RISK REDUCTION BY CONDITIONAL MEAN RISK SHARING WITH APPLICATION TO COLLABORATIVE INSURANCE

Denuit, M. and C. Y. Robert

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MICHEL DENUIT
Institute of Statistics, Biostatistics and Actuarial Science - ISBA
Louvain Institute of Data Analysis and Modeling - LIDAM
UCLouvain
Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

CHRISTIAN Y. ROBERT
Laboratory in Finance and Insurance - LFA
CREST - Center for Research in Economics and Statistics
ENSAE
Paris, France
chrobert@ensae.fr

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Abstract

Consider an economic agent who has to select the optimal pool for a risk to be shared with other participants, adopting the conditional mean risk sharing rule. This paper shows that pointwise comparison of the Lorenz or concentration curves associated to the respective total losses of the pools under consideration allows the agent to decide which pool is preferable. The paper then concentrates on independent losses. The monotonicity of the respective contributions of the participants is established with respect to the convex order, showing that increasing the number of participants is always beneficial provided the conditional mean risk sharing rule is used to allocate independent losses among participants. These results are finally applied to collaborative insurance. It is shown that provided individual losses are mutually independent, there always exists a critical number of participants such that collaborative insurance outperforms commercial one.

Keywords: conditional expectation, risk pooling, Lorenz curve, concentration curve, convex order.

1 Introduction and motivation

In this paper, we consider the conditional mean risk allocation of independent losses, as defined by Denuit and Dhaene (2012). According to this rule, each participant to an insurance pool contributes the conditional expectation of the loss brought to the pool, given the total loss experienced by the entire pool. This risk sharing mechanism is regarded as beneficial by all risk-averse economic agents. If all the conditional expectations involved are non-decreasing functions of the total loss then the conditional mean risk sharing is Pareto-optimal. Denuit and Robert (2020a,b) studied this risk sharing mechanism and established several attractive properties including linear approximations when total losses or the number of participants get large. Denuit and Robert (2020c) proved that the conditional expectation defining the conditional mean risk sharing is asymptotically increasing in the total loss (under mild technical assumptions). This ensures that the risk exchange is Pareto-optimal and that all participants have an interest to keep total losses as small as possible.

The problem investigated in this paper can be summarized as follows. Assume that an economic agent must select a pool where to allocate his or her own loss X. There are two possibilities: pool 1 with total losses S_1 and pool 2 with total losses S_2. No particular assumption is made about X, S_1 and S_2, that may be correlated or obey heavy-tailed distributions, for instance. It will be shown that the comparison of the concentration curves of X with respect to S_1 and S_2 is coherent with second-order stochastic dominance, reflecting the common preferences for all risk-averse decision makers.

We then consider a special case of particular interest, where conditional expectations involved in the sharing rule are linear functions of the total loss experienced by the entire pool. This case has been thoroughly investigated by Furman et al. (2018). In this setting, in turns out that the choice is based on the Lorenz curves of the respective total losses, that is, the choice agrees with the Lorenz order. The Lorenz order, or dominance relation is associated with the Lorenz curve that has been widely used to assess income inequalities. A general account can be found in Arnold and Sarabia (2018). When the insurer adopts the expected value principle for premium calculation, it is equivalent to comparing loss ratios rather than the risks themselves. For more information, we refer the interested reader to Denuit and Vermandele (1999) and the references therein.

The paper then concentrates on independent individual losses. In that setting, our main result shows that increasing the number of participants is always beneficial under conditional mean risk sharing. This comes from the fact that each participant's contribution forms a reverse martingale when the size of the pool increases. As a consequence, there exists a limiting contribution when the number of participants tends to infinity. The limit may be the pure premium or remain random in case the risk cannot be fully diversified inside the pool. These results demonstrate that conditional mean risk sharing is tailored to insurance applications: provided losses are allocated accordingly, it is always beneficial to increase the size of the pool to ensure diversification. The limiting case also identifies residual risk.

As an application, we consider collaborative, or Peer-to-Peer (P2P) insurance schemes. Denuit (2019, 2020) demonstrated that conditional mean risk sharing is the appropriate theoretical tool to share losses in such schemes. Denuit and Robert (2020d) designed a flexible system where entry prices can be made attractive compared to the premium of a regular, commercial insurance contract. The present paper complements their study by establishing the feasibility of collaborative insurance. Precisely, we show that provided individual losses are mutually independent and participants are risk-averse expected-utility maximizers, collaborative insurance always outperforms commercial one provided the pool becomes large enough.

