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ORIGINAL RESEARCH PAPER

The Three-step method in a dynamic setting

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Abstract

A crucial issue in a dynamic framework is how risk valuations at different times are interrelated. In this regard, the notion of time consistency was widely introduced and discussed in the literature. A time-consistent dynamic valuation states that a future payoff preferred to another payoff at some future time point should already be preferred to this payoff today. This paper aims to construct a time-consistent, dynamic version of the Three-step method introduced in Deelstra et al. ((2020). *ASTIN Bulletin: The Journal of the IAA*, 50(3), 709–742.) for hybrid life Pure Endowment products, employing a backward iteration scheme. The backward scheme is illustrated in a dual-iteration approach using a Pure Endowment product without profit sharing. Furthermore, we explore the continuous-time limit of the backward scheme, incorporating profit-sharing into the Pure Endowment to investigate a hybrid life payoff. Our analysis demonstrates that the presence of the diversifiable component undermines the time-consistency of the dynamic three-step method. Consequently, the time-consistent price of the actuarial part shows a notable increase. To address this, and in accordance with Devolder and Lebègue ((2016). *Risks*, 4(4), 49.), we present a reduced time-consistent variant by decreasing the safety loads in each iterative step of the backward scheme.

Keywords: backward iteration scheme; fair dynamic valuation; hybrid life payoff; premium principles; time-consistent

1. Introduction

Valuation methods serve as quantitative instruments for assessing the liabilities of an insurance company and must comply with regulatory frameworks. In particular, Solvency II requires insurance liabilities to be assessed based on both the market-consistent and actuarial-consistent properties. In a complete financial market without arbitrage opportunities, the market-consistent property guarantees the valuation of purely financial payoffs by employing the risk-neutral measure. Market-consistent valuations have been largely studied, and we refer to Malamud et al. (2008) for an overview. In continuation of this theoretical concept, Pelsser and Stadje (2014) have proposed a valuation technique referred to as the two-step method. This method systematically converts individual conditional valuations into market-consistent valuations, thereby enabling the generation of a diverse set of market-consistent valuations. Moreover, it was shown in Dhaene et al. (2017) that market consistent valuations can be characterized by the hedge-based valuations, which under the assumption of market completeness correspond with the two-step valuations.

Nonetheless, there has been recent critique in the literature regarding market-consistency in the evaluation of long-term insurance liabilities, as discussed, for example, by Le Courtois et al. (2020). Based on pooling and diversification arguments, the actuarial-consistent property asserts that applying insurance premium principles is essential in the valuation of purely actuarial products; see also Barigou et al. (2023).

The life insurance industry is increasingly directing its attention towards hybrid products distinguished by the presence of both hedgeable financial risks and non-hedgeable actuarial risks. An illustrative category of hybrid products includes participating life insurance contracts, wherein the insurer initially extends a return guarantee and subsequently offers the policyholder a share in the event that the realized return surpasses the guaranteed return. The profit-sharing component may be earmarked for maturity or scheduled periodically, as illustrated in Bacinello (2001). Additionally, the apportionment of profits could be contingent upon financial risk, actuarial risk, or a combination of both, as elaborated in Devolder and Azizieh (2013). The literature extensively covers valuation operators designed for the assessment of hybrid payoffs. While brief mention of works such as Chen et al. (2020) and (2021) is by no means exhaustive, our study is particularly focused on exploring the Three-step method elucidated within a static framework in Deelstra et al. (2020).

A crucial issue in the dynamic framework is how risk valuations at different times are interrelated. This inquiry prompts the exploration of the time-consistent property. In essence, this property asserts that a future payoff preferred to another payoff at some future time point should already be preferred to this payoff today, as discussed in Acciaio and Penner (2011) in the context of random variables representing purely financial positions. Although time consistency is present in the financial literature, it remains comparatively limited in actuarial literature. Devolder and Lebègue (2016) addressed time consistency in an actuarial context, specifically for the computation of the solvency capital requirement (SCR), but relaxed the assumption concerning mortality risk. Barigou et al. (2019) examined time consistency for hybrid products using a two-step method; however, that study neglects diversifiable risk. Our contribution addresses this gap by treating time consistency for hybrid products, focusing on hybrid life Pure Endowment products, while explicitly incorporating diversifiable risk via the Three-step method, thereby pinpointing the valuation of diversifiable risk as the principal source of non-time consistency. The main contribution of this paper is twofold. First, we extend the Three-step valuation of Deelstra et al. (2020) to a dynamic context, thereby providing updated valuations aligned with the available information. In common with the majority of non-linear valuation operators, the dynamic Three-step method is not time-consistent. By adapting the iterative version of time-consistency outlined in Bion-Nadal (2009) for hybrid life Pure Endowment products, against the backdrop of our target dynamic valuation, namely the Three-step method, we progressively derive a succession of time-consistent values, evolving from the initially non-time-consistent version. Secondly, to alleviate the risk premium of time-consistency, we adopt the approach proposed by Devolder and Lebègue (2016), proven to reduce the time-consistent SCR. The approach involves reducing the safety loads employed in the backward scheme.

The remainder of the paper is organized as follows. In Section 2, we present the main assumptions used throughout the paper and an exposition of the Three-step method within a dynamic framework. Section 3 introduces the definition of the time-consistent property for financial products, establishing its association with a backward iteration scheme. Furthermore, by centering on the Three-step method dynamic, we broaden the scope of the latter backward procedure to encompass hybrid life Pure Endowment products. Using two iterations, the backward procedure is applied to the Three-step method in the context of a Pure Endowment product without profit sharing. In a continuous-time limit framework, Section 4 is dedicated to deriving the partial differential equation (PDE) for the Three-step method time-consistent dynamic. By studying a hybrid life Pure Endowment product, namely the Pure Endowment with terminal bonus, we present a numerical resolution of the PDE. Section 5 concludes the paper.

2. Extending the Three-step method to a dynamic framework

2.1 Assumptions

We consider an arbitrage-free and complete financial market. Moreover, we assume that there are risks that are not traded on the financial market, which we label as "actuarial" risks. These risks are assumed to be independent of the financial market. Such strong assumptions have already been imposed in some papers in the literature, such as Deelstra et al. (2020) and Belhouari et al. (2025). The set of trading dates is $[0, T]$, where T is the maturity. We consider a portfolio of n insured individuals, all of age x at time 0; their remaining lifetimes are denoted by $(\tau_1, \tau_2, \ldots, \tau_n)$. We denote by N_t the number of survivors up to time t, formulated as $N_t = \sum_{j=1}^{n} \mathbb{1}_{\tau_j > t}$. Furthermore, for each $t \in [0, T]$, the remaining lifetimes of the individuals alive at time t, denoted $(\tau_i)_{1 \leq i \leq N_t}$, are assumed to be identically distributed. Assuming that actuarial and financial risks are independent, we consider the following filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}).$$

The sample space is defined as the product $\Omega = \Omega^a \times \Omega^f$, reflecting all the possible financial and mortality events. Here, Ω^a denotes the actuarial space and Ω^f denotes the financial space. The σ-algebra $\mathcal{F}$ is defined as $\mathcal{F} = \mathcal{F}^a \vee \mathcal{F}^f$, where $\mathcal{F}^a$ is the actuarial σ-algebra and $\mathcal{F}^f$ is the financial σ-algebra. The probability measure is $\mathbb{P} = \mathbb{P}^a \times \mathbb{P}^f$, where $\mathbb{P}^a$ and $\mathbb{P}^f$ denote the real-world actuarial and financial probabilities, respectively. We also introduce the risk-neutral financial probability $\mathbb{Q}^f$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ represents the information flow and is defined by $\mathcal{F}_t = \mathcal{F}_t^a \vee \mathcal{F}_t^f$ where $(\mathcal{F}_t^a)_{t \geq 0}$ is the actuarial filtration and $(\mathcal{F}_t^f)_{t \geq 0}$ is the financial filtration. The actuarial filtration $(\mathcal{F}_t^a)_{t \geq 0}$ is composed of the filtration $(\mathfrak{S}_t)_{t \geq 0}$ generated by systematic mortality risk factors and the filtration $(\bigvee_{i=1}^{n} \Im_t^i)_{t \geq 0}$ encompassing information on unsystematic mortality risk. At any given time $t < T$, for each individual $1 \leq i \leq n$, we define $\Im_t^i = \sigma(\mathbb{1}_{\tau_i \leq u} : u \leq t)$. Thus, it follows that $\mathcal{F}_t^a = \mathfrak{S}_t \vee \left(\bigvee_{i=1}^{n} \Im_t^i\right)$. Furthermore, we will assume that the lifetimes of the surviving insured at each instant t display conditional independence, given the systematic mortality risk information at maturity u.[1]

For the actuarial instruments, stochastic mortality rates $(\lambda_t)_{0 \leq t \leq T}$, adapted to the filtration $\mathfrak{S}$ will be used. Thus, the probability of an individual at time $t \leq u$ to survive until maturity $u \leq T$ is[2]

$$_{u-t}p_{x+t} \triangleq \mathbb{E}_{\mathcal{F}_t}\left[e^{-\int_t^u \lambda_s \mathrm{d}s}\right]. \tag{1}$$

In addition to the actuarial risk, we include the financial risk. In the realm of financial instruments, our focus will be on a sole stochastic investment fund denoted as $(S_t)_{0 \leq t \leq T}$, coupled with stochastic interest rates captured by $(r_t)_{0 \leq t \leq T}$.[3] The determination of the value for a zero-coupon bond with maturity $u \leq T$ at a time $t \leq u$ can be accomplished through the following equation

$$P(t, u) \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Q}^f}\left[e^{-\int_t^u r_s \mathrm{d}s}\right]. \tag{2}$$

[1]In a more precise mathematical formulation, this means that for any $s \in [t, u]$, with $0 \leq t \leq u \leq T$, the following equality holds: $\mathbb{P}^a(\tau_1 > s, \tau_2 > s, \ldots, \tau_{N_t} > s | \mathfrak{S}_u) = \prod_{i=1}^{N_t} \mathbb{P}^a(\tau_i > s | \mathfrak{S}_u)$.

[2]At time t, we denote the conditional expectation $\mathbb{E}[.|\mathcal{F}_t]$ and the conditional variance $\mathbb{V}[.|\mathcal{F}_t]$ as $\mathbb{E}_{\mathcal{F}_t}[.]$ and $\mathbb{V}_{\mathcal{F}_t}[.]$, respectively; these are only almost surely unique. Moreover, expectations under the real-world probability will be written without a superscript; all others will carry an explicit superscript.

[3]The interest rate process $(r_t)_{0 \leq t \leq T}$ is a financial risk correlated with the stochastic investment fund $(S_t)_{0 \leq t \leq T}$.

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2.2 Axiomatization

The examination of axioms will take place using a general class of hybrid product described by the following general set of random variables (rv.s)

$$\forall t \in [0, T], \;\; \mathcal{H}_t \triangleq \left\{H \; rv/\mathcal{F}_t - \text{measurable \& payable at } t\right\}. \tag{3}$$

Furthermore, we introduce two more refined subsets, building exclusively on knowing the financial filtration or the actuarial filtration. The precise definitions of these subsets are outlined below

$$\forall t \in [0, T], \;\; \mathcal{H}^f_t \triangleq \{h \; rv/\mathcal{F}^f_t - \text{measurable \& payable at } t\}. \tag{4}$$

$$\forall t \in [0, T], \;\; \mathcal{H}^a_t \triangleq \left\{g \; rv/\mathcal{F}^a_t - \text{measurable \& payable at } t\right\}. \tag{5}$$

According to Barigou et al. (2019), we follow some axioms that a valuation operator must verify in the following; see also Belhouari et al. (2025).

Definition 1 (t^u-Valuation operator).[4] *Let $t \in [0, T]$, $u \in [0, T]$ such as $t \le u$. Let $\pi^u_t : \mathcal{H}_u \to \mathcal{H}_t$, a function which represents the value at time t of the hybrid payoff $H \in \mathcal{H}_u$. We say that π^u_t is a t^u-valuation operator if it verifies the following axioms*

- ***Normalization*** $\pi^u_t(0) = 0$.
- ***Translation invariance*** *$\forall x_t$, $\mathcal{F}_t$ – measurable and payable at the instant u, $\forall H \in \mathcal{H}_u$*

$$\pi^u_t(H + x_t) = \pi^u_t(H) + P(t, u)x_t.$$

Definition 2 (Dynamic valuation). *The sequence $(\pi^u_t)_{0 \le t \le u \le T}$, where each π^u_t is a t^u-valuation operator, is referred to as a dynamic valuation.*

Given that standard asset pricing in finance traditionally determines the value of a purely financial derivative $h \in \mathcal{H}^f_u$ by $\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}^f_t}[e^{-\int_t^u r_s \mathrm{d}s} h]$, one might naturally consider applying this same methodology to the broader class of hybrid life payoffs $H \in \mathcal{H}_u$ by defining the following t^u-financial valuation operator $\pi^{u,f}_t$

$$\pi^{u,f}_t(H|\mathcal{F}_t) \triangleq \mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^u r_s \mathrm{d}s} H\right], \tag{6}$$

with $H \in \mathcal{H}_u$ and $0 \le t \le u \le T$.

Definition 3 (Market-consistent property). *Let $0 \le t \le u \le T$. A t^u-valuation operator π^u_t is market-consistent with respect to the t^u-financial operator $\pi^{u,f}_t$ defined in (6) if and only if*

$$\forall h \in \mathcal{H}^f_u, \forall H \in \mathcal{H}_u : \pi^u_t(H + h|\mathcal{F}_t) = \pi^u_t(H|\mathcal{F}_t) + \pi^{u,f}_t(h|\mathcal{F}_t).$$

On the actuarial front, we assume that establishing actuarial consistency will draw upon two actuarial principles, namely, the standard deviation premium principle and the variance premium principle. Given the stochastic nature of interest rates in our study, we will adopt the framework proposed by Belhouari *et al.* (2025), albeit with minor modifications.

The standard deviation premium principle is

$$\pi^{u,SD}_t[H|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^u r_s \mathrm{d}s} H\right] + \alpha_t \sqrt{\mathbb{V}_{\mathcal{F}_t}[H]} \times \sqrt{u - t} \times P(t, u), \tag{7}$$

[4] To avoid heavy notation, we will use $\pi_t[H]$ to represent the t^T-valuation operator, indicating the value at time t of the hybrid payoff $H \in \mathcal{H}_T$.

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for $0 \le t \le u \le T$, $H \in \mathcal{H}_u$, and $\alpha_t > 0$ is the portion of the load used at time t. The standard deviation term is multiplied by the factor $\sqrt{u-t}$; this small change is necessary to solve the dimensionality problem that arises from the different time-scales produced by the expectation and the standard deviation, see Pelsser and Ghalehjooghi (2016). The variance premium principle is

$$\pi_t^{u,V}[H|\mathcal{F}_t] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f}\left[e^{-\int_t^u r_s \mathrm{d}s} H\right] + \gamma_t \times \mathbb{V}_{\mathcal{F}_t}[H] \times P(t,u), \tag{8}$$

for $0 \le t \le u \le T$, $H \in \mathcal{H}_u$. As the expectation is linear in monetary units while the variance is quadratic, a dimensional mismatch arises. To address this, we define $\gamma_t > 0$ as a loading parameter expressed in inverse monetary units. In our market-fair valuation context, detached from specific regulatory constraints, the insurer sets the parameters α_t and γ_t based on its own risk appetite.

