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Locally stationary volatility modelling

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Abstract

The increasing works on parameter instability, structural changes and regime switches lead to the natural research question whether the assumption of stationarity is appropriate to model volatility processes. Early econometric studies have provided testing procedures of covariance stationarity and have shown empirical evidence for the unconditional time-variation of the dependence structure of many financial time series.

After a review of several econometric tests of covariance stationarity, this survey paper focuses on several attempts in the literature to model the time-varying second- order dependence of volatility time series. The approaches that are summarized in this discussion paper propose various specification for this time-varying dynamics. In some of them an explicit variation over time is suggested, such as in the spline GARCH model. Larger classes of nonstationary models have also been proposed, in which the variation of the parameters may be more general such as in the so-called locally stationary models. In another approach that is called "adaptive", no explicit global model is assumed and local parametric model are adaptively fitted at each point over time. Multivariate extensions are also visited. A comparison of these approaches is proposed in this paper and some illustrations are provided on the two last decades of data of the Dow Jones Industrial Average index.

Keywords: volatility, locally stationary time series, multiplicative model, adaptive estimation.

JEL Classification: Primary C14, secondary C22, C58

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1 Introduction

The volatility is a measure of variability of an economic time process. One challenging econometric problem is to understand and to model the heterogeneity of the volatility through time. With that respect, conditional models such as ARCH or ARCH-type models have received a high consideration during the last decades.

Conditional models have also been recently questioned because the marginal distribution of the volatility is assumed to be covariance stationary. Several papers, some of them cited below, have suggested that one part of the time heterogeneity of volatility processes might be due to a lack of stationarity, that appears when the unconditional moments or distribution change over time.

The goal of this discussion paper is to present an overview of some approaches that have been proposed to model volatility processes by locally stationary time series. We focus on the modeling of zero mean discrete-time stochastic process X_t $(t = 0, 1, 2, \ldots)$, that sometimes results after removing the trend of an economic time series. One recurring empirical application in this survey is given by the analysis of the volatility of the Dow Jones Industrial Average (DJIA) index presented in Figure 1. The figure shows the observed index as well as its log returns which we assume to be zero mean, as we do for X_t.

In time series analysis most existing models assume that the zero-mean process X_t is covariance stationary, meaning that the covariance between X_s and X_t depends on the lag $|s-t|$ only. This assumption is useful to have some estimators for the autocovariance structure of the process with good statistical properties. It is also a fundamental assumption for forecasting purposes. However, many time series in the applied science are not covariance stationarity and show a time-varying second-order structure. That is, variance and covariance can change over time. The returns of the DJIA index, shown in the second row of Figure 1, are likely to have such an inhomogeneous variance. By this, we mean that the variance of the process is possibly not constant over time. Later in this article, we apply a test of covariance stationarity to these data, confirming this intuition. Many other examples can be found in economics, as well as in many other fields of the applied sciences.

From the sixties gradually more and more attention has been paid to this challenging problem on how to model such processes with an evolutionary autocovariance structure. Among the pioneers, we like to cite the work of Page [39], Silverman [46], Priestley [42] and Loynes [32]. In his paper, Silverman proposed in 1957 the approximation

$$\mathrm{Cov}(X_s, X_t) \approx m\left(\frac{s+t}{2}\right) c^X(s-t)$$

for some function $m(\cdot)$ i.e. the covariance behaves locally as a typical stationary autocovariance function c^X but is allowed to vary from time to time depending on the midpoint between s and t. As in Silverman's definition, each model of nonstationary covariance has to define explicitly its departure from stationarity. However, constructing a statistical theory for these locally stationary processes is hampered by serious methodological problems. For instance, with this lack of invariant second-order structure, how can one estimate the time-varying covariance accurately? Even if some regularity assumptions are imposed on the function m, a serious problem here is that one cannot build an asymptotic theory for

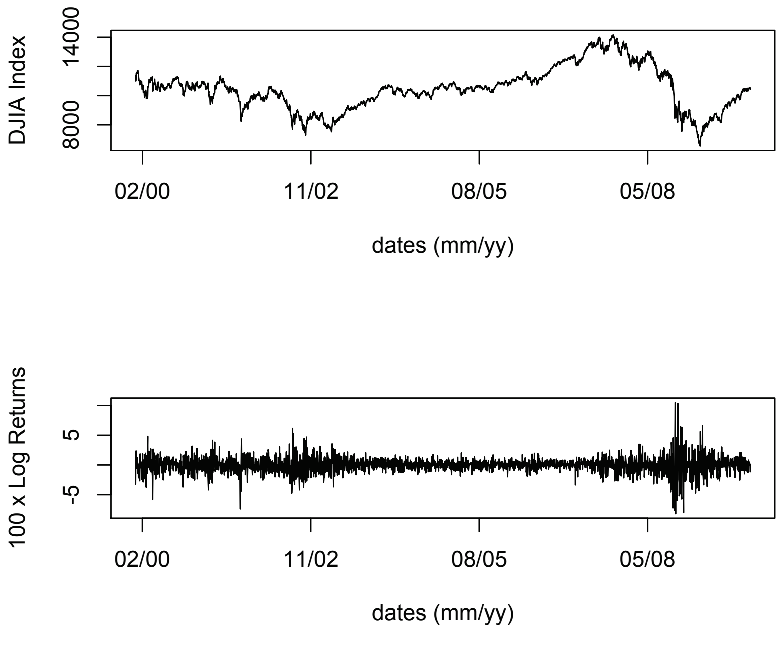
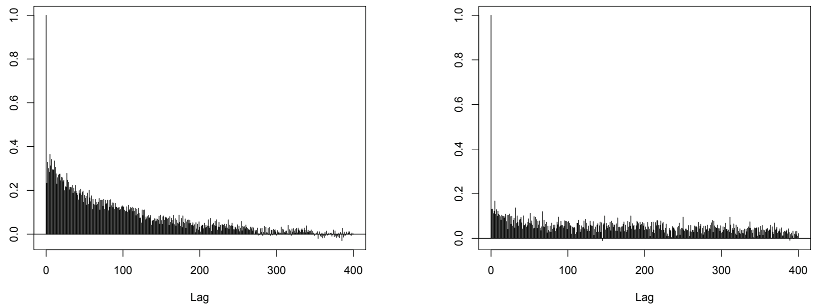


