjunction (B). Two of them are the so-called *regular closure operator* [Sal76] and *pullback closure operator* [Hol96]. These two closure operators actually coincide when the injective homomorphisms in the subvariety $\mathcal{X}$ are *regular monomorphisms*,

which is the case in the subvariety $\mathbf{Qnd}^*$ of trivial quandles. We recall the construction of the pullback closure operator here: if $m: M \rightarrow A$ is a subobject in a variety $\mathcal{C}$, the pullback closure operator $c_A(m)$ is constructed as the pullback of i along η_A

$$\begin{array}{ccccc}
 M & \xrightarrow{m} & A & & \\
 \eta_M \downarrow & \nearrow c_A(m) & \downarrow \eta_A & & \\
 & c_A(M) & & & \\
 \downarrow & & \downarrow & & \\
 HI(M) & \xrightarrow{HI(m)} & HI(A) & & \\
 \searrow e & & \nearrow i & & \\
 & I & & &
 \end{array}$$

where i is the injective homomorphism coming from the surjective-injective factorization of $HI(m)$.

In the category $\mathbf{Qnd}$ of quandles, this closure operator has a very simple description: the closure of a subobject $m: M \rightarrow A$ is the union of the connected components which contain elements of M . It can be represented by the following diagram



where the closure of M in A is the light grey part (which is here the union of three connected components). This closure operator is *idempotent*, as any pullback closure operator is, but it is also *finitely productive*, *fully additive* and has the property that the image by $f: A \rightarrow B$ of the closure of M in A is the closure of the image $f(M)$ in B when f is surjective. As expected, the *connected quandles* are exactly the *c-connected objects* relative to this closure operator, while the trivial quandles are precisely the *c-separated objects* for this closure operator. The interaction between trivial and connected quandles actually partially lies in a *Herrlich-Preuss-Arhangel'skiĭ-Wiegandt correspondence* [AW75, Her68, Pre71]: the class of connected quandles is a *connectedness* of the class of trivial quandles, although it is not true that the class of trivial quandles forms a *disconnectedness* of the class of connected quandles. Indeed, the disconnectedness of the class of connected quandles contains the class of trivial quandles but also the larger

class of *quasi-trivial quandles* [Ino13], which are the quandles A satisfying

$$a_0 \triangleleft^{\alpha_0} (a_0 \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n) = a_0$$

for all $a_i \in A$ and $\triangleleft^{\alpha_i} \in \{\triangleleft^{-1}, \triangleleft\}$ with $0 \leq i \leq n$.

Structure of the text and contributions

Chapter 1 serves as a (quick) introduction to the categorical concepts and results we shall need in the following chapters. After setting the context we will be working in, we review the Categorical covering theory, as well as the concepts of factorization systems and of closure operators for both subobjects and effective equivalence relations. We also recall the construction of the *fundamental group* of a *perfect* object due to Janelidze.

In chapter 2, we recall some basic properties of the structure of quandle as well as the definition of *algebraic quandle covering* due to Eisermann [Eis03]. We give a counter-example showing that the adjunction between the category $\mathbf{Qnd}$ of quandles and the category $\mathbf{Grp}$ of groups is not admissible. Also, we show that the category $\mathbf{Qnd}$ is not an *extensive category* [CLW93].

The chapter 3 is the core of this work. We study the adjunction between the category of quandles and its subcategory of trivial quandles. We show that this adjunction is admissible with respect to the class of surjective quandle homomorphisms by investigating some permutability property that holds in $\mathbf{Qnd}$. We then give a description of trivial and of normal coverings. To describe the coverings related to this adjunction, we use the construction of *weakly universal quandle covering* of a quandle due to Eisermann [Eis14] and show that the coverings are exactly the algebraic quandle coverings introduced by Eisermann. We also show that there exist coverings that are not normal, and normal coverings that are not trivial. In section 3, we prove that the fundamental group of a connected quandle corresponds to the *algebraic fundamental group* of a connected quandle, this latter having been introduced by Eisermann in [Eis14]. We describe the factorization system for surjective quandle homomorphisms arising from the adjunction and we prove that there exists another factorization system for surjective quandle homomorphisms that can be derived from the work of Bunch, Lofgren, Rapp and Yetter [BLRY10]. We compare these two factorization systems and show that the latter does not satisfy the typical property of reflective ones. In section 5, we prove that the category of coverings of a quandle A is a reflective subcategory of the category of surjective quandle homomorphisms over A . This could be used in the future as a starting point for the study of higher dimensional coverings, on the model of the study of

higher central extensions [EGV08]. We finally describe the pullback closure operator and investigate some of its properties in section 7. We show that connected quandles correspond to the *c-connected objects* for the pullback closure operator and that trivial quandles correspond to the *c-separated objects* for the same closure operator. We also investigate the existence of a *Herrlich-Preuss-Arhangel'skiĭ-Wiegandt correspondence* [AW75, Her68, Pre71] between connected quandles and trivial quandles. We finish this chapter by describing the closure operator for effective equivalence relations as constructed by Borceux, Gran and Mantovani [BGM07]. Many results in this chapter are joint work with Marino Gran.

The last chapter deals with another adjunction in the category of quandles: the one with *abelian symmetric quandles*. Basing ourselves on the work of Bourn, Montoli, Nelson-Ferreira and Sobral [BMFMS13, BMFMS15] as well as the work of Bourn [Bou15], we investigate the coverings arising from this adjunction, after showing that it is admissible. We give a description of normal coverings, and show that normal coverings coincide with coverings in this case. This is joint work with Marino Gran and Andrea Montoli.

Chapter 1

Categorical concepts

In this first chapter, we introduce the categorical concepts that we are going to investigate for the specific case of quandles later on.

1.1 Exact categories

We recall here the notion of exact categories as well as some properties of relations. We recommend [CKP93, Section 2] and [Bor94], for instance, for a good introduction to the topic.

Subobjects and quotient objects

Definition 1.1.1. A morphism $f: A \rightarrow B$ is a *monomorphism* when, for any $g, h: C \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$.

1.1.2. The notion of monomorphism can be seen as a generalization of the notion of injective map of sets to arbitrary categories. Monomorphisms have the following properties: if $f: A \rightarrow B$ and $g: B \rightarrow C$ are monomorphisms then $f \circ g: A \rightarrow C$ is a monomorphism; if $f \circ g: A \rightarrow C$ is a monomorphism, then $g: B \rightarrow C$ is a monomorphism; every pullback of a monomorphism is a monomorphism.

Let $\mathcal{C}$ be a category with finite limits. For an object A in $\mathcal{C}$ and two monomorphisms $m: M \rightarrow A$ and $n: N \rightarrow A$, we say that m *factors through* n (and we will write $m \leq n$) when there exists $t: M \rightarrow N$ such that $m = n \circ t$. A *subobject* of an object A in $\mathcal{C}$ refers to an equivalence class of monomorphisms with codomain A under the following equivalence relation: $m \equiv n$ if and only if $m \leq n$ and $n \leq m$. The subobjects of an object A form an ordered class $\text{Sub } A$ with finite meets (which are computed by taking pullbacks).

1.1.3. The dual notion of monomorphism is that of epimorphism.

Definition 1.1.4. A morphism $f: A \rightarrow B$ is an *epimorphism* when, for any $g, h: B \rightarrow C$, $g \circ f = h \circ f$ implies $g = h$.

Although monomorphisms still capture the notion of injective morphism in any variety of universal algebra (see [BS81] for instance to learn more about universal algebra), epimorphisms fail to do so for surjective morphisms in pretty standard categories like the category of rings for example. It is thus useful to consider a smaller class of epimorphisms.

Definition 1.1.5. A morphism $f: A \rightarrow B$ is a *regular epimorphism* if it is the coequalizer of a pair of morphisms $C \rightrightarrows A$.

For an object A in $\mathcal{C}$ and two regular epimorphisms $p: A \rightarrow P$ and $q: A \rightarrow Q$, we write $p \leq q$ when there exists $l: Q \rightarrow P$ such that $p = l \circ q$. A *quotient object* of an object A in $\mathcal{C}$ is then an equivalence class of regular epimorphisms with domain A under the equivalence relation: $p \equiv q$ if and only if $p \leq q$ and $q \leq p$. Quotient objects of an object A also form an ordered class $\text{Quot } A$.

Regular categories and relations

1.1.6. A category $\mathcal{C}$ is said to *admit images* (or *factorizations*) if every morphism $f: A \rightarrow B$ can be decomposed as $f = i \circ p$ where p is a regular epimorphism and i is a monomorphism. The monomorphism i is called the *regular image* of f . This factorization is unique up to isomorphism, and the subobject $[i]$ of B is called the *image of f* .

Definition 1.1.7. A category $\mathcal{C}$ is *regular* if it is finitely complete, it admits images and regular epimorphisms are stable under pullbacks, that is, in any pullback

$$\begin{array}{ccc} C \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

the morphism $p_1: C \times_B A \rightarrow C$ is a regular epimorphism whenever $f: A \rightarrow B$ is a regular epimorphism.

Example 1.1.8. The category **Set** of sets, any abelian category and any variety of universal algebra are examples of regular categories. The category **Top** of topological spaces, for instance, is not a regular category, since regular epimorphisms are not stable under pullbacks.

1.1.9. Let $\mathcal{C}$ be a finitely complete category, the pullback of a morphism $f: A \rightarrow B$ along itself yields a pair of morphisms $\text{Eq}(f) \xrightarrow[f_2]{f_1} A$ universal among pairs $C \xrightarrow[h]{g} A$ such that $f \circ g = f \circ h$. The pair of morphisms f_1, f_2 together with $\text{Eq}(f)$ is called the *kernel pair* of f . We will often call $\text{Eq}(f)$ the kernel pair of f , dropping the projections from the notation. Note that if f factorizes as $f = i \circ p$ where i is a monomorphism (no assumption on p here), then $\text{Eq}(f)$ is also the kernel pair of p . Moreover, if a regular epimorphism $f: A \rightarrow B$ has a kernel pair, it is the coequalizer of that kernel pair.

1.1.10. In a regular category $\mathcal{C}$, regular epimorphisms have the following properties: if $f: A \rightarrow B$ and $g: B \rightarrow C$ are regular epimorphisms then $f \circ g$ is a regular epimorphism; if $f \circ g$ is a regular epimorphism, then f is a regular epimorphism; every pushout of a regular epimorphism is a regular epimorphism.

1.1.11. In any finitely complete category, a *relation* R from A to B is a subobject $(r_1, r_2): R \rightarrow A \times B$ of the product. A relation R on A is a relation from A to A . For any relation $(r_1, r_2): R \rightarrow A \times B$ there is an *opposite relation* R^o given by the subobject $(r_2, r_1): R \rightarrow B \times A$. A morphism $f: A \rightarrow B$ can be seen as a relation by identifying it with its *graph* $(1_A, f): A \rightarrow A \times B$. The identity morphism $1_A: A \rightarrow A$ yields in particular the *identity relation* Δ_A given by $(1_A, 1_A): A \rightarrow A \times A$. A pair of morphisms $a: C \rightarrow A, b: C \rightarrow B$ is said to *belong to* R if $(a, b): C \rightarrow A \times B$ factorizes through $(r_1, r_2): R \rightarrow A \times B$, which we write $(a, b) \in R$ or aRb .

Regular categories allow us to compose relations. Given two relations R and S from A to B and from B to C respectively, represented by the subobjects $(r_1, r_2): R \rightarrow A \times B$ and $(s_1, s_2): S \rightarrow B \times C$, we form the diagram

$$\begin{array}{ccccc}
 & & R \times_B S & & \\
 & p_1 \swarrow & & \searrow p_2 & \\
 & R & & S & \\
 r_1 \swarrow & & & & \searrow s_2 \\
 A & & B & & C
 \end{array} \quad (1.1.11.1)$$

where the diamond is a pullback. We define the *composite relation* $S \circ R$ from A to C by taking the image of $(r_1 \circ p_1, s_2 \circ p_2): R \times_B S \rightarrow A \times C$. In the category of sets,

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B, aRb, bSc\}.$$

Moreover, this composition is associative, the identity relation is an identity for the composition, and the composition satisfies

$$(S \circ R)^o = R^o \circ S^o.$$

Proposition 1.1.12. *Let $\mathcal{C}$ be a regular category. Given equivalence relations R, S, T and U on an object A , if $R \leq T$ and $S \leq U$, then*

$$S \circ R \leq U \circ T.$$

Note that, for a morphism $f: A \rightarrow B$, the relation $f^o \circ f$ is the kernel pair of f , while the *direct image* $f(R)$ by f of a relation R on A is given by the composite $f \circ R \circ f^o$. An arrow f is a regular epimorphism if and only if $f \circ f^o = \Delta_B$. Also,

$$f \circ f^o \circ f = f \text{ and } f^o \circ f \circ f^o = f^o, \quad (1.1.12.1)$$

and any relation R from A to B given by $(r_1, r_2): R \rightarrow A \times B$ can be also expressed as $r_2 \circ r_1^o$. Finally, we recall that $f: A \rightarrow B$ is a monomorphism if and only if its kernel pair $\text{Eq}(f)$ is isomorphic to the relation Δ_A .

Let us end this section by giving some nice properties that hold in regular categories and will be important later.

Proposition 1.1.13. *Let $\mathcal{C}$ be a regular category and $f: A \rightarrow B$ be a regular epimorphism. Then $f(\Delta_A) = \Delta_B$.*

Proposition 1.1.14. *In a regular category $\mathcal{C}$, suppose that the exterior and left square of the following commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ a \downarrow & & b \downarrow & & \downarrow c \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array} \quad (1.1.14.1)$$

are pullbacks. Then the right square is a pullback if f' is a regular epimorphism.

Proof. First consider the pullback

$$\begin{array}{ccc} P & \xrightarrow{v} & C \\ u \downarrow & & \downarrow c \\ B' & \xrightarrow{g'} & C' \end{array} \quad (1.1.14.2)$$

and consider then the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \downarrow a & \searrow & \downarrow b & \searrow w & \nearrow v \\
 & A & \xrightarrow{w \circ f} & P & \\
 & \swarrow a & \downarrow & \swarrow u & \\
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
 \end{array} \quad (1.1.14.3)$$

The exterior and right squares in the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{w \circ f} & P & \xrightarrow{v} & C \\
 a \downarrow & & u \downarrow & & \downarrow c \\
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
 \end{array} \quad (1.1.14.4)$$

are pullbacks so that the left square is a pullback. Now for the same reasons applied to

$$\begin{array}{ccccc}
 A & \xlongequal{\quad} & A & \xrightarrow{a} & A' \\
 f \downarrow & & w \circ f \downarrow & & \downarrow f' \\
 B & \xrightarrow{w} & P & \xrightarrow{u} & B'
 \end{array} \quad (1.1.14.5)$$

we have that

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 f \downarrow & & \downarrow w \circ f \\
 B & \xrightarrow{w} & P
 \end{array} \quad (1.1.14.6)$$

is a pullback. We thus have

$$\begin{array}{ccccc}
 \Delta_A & \xlongequal{\quad} & A & \xlongequal{\quad} & A \\
 \bar{f} \downarrow & & \downarrow f & & \downarrow w \circ f \\
 \text{Eq}(w) & \xrightarrow[w_2]{w_1} & B & \xrightarrow{w} & P
 \end{array} \quad (1.1.14.7)$$

where $\bar{f}: \Delta_A \rightarrow \text{Eq}(w)$ is a regular epimorphism thanks to pullback stability of regular epimorphisms, and thus $\text{Eq}(w) = f(\Delta_A) = \Delta_B$ thanks to Proposition 1.1.13. This shows that w is a monomorphism. It is easy to see that it is also a regular epimorphism, making it an isomorphism, as desired. ■

Equivalence relations and exact categories

1.1.15. A relation R on A is said to be *reflexive* when $\Delta_A \leq R$. It is *symmetric* when $R^o \leq R$. It is *transitive* when $R \circ R \leq R$.

Definition 1.1.16. A relation R on A is said to be an *equivalence relation* when it is reflexive, symmetric and transitive.

The kernel pair of any morphism $f: A \rightarrow B$ is an equivalence relation.

If S is an equivalence relation on B , we write $f^{-1}(S)$ for the equivalence relation on A which is the inverse image of S along f : it is obtained by the dotted arrow in the following pullback:

$$\begin{array}{ccc} f^{-1}(S) & \longrightarrow & S \\ \vdots \downarrow & & \downarrow (s_1, s_2) \\ Y \times Y & \xrightarrow{f \times f} & X \times X. \end{array} \quad (1.1.16.1)$$

Definition 1.1.17. An equivalence relation R on an object A is said to be *effective* when it is the kernel pair of some morphism $f: A \rightarrow B$.

Definition 1.1.18 ([BGO71]). A category $\mathcal{C}$ is an *exact category* when it is a regular one in which every equivalence relation is effective.

Example 1.1.19. Examples of exact categories include any variety in the sense of universal algebra where monomorphisms, regular epimorphisms and effective equivalence relations correspond respectively to injective homomorphisms, surjective homomorphisms and *congruences*. Congruences on an object A are equivalence relations on A compatible with the algebraic structure. The category of topological groups for example is a regular category that is not an exact category.

In a regular category $\mathcal{C}$, it is in general not true that given two equivalence relations R and S on an object A , their composition $S \circ R$ is again an equivalence relation. In fact, the composite $S \circ R$ is a reflexive and symmetric relation, but usually fails to be transitive. Actually, the transitivity of $S \circ R$ is equivalent to the permutability of equivalence relations: $S \circ R = R \circ S$.

Definition 1.1.20. An exact category $\mathcal{C}$ is a *Mal'tsev category* when

$$R \circ S = S \circ R$$

for all equivalence relations on any object A .

Example 1.1.21. The category $\mathbf{Grp}$ of groups is a Mal'tsev category. It has been shown by Mal'tsev [Mal54] that a variety $\mathcal{V}$ of universal algebra has permutable congruences if and only if we can find a term p in the theory of the variety such that $p(a, a, b) = b$ and $p(a, b, b) = a$ for all $a, b \in A$. In the case of the category $\mathbf{Grp}$ of groups, $p(a, b, c) = a.b^{-1}.c$.

In a regular category, we always have $R \leq S \circ R$ and $S \leq S \circ R$ for equivalence relations on an object A . Because of this, it is easy to see that, when $S \circ R$ is an equivalence relation, it is the smallest one containing both R and S (thanks to Proposition 1.1.12) and is thus $R \vee S$.

Moreover, we have the following important result:

Proposition 1.1.22. *Let $\mathcal{C}$ be a regular category. Let $(r_1, r_2): R \rightarrow A \times A$ be an equivalence relation on an object A and $f: A \rightarrow B$. If $R \circ \text{Eq}(f) = \text{Eq}(f) \circ R$, then the image $f(R)$ is an equivalence relation.*

Proof. It is easy to show the $f(R)$ is both reflexive and symmetric. To show transitivity, recall that $f(R) = f \circ R \circ f^o$ and $\text{Eq}(f) = f^o \circ f$, thus

$$\begin{aligned} f(R) \circ f(R) &= f \circ R \circ f^o \circ f \circ R \circ f^o \\ &= f \circ f^o \circ f \circ R \circ R \circ f^o \\ &= f \circ R \circ f^o \\ &= f(R). \end{aligned}$$

■

1.2 Admissible subcategories

1.2.1. From now on, we will consider a full replete reflective subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$,

$$\begin{array}{ccc} & I & \\ \mathcal{C} & \xrightarrow{\quad} & \mathcal{X} \\ & \perp & \\ & H & \end{array} \quad (1.2.1.1)$$

where $H: \mathcal{X} \rightarrow \mathcal{C}$ is the inclusion functor, $I: \mathcal{C} \rightarrow \mathcal{X}$ is its left adjoint and $\eta: 1_{\mathcal{C}} \Rightarrow HI$ is the unit of the adjunction. Replete means that for any object X in $\mathcal{X}$ and any isomorphism $f: A \rightarrow H(X)$ in $\mathcal{C}$, A is in $\mathcal{X}$. Furthermore, we will choose the counit of the adjunction $\epsilon: IH \Rightarrow 1_{\mathcal{X}}$ to be a natural identity transformation, which is not a restriction, since $\mathcal{X}$ is a full subcategory of $\mathcal{C}$.

The subcategory $\mathcal{X}$ is said to be *closed under subobjects* if every subobject in $\mathcal{C}$ of an object of $\mathcal{X}$ lies in $\mathcal{X}$. In presence of a replete subcategory $\mathcal{X}$, one can show that being closed under subobjects is equivalent to the fact that the A -component of the unit of the adjunction $\eta_A: A \rightarrow HI(A)$ is a regular epimorphism for all A in $\mathcal{C}$. When this is the case, we call $\mathcal{X}$ a *regular-epireflective subcategory* of $\mathcal{C}$. A consequence of this is that $HI(f)$ is a regular epimorphism

if f is a regular epimorphism due to the naturality of η .

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ f \downarrow & & \downarrow HI(f) \\ B & \xrightarrow{\eta_B} & HI(B) \end{array} \quad (1.2.1.2)$$

Also, the regular image in $\mathcal{C}$ of a morphism $H(f): H(X) \rightarrow H(Y)$ coming from $\mathcal{X}$ lies in $\mathcal{X}$, so that f is a regular epimorphism in $\mathcal{X}$ if and only if $H(f)$ is a regular epimorphism in $\mathcal{C}$.

Moreover, the subcategory $\mathcal{X}$ is a *Birkhoff subcategory* of $\mathcal{C}$ if it is closed under both subobjects and quotient objects in $\mathcal{C}$. In this particular context, being closed under quotients is equivalent to the fact that the naturality square (1.2.1.2) is a pushout when f is a regular epimorphism. When $\mathcal{C}$ is a variety, the Birkhoff subcategories $\mathcal{X}$ of $\mathcal{C}$ are exactly its subvarieties.

Let us now recall the definition of Galois structure by Janelidze [Jan90]. Although this definition is not the most general one, it is the appropriate one for our context.

Definition 1.2.2 (Janelidze). A *Galois structure* $\Gamma = \{\mathcal{C}, \mathcal{X}, H, I, \eta, \mathcal{F}\}$ consists of a category $\mathcal{C}$, a full replete reflective subcategory $\mathcal{X}$ with inclusion functor $H: \mathcal{X} \rightarrow \mathcal{C}$, a left adjoint $I: \mathcal{C} \rightarrow \mathcal{X}$ and unit $\eta: 1_{\mathcal{C}} \rightarrow HI$, and a class of morphisms $\mathcal{F}$ such that:

1. $\mathcal{F}$ contains all isomorphisms;
2. $\mathcal{F}$ is pullback stable in the sense that pullbacks of morphisms in $\mathcal{F}$ along morphisms of $\mathcal{C}$ exists and lie in $\mathcal{F}$;
3. $\mathcal{F}$ is closed under composition;
4. $HI(\mathcal{F}) \subset \mathcal{F}$.

When taking $\mathcal{C}$ an exact category, $\mathcal{X}$ a full (replete) reflective Birkhoff subcategory of $\mathcal{C}$ and $\mathcal{F}$ the class of regular epimorphisms in $\mathcal{C}$, we obtain a Galois structure $\Gamma = \{\mathcal{C}, \mathcal{X}, H, I, \eta, \mathcal{F}\}$.

From now on, this is the setting that will always be assumed, unless otherwise stated.

1.2.3. Given a Galois structure $\Gamma = \{\mathcal{C}, \mathcal{X}, H, I, \eta, \mathcal{F}\}$, there is an induced adjunction between the category of regular epimorphisms $f: A \rightarrow B$ with codomain B in $\mathcal{C}$ and the category of regular epimorphisms $\phi: X \rightarrow I(B)$ with codomain $I(B)$ in $\mathcal{X}$.

For an object B in the category $\mathcal{C}$, we have the category $\mathcal{C} \downarrow B$ in which objects, that we denote by (A, f) , are the regular epimorphisms $f: A \rightarrow B$ in $\mathcal{C}$ with codomain B and morphisms $(A, f) \rightarrow (A', f')$ are morphisms $g: A \rightarrow A'$ such that the following triangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ & \searrow f & \swarrow f' \\ & B. & \end{array} \quad (1.2.3.1)$$

It follows from 1.2.1 that the left adjoint functor $I: \mathcal{C} \rightarrow \mathcal{X}$ yields a functor $I^B: \mathcal{C} \downarrow B \rightarrow \mathcal{X} \downarrow I(B)$ sending a regular epimorphism $f: A \rightarrow B$ to the regular epimorphism $I(f): I(A) \rightarrow I(B)$.

The functor I^B has a right adjoint $H^B: \mathcal{X} \downarrow I(B) \rightarrow \mathcal{C} \downarrow B$ sending a regular epimorphism $f: X \rightarrow I(B)$ to $H^B(f)$ constructed in the following pullback:

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{p_2} & H(X) \\ H^B(f) := p_1 \downarrow & & \downarrow H(f) \\ B & \xrightarrow{\eta_B} & HI(B). \end{array} \quad (1.2.3.2)$$

The (A, f) -component of the unit of the adjunction

$$\eta_{(A,f)}^B: (A, f) \rightarrow (B \times_{HI(B)} HI(A), p_1)$$

is thus the arrow (f, η_A) , which is the following factorization into the pull-back $B \times_{HI(B)} HI(A)$:

$$\begin{array}{ccccc} A & & \xrightarrow{\eta_A} & & HI(A) \\ & \searrow (f, \eta_A) & & \searrow p_2 & \downarrow H(f) \\ & B \times_{HI(B)} HI(A) & \xrightarrow{p_2} & HI(A) & \\ & \downarrow p_1 & & \downarrow H(f) & \\ & B & \xrightarrow{\eta_B} & HI(B). & \end{array}$$

Definition 1.2.4 (Janelidze). The Galois structure Γ is an *admissible* Galois structure when, for any $B \in \mathcal{C}$, the functor $H^B: \mathcal{X} \downarrow I(B) \rightarrow \mathcal{C} \downarrow B$ is fully faithful.

We will say that a Birkhoff subcategory $\mathcal{X}$ of an exact category $\mathcal{C}$ is admissible when the previous Galois structure is admissible.

Here are some equivalent statements for admissibility.

Proposition 1.2.5. *The following assertions are equivalent:*

1. *the Birkhoff subcategory $\mathcal{X}$ is admissible;*
2. *the functor I preserves pullbacks of the form (1.2.3.2), where $f: X \rightarrow I(B)$ is a regular epimorphism;*
3. *the functor I preserves pullbacks of the form*

$$\begin{array}{ccc}
 B \times_{H(Y)} H(X) & \xrightarrow{p_2} & H(X) \\
 p_1 \downarrow & & \downarrow H(f) \\
 B & \xrightarrow{g} & H(Y)
 \end{array} \tag{1.2.5.1}$$

where $f: X \rightarrow Y$ is a regular epimorphism.

Remark 1.2.6. When the functor I preserves all pullbacks of the form (1.2.3.2) (where $f: X \rightarrow I(B)$ is not required to be a regular epimorphism), the functor I is called *semi-left-exact* [CHK85].

Here we state a useful result [JK94] concerning exact Mal'tsev categories.

Theorem 1.2.7. *Any Birkhoff subcategory $\mathcal{X}$ of an exact Mal'tsev category $\mathcal{C}$ is admissible.*

Example 1.2.8. Consider the adjunction between the category of groups $\mathbf{Grp}$ and its subcategory $\mathbf{Ab}$:

$$\begin{array}{ccc}
 & \xrightarrow{ab} & \\
 \mathbf{Grp} & \begin{array}{c} \perp \\ \leftarrow U \end{array} & \mathbf{Ab}
 \end{array} \tag{1.2.8.1}$$

The subcategory $\mathbf{Ab}$ is admissible.

1.3 Covering theory

1.3.1. The notions of trivial coverings, normal coverings and coverings can be defined with respect to an admissible subcategory $\mathcal{X}$ of $\mathcal{C}$.

Definition 1.3.2. A regular epimorphism $f: A \rightarrow B$ is a *trivial covering* when the diagram (1.2.1.2) is a pullback.

Trivial coverings are thus objects of $\mathcal{C} \downarrow B$ lying in the image of the functor $H^B: \mathcal{X} \downarrow I(B) \rightarrow \mathcal{C} \downarrow B$. Equivalently, they are objects (A, f) such that $\eta_{(A, f)}$ is an isomorphism.