Definition 4 (Actuarial-consistent property). *Let $0 \le t \le u \le T$. A t^u-valuation operator π_t^u is actuarial-consistent with respect to the t^u-actuarial operator $\pi_t^{u,a}$ if and only if $\forall g \in \mathcal{H}_u^a$,*

$$\pi_t^u(g|\mathcal{F}_t) = \pi_t^{u,a}(g|\mathcal{F}_t),$$

with $\pi_t^{u,a}$ is given by (7) or (8).

Definition 5 (t^T-Fair valuation). *Let $0 \le t \le u \le T$. A t^u-fair valuation is a t^u-valuation operator that is both market-consistent and actuarial-consistent.*

Definition 6 (Market-consistent dynamic valuation). *A market-consistent dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \le t \le u \le T}$, where for each $0 \le t \le u \le T$, π_t^u is a t^u-market-consistent valuation operator.*

Definition 7 (Actuarial-consistent dynamic valuation). *An actuarial-consistent dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \le t \le u \le T}$, where for each $0 \le t \le u \le T$, π_t^u is an t^u-actuarial-consistent valuation operator.*

Definition 8 (Fair dynamic valuation). *A fair dynamic valuation is a dynamic valuation $(\pi_t^u)_{0 \le t \le u \le T}$, where for each $0 \le t \le u \le T$, π_t^u is a t^u-fair valuation.*

Note that the operator $\pi_t^{u,f}$ is market-consistent but not actuarial-consistent. In contrast, the standard deviation $\pi_t^{u,SD}$ and variance $\pi_t^{u,V}$ premium principles are actuarial-consistent but do not satisfy market-consistency. Consequently, the remainder of this paper focuses on the Three-step method, which defines a fair valuation operator for assessing hybrid products.

2.3 Three-step decomposition in a dynamic setting

Within the scope of our study, we investigate the Three-step method introduced in Deelstra et al. (2020). To this end, we extend the natural dynamic approach of the Three-step decomposition. We consider the aggregate hybrid payoff for the insured entities, outlined as

$${}_tH_u^c = h(S_u) \sum_{j=1}^{N_t} g(\tau_j), \tag{9}$$

where $0 \le t \le u \le T$, $g\colon]0,\infty[\to [0,\infty[$ and $h\colon]0,\infty[\to [0,\infty[$ are two $\mathcal{F}_u^a$-measurable functions and $\mathcal{F}_u^f$-measurable, respectively.[5] We decompose $\frac{{}_tH_u^c}{N_t} \times \mathbb{1}_{N_t>0}$, with respect to the full

[5]It is postulated that for each $0 \le t \le u \le T$, for all $j = 1, \ldots, N_t$, the quantities $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_t^u r_s \mathrm{d}s} h(S_u) g(\tau_j)]$, $\mathbb{E}_{\mathcal{F}_t}[e^{-\int_t^u r_s \mathrm{d}s} h(S_u) g(\tau_j)]$, $\mathbb{V}_{\mathcal{F}_t}^{\mathbb{P}^a \times \mathbb{Q}^f}[e^{-\int_t^u r_s \mathrm{d}s} h(S_u) g(\tau_j)]$ and $\mathbb{V}_{\mathcal{F}_t}[e^{-\int_t^u r_s \mathrm{d}s} h(S_u) g(\tau_j)]$ are finite.

spectrum of information, as follows[6]

$$ {}_tH_u = \frac{{}_tH_u^c}{N_t} \times \mathbb{1}_{N_t>0} = {}_tH_u^1 + {}_tH_u^2 + {}_tH_u^3, \tag{10} $$

where

$$ {}_tH_u^1 \triangleq \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}[{}_tH_u] = \frac{\sum_{j=1}^{N_t} \mathbb{E}_{\mathcal{F}_t}[g(\tau_j)]h(S_u)}{N_t} \times \mathbb{1}_{N_t>0}. \tag{11} $$

$$ {}_tH_u^2 \triangleq {}_tH_u - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u \vee \mathcal{F}_u^f}[{}_tH_u] = \frac{\sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u}[g(\tau_j)])h(S_u)}{N_t} \times \mathbb{1}_{N_t>0}. \tag{12} $$

$$ \begin{aligned} {}_tH_u^3 &\triangleq \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u \vee \mathcal{F}_u^f}[{}_tH_u] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_u^f}[{}_tH_u] \\ &= \frac{1}{N_t} \sum_{j=1}^{N_t} (\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u}[g(\tau_j)] - \mathbb{E}_{\mathcal{F}_t}[g(\tau_j)])h(S_u)\, \mathbb{1}_{N_t>0}. \end{aligned} \tag{13} $$

The term $\frac{\mathbb{E}_{\mathcal{F}_t}[g(\tau_j)]}{N_t} \times \mathbb{1}_{N_t>0}$ exhibits $\mathcal{F}_t$-measurability, then ${}_tH_u^1$ is a financial claim; consequently $\pi_t^{u,f}$ serves as the suitable operator for its assessment. The component (12) is diversifiable since this assertion holds

$$ \begin{aligned} {}_tH_u^2\Big|_{\mathcal{F}_t \vee \mathcal{F}_u^f \vee \mathfrak{G}_u} &= \frac{1}{N_t} \sum_{j=1}^{N_t} (g(\tau_j) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u}[g(\tau_j)])h(S_u)\Big|_{\mathcal{F}_t \vee \mathcal{F}_u^f \vee \mathfrak{G}_u} \\ &\xrightarrow{N_t \to \infty} \mathbb{E}\big[(g(\tau_1) - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{G}_u}[g(\tau_1)]\big)h(S_u) \mid \mathcal{F}_t \vee \mathcal{F}_u^f \vee \mathfrak{G}_u\big] = h(S_u) \times 0 = 0 \quad \text{a.s.} \end{aligned} $$

Hence, we assess it by applying actuarial premium principles. Furthermore, both the standard deviation premium principle (7) and the variance principle (8) conform to pooling arguments in relation to ${}_tH_u^2$. Indeed, they approach a zero value in an asymptotic sense.[7] Concerning the final part ${}_tH_u^3$, it represents the systematic risk which is unaffected by both hedging and diversification. To compute its valuation, we propose an Esscher transform following Deelstra et al. (2020), detailed as follows

$$ \pi_t^{u,n}(H|\mathcal{F}_t) \triangleq \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}\big[e^{-\int_t^u r_s \mathrm{d}s} \times H\big], \tag{14} $$

$H \in \mathcal{H}_u$, $0 \le t \le u \le T$ and

$$ \frac{d\mathbb{Z}_\theta^a}{d\mathbb{P}^a}\bigg|_{\mathcal{F}_t} \triangleq \frac{e^{\int_t^u \theta_{t,u}^{(1)} \lambda_s \mathrm{d}s}}{\mathbb{E}_{\mathcal{F}_t}\big[e^{\int_t^u \theta_{t,u}^{(1)} \lambda_s \mathrm{d}s}\big]}, \tag{15} $$

The explicit formula for $\theta_{t,u}^{(1)}$ depends on the specified dynamics of stochastic mortality rates and will therefore be provided later; see Subsection 3.2 for further details on this point.

Definition 9 (The dynamic Three-step method). *Let $0 \le t \le u \le T$. The dynamic Three-step method is defined by*

$$ \pi_t^{u,3}\big({}_tH_u|\mathcal{F}_t\big) \triangleq \pi_t^{u,f}\big({}_tH_u^1|\mathcal{F}_t\big) + \pi_t^{u,a}\big({}_tH_u^2|\mathcal{F}_t\big) + \pi_t^{u,n}\big({}_tH_u^3|\mathcal{F}_t\big), $$

where $\pi_t^{u,f}$ is given by (6), $\pi_t^{u,a}$ is given by (7) or (8) and $\pi_t^{u,n}$ is given by (14).

It is easy to check that the dynamic Three-step method is fair according to our axiomatization.

[6]Inherently, the subscript t appended to ${}_tH_u^1$, ${}_tH_u^2$ and ${}_tH_u^3$ also determines the evaluation instance.

[7]In this formulation $\frac{\pi_t^{u,V}({}_tH_u^{2-c}|\mathcal{F}_t)}{N_t} \underset{N_t \to +\infty}{\not\longrightarrow} 0$, the variance principle cannot be diversified and differs from our valuation methodology, i.e. $\pi_t^{u,V}({}_tH_u^2|\mathcal{F}_t) \underset{N_t \to +\infty}{\longrightarrow} 0$.

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3. Time-consistent dynamic valuations

3.1 Time consistency properties for financial products

Once a dynamic valuation is defined, the inquiry naturally extends to the temporal behavior of valuations across distinct time instances. The challenge lies in maintaining the hierarchical sequence of position preferences for products evaluated through dynamic valuation to safeguard rational behavior and thereby mitigate the emergence of financial arbitrage opportunities. We will first establish a mathematical formulation of this fundamental property for purely financial products, then broaden the scope to encompass hybrid life Pure Endowment products, with a particular focus on the dynamic Three-step method. Based on Acciaio and Penner (2011), we state the following definition of time-consistency.

Definition 10 (Time-consistent property for financial products). *A dynamic valuation* $(\pi_t^u)_{0\leq t\leq u\leq T}$ *is time-consistent if and only if,* $\forall 0\leq t\leq u, v\leq T$, $\forall\ 0<s\leq u-t,\ 0<s\leq v-t$[8], $\forall\ h\in\mathcal{H}_u^f$, $g\in\mathcal{H}_v^f$,

$$\pi_{t+s}^u[h]\leq\pi_{t+s}^v[g]\ \Rightarrow\pi_t^u[h]\leq\pi_t^v[g]. \tag{16}$$

For instance, the mathematical expectation is a time-consistent dynamic because of the tower rule property. Notably, $(\pi_t^{u,f})_{0\leq t\leq u\leq T}$ is time-consistent. Alternatively, an iteratively backward scheme, closely associated to time-consistency, is frequently used in the literature.

Definition 11 (Iterated property for financial products). *A dynamic valuation* $(\pi_t^u)_{0\leq t\leq u\leq T}$ *is iterated if and only if,* $\forall 0\leq t\leq u\leq T$, $\forall 0<s\leq u-t$, $\forall h\in\mathcal{H}_u^f$,

$$\pi_t^{t+s}\left[\pi_{t+s}^u[h]\right]=\pi_t^u[h]. \tag{17}$$

Remark 1. Each time-consistent dynamic valuation satisfies the equation (17). Moreover, under the monotonicity[9] hypothesis, Definitions 10 and 11 are equivalent, see details in the Appendix.

Through Definition 11, we can progressively devise a backward-looking procedure in a discrete configuration, enabling, when it is not the case initially, the establishment of time-consistency for dynamic valuations that exhibit monotonicity.

Corollary 1. *Let* $\Delta t\in\ ]0,T[$. *Let* $(\pi_t^u)_{0\leq t\leq u\leq T}$ *be a dynamic valuation that adheres to monotonicity but not necessarily time-consistency, then the resulting sequence* $(\tilde{\pi}_t)_t$[10] *constructed according to the outlined procedure satisfies time-consistency.*

1. *First step.*

$$\tilde{\pi}_{T-\Delta t}[h]=\pi_{T-\Delta t}[h].$$

2. *Second step. For* $s=\Delta t,\ 2\Delta t,\ 3\Delta t,\ \ldots$ *and while* $\tilde{s}\stackrel{\Delta}{=}s+\Delta t\leq T$

$$\tilde{\pi}_{T-\tilde{s}}[h]=\pi_{T-\tilde{s}}^{T-s}[\tilde{\pi}_{T-s}[h]],$$

where $h\in\mathcal{H}_T^f$.

[8] The exclusive occurrence of either $s=u-t$ or $s=v-t$ is permissible, prohibiting their simultaneous fulfillment.

[9] A dynamic valuation $(\pi_t^u)_{0\leq t\leq u\leq T}$ satisfies the condition of monotonicity if $\forall 0\leq t\leq u\leq T$, $\forall\ h,g\in\mathcal{H}_u^f$,

$$h\leq g\Rightarrow\pi_t^u[h]\leq\pi_t^u[g].$$

[10] The newly generated time-consistent dynamic $(\tilde{\pi}_t)_t$ is, by construction, assumed to be a sequence of t^T-valuation operators.

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The following subsection will examine time-consistency for hybrid-life Pure Endowment products. Fair – i.e., market-actuarial consistent – valuation operators for these products are frequently neither monotone nor time-consistent; see, for example, the operator of Møller (2002), the Two-step operator of Pelsser and Stadje (2014), the Two-step operator of Barigou *et al.* (2019), and the Three-step operator considered in our work. Consequently, the usual equivalence between the two versions of time-consistency (the two formulations in Definitions 10 and 11) no longer holds. Nonetheless, such counterexamples typically arise only in extreme contexts and thus have limited practical relevance. For the remainder of the paper, we will therefore adopt the iterative version of time-consistency as the reference definition. Moreover, some contributions in the financial literature consider the formulation presented in Definition 11 as the primary definition of time-consistency; see, for instance, Bion-Nadal (2009).

3.2 Time-consistency in hybrid life Pure Endowment products

We investigate a hybrid life insurance product, where the longevity risk corresponds to what is commonly referred to in the literature as a Pure Endowment product. The benefit consists of generating a payment only in case of survival of the insured at a fixed maturity. No payout is made in the event of premature death. For the financial payoff, we consider a broad framework involving any measurable function h of the financial investment fund. Formally, the collective payoff for a maturity $u \leq T$ is

$$ {}_tH_u^c = \left(\sum_{j=1}^{N_t} \mathbb{1}_{\tau_j > u} \right) \times h(S_u). \tag{18} $$

Under mortality risk, the number of survivors decreases over time from the initial instant to maturity. This necessitates ensuring price equilibrium across different time points, taking into account the number of survivors at each point in time. When dealing with such hybrid life Pure Endowment products, this adjustment should be integrated into the iterative time consistency property specified in Definition 11. Hence, Definition 11 extends to hybrid life Pure Endowment products as follows.

Definition 12 (Time-consistency for hybrid life Pure Endowment products). *Consider the collective payoff given by (18), and let ${}_tH_u$ be its individual payoff. A dynamic valuation $(\pi_t^u)_{0 \leq t \leq u \leq T}$ is time-consistent if and only if,*

$$ \pi_t^{t+s}\left[\frac{N_{t+s}}{N_t} \pi_{t+s}^u[{}_{t+s}H_u] \right] = \pi_t^u[{}_tH_u]. \tag{19} $$

In the context of hybrid products embedding non-tradable longevity risk, time-inconsistent valuation does not give rise to exploitable arbitrage opportunities, as the difficulty of trading such risk prevents their effective implementation. Nevertheless, time consistency remains essential for insurers' rational decision-making about the products they offer, as it ensures that choices made today remain viable over time.