The remainder of this paper is organized as follows. In Section 2, we recall the definition of the conditional mean risk sharing rule and formalize the comparison between pools. The choice of the optimal pool is then related to concentration curves. Section 3 is devoted to the particular case where the conditional expectation defining the risk sharing rule is proportional to total losses. In this case, Lorenz curves can be used to perform the comparison. Section 4 deals with independent losses. The main result there is that under conditional mean risk sharing, it is always beneficial to increase the number of participants. In Section 5, these results are applied to the formation of P2P insurance communities to show that collaborative insurance outperforms commercial insurance provided the size of the pool is large enough.

2 Comparing pools

2.1 Conditional mean risk sharing

In the design of a risk pooling scheme, it is important that the sharing rule is both intuitively acceptable and transparent. In that respect, the conditional mean risk sharing (or allocation) proposed by Denuit and Dhaene (2012) turns out to be particularly attractive. Precisely, consider n participants to an insurance pool, numbered $i = 1, 2, \dots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing a monetary loss caused by some insurable peril over one period (a calendar year, say). No particular assumption is made about $X_1, X_2, \dots, X_n$ that can be correlated or obey heavy-tailed distribution, for instance. Henceforth, we use the notation

$$S = \sum_{i=1}^{n} X_i$$

for the total risk of the pool.

Throughout this paper, we assume that the risks X_i to be shared have a probability mass at 0 and possess a probability density function over $(0, \infty)$. Such a distribution is said to be zero-augmented and it appears to be appropriate for modeling insurance losses X_i. Precisely, we assume that each X_i is a zero-augmented risk, i.e. it is equal to 0 with probability $\mathrm{P}[X_i = 0] > 0$ or strictly positive with probability $\mathrm{P}[X_i > 0]$ and possesses the probability density function $f_{X_i|X_i>0}$ over $(0, \infty)$.

In the risk pooling scheme proposed by Denuit and Dhaene (2012), each participant contributes ex-post the amount $\mathrm{E}[X_i|S = s]$ where $s = \sum_{i=1}^{n} x_i$ is the sum of the realizations $x_1, x_2, \dots, x_n$ of $X_1, X_2, \dots, X_n$. Clearly, the conditional mean risk sharing allocates the full risk S as we obviously have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|S] = S.$$

In words, participant i must contribute the expected value of the risk X_i he or she brings to the pool, given the total loss S experienced by the entire community. Stated differently, the contribution of each participant is the average part of the total loss S that can be attributed to the risk he or she brought to the pool. Explained in this way, the conditional mean risk sharing rule appears to be very intuitive and easily understandable.

The key point here is that the actual realization of X_i is mostly due to chance, and only partly reveals the underlying magnitude of the risk brought to the community. Over one period, the conditional mean risk sharing splits the risk X_i into a random part $X_i - \mathrm{E}[X_i|S]$ shared among participants by virtue of the mutuality (or random solidarity) principle and a structural part $\mathrm{E}[X_i|S]$ to be supported by participant i, individually.

Remark 2.1. *Sometimes, no risk reduction can be obtained with conditional mean risk sharing. This is typically the case with comonotonic (or perfectly correlated) losses $X_1, X_2, \ldots$ but may even arise with independent losses. For instance, assume that X_1 is uniformly distributed over the interval $[0,1]$ and that X_2 is integer valued. Then, we have $\mathrm{E}[X_1|X_1+X_2] = X_1$ and $\mathrm{E}[X_2|X_1+X_2] = X_2$. Conditional mean risk sharing thus leaves each participant with his or her initial loss, despite mutual independence. This phenomenon disappears under appropriate condition placed on the respective supports (such as the assumption of zero-augmented distributions made throughout this paper).*

2.2 Optimality condition

2.2.1 Convex order

The convex order, henceforth denoted as $\preceq_{\text{convex}}$, expresses the common preferences for all risk-averse economic agents about losses with the same expected value. Formally, given two losses X and Y, X is said to be smaller than Y in the convex order, henceforth denoted by $X \preceq_{\text{convex}} Y$ if

$$\mathrm{E}[u(w-X)] \geq \mathrm{E}[u(w-Y)] \text{ for all concave utility function } u \text{ and wealth level } w,$$

provided the expectations exist. Hence, $\preceq_{\text{convex}}$ corresponds to the mean-preserving increase in risk in economics when gains are replaced with losses.