Example 1.3.3. In the case of the adjunction (1.2.8.1), $f: A \rightarrow B$ is a trivial covering if and only if the restriction $\bar{f}: [A, A] \rightarrow [B, B]$ of $f: A \rightarrow B$ to the commutator subgroup is an isomorphism.

An immediate consequence of admissibility is the following proposition.

Proposition 1.3.4. *The functor $I: \mathcal{C} \rightarrow \mathcal{X}$ preserves pullbacks along trivial coverings.*

Proof. Suppose the following diagram is a pullback with $t: E \rightarrow B$ a trivial covering:

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{t} & B. \end{array}$$

Now consider the following cube:

$$\begin{array}{ccccc} E \times_B A & \xrightarrow{p_2} & A & & \\ \downarrow p_1 & \searrow \eta_{E \times_B A} & \downarrow f & \searrow \eta_A & \\ & HI(E \times_B A) & \xrightarrow{HI(p_2)} & HI(A) & \\ & \downarrow HI(p_1) & \downarrow & \downarrow HI(f) & \\ E & \xrightarrow{t} & B & & \\ \downarrow \eta_E & \searrow & \downarrow \eta_B & \searrow & \\ & HI(E) & \xrightarrow{HI(t)} & HI(B) & \end{array}$$

where the bottom face is a pullback by definition of trivial covering and the rear one is also a pullback, so that the exterior square of

$$\begin{array}{ccccc} E \times_B A & \xrightarrow{\eta_{E \times_B A}} & HI(E \times_B A) & \xrightarrow{HI(p_1)} & HI(E) \\ p_2 \downarrow & & \downarrow HI(p_2) & & \downarrow HI(t) \\ A & \xrightarrow{\eta_A} & HI(A) & \xrightarrow{HI(f)} & HI(B) \end{array}$$

is a pullback. But since $\mathcal{X}$ is admissible, this square is preserved by I by Proposition 1.2.5, making

$$\begin{array}{ccc} I(E \times_B A) & \xrightarrow{I(p_2)} & I(A) \\ I(p_1) \downarrow & & \downarrow I(f) \\ I(E) & \xrightarrow{I(t)} & I(B) \end{array}$$

a pullback, as desired. $\blacksquare$

This result is useful to show that trivial coverings are stable under pullbacks.

Proposition 1.3.5. *Trivial coverings are stable under pullbacks, i.e. if the morphism $f: A \rightarrow B$ is a trivial covering then the morphism $p_1: C \times_B A \rightarrow C$ in the following pullback is a trivial covering.*

$$\begin{array}{ccc} C \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array} \quad (1.3.5.1)$$

Proof. Just consider the cube

$$\begin{array}{ccccc} C \times_B A & \xrightarrow{p_2} & A & & \\ \downarrow p_1 & \searrow \eta_{C \times_B A} & \downarrow f & \searrow \eta_A & \\ & HI(C \times_B A) & \xrightarrow{HI(p_2)} & HI(A) & \\ & \downarrow HI(p_1) & \downarrow & \downarrow HI(f) & \\ E & \xrightarrow{g} & B & & \\ \downarrow \eta_E & \searrow & \downarrow \eta_B & \searrow & \\ & HI(C) & \xrightarrow{HI(g)} & HI(B) & \end{array} \quad (1.3.5.2)$$

where the rear face is a pullback by assumption, the right face is a pullback by definition of trivial covering and the front face is a pullback thanks to Proposition 1.3.4. One concludes that the left face of the cube is a pullback as well, proving the statement. $\blacksquare$

Definition 1.3.6. A regular epimorphism $f: A \rightarrow B$ is a *covering* when there exists a regular epimorphism $p: E \rightarrow B$ such that $p_1: E \times_B A \rightarrow E$ in the

following pullback is a trivial covering.

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{p} & B \end{array} \quad (1.3.6.1)$$

We also say that (A, f) is *split* by the regular epimorphism p .

Example 1.3.7. Again in the case of the adjunction (1.2.8.1), we can show that $f: A \rightarrow B$ is a covering if and only if it is a central extension of groups, meaning that the kernel of f is in the center of its domain: $\ker(f) \subset Z(A)$.

A consequence of Proposition 1.3.5 is that coverings are pullback stable.

Proposition 1.3.8. *Coverings are stable under pullbacks.*

Proof. Consider the following pullback where $f: A \rightarrow B$ is a covering:

$$\begin{array}{ccc} C \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B. \end{array}$$

Now since f is a covering, there exists $p: E \rightarrow B$ such that the pullback of f along p is a trivial covering. Now take also the pullback of p along g :

$$\begin{array}{ccccc} & & E \times_B A & & \\ & & \downarrow & \searrow & \\ C \times_B A & \xrightarrow{p_2} & A & & \\ p_1 \downarrow & & \downarrow f & & \\ C \times_B E & \xrightarrow{\quad} & E & \xrightarrow{p} & B \\ q \searrow & & \downarrow p & & \downarrow f \\ C & \xrightarrow{g} & B & & B \end{array}$$

Finally, finish the cube by taking the pullback of p_1 along q :

$$\begin{array}{ccccc}
 P & \xrightarrow{\quad} & E \times_B A & & \\
 \downarrow t & \searrow & \downarrow s & \searrow & \\
 & & C \times_B A & \xrightarrow{p_2} & A \\
 & & \downarrow p_1 & \downarrow & \downarrow f \\
 C \times_B E & \xrightarrow{\quad} & E & \xrightarrow{p} & B \\
 & \searrow q & \downarrow g & & \\
 & & C & \xrightarrow{\quad} & B
 \end{array}$$

Since the left and front faces are pullbacks, and the right face is a pullback too, one deduces that the rear face is pullback. By Proposition 1.3.5, one concludes that $t: P \rightarrow C \times_B E$ is a trivial covering (since $s: E \times_B A \rightarrow E$ is a trivial covering). This shows that p_1 is a covering. ■

Definition 1.3.9. A regular epimorphism $f: A \rightarrow B$ is a *normal covering* when $f_1: \text{Eq}(f) \rightarrow A$ in the following pullback is a trivial covering.

$$\begin{array}{ccc}
 \text{Eq}(f) & \xrightarrow{f_2} & A \\
 f_1 \downarrow & & \downarrow f \\
 A & \xrightarrow{f} & B
 \end{array} \tag{1.3.9.1}$$

Example 1.3.10. In the case of the adjunction (1.2.8.1), normal coverings coincide with coverings.

Again, normal coverings are stable under pullbacks.

Proposition 1.3.11. *Normal coverings are stable under pullbacks.*

Proof. The proof is similar to the proof of Proposition 1.3.8. ■

Note that we always have

$$\text{Triv}(B) \subset \text{Norm}(B) \subset \text{Cov}(B)$$

where $\text{Triv}(B)$ is the category of trivial coverings over B and $\text{Norm}(B)$, and $\text{Cov}(B)$ are the categories of normal coverings and coverings over B .

1.4 Fundamental group

1.4.1. In this section, we recall the definition of fundamental group in our context. Recall that we consider the adjunction

$$\begin{array}{ccc} & I & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathcal{X} \\ & H & \end{array} \quad (1.4.1.1)$$

where $\mathcal{X}$ is an admissible subcategory of $\mathcal{C}$.

Let us introduce one of the most essentials concepts of categorical Galois theory, the so-called Galois groupoid. This groupoid, like the admissibility condition, depends on the Galois structure Γ .

Any morphism $p: E \rightarrow B$ gives a (internal) groupoid:

$$\text{Eq}(p) \times_E \text{Eq}(p) \begin{array}{c} \xrightarrow{q_1} \\ \xleftarrow{q_2} \end{array} \text{Eq}(p) \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_2} \end{array} E$$

$\begin{array}{c} i \\ \curvearrowright \end{array}$

where $\text{Eq}(p) \times_E \text{Eq}(p)$ is the pullback of p_1 along p_2

$$\begin{array}{ccc} \text{Eq}(p) \times_E \text{Eq}(p) & \xrightarrow{q_2} & \text{Eq}(p) \\ q_1 \downarrow & & \downarrow p_1 \\ \text{Eq}(p) & \xrightarrow{p_2} & E \end{array} \quad (1.4.1.2)$$

and m is induced by the fact that $p \circ p_2 \circ q_2 = p \circ p_1 \circ q_1$.

$$\begin{array}{ccccc} \text{Eq}(p) \times_E \text{Eq}(p) & & & & \\ & \searrow m & & \searrow p_2 \circ q_2 & \\ & & \text{Eq}(p) & \xrightarrow{p_2} & E \\ & \searrow p_1 \circ q_1 & \downarrow p_1 & & \downarrow p \\ & & E & \xrightarrow{p} & B \end{array}$$

The elements of the groupoid are the elements of $\text{Eq}(p)$, and the partial multiplication is given by the morphism m . In general, applying the functor I to this groupoid does not yield back a groupoid but a "pregroupoid", we won't go into these details here. In order for the image under I of the previous groupoid to be again a groupoid, one shall take for $p: E \rightarrow B$ a normal covering. Indeed, when p is a normal covering, p_1 , p_2 , q_1 and q_2 are trivial coverings, so that by the admissibility of the Galois structure, the pullback (1.4.1.2) is preserved by

the functor I : $I(\text{Eq}(p) \times_E \text{Eq}(p)) = I(\text{Eq}(p)) \times_{I(E)} I(\text{Eq}(p))$. Therefore,

$$I(\text{Eq}(p)) \times_{I(E)} I(\text{Eq}(p)) \begin{array}{c} \xrightarrow{I(q_1)} \\ \xleftarrow{I(m)} \\ \xrightarrow{I(q_2)} \end{array} \begin{array}{c} I(i) \\ \downarrow \\ \text{Eq}(p) \end{array} \begin{array}{c} \xleftarrow{I(p_1)} \\ \xleftarrow{I(e)} \\ \xrightarrow{I(p_2)} \end{array} I(E)$$

is again a groupoid.

Definition 1.4.2 (Janelidze). Let $p: E \rightarrow B$ be a normal covering of B . The *Galois groupoid* $\text{Gal}_\Gamma(p)$ of p is the image by the left adjoint I of the kernel pair $\text{Eq}(p)$ of p .

$$\begin{array}{ccccc} \text{Eq}(p) \times_E \text{Eq}(p) & \begin{array}{c} \xrightarrow{q_1} \\ \xleftarrow{m} \\ \xrightarrow{q_2} \end{array} & \begin{array}{c} i \\ \downarrow \\ \text{Eq}(p) \end{array} & \begin{array}{c} \xleftarrow{p_1} \\ \xleftarrow{e} \\ \xrightarrow{p_2} \end{array} & E \\ \downarrow I & \begin{array}{c} \xrightarrow{I(q_1)} \\ \xleftarrow{I(m)} \\ \xrightarrow{I(q_2)} \end{array} & \begin{array}{c} I(i) \\ \downarrow \\ I(\text{Eq}(p)) \end{array} & \begin{array}{c} \xleftarrow{I(p_1)} \\ \xleftarrow{I(e)} \\ \xrightarrow{I(p_2)} \end{array} & \downarrow I \\ I(\text{Eq}(p)) \times_{I(E)} I(\text{Eq}(p)) & & I(\text{Eq}(p)) & & I(E) \end{array}$$

When the object E is *perfect* (meaning that $I(E)$ is the terminal object), the Galois groupoid $\text{Gal}_\Gamma(p)$ becomes a group (since then the pullback $I(\text{Eq}(p)) \times_{I(E)} I(\text{Eq}(p)) = I(\text{Eq}(p)) \times I(\text{Eq}(p))$), called the *Galois group* of p .

Definition 1.4.3. A normal covering $p: E \rightarrow B$ is *weakly universal* if, for each normal covering $f: A \rightarrow B$, there exists a morphism $g: E \rightarrow A$ such that $f \circ g = p$.

When, moreover, the normal covering $p: E \rightarrow B$ is weakly universal, the Galois group of p is called the *fundamental group* of B , and we denote it by $\pi_1^{\text{Gal}}(B)$. Note that $\pi_1^{\text{Gal}}(B)$ is independent (up to isomorphism) of the choice of weakly universal normal covering $p: E \rightarrow B$.

1.5 Factorization systems

This section is devoted to factorization systems and, in particular, to factorization systems arising from an adjunction. More information on factorization systems can be found in [CHK85, CJKP97].

1.5.1. First let us recall the definition of factorization system. Let $\mathcal{F}$ be a class of morphisms in $\mathcal{C}$ containing identities, closed under composition, and pullback stable.

Definition 1.5.2 ([FK72, Chi04]). A pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in $\mathcal{C}$ constitutes a *factorization system* for $\mathcal{F}$ if

- (i) $\mathcal{E}$ and $\mathcal{M}$ contain the identities and are closed under composition with isomorphisms;
- (ii) every map in $\mathcal{F}$ can be written as $m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;
- (iii) the class $\mathcal{M}$ is contained in $\mathcal{F}$;
- (iv) given any commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{m} & D \end{array}$$

with e in $\mathcal{E}$ and m in $\mathcal{M}$, there is a unique arrow $w: B \rightarrow C$ such that $w \circ e = u$ and $m \circ w = v$.

Example 1.5.3. The first straightforward example is obtained by taking for $\mathcal{F}$ the class of all morphisms in any regular category $\mathcal{C}$. Then we have a factorization system with $\mathcal{E}$ the class of regular epimorphisms and $\mathcal{M}$ the class of monomorphisms. This factorization is also *stable* since the class $\mathcal{E}$ is stable under pullbacks.

Remark 1.5.4. When we consider an admissible Galois structure $\Gamma = \{\mathcal{C}, \mathcal{X}, H, I, \eta, \mathcal{F}\}$, one has an induced factorization system for the class $\mathcal{F}$. The class $\mathcal{E}$ consists then of morphisms of $\mathcal{F}$ inverted by I , while $\mathcal{M}$ is the class of trivial coverings.

We recall the construction here. First start with $f: A \rightarrow B$ in $\mathcal{F}$ and consider the natural square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta_A \downarrow & & \downarrow \eta_B \\ HI(A) & \xrightarrow{HI(f)} & HI(B). \end{array}$$

Now take the pullback of $HI(f)$ along η_B , and we have thus a unique morphism $e: A \rightarrow HI(A) \times_{HI(B)} B$ such that $f = p_2 \circ e$ and $p_1 \circ e = \eta_A$.

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \eta_A \downarrow & \searrow e & \nearrow p_2 \\
 & HI(A) \times_{HI(B)} B & \\
 & \swarrow p_1 & \downarrow \eta_B \\
 HI(A) & \xrightarrow{HI(f)} & HI(B)
 \end{array} \tag{1.5.4.1}$$

It is easy to see that p_2 is a trivial covering by Proposition 1.3.5 (since any morphism in the class $HI(\mathcal{F})$ is a trivial covering). Moreover, the pullback in (1.5.4.1) is preserved by I since $\mathcal{X}$ is an admissible subcategory of $\mathcal{C}$, so that $I(e)$ is actually an isomorphism.

We finish this section by recalling a property of factorization systems arising from a full reflective subcategory $\mathcal{X}$ when $\mathcal{F}$ is the class of all morphisms, called *reflective factorization systems*.

Proposition 1.5.5. *The factorization system $(\mathcal{E}, \mathcal{M})$ is reflective if and only if $g \in \mathcal{E}$ whenever $f \circ g \in \mathcal{E}$ and $f \in \mathcal{E}$.*

1.6 Closure operators for subobjects

In this section, we recall the definition of closure operator for subobjects and describe two particular closure operators.

1.6.1. Let $\mathcal{C}$ be a regular category. We shall first fix some notations.

As mentioned in Section 1.5, the classes $\mathcal{E}$ of regular epimorphisms and $\mathcal{M}$ of monomorphisms form a stable factorization system in $\mathcal{C}$. Given any arrow $f: A \rightarrow B$ and a subobject $m: M \rightarrow A$, we denote by $f(m)$ the subobject of B obtained by taking the regular image of m along f , that is, the regular image of the composite $f \circ m$. When $n: N \rightarrow B$ is a subobject of B we write $f^{-1}(n)$ for the subobject of A which is the *inverse image* of n along f , that is, the pullback of n along f .

Definition 1.6.2 ([DG87, DT95]). A *closure operator* c in $\mathcal{C}$ associates, with any subobject $m: M \rightarrow A$, another subobject $c_A(m): c_A(M) \rightarrow A$, the closure of m in A . This application satisfies the following properties for any $m, n \in \text{Sub}(A)$, and $f: A \rightarrow B$:

1. $m \leq c_A(m)$;

2. if $m \leq n$, then $c_A(m) \leq c_A(n)$;
3. $f(c_A(m)) \leq c_B(f(m))$.

Moreover, we say that a subobject $m: M \rightarrow A$ is *closed* if $m = c_A(m)$, and that it is *dense* if $c_A(m) = 1_A$, for all $A \in \mathcal{C}$. The closure operator factors every subobject $m: M \rightarrow A$ as

$$\begin{array}{ccc} M & \xrightarrow{m} & A \\ & \searrow m/c_A(m) & \nearrow c_A(m) \\ & c_A(M) & \end{array}$$

The closure operator c is said to be *idempotent* if $c_A(m)$ is closed for all $m \in \mathcal{M}$, and *weakly hereditary* if $m/c_A(m)$ is dense for all $m \in \mathcal{M}$.

An important result that can be found in [DG87] concerning idempotent and weakly hereditary closure operators is the following:

Theorem 1.6.3. *There is a Galois equivalence between the conglomerate of subclasses of $\mathcal{M}$ which are part of a factorization system, and the conglomerate of idempotent and weakly hereditary closure operators of $\mathcal{C}$ with respect to $\mathcal{M}$.*

Pullback closure operator

1.6.4. We recall that a pointed endofunctor (R, r) is given by an endofunctor $R: \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $r: 1_{\mathcal{C}} \rightarrow R$. Any pointed endofunctor in a category with a stable factorization system $(\mathcal{E}, \mathcal{M})$ induces a corresponding closure operator, called the *pullback closure operator* by Holgate [Hol96]. The construction is the following: let $m: M \rightarrow A$ be a morphism in the class $\mathcal{M}$ and construct the following diagram where $i \circ e = R(m)$ is the $(\mathcal{E}, \mathcal{M})$ -factorization of $R(m)$, and $c_A(m)$ is the pullback of i along r_A :

$$\begin{array}{ccccc} M & \xrightarrow{m} & & A & \\ & \searrow c_A(m) & & \nearrow & \\ & c_A(M) & & & \\ \downarrow r_M & & \downarrow & & \downarrow r_A \\ R(M) & \xrightarrow{R(m)} & & R(A) & \\ & \searrow e & & \nearrow i & \\ & I & & & \end{array}$$

Then the assignment $c_A: \text{Sub}(A) \rightarrow \text{Sub}(A)$ defined by $c_A(m) = r_A^{-1}(i)$ is a closure operator, called the pullback closure operator corresponding to (R, r) .

This closure operator is in particular idempotent when $\mathcal{X}$ is a Birkhoff subcategory, where in this case (R, r) is given by (HI, η) .

Example 1.6.5. In the case of the adjunction (1.2.8.1), the pullback closure $c_G(M)$ is the group consisting of elements $g \in G$ for which there exists $h \in M$ such that $g[G, G] = h[G, G]$ where $[G, G]$ is the commutator subgroup of G . It is idempotent but not weakly hereditary.

Regular closure operator

1.6.6. These particular closure operators were introduced by Salbany [Sal76] first in the category **Top** of topological spaces (see also [Cle93, CT98]). Other authors extended the construction to arbitrary categories, leading to the general concept of categorical closure operator.

Let $\mathcal{X}$ be a full reflective subcategory of $\mathcal{C}$. We recall the construction of the so-called *regular closure operator*. Given a subobject $m: M \rightarrow A$, consider its cokernel pair $A \begin{smallmatrix} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{smallmatrix} A +_M A$ and compose these morphisms with the unit of the adjunction $\eta_{A+_MA}: A +_M A \rightarrow HI(A +_M A)$. The closure of m is then the equalizer of the pair $A \begin{smallmatrix} \xrightarrow{\eta_{A+_MA} \circ i_1} \\ \xrightarrow{\eta_{A+_MA} \circ i_2} \end{smallmatrix} HI(A +_M A)$. We denote the regular closure operator of $m: M \rightarrow A$ by $c_A^{reg}(m): c_A^{reg}(M) \rightarrow A$.

The regular closure operator is always idempotent.

Example 1.6.7. In the case of the adjunction (1.2.8.1), the regular closure operator $c_G^{reg}(M)$ is the intersection of all normal subgroups H containing M such that G/H is abelian. This is exactly the same as the pullback closure operator in this case.

In general, the regular and pullback closure operators are not the same (see [Hol96, Example 8 (2)] for example) but they are the same when the monomorphisms in $\mathcal{X}$ are regular monomorphisms (equalizers of some pair of morphisms). We recall the proof of this latter statement.

Proposition 1.6.8. *Let $\mathcal{X}$ be a full regular-epireflective subcategory of a regular category $\mathcal{C}$. The regular closure operator coincide with the pullback closure operator when monomorphisms in $\mathcal{X}$ are regular.*

Proof. Consider the following diagram

$$\begin{array}{ccccc}
 & & c_A^{reg}(M) & & \\
 & \nearrow & & \searrow c_A^{reg}(m) & \\
 M & \xrightarrow{m} & A & \xrightleftharpoons[i_2]{i_1} & A +_M A \\
 & \searrow & \nearrow c_A(m) & & \downarrow \eta_{A+M A} \\
 & & c_A(M) & & \\
 \downarrow \eta_M & & \downarrow \eta_A & & \\
 HI(M) & \xrightarrow{HI(m)} & HI(A) & \xrightleftharpoons[HI(i_2)]{HI(i_1)} & HI(A +_M A) \\
 & \searrow e & \nearrow n & & \\
 & & N & &
 \end{array}$$

p is the pullback of n along η_A and $c_A^{reg}(m)$ is the equalizer of $\eta_{A+M A} \circ i_1$ and $\eta_{A+M A} \circ i_2$.

where $n \circ e$ is the regular epimorphism-monomorphism factorization of $HI(m)$, $A +_M A$ is the cokernel pair of m , $c_A(m)$ is the pullback of n along η_A and $c_A^{reg}(m)$ is the equalizer of $\eta_{A+M A} \circ i_1$ and $\eta_{A+M A} \circ i_2$.

It is easy to see that $c_A(m) \leq c_A^{reg}(m)$ is always true since

$$HI(i_1) \circ HI(m) = HI(i_2) \circ HI(m)$$

implies that

$$HI(i_1) \circ n \circ e = HI(i_2) \circ n \circ e$$

and thus

$$HI(i_1) \circ n = HI(i_2) \circ n.$$

Consequently we have

$$\begin{aligned}
 \eta_{A+M A} \circ i_1 \circ c_A(m) &= HI(i_1) \circ \eta_A \circ c_A(m) \\
 &= HI(i_1) \circ n \circ p \\
 &= HI(i_2) \circ n \circ p \\
 &= HI(i_2) \circ \eta_A \circ c_A(m) \\
 &= \eta_{A+M A} \circ i_2 \circ c_A(m)
 \end{aligned}$$

from which there exists $\alpha: c_A(M) \rightarrow c_A^{reg}(M)$ such that $c_A^{reg}(m) \circ \alpha = c_A(m)$.

Now if n is a regular monomorphism, it is the equalizer of $HI(i_1)$ and $HI(i_2)$. But since

$$\begin{aligned} HI(i_1) \circ \eta_A \circ c_A^{reg}(m) &= \eta_{A+MA} \circ i_1 \circ c_A^{reg}(m) \\ &= \eta_{A+MA} \circ i_2 \circ c_A^{reg}(m) \\ &= HI(i_2) \circ \eta_A \circ c_A^{reg}(m) \end{aligned}$$

there exists $q: c_A^{reg}(M) \rightarrow N$ such that $n \circ q = \eta_A \circ c_A^{reg}(m)$. By the universal property of the pullback, we have thus a unique morphism $s: c_A^{reg}(M) \rightarrow c_A(M)$ such that $p \circ s = q$ and

$$c_A(m) \circ s = c_A^{reg}(m),$$

which shows that $c_A^{reg}(M) \leq c_A(M)$. ■

1.7 Closure operators for effective equivalence relations

We recall some results about closure operators on effective equivalence relations in regular categories, which can be found in [BGM07].

Definition 1.7.1. An *effective closure operator* c on effective equivalence relations in $\mathcal{C}$ consists in giving, for every effective equivalence relation R on an object A , another effective equivalence relation $c_A(R)$, called the *closure* of R . This assignment has to satisfy the following properties, where R and S are effective equivalence relations on A , $f: B \rightarrow A$ is a morphism in $\mathcal{C}$:

- (1) $R \leq c_A(R)$;
- (2) $R \leq S$ implies $c_A(R) \leq c_A(S)$;
- (3) $c_B(f^{-1}(R)) \leq f^{-1}(c_A(R))$;
- (4) $c_A(c_A(R)) = c_A(R)$;
- (5) if $f: B \rightarrow A$ is a *regular epimorphism*, one then has the equality

$$c_B(f^{-1}(R)) = f^{-1}(c_A(R)).$$

1.7.2. Consider again the adjunction (1.2.1.1) where this time $\mathcal{C}$ is a regular category and $\mathcal{X}$ is a regular-epireflective subcategory of $\mathcal{C}$.

Theorem 1.7.3 ([BGM07]). *There is a bijection between regular-epireflective subcategories of $\mathcal{C}$ and effective closure operators in $\mathcal{C}$.*

We recall how the closure $c_A(R)$ of an effective equivalence relation R on A is defined starting from a regular-epireflective subcategory $\mathcal{X}$ of $\mathcal{C}$ as in (1.2.1.1). One first takes the canonical quotient $f: A \rightarrow A/R$, and then considers the inverse image $f^{-1}(\text{Eq}(\eta_{A/R}))$ along f of the kernel pair $\text{Eq}(\eta_{A/R})$ of the A/R -component $\eta_{A/R}$ of the unit of the adjunction. Equivalently, the closure $c_A(R)$ can be defined as the kernel pair of the arrow $\eta_{A/R} \circ f: A \rightarrow HI(A/R)$ (this also shows that the equivalence relation $c_A(R)$ is effective).

Finally, we give a result characterizing the effective closure operator in Mal'tsev categories.

Proposition 1.7.4. *Let $\mathcal{C}$ be a Mal'tsev category. Then the effective closure operator on an effective equivalence relation R on A is given by*

$$c_A(R) = R \circ c_A(\Delta_A).$$

Example 1.7.5. We return again to the example of the adjunction (1.2.8.1) between groups and abelian groups, where $c_G(R) = R \circ [G, G]$ where $[G, G]$ is the congruence on G defined by $(g, h) \in [G, G]$ if $h^{-1} \cdot g$ belongs to the commutator subgroup of G .

Chapter 2

The structure of a quandle

In this chapter we recall the definition of quandle introduced by Joyce [Joy79, Joy82] and independently by Matveev [Mat82]. We investigate some of its properties and introduce the definition of algebraic quandle covering, which is due to Eisermann [Eis14].

2.1 Quandles

2.1.1. The structure of quandles has been mainly studied in knot theory, as one can associate with any knot an invariant, called the *knot quandle*.

Definition 2.1.2 ([Joy79, Joy82, Mat82]). A *quandle* is a set A equipped with two binary operations $\triangleleft$ and $\triangleleft^{-1}$ such that for all $a, b, c \in A$:

$$(Q1) \quad a \triangleleft a = a \quad (\text{idempotency});$$

$$(Q2) \quad (a \triangleleft b) \triangleleft^{-1} b = a = (a \triangleleft^{-1} b) \triangleleft b \quad (\text{right invertibility});$$

$$(Q3) \quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad (\text{self-distributivity}).$$

Definition 2.1.3. A *quandle homomorphism* is a function $f: A \rightarrow B$ such that

$$f(a \triangleleft a') = f(a) \triangleleft f(a')$$

for all $a, a' \in A$.

As for groups, one can ask for the preservation of only one operation, the other following from this one. We will take some time to prove every result of this chapter in order to get used to calculation technique in quandles.