3.3 The Three-step method time-consistent dynamic for hybrid life Pure Endowment products

Definition 13 (Time-consistency of the Three-step method). *Consider the collective payoff given by (18). Let ${}_tH_u^1$, ${}_tH_u^2$, and ${}_tH_u^3$ be its Three-step decomposition, as defined respectively in (10), (11), and (12). Then, the dynamic Three-step method is time-consistent if and only if the equation*

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below holds

$$\pi_t^{t+s,3}\left[\frac{N_{t+s}}{N_t}\times\left(\pi_{t+s}^{u,f}\left[{}_{t+s}H_u^1\right]+\pi_{t+s}^{u,a}\left[{}_{t+s}H_u^2\right]+\pi_{t+s}^{u,n}\left[{}_{t+s}H_u^3\right]\right)\right]$$
$$=\pi_t^{u,f}\left[{}_tH_u^1\right]+\pi_t^{u,a}\left[{}_tH_u^2\right]+\pi_t^{u,n}\left[{}_tH_u^3\right], \tag{20}$$

for all $0\le t\le u\le T, 0<s\le u-t$.

We cannot proceed by applying the iterative time-consistency formula to each Three-step component in isolation. This comes from the fact that a unique initial decomposition at maturity u cannot be preserved as time progresses. For instance, while the risk ${}_{t+s}H_u^1$ is hedgeable at time $t+s$, the risk $\left(\pi_{t+s}^{u,f}[{}_{t+s}H_u^1]\times\frac{N_{t+s}}{N_t}\right)$ ceases to be hedgeable at time t. Consequently, applying the financial operator $\pi_t^{t+s,f}$ to $\left(\pi_{t+s}^{u,f}[{}_{t+s}H_u^1]\times\frac{N_{t+s}}{N_t}\right)$ would be inappropriate. Likewise, there is no assurance that the risk $\left(\pi_{t+s}^{u,a}[{}_{t+s}H_u^2]\times\frac{N_{t+s}}{N_t}\right)$ will remain diversifiable at time t, excluding its revaluation through a t^{t+s}-actuarial operator. Viewed from time t,

$${}_tLY_{t+s}\triangleq\frac{N_{t+s}}{N_t}\times\left(\pi_{t+s}^{u,f}\left[{}_{t+s}H_u^1\right]+\pi_{t+s}^{u,a}\left[{}_{t+s}H_u^2\right]+\pi_{t+s}^{u,n}\left[{}_{t+s}H_u^3\right]\right) \tag{21}$$

is a payoff that matures at $t+s$. Hence, once again, it must be decomposed into a hedgeable component, a diversifiable component, and a systematic component to apply the Three-step method and determine the valuation at time t.

As an introductory step, we aim to assess the time-consistency of the Three-step approach, disregarding the diversifiable risk. This results in the Three-step decomposition being simplified to a 'Two-step' model, comprising a hedgeable part and a systematic part, as detailed below

$$\begin{Bmatrix} {}_tH_u^h\triangleq\mathbb{E}_{\mathcal{F}_t\vee\mathcal{F}_u^f}[{}_tH_u]=\mathbb{E}_{\mathcal{F}_t}\left[e^{-\int_t^u\lambda_s\mathrm{d}s}\right]\times h(S_u)\times\mathbb{1}_{N_t>0}, \\ {}_tH_u^{sy}\triangleq{}_tH_u-\mathbb{E}_{\mathcal{F}_t\vee\mathcal{F}_u^f}[{}_tH_u]=\left(\dfrac{N_u}{N_t}-\mathbb{E}_{\mathcal{F}_t}\left[e^{-\int_t^u\lambda_s\,\mathrm{d}s}\right]\right)\times h(S_u)\times\mathbb{1}_{N_t>0}. \end{Bmatrix}$$

Therefore, $\pi_t^{u,3}[{}_tH_u]$ introduced in Definition 9, is reduced to

$$\pi_t^{u,2}[{}_tH_u]\triangleq\pi_t^{u,f}\left[{}_tH_u^h\right]+\pi_t^{u,n}\left[{}_tH_u^{sy}\right]. \tag{22}$$

Lemma 1. *The dynamic* $(\pi_t^{u,2})_{0\le t\le u\le T}$ *is time-consistent; consequently, its time-consistent valuation is given, by the one-period valuation, as follows*

$$\tilde{\pi}_t^2[{}_tH_T|\mathcal{F}_t]=\pi_t^2[{}_tH_T|\mathcal{F}_t]=\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Q}^f}\left[e^{-\int_t^T r_v dv}h(S_T)\right]\times\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a}\left[e^{-\int_t^T\lambda_v dv}\right],$$

with $t\in[0,T]$.

Proof. See the Appendix. □

Next, we include the diversifiable risk into the previous decomposition, finding the full Three-step decomposition as provided below

$$\begin{Bmatrix} {}_tH_u^1=\mathbb{E}_{\mathcal{F}_t}\left[e^{-\int_t^u\lambda_s\mathrm{d}s}\right]\times h(S_u)\times\mathbb{1}_{N_t>0}, \\ {}_tH_u^2=\left(\dfrac{N_u}{N_t}-e^{-\int_t^u\lambda_s\mathrm{d}s}\right)\times h(S_u)\times\mathbb{1}_{N_t>0}, \\ {}_tH_u^3=\left(e^{-\int_t^u\lambda_s\mathrm{d}s}-\mathbb{E}_{\mathcal{F}_t}\left[e^{-\int_t^u\lambda_s\mathrm{d}s}\right]\right)\times h(S_u)\times\mathbb{1}_{N_t>0}. \end{Bmatrix}$$

Lemma 2. *The dynamic Three-step method* $(\pi_t^{u,3})_{0\le t\le u\le T}$ *introduced in Definition 9, is not time-consistent.*

Proof. See the Appendix. □

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Since the dynamic Three-step method is not time-consistent, we strive to iteratively construct a sequence of time-consistent valuations by applying Corollary 1 in line with Definition 13.

Corollary 2. *Let $\Delta t \in]0, T[$. Let $(\pi_t^{u,3})_{0\leq t\leq u\leq T}$ be the dynamic Three-step method defined in Definition 9, then the resulting sequence $(\tilde{\pi}_t^3)_t$ constructed according to the outlined procedure is time-consistent*

1. *First step.*

$$\tilde{\pi}^3_{T-\Delta t}[{}_{T-\Delta t}H_T] = \pi^3_{T-\Delta t}[{}_{T-\Delta t}H_T].$$

2. *Second step. For $s = \Delta t,\ 2\Delta t,\ 3\Delta t,\ \ldots$ and while $\tilde{s} \triangleq s + \Delta t \leq T$*

$$\tilde{\pi}^3_{T-\tilde{s}}[{}_{T-\tilde{s}}H_T] = \pi^{T-s,3}_{T-\tilde{s}}\left[\frac{N_{T-s}}{N_{T-\tilde{s}}} \times \tilde{\pi}^3_{T-s}[{}_{T-s}H_T]\right], \tag{23}$$

for $_tH_u = \frac{\left(\sum_{j=1}^{N_t} \mathbb{1}_{\tau_j > u}\right)\times h(S_u)}{N_t}$ *and where* $0 \leq t \leq u \leq T$.

To derive explicit time-consistent valuations based on Corollary 2, it is essential to specify the dynamics of $(\lambda_t)_{0\leq t\leq T}$, $(S_t)_{0\leq t\leq T}$ and $(r_t)_{0\leq t\leq T}$. Under $\mathbb{P}^a$, the stochastic mortality rate λ_t follows a stochastic solution of the following equation commonly recognized as the Ornstein–Uhlenbeck intensity process (see Luciano & Vigna, 2008)

$$d\lambda_t = \mu_m\lambda_t dt + \sigma_m dW_m(t),$$

where $(W_m(t))_{t\geq 0}$ is a Brownian motion and $\mu_m > 0, \sigma_m > 0$. Thus, for $0 \leq t \leq u \leq T$, ${}_{u-t}p_{x+t} = e^{-\lambda_t l_{t,u} + \frac{1}{2}s^2_{t,u}}$, where $l_{t,u} = \frac{e^{\mu_m(u-t)}-1}{\mu_m}$, and $s^2_{t,u} = \frac{\sigma_m^2}{2\mu_m^3}(1-e^{\mu_m(u-t)})^2 + \frac{\sigma_m^2}{\mu_m^2}((u-t) - \frac{e^{\mu_m(u-t)}-1}{\mu_m})$. The parameter $\theta^{(1)}_{t,u}$ present in (15) is adjusted in accordance with the Cost of Capital approach detailed in Belhouari et al. (2025), producing $\theta^{(1)}_{t,u} = \frac{-0.16}{s_{t,u}}$.

The dynamics of the couple stochastic investment fund and stochastic interest rates will be characterized using the following Black-Scholes-Vasicek model, described under $\mathbb{P}^f$ by

$$\left\{\begin{array}{l} dr_t = (b - ar_t)dt + \sigma_r d\widetilde{W}_r(t), \\ dS_t = \mu_S S_t dt + S_t\sigma_S(\rho d\widetilde{W}_r(t) + \sqrt{1-\rho^2}d\widetilde{W}_S(t)). \end{array}\right\}.$$

where $(\tilde{W}_S(t))_{t\geq 0}$ is a Brownian motion independent from $(\tilde{W}_r(t))_{t\geq 0}$, a is the constant speed of mean reversion, b represents a strictly positive scalar such that the ratio $\frac{b}{a}$ is the long-term mean value and σ_r is the constant volatility of interest rates. μ_S is the instantaneous return of $(S_t)_{t\geq 0}$, σ_S is the volatility of the financial investment fund, and ρ defines the correlation between the financial investment fund and interest rates. Switching to the risk-neutral probability involves the application of Girsanov's theorem, which defines the Brownian motions $(W_r(t))_t$ and $(W_S(t))_t$ in the following manner $\mathbb{P}^f$

$$\left\{\begin{array}{l} W_r(t) \triangleq \widetilde{W}_r(t) - \nu t, \\ W_S(t) \triangleq \widetilde{W}_S(t) + \int_0^t \left(\dfrac{\mu_S - r_u}{\sqrt{1-\rho^2}\sigma_S} + \dfrac{\nu\rho}{\sqrt{1-\rho^2}}\right) du. \end{array}\right\}.$$

Drawing upon this, the Radon-Nikodym derivative of $\mathbb{Q}^f$ with respect to $\mathbb{P}^f$ is

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}^f}\bigg|_{\mathcal{F}_t} = e^{\nu\widetilde{W}_r(T) - \int_t^T \left(\frac{\mu_S - r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu\rho}{\sqrt{1-\rho^2}}\right) d\widetilde{W}_S(u) - \frac{1}{2}\int_t^T \left(\nu^2 + \left(\frac{\mu_S - r_u}{\sqrt{1-\rho^2}\sigma_S} + \frac{\nu\rho}{\sqrt{1-\rho^2}}\right)^2\right) du}.$$

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Hence, the following dynamics are obtained under $\mathbb{Q}^f$

$$\begin{cases} dr_t = (\theta - ar_t)dt + \sigma_r dW_r(t), \\ dS_t = r_t S_t dt + S_t \sigma_S(\rho dW_r(t) + \sqrt{1-\rho^2} dW_S(t)), \end{cases}$$

with $\theta = b + \sigma_r \nu$.

3.4 The backward scheme in a dual-iteration approach

The achievement of closed forms through the recursive equation (23) with a substantial number of iterations, given the stochastic risks on both the financial and mortality fronts, as assumed in our study, poses a formidable challenge, if not an unattainable one. To simplify the problem, we consider two periods $[0, b]$ and $[b, T]$ in a constant interest rate framework,[11] and where we set $h(S_u) = 1$. In this case, we are delving into the purely actuarial, Pure Endowment product. Owing to the requirements of derivation and computation, we will use the variance premium principle, defined in (8), as the operator for evaluating the diversifiable risk component of the Three-step method. This choice enables the derivation of a closed-form solution, unlike the standard deviation premium principle. In light of the newly introduced denotations

$$\begin{aligned} {}_i\tilde{f}_j &= e^{-i\lambda_0 l_{b,T} e^{\mu_m b} + j\frac{(e^{\mu_m b}+1)}{2}\frac{l_{b,T}^2 l_{0,b}\sigma_m^2}{2} + j\frac{\theta_{0,b}^{(1)} l_{0,b}^2 l_{b,T}\sigma_m^2}{2}}, \\ {}_if_k^j &= e^{-i\lambda_0 l_{0,b} + j\frac{l_{b,T} l_{0,b}^2 \sigma_m^2}{2} + k\frac{1}{2}s_{0,b}^2}, \end{aligned}$$

we hereby present the following lemma.

Lemma 3. *The Three-step valuation of the Pure Endowment product at the instant b is* $\tilde{\pi}_b^3[{}_bH_T|\mathcal{F}_b] = P(b, T)\left(e^{-\lambda_b l_{b,T} + \frac{1}{2}s_{b,T}^2} e^{-\theta_{b,T}^{(1)} s_{b,T}^2} + \gamma_b \times \frac{e^{-\lambda_b l_{b,T} + \frac{1}{2}s_{b,T}^2} - e^{-2\lambda_b l_{b,T} + 2s_{b,T}^2}}{N_b}\right)$. *While the time-consistent valuation at the instant 0 is*

$$\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] = \Gamma_0^{(h,sy)} + \Gamma_0^{(div)},$$

where

$$\begin{aligned} \Gamma_0^{(h,sy)} &= P(0, T)\left(e^{-\lambda_0 l_{0,T} - s_{0,T}^2(\theta_{0,T}^{(1)} - \frac{1}{2})} + \gamma_b \frac{e^{\frac{1}{2}s_{b,T}^2}\,{}_1\tilde{f}_1 - e^{2s_{b,T}^2}\,{}_2\tilde{f}_4}{n}\right), \\ \Gamma_0^{(div)} &= \frac{1}{n} \times \gamma_0 \times P(0, b)P(b, T)^2 \times e^{-2s_{b,T}^2\left(\theta_{b,T}^{(1)} - \frac{1}{2}\right)}\left({}_1f_1^2 - {}_2f_4^8\right), \\ \theta_{b,T}^{(1)} &= \frac{-0.16}{s_{b,T}}, \\ \theta_{0,b}^{(1)} &= \frac{-0.16}{s_{0,b}}. \end{aligned}$$

Proof. See the Appendix. □

A notable aspect is that the aggregated non-diversifiable risks, including systematic and hedgeable components of the time-consistent formula at time 0, incorporate loading terms arising from the valuation of the diversifiable component at time b; as shown in $\Gamma_0^{(h,sy)}$. This implies that, during the backward scheme, the valuation of the diversifiable component from the prior iteration impacts the valuation of the subsequent non-diversifiable components. Moreover, observe that, under perfect diversification of the portfolio, the diversifiable risk disappears. Correspondingly,

[11] i.e $P(t, T) = e^{-r_0(T-t)}$.

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the closed-form Three-step time-consistent price $\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0]$ in Lemma 3 converges to the Two-step static price, where it reduces to the best estimate, i.e., $(P(0,T)\times e^{-\lambda_0 l_{0,T}-s_{0,T}^2(\theta_{0,T}^{(1)}-\frac{1}{2})})$; thereby explicitly confirming that the diversifiable risk component operates to undermine time-consistency.

4. Partial differential equation for the Three-step method time-consistent dynamic

4.1 Derivation

In the preceding section, several simplistic measures were used to derive the time-consistent valuation across two periods. To comprehensively unveil the repercussions of time consistency on pricing, a more sophisticated numerical approach surpassing two iterations is imperative. This section focuses on studying the continuous-time limit of the Three-step method when integrated into the backward scheme. In order to accomplish this, and following Pelsser and Ghalehjooghi (2016), we will derive PDE[12] of the Three-step method time-consistent dynamic, incorporating all the stochastic processes of mortality and finance considered in our study. Considering an infinitesimally small time increment dt that tends towards 0, we position ourselves within the $(t, t+dt)$ interval and employ

$$\tilde{\pi}_t^3 = \pi_t^{t+dt,3}\left[\frac{N_{t+dt}}{N_t}\times\tilde{\pi}_{t+dt}^3\right] \tag{24}$$

This time, we purposefully adopt the standard deviation premium principle to assess the actuarial component of the Three-step method. Interestingly, the variance principle enables the generation of a similar PDE, reflecting also a time-consistent price.