It can be shown that $X \preceq_{\text{convex}} Y$ holds if, and ony if, $\mathrm{E}[X] = \mathrm{E}[Y]$ and $\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+]$ for all $d \geq 0$. Thus, the respective stop-loss premiums are ordered for all possible levels d of the deductible. The term "convex" is used since $X \preceq_{\text{convex}} Y \Leftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all convex functions g for which the expectations exist. The stochastic inequality $X \preceq_{\text{convex}} Y$ intuitively means that X and Y have the same "size" (as $\mathrm{E}[X] = \mathrm{E}[Y]$ holds) but that Y is "more variable" than X. For instance, the variance of Y is larger than the variance of X. The next property will be useful in the remainder of this text:

$$X \preceq_{\text{convex}} Y \Leftrightarrow aX + b \preceq_{\text{convex}} aY + b \text{ for all } a,\, b \in \mathbb{R}. \tag{2.1}$$

For a thorough description of the convex order and its applications in an actuarial context, we refer the reader to Denuit et al. (2005).

2.2.2 Comparison of pools

Assume that the economic agent must select a pool where to allocate his or her own loss X. There are two possibilities:

pool 1 with total losses $S_1 = X + Y_1$ and

pool 2 with total losses $S_2 = X + Y_2$.

Here, Y_k stands for the risks in pool k before agent's entrance.

Since the convex order $\preceq_{\text{convex}}$ reflects the preferences about monetary losses shared by all risk-averse agents in the expected utility setting, we take as criterion

$$\text{pool 1 is preferred over pool 2 } \Leftrightarrow \mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2].$$

Denuit (2010) established conditions under which conditional expectations are ranked in the sense of the convex order $\preceq_{\text{convex}}$ when S_1 and S_2 are identically distributed. Here, we adapt the reasoning to the present setting. To this end, we need to introduce concentration curves that are intimately related to Lorenz curves. The concentration curve of X with respect to S is defined as

$$C_{(X,S)}(p) = \frac{\mathrm{E}\big[X\mathrm{I}[S \leq F_S^{-1}(p)]\big]}{\mathrm{E}[X]}, \;\; p \in (0,1),$$

where $\mathrm{I}[\cdot]$ is the indicator function (equal to 1 if the condition appearing within the brackets is fulfilled and to 0 otherwise) and F_S^{-1} is the quantile function corresponding to the distribution function F_S for S. For an exhaustive review of the properties of a concentration curve we refer the interested reader to the book by Yitzhaki and Schechtman (2013).

The next result gives a simple criterion ensuring that $\mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2]$ holds true so that pool 1 is preferred over pool 2.

Proposition 2.2. *Consider three, possibly dependent risks X, Y_1 and Y_2. Define $S_1 = X+Y_1$ and $S_2 = X + Y_2$. Assume that $s \mapsto \mathrm{E}[X|S_k = s]$ is continuously increasing for $k = 1, 2$. Then,*

$$\mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2] \Leftrightarrow C_{(X,S_1)}(p) \geq C_{(X,S_2)}(p) \textit{ for all probability level } p.$$

Proof. Considering formula (3.A.41) in Shaked and Shanthikumar (2007), $Y \preceq_{\text{convex}} Z$ if, and only if, $\mathrm{E}[Y] = \mathrm{E}[Z]$ and $\mathrm{E}[Y|Y \geq F_Y^{-1}(p)] \leq \mathrm{E}[Z|Z \geq F_Z^{-1}(p)]$ for all $p \in [0, 1)$. Since

$$\begin{aligned}\mathrm{E}\Big[\mathrm{E}[X|S_1]\Big|\mathrm{E}[X|S_1] \geq F^{-1}_{\mathrm{E}[X|S_1]}(p)\Big] &= \mathrm{E}\Big[\mathrm{E}[X|S_1]\Big|S_1 \geq F_{S_1}^{-1}(p)\Big] \\ &= \mathrm{E}[X|S_1 \geq F_{S_1}^{-1}(p)],\end{aligned}$$

we see that $\mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2]$ holds true if, and only if, the inequality

$$\mathrm{E}[X|S_1 \geq F_{S_1}^{-1}(p)] \leq \mathrm{E}[X|S_2 \geq F_{S_2}^{-1}(p)] \text{ is valid for all } p \in (0,1). \tag{2.2}$$

Inequality (2.2) corresponds to the announced pointwise comparison of the respective concentration curves. This ends the proof. □