Proposition 2.1.4. *If $f: A \rightarrow B$ be a quandle homomorphism, then it also satisfies*

$$f(a \triangleleft^{-1} a') = f(a) \triangleleft^{-1} f(a')$$

for all $a, a' \in A$.

Proof. We have

$$f(a) = f((a \triangleleft^{-1} a') \triangleleft a') = f(a \triangleleft^{-1} a') \triangleleft f(a').$$

So that

$$f(a) \triangleleft^{-1} f(a') = (f(a \triangleleft^{-1} a') \triangleleft f(a')) \triangleleft^{-1} f(a') = f(a \triangleleft^{-1} a'). \quad \blacksquare$$

Moreover, a quandle satisfies similar identities than (Q1) and (Q3) involving $\triangleleft^{-1}$ as well as a few others.

Proposition 2.1.5. *Let A be a quandle. Then*

- (1) $a \triangleleft^{-1} a = a$;
- (2) $a \triangleleft (b \triangleleft c) = ((a \triangleleft^{-1} c) \triangleleft b) \triangleleft c$;
- (3) $a \triangleleft (b \triangleleft^{-1} c) = ((a \triangleleft c) \triangleleft b) \triangleleft^{-1} c$;
- (4) $(a \triangleleft b) \triangleleft^{-1} c = (a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c)$;
- (5) $a \triangleleft^{-1} (b \triangleleft^{-1} c) = ((a \triangleleft c) \triangleleft^{-1} b) \triangleleft^{-1} c$;
- (6) $a \triangleleft^{-1} (b \triangleleft c) = ((a \triangleleft^{-1} c) \triangleleft^{-1} b) \triangleleft c$;
- (7) $(a \triangleleft^{-1} b) \triangleleft c = (a \triangleleft c) \triangleleft^{-1} (b \triangleleft c)$;
- (8) $(a \triangleleft^{-1} b) \triangleleft^{-1} c = (a \triangleleft^{-1} c) \triangleleft^{-1} (b \triangleleft^{-1} c)$

for all $a, b, c \in A$.

Proof. (1) The first identity is simply a consequence of (Q1) and (Q2) since

$$a \stackrel{(Q2)}{=} (a \triangleleft a) \triangleleft^{-1} a \stackrel{(Q1)}{=} a \triangleleft^{-1} a.$$

(2) We have

$$\begin{aligned} a \triangleleft (b \triangleleft c) &= ((a \triangleleft^{-1} c) \triangleleft c) \triangleleft (b \triangleleft c) && \text{(by (Q2))} \\ &= ((a \triangleleft^{-1} c) \triangleleft b) \triangleleft c && \text{(by (Q3)).} \end{aligned}$$

(3) The identity follows from

$$\begin{aligned} (a \triangleleft (b \triangleleft^{-1} c)) \triangleleft c &= (a \triangleleft c) \triangleleft ((b \triangleleft^{-1} c) \triangleleft c) && \text{(by (Q3))} \\ &= (a \triangleleft c) \triangleleft b && \text{(by (Q2)).} \end{aligned}$$

(4) Thanks to (3), it is easy to see that

$$\begin{aligned} (a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c) &= (((a \triangleleft^{-1} c) \triangleleft c) \triangleleft b) \triangleleft^{-1} c && \text{(by (3))} \\ &= (a \triangleleft b) \triangleleft^{-1} c && \text{(by (Q2)).} \end{aligned}$$

(5) Using (4), we see that

$$\begin{aligned} (((a \triangleleft c) \triangleleft^{-1} b) \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c) & \\ &= (((a \triangleleft c) \triangleleft^{-1} b) \triangleleft b) \triangleleft^{-1} c && \text{(by (4))} \\ &= a && \text{(by (Q2)).} \end{aligned}$$

(6) It is a straightforward application of (Q2) and (Q3):

$$\begin{aligned} (((a \triangleleft^{-1} c) \triangleleft^{-1} b) \triangleleft c) \triangleleft (b \triangleleft c) & \\ &= (((a \triangleleft^{-1} c) \triangleleft^{-1} b) \triangleleft b) \triangleleft c && \text{(by (Q3))} \\ &= a && \text{(by (Q2)).} \end{aligned}$$

(7) Just like point (6):

$$\begin{aligned} ((a \triangleleft^{-1} b) \triangleleft c) \triangleleft (b \triangleleft c) &= ((a \triangleleft^{-1} b) \triangleleft b) \triangleleft c && \text{(by (Q3))} \\ &= a \triangleleft c && \text{(by (Q2)).} \end{aligned}$$

(8) Thanks to (4),

$$\begin{aligned} ((a \triangleleft^{-1} b) \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c) &= ((a \triangleleft^{-1} b) \triangleleft b) \triangleleft^{-1} c && \text{(by (4))} \\ &= a \triangleleft^{-1} c && \text{(by (Q2)).} \end{aligned}$$

■

If we use the convention that $a \triangleleft^1 b = a \triangleleft b$, the previous proposition tells us in particular that

$$(a \triangleleft^\alpha b) \triangleleft^\beta c = (a \triangleleft^\beta c) \triangleleft^\alpha (b \triangleleft^\beta c),$$

and

$$a \triangleleft^\alpha (b \triangleleft^\beta c) = ((a \triangleleft^{-\beta} c) \triangleleft^\alpha b) \triangleleft^\beta c, \quad (2.1.5.1)$$

hold for all $\alpha, \beta \in \{1, -1\}$ and $a, b, c \in A$.

The identity (2.1.5.1) can be generalized naturally.

Corollary 2.1.6. *Given a quandle A ,*

$$\begin{aligned} a_0 \triangleleft^{\alpha_0} ((\dots ((a_1 \triangleleft^{\alpha_1} a_2) \triangleleft^{\alpha_2} a_3) \dots) \triangleleft^{\alpha_n} a_n) = \\ (\dots (((\dots (a_0 \triangleleft^{-\alpha_n} a_n) \triangleleft^{-\alpha_{n-1}} \dots) \triangleleft^{-\alpha_1} a_1) \triangleleft^{\alpha_0} a_1) \triangleleft^{\alpha_1} a_2) \dots) \triangleleft^{\alpha_n} a_n, \end{aligned}$$

for all $\alpha_i \in \{-1, 1\}$ and $a_i \in A$ with $0 \leq i \leq n$.

Proof. We first apply the identity (2.1.5.1):

$$\begin{aligned} a_0 \triangleleft^{\alpha_0} ((\dots ((a_1 \triangleleft^{\alpha_1} a_2) \triangleleft^{\alpha_2} a_3) \dots) \triangleleft^{\alpha_n} a_n) \\ = ((a_0 \triangleleft^{-\alpha_n} a_n) \triangleleft^{\alpha_0} (\dots ((a_1 \triangleleft^{\alpha_1} a_2) \triangleleft^{\alpha_2} a_3) \dots \triangleleft^{\alpha_{n-1}} a_{n-1})) \triangleleft^{\alpha_n} a_n. \end{aligned}$$

And we apply (2.1.5.1) $n - 1$ times to get

$$(\dots (((\dots (a_0 \triangleleft^{-\alpha_n} a_n) \triangleleft^{-\alpha_{n-1}} \dots) \triangleleft^{-\alpha_1} a_1) \triangleleft^{\alpha_0} a_1) \triangleleft^{\alpha_1} a_2) \dots) \triangleleft^{\alpha_n} a_n. \blacksquare$$

Convention 2.1.7. In order to simplify the notations, from now on we shall write

$$a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_n} a_n := (\dots ((a \triangleleft^{\alpha_1} a_1) \triangleleft^{\alpha_2} a_2) \dots) \triangleleft^{\alpha_n} a_n$$

with $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq n$ where

$$a \triangleleft^\alpha b = a \underbrace{\triangleleft b \triangleleft b \dots \triangleleft b}_{\alpha \text{ times}}$$

if α is positive. Similarly,

$$a \triangleleft^\alpha b = a \underbrace{\triangleleft^{-1} b \triangleleft^{-1} b \dots \triangleleft^{-1} b}_{-\alpha \text{ times}}$$

if α is negative.

Let us write $\mathbf{Qnd}$ for the category of quandles. It is in particular a variety of universal algebras.

When the operations of a quandle A satisfy $\triangleleft = \triangleleft^{-1}$, the quandle A is called an *involution quandle*.

We list some examples of quandles here below.

Example 2.1.8. 1. Let A be a set and define

$$a \triangleleft a' = a = a \triangleleft^{-1} a'$$

for all $a, a' \in A$, then A becomes an involutive quandle called *trivial quandle*.

2. Let G be a multiplicative group and define

$$g \triangleleft h = h^{-1} \cdot g \cdot h$$

and

$$g \triangleleft^{-1} h = h \cdot g \cdot h^{-1}.$$

The group G becomes a quandle with this structure, called the *conjugation quandle* and denoted by $\text{Conj}(G)$. Actually, this correspondence yields a functor $\text{Conj}: \mathbf{Grp} \rightarrow \mathbf{Qnd}$ from the category $\mathbf{Grp}$ of groups to the category $\mathbf{Qnd}$ of quandles.

3. Let n be a positive integer and define on $\mathbb{Z}_n$ the following operation

$$i \triangleleft j = 2j - i = i \triangleleft^{-1} j$$

for all $i, j \in \mathbb{Z}_n$. This defines a quandle called *dihedral quandle*.

4. More generally, let G be a multiplicative group and define

$$g \triangleleft h = h \cdot g^{-1} \cdot h = g \triangleleft^{-1} h.$$

This defines a quandle called *core quandle*.

5. Let M be a module over the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials. Define

$$x \triangleleft y = t(x - y) + y$$

and

$$x \triangleleft^{-1} y = t^{-1}(x - y) + y.$$

This defines a quandle called *Alexander quandle*. These quandles can be used to compute the Alexander polynomial of a knot.

6. Let $\langle \cdot, \cdot \rangle: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a symmetric bilinear form on $\mathbb{R}^n$. Let X be a subset of $\mathbb{R}^n$ with elements x such that $\langle x, x \rangle \neq 0$. Define on X the

following operation

$$x \triangleleft y = \frac{2\langle x, y \rangle}{\langle x, x \rangle} y - x = x \triangleleft^{-1} y.$$

This defines a quandle called *Coxeter quandle*.

Remark 2.1.9. First remark that the category $\mathbf{Qnd}^*$ of trivial quandles is isomorphic to the category $\mathbf{Set}$ of sets.

Note also that a trivial quandle is a quandle satisfying an additional identity, making the corresponding category, $\mathbf{Qnd}^*$, a subvariety, and thus a Birkhoff subcategory of $\mathbf{Qnd}$.

2.2 Limits and colimits

2.2.1. As in any variety of universal algebras, limits are constructed in $\mathbf{Qnd}$ objectwise in the category $\mathbf{Set}$ of sets.

For example, the *product* of two quandles A and B will be the set-theoretic product $A \times B$ with the operations

$$(a, b) \triangleleft (a', b') = (a \triangleleft a', b \triangleleft b')$$

and

$$(a, b) \triangleleft^{-1} (a', b') = (a \triangleleft^{-1} a', b \triangleleft^{-1} b').$$

The *pullback* of $f: A \rightarrow B$ along $g: C \rightarrow B$ yields a subset of the product $A \times C$:

$$A \times_B C = \{(a, c) \in A \times C \mid f(a) = g(c)\},$$

equipped with the same algebraic structure as the one on the product.

The *terminal object* is the trivial quandle with one element $\{*\}$.

2.2.2. Colimits are trickier to construct.

Let A and B be two quandles. Then their *coproduct* $A \coprod B$ is given by elements

$$(i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \cdots \triangleleft^{\alpha_n} (i_n, a_n)$$

of the free quandle generated by the set-theoretic coproduct

$$A + B = \bigcup_{i \in \{1, 2\}} \{(i, x) \mid x \in A \text{ if } i = 1 \text{ and } x \in B \text{ if } i = 2\}$$

modulo the following identities (I):

$$(i_0, a_0) \triangleleft^{\alpha_1} (i_0, a_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} (i_n, a_n) = (i_0, a_0 \triangleleft^{\alpha_1} a_1) \triangleleft^{\alpha_2} (i_2, a_2) \dots \triangleleft^{\alpha_n} (i_n, a_n)$$

and

$$\begin{aligned} (i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \dots \triangleleft^{\alpha_j} (i_j, a_j) \triangleleft^{-\alpha_j} (i_j, a_j) \triangleleft^{\alpha_{j+2}} \dots \triangleleft^{\alpha_n} (i_n, a_n) = \\ (i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \dots \triangleleft^{\alpha_{j-1}} (i_{j-1}, a_{j-1}) \triangleleft^{\alpha_{j+2}} (i_{j+2}, a_{j+2}) \dots \triangleleft^{\alpha_n} (i_n, a_n). \end{aligned}$$

The quandle operations on $A \amalg B$ are given by

$$\begin{aligned} ((i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \dots \triangleleft^{\alpha_n} (i_n, a_n)) \triangleleft^{\alpha} ((j_0, b_0) \triangleleft^{\beta_1} (j_1, b_1) \dots \triangleleft^{\beta_m} (j_m, b_m)) \\ = (i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \dots \triangleleft^{\alpha_n} (i_n, a_n) \triangleleft^{-\beta_m} (j_m, b_m) \triangleleft^{-\beta_{m-1}} (j_{m-1}, b_{m-1}) \dots \\ \triangleleft^{-\beta_1} (j_1, b_1) \triangleleft^{\alpha} (j_0, b_0) \triangleleft^{\beta_1} (j_1, b_1) \dots \triangleleft^{\beta_m} (j_m, b_m). \end{aligned}$$

Let us check the universality of the coproduct. Let $f_1: A \rightarrow C$ and $f_2: B \rightarrow C$ two quandle homomorphisms. Now define $t: A \amalg B \rightarrow C$ by

$$t((i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \dots \triangleleft^{\alpha_n} (i_n, a_n)) = f_{i_0}(a_0) \triangleleft^{\alpha_1} f_{i_1}(a_1) \dots \triangleleft^{\alpha_n} f_{i_n}(a_n)$$

with $(i_j, a_j) \in A + B$ for all $0 \leq j \leq n$. It is well-defined since t preserves the identities (I) and it is easy to see that it is a quandle homomorphism.

The *initial object* is the empty quandle $\emptyset$.

Remark 2.2.3. Unlike a category of G -sets, which is a *topos* (see [AGV72]), the category of quandles is not *extensive* (see [Law91, CLW93]). A finitely complete category is *extensive* if, given the following commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A \amalg B & \longleftarrow & B \end{array}$$

the top row is a coproduct if and only if both squares are pullbacks. In the category of quandles, coproducts do not interact well with pullbacks as one can see from the following example where $\{*\}$ is the one-element trivial quandle and $\{a, b\}$ is the trivial quandle with 2 elements:

$$\begin{array}{ccccc} \{*\} & \xrightarrow{i_1} & \{*\} \amalg \{a, b\} & \xleftarrow{i_2} & \{a, b\} \\ \parallel & & \downarrow t & & \downarrow ! \\ \{*\} & \xrightarrow{i'_1} & \{*\} \amalg \{*\} & \xleftarrow{i'_2} & \{*\}. \end{array}$$

The homomorphism $t: \{*\} \coprod \{a, b\} \rightarrow \{*\} \coprod \{*\}$ is the unique universal homomorphism defined by

$$t((i_0, a_0) \triangleleft^{\alpha_1} (i_1, a_1) \cdots \triangleleft^{\alpha_n} (i_n, a_n)) = (i_0, *) \triangleleft^{\alpha_1} (i_1, *) \cdots \triangleleft^{\alpha_n} (i_n, *)$$

with $(i_j, a_j) \in \{*\} + \{a, b\}$ for all $0 \leq j \leq n$. Here, the top row is a coproduct but the left-hand square is not a pullback since the pullback of i'_1 along t contains at least two distinct elements. Indeed, we have

$$i'_1(*) = (1, *)$$

and

$$t(1, *) = (1, *) = t((1, *) \triangleleft (2, a) \triangleleft^{-1} (2, b))$$

with $(1, *) \neq (1, *) \triangleleft (2, a) \triangleleft^{-1} (2, b)$. This shows that at least $(*, (1, *))$ and $(*, (1, *) \triangleleft (2, a) \triangleleft^{-1} (2, b))$ belong to the pullback of t along i'_1 .

2.3 Inner automorphisms and connected components

2.3.1. We see from Proposition 2.1.5 and the second quandle axiom (both depending only on (Q2) and (Q3)) that the *right translations* $- \triangleleft b: A \rightarrow A$ sending an element a to $a \triangleleft b$, for all $a \in A$, are bijective quandle homomorphisms with inverse maps $- \triangleleft^{-1} b$.

Definition 2.3.2. The group $\text{Inn}(A)$ of *inner automorphisms* of a quandle A is the subgroup of $\text{Aut}(A)$ (the group of all automorphisms of A) generated by all right translations $(-)^{\rho_b} = - \triangleleft b$ with $b \in A$.

Remark 2.3.3. Let a and b be two elements of A . The multiplication $(-)^{\rho_a}$ and $(-)^{\rho_b}$ in $\text{Inn}(A)$ is given by the element $(-)^{\rho_a \rho_b} = - \triangleleft a \triangleleft b$.

Definition 2.3.4. A *right action* of a group G by quandle automorphisms on A is a group homomorphism $h: G \rightarrow \text{Aut}(A)$ with $h(g): A \rightarrow A$, $a \mapsto a^g$. We say that G *acts by inner automorphisms* if $h(G) \subset \text{Inn}(A)$.

Remark 2.3.5. Note that there is a function $\text{inn}: A \rightarrow \text{Conj}(\text{Inn}(A))$ (where Conj is the functor introduced in Example 2.1.8) sending $a \in A$ to $(-)^{\rho_a}$ that is in fact a quandle homomorphism. Indeed, let a and a' be two elements of A and consider $\text{inn}(a \triangleleft a') = (-)^{\rho_{a \triangleleft a'}}$. By Proposition 2.1.5, we know that $(-)^{\rho_{a \triangleleft a'}} = - \triangleleft (a \triangleleft a')$ is equal to $- \triangleleft^{-1} a' \triangleleft a \triangleleft a' = (-)^{\rho_{a'}^{-1} \rho_a \rho_{a'}}$ which is equal to $(-)^{\rho_{a'}^{-1}} (-)^{\rho_a} (-)^{\rho_{a'}} = (-)^{\rho_a} \triangleleft (-)^{\rho_{a'}}$ in $\text{Conj}(\text{Inn}(A))$ (be careful that the operation $\triangleleft$ in $\text{Conj}(\text{Inn}(A))$ is not the same as the one of A).

Definition 2.3.6. A *connected component* of A is an orbit under the action of $\text{Inn}(A)$. We shall denote by $[a]_{\text{Inn}(A)}$ the connected component of a in A . Two elements a and a' are in the same orbit if there exist $a_1, a_2, \dots, a_n \in A$ such that

$$a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n = a'.$$

The set of connected components of a quandle A is denoted by $\pi_0(A)$. Note that $\pi_0(A)$ is actually a trivial quandle (as any set). In fact, this assignment extends to a functor $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$, which is the left adjoint of the inclusion functor $U: \mathbf{Qnd}^* \rightarrow \mathbf{Qnd}$. The A -component η_A of the unit of this adjunction is defined by

$$\eta_A(a) = [a]_{\text{Inn}(A)}.$$

$$\begin{array}{ccc} & \xrightarrow{\pi_0} & \\ \mathbf{Qnd} & \begin{array}{c} \perp \\ \xleftarrow{U} \end{array} & \mathbf{Qnd}^* \end{array} \quad (2.3.6.1)$$

From now on, we will suppress U from the notations, writing for example $\eta_A: A \rightarrow \pi_0(A)$ for the A -component of the unit.

One could wonder if Inn is a functor. It turns out that this is not the case, as the following counter-example shows:

Example 2.3.7. Consider the following involutive quandle A given by the following table

$\triangleleft$	a	b	c
a	$a \triangleleft a = a$	$a \triangleleft b = a$	$a \triangleleft c = b$
b	$b \triangleleft a = b$	$b \triangleleft b = b$	$b \triangleleft c = a$
c	$c \triangleleft a = c$	$c \triangleleft b = c$	$c \triangleleft c = c$

and consider the quandle homomorphism $f: \{*\} \rightarrow A$ defined by $f(*) = c$. It is easy to see that $\text{Inn}(\{*\}) = \{\text{id}_{\{*\}}\}$ while $\text{Inn}(A) = \{\text{id}_A, (-)^{\rho_c}\}$. But then $\text{Inn}(f): \text{Inn}(\{*\}) \rightarrow \text{Inn}(A)$ is not a group homomorphism since

$$\text{Inn}(f)(\text{id}_{\{*\}}) = (-)^{\rho_c} \neq \text{id}_A.$$

However, Inn behaves well with respect to *surjective* quandle homomorphisms.

Proposition 2.3.8. *If $f: A \rightarrow B$ is a surjective quandle homomorphism, then $\text{Inn}(f): \text{Inn}(A) \rightarrow \text{Inn}(B)$ is a surjective group homomorphism.*

Proof. Define $\text{Inn}(f): \text{Inn}(A) \rightarrow \text{Inn}(B)$ by

$$\text{Inn}(f)(-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n) = -\triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} f(a_2) \cdots \triangleleft^{\alpha_n} f(a_n).$$

First let us see that $\text{Inn}(f): \text{Inn}(A) \rightarrow \text{Inn}(B)$ is well-defined when $f: A \rightarrow B$ is a surjective quandle homomorphism: if

$$-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n = -\triangleleft^{\beta_1} a'_1 \triangleleft^{\beta_2} a'_2 \cdots \triangleleft^{\beta_m} a'_m$$

in $\text{Inn}(A)$, then

$$a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n = a \triangleleft^{\beta_1} a'_1 \triangleleft^{\beta_2} a'_2 \cdots \triangleleft^{\beta_m} a'_m$$

holds in A for all $a \in A$. Thus

$$f(a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n) = f(a \triangleleft^{\beta_1} a'_1 \triangleleft^{\beta_2} a'_2 \cdots \triangleleft^{\beta_m} a'_m),$$

which is equivalent to

$$f(a) \triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} f(a_2) \cdots \triangleleft^{\alpha_n} f(a_n) = f(a) \triangleleft^{\beta_1} f(a'_1) \triangleleft^{\beta_2} f(a'_2) \cdots \triangleleft^{\beta_m} f(a'_m)$$

for all $f(a) \in B$, but since f is surjective, the last equality is true for all elements of B , thus

$$-\triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} f(a_2) \cdots \triangleleft^{\alpha_n} f(a_n) = -\triangleleft^{\beta_1} f(a'_1) \triangleleft^{\beta_2} f(a'_2) \cdots \triangleleft^{\beta_m} f(a'_m),$$

or equivalently,

$$\text{Inn}(f)(-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n) = \text{Inn}(f)(-\triangleleft^{\beta_1} a'_1 \triangleleft^{\beta_2} a'_2 \cdots \triangleleft^{\beta_m} a'_m).$$

The surjectivity of $\text{Inn}(f)$ is trivial since f is surjective.

To show it is a group homomorphism, let $-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n$ and $-\triangleleft^{\beta_1} b_1 \triangleleft^{\beta_2} b_2 \cdots \triangleleft^{\beta_n} b_n$ be two elements of $\text{Inn}(A)$, and consider their "product"

$$-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n \triangleleft^{\beta_1} b_1 \triangleleft^{\beta_2} b_2 \cdots \triangleleft^{\beta_n} b_n.$$

Now

$$\text{Inn}(f)(-\triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \cdots \triangleleft^{\alpha_n} a_n \triangleleft^{\beta_1} b_1 \triangleleft^{\beta_2} b_2 \cdots \triangleleft^{\beta_n} b_n)$$

is equal to

$$-\triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} f(a_2) \cdots \triangleleft^{\alpha_n} f(a_n) \triangleleft^{\beta_1} f(b_1) \triangleleft^{\beta_2} f(b_2) \cdots \triangleleft^{\beta_n} f(b_n),$$

this latter being the "product" of $- \triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} f(a_2) \cdots \triangleleft^{\alpha_n} f(a_n)$ and $- \triangleleft^{\beta_1} f(b_1) \triangleleft^{\beta_2} f(b_2) \cdots \triangleleft^{\beta_n} f(b_n)$. ■

Orbit congruence

Inner automorphisms are really important in the study of quandles. One can define equivalence relations on a quandle A by using the subgroups of $\text{Inn}(A)$: let N be a subgroup of $\text{Inn}(A)$ and define $\sim_N \subset A \times A$ by

$$a \sim_N b \text{ if and only if there exists } n \in N \text{ such that } a^n = b.$$

It is obviously a reflexive relation, it is symmetric since elements of the subgroup N can be inverted, and it is transitive since the product of two elements in N stays in N , thus $\sim_N$ is an equivalence relation on A . Bunch, Lofgren, Rapp and Yetter [BLRY10] found that these equivalence relations become congruences when N is a *normal subgroup* of $\text{Inn}(A)$.

Proposition 2.3.9. *Let A be a quandle and N a normal subgroup of $\text{Inn}(A)$. Then the relation $\sim_N$ is a congruence on A .*

Proof. Let $a, b, c, d \in A$. If $a \sim_N b$ and $c \sim_N d$, we know there exist n and m in N such that $a^n = b$ and $c^m = d$. We can thus consider $\rho_c^{-1} n \rho_d \in \text{Inn}(A)$, which gives

$$\begin{aligned} (a \triangleleft c)^{\rho_c^{-1} n \rho_d} &= ((a \triangleleft c) \triangleleft^{-1} c)^{n \rho_d} \\ &= a^{n \rho_d} = a^n \triangleleft d = b \triangleleft d. \end{aligned}$$

Now to see that $\rho_c^{-1} n \rho_d$ is in N , remark that $\rho_d = \rho_{c^m}$, where $\rho_{c^m} = m^{-1} \rho_c m$ by Corollary 2.1.6. Thus

$$\rho_c^{-1} n \rho_d = \rho_c^{-1} n m^{-1} \rho_c m.$$

But the subgroup N is normal, so $\rho_c^{-1} n m^{-1} \rho_c$ is in N , from which it follows that $\rho_c^{-1} n \rho_d \in N$. ■

Definition 2.3.10 (Bunch, Lofgren, Rapp, Yetter). Let A be a quandle and N be a normal subgroup of $\text{Inn}(A)$. Then $\sim_N \subset A \times A$ is the *N -orbit congruence* on A .

Notation 2.3.11. Let A be a quandle and N be a normal subgroup of $\text{Inn}(A)$. We shall write $[a]_N$ for an element of $A / \sim_N$. The class $[a]_N$ is the orbit of $a \in A$ under the action of $N \subseteq \text{Inn}(A)$.

2.4 Adjoint group of a quandle

2.4.1. We saw in Example 2.1.8.2. that there is a functor Conj that assigns a quandle to any group via the conjugation of its elements. This functor actually has a left adjoint.

For a quandle A , we define the *adjoint group* $\text{Adj}(A)$ of A as the quotient group of the group $F(A) = \langle \{e_a \mid a \in A\} \rangle$ freely generated by the set A modulo the relations $R = \{e_{(a \triangleleft b)} e_b^{-1} e_a^{-1} e_b = 1 \mid a, b \in A\}$.

Notation 2.4.2. We shall simply write e_a for an element of the adjoint group instead of $[e_a]$.

This defines a functor $\text{Adj}: \text{Qnd} \rightarrow \text{Grp}$ that is left adjoint to the functor $\text{Conj}: \text{Grp} \rightarrow \text{Qnd}$.

$$\begin{array}{ccc}
 & \text{Adj} & \\
 \text{Qnd} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \text{Grp} \\
 & \text{Conj} &
 \end{array} \tag{2.4.2.1}$$

We shall write $\zeta_A: A \rightarrow \text{Conj}(\text{Adj}(A))$ for the A -unit of the adjunction (2.4.2.1).