Figure 1: Top: Dow Jones Industrial Average (DJIA) index, from 3rd Jan 2000 to 31st Dec 2009 (2,528 daily observations). Bottom: returns of the DJIA index.

the estimation of m. Consequently the standard statistical properties such as consistency, efficiency or central limit theorems cannot be used to measure and compare the quality of different estimators.

To answer these questions, a decisive idea was introduced by Dahlhaus [7, 8] with his concept of "local stationarity". This concept allows the modelling of a time-varying auto-covariance structure which can be estimated rigorously. An appropriate asymptotic theory can be developed for those processes and the usual statistical properties of estimators can be derived.

This paper is organized as follows. After presenting some economic justifications and empirical evidences why volatility time series can sometimes be considered as non stationary (Section 2), we present the definition of local stationarity in Section 3 below. The formal use of locally stationary processes to model volatility time series is a recent advance of the last decade, and is considered in Section 4. Multivariate extensions of locally stationary models of volatility is an active topic of research, and we survey in Section 5 two recent approaches.



(a) From 2nd Jan 1990 to 31st Dec 1999 (b) From 3rd Jan 2000 to 31st Dec 2009

Figure 2: The empirical autocorrelation function of the Dow Jones Industrial Average (DJIA) absolute index returns, computed over two segments of time

2 Empirical evidences

2.1 Structural breaks, non stationarity and persistence

In the early eighties the modeling of trends and business cycles in economic time series showed new developments. The common additive decomposition of a time series as a (linear) deterministic trend, a cyclical component and a stationary stochastic process was questioned in both empirical and theoretical research [23, 36, 40, 43]. Among the developments of this research,[1] an important branch of the macroeconomic literature emerged on testing the presence of structural breaks in time series. In this approach, the considered time series is supposed to be stationary over segments of time, and the limit of those segments define the breakpoints. Before and after a breakpoint, the process is not supposed to have the same parameter levels, or even not the same structure.

This debate on structural break naturally appeared in the literature on volatility modeling. Modelling structural changes has then been studied e.g. by the Markov regime-switching models (e.g. [24] for ARCH; [21] for GARCH). Also, the literature on break identification in volatility processes has been very active during the last decade (e.g. [26, 18, 2]).

Models with structural changes in the variance or autocovariance provide a natural framework to analyze volatility if one considers the historical variations that have been observed in the volatility of many macroeconomic time series. A decline in volatility of the U.S. output growth appeared in the early 1980s, as documented in [28, 33], and breaks in the unconditional volatility has been detected in many macroeconomic time series over the last sixty years (e.g. [44]). Volatility time series with one or more structural breaks is a particular case of locally stationary process (a precise definition is to be found below). It is not a weakly stationary time series since its variance, or its autocorrelation function changes between the breaks. Nonstationary models of volatility have then a strong empirical justification, and we recall below basic tests of stationarity that have been implemented to support this observation.

[1] A more complete historical view can be found e.g. in [4]

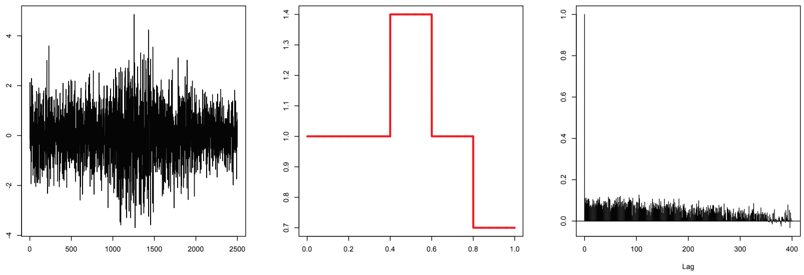


Figure 3: The first graph is a simulated time series of 2,500 observations that has a time-varying standard deviation as plotted in the second graph. The last graph is the empirical autocorrelation function of the absolute series.

Another motivation for modeling volatility by means of nonstationary processes is related to the high persistence that is commonly observed in the squared or absolute returns. This persistence refers to the typical pattern for the autocorrelation function (ACF) of the squared/absolute returns, that are positive and slowly decreasing. In a discussion on the IGARCH model of Engle and Bollerslev [15] the point was raised that it may not be possible to identify persistence effect from structural changes [14]. An illustration on the Dow Jones index is given in Figure 2, where the ACF of the absolute returns is given for two decades of the index. The first segment is over the years 1990 to 1999, and second segment is over the years 2000 to 2009. Both ACFs show persistence, although it is remarkable to see how the ACF has changed over the two last decades. The last decade shows more persistence and quicker convergence to a positive value.

One surprising feature of models with structural changes in their variance or ACF is their ability to capture the persistence in the absolute returns. This phenomenon has been pointed out in [35] and is illustrated by means of a simulation in Figure 3. In this figure we simulate the time series $X_t = \sigma_t Z_t$ with Z_t iid $N(0,1)$, and σ_t is a piecewise constant function of t given in the second graph of Figure 3. The resulting process Y_t is not stationary since it contains three breaks in its unconditional variance. Between each break, the time series is a Gaussian white noise. The ACF of the resulting concatenation of white noises is empirically estimated in Figure 3 (last graph). The pattern of the ACF is similar to the pattern of long range dependent data.