Notice that (2.2) can be equivalently rewritten as follows:

$$\begin{aligned}(2.2) \;\;\Leftrightarrow\;\; & \mathrm{E}\Big[X\mathrm{I}\big[S_1 \geq F_{S_1}^{-1}(p)\big]\Big] \leq \mathrm{E}\Big[X\mathrm{I}\big[S_2 \geq F_{S_2}^{-1}(p)\big]\Big] \text{ for all } p \in (0,1) \\ \Leftrightarrow\;\; & \mathrm{Cov}\Big[X, \mathrm{I}\big[S_1 \geq F_{S_1}^{-1}(p)\big]\Big] \leq \mathrm{Cov}\Big[X, \mathrm{I}\big[S_2 \geq F_{S_2}^{-1}(p)\big]\Big] \text{ for all } p \in (0,1).\end{aligned}$$

The latter inequality means that, to be preferred, pool 1 must be such that X is less correlated to the payoff $\mathrm{I}[S_1 \geq F_{S_1}^{-1}(p)]$ of every digital option written on S_1 (paying 1 if $S_1 \geq F_{S_1}^{-1}(p)$ and nothing otherwise). In an insurance setting, we see that (2.2) ensures that the contribution of X to the total capital measured by the conditional tail expectation (CTE) is smaller for S_1 compared to S_2.

3 Proportional case

3.1 Characterization

When $X_1, X_2, \ldots, X_n$ are exchangeable, in particular if they are independent and identically distributed, it is well known that

$$\mathrm{E}[X_i|S] = \frac{S}{n} \tag{3.1}$$

so that the conditional expectation defining the risk sharing rule appears to be proportional to S. We refer the reader to Albrecht and Huggenberger (2017) for a thorough investigation of the exchangeable case.

A proportional relation similar to (3.1) also holds in other situations, as established by Furman et al. (2018). Before stating the characterization derived by these authors, let us give a simple extension of (3.1) where proportionality holds. Assume that losses are decomposed into their frequency component N_i and the corresponding severities $C_{i1}, C_{i2}, \ldots$, that is, $X_i = \sum_{k=1}^{N_i} C_{ik}$. Severities C_{ik} are independent and identically distributed for all i and k, and independent of the counts N_i. Frequencies N_i are also assumed to be mutually independent and such that

$$N_i = \sum_{k=1}^{\nu_i} M_{ij}$$

where the integer-valued random variables M_{ij} are independent and identically distributed for all i and j and $\nu_1, \nu_2, \ldots, \nu_n$ are positive integers. In this model, losses X_i only differ by their "volume" reflected in ν_i. Then, X_i can be rewritten as

$$X_i = \sum_{j=1}^{\nu_i} T_{ij} \text{ with } T_{ij} = \sum_{k=1}^{M_{ij}} C_{ik}^{(j)}$$

where $C_{ik}^{(j)}$ are independent and identically distributed for all i, j and k, and distributed as C_{ik}. Since the random variables T_{ij} are independent and identically distributed for all i and j, it directly follows from (3.1) that

$$\begin{aligned}
\mathrm{E}[X_i|S] &= \sum_{j=1}^{\nu_i} \mathrm{E}[T_{ij}|T_{\bullet\bullet}] \text{ where } T_{\bullet\bullet} = \sum_{i=1}^{n}\sum_{j=1}^{\nu_i} T_{ij} = S \\
&= \nu_i \mathrm{E}[T_{11}|T_{\bullet\bullet}] \\
&= \frac{\nu_i}{\nu_\bullet} S \text{ where } \nu_\bullet = \sum_{i=1}^{n} \nu_i
\end{aligned}$$

and we end up with a conditional expectation proportional to S.

This construction works when N_i obeys some infinitely divisible distribution (such as Poisson or Negative Binomial, provided expected frequencies are multiples of the same unit) or Binomial, for instance. The entire Panjer-Katz family widely used in insurance studies is thus covered. We refer the reader to Denuit and Robert (2020a) for the study of conditional mean risk sharing for compound sums with frequency components in that family.

Furman et al. (2018) characterized the situation where the conditional expectation $\mathrm{E}[X|S]$ is proportional to $S = X + Y$. Let $L(u, v) = \mathrm{E}[\exp(-uX - vY)]$ be the Laplace transform of the random couple (X, Y). Then,

$$\mathrm{E}[X|S] = \frac{\mathrm{E}[X]}{\mathrm{E}[S]} S \Leftrightarrow \left.\frac{\frac{\mathrm{d}}{\mathrm{d}u} L(u, v)}{\frac{\mathrm{d}}{\mathrm{d}v} L(u, v)}\right|_{(u,v)=(t,t)} = \frac{\mathrm{E}[X]}{\mathrm{E}[Y]} \text{ for all } t. \tag{3.2}$$

If X and Y are mutually independent then the requirement in (3.2) is equivalent to the constantness of the function $t \mapsto \ln L_X(t)/\ln L_Y(t)$ over $[0, \infty)$, where $L_X(t) = \mathrm{E}[\exp(-tX)]$ and $L_Y(t) = \mathrm{E}[\exp(-tY)]$ are the respective Laplace transforms of X and Y. The constant must be $\mathrm{E}[X]/\mathrm{E}[Y]$.