Theorem 1. *Let $t^{(1)}$ be a strictly positive real scalar. The Three-step method time-consistent dynamic $(\tilde{\pi}_t^3)_{t\in[0,T-t^{(1)}]}$ is a solution of the following PDE*

$$r_t\tilde{\pi}_t^3 = \Lambda_t^{(1)} + \alpha_t \times \sqrt{\Lambda_t^{(2)}}, \tag{25}$$

with

$$\begin{aligned}
\Lambda_t^{(1)} &= \partial_t\tilde{\pi}_t^3 + r_tS_t\partial_S\tilde{\pi}_t^3 + (\theta - ar_t)\partial_r\tilde{\pi}_t^3 + \left(\overline{\theta_{t,T}^{(1)}} + \mu_m\lambda_t\right)\partial_\lambda\tilde{\pi}_t^3 + \frac{1}{2}\sigma_S^2S_t^2\partial_S^2\tilde{\pi}_t^3 + \frac{1}{2}\sigma_m^2\partial_\lambda^2\tilde{\pi}_t^3\\
&\quad + \frac{1}{2}\sigma_r^2\partial_r^2\tilde{\pi}_t^3 + \rho\sigma_r\sigma_SS_t\partial_{rS}^2\tilde{\pi}_t^3 + \left(\tilde{\pi}_t^{3*} - \tilde{\pi}_t^3\right)\lambda_t,\\
\Lambda_t^{(2)} &= \sigma_S^2S_t^2\left(\partial_S\tilde{\pi}_t^3\right)^2 + \sigma_m^2\left(\partial_\lambda\tilde{\pi}_t^3\right)^2 + \sigma_r^2\left(\partial_r\tilde{\pi}_t^3\right)^2 + 2\rho\sigma_S\sigma_rS_t\partial_S\tilde{\pi}_t^3\partial_r\tilde{\pi}_t^3 + \left(\tilde{\pi}_t^3 - \tilde{\pi}_t^{3*}\right)^2\lambda_t,\\
\overline{\theta_{t,T}^{(1)}} &= \theta_{0,T}^{(1)}\sigma_m^2 \times \frac{e^{\mu_m(T-t)}-1}{\mu_m},\\
\tilde{\pi}_t^{3*}[.] &= \tilde{\pi}_t^3[.|N_t - 1].
\end{aligned}$$

Proof. See the Appendix. □

Represented by $\Lambda_t^{(1)}$, the first component of the PDE captures the time-consistent price of the aggregate replicable and systematic risks. The subsequent part $\left(\alpha_t\times\sqrt{\Lambda_t^{(2)}}\right)$, relates to the time-consistent value of the diversifiable risk.

Contrary to the time-consistent case, the static continuous-time valuation framework circumvents the requirement for a PDE. Indeed, a closed-form expression is readily tractable and is given

[12] We will proceed without including the payoff in the PDE.

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by the following formula

$$\pi_t^3[{}_tH_T] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}\left[e^{-\int_t^T r_s \mathrm{d}s}\, {}_tH_T\right] + \alpha_t \times \sqrt{\mathbb{E}_{\mathcal{F}_t}[h(S_T)^2]} \times \sqrt{\frac{\mathbb{E}_{\mathcal{F}_t}[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_T}[g(\tau_1)]]}{N_t}} \times \sqrt{T-t} \times P(t,T), \tag{26}$$

for all $t \in [0, T - t^{(1)}]$, with $t^{(1)}$ denoting a strictly positive real scalar.

Let us remark that our analysis of time-consistency shall be conducted over the interval $[0, T - t^{(1)}]$. Consequently, this entails a static transition from time T to time $T - t^{(1)}$. Moreover, we shall exploit the Three-step closed-form solution in the static framework and adopt, as the terminal condition, the Three-step valuation at time $T - t^{(1)}$ of the payoff ${}_{T-t^{(1)}}H_T$ maturing at T. The rationale behind this circumvention will be explained in detail in the subsequent subsection, which outlines the numerical resolution scheme.

4.2 Numerical scheme for the resolution of the PDE using a Pure Endowment product with terminal bonus

We will concentrate our inquiry on a hybrid life product identified as the Pure Endowment with terminal bonus. Within this setting, a financial profit-sharing element is incorporated in addition to the actuarial yield of the Pure Endowment product. In the event that the financial investment fund exceeds the capitalized premium P by the insurer's technical rate i, profit-sharing between the insurer and the insured will occur, given by $h(S_T) = C \times (1 + \beta(S_T - P(1+i)^T)^+)$, where β represents the profit-sharing rate, and $C > 0$ is the guaranteed minimum capital. Therefore, the defined formulation for the hybrid collective payoff is as follows $H_T^c = N_T \times C \times (1 + \beta(S_T - P(1 + i)^T)^+)$. Working on a first-order basis, the premium P is calculated through $P = (\frac{1}{1+i})^T \times {}_T\tilde{p}_x$, where ${}_T\tilde{p}_x$[13] signifies the prudent survival probability, and it is entirely allocated to the financial investment fund, resulting in $S_0 = P$. Here is the unfolding of the Three-step decomposition of the Pure Endowment product with a terminal bonus

$$\begin{cases} {}_{T-t^{(1)}}H_T^1 = \mathbb{E}_{\mathcal{F}_{T-t^{(1)}}}\left[e^{-\int_{T-t^{(1)}}^T \lambda_s \mathrm{d}s}\right] \times h(S_T) \times \mathbb{1}_{N_{T-t^{(1)}}>0}, \\ {}_{T-t^{(1)}}H_T^2 = \left(\frac{N_T}{N_{T-t^{(1)}}} - e^{-\int_{T-t^{(1)}}^T \lambda_s \mathrm{d}s}\right) \times h(S_T) \times \mathbb{1}_{N_{T-t^{(1)}}>0}, \\ {}_{T-t^{(1)}}H_T^3 = \left(e^{-\int_{T-t^{(1)}}^T \lambda_s \mathrm{d}s} - \mathbb{E}_{\mathcal{F}_{T-t^{(1)}}}\left[e^{-\int_{T-t^{(1)}}^T \lambda_s \mathrm{d}s}\right]\right) \times h(S_T) \times \mathbb{1}_{N_{T-t^{(1)}}>0}, \end{cases}$$

As stated in Lemma 2, the dynamic Three-step method is not time-consistent because of the presence of diversifiable actuarial risks. Consequently, we will numerically address the PDE stated in Theorem 1 using the finite difference method to obtain the time-consistent price. Firstly, we discretize $(t)_t$, $(r_t)_t$, $(S_t)_t$, $(\lambda_t)_t$ and $(N_t)_t$ on the following finite grids $[0, T - t^{(1)}]$, $[r_{\min}, r_{\max}]$, $[0, S_{\max}]$, $[0, \lambda_{\max}]$, and $[0, n]$, respectively. The parameters $r_{\min}$, $r_{\max}$, $S_{\max}$, and $\lambda_{\max}$ are configured to be attained with a probability that is very low.[14] We define the discretization step by $\Delta\zeta = \frac{\zeta_{\max} - \zeta_{\min}}{M_\zeta}$, where $\zeta \in \{t, S, r, \lambda, N\}$ and M_ζ denotes a strictly positive integer that represents the number of steps in the grid for ζ. Furthermore, we denote the time-consistent option price at

[13] The survival probability ${}_T\tilde{p}_x$ is computed according to Makeham's formula using the parameters from the Belgian Royal Decree.

[14] $\lambda_{\max} = 0.02$, $r_{\min} = -0.1$, $r_{\max} = 0.03$, $S_{\max} = 7$.

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time $i\Delta t$, for an underlying asset price of $j_S\Delta S$, with a mortality intensity value of $j_\lambda\Delta\lambda$, an interest rate value of $(r_{\min}+j_r\Delta r)$, and a current count of surviving policyholders $j_N\Delta N$ by $\tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda}$, where $j_{\tilde{\zeta}} \in \{0,1,\ldots,M_{\tilde{\zeta}}\}$ for $\tilde{\zeta} \in \{S,r,\lambda,N\}$ and $i \in \{0,1,\ldots,M_t\}$.

Given that the risk factors $\tilde{M_t}^{(1)} = \frac{N_t}{N_{t-t^{(1)}}}$ and $\tilde{M_t}^{(2)} = e^{-\int_{t-t^{(1)}}^{t}\lambda_s \mathrm{d}s}$ are not Markovian processes,[15] we establish the terminal condition at the instant $T-t^{(1)}$ instead of at T. The definitive formula below follows from the closed-form expression in (26), the valuation of the call option under stochastic interest rates, and the log-normal distribution of the investment fund under $\mathbb{P}^f$

$$
\begin{aligned}
\tilde{\pi}^3_{T-t^{(1)}}[{}_{T-t^{(1)}}H_T] = {} & C \times \Big(P\big(T-t^{(1)},T\big) + \beta\Big(S_{T-t^{(1)}}\Phi\big(d_{T-t^{(1)}}\big) - P\big(T-t^{(1)},T\big)PG \\
& \times \Phi\Big(d_{T-t^{(1)}} - n\big(T-t^{(1)},T\big)\Big)\Big)\Big) \times e^{-\lambda_{T-t^{(1)}} l_{T-t^{(1)},T} + \frac{1}{2}s^2_{T-t^{(1)},T}} \\
& \times e^{-\theta^{(1)}_{T-t^{(1)},T} s^2_{T-t^{(1)},T}} + C \times \alpha_{T-t^{(1)}}\sqrt{t^{(1)}}P\big(T-t^{(1)},T\big)\Big[1 + 2\beta\Big(S_{T-t^{(1)}} \\
& \times e^{\mu_S t^{(1)}}\Phi(\delta_{T-t^{(1)}}) - PG\Phi\Big(\delta_{T-t^{(1)}} - \sigma_S\sqrt{t^{(1)}}\Big)\Big) + \beta^2\Big(S^2_{T-t^{(1)}}e^{2\left(\mu_S + \frac{1}{2}\sigma_S^2\right)t^{(1)}} \\
& \times \Phi\Big(\delta_{T-t^{(1)}} + \sigma_S\sqrt{t^{(1)}}\Big) + (PG)^2\Phi\Big(\delta_{T-t^{(1)}} - \sigma_S\sqrt{t^{(1)}}\Big) - 2PGS_{T-t^{(1)}}e^{\mu_S t^{(1)}} \\
& \times \Phi\Big(\delta_{T-t^{(1)}}\Big)\Big)\Big]^{\frac{1}{2}} \times \mathbb{1}_{N_{T-t^{(1)}}>0} \\
& \times \sqrt{\frac{1}{N_{T-t^{(1)}}}\Big(e^{-\lambda_{T-t^{(1)}} l_{T-t^{(1)},T} + \frac{1}{2}s^2_{T-t^{(1)},T}} - e^{-2\lambda_{T-t^{(1)}} l_{T-t^{(1)},T} + 2s^2_{T-t^{(1)},T}}\Big)},
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_{T-t^{(1)}} &= \frac{\left(\mu_S + \frac{1}{2}\sigma_S^2\right)t^{(1)} + ln\left(\frac{S_{T-t^{(1)}}}{PG}\right)}{\sigma_S\sqrt{t^{(1)}}}, \\
\theta^{(1)}_{T-t^{(1)},T} &= \frac{-0.16}{s_{T-t^{(1)},T}}, \\
d_{T-t^{(1)}} &= \frac{\frac{1}{2}n\big(T-t^{(1)},T\big)^2 + ln\left(\frac{S_{T-t^{(1)}}}{PG}\right) - ln\left(P\big(T-t^{(1)},T\big)\right)}{n\big(T-t^{(1)},T\big)}, \\
n\big(T-t^{(1)},T\big)^2 &= \sigma_S^2 t^{(1)} + \frac{\sigma_r^2}{a^2}t^{(1)} + \frac{\sigma_r^2}{a^3}\left(\frac{-3}{2} - \frac{1}{2}e^{-2at^{(1)}} + 2e^{-at^{(1)}}\right) + \frac{2\rho\sigma_S\sigma_r t^{(1)}}{a} \\
&\quad - \frac{2\rho\sigma_S\sigma_r\big(1-e^{-at^{(1)}}\big)}{a^2}.
\end{aligned}
$$

The approximations for the partial derivatives existing in Theorem 1 are detailed as follows

$$
\partial_t\tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \frac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda}}{\Delta t},\ i < M_t
$$

[15]The derivation of the PDEs is based on Ito's lemma, whose formulation is predicated upon the foundational tenets of Markovian assumptions governing the risk factors $(S_t)_t$, $(r_t)_t$, $(\lambda_t)_t$, $(N_t)_t$.

$$\partial_S \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \begin{cases} \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S-2,j_r,j_\lambda}}{2\Delta S} & \text{if } j_S = M_S \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S+2,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}}{2\Delta S} & \text{if } j_S = 0 \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda}}{2\Delta S} & \text{else} \end{cases}$$

$$\partial_r \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \begin{cases} \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r-2,j_\lambda}}{2\Delta r} & \text{if } j_r = M_r \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r+2,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}}{2\Delta r} & \text{if } j_r = 0 \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda}}{2\Delta r} & \text{else} \end{cases}$$

$$\partial_\lambda \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \begin{cases} \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda-2}}{2\Delta \lambda} & \text{if } j_\lambda = M_\lambda \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda+2} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}}{2\Delta \lambda} & \text{if } j_\lambda = 0 \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda+1} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda-1}}{2\Delta \lambda} & \text{else} \end{cases}$$

$$\partial^2_S \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \begin{cases} \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S-2,j_r,j_\lambda}}{\Delta S^2} & \text{if } j_S = M_S \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S+2,j_r,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}}{\Delta S^2}, & \text{if } j_S = 0 \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda}}{\Delta S^2} & \text{else} \end{cases}$$

$$\partial^2_r \tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} = \left\{ \begin{array}{ll} \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S,j_r-2,j_\lambda}}{\Delta r^2} & \text{if } j_r = M_r \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r+2,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}}{\Delta r^2} & \text{if } j_r = 0 \\ \dfrac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} - 2\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda}}{\Delta r^2} & \text{else} \end{array} \right\}$$