This observation is probably one of the most important motivations why it would be valuable to analyze volatility data using nonstationary models. In the above exercise, the nonstationary model has a very simple form with only three breakpoints in the unconditional variance. Before generalizing this approach to more involved models and to present a rigorous framework to construct nonstationary models, we briefly recall some useful tests to detect nonstationarity in volatility time series.

2.2 Testing stationarity

In this section, we review some tests of nonstationarity. A wide class of tests are based on the cumulative sum of squares. Certainly the simplest version of this test is the *post-sample* *prediction test*. Suppose we observe the zero-mean process $X_0, \ldots, X_{T-1}$ and split the time axis in $T = T_1 + T_2$ with $T_1 = T_2$. If we want to test the hypothesis that the variance on $X_0, \ldots, X_{T_1-1}$ is equal to the variance on $X_{T_1}, \ldots, X_{T-1}$, a suitable test statistic is

$$\hat{\tau} = \hat{\sigma}_1^2 - \hat{\sigma}_2^2$$

where $\hat{\sigma}_i^2$ is the sample variance on the ith segment. Under the null hypothesis, the distribution of $T_1^{1/2}\hat{\tau}$ is asymptotically normal if X_t^2 is a stationary process with autocovariance γ_j [27]:

$$T_1^{1/2}\hat{\tau} \xrightarrow{\mathrm{d}} \mathcal{N}(0, 2\nu) \tag{2.1}$$

as T tends to infinity, where

$$\nu = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \ .$$

ν is estimated using the kernel-based estimate

$$\hat{\nu}_\ell = \hat{\gamma}_0 + 2\sum_{j=1}^{\ell} \left(1 - \frac{j}{\ell + 1}\right) \hat{\gamma}_j$$

where $\hat{\gamma}_j$ is the jth serial covariance of X_t^2 and ℓ is a truncation number. A discussion on this estimator can be found in [38], where a consistency result is established when $\ell = \ell(T)$ tends to infinity with T and is such that $\ell(T) = O(T^{-1/4})$. Discussions about the choice of ℓ can be found in [41] and [53].

Note that the post-sample prediction test crucially depends on the time point where we split the series into two parts. As in practice this time point is arbitrary, we recall another test for covariance stationarity, the *CUSUM test*. This test does not require to split the time series into two parts. Define

$$\psi(r) = \frac{1}{\sqrt{T\nu}} \sum_{t=1}^{[Tr]} \left(X_t^2 - \hat{\sigma}_T^2\right) \tag{2.2}$$

where $0 < r < 1$ and $\hat{\sigma}_T^2$ is the classical variance estimate over the whole segment of length T. This test compares the global variance estimate with the partial sum of the squared process (recall that we assume the process to be zero-mean). If the X_t obey the moment and mixing conditions in [41], then [30] proves that, under the null, $\psi(r)$ converges in distribution to a Brownian bridge.

Figure 4 illustrates the test (2.2) on three segments of the Dow Jones index (first segment from 2nd Jan 1990 to 31st Dec 1999, second segment from 3rd Jan 2000 to 31st Dec 2009 and third segment from 3rd Jan 2000 to 31 Dec 2001). For each segment the curve $\psi(r)$ is displayed together with the percentiles of the Brownian bridge. The two first figures lead to a rejection of the covariance stationary hypothesis over the two decades. The last figure does not reject the hypothesis over the two years 2000-2001.

Tests for covariance stationarity are not limited to the above basic tests, and a survey of more recent approaches is beyond the scope of this study. We refer the interested reader to [31, 13, 6, 54] to cite but a few references in the topic.

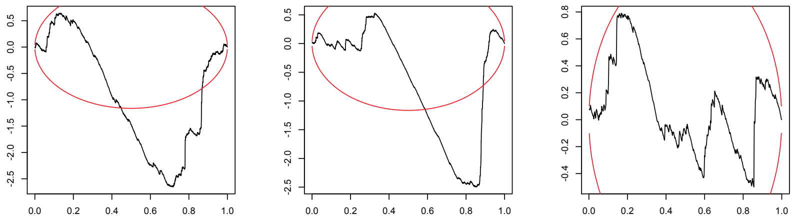


Figure 4: The result of the CUSUM test (2.2) for three segments of the DJIA index. First segment from 2nd Jan 1990 to 31st Dec 1999, second segment from 3rd Jan 2000 to 31st Dec 2009 and third segment from 3rd Jan 2000 to 31 Dec 2001.

3 Locally stationary processes and their time-varying autocovariance function

The first definition of locally stationary processes proposed by Dahlhaus [7, 8] was written in the frequency domain. In this definition the spectral density is allowed to depend on time, with some degree of regularity with respect to time. Dahlhaus [7, 8] assumes this variation to be smooth on time (existing second order derivatives). This assumption has been relaxed in [37] to allow jumps in the spectral density over time.

The definition presented below allows jumps and contains milder structural assumptions than the initial definition of locally stationary processes. It is written in the time domain and is taken from [10].