In essence, (3.2) means that all risk-averse agents recognize that there is some diversification between X and Y, in the sense that the sum S is smaller compared to both X and Y in the Lorenz order, whose definition is recalled next.

3.2 Lorenz order

Lorenz order is defined by means of pointwise comparison of the Lorenz curves. These curves are widely used in economics to measure income inequality. Precisely, the Lorenz curve L_X associated with the non-negative random variable X is defined at probability level $p \in (0, 1)$ by

$$L_X(p) = \frac{\int_0^p F_X^{-1}(\xi)\mathrm{d}\xi}{\int_0^1 F_X^{-1}(\xi)\mathrm{d}\xi} = \frac{\mathrm{E}\left[X\mathrm{I}\left[X \leq F_X^{-1}(p)\right]\right]}{\mathrm{E}[X]}. \tag{3.3}$$

Formula (3.3) shows that the Lorenz curve of X is just the concentration curve of X with respect to itself. We refer the interested reader to Arnold and Sarabia (2018) for an extensive presentation or Lorenz curves.

Given two risks X and Y, X is said to be smaller than Y in the Lorenz order, which is henceforth denoted as $X \preceq_{\text{Lorenz}} Y$ when the graph of the Lorenz curve for X lies above the graph of the Lorenz curve for Y. Formally,

$$X \preceq_{\text{Lorenz}} Y \Leftrightarrow L_X(p) \geq L_Y(p) \text{ for all } p \in [0, 1]. \tag{3.4}$$

It is known from formula (3.A.33) in Shaked and Shanthikumar (2007) that given two risks X and Y,

$$X \preceq_{\text{Lorenz}} Y \Leftrightarrow \frac{X}{\mathrm{E}[X]} \preceq_{\text{convex}} \frac{Y}{\mathrm{E}[Y]}. \tag{3.5}$$

Considering $\mathrm{E}[X]$ and $\mathrm{E}[Y]$ as the pure premiums for X and Y, Lorenz order is thus equivalent to compare loss ratios $\frac{X}{\mathrm{E}[X]}$ and $\frac{Y}{\mathrm{E}[Y]}$ with the help of convex order. Whereas only risks possessing the same means can be compared through $\preceq_{\text{convex}}$, ordering risks through $\preceq_{\text{Lorenz}}$ does not require that X and Y have the same first moment. Obviously, when $\mathrm{E}[X] = \mathrm{E}[Y]$, we then have $X \preceq_{\text{Lorenz}} Y \Leftrightarrow X \preceq_{\text{convex}} Y$ from (3.5).

Coming back to (3.2), we see that it can be equivalently rewritten as

$$\mathrm{E}\left[\frac{X}{\mathrm{E}[X]}\middle|\frac{S}{\mathrm{E}[S]}\right] = \frac{S}{\mathrm{E}[S]} \text{ and } \mathrm{E}\left[\frac{Y}{\mathrm{E}[Y]}\middle|\frac{S}{\mathrm{E}[S]}\right] = \frac{S}{\mathrm{E}[S]}.$$

By Jensen inequality, this implies that

$$\frac{S}{\mathrm{E}[S]} \preceq_{\text{convex}} \frac{X}{\mathrm{E}[X]} \text{ and } \frac{S}{\mathrm{E}[S]} \preceq_{\text{convex}} \frac{Y}{\mathrm{E}[Y]}.$$

Hence, (3.2) ensures that the sum S is smaller compared to both X and Y in the Lorenz order. In other words, aggregating the risks X and Y is considered to be beneficial in terms of loss ratios by all risk-averse agents when (3.2) holds true.

3.3 Comparison of pools

The next result shows that agents prefer the pool with the smaller total loss in the sense of the Lorenz order, when conditional expectations involved in the risk sharing are proportional to total losses.