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$$\partial_{\lambda}^{2}\tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N}=\left\{\begin{array}{ll}\dfrac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}-2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{j_N}+\tilde{\pi}_{i+1,j_S,j_r,j_\lambda-2}^{j_N}}{\Delta\lambda^2} & \text{if } j_\lambda=M_\lambda\\ \dfrac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+2}^{j_N}-2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{j_N}+\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}}{\Delta\lambda^2} & \text{if } j_\lambda=0\\ \dfrac{\tilde{\pi}_{i+1,j_S,j_r,j_\lambda+1}^{j_N}-2\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}+\tilde{\pi}_{i+1,j_S,j_r,j_\lambda-1}^{j_N}}{\Delta\lambda^2} & \text{else}\end{array}\right\}$$

$$(\tilde{\pi}^*-\tilde{\pi})_{i,j_S,j_r,j_\lambda}^{j_N}=\left\{\begin{array}{ll}0 & \text{if } j_N=0\\ \left(\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N-1}-\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}\right) & \text{else}\end{array}\right\}$$

$$\partial_{rS}^{2}\tilde{\pi}_{i,j_S,j_r,j_\lambda}^{j_N}=\left\{\begin{array}{ll}
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-2,j_r-2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=M_S\ \&\ j_r=M_r\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-2,j_r-1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-2,j_r+1,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=M_S\ \&\ j_r\neq M_r,0\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+1,j_r-2,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-1,j_r-2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S\neq M_S,0\ \&\ j_r=M_r\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+2,j_r-2,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S,j_r-2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=0\ \&\ j_r=M_r\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-2,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-2,j_r+2,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=M_S\ \&\ j_r=0\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+2,j_r+2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+2,j_r,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r+2,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=0\ \&\ j_r=0\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+2,j_r+1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+2,j_r-1,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S,j_r-1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S,j_r+1,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S=0\ \&\ j_r\neq 0,M_r\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+1,j_r+2,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+1,j_r,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-1,j_r,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-1,j_r+2,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{if } j_S\neq 0,M_S\ \&\ j_r=0\\
\dfrac{\begin{array}{c}\tilde{\pi}_{i+1,j_S+1,j_r+1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S+1,j_r-1,j_\lambda}^{j_N}\\+\tilde{\pi}_{i+1,j_S-1,j_r-1,j_\lambda}^{j_N}-\tilde{\pi}_{i+1,j_S-1,j_r+1,j_\lambda}^{j_N}\end{array}}{4\Delta S\Delta r} & \text{else}
\end{array}\right.$$

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Excluding the boundary conditions, the backward recursive scheme assumes the ensuing structure for $i \in \{0, 1, \ldots, M_t - 1\}$

$$\begin{aligned}\tilde{\pi}^{j_N}_{i,j_S,j_r,j_\lambda} &= b_{j_S,j_r}\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda} + c^{+}_{j_S,j_r}\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} - c^{-}_{j_S,j_r}\tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda} + d^{+}_{j_r}\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} \\ &\quad - d^{-}_{j_r}\tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda} + \left(\overline{\theta^{(1)}_{i\Delta t,T-t^{(1)}}}\frac{g_{j_r}}{2\Delta\lambda} + e^{+}_{j_r,j_\lambda}\right)\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda+1} - \left(\overline{\theta^{(1)}_{i\Delta t,T-t^{(1)}}}\frac{g_{j_r}}{2\Delta\lambda} + e^{-}_{j_r,j_\lambda}\right) \\ &\quad \times \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda-1} + g_{j_r}P^{j_N}_{i+1,j_S,j_r,j_\lambda} + g_{j_r}\left(\tilde{\pi}^{j_N-1}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}\right) j_\lambda \Delta\lambda \\ &\quad + \alpha_{i\Delta t} g_{j_r}\sqrt{S^{j_N}_{i+1,j_S,j_r,j_\lambda} + \left(\tilde{\pi}^{j_N-1}_{i+1,j_S,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda}\right)^2 j_\lambda \Delta\lambda}.\end{aligned}$$

where $b_{j_S,j_r} = \frac{1-\sigma_S^2 j_S^2\Delta t - \frac{\sigma_m^2}{\Delta\lambda^2}\Delta t - \frac{\sigma_r^2\Delta t}{\Delta r^2}}{1+(r_{\min}+j_r\Delta r)\Delta t}$, $c^{+-}_{j_S,j_r} = \frac{\frac{1}{2}(r_{\min}+j_r\Delta r)j_S\Delta t +- \frac{1}{2}\sigma_S^2 j_S^2\Delta t}{1+(r_{\min}+j_r\Delta r)\Delta t}$, $d^{+-}_{j_r} = \frac{\frac{(\theta - a(r_{\min}+j_r\Delta r)\Delta t}{2\Delta r} +- \frac{\sigma_r^2\Delta t}{2\Delta r^2}}{1+(r_{\min}+j_r\Delta r)\Delta t}$, $e^{+-}_{j_r,j_\lambda} = \frac{\frac{1}{2}\mu_m j_\lambda\Delta t +- \frac{1}{2}\frac{\sigma_m^2\Delta t}{\Delta\lambda^2}}{1+(r_{\min}+j_r\Delta r)\Delta t}$, $g_{j_r} = \frac{\Delta t}{1+(r_{\min}+j_r\Delta r)\Delta t}$, $P^{j_N}_{i+1,j_S,j_r,j_\lambda} = \rho\sigma_r\sigma_S j_S \frac{\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r+1,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S+1,j_r-1,j_\lambda} + \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r-1,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r+1,j_\lambda}}{4\Delta r}$ and $S^{j_N}_{i+1,j_S,j_r,j_\lambda} = (\sigma_S j_S)^2 \frac{\left(\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda}\right)^2}{4} + \sigma_m^2 \frac{\left(\tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda+1} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r,j_\lambda-1}\right)^2}{4\Delta\lambda^2} + \sigma_r^2 \frac{\left(\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda}\right)^2}{4\Delta r^2} + \rho\sigma_r\sigma_S j_S \frac{\tilde{\pi}^{j_N}_{i+1,j_S,j_r+1,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S,j_r-1,j_\lambda}}{2\Delta r}\left(\tilde{\pi}^{j_N}_{i+1,j_S+1,j_r,j_\lambda} - \tilde{\pi}^{j_N}_{i+1,j_S-1,j_r,j_\lambda}\right)$.

It is important to note that enforcing time consistency through the backward-iterative scheme defined in (23) does not require the same risk measure operator to be applied at every iteration. Hence, the $\alpha_{i\Delta t}$-loading parameter of this backward algorithm does not need to remain identical across periods and can be recalibrated over time between periods.

4.3 Numerical results

We consider a cohort of $n = 100$ females aged 35 contracting a Pure Endowment with a terminal bonus. Our numerical application follows the assumptions used in Belhouari et al. (2025), where the parameters characterizing our financial-actuarial models can be summarized as follows (see Table 1).

Prior to elucidating the numerical results, we must preliminarily verify the convergence of the finite difference method. Since the discretization step ΔN remains invariably equal to 1,[16] it obviates the need for any calibration of the number of steps concerning the number of survivors. As depicted in Figure 1, the convergence of time is reached after 989 iterations, implying that approximately one iteration occurs every three days and one tier. Figures 2–4 respectively investigate the convergence of interest rates, mortality rates, and the financial investment fund. The stability of the numerical values is deemed sufficient for $M_S = 35$, $M_r = 40$ and $M_\lambda = 40$. Figure 5 contrasts the time-consistent values of the Three-step method with its static counterparts and the best estimate calculated under $\mathbb{Z}^a_\theta \times \mathbb{Q}^f$. The dynamic Three-step method follows a monotonic trend analogous to that of the best estimate, reflecting a price upward over time. Furthermore, it demonstrates that the static values of the dynamic Three-step method exhibit a modest increase, ultimately resulting in a 3.7% cumulative growth by time 0. By including the diversifiable part, the dynamic Three-step method loses its time-consistent property, which it previously maintained with a valuation formula equivalent to $\mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}[e^{-\int_t^T r_s ds}\, {}_tH_T]$, as referenced in Lemma 1. Therefore, to better understand the effect of time-consistency, it is important to compare the time-consistent Three-step prices, net of this best estimate, with the one-period Three-step prices,

[16] i.e. $M_N = n$.

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Table 1. Numerical inputs

Parameters	Numerical Values
α_t	15%
i	0.9%
C	100, 000
T	10
P	0.9070291
r_0	1.6%
a	40%
θ	0.8%
σ_r	0.5%
σ_S	15%
ρ	20%
μ_S	5%
β	40.19%
λ_0	0.001080584
μ_m	0.0745323698
σ_m	0.0002248628

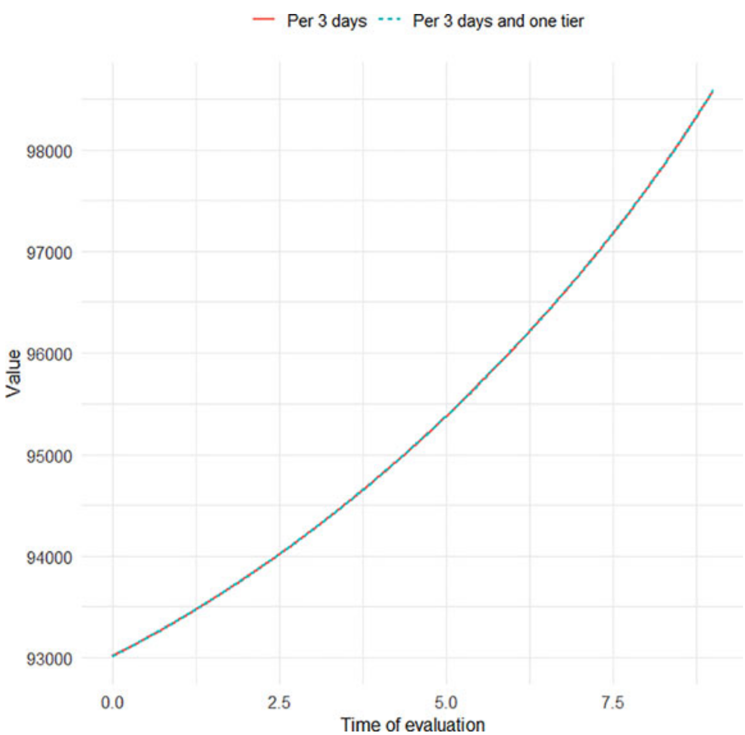

Figure 1. The Three-step method time-consistent dynamic with $M_S = 35$, $M_r = 40$, $M_\lambda = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0, \lambda_t = \lambda_0, N_t = n$. $(t^{(1)} = 1)$.

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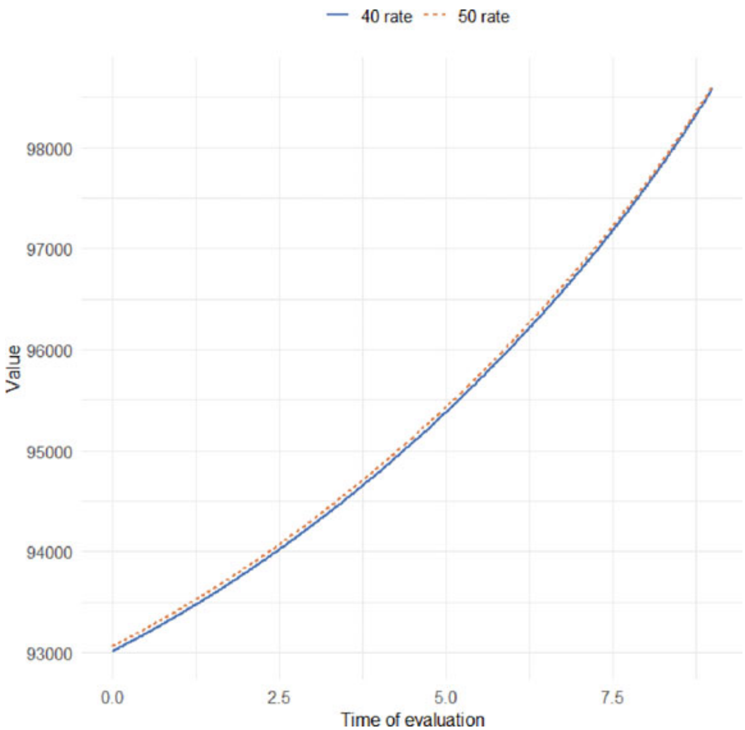

Figure 2. The Three-step method time-consistent dynamic with $M_t = 989$, $M_S = 35$, $M_\lambda = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$).

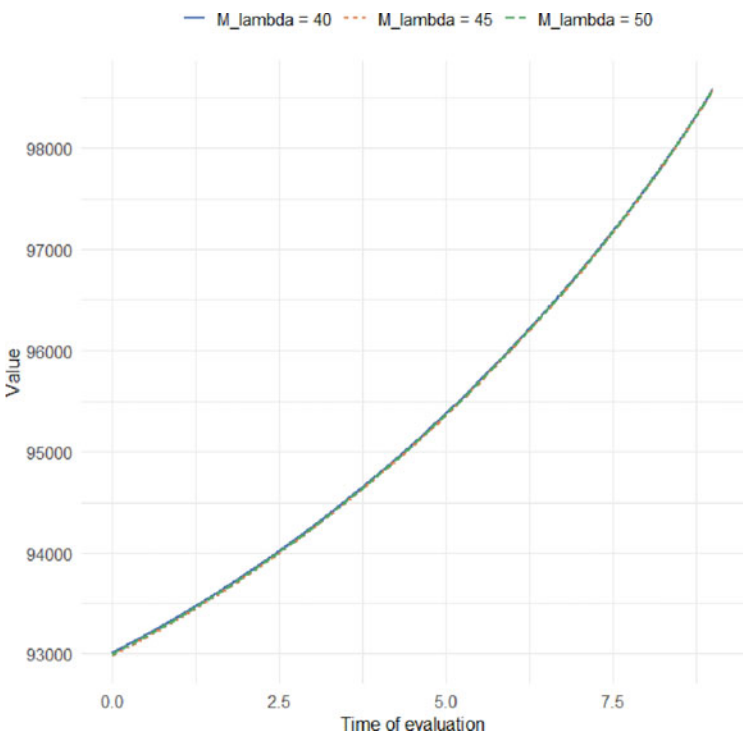

Figure 3. The Three-step method time-consistent dynamic with $M_t = 989$, $M_S = 35$, $M_r = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0$, $\lambda_t = \lambda_0$, $N_t = n$. ($t^{(1)} = 1$).

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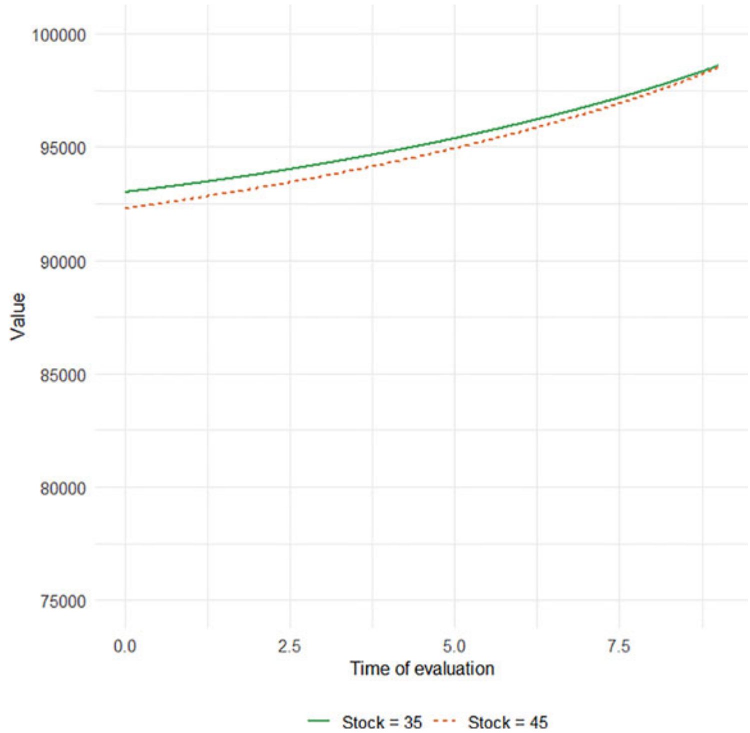

Figure 4. The three-step method time-consistent dynamic with $M_t = 989$, $M_\lambda = 40$, $M_r = 40$, $M_N = n = 100$, and, $S_t = S_0$, $r_t = r_0, \lambda_t = \lambda_0, N_t = n$. ($t^{(1)} = 1$).