To define the regularity of the (co)variance over time, we first recall the total variation divergence of a function g defined on the interval $[0, 1]$:

$$TV(g) = \sup \left\{ \sum_{i=1}^{I} |g(x_i) - g(x_{i-1})| \;:\; 0 \leq x_0 < \ldots < x_I \leq 1, I \in N \right\} .$$

Functions with finite total variation can have a countable number of breaks of a limited size. For some $\kappa > 0$, we also need to define the function

$$\ell(j) := \begin{cases} 1 & |j| \leq 1 \\ |j| \log^{1+\kappa} |j| & |j| > 1. \end{cases} \tag{3.1}$$

Definition 3.1 ([10]). *The process $X_{t,T}$ $(t = 1, \ldots, T)$ is a locally stationary process if it is such that*

$$X_{t,T} = \sum_{j=-\infty}^{\infty} a_{t,T}(j)\epsilon_{t-j}$$

with the following conditions: for some finite generic constant K,

$$\sup_{t,T} |a_{t,T}(j)| \leq \frac{K}{\ell(j)}$$

and there exist functions $a(\cdot, j) : [0,1) \to \mathrm{R}$ *such that*

$$\sup_u |a(u,j)| \leq \frac{K}{\ell(j)}$$

$$\sup_j \sum_{t=1}^{n} \left| a_{t,n}(j) - a\left(\frac{t}{n}, j\right) \right| \leq K$$

$$TV(a(\cdot, j)) \leq \frac{K}{\ell(j)} \ .$$

The ϵ_t *are assumed to be independent and identically distributed with all existing moments and such that* $\mathbb{E}(\epsilon_t) = 0$ *and* $\mathbb{E}(\epsilon_t^2) = 1$.

According to this definition, locally stationary processes have an MA(∞) representation with time-varying filters satisfying some regularity conditions.

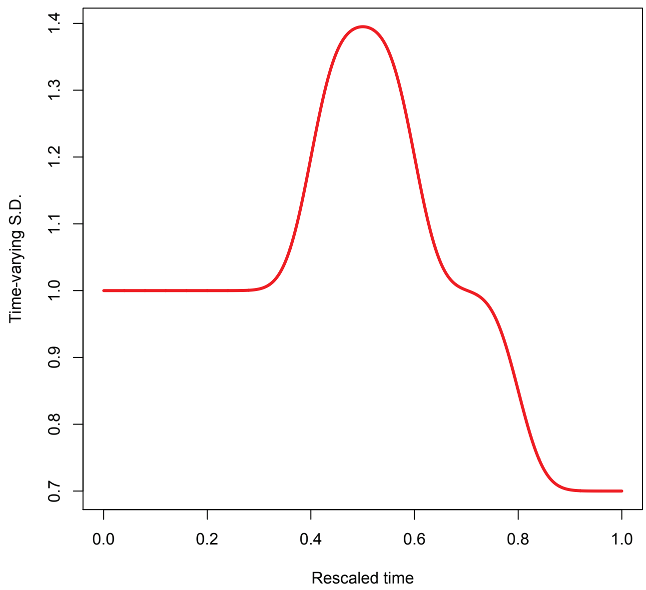
In order to understand the above definition it is useful to compute the autocovariance function of the resulting process. Consider the function $c_T(t,s) = \mathrm{Cov}(X_{[t-s/2],T}, X_{[t+s/2],T})$ for fixed T and $u \in [0,1)$, and for $s \in \{0, 1, \dots, T\}$. It can be shown that this covariance converges (in a sense defined below) to the function $c(u,s) := \sum_{j=-\infty}^{\infty} a(u, s+j)a(u,j)$. More precisely, Proposition 5.4 in [10] implies that

$$\sum_{t=1}^{T} \left| c_T(t,s) - c\left(\frac{t}{T}, s\right) \right|$$

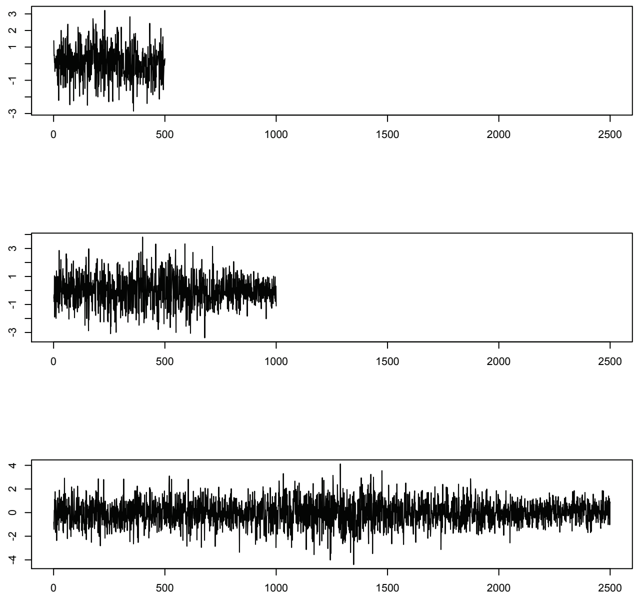
is uniformly bounded as $T \to \infty$. This result is important because it shows the uniqueness of the limit function $c(u,s)$, that can then be called the *time-varying autocovariance* of the locally stationary time series.

The class of locally stationary time series contains many processes of interest. It contains of course stationary MA(∞) for which the coefficients $a_{t,T}(j)$ in the definition do not depend on t. In such a case, the autocovariance function is the usual ACF for stationary processes. ARMA processes with time-varying coefficients are also in this family under some constraints on the regularity of their coefficients (Proposition 2.4 in [10]). This fact is a reason why the definition of locally stationary processes involves two different objects $a_{t,T}(j)$ for $t = 1, \dots, T$ and $a(u,j)$ for $u \in [0,1)$.