Proposition 3.1. *Consider three possibly correlated risks X, Y_1 and Y_2. Define $S_1 = X + Y_1$ and $S_2 = X + Y_2$ and assume that both (X, Y_1) and (X, Y_2) satisfy* (3.2)*. Then,*

$$\mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2] \Leftrightarrow S_1 \preceq_{\text{Lorenz}} S_2.$$

Proof. We see that

$$\begin{aligned} \mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2] \quad &\Leftrightarrow \quad \frac{\mathrm{E}[X]}{\mathrm{E}[S_1]}S_1 \preceq_{\text{convex}} \frac{\mathrm{E}[X]}{\mathrm{E}[S_2]}S_2 \\ &\Leftrightarrow \quad \frac{S_1}{\mathrm{E}[S_1]} \preceq_{\text{convex}} \frac{S_2}{\mathrm{E}[S_2]} \text{ by (2.1).} \end{aligned}$$

It turns out from (3.5) that the latter condition corresponds to the Lorenz order. This ends the proof. □

Thus we see that under (3.2), pool 1 is preferred over pool 2 by all risk-averse agents if, and only if, $S_1 \preceq_{\text{Lorenz}} S_2$.

4 Pooling independent losses

In the remainder of this paper, we assume that risks $X_1, X_2, \ldots$ are independent. This assumption is not restrictive for the application to insurance, including collaborative insurance considered in the next section. Henceforth, we use the notation $S_n = \sum_{i=1}^{n} X_i$ for the total risk of the pool comprising n participants, emphasizing the dependence on n. Often in the literature devoted to insurance, the random variables X_i are assumed to be identically distributed. Here, we depart from the homogeneous situation and explicitly allow for different distributions.

4.1 Recruiting new participants

4.1.1 Proportional case

We know that (3.2) in essence ensures the existence of risk diversification by pooling. Indeed, assume that condition (3.2) holds true with $X = X_i$ and $Y = S_n - X_i$ for all n and $i \in \{1, 2, \ldots, n\}$. Thus,

$$\mathrm{E}[X_i|S_n] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n \text{ for all } n \text{ and } i \in \{1, 2, \ldots, n\},$$

which can be rewritten as

$$\mathrm{E}\left[\frac{X_i}{\mathrm{E}[X_i]} \Big| \frac{S_n}{\mathrm{E}[S_n]}\right] = \frac{S_n}{\mathrm{E}[S_n]} \quad \Rightarrow \quad \frac{S_n}{\mathrm{E}[S_n]} \preceq_{\text{convex}} \frac{X_i}{\mathrm{E}[X_i]} \Leftrightarrow S_n \preceq_{\text{Lorenz}} X_i.$$

Now, summing $\mathrm{E}[X_i|S_{n+1}] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_{n+1}]} S_{n+1}$ over $i \in \{1, 2, \ldots, n\}$, we also have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|S_{n+1}] = \mathrm{E}[S_n|S_{n+1}] = \frac{\mathrm{E}[S_n]}{\mathrm{E}[S_{n+1}]} S_{n+1} \Rightarrow S_{n+1} \preceq_{\text{Lorenz}} S_n.$$

This means that recruiting a new member bringing the loss X_{n+1} is beneficial for all participants, in the sense that $\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{convex}} \mathrm{E}[X_i|S_n]$ for every $i = 1, 2, \ldots, n$. Whatever the distributions of the risks X_i, the inclusion of new participants within the pool is always considered as beneficial by all risk-averse economic agents provided independent losses are allocated according to the conditional mean risk sharing. The next section shows that this result holds true beyond the proportional case (3.2).

4.1.2 Monotonicity of participants' contributions within increasing pools

Recall that when risks X_i are independent and identically distributed, every participant pays $\mathrm{E}[X_1|S_n] = \overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Denote as μ the common mean of the risks X_i. Since

$$\frac{S_n}{\mathrm{E}[S_n]} = \frac{1}{\mu} \overline{X}_n$$

in this case, and we know from Example 3.A.29 in Shaked and Shanthikumar (2007) that

$$\overline{X}_{n+1} \preceq_{\text{convex}} \overline{X}_n \tag{4.1}$$

holds for all n when risks X_i are independent and identically distributed, we see that recruiting more members is always beneficial. Let us now establish that a similar result holds true for independent but not identically distributed risks, replacing $\overline{X}_n$ with $\mathrm{E}[X_1|S_n]$.

Proposition 4.1. *Consider independent risks $X_1, X_2, \ldots$ with partial sums $S_n = X_1 + \ldots + X_n$. Then, the stochastic inequality*

$$\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{convex}} \mathrm{E}[X_i|S_n]$$

holds for every integer n and $i \in \{1, 2, \ldots, n\}$.