Remark 2.4.3. First remark that by the universal property of the adjunction (2.4.2.1), there is a group homomorphism $\tau_A: \text{Adj}(A) \rightarrow \text{Inn}(A)$ for any quandle A such that the following triangle commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{\zeta_A} & \text{Conj}(\text{Adj}(A)) \\
 \text{inn} \searrow & & \swarrow \text{Conj}(\tau_A) \\
 & \text{Conj}(\text{Inn}(A)) &
 \end{array}$$

This implies that $\text{Adj}(A)$ acts on A by inner automorphisms. We write the action

$$A \times \text{Adj}(A) \rightarrow A, (a, g) \mapsto a^g \tag{2.4.3.1}$$

and thus, $a^{e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n}} = a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_n} a_n$ for any $a_i \in A$ and $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq n$. It is easy to see that the connected components of A are also the orbits under the action of $\text{Adj}(A)$.

For every quandle A , consider the quandle homomorphism $\chi: A \rightarrow \text{Conj}(\mathbb{Z})$ defined by $\chi(a) = 1$ for all $a \in A$, then the universal property of the adjunction (2.4.2.1) implies the existence of a unique group homomorphism $\sigma: \text{Adj}(A) \rightarrow \mathbb{Z}$ defined by $\sigma(e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n}) = \sum_{i=1}^n \alpha_i$ for all $a_i \in A$ for

$1 \leq i \leq n$.

$$\begin{array}{ccc} A & \xrightarrow{\zeta_A} & \text{Conj}(\text{Adj}(A)) \\ & \searrow \chi & \swarrow \text{Conj}(\sigma) \\ & \text{Conj}(\mathbb{Z}) & \end{array}$$

Its kernel $\text{Adj}(A)^\circ = \ker(\sigma)$ is generated by all products of the form $e_a e_b^{-1}$ with $a, b \in A$. The action (2.4.3.1) can be restricted to $\text{Adj}(A)^\circ$, and we obtain the same orbits with respect to this subgroup since for $a \in A$ and $g \in \text{Adj}(A)$, we have $a^g = a^{e_a^{-\sigma(g)} g}$.

Proposition 2.4.4. *The group homomorphism $\tau_A: \text{Adj}(A) \rightarrow \text{Inn}(A)$ is a central extension.*

Proof. Consider $e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n} \in \text{Adj}(A)$ such that

$$\tau_A(e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n}) = \text{id}_A$$

then $a^{\rho_{a_1}^{\alpha_1} \rho_{a_2}^{\alpha_2} \dots \rho_{a_n}^{\alpha_n}} = a$ for all $a \in A$. Thus

$$e_{a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_n} a_n} = e_a$$

for all $a \in A$ which by the definition of Adj is the same as

$$e_{a_n}^{-\alpha_n} \dots e_{a_2}^{-\alpha_2} e_{a_1}^{-\alpha_1} e_a e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n} = e_a$$

for all $a \in A$. ■

We will end this section by showing that the adjunction (2.4.2.1) is not admissible even for surjective homomorphisms. Here, we are not dealing with an adjunction between a variety and its subvariety, but the Definition 1.2.4 of admissibility is still valid in this case (see for example [MRV14] where the adjunction considered resembles the adjunction (2.4.2.1)).

Example 2.4.5. Consider the empty quandle $\emptyset$, then $\zeta_\emptyset: \emptyset \rightarrow \{*\}$ (since $\text{Adj}(\emptyset) = \{*\}$). We will show that the functor $\text{Conj}^\emptyset: \text{Grp} \downarrow \text{Adj}(\emptyset) \rightarrow \text{Qnd} \downarrow \emptyset$ is not faithful. Take the surjective group homomorphisms $\mathbb{Z} \rightarrow \{*\}$ and $\mathbb{Z} \times \mathbb{Z} \rightarrow \{*\}$, and consider the two group homomorphisms $\langle 0, 1 \rangle: \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ and $\langle 1, 0 \rangle: \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$, applying the functor $\text{Conj}^\emptyset$ to these two group homo-

morphisms yields the following commutative diagram

$$\begin{array}{ccccc}
 \emptyset & \xrightarrow{\quad} & \text{Conj}(\mathbb{Z}) & & \\
 \parallel & \searrow & \downarrow & \swarrow \text{Conj}(\langle 1,0 \rangle) & \\
 \emptyset & \xrightarrow{\quad} & \emptyset & \xrightarrow{\quad} & \text{Conj}(\mathbb{Z} \times \mathbb{Z}) \\
 \parallel & \nearrow & \downarrow & \nwarrow \text{Conj}(\langle 0,1 \rangle) & \\
 \emptyset & \xrightarrow{\quad \zeta_\emptyset \quad} & \{*\} & \xleftarrow{\quad} &
 \end{array}$$

where

$$\text{Conj}^\emptyset(\langle 0,1 \rangle) = \text{id}_\emptyset = \text{Conj}^\emptyset(\langle 1,0 \rangle)$$

but

$$\langle 0,1 \rangle \neq \langle 1,0 \rangle.$$

2.5 Algebraic quandle coverings

2.5.1. In this section, we recall the definition of algebraic quandle covering introduced by Eisermann [Eis03, Eis14] and give some connections with the central extensions of groups.

Definition 2.5.2. A quandle homomorphism $f: A \rightarrow B$ is an *algebraic quandle covering* if it is surjective and satisfies the following condition (C):

$$f(a) = f(a') \text{ implies } c \triangleleft a = c \triangleleft a' \text{ for all } a, a', c \in A.$$

Remark 2.5.3. Remark that, given an algebraic quandle covering $f: A \rightarrow B$, one can define a quandle homomorphism $s: B \rightarrow \text{Conj}(\text{Inn}(A))$ by $s(b) = (-)^{\rho_a}$ with $a \in A$ such that $f(a) = b$. Since $f: A \rightarrow B$ is an algebraic quandle covering, it is easy to show that it is well defined. It is also a quandle homomorphism: given b and b' in B , $s(b \triangleleft b') = (-)^{\rho_{a \triangleleft a'}}$ for some $a, a' \in A$ such that $f(a) = b$ and $f(a') = b'$. By using the same argument as in Remark 2.3.5, one can show that $(-)^{\rho_{a \triangleleft a'}} = (-)^{\rho_a} \triangleleft (-)^{\rho_{a'}} = s(b) \triangleleft s(b')$. By the universality of the adjunction (2.4.2.1), there is then an induced group homomorphism $t: \text{Adj}(B) \rightarrow \text{Inn}(A)$ defining an action of the adjoint group $\text{Adj}(B)$ of B on the quandle A . This action $A \times \text{Adj}(B) \rightarrow A$ with $(a, g) \mapsto a^g$ is defined as follows: if $g = e_b$ with $b \in B$ then $a^{e_b} := a \triangleleft a'$ where a' is an element of A such that $f(a') = b$.

Now let us see how the notion of algebraic quandle covering is in fact strongly related to the notion of central extension of groups.

Proposition 2.5.4. *Let $f: G \rightarrow H$ be a surjective group homomorphism. The group homomorphism $f: G \rightarrow H$ is a central extension if and only if $\text{Conj}(f): \text{Conj}(G) \rightarrow \text{Conj}(H)$ is an algebraic quandle covering.*

Proof. Start with a central extension $f: G \rightarrow H$, then $\text{Conj}(f)$ is surjective. Now if $\text{Conj}(f)(g) = \text{Conj}(f)(g')$ then in particular $f(g) = f(g')$ in the group H , or $f(gg'^{-1}) = 1_H$. Since f is a central extension, $gg'^{-1} \in Z(G)$, or $gg'^{-1}l = lgg'^{-1}$ for all $l \in G$, which is equivalent to $g'^{-1}lg' = g^{-1}lg$ in G or $l \triangleleft g' = l \triangleleft g$ in $\text{Conj}(G)$.

Now let $\text{Conj}(f): \text{Conj}(G) \rightarrow \text{Conj}(H)$ be an algebraic quandle covering and let $g \in \ker(f)$. Then in particular $\text{Conj}(f)(g) = 1_H = \text{Conj}(f)(1_G)$ which implies that $l \triangleleft g = l \triangleleft 1_G$ in $\text{Conj}(G)$ for all $l \in G$. This latter equality translates in G by $g^{-1}lg = l$ for all $l \in G$ so that $g \in Z(G)$. ■

Proposition 2.5.5. *Let $f: A \rightarrow B$ be an algebraic quandle covering. Then $\text{Inn}(f): \text{Inn}(A) \rightarrow \text{Inn}(B)$ is a central extension.*

Proof. If $\text{Inn}(f)(g) = \text{id}_B$ then $f(a^g) = f(a)$ for all $a \in A$ which implies that $a' \triangleleft a^g = a' \triangleleft a$ for all $a' \in A$. So by Corollary 2.1.6, we get $a'^{\rho_{a^g}} = a'^{g^{-1}\rho_a g}$ which is equal by assumption to a'^{ρ_a} for all $a' \in A$, so that $g^{-1}\rho_a g = \rho_a$ for all $a \in A$. ■

Proposition 2.5.6. *Let $f: A \rightarrow B$ be an algebraic quandle covering. Then $\text{Adj}(f): \text{Adj}(A) \rightarrow \text{Adj}(B)$ is a central extension.*

Proof. Since $f: A \rightarrow B$ is an algebraic quandle covering, there is an induced group homomorphism $t: \text{Adj}(B) \rightarrow \text{Inn}(A)$. We then have the following commutative diagram:

$$\begin{array}{ccc} \text{Adj}(A) & \xrightarrow{\tau_A} & \text{Inn}(A) \\ & \searrow \text{Adj}(f) & \nearrow t \\ & \text{Adj}(B) & \end{array}$$

with τ_A a central extension so that $\text{Adj}(f)$ is a central extension too. ■

Chapter 3

Study of the functor

$$\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$$

We study the concepts introduced in Chapter 1 for the adjunction between the category of quandles and its subcategory of trivial quandles. We first show that this adjunction is admissible by investigating some local permutability property in the category $\mathbf{Qnd}$ of quandles. We then characterize trivial and normal coverings and relate the notion of algebraic quandle covering to the notion of categorical covering introduced in Chapter 1. We then study factorization systems for surjective homomorphisms and compare the one induced by the adjunction with another one that can be deduced from the work of Bunch, Lofgren, Rapp and Yetter [BLRY10]. Finally, we study closure operators for subobjects and for congruences.

3.1 Admissibility

3.1.1. We first investigate some properties of orbit congruences.

Although the category of quandles is not a Mal'tsev category, a very important result concerning orbit congruences is that they permute with any other reflexive (internal) relation in the category of quandles.

Lemma 3.1.2. *Let A be a quandle, R a reflexive relation on A , and N a normal subgroup of $\text{Inn}(A)$. Then the relations R and $\sim_N$ permute:*

$$\sim_N \circ R = R \circ \sim_N .$$

Proof. Let $(a, b) \in \sim_N \circ R$, so that there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in \sim_N$. In particular, there is an automorphism $n \in N$ such that $c^n = b$.

It follows that $(a, b) \in R \circ \sim_N$, since $(a, a^n) \in \sim_N$ and $(a^n, b) = (a^n, c^n) \in R$. Accordingly, one has the inclusion $\sim_N \circ R \subset R \circ \sim_N$.

We also have the other inclusion $R \circ \sim_N \subset \sim_N \circ R$: if $(a, b) \in R \circ \sim_N$, then there exists $c \in A$ such that $a \sim_N c$ and cRb . The first relation implies the existence of some $n \in N$ such that $a^n = c$, which is equivalent to $a = c^{n^{-1}}$. Since cRb , one has that $a = c^{n^{-1}}Rb^{n^{-1}}$ and it is always true that $b^{n^{-1}} \sim_N b$.

This implies the equality

$$\sim_N \circ R = R \circ \sim_N . \quad \blacksquare$$

A straightforward observation can be made about the adjunction (2.3.6.1) between the category of quandles and the category of trivial quandles: the kernel pair of the A -component of the unit $\eta_A: A \rightarrow \pi_0(A)$ is an $\text{Inn}(A)$ -orbit congruence: $\sim_{\text{Inn}(A)}$.

The Lemma 3.1.2 implies a useful property of a special type of pushouts in $\mathbf{Qnd}$ [CKP93].

Lemma 3.1.3. *Let*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \downarrow \bar{f} \\ B & \xrightarrow{\bar{g}} & D \end{array}$$

be a pushout of surjective homomorphisms in $\mathbf{Qnd}$ such that $\text{Eq}(f)$ is an orbit congruence. Then the canonical factorization $(f, g): A \rightarrow B \times_D C$ to the pullback of $\bar{f}$ and $\bar{g}$ is a surjective homomorphism.

Proof. The fact that $\text{Eq}(f) = \sim_N$ for some normal subgroup $N \subset \text{Inn}(A)$ implies that

$$\text{Eq}(f) \circ \text{Eq}(g) = \text{Eq}(g) \circ \text{Eq}(f) = \text{Eq}(f) \vee \text{Eq}(g)$$

is the supremum $\text{Eq}(f) \vee \text{Eq}(g)$ of $\text{Eq}(f)$ and $\text{Eq}(g)$ as congruences on A . Moreover, the fact that the square is a pushout implies that $\text{Eq}(t) = \text{Eq}(f) \vee \text{Eq}(g)$, with $t = \bar{f} \circ g$. Consequently,

$$t^o \circ t = f^o \circ f \circ g^o \circ g. \quad (3.1.3.1)$$

The regular image of $(f, g): A \rightarrow B \times_D C$ is given by the relation $f \circ g^o$, whereas the relation $(B \times_D C, p_1, p_2)$ given by the pullback projections is $\bar{g}^o \circ \bar{f}$. Finally, by composing (3.1.3.1) on the left by f and on the right by g^o one obtains the equality

$$f \circ t^o \circ t \circ g^o = f \circ f^o \circ f \circ g^o \circ g \circ g^o,$$

which becomes, thanks to the identities (3.1.3.1) and (1.1.12.1),

$$f \circ f^o \circ \bar{g}^o \circ \bar{f} \circ g \circ g^o = f \circ g^o.$$

Since $f \circ f^o = \Delta_B$ and $g \circ g^o = \Delta_C$, it follows that

$$\bar{g}^o \circ \bar{f} = f \circ g^o,$$

as desired. ■

In particular, the following useful result holds:

Corollary 3.1.4. *For any surjective homomorphism $f: A \rightarrow B$ in $\mathbf{Qnd}$ the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \pi_0(A) \\ f \downarrow & & \downarrow \pi_0(f) \\ B & \xrightarrow{\eta_B} & \pi_0(B) \end{array} \quad (3.1.4.1)$$

where η is the unit of the adjunction (2.3.6.1) has the property that the canonical arrow $(f, \eta_A): A \rightarrow B \times_{\pi_0(B)} \pi_0(A)$ to the pullback (of $\pi_0(f)$ and η_B) is surjective.

Proof. This follows immediately from Lemma 3.1.3, the fact that the kernel pair of η_A is an orbit congruence and the fact that the square (3.1.4.1) is a pushout as mentioned in 1.2.1. ■

This leads us to the proof of the admissibility of the subcategory $\mathbf{Qnd}^*$ of the category $\mathbf{Qnd}$ of quandles.

Theorem 3.1.5. *In the adjunction (2.3.6.1), the reflector $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ preserves all pullbacks in $\mathbf{Qnd}$ of the form*

$$\begin{array}{ccc} B \times_{\pi_0(B)} X & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow \phi \\ B & \xrightarrow{\eta_B} & \pi_0(B) \end{array} \quad (3.1.5.1)$$

where $\phi: X \rightarrow \pi_0(B)$ is a surjective homomorphism lying in the subcategory $\mathbf{Qnd}^*$.

Proof. Consider the following commutative diagram where:

- the exterior rectangle is the pullback (3.1.5.1), where $\phi: X \rightarrow \pi_0(B)$ is a surjective homomorphism in the subcategory $\mathbf{Qnd}^*$;

- Q is the pullback of $\pi_0(p_1)$ and η_B ;
- the universal property of the unit

$$\eta_{B \times_{\pi_0(B)} X}: B \times_{\pi_0(B)} X \rightarrow \pi_0(B \times_{\pi_0(B)} X)$$

induces a unique morphism $\psi: \pi_0(B \times_{\pi_0(B)} X) \rightarrow X$ such that

$$\psi \circ \eta_{B \times_{\pi_0(B)} X} = p_2;$$

- the arrow $\gamma: B \times_{\pi_0(B)} X \rightarrow Q$ is the one induced by the universal property of the pullback of η_B along $\pi_0(p_1)$.

$$\begin{array}{ccccc}
 B \times_{\pi_0(B)} X & \xrightarrow{p_2} & X & & \\
 \downarrow p_1 & \searrow \gamma & \nearrow \psi & & \downarrow \phi \\
 & Q & \xrightarrow{q_2} & \pi_0(B \times_{\pi_0(B)} X) & \\
 & \nwarrow q_1 & & \searrow \pi_0(p_1) & \\
 B & \xrightarrow{\eta_B} & \pi_0(B) & &
 \end{array}$$

By Corollary 3.1.4, we know that the homomorphism γ is surjective. The fact that $q_1 \circ \gamma = p_1$ and $\psi \circ q_2 \circ \gamma = p_2$ implies that γ is also injective. Indeed, this latter property follows from the fact that the pullback projections p_1 and p_2 are jointly monomorphic, i.e. if $p_i \circ u = p_i \circ v$ (for $i \in \{1, 2\}$) then $u = v$. Accordingly, the arrow γ is bijective, thus it is an isomorphism. We can then consider the following diagram

$$\begin{array}{ccccc}
 B \times_{\pi_0(B)} X & \xrightarrow{\eta_{B \times_{\pi_0(B)} X}} & \pi_0(B \times_{\pi_0(B)} X) & \xrightarrow{\psi} & X \\
 \downarrow p_1 & & \downarrow \pi_0(p_1) & & \downarrow \phi \\
 B & \xrightarrow{\eta_B} & \pi_0(B) & \xlongequal{\quad} & \pi_0(B)
 \end{array}
 \quad \begin{array}{c} \\ (1) \\ \\ \end{array} \quad \begin{array}{c} \\ (2) \\ \\ \end{array}$$

where both the outer rectangle (1) + (2) and the square (1) are pullbacks. Since η_B is a surjective homomorphism it follows from Proposition 1.1.14 that (2) is a pullback. This shows that the pullback (3.1.5.1) is preserved by the functor π_0 , as desired. $\blacksquare$

Remark 3.1.6. We are now going to show that the surjectivity of the morphism $\phi: X \rightarrow \pi_0(B)$ is crucial in the proof of Theorem 3.1.5: the functor π_0 no longer preserves pullbacks of the form (3.1.5.1) when $\phi: X \rightarrow \pi_0(B)$ is not assumed to be surjective. In other words, the functor $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ is not semi-left-exact in the sense of [CHK85].

Let $X = \{*\}$ be the trivial quandle on the one-element set, and take for B the quandle having three elements a , b and c with $\triangleleft = \triangleleft^{-1}$ defined by the following table:

$\triangleleft$	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

By setting $\phi(*) = [a]_{\text{Inn}(B)} = [b]_{\text{Inn}(B)}$, we define a quandle homomorphism $\phi: X \rightarrow \pi_0(B)$ that is not surjective. If we write P for the pullback $B \times_{\pi_0(B)} X$ of ϕ and η_B , we thus have

$$P = \{ (p, *) \mid p \in \{a, b\} \},$$

and we want to show that the morphism $i: \pi_0(P) \rightarrow \pi_0(B) \times_{\pi_0(B)} X$ defined by $i\left([(p, *)]_{\text{Inn}(P)}\right) = ([p]_{\text{Inn}(B)}, *)$ is not an isomorphism. For this, it suffices to remark that

$$i([(a, *)]_{\text{Inn}(P)}) = ([a]_{\text{Inn}(B)}, *) = ([b]_{\text{Inn}(B)}, *) = i([(b, *)]_{\text{Inn}(P)})$$

with $[(a, *)]_{\text{Inn}(P)} \neq [(b, *)]_{\text{Inn}(P)}$. In other words the functor π_0 is not semi-left-exact.

Remark 3.1.7. One might wonder if, in general, the functor π_0 preserves pullbacks of surjective homomorphisms along surjective homomorphisms. The answer is negative again, as the following counter-example shows: π_0 does not even preserve kernel pairs of split epimorphisms, in general. This shows that the category **Qnd** behaves very differently compared to a *semi-abelian category* [JMT02] (see also [Gra04] for the result).

Let us consider the involutive quandle A with four elements $\{a, b, c, d\}$ defined by the following table

$\triangleleft$	a	b	c	d
a	a	a	a	b
b	b	b	b	a
c	c	c	c	c
d	d	d	d	d

and the trivial quandle B with two elements $\{x, y\}$. Let $f: A \rightarrow B$ be defined by $f(a) = f(b) = f(c) = x$ and $f(d) = y$. This quandle homomorphism is surjective, and it is even split by the quandle homomorphism $s: B \rightarrow A$

defined by $s(x) = c$ and $s(y) = d$. Its kernel pair $\text{Eq}(f)$, however, is not preserved by the functor π_0 . Indeed, $[(a, b)]_{\text{Inn}(\text{Eq}(f))}$ and $[(a, a)]_{\text{Inn}(\text{Eq}(f))}$ are distinct elements in $\pi_0(\text{Eq}(f))$ (since (d, d) is the only member of $\text{Eq}(f)$ acting non trivially on $\text{Eq}(f)$), while the corresponding images $([a]_{\text{Inn}(A)}, [b]_{\text{Inn}(A)})$ and $([a]_{\text{Inn}(A)}, [a]_{\text{Inn}(A)})$ are equal in $\text{Eq}(\pi_0(f))$. Accordingly, $\text{Eq}(\pi_0(f))$ is not isomorphic to $\pi_0(\text{Eq}(f))$.

3.2 Coverings

3.2.1. In this section, we give algebraic characterizations of trivial coverings, normal coverings and coverings. We also give examples showing that there exists coverings that are not normal and normal coverings that are not trivial.

Proposition 3.2.2. *A surjective homomorphism $f: A \rightarrow B$ is a trivial covering if and only if the following condition (T) holds:*

$$\forall a, a' \in A, \text{ if } f(a) = f(a') \text{ and } [a]_{\text{Inn}(A)} = [a']_{\text{Inn}(A)}, \text{ then } a = a'.$$

Proof. This follows directly from the definition of $\gamma: A \rightarrow B \times_{\pi_0(B)} \pi_0(A)$, together with the fact that γ is always surjective by Corollary 3.1.4. ■

Proposition 3.2.3. *A surjective homomorphism $f: A \rightarrow B$ is a normal covering if and only if the following condition (N) holds:*

$$\forall a_i \in A \text{ with } 0 \leq i \leq n, \text{ if}$$

$$a_0 \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n = a_0,$$

then

$$a'_0 \triangleleft^{\alpha_1} a'_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a'_n = a'_0$$

for all $a'_i \in f^{-1}(f(a_i))$.

Proof. By definition, $f: A \rightarrow B$ is a normal covering if and only if the first projection $f_1: \text{Eq}(f) \rightarrow A$ in the following diagram is a trivial covering.

$$\begin{array}{ccc} \text{Eq}(f) & \xrightarrow{f_2} & A \\ f_1 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

But $f_1: \text{Eq}(f) \rightarrow A$ is a trivial covering if and only if the condition (T) holds by Proposition 3.2.2:

$$\forall (a_0, a'_0), (x_0, x'_0) \in \text{Eq}(f), \text{ if}$$

$$f_1(a_0, a'_0) = f_1(x_0, x'_0)$$

or $a_0 = x_0$ and

$$[(a_0, a'_0)]_{\text{Inn}(\text{Eq}(f))} = [(x_0, x'_0)]_{\text{Inn}(\text{Eq}(f))},$$

then

$$(a_0, a'_0) = (a_0, x'_0).$$

This translates to the following condition:

$\forall a_0, a'_0, x'_0$ such that $f(a_0) = f(a'_0) = f(x'_0)$, if there exists $(a_i, a'_i) \in \text{Eq}(f)$ with $1 \leq i \leq n$ such that

$$(a_0, a'_0) \triangleleft^{\alpha_1} (a_1, a'_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} (a_n, a'_n) = (a_0, x'_0)$$

then $(a_0, a'_0) = (a_0, x'_0)$ or

$$(a_0, a'_0) = (a_0, a'_0) \triangleleft^{\alpha_1} (a_1, a'_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} (a_n, a'_n).$$

Clearly, condition (N) implies the previous condition, but it is also true that the previous condition implies (N) since we can therefore take x'_0 to be

$$x'_0 = a'_0 \triangleleft^{\alpha_1} a'_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a'_n. \quad \blacksquare$$

Although it is always true that $\text{Triv}(B) \subset \text{Norm}(B)$ when the subcategory $\mathcal{X}$ of $\mathcal{C}$ is admissible, we will take some time to show how the condition (T) implies condition (N). Suppose having a surjective homomorphism that satisfies condition (T), and suppose

$$a_0 \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n = a_0$$

for some $a_i \in A$ with $0 \leq i \leq n$. For any $a'_i \in A$ such that $f(a'_i) = f(a_i)$, we have that

$$\begin{aligned} f(a'_0 \triangleleft^{\alpha_1} a'_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a'_n) &= f(a'_0) \triangleleft^{\alpha_1} f(a'_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} f(a'_n) \\ &= f(a_0) \triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} f(a_n) \\ &= f(a_0 \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n) \\ &= f(a_0) = f(a'_0) \end{aligned}$$

and clearly $[a'_0 \triangleleft^{\alpha_1} a'_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a'_n]_{\text{Inn}(A)} = [a'_0]_{\text{Inn}(A)}$ so that, by condition (T),

$$a'_0 \triangleleft^{\alpha_1} a'_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a'_n = a'_0.$$

The converse is not true however, there exist normal coverings that are not trivial.

Example 3.2.4. Consider the involutive quandle A given by the following table

$\triangleleft$	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

and consider $\eta_A: A \rightarrow \pi_0(A)$. Since $\pi_0(A) = \{[a]_{\text{Inn}(A)}, [b]_{\text{Inn}(A)}, [c]_{\text{Inn}(A)}\}$, it is obvious that η_A is not a trivial covering since $\eta_A(a) = \eta_A(b)$ but $a \neq b$. However, η_A satisfies the condition (N): since a and b act trivially on all members of A , the only interesting case is if

$$x \triangleleft c \triangleleft c \cdots \triangleleft c = x$$

for $x = a$ or b . This latter equality is only true when there is an even number of $-\triangleleft c$ in the "chain", so that it acts trivially on both a and b , showing (N).

3.2.5. We will now show that coverings correspond to algebraic covering introduced in Chapter 2.

First let us show that algebraic quandle coverings behave well with respect to pullbacks.

Lemma 3.2.6. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{p} & B. \end{array} \quad (3.2.6.1)$$

If f is an algebraic quandle covering then p_1 is an algebraic quandle covering.

Moreover, if $p: E \rightarrow B$ is a surjective homomorphism, f is an algebraic quandle covering whenever p_1 is an algebraic quandle covering.

Proof. Algebraic quandle coverings are stable under pullbacks. Let (e, a) and $(e', a') \in E \times_B A$ such that $p_1(e, a) = p_1(e', a')$, i.e. $e = e'$. Then we have

$$f(a) = p(e) = p(e') = f(a'),$$

and by assumption we know that $c \triangleleft a = c \triangleleft a'$ for all $c \in A$, so that p_1 is an algebraic quandle covering since

$$(x, c) \triangleleft (e, a) = (x, c) \triangleleft (e, a')$$

for all $(x, c) \in E \times_B A$.