In the above definition the process $X_{t,T}$ is doubly-indexed, meaning that data arises in a triangular array. This setting is needed in order to study the asymptotic behavior of any estimator of the time-varying autocovariance function $c(u,k)$. Due to the triangular device, growing T does not mean to look into the future of the process. Instead it means that one observes another realization of the entire time series $X_{t,T}$ from which we can expect to construct a more accurate estimator of the time-varying autocovariance function $c(u,s)$. This phenomenon is illustrated in Figure 5 on the particular case of a process with a time-varying standard deviation. The standard deviation $\sqrt{c(u,0)}$ is plotted in the rescaled time $[0,1)$ in Figure 5(a). This function is then used to construct the locally stationary process $X_{t,T} = \sqrt{c(t/T,0)}\epsilon_t$ with a Gaussian white noise ϵ_t for three values of T. Each of the three simulated time series contain an information about the entire time-varying variance of Figure 5(a), with a starting constant variance, then a peak of volatility, then stepwise



(a)



(b)

Figure 5: Figure (a) is a given time-varying standard deviation (noted $c^{1/2}(u, 0)$ in the text) from which we give in figure (b) three variations of the locally stationary process to the right $(X_{t,T})$ with $T = 500, 1000$ and 2500.

decline of the volatility. The gain when T increases is now that the variance curve from which the process is simulated, $c(t/T, 0)$, is sampled on a finer grid, and therefore one can expect to construct estimators with more accuracy for larger sample sizes.

The introduction of rescaled time and triangular process is only theoretical and needed when asymptotic properties are considered. It is not a structural assumption on the observed process. Although it implies that future observations are not provided for growing T, it is nevertheless possible to reconcile this view with a prediction theory for locally stationary times series. This point has been formalized in [20, 50].

4 Locally stationary volatility models

4.1 Multiplicative models

A particular locally stationary model for volatility consists of a second-order stationary process that is modulated by a deterministic time-varying variance. If Y_t, $t = 1, 2, \ldots, T$ denotes a zero mean stationary process (e.g. a GARCH process), and if $\sigma(\cdot)$ is a function on the interval $[0, 1)$, the time-modulated model is

$$X_{t,T} = \sigma\left(\frac{t}{T}\right) Y_t \ .$$

for $t = 1, 2, \ldots, T$. This model satisfies the definition of a locally stationary process, provided that the function $\sigma(\cdot)$ has a finite total variation norm, and all moments of Y_t exist. For identification reasons, we also assume that the unconditional variance of Y_t is normalized to one. Let $c^Y(\cdot)$ be the autocovariance function of Y. Then the time-varying covariance of $X_{t,T}$ is

$$\begin{aligned} c^X(u,\tau) &= \lim_{T\to\infty} \mathrm{Cov}\left(X_{[uT-\tau/2]}, X_{[uT+\tau/2]}\right) \qquad u \in [0,1], \tau \in N \\ &= \lim_{T\to\infty} \sigma\left(\frac{[uT-\tau/2]}{T}\right) \sigma\left(\frac{[uT+\tau/2]}{T}\right) c^Y(\tau) \\ &= \sigma(u)^2 c^Y(\tau) \end{aligned}$$

if we assume the function $\sigma(\cdot)$ to be continuous.[2] Therefore, with the normalisation $c^Y(0) \equiv 1$, $\sigma(u)^2$ models the time-varying unconditional variance.

The gain of this variance modulation in terms of forecasting economic data (stock returns and exchange rates) has been studied in [50], where various specification of Y_t are tested (ARMA, GARCH and EGARCH) as well as two regularity conditions on the function $\sigma(\cdot)$ (Lipschitz continuous or piecewise constant).

Modeling a time-varying unconditional variance can be beneficial to account for more permanent or slowly varying patterns in volatility. This aspect is not modeled by standard GARCH models, that are therefore more adequate for shorter run forecasts. In [16] a multiplicative model is proposed, with more structural assumptions on the function $\sigma(\cdot)$ and on Y_t. They introduce a class of models, called spline-GARCH, that can be written

[2]The argument can be extended to the case of non continuous functions $\sigma(u)$, see [50].

with the following constraints:

$$X_{t,T} = \sigma\left(\frac{t}{T}\right) g_t Z_t, \qquad \text{where } Z_t \text{ is conditionally } N(0,1),$$

$$\begin{aligned} g_t^2 &= (1-\alpha-\beta) + \alpha g_{t-1}^2 Z_{t-1}^2 + \beta g_{t-1}^2, \\ \sigma(u)^2 &= c \ \exp\left(w_0 u + \sum_{i=0}^{k} w_i ((u-u_{i-1})_+)^2 + h(u)\gamma \right) \end{aligned} \tag{4.1}$$

where h is a deterministic function and where $x_+ = x$ if $x > 0$ and 0 otherwise. The nodes $\{u_0 = 0, u_1, u_2, \ldots, u_k = 1\}$ denote a partition of $[0,1]$ in k equally spaced segments. The original definition of [16] is not written in the rescaled time. Moreover, they allow the function h to depend on exogenous variables, a case that we do not discuss in this section. The function $\sigma(\cdot)$ is interpreted as the low frequency component of the volatility. If $\sigma(u)^2$ is a constant then the process Y_t is a GARCH process.

Figure 6 shows an estimator of $\sigma(u)^2$ for two segments of the Dow Jones Industrial Average index, with $k = 3$. Spline-GARCH can be viewed as a particular case of the time-modulated processes with a particular spline structure for $\sigma(u)$. In [16], the dimension of the spline, k, is supposed to be known. An adaptive, data-driven procedure to select the dimensionality of a spline is studied in [49] in the general context of the approximation of time varying parameters by the method of sieves.