Proof. Without loss of generality, let us establish the result for $i = 1$. Since the random variables $X_1, X_2, \ldots$ are mutually independent, we can write

$$\begin{aligned} \mathrm{E}[X_1|S_n] &= \mathrm{E}[X_1|S_n, X_{n+1}, X_{n+2}, \ldots] \\ &= \mathrm{E}[X_1|S_n, S_{n+1}, S_{n+2}, \ldots] \end{aligned}$$

because the sigma-algebra $\sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$ generated by $S_n, X_{n+1}, X_{n+2}, \ldots$ coincides with $\sigma(S_n, S_{n+1}, S_{n+2}, \ldots)$. Similarly, we get

$$\mathrm{E}[X_1|S_{n+1}] = \mathrm{E}[X_1|S_{n+1}, S_{n+2}, S_{n+3}, \ldots].$$

Thus, we see that

$$\begin{aligned} \mathrm{E}[X_1|S_{n+1}] &= \mathrm{E}\big[\mathrm{E}[X_1|S_n, S_{n+1}, S_{n+2}, \ldots]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big] \\ &= \mathrm{E}\big[\mathrm{E}[X_1|S_n]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big] \end{aligned}$$

for every integer n. Considering a convex function g, Jensen's inequality allows us to write

$$\begin{aligned} \mathrm{E}\Big[g\big(\mathrm{E}[X_1|S_n]\big)\Big] &= \mathrm{E}\Big[\mathrm{E}\big[g\big(\mathrm{E}[X_1|S_n]\big)\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big]\Big] \\ &\geq \mathrm{E}\Big[g\big(\mathrm{E}\big[\mathrm{E}[X_1|S_n]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big]\big)\Big] \\ &= \mathrm{E}\Big[g\big(\mathrm{E}[X_1|S_{n+1}]\big)\Big]. \end{aligned}$$

This ends the proof. □

Let us mention that Proposition 4.1 is related to peacock theory, that is, to processes with marginals that are monotonic in the sense of the convex order. We refer the reader to Hirsch et al. (2011) for a detailed account. Notice also that no monotonicity condition is imposed on $s \mapsto \mathrm{E}[X_i|S_n = s]$ in Proposition 4.1.

In the expected utility setting, Proposition 4.1 shows that increasing the number of participants is always regarded as beneficial by all risk-averse economic agents, whatever the distribution of the risks they bring to the pool as long as these risks are mutually independent and the sharing is operated according to the conditional mean risk allocation rule. The conditional mean risk sharing rule thus provides the appropriate extension to simple averaging in the homogeneous case.

Recruiting a new member bringing the loss X_{n+1} is beneficial for all participants, in the sense that $\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{convex}} \mathrm{E}[X_i|S_n]$ is valid for every $i = 1, 2, \ldots, n$. Provided that $s \mapsto \mathrm{E}[X_i|S_n = s]$ is continuously increasing for $i = 1, 2, \ldots, n$, the stochastic inequality in Proposition 4.1 ensures that

$$\mathrm{E}[X_i|S_{n+1} \geq F^{-1}_{S_{n+1}}(p)] \leq \mathrm{E}[X_i|S_n \geq F^{-1}_{S_n}(p)] \text{ for all probability level } p \text{ and } i = 1, 2, \ldots, n.$$

Notice that this inequality always holds true on an aggregate basis as summing them over i gives

$$\mathrm{E}[S_n|S_{n+1} \geq F^{-1}_{S_{n+1}}(p)] \leq \mathrm{E}[S_n|S_n \geq F^{-1}_{S_n}(p)]$$

which is known to be true for all p since $\mathrm{E}[Z|A] \leq \mathrm{E}[Z|Z \geq F^{-1}_Z(p)]$ is valid for all random event A with probability $1 - p$, whatever the continuous random variable Z.

4.1.3 Residual risk

From the proof of Proposition 4.1, we see that the sequence $\{\mathrm{E}[X_1|S_n],\ n = 1, 2, \ldots\}$, forms a reverse martingale. As a consequence, $\mathrm{E}[X_1|S_n]$ converges to the conditional expectation $\mathrm{E}[X_1|\mathcal{T}]$ where $\mathcal{T}$ is the tail sigma-field

$$\mathcal{T} = \cap_{n=1}^{\infty} \sigma(S_n, S_{n+1}, S_{n+2}, \ldots).$$

If $\mathcal{T}$ is a trivial sigma-field then $\mathrm{E}[X_1|\mathcal{T}] = \mathrm{E}[X_1]$ holds with probability 1. This is the case for instance when the random variables X_i are independent and identically distributed and corresponds to Proposition 1.1 in Zabell (1980). However, this is not always true, as shown by the next example (inspired from Example 2.3 in Denuit (2019)).