Now let us see that this property is reflected by pullbacks along surjective homomorphisms. Suppose that $f(a) = f(a')$: by surjectivity of p , there exists $e \in E$ such that

$$f(a) = f(a') = p(e),$$

so that both (e, a) and (e, a') belong to $E \times_B A$. Moreover, these elements have the same image by p_1 and thus, by assumption, we know that

$$(x, c) \triangleleft (e, a) = (x, c) \triangleleft (e, a')$$

or

$$(x \triangleleft e, c \triangleleft a) = (x \triangleleft e, c \triangleleft a')$$

for all $(x, c) \in E \times_B A$. Now since p_2 is surjective, there exists such an element (x, c) for all $c \in A$, so that, in particular, $c \triangleleft a = c \triangleleft a'$ for all $c \in A$. ■

From this lemma we get the following:

Corollary 3.2.7. *If $f: A \rightarrow B$ is a covering then $f: A \rightarrow B$ is an algebraic quandle covering.*

Proof. If f is a covering then there exists a surjective homomorphism p such that the pullback of f along p as in the diagram (3.2.6.1), namely p_1 , in the following diagram is a trivial covering.

$$\begin{array}{ccccc} \pi_0(E \times_B A) & \xleftarrow{\eta_{E \times_B A}} & E \times_B A & \xrightarrow{p_2} & A \\ \pi_0(p_1) \downarrow & & p_1 \downarrow & & \downarrow f \\ \pi_0(E) & \xleftarrow{\eta_E} & E & \xrightarrow{p} & B \end{array} \quad (3.2.7.1)$$

Since p_1 is a trivial covering, the left hand square in (3.2.7.1) is a pullback. Of course, in $\mathbf{Qnd}^*$, every surjective homomorphism is an algebraic quandle covering. By using the previous lemma twice, one can lift the algebraic quandle covering property from $\pi_0(p_1)$ to p_1 , and then to f . ■

To prove the converse, given an algebraic quandle covering $f: A \rightarrow B$, we need to construct $p: E \rightarrow B$ such that the pullback of f along p is a trivial covering. It appears that a good candidate for this has already been introduced by Eisermann in [Eis14].

Definition 3.2.8 (Eisermann). Let A be a quandle with connected components $(A_i)_{i \in I}$ and choose a base point q_i in each connected component A_i . Let $\text{Adj}(A)^\circ$ be the kernel of the group homomorphism $\sigma: \text{Adj}(A) \rightarrow \mathbb{Z}$. For each $i \in I$, we define

$$\tilde{A}_i := \{ (a, g) \in A_i \times \text{Adj}(A)^\circ \mid a = q_i^g \}.$$

We define $\tilde{A} = \sqcup_{i \in I} \tilde{A}_i$ as the disjoint union of the $\tilde{A}_i$.

The following lemma serves to recall the quandle structure on $\tilde{A}$. The quandle homomorphism $p: \tilde{A} \rightarrow A$ is also defined. It will be proved to be a weakly universal algebraic quandle covering of A . Both the following two results can be found in [Eis14] (Lemma 7.11 and Proposition 7.12, respectively).

Lemma 3.2.9. 1. The set $\tilde{A} = \sqcup_{i \in I} \tilde{A}_i$ becomes a quandle by defining:

$$\begin{aligned} (i, (a, g)) \triangleleft (j, (b, h)) &= (i, (a \triangleleft b, g e_a^{-1} e_b)) \\ (i, (a, g)) \triangleleft^{-1} (j, (b, h)) &= (i, (a \triangleleft^{-1} b, g e_a e_b^{-1})) \end{aligned}$$

for any $(a, g) \in \tilde{A}_i$ and $(b, h) \in \tilde{A}_j$.

2. The quandle $\tilde{A}$ is equipped with an action $\tilde{A} \times \text{Adj}(A) \rightarrow \tilde{A}$ defined by

$$(i, (a, g))^h := (i, (a^h, e_{q_i}^{-\sigma(h)} g h))$$

for all $h \in \text{Adj}(A)$, $(a, g) \in \tilde{A}_i$ and $q_i \in A_i$. There is thus a restricted action of $\text{Adj}(A)$ on each $\tilde{A}_i$, defined by

$$(a, g)^h = (a^h, e_{q_i}^{-\sigma(h)} g h).$$

The subgroup $\text{Adj}(A)^\circ$ acts freely and transitively on each $\tilde{A}_i$. As a consequence, the connected components of $\tilde{A}$ are the sets $\tilde{A}_i$.

3. The arrow $p: \tilde{A} \rightarrow A$ defined by $p(i, (a, g)) = a$ is a surjective quandle homomorphism and is equivariant under the action of $\text{Adj}(A)$.

Proof. 1. The first quandle axiom is obvious:

$$(i, (a, g)) \triangleleft (i, (a, g)) = (i, (a \triangleleft a, g e_a^{-1} e_a)) = (i, (a, g)).$$

We will only give the proof of one identity for the second axiom, the other identity being proved in the same way by exchanging the roles of $\triangleleft$ and $\triangleleft^{-1}$. The second axiom follows from the equality $e_{a \triangleleft b} = e_b^{-1} e_a e_b$:

$$\begin{aligned} ((i, (a, g)) \triangleleft (j, (b, h))) \triangleleft^{-1} (j, (b, h)) &= (i, (a \triangleleft b, g e_a^{-1} e_b)) \triangleleft^{-1} (j, (b, h)) \\ &= (i, ((a \triangleleft b) \triangleleft^{-1} b, g e_a^{-1} e_b e_{a \triangleleft b} e_b^{-1})) \\ &= (i, (a, g e_a^{-1} e_b e_b^{-1} e_a e_b e_b^{-1})) \\ &= (i, (a, g)). \end{aligned}$$

The third axiom also results from the previous equality:

$$\begin{aligned}
((i, (a, g)) \triangleleft (j, (b, h))) \triangleleft (k, (c, l)) \\
&= (i, (a \triangleleft b, ge_a^{-1}e_b)) \triangleleft (k, (c, l)) \\
&= (i, ((a \triangleleft b) \triangleleft c, ge_a^{-1}e_b e_a^{-1}e_b e_c)) \\
&= (i, ((a \triangleleft b) \triangleleft c, ge_a^{-1}e_b e_b^{-1}e_a^{-1}e_b e_c)) \\
&= (i, ((a \triangleleft b) \triangleleft c, ge_a^{-1}e_a^{-1}e_b e_c)),
\end{aligned}$$

and

$$\begin{aligned}
((i, (a, g)) \triangleleft (k, (c, l))) \triangleleft ((j, (b, h)) \triangleleft (k, (c, l))) \\
&= (i, (a \triangleleft c, ge_a^{-1}e_c)) \triangleleft (j, (b \triangleleft c, he_b^{-1}e_c)) \\
&= (i, ((a \triangleleft c) \triangleleft (b \triangleleft c), ge_a^{-1}e_c e_a^{-1}e_b e_c)) \\
&= (i, ((a \triangleleft b) \triangleleft c, ge_a^{-1}e_a^{-1}e_b e_c)).
\end{aligned}$$

2. Now let us prove that $\text{Adj}(A)^\circ$ acts transitively via the action defined in the lemma: if (a, g) and (b, h) are in $\tilde{A}_i$, then $a = q_i^g$ and $b = q_i^h$ where $g, h \in \text{Adj}(A)^\circ$. By taking $g^{-1}h \in \text{Adj}(A)^\circ$ we see that $(a, g)^{g^{-1}h} = (b, h)$. It also acts freely on each $\tilde{A}_i$: if $(a, g)^h = (a, g)^k$ for some $h, k \in \text{Adj}(A)^\circ$, then $(a^h, gh) = (a^k, gk)$ and so $h = k$.
3. It is easy to see that the arrow $p: \tilde{A} \rightarrow A$ is a surjective quandle homomorphism. And we have

$$p((i, (a, g))^h) = p(i, (a^h, e_{q_i}^{-\sigma(h)}gh)) = a^h = p(i, (a, g))^h$$

for any $(i, (a, g)) \in \tilde{A}$ and $h \in \text{Adj}(A)$. ■

Proposition 3.2.10. *Let A be a quandle. Then the construction of $\tilde{A}$ does not depend on the choice of points $q_i \in A_i$, i.e. another choice of base points would just lead to an isomorphic structure.*

Proof. Let $\tilde{A}$ and $\hat{A}$ be the quandles from Definition 3.2.8 with base points $(q_i)_{i \in I}$ and $(r_i)_{i \in I}$ in the connected components $(A_i)_{i \in I}$ of A respectively. Then, for each $i \in I$, there exists $h_i \in \text{Adj}(A)^\circ$ such that $q_i^{h_i} = r_i$ (or $r_i^{h_i^{-1}} = q_i$) via the action of $\text{Adj}(A)^\circ$ on A (see Remark 2.4.3 and Lemma 3.2.9). Now define the map $\gamma: \hat{A} \rightarrow \tilde{A}$ by

$$\gamma((i, (a, g))) = (i, (a, h_i g)).$$

It is well-defined since if $(i, (a, g)) = (j, (b, h))$ in $\hat{A}$, then $i = j$, $a = b$ and $g = h$ so that $(i, (a, h_i g)) = (j, (b, h_i h))$ in $\tilde{A}$. The function γ is injective: if

$$\gamma((i, (a, g))) = \gamma((j, (b, h)))$$

for some $(i, (a, g))$ and $(j, (b, h))$ in $\hat{A}$ then

$$(i, (a, h_i g)) = (j, (b, h_i h))$$

in $\tilde{A}$. The last equality is true when $i = j$ (so that $h_i = h_j$), $a = b$ and $h_i g = h_j h = h_i h$ which implies that $g = h$. It is clearly surjective for if $(i, (a, g))$ is in $\tilde{A}$ (so $q_i^g = a$, thus $r_i^{h_i^{-1}g} = a$), one can take $(i, (a, h_i^{-1}g))$ in $\hat{A}$, whose image by γ is $(i, (a, h_i h_i^{-1}g)) = (i, (a, g))$. Now let us show it is a quandle homomorphism: if $(i, (a, b))$ and $(j, (b, h))$ are elements of $\hat{A}$, then

$$\begin{aligned} \gamma((i, (a, g)) \triangleleft (j, (b, h))) &= \gamma(i, (a \triangleleft b, g e_a^{-1} e_b)) \\ &= (i, (a \triangleleft b, h_i g e_a^{-1} e_b)) \\ &= (i, (a, h_i g)) \triangleleft (j, (b, h_i h)) \\ &= \gamma((i, (a, g))) \triangleleft \gamma((j, (b, h))). \end{aligned}$$

The application γ is thus a quandle isomorphism, as desired. ■

Definition 3.2.11. An algebraic quandle covering $f: X \rightarrow A$ is *weakly universal* if for each algebraic quandle covering $g: C \rightarrow A$, there exists a quandle homomorphism $\phi: X \rightarrow C$ such that $g \circ \phi = f$.

Proposition 3.2.12. Let A be a quandle and let $\tilde{A}$ be defined as in Lemma 3.2.9. Then the arrow $p: \tilde{A} \rightarrow A$ defined by

$$p(i, (a, g)) = a \quad \forall (i, (a, g)) \in \tilde{A}$$

is a weakly universal algebraic quandle covering of A .

Proof. It is easy to see that p is an algebraic quandle covering for if

$$p(i, (a, g)) = p(j, (b, h))$$

then $a = b$ and

$$\begin{aligned} (k, (c, l)) \triangleleft (i, (a, g)) &= (k, (c \triangleleft a, l e_c^{-1} e_a)) \\ &= (k, (c \triangleleft b, l e_c^{-1} e_b)) \\ &= (k, (c, l)) \triangleleft (j, (b, h)) \end{aligned}$$

for all $(k, (c, l)) \in \tilde{A}$.

Now we need to prove that if $f: X \rightarrow A$ is an algebraic quandle covering then there exists a homomorphism $\phi: \tilde{A} \rightarrow X$ such that $f \circ \phi = p$. First recall that since $f: X \rightarrow A$ is an algebraic quandle covering, there is an action of $\text{Adj}(A)$ on X . So we define $\phi: \tilde{A} \rightarrow X$ by $\phi(i, (a, g)) = x_i^g$ for x_i such that $f(x_i) = q_i$, so that $f(x_i^g) = q_i^g = a$. Remark that the x_i 's are in different connected components since the q_i 's would be in the same connected components otherwise. Thus $f \circ \phi = p$ on each connected component, so that this is true in general. It suffices now to show that ϕ is equivariant under the action of $\text{Adj}(A)^\circ$ in order to prove that it is a quandle homomorphism because

$$\begin{aligned} \phi((i, (a, g)) \triangleleft (j, (b, h))) &= \phi((i, (a \triangleleft b, g e_a^{-1} e_b))) \\ &= \phi((i, (a, g))^{e_a^{-1} e_b}) \end{aligned}$$

and

$$\phi((i, (a, g)) \triangleleft \phi(j, (b, h))) = \phi((i, (a, g)))^{e_a^{-1} e_b}.$$

But indeed, if $h \in \text{Adj}(A)^\circ = \ker(\sigma)$, then

$$\begin{aligned} \phi((i, (a, g))^h) &= \phi((i, (a^h, e_q^{-\sigma(h)} g h))) \\ &= \phi((i, (a^h, g h))) \\ &= x_i^{gh} \\ &= \phi((i, (a, g)))^h. \end{aligned} \quad \blacksquare$$

Before proving the main result of this section, we shall need one more lemma.

Lemma 3.2.13. *Let A be a quandle with $a = b^g$ for some $a, b \in A$ and $g \in \text{Adj}(A)$. Then*

$$e_a = g^{-1} e_b g.$$

In particular, $g e_a^\alpha = e_b^\alpha g$ with $\alpha \in \mathbb{Z}$.

Proof. Since $g \in \text{Adj}(A)$,

$$g = e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n}$$

for some $a_i \in A$ and $\alpha_i \in \{-1, 1\}$ with $1 \leq i \leq n$. So

$$\begin{aligned} b^g &= b^{e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n}} \\ &= b \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_n} a_n. \end{aligned}$$

And by using the identity $e_{a \triangleleft b} = e_b^{-1} e_a e_b$, one finds that

$$\begin{aligned}
 e_a &= e_{b \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_n} a_n} \\
 &= e_{a_n}^{-\alpha_n} (e_{b \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} a_2 \dots \triangleleft^{\alpha_{n-1}} a_{n-1}}) e_{a_n}^{\alpha_n} \\
 &= \dots \\
 &= e_{a_n}^{-\alpha_n} \dots e_{a_2}^{-\alpha_2} e_{a_1}^{-\alpha_1} e_b e_{a_1}^{\alpha_1} e_{a_2}^{\alpha_2} \dots e_{a_n}^{\alpha_n} \\
 &= g^{-1} e_b g. \quad \blacksquare
 \end{aligned}$$

Remark 3.2.14. This lemma allows one to redefine the quandle operations on $\tilde{A}$ this way:

$$\begin{aligned}
 (i, (a, g)) \triangleleft (j, (b, h)) &= (i, (a \triangleleft b, e_{q_i}^{-1} g e_b)) \\
 (i, (a, g)) \triangleleft^{-1} (j, (b, h)) &= (i, (a \triangleleft^{-1} b, e_{q_i} g e_b^{-1})).
 \end{aligned}$$

Finally we can prove that algebraic quandle coverings correspond to coverings for the adjunction between the category of quandles and the category of trivial quandles [Eve14a].

Theorem 3.2.15. *A quandle homomorphism $f: X \rightarrow A$ is an algebraic quandle covering if and only if it is a covering.*

Proof. By Corollary 3.2.7 one only needs to show that any algebraic quandle covering is a covering. More precisely we are going to show that any algebraic quandle covering is split by the weakly universal algebraic quandle covering $p: \tilde{A} \rightarrow A$ constructed in Proposition 3.2.12. Let us then consider the pullback

$$\begin{array}{ccc}
 \tilde{A} \times_A X & \xrightarrow{p_2} & X \\
 p_1 \downarrow & & \downarrow f \\
 \tilde{A} & \xrightarrow{p} & A
 \end{array}$$

and check that the surjective homomorphism p_1 is a trivial covering. For this, suppose that

$$p_1((i, (a, g)), y) = p_1((j, (b, h)), z)$$

and

$$[(i, (a, g)), y]_{\text{Inn}(\tilde{A} \times_A X)} = [(j, (b, h)), z]_{\text{Inn}(\tilde{A} \times_A X)},$$

we have to prove that $((i, (a, g)), y) = ((j, (b, h)), z)$ (by Proposition 3.2.2).

The first equality already gives $(i, (a, g)) = (j, (b, h))$.

The second one guarantees the existence of $((i_k, (a_{i_k}, g_{i_k})), y_{i_k}) \in \tilde{A} \times_A X$ with $1 \leq k \leq n$ such that

$$\begin{aligned} ((i, (a, g)), y) \triangleleft^{\alpha_1} ((i_1, (a_{i_1}, g_{i_1})), y_{i_1}) \cdots \triangleleft^{\alpha_n} ((i_n, (a_{i_n}, g_{i_n})), y_{i_n}) \\ = ((j, (b, h)), z) \\ = ((i, (a, g)), z). \end{aligned}$$

This implies that

$$\begin{aligned} ((i, (a, g)) \triangleleft^{\alpha_1} (i_1, (a_{i_1}, g_{i_1})) \cdots \triangleleft^{\alpha_n} (i_n, (a_{i_n}, g_{i_n})), y) \triangleleft^{\alpha_1} y_{i_1} \cdots \triangleleft^{\alpha_n} y_{i_n} \\ = ((i, (a, g)), z), \end{aligned}$$

and one then gets the following equality by using the alternative definitions of the quandle operations given in Remark 3.2.14:

$$\begin{aligned} ((i, (a \triangleleft^{\alpha_1} a_{i_1} \cdots \triangleleft^{\alpha_n} a_{i_n}, e_{q_i}^{-\alpha_n} \cdots e_{q_i}^{-\alpha_1} g e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n})), y) \triangleleft^{\alpha_1} y_{i_1} \cdots \triangleleft^{\alpha_n} y_{i_n} \\ = ((i, (a, g)), z) \end{aligned}$$

so if we write $\alpha := \sum_{k=1}^n \alpha_k$, then

$$\begin{aligned} ((i, (a \triangleleft^{\alpha_1} a_{i_1} \cdots \triangleleft^{\alpha_n} a_{i_n}, e_{q_i}^{-\alpha} g e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n})), y) \triangleleft^{\alpha_1} y_{i_1} \cdots \triangleleft^{\alpha_n} y_{i_n} \\ = ((i, (a, g)), z). \end{aligned}$$

From this and Lemma 3.2.13, one deduces that

$$(i, (a \triangleleft^{\alpha_1} a_{i_1} \cdots \triangleleft^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n})) = (i, (a, g))$$

and

$$y \triangleleft^{\alpha_1} y_{i_1} \cdots \triangleleft^{\alpha_n} y_{i_n} = z.$$

Accordingly:

$$\begin{aligned} (a, g) &= (a \triangleleft^{\alpha_1} a_{i_1} \cdots \triangleleft^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n}) \\ &= (a \triangleleft^{-\alpha} a \triangleleft^{\alpha_1} a_{i_1} \cdots \triangleleft^{\alpha_n} a_{i_n}, g e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n}) \\ &= (a, g)^{e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n}}. \end{aligned}$$

But because $e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n} \in \ker(\sigma) = \text{Adj}(A)^\circ$ acts freely on $\tilde{A}_i$, one has that

$$e_a^{-\alpha} e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n} = 1,$$

which means

$$e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n} = e_a^\alpha.$$

Since f is an algebraic quandle covering, there is an action of $\text{Adj}(A)$ on X (Remark 2.4.3), which gives the following since $f(y_{i_k}) = a_{i_k}$ for all $1 \leq k \leq n$ and $f(y) = a$:

$$\begin{aligned} z &= y \triangleleft^{\alpha_1} y_{i_1} \cdots \triangleleft^{\alpha_n} y_{i_n} \\ &= y^{e_{a_{i_1}}^{\alpha_1} \cdots e_{a_{i_n}}^{\alpha_n}} \\ &= y^{e_a^\alpha} \\ &= y \triangleleft^\alpha y = y. \end{aligned} \quad \blacksquare$$

Corollary 3.2.16. *The weakly universal covering $p: \tilde{A} \rightarrow A$ is a normal covering.*

We finish this section by showing that $\text{Norm}(B) \subset \text{Cov}(B)$ in terms of the conditions (N) and (C). We also give an example of a covering that is not a normal covering.

Suppose having a surjective homomorphism $f: A \rightarrow B$ satisfying condition (N) and suppose $f(a) = f(a')$. Certainly, given any $c \in A$, we have $c \triangleleft a \triangleleft^{-1} a = c$ but then condition (N) says that in particular, $c \triangleleft a \triangleleft^{-1} a' = c$ which translates into $c \triangleleft a = c \triangleleft a'$.

Example 3.2.17. Consider the involutive quandle A given by

$\triangleleft$	a	b	c	d
a	a	a	a	a
b	b	b	d	b
c	c	c	c	c
d	d	d	b	d

and the two elements trivial quandle $X = \{x, y\}$. Now consider $f: A \rightarrow X$ defined by $f(a) = f(b) = f(d) = x$ and $f(c) = y$. It is not a normal covering since $f(a) = f(b)$ and $a \triangleleft c = a$ but $b \triangleleft c = d \neq b$. To see that it satisfies condition (C), it suffices to remark that elements with same image by f have the same column in the table.

3.3 Fundamental group of connected quandles

We prove that the notion of fundamental group of a connected quandle introduced by Eisermann in [Eis14], that we shall call *algebraic fundamental group* in

the following, is also a particular case of the categorical notion of fundamental group recalled in Chapter 1.

Definition 3.3.1. Let A be a connected quandle with base point q . The *algebraic fundamental group* $\pi_1(A, q)$ is defined as

$$\pi_1(A, q) = \{g \in \text{Adj}(A)^o \mid q^g = q\}.$$

Theorem 3.3.2. The fundamental group $\pi_1^{Gal}(A)$ of a connected quandle A is isomorphic to the algebraic fundamental group $\pi_1(A, q)$ for any $q \in A$.

Proof. As explained in Section 1.4, one should consider the weakly universal normal covering $p: \tilde{A} \rightarrow A$ of A as constructed in the previous section. We construct the kernel pair of p :

$$\text{Eq}(p) = \{((a, g), (a', g')) \mid a = a'\} = \{((a, g), (a, g'))\}.$$

Since A is a connected quandle, the internal groupoid

$$\text{Eq}(p) \times_{\tilde{A}} \text{Eq}(p) \begin{array}{c} \xrightarrow{s_1} \\ \xrightarrow{m} \\ \xrightarrow{s_2} \end{array} \text{Eq}(p) \begin{array}{c} \xleftarrow{p_1} \\ \xleftarrow{e} \\ \xleftarrow{p_2} \end{array} \tilde{A}$$

$\begin{array}{c} i \\ \curvearrowright \end{array}$

is sent to the Galois group π_1^{Gal} by π_0 :

$$\pi_0(\text{Eq}(p)) \times \pi_0(\text{Eq}(p)) \begin{array}{c} \xrightarrow{\pi_0(s_1)} \\ \xrightarrow{\pi_0(m)} \\ \xrightarrow{\pi_0(s_2)} \end{array} \pi_0(\text{Eq}(p)) \begin{array}{c} \xleftarrow{\pi_0(p_1)} \\ \xleftarrow{\pi_0(e)} \\ \xleftarrow{\pi_0(p_2)} \end{array} \{*\}$$

$\begin{array}{c} \pi_0(i) \\ \curvearrowright \end{array}$

whose elements are the elements of

$$\pi_0(\text{Eq}(p)) = \left\{ [((a, g), (a, g'))]_{\text{Inn}(\text{Eq}(p))} \mid ((a, g), (a, g')) \in \text{Eq}(p) \right\}.$$

Thanks to Lemma 3.2.9, we know that since A is algebraically connected, so is $\tilde{A}$. We can thus deduce that for any two elements (a, g) and (b, h) in $\tilde{A}$ there exist $(c_i, k_i) \in \tilde{A}$ with $1 \leq i \leq n$ such that

$$\begin{aligned} (a, g) &= (b, h) \triangleleft^{\alpha_1} (c_1, k_1) \triangleleft^{\alpha_2} (c_2, k_2) \cdots \triangleleft^{\alpha_n} (c_n, k_n) \\ &= (b \triangleleft^{\alpha_1} c_1 \triangleleft^{\alpha_2} c_2 \cdots \triangleleft^{\alpha_n} c_n, e_q^{-\alpha} h e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \cdots e_{c_n}^{\alpha_n}) \end{aligned}$$

where $\alpha = \sum_{i=1}^n \alpha_i$. Note that this implies that

$$a = b \triangleleft^{\alpha_1} c_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} c_n$$

and

$$e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n} = h^{-1} e_q^\alpha g. \quad (3.3.2.1)$$

Let us now clarify the different group operations on $\pi_0(\text{Eq}(p))$ in order to prove that the algebraic fundamental group is isomorphic to the fundamental group $\pi_1^{\text{Gal}}(A)$:

- the neutral element of the group comes from the reflexivity of the relation $\text{Eq}(p)$ and is thus given by the class

$$[((a, g), (a, g))]_{\text{Inn}(\text{Eq}(p))}$$

for any $(a, g) \in \tilde{A}$;

- the inverse of an element should come from the symmetry of $\text{Eq}(p)$: the inverse of $[((a, g), (a, g'))]_{\text{Inn}(\text{Eq}(p))} \in \pi_0(\text{Eq}(f))$ is given by

$$[((a, g), (a, g'))]_{\text{Inn}(\text{Eq}(p))}^{-1} = [((a, g'), (a, g))]_{\text{Inn}(\text{Eq}(p))};$$

- the multiplication of two elements is given by the transitivity of $\text{Eq}(p)$, so given $[((a, g), (a, g'))]_{\text{Inn}(\text{Eq}(p))}$ and $[(b, h), (b, h')]_{\text{Inn}(\text{Eq}(p))}$ in $\pi_0(\text{Eq}(p))$, we have to find how to transform (b, h) into (a, g') in order to use the transitivity of $\text{Eq}(p)$. Since $\tilde{A}$ is connected, there exist $(c_i, k_i) \in \tilde{A}$ with $1 \leq i \leq n$ such that

$$(a, g') = (b, h) \triangleleft^{\alpha_1} (c_1, k_1) \dots \triangleleft^{\alpha_n} (c_n, k_n),$$

and thus, since $((c_i, k_i), (c_i, k_i)) \in \text{Eq}(p)$ for all $1 \leq i \leq n$, we have

$$\begin{aligned} & ((b, h), (b, h')) \triangleleft^{\alpha_1} ((c_1, k_1), (c_1, k_1)) \dots \triangleleft^{\alpha_n} ((c_n, k_n), (c_n, k_n)) = \\ & ((b, h) \triangleleft^{\alpha_1} (c_1, k_1) \dots \triangleleft^{\alpha_n} (c_n, k_n), (b, h') \triangleleft^{\alpha_1} (c_1, k_1) \dots \triangleleft^{\alpha_n} (c_n, k_n)) \end{aligned}$$

where the first projection is

$$(b \triangleleft^{\alpha_1} c_1 \dots \triangleleft^{\alpha_n} c_n, e_q^{-\alpha} h e_{c_1}^{\alpha_1} \dots e_{c_n}^{\alpha_n}) = (a, g')$$

and the second projection yields thanks to the identity (3.3.2.1)

$$(b \triangleleft^{\alpha_1} c_1 \dots \triangleleft^{\alpha_n} c_n, e_q^{-\alpha} h' e_{c_1}^{\alpha_1} \dots e_{c_n}^{\alpha_n}) = (a, e_q^{-\alpha} h' h^{-1} e_q^\alpha g').$$

Now by Lemma 3.2.13,

$$e_q^{-\alpha} h' = h' e_{q^{h'}}^{-\alpha} = h' e_b^{-\alpha}$$

and

$$h^{-1} e_q^\alpha = e_{q^h}^\alpha h^{-1} = e_b^\alpha h^{-1}.$$

This implies that

$$(a, e_q^{-\alpha} h' h^{-1} e_q^\alpha g') = (a, h' e_b^{-\alpha} e_b^\alpha h^{-1} g') = (a, h' h^{-1} g').$$

To sum up, we just showed that

$$[(b, h), (b, h')]_{\text{Inn}(\text{Eq}(p))} = [((a, g'), (a, h' h^{-1} g'))]_{\text{Inn}(\text{Eq}(p))}.$$

Now the multiplication of the class $[((a, g'), (a, h' h^{-1} g'))]_{\text{Inn}(\text{Eq}(p))}$ and the class $[((b, h), (b, h'))]_{\text{Inn}(\text{Eq}(p))}$ is equal to the multiplication of the classes $[((a, g'), (a, g'))]_{\text{Inn}(\text{Eq}(p))}$ and $[((a, g'), (a, h' h^{-1} g'))]_{\text{Inn}(\text{Eq}(p))}$ which yields

$$\begin{aligned} [((a, g'), (a, g'))]_{\text{Inn}(\text{Eq}(p))} [((b, h), (b, h'))]_{\text{Inn}(\text{Eq}(p))} \\ = [((a, g'), (a, h' h^{-1} g'))]_{\text{Inn}(\text{Eq}(p))}. \end{aligned}$$

Consider $\iota: \pi_1^{Gal}(A) \rightarrow \pi_1(A, q)$ that sends $[((a, g'), (a, g'))]_{\text{Inn}(\text{Eq}(p))}$ to gg'^{-1} . It is well-defined: if

$$[((a, g'), (a, g'))]_{\text{Inn}(\text{Eq}(p))} = [((b, h), (b, h'))]_{\text{Inn}(\text{Eq}(p))},$$

then there exist $((c_i, k_i), (c_i, k'_i)) \in \text{Eq}(p)$ with $1 \leq i \leq n$ such that

$$\begin{aligned} ((a, g'), (a, g')) \\ = ((b, h), (b, h')) \triangleleft^{\alpha_1} ((c_1, k_1), (c_1, k'_1)) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} ((c_n, k_n), (c_n, k'_n)) \\ = ((a, e_q^{-\alpha} h e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n}), (a, e_q^{-\alpha} h' e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n})). \end{aligned}$$

Thus we get

$$\begin{aligned} gg'^{-1} &= e_q^{-\alpha} h e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n} (e_q^{-\alpha} h' e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n})^{-1} \\ &= e_q^{-\alpha} h e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \dots e_{c_n}^{\alpha_n} e_{c_n}^{-\alpha_n} \dots e_{c_2}^{-\alpha_2} e_{c_1}^{-\alpha_1} h'^{-1} e_q^\alpha \\ &= e_q^{-\alpha} h h'^{-1} e_q^\alpha \end{aligned}$$

and by Lemma 3.2.13, the last equality becomes

$$gg'^{-1} = e_q^{-\alpha} h h'^{-1} e_q^\alpha = h e_b^{-\alpha} e_b^\alpha h'^{-1} = h h'^{-1}.$$

The application ι is clearly surjective since given $g \in \pi_1(A, q)$ then we can take $[(q, g), (q, 1)]_{\text{Inn}(\text{Eq}(p))}$ in $\pi_0(\text{Eq}(p))$.