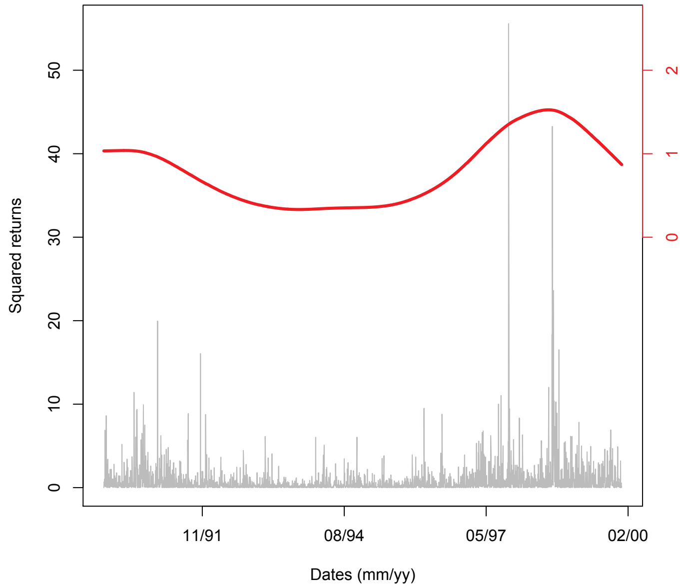
Recent extension of the spline GARCH approach for modeling volatility include the use of large dimensional B splines [3] and the use of more flexible parametric forms for $\sigma(u)$ [1, 48]. It is possible to go beyond this limitation and to estimate $\sigma(u)$ by arbitrary basis functions. In the context of locally stationary time series, [49] have developed a method of sieve to estimate time-varying second-order structure. The dimensionality of the sieve is not known and adaptively given from the observed process.

4.2 Time-varying ARCH processes

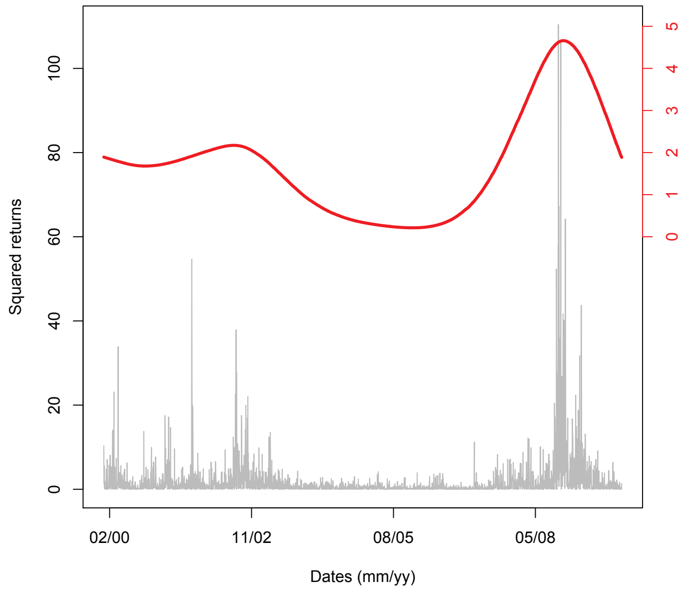
The class of ARCH(p) processes has been generalized to allow their parameters to change through time [11]. The resulting time-varying ARCH(p) model, is defined as

$$\begin{aligned} X_{t,T} &= \sigma_{t,T} Z_t, \\ \sigma_{t,T}^2 &= a_0\left(\frac{t}{T}\right) + \sum_{j=1}^{p} a_j\left(\frac{t}{T}\right) X_{t-j,T}^2 \end{aligned}$$

$t = 1, 2, \ldots, T$, where Z_t are independent and identically distributed variables with zero mean and such that $\mathbb{E}(Z_t^2) = 1$. Analogously to the stationary ARCH model, the time-varying parameters need to satisfy some constraints: there exists positive, finite constants c, d such that $\inf_u a_0(u) > c$ and $\sup_u a_j(u) \leq d/\ell(j)$, where $\ell(j)$ is defined in (3.1). The regularity of curves $a_j(\cdot)$ over time is also supposed to follow a Lipschitz constraint, that is there exists constant K such that $|a_j(u) - a_j(v)| \leq K|u-v|/\ell(j)$ which, in particular, implies that the time-varying curves are continuous. The conditions on the coefficients also imply that $\mathbb{E}(X_{t,T}^2)$ is uniformly bounded over t and T.



(a) Index from 2nd Jan 1990 to 31st Dec 1999



(b) Index from 3rd Jan 2000 to 31st Dec 2009

Figure 6: Spline GARCH fits to two segments of the DJIA index. The squared returns (in gray) is superimposed with a cubic spline estimator of the time-varying unconditional variance (black curve and scale on the right side)

The persistence effect of time-varying ARCH(p) has been studied in detail by [19]. In Proposition 2 of [19] and under technical conditions, they establish that the sample autocovariance function of $X_{t,T}^2$ evaluated under the wrong premise of stationarity, i.e.

$$\frac{1}{T-h}\sum_{t=1}^{T-h} X_{t,T}^2 X_{t+h,T}^2 - \left(\frac{1}{T-h}\sum_{t=1}^{T-h} X_{t,T}^2\right)^2$$

converges in probability, for fixed h and as $T \to \infty$, to

$$\int_0^1 c^{X^2}(u,h)du + \iint_{\{0\leq u<v\leq 1\}} \{\mu^{X^2}(u) - \mu^{X^2}(v)\}^2 dudv$$

where $\mu^{X^2}(u)$ and c^{X^2} are respectively the time-varying mean and autocovariance function of the squared process. The ACF of the squares of a time-varying ARCH process is shown to decay exponentially to zero. The persistence effect appears from the second integral in the limit, which does not depend on the lag h and is typically non zero (except for stationary ARCH in which case μ^{X^2} is constant).