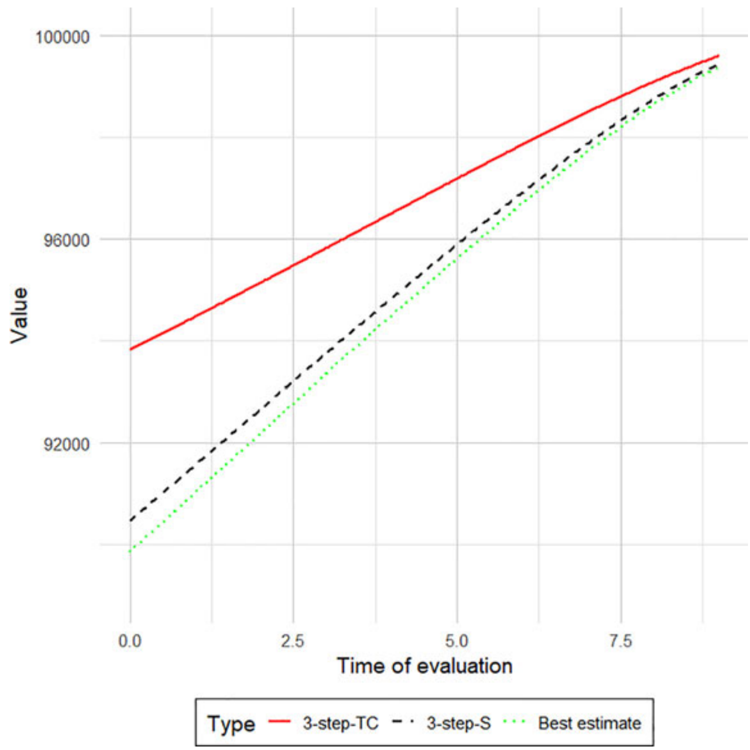

Figure 5. The dynamic Three-step method in the static versus time-consistent case and the best estimate under $\mathbb{Z}^a_\theta \times \mathbb{Q}^f$. ($t^{(1)} = 1$).

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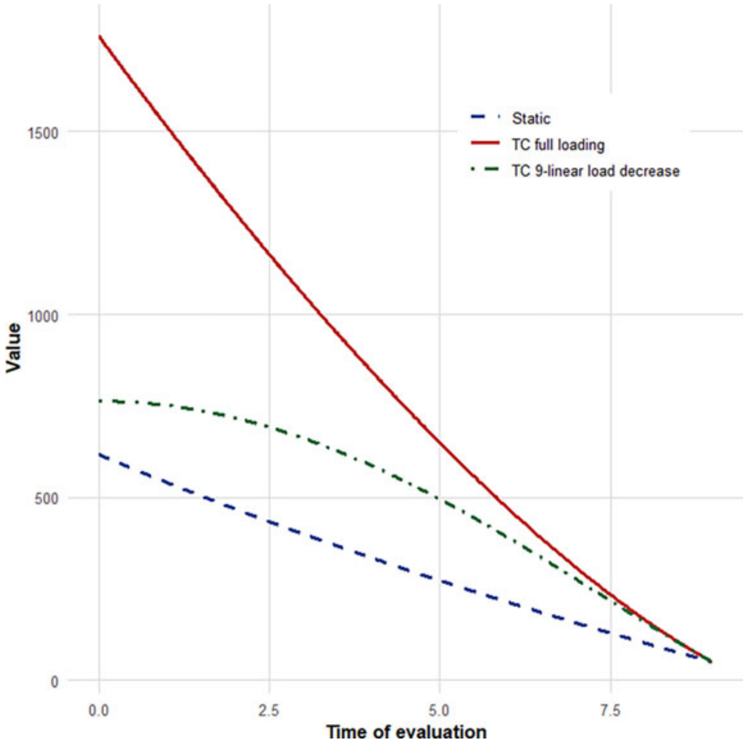

Figure 6. The dynamic Three-step method net of the best estimate $\mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}[e^{-\int_t^T r_s \mathrm{d}s}\, {}_tH_T]$ in the static versus time-consistent cases. ($t^{(1)} = 1$).

net again of this best estimate.[17] As we can see in Figure 6, the static values undergo a substantial increase, ending in an 185% increase by time 0. To address this issue, we follow Devolder and Lebègue (2016) and adjust the $\alpha_{i\Delta t}$ safety loads portion applied in the backward recursive scheme. In our study, we propose a linear decrease of the safety loads portion over the entire time frame, within the backward algorithm,[18] as detailed below

$$\alpha_{i\Delta t} = \left\{ \frac{0.15}{989}\, i.\ 0 \le i \le 989 \right\}.$$

As illustrated in Figure 6, the linear recalibration of the safety loading portion successfully reduces the percentage increase induced by time consistency at time 0 to a value below 100%, namely 23%. Our decision to implement a backward-decreasing safety loading portion is primarily motivated by the objective of reducing the time-consistent price, which would otherwise have been prohibitively high. Moreover, this choice of a monotonically decreasing safety loading portion is also supported in the actuarial literature, see, for example, Hürlimann (2010) and Devolder and Lebègue (2016), where in the context of dynamic capital requirement calculation, the analog of the safety loading portion corresponds to the confidence level applied to Value-at-Risk. With this portion loading recalibration approach, we achieve a good compromise between a time-consistent evaluation methodology and a value that is not overly costly relative to the

[17] In the one-period framework, the difference between the Three-step price and this best estimate is equal to the static value of the diversifiable component.

[18] Devolder and Lebègue (2016) examined the impact of various time-dependent evolutions, including linear, quadratic, and exponential, of security loads on the SCR.

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benchmark static value. This therefore proves that it is possible to switch to a time-consistent valuation without generating an unreasonable risk margin.

5. Conclusion

This paper explores the extension of the Three-step method to the dynamic context. To achieve this, we first refine the axiomatization of valuation operators, as well as the properties of market-consistency and actuarial-consistency, ultimately deriving a dynamic Three-step decomposition. We defined a crucial concept from the literature known as time-consistency. Furthermore, we presented an iterative backward procedure in a discrete model, enabling the enforcement of time-consistent dynamics where such consistency is previously absent. With a combination of any hybrid Pure Endowment product, we identified that the diversifiable component of the Three-step method detrimentally induces the non-verification of the time-consistent property. Therefore, we applied the backward procedure through a dual approach to the Three-step method to derive closed, time-consistent formulas using the purely actuarial Pure Endowment product. Thanks to the two-iteration closed-form solution, we observed that the diversifiable part influences the valuation of the non-diversifiable components of the subsequent iteration. Thus, achieving time-consistency by relying on traditional non-linear actuarial operators, such as the variance/standard-deviation premium principle, to assess the diversifiable part is not entirely viable. Immersed in the continuous-time limit, a PDE governing the Three-step method time-consistent dynamic was derived. By specifying a financial terminal bonus as the financial payoff of the hybrid life Pure Endowment product, the PDE was numerically solved using the finite difference method. The obtained numerical results indicated that the effect of time led to a slight rise in the one-step prices of the full Three-step method. Nonetheless, the static prices of the diversifiable actuarial component have surged notably when compared to the time-consistent Three-step prices, net of the best estimate under $\mathbb{Z}_\theta^a \times \mathbb{Q}^f$. That being said, the time-consistent property, though essential for ensuring the consistency of the insurer's internal decisions over time, is not immune to criticism and may have specific downsides. To mitigate the resulting impact, the safety charges applied in the backward scheme were progressively reduced through a linear approach. In most cases, the actuary managing liability assessments has considerable flexibility in setting the level of security measures to apply. When necessary, a more risk-averse strategy can be employed by increasing the security proportion loading, causing prices to increase. Conversely, a less cautious approach permits a reduction in the loading, resulting in lower prices. In our case, moderating the security loadings becomes a rational decision to ensure a suitable trade-off between time-consistency and the smooth progression of price changes. This indicates that careful consideration is required concerning the magnitude of security charges used in the backward scheme when applying non-linear actuarial premium principles to the valuation of the diversifiable part of the Three-step method.

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References

Acciaio, B., & **Penner, I.** (2011). Dynamic risk measures. In *Advanced mathematical methods for finance* (pp. 1–34). Springer.
Bacinello, A. R. (2001). Fair pricing of life insurance participating policies with a minimum interest rate guaranteed. *ASTIN Bulletin: The Journal of the IAA*, **31**(2), 275–297.

Barigou, K., Chen, Z., & Dhaene, J. (2019). Fair dynamic valuation of insurance liabilities: Merging actuarial judgement with market-and time-consistency. *Insurance: Mathematics and Economics*, **88**, 19–29.
Barigou, K., Linders, D., & Yang, F. (2023). Actuarial-consistency and two-step actuarial valuations: A new paradigm to insurance valuation. *Scandinavian Actuarial Journal*, **2023**(2), 191–217.
Belhouari, O., Deelstra, G., & Devolder, P. (2025). Hybrid life insurance valuation based on a new standard deviation premium principle in a stochastic interest rate framework. *European Actuarial Journal*, **15**(1), 45–72.
Bion-Nadal, J. (2009). Time consistent dynamic risk processes. *Stochastic Processes and their Applications*, **119**(2), 633–654.
Chen, Z., Chen, B., & Dhaene, J. (2020). Fair dynamic valuation of insurance liabilities: A loss averse convex hedging approach. *Scandinavian Actuarial Journal*, **2020**(9), 792–818.
Chen, Z., Chen, B., Dhaene, J., & Yang, T. (2021). Fair dynamic valuation of insurance liabilities via convex hedging. *Insurance: Mathematics and Economics*, **98**, 1–13.
Deelstra, G., Devolder, P., Gnameho, K., & Hieber, P. (2020). Valuation of hybrid financial and actuarial products in life insurance by a novel three-step method. *ASTIN Bulletin: The Journal of the IAA*, **50**(3), 709–742.
Devolder, P., & Azizieh, C. (2013). Margrabe option and life insurance with participation. *Bulletin Français d'Actuariat*, **13**(26), 47–77.
Devolder, P., & Lebègue, A. (2016). Compositions of conditional risk measures and solvency capital. *Risks*, **4**(4), 49.
Dhaene, J., Stassen, B., Barigou, K., Linders, D., & Chen, Z. (2017). Fair valuation of insurance liabilities: Merging actuarial judgement and market-consistency. *Insurance: Mathematics and Economics*, **76**, 14–27.
Hürlimann, W. (2010). Solvency risk capital for the short and long term: Probabilistic versus stability criterion. *Belgian Actuarial Bulletin*, **9**(1), 8–16.
Le Courtois, O., Majri, M., & Shen, L. (2020). Utility-consistent valuation schemes for the own risk and solvency assessment of life insurance companies. *Asia-Pacific Journal of Risk and Insurance*, **15**(1), 47–79.
Luciano, E., & Vigna, E. (2008). Mortality risk via affine stochastic intensities: Calibration and empirical relevance. *Belgian Actuarial Journal*, **8**, 5–16.
Malamud, S., Trubowitz, E., & Wüthrich, M. V. (2008). Market consistent pricing of insurance products. *ASTIN Bulletin: The Journal of the IAA*, **38**(2), 483–526.
Møller, T. (2002). On valuation and risk management at the interface of insurance and finance. *British Actuarial Journal*, **8**(4), 787–827.
Pelsser, A., & Ghalehjooghi, A. S. (2016). Time-consistent actuarial valuations. *Insurance: Mathematics and Economics*, **66**, 97–112.
Pelsser, A., & Stadje, M. (2014). Time-consistent and market-consistent evaluations. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, **24**(1), 25–65.

Appendices

A. Asymptotic behavior of the standard deviation premium-variance principle

Details about

$$\pi_t^{u,SD}\left({}_tH_u^2|\mathcal{F}_t\right) \underset{N_t\to+\infty}{\longrightarrow} 0,$$
$$\pi_t^{u,V}\left({}_tH_u^2|\mathcal{F}_t\right) \underset{N_t\to+\infty}{\longrightarrow} 0,$$

for any $t \in [0, T]$.

Proof. Let $0 \le t \le u \le T$, such that $N_t > 0$. By construction, the actuarial component in ${}_tH_u^2$ involves a sum of actuarial risks centered under the real-world actuarial probability. Consequently, relying on the independence between financial and actuarial risks, the best estimate is immediately zero:

$$\mathbb{E}_{\mathcal{F}_t}^{\mathbb{P}^a\times\mathbb{Q}^f}\left[e^{-\int_t^u r_s \mathrm{d}s}\ {}_tH_u^2\right] = 0.$$

Using the same argument of independence and centering, we obtain for the loading term

$$\mathbb{V}_{\mathcal{F}_t}\left[{}_tH_u^2\right] = \frac{1}{N_t^2}\mathbb{E}_{\mathcal{F}_t}[h(S_u)^2] \times \mathbb{V}_{\mathcal{F}_t}\left[\sum_{j=1}^{N_t}\left(g(\tau_j) - \mathbb{E}_{\mathcal{F}_t\vee\mathfrak{G}_u}[g(\tau_j)]\right)\right].$$

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Applying the law of total variance to $\mathbb{V}_{\mathcal{F}_t}\big[\sum_{j=1}^{N_t}(g(\tau_j)-\mathbb{E}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_j)])\big]$ conditioned on $\mathfrak{S}_u\vee\mathcal{F}_t$, and exploiting the i.i.d. property of remaining lifetimes conditional on $\mathfrak{S}_u$, we obtain:

$$\mathbb{V}_{\mathcal{F}_t}\left[\sum_{j=1}^{N_t}(g(\tau_j)-\mathbb{E}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_j)])\right]=N_t\times\mathbb{E}_{\mathcal{F}_t}[\mathbb{V}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_1)]].$$

Thus,

$$\mathbb{V}_{\mathcal{F}_t}[{}_tH_u^2]=\mathbb{E}_{\mathcal{F}_t}[h(S_u)^2]\times\frac{1}{N_t}\times\mathbb{E}_{\mathcal{F}_t}[\mathbb{V}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_1)]].$$

Finally

$$\left\{\begin{array}{l}\pi_t^{u,V}\big({}_tH_u^2|\mathcal{F}_t\big)=\gamma_t\times\mathbb{E}_{\mathcal{F}_t}[h(S_u)^2]\times\frac{\mathbb{E}_{\mathcal{F}_t}[\mathbb{V}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_1)]]}{N_t}\times P(t,u).\\ \pi_t^{u,SD}\big({}_tH_u^2|\mathcal{F}_t\big)=\alpha_t\times\sqrt{\mathbb{E}_{\mathcal{F}_t}[h(S_u)^2]}\times\sqrt{\frac{\mathbb{E}_{\mathcal{F}_t}[\mathbb{V}_{\mathcal{F}_t\vee\mathfrak{S}_u}[g(\tau_1)]]}{N_t}}\times\sqrt{u-t}\times P(t,u).\end{array}\right\}.$$

Therefore $\pi_t^{u,SD}({}_tH_u^2|\mathcal{F}_t)\underset{N_t\to+\infty}{\longrightarrow}0$, and $\pi_t^{u,V}({}_tH_u^2|\mathcal{F}_t)\underset{N_t\to+\infty}{\longrightarrow}0$. □

B. Equivalence of time-consistency definitions

Details about Remark 1.