It is injective: let $[(a, g), (a, g')]_{\text{Inn}(\text{Eq}(p))}$ and $[(b, h), (b, h')]_{\text{Inn}(\text{Eq}(p))}$ in $\pi_0(\text{Eq}(p))$ such that

$$gg'^{-1} = hh'^{-1}$$

or

$$h' h^{-1} g = g'.$$

Since $\tilde{A}$ is algebraically connected, there exist $(c_i, k_i) \in \tilde{A}$ with $1 \leq i \leq n$ such that

$$(b, h) \triangleleft^{\alpha_1} (c_1, k_1) \triangleleft^{\alpha_2} (c_2, k_2) \cdots \triangleleft^{\alpha_n} (c_n, k_n) = (a, g),$$

which implies that

$$e_q^{-\alpha} h e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \cdots e_{c_n}^{\alpha_n} = g$$

or, equivalently,

$$e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \cdots e_{c_n}^{\alpha_n} = h^{-1} e_q^\alpha g.$$

Now let us see that the same "chain" links (a, g') to (b, h') : indeed

$$\begin{aligned} e_q^{-\alpha} h' e_{c_1}^{\alpha_1} e_{c_2}^{\alpha_2} \cdots e_{c_n}^{\alpha_n} &= e_q^{-\alpha} h' h^{-1} e_q^\alpha g \\ &= h' e_{q h'}^{-\alpha} e_{q h}^\alpha h^{-1} g \\ &= h' h^{-1} g \\ &= g'. \end{aligned}$$

This implies that

$$[(a, g), (a, g')]_{\text{Inn}(\text{Eq}(p))} = [(b, h), (b, h')]_{\text{Inn}(\text{Eq}(p))}.$$

Let us then verify that ι is a group homomorphism:

$$\iota([(a, g), (a, g')]_{\text{Inn}(\text{Eq}(p))}) \iota([(b, h), (b, h')]_{\text{Inn}(\text{Eq}(p))}) = gg'^{-1} hh'^{-1}$$

while

$$\begin{aligned}
& \iota([((a, g), (a, g'))]_{\text{Inn}(\text{Eq}(p))} [((b, h), (b, h'))]_{\text{Inn}(\text{Eq}(p))}) \\
&= \iota([((a, g), (a, h'h^{-1}g'))]_{\text{Inn}(\text{Eq}(p))}) \\
&= g(h'h^{-1}g')^{-1} \\
&= gg'^{-1}hh'^{-1}.
\end{aligned}$$

In conclusion, $\iota: \pi_1^{\text{Gal}}(A) \rightarrow \pi_1(A, q)$ is a group isomorphism. ■

3.4 Factorization systems for surjective homomorphisms

In this section we describe the factorization system of surjective homomorphisms induced by the reflective subcategory $\mathbf{Qnd}^*$ of $\mathbf{Qnd}$ and we show the existence of another factorization system derived from the work of Bunch, Lofgren, Rapp and Yetter [BLRY10]. Finally, we compare these two factorization systems and we show that the last one does not satisfy the typical property of the factorization systems arising from an adjunction (see Proposition 1.5.5). This is joint work with my supervisor M. Gran [EG14].

The induced factorization system

3.4.1. Consider the class $\mathcal{F}$ of surjective quandle homomorphisms and consider the pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms, where $\mathcal{E}$ is given by the morphisms in $\mathcal{F}$ inverted by the functor π_0 , and $\mathcal{M}$ is the class of trivial coverings. The morphisms belonging to the class $\mathcal{E}$ can be described as follows:

Proposition 3.4.2. *A surjective homomorphism $f: A \rightarrow B$ belongs to the class $\mathcal{E}$ if and only if $\text{Eq}(f) \subset \sim_{\text{Inn}(A)}$.*

Proof. The fact that π_0 inverts a surjective homomorphism $f: A \rightarrow B$ obviously implies that $\text{Eq}(f) \subset \sim_{\text{Inn}(A)}$.

Conversely, suppose now that $\text{Eq}(f) \subset \sim_{\text{Inn}(A)}$, so that we have the following commutative diagram

$$\begin{array}{ccc}
 \sim_{\text{Inn}(A)} & \xrightarrow{\bar{f}} & \sim_{\text{Inn}(B)} \\
 \downarrow & & \downarrow p_1 \downarrow p_2 \\
 \text{Eq}(f) \rightrightarrows & A \xrightarrow{f} & B \\
 \eta_A \downarrow & \nearrow \phi & \downarrow \eta_B \\
 \pi_0(A) & \xrightarrow{\pi_0(f)} & \pi_0(B)
 \end{array}$$

where the induced dotted homomorphism $\bar{f}$ is a surjective homomorphism ($f(\sim_{\text{Inn}(A)}) = \sim_{\text{Inn}(B)}$ by Lemma 3.1.2 and Proposition 1.1.22), and the induced dotted homomorphism ϕ is such that $\phi \circ f = \eta_A$. It follows that $\phi \circ p_1 = \phi \circ p_2$, so that there exists a unique morphism $\psi: \pi_0(B) \rightarrow \pi_0(A)$ with $\psi \circ \eta_B = \phi$, which is the inverse of $\pi_0(f)$. ■

Remark 3.4.3. For a surjective homomorphism $f: A \rightarrow B$ the condition

$$\text{Eq}(f) \subset \sim_{\text{Inn}(A)}$$

says the following: if $f(a) = f(a')$, then there is an automorphism $n \in \text{Inn}(A)$ such that $a^n = a'$. In other words, f can only identify elements of A belonging to the same connected component.

Corollary 3.4.4. *The classes*

$$\mathcal{E} = \{f: A \rightarrow B \mid f \in \mathcal{F} \text{ and } \text{Eq}(f) \subset \sim_{\text{Inn}(A)}\}$$

and

$$\mathcal{M} = \{f: A \rightarrow B \mid f \in \mathcal{F} \text{ and } \text{Eq}(f) \cap \sim_{\text{Inn}(A)} = \Delta_A\}$$

form a factorization system for $\mathcal{F}$ in the category $\mathbf{Qnd}$ of quandles.

Proof. The condition (i) in Definition 1.5.2 is easily checked, while condition (ii) is the Remark 1.5.4. To check the condition (iii) in the definition of a factorization system for $\mathcal{F}$ consider any commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow h \\
 C & \xrightarrow{m} & D
 \end{array}$$

in $\mathbf{Qnd}$ where $f \in \mathcal{E}$ and $m \in \mathcal{M}$. We have to show the existence of a unique morphism $t: B \rightarrow C$ such that $t \circ f = g$ and $m \circ t = h$. Consider the cube

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 \eta_A \searrow & & \eta_B \searrow & & \\
 & \pi_0(A) & \xleftarrow{\varphi} & \pi_0(B) & \\
 & \downarrow \varphi^{-1} & & \downarrow & \\
 C & \xrightarrow{m} & D & & \\
 \eta_C \searrow & & \eta_D \searrow & & \\
 & \pi_0(C) & \xrightarrow{\pi_0(m)} & \pi_0(D) & \\
 & \downarrow \pi_0(g) & & \downarrow \pi_0(h) & \\
 & & & &
 \end{array}$$

where the bottom face is a pullback since m belongs to $\mathcal{M}$, and $\pi_0(f) = \varphi$ is an isomorphism since $f \in \mathcal{E}$. The universal property of the pullback and the equality

$$\pi_0(m) \circ \pi_0(g) \circ \varphi^{-1} \circ \eta_B = \pi_0(h) \circ \eta_B = \eta_D \circ h$$

induce a unique morphism $t: B \rightarrow C$ such that, in particular, $m \circ t = h$. The equality $t \circ f = g$ follows from the fact that the morphisms η_C and m are jointly monomorphic. $\blacksquare$

Comparison with another factorization system

3.4.5. When $\text{Eq}(f) = \sim_N$ for a normal subgroup N of $\text{Inn}(A)$, one always has that

$$\sim_N = \sim_{\text{Ker}(\text{Inn}(f))}$$

(see Theorem 7.1 in [BLRY10]). This observation suggests to consider the following class of morphisms of $\mathcal{F}$:

Definition 3.4.6.

$$\mathcal{E}_1 = \{f: A \rightarrow B \in \mathcal{F} \mid \text{Eq}(f) = \sim_{\text{Ker}(\text{Inn}(f))}\}$$

Thanks to Lemma 3.1.2 we know that the kernel congruences of the arrows in the class $\mathcal{E}_1$ have the strong property that they permute with any other congruence.

Let us compare the previous factorization system with another one. In their paper, Bunch, Lofgren, Rapp and Yetter showed that every surjective homomorphism in $\mathbf{Qnd}$ has a canonical factorization whose first component belongs to $\mathcal{E}_1$ and second component is what the authors call a *rigid quotient*, namely a surjective homomorphism h such that $\text{Inn}(h)$ is an isomorphism.

Proposition 3.4.7 ([BLRY10]). *Let $f: A \rightarrow B$ be a surjective homomorphism in $\mathbf{Qnd}$. Then f has a factorization as $f = h \circ g$, where $g: A \rightarrow A / \sim_{\text{Ker}(\text{Inn}(f))}$ and $h: A / \sim_{\text{Ker}(\text{Inn}(f))} \rightarrow B$ is such that $\text{Inn}(h)$ is an isomorphism.*

By using this result, we now show that the subclasses of $\mathcal{F}$

$$\mathcal{E}_1 = \{f: A \rightarrow B \mid \text{Eq}(f) = \sim_{\text{Ker}(\text{Inn}(f))}\}$$

and

$$\mathcal{M}_1 = \{f: A \rightarrow B \mid \text{Inn}(f) \text{ is an isomorphism}\}$$

form a factorization system for surjective homomorphisms:

Proposition 3.4.8. $(\mathcal{E}_1, \mathcal{M}_1)$ is a factorization system for $\mathcal{F}$ in $\mathbf{Qnd}$.

Proof. The first axiom in the definition of factorization system is easy to check, while (ii) is precisely Proposition 3.4.7 here above. To check the validity of property (iii) consider a commutative square of surjective homomorphisms

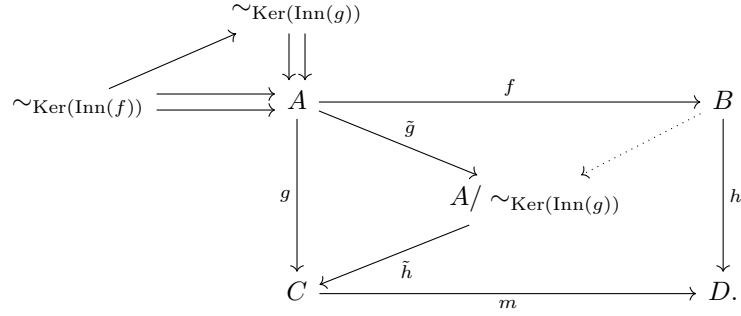
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{m} & D \end{array}$$

with $f \in \mathcal{E}_1$ and $m \in \mathcal{M}_1$. By applying Inn to this commutative square (this is possible only because it is a square of surjective homomorphisms) we get the commutative diagram of surjective group homomorphisms

$$\begin{array}{ccccc} & & \text{Ker}(\text{Inn}(g)) & & \\ & \nearrow \iota & \downarrow k & \nearrow \gamma & \\ \text{Ker}(\text{Inn}(f)) & \xrightarrow{k'} & \text{Inn}(A) & \xrightarrow{\text{Inn}(f)} & \text{Inn}(B) \\ & \text{Inn}(g) \downarrow & & & \downarrow \text{Inn}(h) \\ & \text{Inn}(C) & \xrightarrow{\text{Inn}(m)} & \text{Inn}(D) & \end{array}$$

with $\text{Inn}(m)$ an isomorphism. Accordingly, there is an induced inclusion $\iota: \text{Ker}(\text{Inn}(f)) \hookrightarrow \text{Ker}(\text{Inn}(g))$ between the kernels such that $k \circ \iota = k'$. This induces an inclusion $\iota': \sim_{\text{Ker}(\text{Inn}(f))} \rightarrow \sim_{\text{Ker}(\text{Inn}(g))}$ of the corresponding kernel congruences in $\mathbf{Qnd}$. Using Proposition 3.4.7, one obtains an $(\mathcal{E}_1, \mathcal{M}_1)$

factorization $\tilde{h} \circ \tilde{g}$ of g as in the diagram

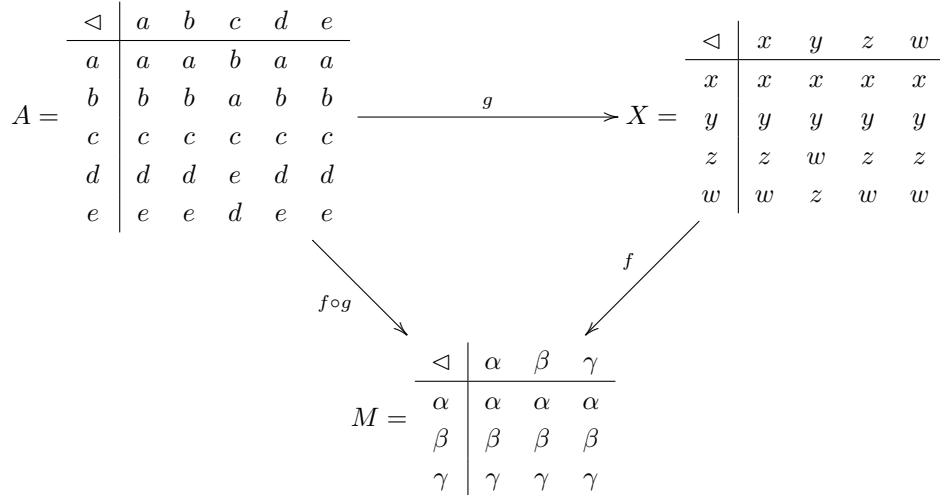


This induces a homomorphism $\phi: B \rightarrow A/\sim_{\text{Ker}(\text{Inn}(g))}$ such that $\phi \circ f = \tilde{g}$. The arrow $\tilde{h} \circ \phi$ is the desired factorization showing the orthogonality of $\mathcal{E}_1$ and $\mathcal{M}_1$. ■

By comparing this factorization system with the one considered in the previous section one remarks that $\mathcal{E}_1 \subset \mathcal{E}$, since $\text{Ker}(\text{Inn}(f)) \subset \text{Inn}(A)$ and, consequently, $\mathcal{M} \subset \mathcal{M}_1$.

Remark 3.4.9. We finally observe that the factorization system $(\mathcal{E}_1, \mathcal{M}_1)$ does not have the property that g belongs to $\mathcal{E}_1$ whenever $f \circ g$ and f belong to $\mathcal{E}_1$. This shows a difference with the factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathcal{F}$ in Qnd considered in the previous subsection.

Consider the following commutative diagram of involutive quandles:



Let $g: A \rightarrow X$ be defined by $g(a) = g(b) = x$, $g(c) = y$, $g(d) = z$ and $g(e) = w$, and let $f: X \rightarrow M$ be defined by $f(x) = \alpha$, $f(y) = \beta$ and $f(z) = f(w) = \gamma$. One can show that $f = \eta_X$ and $f \circ g = \eta_A$ so that both f and $f \circ g$ are in $\mathcal{E}_1$. To see that g is not in $\mathcal{E}_1$, remark that $(a, b) \in \text{Eq}(g)$

but the only member of $\text{Inn}(A)$ linking them is ρ_c which does not belong to $\text{Ker}(\text{Inn}(g))$ (since $\text{Inn}(g)(\rho_c) = \rho_y$).

3.5 From surjective homomorphisms to coverings

We will now describe how to turn surjective homomorphisms into coverings in the category of quandles. The results in this section come from a joint work with Andrea Montoli and Mathieu Duckerts-Antoine.

3.5.1. First let us prove a result that is key to the construction.

Lemma 3.5.2. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{p} & B \end{array} \quad (3.5.2.1)$$

where $f: A \rightarrow B$ is a surjective quandle homomorphism and $p: E \rightarrow B$ is a normal covering. Then

$$\text{Eq}(p_2) \circ (\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)}) = (\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)}) \circ \text{Eq}(p_2).$$

Proof. If $((e, a), (e', a')) \in \text{Eq}(p_2) \circ (\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)})$ then there exists $(\epsilon, \alpha) \in E \times_B A$ such that

$$(e, a)(\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)})(\epsilon, \alpha) \text{Eq}(p_2)(e', a').$$

Thus we have $(\epsilon, \alpha) = (e, a')$ with $p(e) = f(a')$ and

$$[(e, a')]_{\text{Inn}(E \times_B A)} = [(e, a)]_{\text{Inn}(E \times_B A)}.$$

The first condition implies that $p(e) = f(a') = p(e')$ while the second condition implies the existence of elements $(e_i, a_i) \in E \times_B A$ and $\alpha_i \in \mathbb{Z}$ with $1 \leq i \leq n$ such that

$$(e, a) \triangleleft^{\alpha_1} (e_1, a_1) \triangleleft^{\alpha_2} (e_2, a_2) \cdots \triangleleft^{\alpha_n} (e_n, a_n) = (e, a').$$

In particular, this shows that

$$p(e) = p(e')$$

and

$$e \triangleleft^{\alpha_1} e_1 \triangleleft^{\alpha_2} e_2 \cdots \triangleleft^{\alpha_n} e_n = e.$$

Since $p: E \rightarrow B$ is a normal covering, Proposition 3.2.3 implies that

$$e' \triangleleft^{\alpha_1} e_1 \triangleleft^{\alpha_2} e_2 \cdots \triangleleft^{\alpha_n} e_n = e'.$$

Now we have that

$$(e, a) \text{Eq}(p_2)(e', a)$$

and

$$(e', a) \triangleleft^{\alpha_1} (e_1, a_1) \triangleleft^{\alpha_2} (e_2, a_2) \cdots \triangleleft^{\alpha_n} (e_n, a_n) = (e', a')$$

which implies that

$$(e', a)(\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)})(e', a')$$

and thus

$$((e, a), (e', a')) \in (\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)}) \circ \text{Eq}(p_2). \quad \blacksquare$$

Corollary 3.5.3. *Consider the pullback (3.5.2.1) where $f: A \rightarrow B$ is a surjective quandle homomorphism and $p: E \rightarrow B$ is a normal covering. Then $p_2(\text{Eq}(p_1) \cap \sim_{\text{Inn}(E \times_B A)})$ is a congruence on the quandle A .*

Proof. This follows directly from Lemma 3.5.2 and Proposition 1.1.22. $\blacksquare$

Although the following result is already known by the work of Janelidze and Kelly [JK97], we give a constructive proof of the result by following the idea of Everaert [Eve14b].

Theorem 3.5.4. *The category $\text{Cov}(B)$ of coverings of B is a reflective subcategory of the category $\mathbf{Qnd} \downarrow B$.*

Proof. Let $f: A \rightarrow B$ be a surjective quandle homomorphism and consider the weakly universal quandle covering $p: \tilde{B} \rightarrow B$ described in Section 3.2. Take the pullback of f along p :

$$\begin{array}{ccc} \tilde{B} \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ \tilde{B} & \xrightarrow{p} & B. \end{array}$$

In order to simplify the notations, let us write

$$\cap := \text{Eq}(p_1) \cap \sim_{\text{Inn}(\tilde{B} \times_B A)}.$$

Recall by Corollary 3.4.4 that $\cap$ is the congruence such that $l: (\tilde{B} \times_B A)/\cap \rightarrow \tilde{B}$ is a trivial covering coming from the $(\mathcal{E}, \mathcal{M})$ factorization of p_1 . Now since $p: \tilde{B} \rightarrow B$ is a normal covering by Corollary 3.2.16, we already know from Corollary 3.5.3 that $p_2(\cap)$ is a congruence on the quandle A . Now consider the following diagram

$$\begin{array}{ccccc}
 \cap & \xrightarrow{\phi} & p_2(\cap) & & \\
 t_1 \downarrow & & q_1 \downarrow & & \\
 \tilde{B} \times_B A & \xrightarrow{p_2} & A & \xrightarrow{q} & A/p_2(\cap) \\
 p_1 \downarrow & t \searrow & \downarrow f & & \\
 & (\tilde{B} \times_B A)/\cap & & & \\
 & \swarrow l & & & \\
 \tilde{B} & \xrightarrow{p} & B & &
 \end{array}$$

First remark that since

$$\begin{aligned}
 q \circ p_2 \circ t_1 &= q \circ q_1 \circ \phi \\
 &= q \circ q_2 \circ \phi \\
 &= q \circ p_2 \circ t_2
 \end{aligned}$$

there exists a unique quandle homomorphism $h: (\tilde{B} \times_B A)/\cap \rightarrow A/p_2(\cap)$ such that $h \circ t = q \circ p_2$.

Also, since $\phi: \cap \rightarrow p_2(\cap)$ is a surjective quandle homomorphism and $f \circ q_1 \circ \phi = f \circ q_2 \circ \phi$, we have a unique quandle homomorphism $c_f: A/p_2(\cap) \rightarrow B$ such that $c_f \circ q = f$.

Now remark that the square of surjective homomorphisms

$$\begin{array}{ccc}
 \tilde{B} \times_B A & \xrightarrow{p_2} & A \\
 t \downarrow & & \downarrow q \\
 (\tilde{B} \times_B A)/\cap & \xrightarrow{h} & A/p_2(\cap)
 \end{array} \tag{3.5.4.1}$$

is a pushout since $\phi: \cap \rightarrow p_2(\cap)$ and $t: \tilde{B} \times_B A \rightarrow (\tilde{B} \times_B A)/\cap$ are surjective quandle homomorphisms. This implies that the square is a pullback by Lemma 3.5.2 and Lemma 3.1.3.

Now we have the following situation

$$\begin{array}{ccccc}
 \tilde{B} \times_B A & \xrightarrow{p_2} & A & & \\
 \downarrow p_1 & \searrow t & \downarrow f & \searrow q & \\
 & (\tilde{B} \times_B A)/\cap & \xrightarrow{h} & A/p_2(\cap) & \\
 & \swarrow l & \downarrow & \swarrow c_f & \\
 \tilde{B} & \xrightarrow{p} & B & &
 \end{array}$$

where the back square and the top square are pullbacks. Since $q: A \rightarrow A/p_2(\cap)$ is a surjective quandle homomorphism, the square

$$\begin{array}{ccc}
 (\tilde{B} \times_B A)/\cap & \xrightarrow{h} & A/p_2(\cap) \\
 l \downarrow & & \downarrow c_f \\
 \tilde{B} & \xrightarrow{p} & B
 \end{array}$$

is also a pullback by Proposition 1.1.14. This means that $c_f: A/p_2(\cap) \rightarrow B$ is a covering (Definition 1.3.6).

All is left to show now is the universality of the construction. For this consider the following factorization

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow u & \swarrow c \\
 & C &
 \end{array}$$

where $c: C \rightarrow B$ is a covering. Take the pullback of c along p

$$\begin{array}{ccc}
 \tilde{B} \times_B C & \xrightarrow{s_2} & C \\
 s_1 \downarrow & & \downarrow c \\
 B & \xrightarrow{p} & B
 \end{array}$$

where $s_1: \tilde{B} \times_B C \rightarrow \tilde{B}$ is then a trivial covering. Since the diagram

$$\begin{array}{ccc}
 \tilde{B} \times_B A & \xrightarrow{u \circ p_2} & C \\
 p_1 \downarrow & & \downarrow c \\
 \tilde{B} & \xrightarrow{p} & B
 \end{array}$$

commutes, there is a unique quandle homomorphism $\beta: \tilde{B} \times_B A \rightarrow \tilde{B} \times_B C$ such that $u \circ p_2 = s_2 \circ \beta$ and $s_1 \circ \beta = p_1$. By universality of the factorization

$l \circ t$, there exists a unique quandle homomorphism $\gamma: (\tilde{B} \times_B A)/\cap \rightarrow \tilde{B} \times_B C$ such that $\beta = \gamma \circ t$ and $l = s_1 \circ \gamma$. Thus $s_2 \circ \gamma \circ t = s_2 \circ \beta = u \circ p_2$ but since the square (3.5.4.1) is a pushout, this yields a unique quandle homomorphism $\alpha: A/p_2(\cap) \rightarrow C$ such that $\alpha \circ h = s_2 \circ \gamma$ and $\alpha \circ q = u$. The latter equality implies that

$$c \circ \alpha \circ q = c \circ u = f = c_f \circ q$$

but since q is a surjective homomorphism, we have $c_f = c \circ \alpha$. $\blacksquare$

3.6 Closure operators

In this section, we describe and investigate the pullback closure operator corresponding to the reflection (2.3.6.1) between the category $\mathbf{Qnd}$ of quandles and its full reflective subcategory $\mathbf{Qnd}^*$ of trivial quandles. This section is again a joint work with Professor Marino Gran [EG16].

3.6.1. From now on, in the category $\mathbf{Qnd}$ of quandles, we shall take as representative of a subobject $m: M \rightarrow A$ the corresponding subquandle inclusion. In particular the regular image I of an arrow $f: A \rightarrow B$ will be the inclusion of the regular image $f(A)$ as a subquandle of B .