Advanced inference for time-varying ARCH(p) has been recently studied. In the seminal work [11], a localized version of the quasi maximum likelihood is studied. A recursive online algorithm of estimation is proposed in [12]. An alternative least-squares type estimator is studied in [19]. This estimator considers the following contrast function:

$$L_{t,T}(a_0,\ldots,a_p) = \sum_{k=p+1}^{T} W_{bT}(t-k)\frac{(X_{k,T}^2 - a_0 - \sum_{j=1}^p a_j X_{k-j,T}^2)^2}{g(u, X_{k-1,T}^2,\ldots,X_{k-p,T}^2)}$$

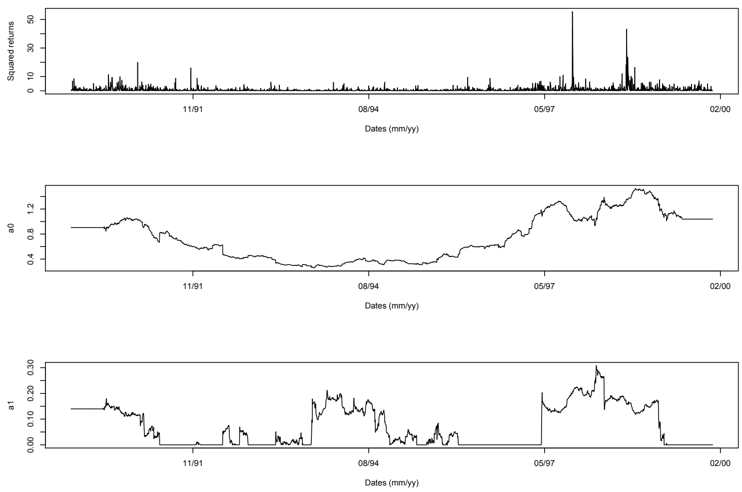
where $W_{bT}()$ is a given kernel function with bandwidth b and g is a weight function. If u is such that $|u - t/T| < 1/T$, then the weighted least-squares estimator of $(a_0(u),\ldots,a_p(u))$ is given by the arg min of $L_{t,T}(a_0,\ldots,a_p)$. The asymptotic normality of the estimator is established in [19] and it is shown that the performance of the estimator depends on the weight function g. A two-stage estimator is thus also studied where in the first stage the weight function is estimated.

An interesting connection between time-varying ARCH and spline GARCH can be established if we restrict the latter to spline ARCH processes (that is $\beta \equiv 0$ in equation (4.1)). Assuming the conditional distribution is Normal in both models, a simple calculation leads to

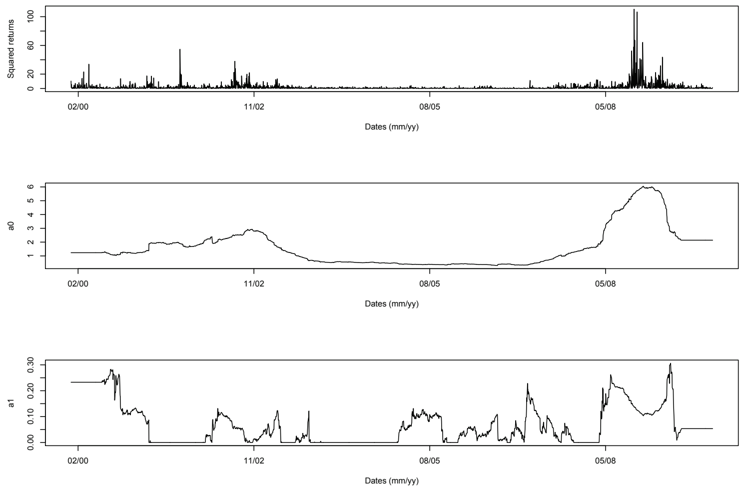
$$a_0\left(\frac{t}{T}\right) = (1-\alpha)\sigma_S\left(\frac{t}{T}\right)^2 \quad \text{and} \quad a_1\left(\frac{t}{T}\right) = \frac{\alpha\sigma_S(t/T)^2}{\sigma_S((t-1)/T)^2}\,. \tag{4.2}$$

where σ_S denotes the low frequency component of the spline-ARCH. Spline ARCH is therefore a specific case of time-varying ARCH in which the variation of a_0 is, up to a constant, given by the variation of $\sigma_S(\cdot)$. Since $\sigma_S(\cdot)$ is a smooth function of time, the curve $a_1(t/T)$ is not supposed to show high unconditional variations and is approximately equal to α.

The weighted least-square estimator of time-varying ARCH curves is illustrated in Figure 7 where a time-varying ARCH(1) model is fitted on two segments of the Dow Jones Industrial Average index. In this estimation exercise the bandwidth is set to $bT = 250$ data.



(a) Index from 2nd Jan 1990 to 31st Dec 1999



(b) Index from 3rd Jan 2000 to 31st Dec 2009

Figure 7: Time-varying ARCH(1) fits to two segments of the DJIA index. In each subfigure, the first row is the square returns, the second and third rows are weighted least-squares estimators of $a_0(u)$ and $a_1(u)$ with bandwidth $b = 250$ data.

The qualitative aspect of the curves do not vary when this bandwidth is taken between 150 and 800. The estimated $a_0(u)$ in Figure 7 behaves very similarly to the what we obtained with the spline GARCH in Figure 6. According to equations 4.2 and under the assumption of spline ARCH, they only differ in theory from a multiplicative constant.

4.3 Adaptive approaches

The above procedures are based on the structural assumption that the volatility belongs to a precise class of locally stationary models. The class of spline GARCH provides the most restrictive formulation. An alternative approach to handle nonstationary volatilities is provided by adaptive approaches, in which one only assumes that volatility can be locally approximated by a constant volatility. Even if the volatility is locally constant, it is not globally and this new approach aims to evaluate pointwise estimators of volatility by fitting constant volatilities at each time point.