Example 4.2. *Consider the particular case where independent Gamma losses are shared among participants. Precisely, the probability density function of X_i is given by*

$$f_{X_i}(x) = \begin{cases} \frac{x^{\alpha_i - 1}\tau^{\alpha_i} \exp(-x\tau)}{\Gamma(\alpha_i)}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

for some positive parameters α_i and τ. Here, $\mathrm{E}[X_i] = \alpha_i/\tau$ and $\mathrm{Var}[X_i] = \alpha_i/\tau^2$. This is henceforth denoted as $X_i \sim \mathcal{G}am(\alpha_i, \tau)$. Then, $S_n \sim \mathcal{G}am(\alpha_\bullet, \tau)$ where $\alpha_\bullet = \sum_{i=1}^{n} \alpha_i$. It is easy to see that (3.2) *applies to this situation so that*

$$\mathrm{E}[X_i|S_n] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n = \frac{\alpha_i}{\alpha_\bullet} S_n \sim \mathcal{G}am\left(\alpha_i, \frac{\alpha_\bullet \tau}{\alpha_i}\right).$$

Letting n tend to infinity with α_i such that $\alpha_\bullet$ converges to some constant $\alpha_\infty < \infty$, we see that the limit of $\mathrm{E}[X_i|S_n]$ remains random and Gamma distributed. The intuition is as follows: the risks entering the sum tend to become smaller and smaller so that the corresponding series converges to a random variable. This means that X_1 cannot be diluted within the pool and diversification reaches its limit. This situation is certainly not realistic but it shows that the limit may remain random.

5 Application to collaborative insurance

In collaborative, P2P insurance, participants agree to pool the risks they face. P2P insurance thus builds on the fundamental principle of mutuality at the origin of insurance without the equity buffer provided by insurer's capital. Unclaimed money can be paid back, given to a charity or used to fund a common project to which the community adheres. We refer to Clemente and Marano (2020) and the references therein for an overview of P2P insurance.

Setting a collaborative insurance scheme raises the following question: how to fairly share, in an understandable and transparent way, that part of the premium that has not been used to cover the claims? A convenient solution is provided by the conditional mean risk sharing according to which participant i contributes an amount $\mathrm{E}[X_i|S_n]$ corresponding to the average part of the total loss S_n that can be attributed to the risk he or she brought to the pool. Explained in this way, the conditional mean risk sharing rule appears to be very intuitive and easily understandable by the members of the P2P insurance community.

Considering the results derived in Section 4, the conditional mean risk sharing appears to be tailored to insurance applications since it provides the appropriate allocation of total risk S_n to each participant so that increasing the size n of the pool is always regarded as beneficial. This also ensures that provided n is large enough, collaborative insurance finally outperforms commercial insurance, as shown next.

In commercial insurance, the variations in S are absorbed by insurer's capital and individual i must contribute a fixed amount π_i to compensate the cost of insured events. The premium π_i is equal to the pure premium $\mu_i = \mathrm{E}[X_i]$ supplemented with shareholder's profit for providing the risk-bearing capital. Hence, the commercial insurance premium π_i is loaded: $\pi_i > \mu_i$.

Let us first consider the homogeneous case where risks X_i are independent and identically distributed, with common mean μ. The premium π is then equal for all individuals. Assume that participants are expected-utility maximizers. Let w_i stand for the initial wealth for participant i, with utility function u_i. Stochastic inequality (4.1) shows that the expected utility $\mathrm{E}[u_i(w_i - \overline{X}_n)]$ increases in n when u_i is concave (expressing risk aversion), tending to $u_i(w_i - \mu)$. Hence, for some n_i, it must exceed $u_i(w_i - \pi)$. Thus, there must be a critical size $n = \max_i n_i$ such that every agent joins the pool instead of buying insurance.

Now, let us consider the more general setting where risks X_i are mutually independent but not necessarily identically distributed. Since we known from Proposition 4.1 that $\mathrm{E}[X_i|S_n]$ decreases with n in the sense of $\preceq_{\text{convex}}$, a similar reasoning holds provided $\overline{X}_n$ is replaced with $\mathrm{E}[X_i|S_n]$. Under some technical conditions, Denuit and Robert (2020b) established that $\mathrm{E}[X_i|S_n]$ converges to μ_i, almost surely. The expected utility $\mathrm{E}[u_i(w_i - \mathrm{E}[X_i|S_n])]$ thus increases in n when u_i is concave, tending to $u_i(w_i - \mu_i)$. As in the simple case of independent and identically distributed risks, there must be some n_i such that it exceeds $u_i(w_i - \pi_i)$. The expected utility when joining the pool thus becomes larger than the expected utility associated to buying insurance for a loaded premium for some sufficiently large n, precisely $n \geq \max_i n_i$.

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