Lemma 3.6.2. *For a subobject $m: M \rightarrow A \in \mathbf{Qnd}$, its pullback closure operator $c_A(M)$ is given by*

$$c_A(M) = \{a' \in A \mid a' \in [a]_{\text{Inn}(A)} \text{ for some } a \in M\}.$$

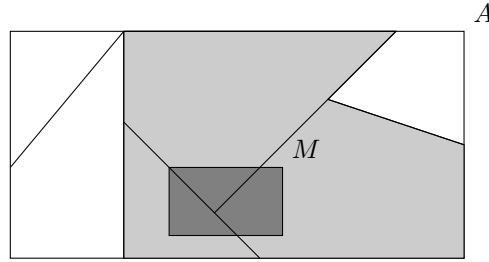
Proof. The adjunction (2.3.6.1) induces the pointed endofunctor (π_0, η) , and the arrow $n: N \rightarrow RA = \pi_0(A)$ in the diagram

$$\begin{array}{ccccc} M & \xrightarrow{m} & A & & \\ \eta_M \downarrow & \searrow m/c_A(m) & \nearrow c_A(m) & & \downarrow \eta_A \\ & c_A(M) & & & \\ \pi_0(M) & \xrightarrow{\pi_0(m)} & \pi_0(A) & & \\ & \searrow & \nearrow n & & \\ & N & & & \end{array}$$

is simply the inclusion of the image of $\pi_0(m)$ in $\pi_0(A)$. Since $c_A(m)$ is a monomorphism, it is the inclusion of the following subquandle of A :

$$c_A(M) = \{a' \in A \mid a' \in [a]_{\text{Inn}(A)} \text{ for some } a \in M\}. \quad \blacksquare$$

The pullback closure operator takes a subquandle M of A and extends it to the union of connected components containing elements of M . For instance, here is a representation of the closure operator of a subobject M of A (represented by the dark grey rectangle here below, while the subdivision of A gives the connected components of A). It gives the light grey part $c_A(M)$ in the following diagram:



Remark 3.6.3. For the adjunction (2.3.6.1) between the category of quandles and the category of trivial quandles, one observes that any monomorphism in the subcategory $\mathbf{Qnd}^*$ is a regular monomorphism (since $\mathbf{Qnd}^*$ is isomorphic to $\mathbf{Set}$). From Proposition 1.6.8, it follows that the regular closure operator and the pullback closure operator coincide: $c_A = c_A^{reg}$, for every $A \in \mathbf{Qnd}$.

Properties of the pullback closure operator

We investigate some properties of the pullback closure operator.

3.6.4. As shown in [Xar13], any reflector I from a variety of universal algebras $\mathcal{C}$ into a subvariety $\mathcal{X}$ preserves finite products when the variety $\mathcal{C}$ is idempotent. We give a proof of this result in the category $\mathbf{Qnd}$ of quandles.

Lemma 3.6.5. *The functor $\pi_0: \mathbf{Qnd} \rightarrow \mathbf{Qnd}^*$ preserves finite products.*

Proof. It suffices to check that the functor π_0 preserve binary products, since it preserves the terminal object. Let $A, B \in \mathbf{Qnd}$. There is a unique quandle homomorphism $\gamma: \pi_0(A \times B) \rightarrow \pi_0(A) \times \pi_0(B)$ such that

$$\gamma([(a, b)]_{\text{Inn}(A \times B)}) = ([a]_{\text{Inn}(A)}, [b]_{\text{Inn}(B)}).$$

It is easy to see that γ is surjective, by using the fact that each component of the unit of the adjunction (2.3.6.1) is surjective.

Let us check that γ is injective: let $[(a, b)]_{\text{Inn}(A \times B)}$ and $[(a', b')]_{\text{Inn}(A \times B)}$ be elements of $\pi_0(A \times B)$ such that $\gamma([(a, b)]_{\text{Inn}(A \times B)}) = \gamma([(a', b')]_{\text{Inn}(A \times B)})$. This means that $[a]_{\text{Inn}(A)} = [a']_{\text{Inn}(A)}$ and $[b]_{\text{Inn}(B)} = [b']_{\text{Inn}(B)}$. There are then

$a_i \in A, b_j \in B$ and $\alpha_i, \beta_j \in \mathbb{Z}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$a \triangleleft^{\alpha_1} a_1 \cdots \triangleleft^{\alpha_n} a_n = a'$$

and

$$b \triangleleft^{\beta_1} b_1 \cdots \triangleleft^{\beta_m} b_m = b'.$$

The idea now is to use the idempotency of $\triangleleft$ and $\triangleleft^{-1}$ in order to pass from a to a' , and then from b to b' , without changing the other component :

$$\begin{aligned} (a, b) \triangleleft^{\alpha_1} (a_1, b) \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_n} (a_n, b) \triangleleft^{\beta_1} (a', b_1) \cdots \triangleleft^{\beta_m} (a', b_m) \\ = (a', b) \triangleleft^{\beta_1} (a', b_1) \cdots \triangleleft^{\beta_m} (a', b_m) \\ = (a', b'). \end{aligned}$$

This shows that $[(a, b)]_{\text{Inn}(A \times B)} = [(a', b')]_{\text{Inn}(A \times B)}$, proving that γ is also injective, thus an isomorphism. $\blacksquare$

Lemma 3.6.6. *Let $s: S \rightarrow A$ and $t: T \rightarrow A$ be two subquandles of $A \in \mathbf{Qnd}$. The smallest subquandle $S \vee T$ containing both S and T is given by the set*

$$\begin{aligned} U = \{a_1 \triangleleft^{\alpha_1} a_2 \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_{n-1}} a_n \mid a_i \in S \cup T \text{ with } 1 \leq i \leq n \\ \text{and } \alpha_j \in \mathbb{Z} \text{ with } 1 \leq j \leq n-1\} \end{aligned}$$

equipped with the quandle operation inherited from A .

Proof. Given elements $a_1 \triangleleft^{\alpha_1} a_2 \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_{n-1}} a_n$ and $b_1 \triangleleft^{\beta_1} b_2 \triangleleft^{\beta_2} \cdots \triangleleft^{\beta_{m-1}} b_m$ of U , Corollary 2.1.6 gives

$$\begin{aligned} (a_1 \triangleleft^{\alpha_1} a_2 \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_{n-1}} a_n) \triangleleft^{\alpha} (b_1 \triangleleft^{\beta_1} b_2 \triangleleft^{\beta_2} \cdots \triangleleft^{\beta_{m-1}} b_m) = \\ a_1 \triangleleft^{\alpha_1} \cdots \triangleleft^{\alpha_{n-1}} a_n \triangleleft^{-\beta_{m-1}} b_m \triangleleft^{-\beta_{m-2}} \cdots \triangleleft^{-\beta_1} b_2 \triangleleft^{\alpha} b_1 \triangleleft^{\beta_1} \cdots \triangleleft^{\beta_{m-1}} b_m \end{aligned}$$

showing that U is stable under $\triangleleft$ and $\triangleleft^{-1}$.

Certainly U contains both S and T , and any quandle containing S and T must contain all chains of the form $a_1 \triangleleft^{\alpha_1} a_2 \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_{n-1}} a_n$ with $a_i \in S \cup T$ so that U is the smallest subquandle containing S and T . $\blacksquare$

Note that the proof of the previous Lemma 3.6.6 can be used to prove the same result for any family I of subobjects $s_i: S_i \rightarrow A$ with $i \in I$. The smallest subobject $\bigvee_{i \in I} S_i$ containing all the S_i 's is given by

$$\begin{aligned} \{a_1 \triangleleft^{\alpha_1} a_2 \triangleleft^{\alpha_2} \cdots \triangleleft^{\alpha_{n-1}} a_n \mid a_k \in \bigcup_{i \in I} S_i \text{ with } 1 \leq k \leq n \\ \text{and } \alpha_j \in \mathbb{Z} \text{ with } 1 \leq j \leq n-1\} \end{aligned}$$

with the quandle operation inherited from A .

Proposition 3.6.7. *The pullback closure operator c for the adjunction between $\mathbf{Qnd}$ and $\mathbf{Qnd}^*$ has the following properties:*

- (1) $c_A(\bigvee_{i \in I} s_i) = \bigvee_{i \in I} c_A(s_i)$ for subobjects $s_i: S_i \rightarrow A$ with $i \in I$ (c is fully additive);
- (2) $c_A(\prod_{1 \leq i \leq n} m_i) = \prod_{1 \leq i \leq n} c_{A_i}(m_i)$, where $A = \prod_{1 \leq i \leq n} A_i$ for any finite family of subobjects $m_i: M_i \rightarrow A_i$ with $1 \leq i \leq n$ (c is finitely productive);
- (3) $f(c_A(m)) = c_B(f(m))$ for any surjective homomorphism $f: A \rightarrow B$ and subobject $m: M \rightarrow A$.

Proof. To check (1), first observe that the class $[a]_{\text{Inn}(A)}$ of an element a in $\bigvee_{i \in I} S_i$ under the action of $\text{Inn}(A)$ does not depend on the chain, so that it is equivalent to take $a \in \bigcup_{i \in I} S_i$. Accordingly:

$$\begin{aligned} c_A\left(\bigvee_{i \in I} S_i\right) &= \{a' \in A \mid a' \in [a]_{\text{Inn}(A)} \text{ with } a \in \bigvee_{i \in I} S_i\} \\ &= \{a' \in A \mid a' \in [a]_{\text{Inn}(A)} \text{ with } a \in \bigcup_{i \in I} S_i\} \\ &= \bigvee_{i \in I} \{a' \in A \mid a' \in [a]_{\text{Inn}(A)} \text{ with } a \in S_i\} \\ &= \bigvee_{i \in I} c_A(S_i). \end{aligned}$$

In order to verify (2), let us write (a_i) for an element of $\prod_{1 \leq i \leq n} M_i = M$ and (a'_i) for an element of $\prod_{1 \leq i \leq n} A_i = A$. Then, by using Lemma 3.6.5,

$$\begin{aligned} c_A(M) &= \{(a'_i) \in A \mid (a'_i) \in [(a_i)]_{\text{Inn}(A)} \text{ for some } (a_i) \in M\} \\ &= \{(a'_i) \in A \mid a'_i \in [a_i]_{\text{Inn}(A_i)} \text{ for some } a_i \in M_i \text{ for all } 1 \leq i \leq n\} \\ &= \prod_{1 \leq i \leq n} \{a'_i \in A_i \mid a'_i \in [a_i]_{\text{Inn}(A_i)} \text{ for some } a_i \in M_i\} \\ &= \prod_{1 \leq i \leq n} c_{A_i}(M_i). \end{aligned}$$

To see (3), we have to check the validity of $c_B(f(M)) \subset f(c_A(M))$ for a surjective quandle homomorphism $f: A \rightarrow B$. If $b \in c_B(f(M))$ then there exist $b_i \in B$ and $\alpha_i \in \mathbb{Z}$ for $1 \leq i \leq n$ such that for some $a \in M$

$$b = f(a) \triangleleft^{\alpha_1} b_1 \cdots \triangleleft^{\alpha_n} b_n.$$

But since f is surjective, there exists $a_i \in A$ such that $f(a_i) = b_i$ for all $1 \leq i \leq n$ so that

$$\begin{aligned} b &= f(a) \triangleleft^{\alpha_1} b_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} b_n \\ &= f(a) \triangleleft^{\alpha_1} f(a_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} f(a_n) \\ &= f(a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n) \\ &\in f(c_A(M)). \end{aligned} \quad \blacksquare$$

Remark 3.6.8. We now show that the pullback closure operator associated with the reflection (2.3.6.1) is not weakly hereditary. Consider the 3-element involutive quandle A , and $\triangleleft$ defined by the following table:

$\triangleleft$	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

(3.6.8.1)

Now, if we look at the closure of the subobject $m: \{a\} \rightarrow A$, where $\{a\}$ is a one-element quandle, we find that $c_A(\{a\}) = \{a, b\}$ equipped with trivial quandle operations. Thus $m/c_A(m)$ is defined as $m/c_A(m)(a) = a$, and

$$c_{c_A(\{a\})}(\{a\}) = c_{\{a,b\}}(\{a\}) = \{a\},$$

which is not isomorphic to $c_A(\{a\})$.

The previous remark enlightened the fact that connected components of a quandle A are not connected in themselves.

Connected quandles

3.6.9. In a category $\mathcal{C}$ equipped with a closure operator c one says that an object A is *c-connected* if the diagonal $\Delta_A: A \rightarrow A \times A$ is dense. We now show that the c -connected quandles for the pullback closure operator associated with (2.3.6.1) are precisely the algebraically connected quandles:

Proposition 3.6.10. *Let c be the pullback closure operator for the adjunction (2.3.6.1). A quandle A is c -connected if and only if it is algebraically connected.*

Proof. When A is algebraically connected, so that $\pi_0(A) = \{*\}$ is the one-element quandle, then $A \times A$ is also algebraically connected, by Lemma 3.6.5.

We then have the commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\Delta_A} & A \times A & & \\
 \eta_A \downarrow & \nearrow c_{A \times A}(\Delta_A) & \downarrow \eta_{A \times A} & & \\
 & c_{A \times A}(A) & & & \\
 \pi_0(A) & \xrightarrow{\pi_0(\Delta_A)} & \pi_0(A \times A) & & \\
 \searrow e & & \nearrow i & & \\
 & N & & &
 \end{array}$$

where i is an isomorphism. Accordingly, $c_{A \times A}(\Delta_A)$ is an isomorphism too, and A is c -connected.

Conversely, assume now that A is c -connected. Since $c_{A \times A}(\Delta_A)$ is an isomorphism and $\eta_{A \times A}$ is a surjective homomorphism, the injective homomorphism i is an isomorphism. It follows that $\pi_0(\Delta_A)$ is surjective: this means that for any $([a]_{\text{Inn}(A)}, [b]_{\text{Inn}(A)}) \in \pi_0(A \times A) = \pi_0(A) \times \pi_0(A)$, there exists $[c]_{\text{Inn}(A)} \in \pi_0(A)$ such that

$$[a]_{\text{Inn}(A)} = [c]_{\text{Inn}(A)} = [b]_{\text{Inn}(A)},$$

showing that A is algebraically connected. ■

From now on we shall call a quandle *connected* when it satisfies the equivalent conditions in Proposition 3.6.10.

A similar result holds for the so-called c -separated objects: these turn out to be exactly the trivial quandles. An object is said to be *c-separated* for a closure operator c when $\Delta_A: A \rightarrow A \times A$ is closed.

Proposition 3.6.11. *Let c be the pullback closure operator for the adjunction (2.3.6.1). A quandle A is c -separated if and only if it is a trivial quandle.*

Proof. By taking into account Lemma 3.6.5 we see that a quandle A is c -separated if and only if the commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta_A} & A \times A \\
 \eta_A \downarrow & & \downarrow \eta_{A \times A} \\
 \pi_0(A) & \xrightarrow{\pi_0(\Delta_A)} & \pi_0(A \times A)
 \end{array}$$

is a pullback. Since this square is a pullback if and only if the kernel pair $\text{Eq}(\eta_A)$ of the unit η_A is the discrete equivalence relation on A , this is also equivalent to the fact that η_A is an injective homomorphism. But η_A is always a surjective

homomorphism, so that A is c -separated if and only if η_A is an isomorphism, as desired. $\blacksquare$

3.6.12. We are now going to show that the connected quandles form a connectedness with respect to the class of trivial quandles. We also give a description of the disconnectedness associated with the class of connected quandles. We will follow [DT15] and define a morphism $f: A \rightarrow B$ to be *constant* if $!_A: A \rightarrow 1$ (where 1 is the terminal object) is a strong epimorphism and a factor of f . In the category $\mathbf{Qnd}$ of quandles, this means that $f: A \rightarrow B$ is constant if and only if it factors through the one-element quandle $\{*\}$.

For a full subcategory $\mathcal{X}$ of $\mathcal{C}$, the class

$$r(\mathcal{X}) := \{C \in \mathcal{C} \mid \text{every } f: X \rightarrow C \text{ is constant for all } X \in \mathcal{X}\}$$

is called a *disconnectedness*, and

$$l(\mathcal{X}) := \{C \in \mathcal{C} \mid \text{every } f: C \rightarrow X \text{ is constant for all } X \in \mathcal{X}\}$$

is called a *connectedness*. There is the Herrlich-Preuss-Arhangel'skiĭ-Wiegandt correspondence [AW75, Her68, Pre71], with $\mathcal{Sub}(\mathcal{C})$ the class of all full subcategories of $\mathcal{C}$ ordered by inclusion:

$$\begin{array}{ccc} & \xrightarrow{r} & \\ \mathcal{Sub}(\mathcal{C}) & \perp & \mathcal{Sub}(\mathcal{C})^{op} \\ & \xleftarrow{l} & \end{array} \quad (3.6.12.1)$$

Example 3.6.13. In the category $\mathbf{Top}$ of topological spaces, we have

$$\begin{aligned} \mathcal{Y} &:= \{\text{connected spaces}\} = l(r(\mathcal{Y})) \\ \mathcal{Z} &:= \{\text{hereditarily disconnected spaces}\} = r(\mathcal{Y}) \end{aligned}$$

We are going to show that there is a similar correspondence in the category $\mathbf{Qnd}$ of quandles. In the following proposition by *trivial subquandle* we shall mean the empty subquandle, and any one-element subquandle of a given quandle.

Theorem 3.6.14. *In the category $\mathbf{Qnd}$, given $\mathcal{X} = \mathbf{Qnd}^*$ we have*

$$\mathcal{Y} := \{\text{connected quandles}\} = l(\mathcal{X}) = l(r(\mathcal{Y}))$$

and

$$\mathcal{Z} := \{A \in \mathbf{Qnd} \mid A \text{ has no non-trivial connected subquandles}\} = r(\mathcal{Y}).$$

Proof. 1. $\mathcal{Y} = l(\mathcal{X})$

If A is connected, then any $f: A \rightarrow X$ with $X \in \mathbf{Qnd}^*$ is constant by commutativity of the following square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \eta_A \downarrow & & \parallel \eta_X \\ \pi_0(A) = \{*\} & \xrightarrow{\pi_0(f)} & X = \pi_0(X). \end{array}$$

To see that $l(\mathcal{X}) \subset \mathcal{Y}$, suppose that every $f: A \rightarrow X$ is constant for all $X \in \mathcal{X}$, and let us prove that $\pi_0(A) = \{*\}$. Note that $\eta_A: A \rightarrow \pi_0(A)$ is constant, but it is a regular epimorphism, thus $\pi_0(A) = \{*\}$.

2. $\mathcal{Z} = r(\mathcal{Y})$

First take $A \in \mathcal{Z}$, C a connected quandle, and $f: C \rightarrow A$. By taking the regular epimorphism-monomorphism factorization $i \circ p$ of f

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ & \searrow p & \nearrow i \\ & f(C) & \end{array}$$

observe that $f(C)$ is connected as a quotient of a connected quandles, but it is also a subquandle of A so it must be trivial, thus $f(C) = \{*\}$.

Now suppose $A \in r(\{\text{connected quandles}\})$, and that

$$A \notin \mathcal{Z} = \{A \in \mathbf{Qnd} \mid A \text{ has no non-trivial connected subquandles}\}.$$

Then A has a non-trivial connected subquandle Γ of cardinality strictly greater than 2 (the only 2-element quandle is trivial), with inclusion $\Gamma \rightarrow A$. But $A \in r(\{\text{connected quandles}\})$ so the inclusion $\Gamma \rightarrow A$ factors through $\{*\}$, thus $\Gamma = \{*\}$, a contradiction.

3. $\mathcal{Y} = l(\mathcal{Z})$

First the unit of the adjunction (3.6.12.1) says exactly that $\mathcal{Y} \subset l(r(\mathcal{Y})) = l(\mathcal{Z})$. But we also have that $\mathcal{X} \subset r(\mathcal{Y})$ so that we get $l(r(\mathcal{Y})) \subset l(\mathcal{X}) = \mathcal{Y}$. ■

Remark 3.6.15. One might wonder whether $\mathcal{Z} = \mathbf{Qnd}^*$. Certainly the class of trivial quandles is contained in $\mathcal{Z}$ but the converse is not true because the larger class of quasi-trivial quandles is also contained in $\mathcal{Z}$. A *quasi-trivial*

quandle [Ino13] is a quandle satisfying $a \triangleleft (a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n) = a$ for all $a, a_i \in A$ with $1 \leq i \leq n$.

To prove that any quasi-trivial quandle A is in $\mathcal{Z}$, let $\gamma: C \rightarrow A$ be an inclusion of a connected subquandle C of A . Then C is also a quasi-trivial quandle since γ is injective and

$$\gamma(a \triangleleft (a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n)) = \gamma(a) \triangleleft (\gamma(a) \triangleleft^{\alpha_1} \gamma(a_1) \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} \gamma(a_n)) = \gamma(a).$$

Since C is connected, for any a, b in C there exist $a_i \in C$ for $1 \leq i \leq n$ such that $a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n = b$. The fact that C is quasi-trivial gives

$$a \triangleleft b = a \triangleleft (a \triangleleft^{\alpha_1} a_1 \triangleleft^{\alpha_2} \dots \triangleleft^{\alpha_n} a_n) = a.$$

The quandle C is then trivial, and then it belongs to $\mathcal{Z}$.

Note that a quasi-trivial quandle is not trivial in general: an example is provided by the following involutive quandle.

$\triangleleft$	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

3.7 Effective closure operator

In this section, we investigate an effective closure operator in the category of quandles. Recall from Section 1.7 that these closure operators are in bijection with regular-epireflective subcategories of $\mathbf{Qnd}$. Although the category $\mathbf{Qnd}$ is not Mal'tsev, we can still describe the closure of any congruence R thanks to Lemma 3.1.2.

Proposition 3.7.1. *Let R be a congruence on a quandle A . Then its effective closure $c_A(R)$ in $\mathbf{Qnd}$ corresponding to the reflection (2.3.6.1) is given by*

$$c_A(R) = \sim_{\text{Inn}(A)} \circ R = R \circ \sim_{\text{Inn}(A)}.$$

Proof. Recall that the closure $c_A(R)$ of a congruence R on a quandle A is constructed as the inverse image $f^{-1}(\sim_{\text{Inn}(A/R)})$ of the congruence $\sim_{\text{Inn}(A/R)}$ along the canonical quotient $f: A \rightarrow A/R$:

$$\begin{array}{ccccc}
& c_A(R) & \longrightarrow & \sim_{\text{Inn}(A/R)} & \\
& \downarrow & & \downarrow p_1 & \downarrow p_2 \\
R & \xrightarrow[f_2]{f_1} & A & \xrightarrow{f} & A/R.
\end{array}$$

The fact that the congruences $\sim_{\text{Inn}(A)}$ and R permute (by Lemma 3.1.2) implies the following equalities then:

$$c_A(R) = R \vee \sim_{\text{Inn}(A)} = R \circ \sim_{\text{Inn}(A)} . \quad \blacksquare$$

Remark 3.7.2. Observe that, for any $A \in \mathbf{Qnd}$, the congruence $\text{Eq}(\eta_A)$ is simply the closure of the equality relation Δ_A on A :

$$c_A(\Delta_A) = \sim_{\text{Inn}(A)} = \text{Eq}(\eta_A).$$

It is not difficult to check that the effective closure operator associated with the adjunction (2.3.6.1) also satisfies the property that

$$f(c_A(\Delta_A)) = c_B(\Delta_B)$$

for any regular epimorphism $f: A \rightarrow B$: this essentially follows from Corollary 3.1.4. One can also show that, for any congruences R and S on the same quandle A ,

$$c_A(R \vee S) = c_A(R) \vee c_A(S).$$

Indeed, since the congruences on an object A form a lattice, we have

$$\begin{aligned}
c_A(R \vee S) &= (R \vee S) \vee \sim_{\text{Inn}(A)} \\
&= (R \vee S) \vee (\sim_{\text{Inn}(A)} \vee \sim_{\text{Inn}(A)}) \\
&= (R \vee \sim_{\text{Inn}(A)}) \vee (S \vee \sim_{\text{Inn}(A)}) \\
&= c_A(R) \vee c_A(S).
\end{aligned}$$

Chapter 4

Coverings corresponding to the adjunction with abelian symmetric quandles

In this chapter, we study the coverings from the adjunction between the category of quandles and its subcategory of abelian symmetric quandles. We prove that this adjunction is admissible for categorical Galois theory and we give a description of the coverings and show that they correspond to normal coverings. This whole chapter is based on a paper written in collaboration with Marino Gran and Andrea Montoli [EGM15].

4.1 Preliminaries

4.1.1. We recall the definition of symmetric quandle and of abelian quandle, and show that the category of symmetric quandles is a Mal'tsev category.

Definition 4.1.2. A quandle A is *symmetric* if it satisfies the additional identity:

$$a \triangleleft b = b \triangleleft a,$$

for all $a, b \in A$.

We write $\mathbf{SymQnd}$ for the corresponding category of symmetric quandles, which is then a subvariety of the variety $\mathbf{Qnd}$ of all quandles. Here below we observe that the category $\mathbf{SymQnd}$ is a Mal'tsev variety [Smi76], which will be shown to be an admissible subcategory of $\mathbf{Qnd}$ for the categorical theory of central extensions.

Proposition 4.1.3. *The category SymQnd is a Mal'tsev variety.*

Proof. Let p be the ternary term defined by

$$p(a, b, c) = (a \triangleleft c) \triangleleft^{-1} b.$$

We then have the identities

$$\begin{aligned} p(a, a, b) &= (a \triangleleft b) \triangleleft^{-1} a = (b \triangleleft a) \triangleleft^{-1} a = b, \\ p(a, b, b) &= (a \triangleleft b) \triangleleft^{-1} b = a. \end{aligned}$$

■

Definition 4.1.4 (Joyce). A quandle is *abelian* when it satisfies the additional identity:

$$(a \triangleleft b) \triangleleft (c \triangleleft d) = (a \triangleleft c) \triangleleft (b \triangleleft d),$$

for all $a, b, c, d \in A$.

Note that this axiom is equivalent to the following one:

$$(a \triangleleft b) \triangleleft^{-1} (c \triangleleft d) = (a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} d). \quad (4.1.4.1)$$

Indeed, if $(a \triangleleft b) \triangleleft^{-1} (c \triangleleft d) = (a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} d)$ then

$$\begin{aligned} ((a \triangleleft b) \triangleleft (c \triangleleft d)) \triangleleft^{-1} (b \triangleleft d) &= ((a \triangleleft b) \triangleleft^{-1} b) \triangleleft ((c \triangleleft d) \triangleleft^{-1} d) \\ &= a \triangleleft c \end{aligned}$$

which proves that

$$(a \triangleleft b) \triangleleft (c \triangleleft d) = (a \triangleleft c) \triangleleft (b \triangleleft d).$$

And if we suppose now that $(a \triangleleft b) \triangleleft (c \triangleleft d) = (a \triangleleft c) \triangleleft (b \triangleleft d)$, then

$$\begin{aligned} ((a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} d)) \triangleleft (c \triangleleft d) &= ((a \triangleleft^{-1} c) \triangleleft c) \triangleleft ((b \triangleleft^{-1} d) \triangleleft d) \\ &= a \triangleleft b \end{aligned}$$

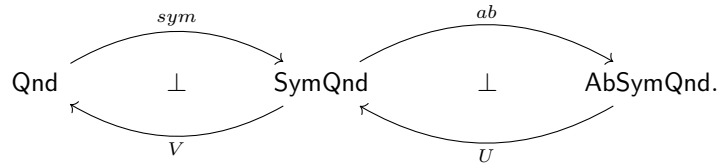
which proves that

$$(a \triangleleft b) \triangleleft^{-1} (c \triangleleft d) = (a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} d).$$

Remark 4.1.5. Not all abelian quandles are symmetric. For instance, any trivial quandle is abelian but not symmetric (as long as it has at least two elements).

Also, not all symmetric quandles are abelian. The smallest symmetric quandle which is not abelian is a quandle of order 81 and is constructed in [Sou71].

Let us write $\mathbf{AbSymQnd}$ for the category of abelian symmetric quandles, $U: \mathbf{AbSymQnd} \rightarrow \mathbf{SymQnd}$ and $V: \mathbf{SymQnd} \rightarrow \mathbf{Qnd}$ for the inclusion functors. Since $\mathbf{AbSymQnd}$ is a subvariety of $\mathbf{SymQnd}$ and $\mathbf{SymQnd}$ is a subvariety of $\mathbf{Qnd}$, both these functors have left adjoints, denoted by $ab: \mathbf{SymQnd} \rightarrow \mathbf{AbSymQnd}$ and $sym: \mathbf{Qnd} \rightarrow \mathbf{SymQnd}$, respectively:



Abelian symmetric quandles are in fact the internal Mal'tsev algebras in $\mathbf{SymQnd}$.

Definition 4.1.6. An *internal Mal'tsev algebra* in a variety $\mathcal{V}$ is an algebra $A \in \mathcal{V}$ with a homomorphism $p_A: A \times A \times A \rightarrow A$ such that $p_A(a, a, b) = b$ and $p_A(a, b, b) = a$.