Proof.

1. *Time-consistent property* $\Rightarrow$ *Iterated property:* Let $0\le t\le u\le T$, $0<s\le u-t$ and $h\in\mathcal{H}_u^f$. The translation invariance axiom implies that,

$$\pi_{t+s}^{t+s}\big[\pi_{t+s}^u[h]\big]=\pi_{t+s}^u[h].$$

Since $\pi_{t+s}^u[h]\in\mathcal{H}_{t+s}^f$, we can therefore apply (16) twice:

$$\pi_{t+s}^{t+s}\big[\pi_{t+s}^u[h]\big]\le\pi_{t+s}^u[h]\implies\pi_t^{t+s}\big[\pi_{t+s}^u[h]\big]\le\pi_t^u[h].$$

$$\pi_{t+s}^{t+s}\big[\pi_{t+s}^u[h]\big]\ge\pi_{t+s}^u[h]\implies\pi_t^{t+s}\big[\pi_{t+s}^u[h]\big]\ge\pi_t^u[h].$$

Hence, we reach the iterative definition.

2. *Iterated property* $\Rightarrow$ *Time-consistent property:* Under the assumption of monotonicity, we embark on the verification of the converse direction. Let $0\le t\le u,v\le T$, $0<s\le u-t$, $0<s\le v-t$, $h\in\mathcal{H}_u^f$, $g\in\mathcal{H}_v^f$ such as $\pi_{t+s}^u[h]\le\pi_{t+s}^v[g]$. Since $\pi_{t+s}^u[h]\in\mathcal{H}_{t+s}^f$, $\pi_{t+s}^v[g]\in\mathcal{H}_{t+s}^f$, then by monotonicity

$$\pi_t^{t+s}\big[\pi_{t+s}^u[h]\big]\le\pi_t^{t+s}\big[\pi_{t+s}^v[g]\big].$$

Hence, using (17),

$$\pi_t^u[h]\le\pi_t^v[g].$$

□

C. Time-inconsistency of the dynamic Three-step method: due to the diversifiable component

C.1 Details about Lemma 1

Proof. **Intuition of why the Two-step method is time-consistent**: The Two-step valuation is identified as a product of expectations under the risk-neutral probability $\mathbb{Q}^f$ and the actuarial

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probability $\mathbb{Z}_\theta^a$. Since expectations are linear and satisfy the tower rule property, the valuation operator satisfies time-consistency.

Let $0 \leq t \leq u \leq T$ such as $0 < s \leq u - t$ and $N_{t+s} > 0$. The one-period Two-step valuation at time $t+s$ with maturity u is given by

$$\pi_{t+s}^{u,2}[{}_{t+s}H_u] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Q}^f}\left[e^{-\int_{t+s}^u r_v \,\mathrm{d}v} h(S_u)\right] \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a}\left[e^{-\int_{t+s}^u \lambda_v \,\mathrm{d}v}\right].$$

Let us set ${}_tLY_{t+s} = \frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2}[{}_{t+s}H_u] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Q}^f}[e^{-\int_{t+s}^u r_v \,\mathrm{d}v} h(S_u)] \times \frac{N_{t+s}}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a}[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}]$, then its two-step decomposition is

$$\left\{\begin{array}{l} {}_tLY_{t+s}^h = \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}_{t+s}^f}[{}_tLY_{t+s}] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Q}^f}\left[e^{-\int_{t+s}^u r_v \mathrm{d}v} h(S_u)\right] \times \mathbb{E}_{\mathcal{F}_t}\left[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a}\left[e^{-\int_t^u \lambda_v \mathrm{d}v}\right]\right] \times \mathbb{1}_{N_t > 0}, \\ {}_tLY_{t+s}^{sy} = \left({}_tLY_{t+s} - \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Q}^f}\left[e^{-\int_{t+s}^u r_v \,\mathrm{d}v} h(S_u)\right] \times \mathbb{E}_{\mathcal{F}_t}\left[\mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a}\left[e^{-\int_t^u \lambda_v \mathrm{d}v}\right]\right]\right) \times \mathbb{1}_{N_t > 0}. \end{array}\right\}$$

The Two-step evaluation between times t and $t+s$ is defined as follows

$$\pi_t^{t+s,2}\left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2}[{}_{t+s}H_u]\right] = \pi_t^{t+s,f}\left[{}_tLY_{t+s}^h\right] + \pi_t^{t+s,n}\left[{}_tLY_{t+s}^{sy}\right].$$

Since the actuarial risks contained in the hedgeable part ${}_tLY_{t+s}^h$ are $\mathcal{F}_t$-measurable, its evaluation is identical in $\pi_t^{t+s,f}[{}_tLY_{t+s}^h]$ and in $\pi_t^{t+s,n}[{}_tLY_{t+s}^{sy}]$; the choice of probability measure (whether $\mathbb{Z}_\theta^a$ or $\mathbb{P}^a$) has no effect. Hence, the valuation of the hedgeable component therefore cancels upon subtraction, and we obtain

$$\pi_t^{t+s,2}\left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2}[{}_{t+s}H_u]\right] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a \times \mathbb{Q}^f}\left[e^{-\int_t^{t+s} r_v \mathrm{d}v}\ {}_tLY_{t+s}\right].$$

Using the independence between mortality and finance and tower rule property under $\mathbb{Z}_\theta^a$ and $\mathbb{Q}^f$, we obtain

$$\begin{aligned} \pi_t^{t+s,2}\left[\frac{N_{t+s}}{N_t} \times \pi_{t+s}^{u,2}[{}_{t+s}H_u]\right] &= \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Q}^f}\left[e^{-\int_t^u r_v \mathrm{d}v} h(S_u)\right] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a}\left[e^{-\int_t^u \lambda_v \mathrm{d}v}\right] \\ &= \pi_t^{u,2}[{}_tH_u]. \end{aligned}$$

Thus, the time-consistent valuation at time $t \in [0, T]$ can be calculated as $\pi_t^2[{}_tH_T] = \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Q}^f}[e^{-\int_t^T r_v \mathrm{d}v} h(S_T)] \times \mathbb{E}_{\mathcal{F}_t}^{\mathbb{Z}_\theta^a}[e^{-\int_t^T \lambda_v \mathrm{d}v}]$. □

C.2 Details about Lemma 2

Proof. **Intuition:** Unlike the Two-step method, which relies solely on linear expectations, the Three-step method assesses diversifiable risk using a variance (or standard deviation) premium principle. Variance is a non-linear operator that does not satisfy the tower rule property. This highlights that the iterative scheme fails to capture the variance of the conditional expectation component, producing the discrepancy between static and iterated valuations.

For the proof, we rely on the variance premium principle as the operator for evaluating the diversifiable risk. Notably, the result holds true under the standard deviation premium principle, and the proof can be readily adapted. Let $0 \leq t \leq u \leq T$ such as $0 < s \leq u - t$ and $N_{t+s} > 0$. The one-period Three-step valuation at time $t+s$ with maturity u is given by

$$\begin{aligned} \pi_{t+s}^{u,3}[{}_{t+s}H_u] = \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Q}^f}\left[e^{-\int_{t+s}^u r_v \mathrm{d}v} h(S_u)\right] \times \mathbb{E}_{\mathcal{F}_{t+s}}^{\mathbb{Z}_\theta^a}\left[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\right] + \gamma_{t+s} \times P(t+s, u) \\ \times \frac{\mathbb{E}_{\mathcal{F}_{t+s}}\left[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\left(1 - e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\right)\right]}{N_{t+s}} \times \mathbb{E}_{\mathcal{F}_{t+s}}[h(S_u)^2]. \end{aligned} \tag{C1}$$

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Let us set ${}_tLY_{t+s} = \frac{N_{t+s}}{N_t} \times \pi^{u,3}_{t+s}[{}_{t+s}H_u] = \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_{t+s}}[e^{-\int_{t+s}^u r_v\,\mathrm{d}v} h(S_u)] \frac{N_{t+s}}{N_t} \times \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}] + \gamma_{t+s} \times P(t+s,u) \times \frac{\mathbb{E}_{\mathcal{F}_{t+s}}[e^{-\int_{t+s}^u \lambda_v\,\mathrm{d}v}(1-e^{-\int_{t+s}^u \lambda_v \mathrm{d}v})]}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}[h(S_u)^2]$, then its Three-step decomposition is

$$\begin{cases} {}_tLY^1_{t+s} = \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+s}}[{}_tLY_{t+s}] = \Big(\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u r_v \mathrm{d}v} h(S_u)\big] \times \mathbb{E}_{\mathcal{F}_t}\Big[\mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_t^u \lambda_v \mathrm{d}v}\big]\Big] \\ +\gamma_{t+s} \times \dfrac{\mathbb{E}_{\mathcal{F}_t}\big[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big(1-e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big)\big]}{N_t} \times \mathbb{E}_{\mathcal{F}_{t+s}}[h(S_u)^2] \times P(t+s,u)\Big) \times \mathbb{1}_{N_t>0}, \\ {}_tLY^2_{t+s} = {}_tLY_{t+s} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+s} \vee \mathcal{F}^f_{t+s}}[{}_tLY_{t+s}] = \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u r_v\,\mathrm{d}v} h(S_u)\big] \Big(\dfrac{N_{t+s}}{N_t} \times \\ \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big] - \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_t^u \lambda_v \mathrm{d}v}\big]\Big) \times \mathbb{1}_{N_t>0}, \\ {}_tLY^3_{t+s} = \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+s} \vee \mathcal{F}^f_{t+s}}[{}_tLY_{t+s}] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+s}}[{}_tLY_{t+s}] = \Big(\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u r_v \mathrm{d}v} h(S_u)\big] \times \\ \Big[\mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_t^u \lambda_v\,\mathrm{d}v}\big] - \mathbb{E}_{\mathcal{F}_t}\Big[\mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_t^u \lambda_v \mathrm{d}v}\big]\Big]\Big] + \gamma_{t+s}\mathbb{E}_{\mathcal{F}_{t+s}}[h(S_u)^2] \times \\ P(t+s,u)\Big[\dfrac{\mathbb{E}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u \lambda_v\,\mathrm{d}v}\big(1-e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big)\big]}{N_t} - \dfrac{\mathbb{E}_{\mathcal{F}_t}\big[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big(1-e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big)\big]}{N_t}\Big]\Big) \times \mathbb{1}_{N_t>0}. \end{cases}$$

The Three-step evaluation between times t and $t+s$ is defined as follows

$$\pi^{t+s,3}_t\left[\frac{N_{t+s}}{N_t} \times \pi^{u,3}_{t+s}[{}_{t+s}H_u]\right] = \pi^{t+s,f}_t\big[{}_tLY^1_{t+s}\big] + \pi^{t+s,V}_t\big[{}_tLY^2_{t+s}\big] + \pi^{t+s,n}_t\big[{}_tLY^3_{t+s}\big].$$

For the same reason stated in the proof of Lemma 1, the valuation of the hedgeable component ${}_tLY^1_{t+s}$ in $\pi^{t+s,f}_t[{}_tLY^1_{t+s}]$ and $\pi^{t+s,n}_t[{}_tLY^3_{t+s}]$ is netted out. Moreover, by applying the law of total variance to $\Big(\frac{N_{t+s}}{N_t} \times \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u \lambda_v\,\mathrm{d}v}\big] - \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}\big[e^{-\int_t^u \lambda_v\,\mathrm{d}v}\big]\Big)$, conditioned on $\mathfrak{S}_{t+s} \vee \mathcal{F}_t$, and using the fact that $N_{t+s}|_{\mathcal{F}_t \vee \mathfrak{S}_{t+s}} \sim \mathrm{Binom}\big(N_t, e^{-\int_t^{t+s} \lambda_v\,\mathrm{d}v}\big)$, we arrive at

$$\begin{aligned} &\pi^{t+s,3}_t\left[\frac{N_{t+s}}{N_t} \times \pi^{u,3}_{t+s}[{}_{t+s}H_u]\right] \\ &= \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_t}\big[e^{-\int_t^u r_v \mathrm{d}v} h(S_u)\big] \times \mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_t}\big[e^{-\int_t^u \lambda_v \mathrm{d}v}\big] + \frac{1}{N_t}\gamma_t \times \mathbb{E}_{\mathcal{F}_t}\Big[\mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u r_v\,\mathrm{d}v} h(S_u)\big]^2\Big] \\ &\quad \times \mathbb{E}_{\mathcal{F}_t}\Big[\big(\mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_{t+s}}[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}]\big)^2 e^{-\int_t^{t+s} \lambda_v \mathrm{d}s} \times \big(1-e^{-\int_t^{t+s} \lambda_v \mathrm{d}v}\big)\Big] \times P(t,t+s) \\ &\quad + \gamma_{t+s} \times \mathbb{E}^{\mathbb{Q}^f}_{\mathcal{F}_t}\Big[e^{-\int_t^{t+s} r_v\,\mathrm{d}v} P(t+s,u) \times \mathbb{E}_{\mathcal{F}_{t+s}}\big[h(S_u)^2\big]\Big] \\ &\quad \times \frac{\mathbb{E}^{\mathbb{Z}^a_\theta}_{\mathcal{F}_t}\Big[\mathbb{E}_{\mathcal{F}_{t+s}}\big[e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big(1-e^{-\int_{t+s}^u \lambda_v \mathrm{d}v}\big)\big]\Big]}{N_t}. \end{aligned} \tag{C2}$$

Because the final derived equation differs from $\big(\pi^{u,f}_t[{}_tH^1_u] + \pi^{u,V}_t[{}_tH^2_u] + \pi^{u,n}_t[{}_tH^3_u]\big)$, it follows that the dynamic Three-step method $(\pi^{u,3}_t)_{0 \le t \le u \le T}$ is not time-consistent. □

D. Closed-form solution in a dual-iteration setting

Details about Lemma 3.