The root of this idea goes back to the paradigm of local polynomial inference in which nonparametric estimator of a regression function is found when constants or polynomials are fitted locally at every point of the regression (e.g. [17]). One crucial aspect of this method is the definition at each estimation point of the neighborhood under which the regression can be supposed to be constant or polynomial. With that respect a fundamental work by Lepski [29] provides an automatic method to determine this neighborhood, which can have a different length from one estimation point to another.

The extension of local regression to estimate time-varying volatilities has been studied by [25, 34]. Suppose X_t denotes some returns from which we want to infer a volatility function. The authors assume that the volatility function is *local time homogenous* meaning that, at time t, there exists an interval $I = [t - m, t]$, with $m > 0$, such that the volatility can be estimated by the empirical average

$$\hat{\sigma}_I^2 = \frac{1}{|I|} \sum_{\tau \in I} X_\tau^2 .$$

(Other forms of estimators are also considered in the above cited literature). Note that the parameter m defining the length of the interval I might differ for various time points t. Therefore the selection of m is the main statistical challenge and, based on the automatic selection of [29], an adaptive procedure for choosing m has been developed [25, 34]). The procedure is a sequence of tests of stationarity on intervals $[t - m, t]$ for growing m. The adaptive interval is found to be the largest interval on which the observed time series is still compatible with the null of stationarity. To concretely implement the procedure, the choice of calibration parameters are needed such as the level of the stationarity tests. Under some assumptions, the necessary calibration parameters are proved to be time-invariant [34], and thus a procedure to find those parameters from some training sets of data can be developed.

Since the mentioned papers have been published, many extensions and improvements of the adaptive approach have been studied, see e.g. [52, 47]. Combining this approach with the above definition of locally stationary time series has been studied in [51].

5 Multivariate models for locally stationarity volatility

The extension of locally stationary time series to multivariate time series has been formally studied by [9]. In this first approach, the multivariate spectral density is allowed to vary with respect to the rescaled time, and this variation is supposed to be modeled by a vector of finite dimensional parameters.

The extension of multivariate locally stationary models of volatility is a very active topic of research today and a lot of work remains to be done. One challenge arises since the resulting process is very complex, difficult to interpret and hard to estimate accurately. One ultimate aim in this context is to specify a multivariate model that achieves a reasonable trade-off between its flexibility (complexity), and the necessity to keep easy interpretation (or identification) and efficient estimation. In the context of stationary models, multivarate extension of volatility models are already challenging. Two surveys are available in this context: [5] focus on parametric multivariate GARCH models, and [45] survey nonparametric and semiparametric multivariate GARCH models.

Below we describe two successful extensions of the above univariate models for local stationarity. One is an extension of the multiplicative model, and the other is an extension of the adaptive approach.

5.1 Multiplicative models

A multivariate extension of the multiplicative model is found when the volatility process $X_{t,T} \in R^N$ satisfies the decomposition

$$X_{t,T} = \Sigma \left(\frac{t}{T}\right)^{1/2} Y_t$$

where Σ now denotes a time-varying $N \times N$ deterministic definite positive matrix and $Y_t \in R^N$ is a stationary time series. Among the many possible specifications for Σ and Y_t, [22] studies the situation where $\Sigma(\cdot)$ is unknown, either smooth or with finite total variation norm components, and Y_t is decomposed as $G_t^{1/2} Z_t$ where Z_t is an N dimensional strictly stationary unit conditional variance martingale difference sequence satisfying $E(Z_t|\mathcal{F}_{t-1}) = 0$, $E(Z_t Z_t'|\mathcal{F}_{t-1}) = I_N$, $\mathcal{F}_{t-1}$ is the sigma algebra generated by $X_{t-1}, X_{t-2}, X_{t-3}, \dots$ and $G_t \in R^N$ is a strictly stationary process in $\mathcal{F}_{t-1}$. In the model of [22],

$$X_{t,T} = \Sigma \left(\frac{t}{T}\right)^{1/2} G_t^{1/2} Z_t \ ,$$

the process G_t is supposed to depend on a finite dimensional vector of parameters. Therefore, it can be viewed as a multivariate generalization of the spline-GARCH model above defined. Local maximum likelihood estimation has been studied in [22], who also establish the asymptotic properties of their estimators under the semi-strong form specification of the errors and show that under the strong form Gaussian distributional specification, their estimators are semiparametrically efficient.

5.2 Adaptive approaches

In principle the adaptive approach is not limited to one-dimensional volatility processes. [25] have considered this extension as follows. Suppose the volatility process $X_{t,T} \in R^N$ satisfies the decomposition

$$X_{t,T} = H\left(\frac{t}{T}\right)^{1/2} Z_t$$

where Z_t is a standard Gaussian multivariate process and $H(\cdot)$ is now an $\mathcal{F}_{t-1}$ measurable process that captures the entire dynamics of $X_{t,T}$. Under the assumption that at time t the volatility is local time homogenous over I (see above), an estimator of $H(t/T)$ is given by

$$\frac{1}{|I|}\sum_{\tau \in I} X_{t,T} X'_{t,T} \ .$$

As before, the main statistical challenge of the approach is to define a data driven selection of the interval I such that the resulting estimator has good properties. To reduce the dimensionality, [25] use the idea of principal component analysis to transform the multivariate process to univariate processes.

6 Conclusion

This paper reviews the concept of local stationarity and its use for modeling volatility. Three main approaches have been discussed. The first approach is a multiplicative decomposition of the volatility as a time-varying unconditional variance that multiplies a stationary ARCH-type process. The time-varying unconditional variance models long term patterns (such as seasonal patterns) of structural breakpoints in the volatility process. The second approach is to consider ARCH(p) models with time-varying coefficients. Feasible nonparametric estimators to estimate the coefficients have been studied in this context. Finally, a third approach, called adaptive, is purely nonparametric and approximates locally the volatility time series by a constant volatility.

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