Let us write $\mathbf{Mal}(\mathcal{V})$ for the category of internal Mal'tsev algebras in $\mathcal{C}$. In a Mal'tsev category, thus in particular in the category $\mathbf{SymQnd}$, any morphism preserves the Mal'tsev operation (see Corollary 4.1 in [Gra01], for instance): this means that the subcategory $\mathbf{Mal}(\mathbf{SymQnd})$ is full in $\mathbf{SymQnd}$. The following observation has been found independently by Bourn [Bou15] and shows that the $\mathbf{AbSymQnd}$ is a naturally Mal'tsev category [Joh89]:

Theorem 4.1.7.

$$\mathbf{AbSymQnd} = \mathbf{Mal}(\mathbf{SymQnd}).$$

Proof. Let $A \in \mathbf{AbSymQnd}$, and let $p_A: A \times A \times A \rightarrow A$ be the Mal'tsev operation on A defined by $p_A(a, b, c) = (a \triangleleft c) \triangleleft^{-1} b$. We have to check that it is a quandle homomorphism. For any $a, b, c, x, y, z \in A$ we have

$$\begin{aligned} p_A((a, b, c) \triangleleft (x, y, z)) &= p_A(a \triangleleft x, b \triangleleft y, c \triangleleft z) \\ &= ((a \triangleleft x) \triangleleft (c \triangleleft z)) \triangleleft^{-1} (b \triangleleft y) \\ &= ((a \triangleleft c) \triangleleft (x \triangleleft z)) \triangleleft^{-1} (b \triangleleft y) \\ &= ((a \triangleleft c) \triangleleft^{-1} b) \triangleleft ((x \triangleleft z) \triangleleft^{-1} y) \\ &= p_A(a, b, c) \triangleleft p_A(x, y, z). \end{aligned}$$

This shows that A belongs to $\mathbf{Mal}(\mathbf{SymQnd})$.

Conversely, when $A \in \mathbf{Mal}(\mathbf{SymQnd})$, the unique internal Mal'tsev operation on A is necessarily given by (any of) the Mal'tsev operations of the theory of the variety $\mathbf{SymQnd}$. Accordingly, it is defined by $p_A(a, b, c) = (a \triangleleft c) \triangleleft^{-1} b$, and it is such that $p_A(a, b, a) = a \triangleleft^{-1} b$. Moreover, $p_A: A \times A \times A \rightarrow A$ preserves the binary operation $\triangleleft$, so that the equality

$$p_A((a, b, a) \triangleleft (x, y, x)) = p_A(a, b, a) \triangleleft p_A(x, y, x)$$

gives

$$(a \triangleleft x) \triangleleft^{-1} (b \triangleleft y) = (a \triangleleft^{-1} b) \triangleleft (x \triangleleft^{-1} y).$$

This is precisely the identity (4.1.4.1), and the quandle A belongs to $\mathbf{AbSymQnd}$. ■

4.1.8. We recall the definition of two classes of morphisms in $\mathbf{Qnd}$, first investigated by Bourn [Bou15].

Definition 4.1.9. Let Σ be the class of split epimorphisms $f: A \rightarrow B$ with a given section $s: B \rightarrow A$ (i.e. $f \circ s = 1_B$) in the category $\mathbf{Qnd}$ such that the map $s(b) \triangleleft -: f^{-1}(b) \rightarrow f^{-1}(b)$ is surjective, for any $b \in B$.

In other words, the split epimorphism f with section s is in Σ if, for any $b \in B$ and $a \in f^{-1}(b)$, there is a $k_a \in f^{-1}(b)$ such that $s(b) \triangleleft k_a = a$.

Remark 4.1.10. This element k_a also depends on b , so that one should write $k_{b,a}$, instead. We shall simply write k_a , however, to simplify the notations.

An equivalence relation (R, r_1, r_2) is said to be a Σ -equivalence relation if the split epimorphism $r_1: R \rightarrow A$ with section $\delta_R: A \rightarrow R$ belongs to the class Σ .

Definition 4.1.11 ([BMFMS13, BMFMS15, Bou15]). A morphism $f: A \rightarrow B$ in $\mathbf{Qnd}$ is Σ -special if $(\text{Eq}(f), f_1, f_2)$ is a Σ -equivalence relation.

The following result is a direct consequence of Theorem 3.9 in [Bou15], and will be useful later on:

Theorem 4.1.12. *Let $f: A \rightarrow B$ be a Σ -special homomorphism in $\mathbf{Qnd}$. Then any congruence R on A permutes with $\text{Eq}(f)$ in the sense of the composition of relations:*

$$R \circ \text{Eq}(f) = \text{Eq}(f) \circ R.$$

Corollary 4.1.13. *Given a pushout of surjective homomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{l} & D \end{array}$$

where f is Σ -special, the induced homomorphism $A \xrightarrow{(g,f)} C \times_D B$ to the pullback is surjective.

Proof. The proof is essentially the same as the one given in Lemma 3.1.3. ■

4.2 Coverings

4.2.1. If $\mathcal{C}$ is a finitely complete category, a double equivalence relation C in $\mathcal{C}$ is an equivalence relation internal in the category of equivalence relations in $\mathcal{C}$. It can be represented by a diagram

$$\begin{array}{ccc} C & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & S \\ \pi_1 \downarrow \quad \pi_2 \downarrow & & s_1 \downarrow \quad s_2 \downarrow \\ R & \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} & A \end{array} \quad (4.2.1.1)$$

where $r_1 \circ \pi_1 = s_1 \circ p_1$, $r_1 \circ \pi_2 = s_2 \circ p_1$, $r_2 \circ \pi_1 = s_1 \circ p_2$ and $r_2 \circ \pi_2 = s_2 \circ p_2$. In this case one usually says that C is a double equivalence relation on the equivalence relations R and S .

Definition 4.2.2. Given equivalence relations R and S on A , a double equivalence relation C on R and S (as in (4.2.1.1)) is called a *centralizing relation* when the square

$$\begin{array}{ccc} C & \xrightarrow{p_2} & S \\ \pi_1 \downarrow & & \downarrow s_1 \\ R & \xrightarrow{r_2} & A \end{array}$$

is a pullback.

Definition 4.2.3. A *connector* between R and S is an arrow $p: R \times_A S \rightarrow A$ such that

- | | |
|---------------------------------------|--|
| 1. $p(x, x, y) = y$ | 1'. $p(x, y, y) = x$ |
| 2. $xSp(x, y, z)$ | 2'. $zRp(x, y, z)$ |
| 3. $p(x, y, p(y, u, v)) = p(x, u, v)$ | 3'. $p(p(x, y, u), u, v) = p(x, u, v)$ |

In the Mal'tsev context [BG02] the existence of a connector between R and S is already guaranteed by the existence of a *partial Mal'tsev operation* $p: R \times_A S \rightarrow A$, i.e. when the identities $p(x, x, y) = y$ and $p(x, y, y) = x$ in Definition 4.2.3 are satisfied. Accordingly, in a Mal'tsev category the existence of a double centralizing relation on R and S is equivalent to the existence of a partial Mal'tsev operation. Moreover, a connector is unique, when it exists: accordingly, for two given equivalence relations, having a connector becomes a property.

In a Mal'tsev variety a congruence R on an algebra A is called *algebraically central* if there is a centralizing double relation on R and $A \times A$, this latter being the largest equivalence relation on A . In terms of commutators, this fact is expressed by the condition $[R, A \times A] = \Delta_A$.

Also, in the variety $\mathbf{Qnd}$ of quandles we shall say that a surjective homomorphism $f: A \rightarrow B$ in $\mathbf{Qnd}$ is a *central extension* if its kernel congruence $\text{Eq}(f)$ is algebraically central: there is a connector between $\text{Eq}(f)$ and $A \times A$.

Given a homomorphism $f: A \rightarrow B$ in $\mathbf{Qnd}$, each fiber $f^{-1}(b)$ (for $b \in B$) is a subquandle of A since the idempotency axiom implies that any one-element quandle is a subquandle of B . We shall say that f has *abelian symmetric fibers* if any $f^{-1}(b) \in \mathbf{AbSymQnd}$.

Lemma 4.2.4. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow f \\ E & \xrightarrow{p} & B. \end{array}$$

If $f: A \rightarrow B$ has abelian symmetric fibers then so does $p_1: E \times_B A \rightarrow E$. Moreover, if $p: E \rightarrow B$ is a surjective homomorphism, then $f: A \rightarrow B$ has abelian symmetric fibers if $p_1: E \times_B A \rightarrow E$ has abelian symmetric fibers.

Proof. The first assertion follows from the fact that if $(e, a) \in E \times_B A$ then the fibers $p_1^{-1}(e)$ and $f^{-1}(f(a))$ are isomorphic. The proof of the second assertion is similar, the surjectivity of p guaranteeing that, for any $a \in A$, there exists $e \in E$ such that $(e, a) \in E \times_B A$. ■

Lemma 4.2.5. *Let $f: A \rightarrow B$ be a split epimorphism, with section $s: B \rightarrow A$, in Σ . Consider the following pullback of f along a split epimorphism $p: E \rightarrow B$*

$$\begin{array}{ccc} E \times_B A & \xleftarrow{(to f, 1_A)} & A \\ p_1 \downarrow \uparrow (1_E, s \circ p) & & f \downarrow \uparrow s \\ E & \xleftarrow[p]{t} & B. \end{array}$$

Then $(1_E, s \circ p)$ and $(t \circ f, 1_A)$ are jointly epimorphic.

Proof. Let $(e, a) \in E \times_B A$; we shall show that (e, a) can be rewritten in terms of two elements in the images of $(1_E, s \circ p)$ and $(t \circ f, 1_A)$, respectively. Since the split epimorphism f is in Σ , there exists an element $k_a \in f^{-1}(f(a))$ such that $sf(a) \triangleleft k_a = a$. Also, we always have $e = (e \triangleleft^{-1} tp(e)) \triangleleft tp(e)$. Accordingly, by using the fact that $f(a) = f(k_a)$ and $p(e) = f(a)$, we see that

$$\begin{aligned} (e, a) &= ((e \triangleleft^{-1} tp(e)) \triangleleft tp(e), sf(a) \triangleleft k_a) \\ &= (e \triangleleft^{-1} tp(e), sf(a)) \triangleleft (tp(e), k_a) \\ &= (e \triangleleft^{-1} tp(e), sp(e)) \triangleleft (tf(k_a), k_a) \\ &= (e \triangleleft^{-1} tp(e), sp(e \triangleleft^{-1} tp(e))) \triangleleft (tf(k_a), k_a) \\ &= (1_E, s \circ p)(e \triangleleft^{-1} tp(e)) \triangleleft (t \circ f, 1_A)(k_a). \end{aligned}$$

■

Corollary 4.2.6. *Let R be an equivalence relation and S be a Σ -equivalence relation on the same quandle A in $\mathbf{Qnd}$. If there is a connector on R and S , then it is unique.*

Proof. This follows directly from Lemma 4.2.5. ■

Lemma 4.2.7. *Let R be an equivalence relation and S be a Σ -equivalence relation on the same quandle A . For a homomorphism $p: R \times_A S \rightarrow A$, the following conditions are equivalent:*

1. p is a partial Mal'tsev operation: $p(x, y, y) = x$ and $p(x, x, y) = y$;
2. p is a connector between R and S .

Proof. We only have to prove that 1. implies 2. Remark that in any variety, in particular in $\mathbf{Qnd}$, the equivalence relation R is the kernel pair of the canonical quotient $r: A \rightarrow A/R$, and S the kernel pair of $s: A \rightarrow A/S$.

$$\begin{array}{ccccc} R \times_A S & \xrightleftharpoons[p_2]{p_1} & S & & \\ \uparrow \uparrow \uparrow & \searrow p & \uparrow \uparrow \uparrow & & \\ \pi_1 \quad i_R \quad \pi_2 & & s_1 \quad \delta_S \quad s_2 & & \\ \downarrow \downarrow \downarrow & \xrightarrow{r_1} & \downarrow \downarrow \downarrow & & \\ R & \xrightleftharpoons[r_2]{\delta_R} & A. & & \end{array}$$

By assumption we have that $p \circ i_S = s_2$ and $p \circ i_R = r_1$. To see that $(x, p(x, y, z)) \in S$, we have to prove that $s \circ p = s \circ r_1 \circ \pi_1$. The equalities

$$\begin{aligned} s \circ p \circ i_S &= s \circ s_2 = s \circ s_1 \\ &= s \circ s_1 \circ p_1 \circ i_S & (p_1 \circ i_S = 1_S) \\ &= s \circ r_1 \circ \pi_1 \circ i_S & (s_1 \circ p_1 = r_1 \circ \pi_1) \end{aligned}$$

and

$$s \circ p \circ i_R = s \circ r_1 = s \circ r_1 \circ \pi_1 \circ i_R,$$

imply that $s \circ p = s \circ r_1 \circ \pi_1$ by Lemma 4.2.5. A similar argument shows that $(z, p(x, y, z)) \in R$.

Now, to see that

$$p(x, y, p(y, u, v)) = p(x, u, v),$$

let us consider $(a, b, c, d) \in R \times_A R \times_A S$ and write

$$\phi(a, b, c, d) = p(a, b, p(b, c, d))$$

and

$$\psi(a, b, c, d) = p(a, c, d).$$

Observe that

$$\phi(a, b, c, c) = p(a, b, p(b, c, c)) = p(a, b, b) = a = p(a, c, c) = \psi(a, b, c, c)$$

for all $(a, b, c, c) \in R \times_A R \times_A S$, and

$$\phi(e, e, e, f) = p(e, e, p(e, e, f)) = p(e, e, f) = \psi(e, e, e, f)$$

for all $(e, e, e, f) \in R \times_A R \times_A S$. Now, let $(x, y, u, v) \in R \times_A R \times_A S$: since the split epimorphism $s_1: S \rightarrow A$ with section $\delta_S: A \rightarrow S$ is in Σ , there exists $k_{(u,v)} = (u, k_v) \in s_1^{-1}(u)$ such that $(u, v) = (u, u) \triangleleft (u, k_v)$. Then one can write

$$(x, y, u, v) = (x \triangleleft^{-1} u, y \triangleleft^{-1} u, u, u) \triangleleft (u, u, u, k_v)$$

for all $(x, y, u, v) \in R \times_A R \times_A S$. It follows that $\phi(x, y, u, v) = \psi(x, y, u, v)$. A similar argument shows that $p(p(x, y, u), u, v) = p(x, u, v)$. ■

Lemma 4.2.8. *Let $f: A \rightarrow B$ be a central extension with abelian symmetric fibers. Then $\text{Eq}(f)$ is isomorphic to a product $Q \times A$, where Q is an abelian symmetric quandle.*

Proof. Let C be the centralizing relation on $\text{Eq}(f)$ and $A \times A$; consider the following diagram

$$\begin{array}{ccccc}
 C & \rightrightarrows & A \times A & & \\
 c_1 \downarrow & & \downarrow c_2 & & \\
 \text{Eq}(f) & \rightrightarrows & A & \xrightarrow{f} & B \\
 q \downarrow & & \downarrow & & \\
 Q & \rightrightarrows & \{*\} & &
 \end{array}$$

where q is the coequalizer of c_1 and c_2 . By the Barr-Kock theorem [BGO71, BG04], the lower squares are pullbacks. By Lemma 4.2.4, both the homomorphisms $Q \rightarrow \{*\}$ have abelian symmetric fibers, hence Q is an abelian symmetric quandle. ■

As a consequence, any central extension $f: A \rightarrow B$ has its kernel pair $\text{Eq}(f)$ isomorphic to a product of an abelian algebra and A .

Proposition 4.2.9. *If $f: A \rightarrow B$ has symmetric fibers, then it is Σ -special.*

Proof. Consider the kernel pair of f

$$\begin{array}{ccc}
 \text{Eq}(f) & \xrightleftharpoons[\delta_f]{f_2} & A \\
 f_1 \downarrow & \uparrow \delta_f & \downarrow f \\
 A & \xrightarrow{f} & B.
 \end{array}$$

One has to check that (f_1, δ_f) is in Σ . If $a \in A$ and $(a, a') \in f_1^{-1}(a)$, then in particular $f(a) = f(a')$, so that $a' \triangleleft^{-1} a$ is such that $f(a) = f(a' \triangleleft^{-1} a)$. It follows that $(a, a' \triangleleft^{-1} a) \in \text{Eq}(f)$, and then

$$(a, a) \triangleleft (a, a' \triangleleft^{-1} a) = (a \triangleleft a, a \triangleleft (a' \triangleleft^{-1} a)) = (a, (a' \triangleleft^{-1} a) \triangleleft a) = (a, a'). \quad \blacksquare$$

Remark 4.2.10. Observe that when a split epimorphism $f: A \rightarrow B$ with section $s: B \rightarrow A$ has symmetric fibers, then $s(b) \triangleleft -: f^{-1}(b) \rightarrow f^{-1}(b)$ is always injective: if $x \in f^{-1}(b)$ and $y \in f^{-1}(b)$ are such that $s(b) \triangleleft x = s(b) \triangleleft y$, since $s(b) \in f^{-1}(b)$, we get $x \triangleleft s(b) = y \triangleleft s(b)$, and hence $x = y$ by right invertibility.

In the following we shall characterize coverings and normal coverings in $\mathbf{Qnd}$ arising from the adjunction between the category of quandles and the category of abelian symmetric quandles:

$$\begin{array}{ccc}
& I & \\
\text{Qnd} & \xrightarrow{\quad} & \text{AbSymQnd} \\
& \xleftarrow{\quad} & \\
& H &
\end{array}
\quad \perp$$

The following theorem shows that the functor I preserves a certain type of pullbacks. This is equivalent to the admissibility condition of the subvariety AbSymQnd of Qnd .

Theorem 4.2.11. *In the previous adjunction, the functor $I: \text{Qnd} \rightarrow \text{AbSymQnd}$ preserves all pullbacks in Qnd of the form*

$$\begin{array}{ccc}
P & \xrightarrow{p_2} & H(X) \\
p_1 \downarrow & & \downarrow \phi \\
A & \xrightarrow{f} & H(Y)
\end{array} \tag{4.2.11.1}$$

where $\phi: H(X) \rightarrow H(Y)$ is a surjective homomorphism lying in the subcategory AbSymQnd and $f: A \rightarrow H(Y)$ is a surjective homomorphism.

Proof. Consider the following commutative diagram where:

- the square on the back is the given pullback, where $\phi: H(X) \rightarrow H(Y)$ is a surjective homomorphism in the subcategory AbSymQnd ;
- the universal property of the unit $\eta_P: P \rightarrow HI(P)$ induces a unique arrow $HI(p_2): HI(P) \rightarrow H(X)$ with $HI(p_2) \circ \eta_P = p_2$;
- the universal property of the unit $\eta_A: A \rightarrow HI(A)$ induces a unique arrow $HI(f): HI(A) \rightarrow H(Y)$ with $HI(f) \circ \eta_A = f$;
- (P', π_1, π_2) is the pullback of $HI(p_1)$ along η_A .

$$\begin{array}{ccccc}
P & \xrightarrow{p_2} & & & H(X) \\
& \searrow \eta_P & & \nearrow HI(p_2) & \downarrow \phi \\
& & P' & \xrightarrow{\pi_2} & HI(P) \\
& \swarrow \pi_1 & & \downarrow HI(p_1) & \\
A & \xrightarrow{f} & & & H(Y) \\
& \searrow \eta_A & & \nearrow HI(f) & \\
& & & HI(A) &
\end{array}$$

The quandle homomorphism p_1 is Σ -special by Lemma 4.2.4 since ϕ has abelian symmetric fibers, thus the homomorphism γ is surjective by Corollary 4.1.13.

The fact that $\pi_1 \circ \gamma = p_1$ and $HI(p_2) \circ \pi_2 \circ \gamma = p_2$ implies that γ is also injective. Indeed, this latter property follows from the fact that the pullback projections p_1 and p_2 are jointly monomorphic. Accordingly, the arrow γ is bijective, thus an isomorphism. Since η_A is a surjective homomorphism it follows that the right face of the diagram is a pullback by Proposition 1.1.14, and the pullback (4.2.11.1) is preserved by the functor I , as desired. ■

Corollary 4.2.12. *The functor I preserves products of the type $A \times Q$ where Q is an abelian symmetric quandle and A is any quandle.*

Proof. Remark that $A \times Q$ is the following pullback

$$\begin{array}{ccc} A \times Q & \xrightarrow{p_2} & Q \\ p_1 \downarrow & & \downarrow \\ A & \longrightarrow & \{*\} \end{array}$$

where $\{*\}$ is the terminal object in $\mathbf{Qnd}$, i.e. the trivial quandle with one element. ■

Lemma 4.2.13. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ E & \xrightarrow[p]{} & B. \end{array} \quad (4.2.13.1)$$

If f is a central extension with abelian symmetric fibers, then π_1 is a central extension with abelian symmetric fibers.

Moreover, if $p: E \rightarrow B$ is a surjective homomorphism, then f is a central extension with abelian symmetric fibers if π_1 is a central extension with abelian symmetric fibers.

Proof. First remark that we already know that the property of having abelian symmetric fibers is preserved and reflected by pullbacks along surjective homomorphisms by Lemma 4.2.4.

Let $f: A \rightarrow B$ be a central extension with abelian symmetric fibers. Write $p_f: A \times \text{Eq}(f) \rightarrow A$ for the connector between $A \times A$ and $\text{Eq}(f)$. Define the quandle homomorphism $p_{\pi_1}: (E \times_B A) \times \text{Eq}(\pi_1) \rightarrow E \times_B A$ as $p_{\pi_1}((e, a), (e', b), (e', c)) = (e, p_f(a, b, c))$. We have

$$p_{\pi_1}((e, a), (e', b), (e', b)) = (e, p_f(a, b, b)) = (e, a)$$

and

$$p_{\pi_1}((e, a), (e, a), (e, b)) = (e, p_f(a, a, b)) = (e, b).$$

It is then a connector by Lemma 4.2.7.

Now let $\pi_1: E \times_B A \rightarrow E$ be a central extension with abelian symmetric fibers. Write $p_{\pi_1}: (E \times_B A) \times \text{Eq}(\pi_1) \rightarrow E \times_B A$ for the connector between $(E \times_B A) \times (E \times_B A)$ and $\text{Eq}(\pi_1)$. The surjectivity of $p: E \rightarrow B$ implies the surjectivity of the homomorphism $\widehat{\pi}_2: (E \times_B A) \times \text{Eq}(\pi_1) \rightarrow A \times \text{Eq}(f)$ defined by

$$\widehat{\pi}_2((e, a), (e', b), (e', c)) = (a, b, c).$$

First let us show that $\text{Eq}(\widehat{\pi}_2) \subset \text{Eq}(\pi_2 \circ p_{\pi_1})$. Let

$$(((e_0, a), (e'_0, b), (e'_0, c)), ((e_1, a), (e'_1, b), (e'_1, c))) \in \text{Eq}(\widehat{\pi}_2).$$

Since f has abelian symmetric fibers by Lemma 4.2.4, it is Σ -special by Proposition 4.2.9. This means that the split epimorphism $\text{Eq}(f) \xrightleftharpoons[f_1]{\delta_f} A$ is in Σ .

In other terms, for all $b \in A$ and all $(b, c) \in f_1^{-1}(b)$ there exists $k_{(b, c)} \in f_1^{-1}(b)$, where $k_{(b, c)} = (b, k_c)$, such that $(b, b) \triangleleft k_{(b, c)} = (b, c)$. Such a $k_{(b, c)} = (b, k_c)$ is unique by Remark 4.2.10: it follows that, for any $(b, c) \in \text{Eq}(f)$, the element $k_c \in A$ such that $f(k_c) = f(b) = f(c)$ and $b \triangleleft k_c = c$ is unique. Then, for $i \in \{0, 1\}$, we have

$$\begin{aligned} & ((e_i, a), (e'_i, b), (e'_i, c)) \\ &= ((e_i, a) \triangleleft^{-1} (e'_i, b), (e'_i, b), (e'_i, b)) \triangleleft ((e'_i, b), (e'_i, b), (e'_i, k_c)). \end{aligned}$$

Consequently we remark that

$$\begin{aligned} & \pi_2 \circ p_{\pi_1}((e_i, a), (e'_i, b), (e'_i, c)) \\ &= \pi_2 \circ p_{\pi_1}(((e_i, a) \triangleleft^{-1} (e'_i, b), (e'_i, b), (e'_i, b)) \triangleleft ((e'_i, b), (e'_i, b), (e'_i, k_c))) \\ &= \pi_2(p_{\pi_1}((e_i, a) \triangleleft^{-1} (e'_i, b), (e'_i, b), (e'_i, b)) \triangleleft p_{\pi_1}((e'_i, b), (e'_i, b), (e'_i, k_c))) \\ &= \pi_2(((e_i, a) \triangleleft^{-1} (e'_i, b)) \triangleleft (e'_i, k_c)) \\ &= \pi_2((e_i \triangleleft^{-1} e'_i) \triangleleft e'_i, (a \triangleleft^{-1} b) \triangleleft k_c) = (a \triangleleft^{-1} b) \triangleleft k_c \end{aligned}$$

for both $i \in \{0, 1\}$. This implies the existence of a quandle homomorphism $p_f: A \times \text{Eq}(f) \rightarrow A$ such that $p_f \circ \widehat{\pi}_2 = \pi_2 \circ p_{\pi_1}$, i.e. $p_f(a, b, c) = (a \triangleleft^{-1} b) \triangleleft k_c$ where k_c is the unique element such that $b \triangleleft k_c = c$ as above. Moreover, we have

$$p_f(a, b, b) = (a \triangleleft^{-1} b) \triangleleft b = a$$

for $(a, b, b) \in A \times \text{Eq}(f)$ and

$$p_f(a, a, b) = (a \triangleleft^{-1} a) \triangleleft k_b = a \triangleleft k_b = b$$

for $(a, a, b) \in A \times \text{Eq}(f)$, so p_f is a connector by Lemma 4.2.7. $\blacksquare$

Theorem 4.2.14. *Given a surjective homomorphism $f: A \rightarrow B$ in Qnd , the following conditions are equivalent:*

1. f is a central extension with abelian symmetric fibers;
2. f is a normal covering;
3. f is a covering.

Proof. Let $f: A \rightarrow B$ be a central extension with abelian symmetric fibers. Then its kernel pair $\text{Eq}(f)$ is isomorphic to a product $Q \times A$ with Q an abelian symmetric quandle by Lemma 4.2.8. Corollary 4.2.12 shows that f is then a normal covering.

Every normal covering is a covering.

Let $f: A \rightarrow B$ be a covering. Then there is a surjective homomorphism $p: E \rightarrow B$ such that the first projection $\pi_1: E \times_A B \rightarrow E$ in the pullback (4.2.13.1) is a trivial extension. Then $f: A \rightarrow B$ is a central extension with abelian symmetric fibers by Lemma 4.2.13, because π_1 is the pullback of a morphism lying in AbSymQnd . $\blacksquare$

Remark 4.2.15. Note that there are surjective homomorphisms with abelian symmetric fibers that are not central. Take for instance the quandle A given by the following table:

$\triangleleft$	a	b	c	d
a	a	c	b	a
b	c	b	a	b
c	b	a	c	c
d	d	d	d	d

and the quandle homomorphism $f: A \rightarrow \{x, y\}$ defined by

$$f(a) = f(b) = f(c) = x \text{ and } f(d) = y.$$

Its kernel pair $\text{Eq}(f)$ has 10 elements, and thus can't be isomorphic to a product $A \times Q$ with Q an abelian symmetric quandles since A has 4 elements.

Remark 4.2.16. There are surjective central extensions that do not have symmetric fibers. Consider the additive group $(\mathbb{Z}/2\mathbb{Z}, +, \bar{0})$ and endow its

underlying set with the trivial quandle structure $a \triangleleft b = a$ for all $a, b \in \mathbb{Z}/2\mathbb{Z}$. Remark that the group operation is a quandle homomorphism:

$$(a \triangleleft b) + (a' \triangleleft b') = a + a' = (a + a') \triangleleft (b + b').$$

It follows that the Mal'tsev operation $p: \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by $p(a, b, c) = a - b + c$ is a connector between the congruences $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The homomorphism $\mathbb{Z}/2 \rightarrow 1$ is then a central extension, whose (unique) fiber is not symmetric, since $\bar{0} \triangleleft \bar{1} = \bar{0} \neq \bar{1} = \bar{1} \triangleleft \bar{0}$.

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