Proof. We implement the backward scheme referenced in (23), specifically concentrating on two iterations. That is, we set $\Delta t = T - b$, $s = \Delta t$ and $\tilde{s} = 2\Delta t = T$. By setting $h(S_u) = 1$ and adapting (C1) from the proof of Lemma 2 to a constant interest rate setting, the value at time b can be

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deduced as follows

$$\tilde{\pi}_b^3[{}_bH_T|\mathcal{F}_b] = P(b,T)\left(e^{-\lambda_b l_{b,T}+\frac{1}{2}s_{b,T}^2}e^{-\theta_{b,T}^{(1)}s_{b,T}^2} + \gamma_b \times \frac{e^{-\lambda_b l_{b,T}+\frac{1}{2}s_{b,T}^2} - e^{-2\lambda_b l_{b,T}+2s_{b,T}^2}}{N_b}\right).$$

By a similar adaptation, equation (C2) in the proof of Lemma 2 yields the following derivation of the time-consistent valuation at time 0

$$\begin{aligned}\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] = P(0,T)&\left(e^{-\lambda_0 l_{0,T}-s_{0,T}^2\left(\theta_{0,T}^{(1)}-\frac{1}{2}\right)} + \gamma_b\frac{e^{\frac{1}{2}s_{b,T}^2}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a}[e^{-\lambda_b l_{b,T}}] - e^{2s_{b,T}^2}\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a}[e^{-2\lambda_b l_{b,T}}]}{n}\right)\\ &+\frac{1}{n}\gamma_0 \times P(0,b)P(b,T)^2 e^{-2s_{b,T}^2\left(\theta_{b,T}^{(1)}-\frac{1}{2}\right)}\left(\mathbb{E}_{\mathcal{F}_0}\left[e^{-2\lambda_b l_{b,T}} \times e^{-\int_0^b \lambda_s \mathrm{d}s}\right]\right.\\ &\left.-\mathbb{E}_{\mathcal{F}_0}\left[e^{-2\lambda_b l_{b,T}} \times e^{-2\int_0^b \lambda_s \mathrm{d}s}\right]\right).\end{aligned}$$

The terms $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a}[e^{-\lambda_b l_{b,T}}]$ and $\mathbb{E}_{\mathcal{F}_0}^{\mathbb{Z}_\theta^a}[e^{-2\lambda_b l_{b,T}}]$ are solved using the following density

$$\left.\frac{d\mathbb{Z}_\theta^a}{d\mathbb{P}^a}\right|_{\mathcal{F}_0} = \frac{e^{\int_0^b \theta_{0,b}^{(1)}\lambda_s \mathrm{d}s}}{\mathbb{E}_{\mathcal{F}_0}\left[e^{\int_0^b \theta_{0,b}^{(1)}\lambda_s\,\mathrm{d}s}\right]} = e^{\int_0^b \theta_{0,b}^{(1)}\sigma_m\frac{e^{\mu_m(b-u)}-1}{\mu_m}dW_m(u) - \frac{1}{2}\int_0^b\left(\theta_{0,b}^{(1)}\sigma_m\frac{e^{\mu_m(b-u)}-1}{\mu_m}\right)^2 du},$$

whereas, the derivation of the terms $\mathbb{E}_{\mathcal{F}_0}[e^{-2\lambda_b l_{b,T}} \times e^{-2\int_0^b \lambda_s \mathrm{d}s}]$ and $\mathbb{E}_{\mathcal{F}_0}[e^{-2\lambda_b l_{b,T}} \times e^{-\int_0^b \lambda_s \mathrm{d}s}]$ involves the corresponding density

$$\left.\frac{d\bar{\mathbb{P}}^a}{d\mathbb{P}^a}\right|_{\mathcal{F}_0} = \frac{e^{-2\lambda_b l_{b,T}}}{\mathbb{E}_{\mathcal{F}_0}[e^{-2\lambda_b l_{b,T}}]} = e^{-\frac{1}{2}\int_0^b F(u)^2\,\mathrm{d}u - \int_0^b F(u)\mathrm{d}W_m(u)},$$

$F(u) = 2 \times l_{b,T}\sigma_m e^{\mu_m(b-u)}$.

After performing the necessary computations, we arrive at the following final formula

$$\begin{aligned}\tilde{\pi}_0^3[{}_0H_T|\mathcal{F}_0] = P(0,T)&\left(e^{-\lambda_0 l_{0,T}-s_{0,T}^2\left(\theta_{0,T}^{(1)}-\frac{1}{2}\right)} + \gamma_b\frac{e^{\frac{1}{2}s_{b,T}^2}\,{}_1\tilde{f}_1 - e^{2s_{b,T}^2}\,{}_2\tilde{f}_4}{n}\right) + \frac{1}{n}\gamma_0 \times P(0,b)\\ &\times P(b,T)^2 e^{-2s_{b,T}^2\left(\theta_{b,T}^{(1)}-\frac{1}{2}\right)}\left({}_1f_1^2 - {}_2f_4^8\right).\end{aligned}$$

□

E. Derivation of the partial differential equation

Proof of Theorem 1.

Proof. Intuitively, the derived PDE represents a local equilibrium condition for the time-consistent valuation, and it will be obtained by taking the continuous-time limit of the backward-recursive formula in (24).

To mitigate notational ambiguity, we opt to exclude the "3" notation from $\tilde{\pi}_t^3$, though the derivation of the PDE, especially for the Three-step method time-consistent dynamic, remains implicit. The risk ${}_tLY_{t+dt} = \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}$, considered at time $t+dt$, is decomposed into the three

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components in order to evaluate it at time t, resulting in the following

$$\left\{\begin{aligned} {}_tLY^1_{t+dt} &= \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right], \\ tLY^2_{t+dt} &= \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right], \\ {}_tLY^3_{t+dt} &= \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right] - \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]. \end{aligned}\right\}$$

Therefore, the constraint $\tilde{\pi}_t = \pi_t^{t+dt}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]$ is equivalent to

$$\begin{aligned} \tilde{\pi}_t &= \pi_t^{t+dt,f}\left[{}_tLY^1_{t+dt}\right] + \pi_t^{t+dt,SD}\left[{}_tLY^2_{t+dt}\right] + \pi_t^{t+dt,n}\left[{}_tLY_{t+dt}\right] \\ &= \mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right] \\ &\quad + \mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right)\right] \\ &\quad + \alpha_t \sqrt{\mathbb{V}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right]} \times \sqrt{dt} \times P(t, t+dt) \\ &\quad + \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \,\mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right] \\ &\quad - \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \,\mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \tilde{\pi}_{t+dt}\right]\right]. \end{aligned}$$

Three points arise, as follows:

- As in the proof of Lemma 2 and Lemma 1, the valuation of the hedgeable component ${}_tLY^1_{t+dt}$ is identical in $\pi_t^{t+dt,f}[{}_tLY^1_{t+dt}]$ and $\pi_t^{t+dt,n}[{}_tLY^3_{t+dt}]$. This follows from the fact that purely actuarial risks in $\mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]$ are $\mathcal{F}_t$-measurable, allowing them to be factored out of the expectations under the $\mathbb{Z}^a_\theta$ or $\mathbb{P}^a$ measure. Hence

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right] \\ &\quad = \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right]. \end{aligned} \tag{E1}$$

- Secondly, the best estimate of the actuarial component is likewise identically zero by construction. The key observation is that $\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \tilde{\pi}_{t+dt}\right]$ does not depend on the choice of probability measure. Consequently,

$$\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \tilde{\pi}_{t+dt}\right] = \mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \tilde{\pi}_{t+dt}\right],$$

and an application of the tower property yields the result:

$$\mathbb{E}^{\mathbb{P}^a \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \left(\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right)\right] = 0. \tag{E2}$$

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- Thirdly, using the same previous argument, we have that $\mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}^3_{t+dt}] = \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}]$. By further applying the tower rule property, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s\,\mathrm{d}s} \times \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right] \\
&= \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \times \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]. \qquad \text{(E3)}
\end{aligned}$$

By combining (E1)–(E3) and stretching dt towards 0, we derive the following

$$lm^{(1)} + \alpha_t \times lm^{(2)} = 0,$$

where

$$lm^{(1)} = \lim_{dt \to 0} \frac{\mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s\,\mathrm{d}s} \frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right] - \tilde{\pi}_t}{dt}, \qquad \text{(E4)}$$

and

$$lm^{(2)} = \lim_{dt \to 0} P(t, t+dt) \times \sqrt{\frac{\mathbb{V}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right]}{dt}}. \qquad \text{(E5)}$$

To resolve the limit $lm^{(1)}$, we apply Itô's lemma to $\tilde{\pi}_{t+dt}$ under the probability measure $\mathbb{Z}^a_\theta \times \mathbb{Q}^f$, where the dynamics of the stochastic mortality rates under $\mathbb{Z}^a_\theta$ are obtained from the density equation (15) and are therefore given by

$$d\lambda_t = \left(\overline{\theta^{(1)}_{t,T}} + \mu_m \lambda_t\right) dt + \sigma_m dW^*_m(t),$$

with $\overline{\theta^{(1)}_{t,T}} = \theta^{(1)}_{0,T} \sigma^2_m \times \frac{e^{\mu_m(T-t)} - 1}{\mu_m}$, and $dW^*_m(t) = dW_m(t) - \theta^{(1)}_{0,T} \sigma_m \frac{e^{\mu_m(T-t)} - 1}{\mu_m}\,\mathrm{d}t$.

Let us denote $dW^\rho(s) = \rho dW_r(t) + \sqrt{1-\rho^2} dW_S(t)$, then by employing Ito's lemma with jump process under $\mathbb{Z}^a_\theta \times \mathbb{Q}^f$, we obtain

$$\begin{aligned}
\tilde{\pi}_{t+dt} = \tilde{\pi}_t &+ \int_t^{t+dt} \partial_t \tilde{\pi}_s ds + \int_t^{t+dt} \partial_S \tilde{\pi}_s r_s S_s ds + \sigma_S \int_t^{t+dt} \partial_S \tilde{\pi}_s S_s dW^\rho(s) \\
&+ \int_t^{t+dt} (\theta - a r_s) \partial_r \tilde{\pi}_s ds + \sigma_r \int_t^{t+dt} \partial_r \tilde{\pi}_s dW_r(s) + \int_t^{t+dt} \left(\overline{\theta^{(1)}_{s,T}} + \mu_m \lambda_s\right) \partial_\lambda \tilde{\pi}_s ds \\
&+ \sigma_m \int_t^{t+dt} \partial_\lambda \tilde{\pi}_s dW^*_m(s) + \frac{1}{2}\sigma^2_S \int_t^{t+dt} \partial^2_S \tilde{\pi}_s S^2_s ds + \frac{1}{2}\sigma^2_m \int_t^{t+dt} \partial^2_\lambda \tilde{\pi}_s ds \\
&+ \frac{1}{2}\sigma^2_r \int_t^{t+dt} \partial^2_r \tilde{\pi}_s ds + \rho \sigma_r \sigma_S \int_t^{t+dt} \partial^2_{rS} \tilde{\pi}_s S_s ds + \int_t^{t+dt} (\tilde{\pi}^*_s - \tilde{\pi}_s) dN_s,
\end{aligned}$$

where $\tilde{\pi}^*_s$ denotes the time-consistent value just before the jump in the number of survivors, i.e. $\tilde{\pi}^*_s[.] = \tilde{\pi}_s[.|N_s - 1]$. By applying the discount factor multiplication $e^{-\int_t^{t+dt} r_s \mathrm{d}s}$, we subsequently engage in the calculation of the mathematical expectation to reach

$$
\begin{aligned}
lm^{(1)} &= \tilde{\pi}_t \lim_{dt\to 0} \frac{\mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s \mathrm{d}s}\right] - 1}{dt} + \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_t \tilde{\pi}_s\right] ds}{dt} \\
&\quad + \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_S \tilde{\pi}_s r_s S_s\right] ds}{dt} \\
&\quad + \sigma_S \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_S \tilde{\pi}_s S_s dW^{\rho}(s)\right]}{dt} \\
&\quad + \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_r \tilde{\pi}_s (\theta - a r_s)\right] ds}{dt} \\
&\quad + \sigma_r \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_r \tilde{\pi}_s dW_r(s)\right]}{dt} \\
&\quad + \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times e^{-\int_t^{t+dt} r_s \mathrm{d}s} \left(\overline{\theta^{(1)}_{s,T}} + \mu_m \lambda_s\right) \partial_\lambda \tilde{\pi}_s\right] ds}{dt} \\
&\quad + \frac{\sigma_S^2}{2} \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_S^2 \tilde{\pi}_s S_s^2\right] ds}{dt} \\
&\quad + \frac{\sigma_m^2}{2} \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[e^{-\int_t^{t+dt} r_s \mathrm{d}s} \frac{N_{t+dt}}{N_t} \partial_\lambda^2 \tilde{\pi}_s\right] ds}{dt} \\
&\quad + \frac{\sigma_r^2}{2} \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_r^2 \tilde{\pi}_s\right] ds}{dt} \\
&\quad + \rho \sigma_S \sigma_r \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_{Sr}^2 \tilde{\pi}_s S_s\right] ds}{dt} \\
&\quad + \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} (\tilde{\pi}_s^* - \tilde{\pi}_s) dN_s\right]}{dt} \\
&\quad + \sigma_m \lim_{dt\to 0} \frac{\int_t^{t+dt} \mathbb{E}^{\mathbb{Z}^a_\theta \times \mathbb{Q}^f}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} e^{-\int_t^{t+dt} r_s \mathrm{d}s} \partial_\lambda \tilde{\pi}_s dW_m^*(s)\right] ds}{dt}. \\
&= -r_t \tilde{\pi}_t + \partial_t \tilde{\pi}_t + r_t S_t \partial_S \tilde{\pi}_t + (\theta - a r_t) \partial_r \tilde{\pi}_t + \left(\overline{\theta^{(1)}_{t,T}} + \mu_m \lambda_t\right) \partial_\lambda \tilde{\pi}_t + \frac{1}{2} \sigma_S^2 S_t^2 \partial_S^2 \tilde{\pi}_t \\
&\quad + \frac{1}{2} \sigma_m^2 \partial_\lambda^2 \tilde{\pi}_t + \frac{1}{2} \sigma_r^2 \partial_r^2 \tilde{\pi}_t + \rho \sigma_r \sigma_S S_t \partial_{rS}^2 \tilde{\pi}_t + (\tilde{\pi}_t^* - \tilde{\pi}_t) \lambda_t.
\end{aligned}
$$

Regarding the second limit $lm^{(2)}$, to resolve it, we use the fact that $\mathbb{V}_{\mathcal{F}_t}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt} - \mathbb{E}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right] = \mathbb{E}_{\mathcal{F}_t}\left[\mathbb{V}_{\mathcal{F}_t \vee \mathfrak{S}_{t+dt} \vee \mathcal{F}^f_{t+dt}}\left[\frac{N_{t+dt}}{N_t} \times \tilde{\pi}_{t+dt}\right]\right]$ and apply Itô's lemma to $\tilde{\pi}_{t+dt}$ and $\tilde{\pi}^2_{t+dt}$ under $\mathbb{P} = \mathbb{P}^a \times \mathbb{P}^f$. Following a similar calculation and algebraic simplification as before to compute $lm^{(1)}$, and considering the continuity property of the root function, and applying the rule that the product of two existing finite limits equals the limit of the product, we then obtain

$$
\begin{aligned}
lm^{(2)} &= \sqrt{\sigma_S^2 S_t^2 (\partial_S \tilde{\pi}_t)^2 + \sigma_m^2 (\partial_\lambda \tilde{\pi}_t)^2 + \sigma_r^2 (\partial_r \tilde{\pi}_t)^2 + 2\rho\sigma_S\sigma_r S_t \times \partial_S \tilde{\pi}_t \partial_r \tilde{\pi}_t + (\tilde{\pi}_t - \tilde{\pi}_t^*)^2 \lambda_t} \\
&\quad \times \lim_{dt \to 0} P(t, t+dt) \\
&= \sqrt{\sigma_S^2 S_t^2 (\partial_S \tilde{\pi}_t)^2 + \sigma_m^2 (\partial_\lambda \tilde{\pi}_t)^2 + \sigma_r^2 (\partial_r \tilde{\pi}_t)^2 + 2\rho\sigma_S\sigma_r S_t \times \partial_S \tilde{\pi}_t \partial_r \tilde{\pi}_t + (\tilde{\pi}_t - \tilde{\pi}_t^*)^2 \lambda_t}. \qquad \square
\end{aligned}
$$

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