Water renewal of a semi-enclosed domain: a generic approach using the concepts of age and residence time

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Over recent years I devoted efforts to the understanding of the water renewal of semi-enclosed domains (Deleersnijder et al. 1997, Tartinville et al. 1997, Deleersnijder et al. 1998). Others published studies on a similar topic, such as the interesting approach of Braunschweig et al. (2003). In my opinion, it is possible to gain more insight into water renewal processes than was achieved in most recent articles by having recourse to a method in which age and residence time are jointly used. This approach is outlined below.

The hydrodynamics of the domain of interest

Let the volume Ω represent the domain of interest, of which the boundary is the surface Γ (Figure 1). The outward unit normal vector to Γ is denoted **n**. The domain boundary may be regarded as made up of three parts, of which the largest, Γ^i, is impermeable. The other two fractions of the boundary, Γ^r and Γ^s, are open. The former is located in a river, while the latter delineates the boundary between the domain of interest and the sea.

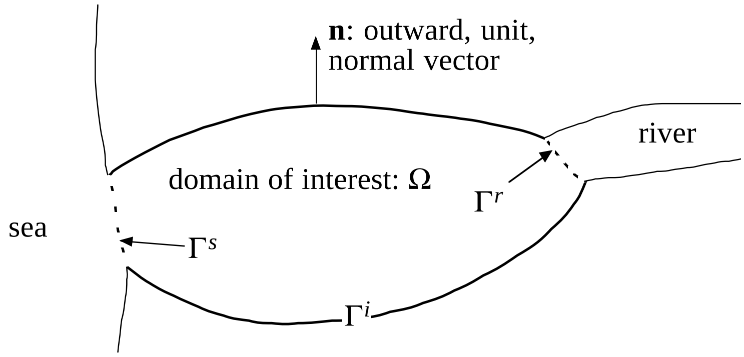

Figure 1. The domain of interest and its boundaries.

The velocity $\mathbf{u}(t,\mathbf{x})$ varies in time and space, but remains divergence-free at any time t and position $\mathbf{x}$:

$$\nabla \bullet \mathbf{u} = 0 \ . \tag{1}$$

There is no water flux through the impermeable boundary:

$$\text{on } \Gamma^i : \ \mathbf{u} \bullet \mathbf{n} = 0 \ . \tag{2}$$

The water is always entering the domain of interest through the riverine boundary:

$$\text{on } \Gamma^r : \ \mathbf{u} \bullet \mathbf{n} \leq 0 \ . \tag{3}$$

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The volume of the domain of interest is assumed to be constant, implying that the water flux entering the domain through the riverine boundary must be balanced by that leaving it through the marine boundary Γ^s. In other words, the integral constraint

$$\int_{\Gamma^s} \mathbf{u} \bullet \mathbf{n} \; d\mathbf{x}^\Gamma \; + \; \int_{\Gamma^r} \mathbf{u} \bullet \mathbf{n} \; d\mathbf{x}^\Gamma \; = \; 0 \tag{4}$$

must be met at any time, which implies that $\mathbf{u} \bullet \mathbf{n}$ must be positive at least on a fraction of Γ^s.

Tracing three types of water

To understand water renewal processes, it is useful to trace different water types which will then be regarded as passive — or inert — tracers (Deleersnijder et al. 2001, Deleersnijder et al. 2002, Braunschweig et al. 2003). Three categories of water may be identified:
— The *river water*: the water that enters the domain through the riverine boundary.
— The *sea water*: the water originating from the sea, which enters the domain through the regions of the boundary Γ^s where $\mathbf{u} \bullet \mathbf{n} < 0$.
— The *original water*: the water which is in the domain at the initial instant t=0, and leaves it progressively through the marine boundary to be replaced by river and sea water.

Let the subscript "*r*", "*s*" and "*o*" be associated with the river, sea and original water, respectively. At any time and position, the concentrations or fractions of these waters must satisfy the inequalities

$$0 \;\le\; C_\mu(t,\mathbf{x}) \;\le\; 1 \;, \quad \mu = r,s,o \;, \tag{5}$$

$$C_w(t,\mathbf{x}) \;=\; C_r(t,\mathbf{x}) \;+\; C_r(t,\mathbf{x}) \;+\; C_o(t,\mathbf{x}) \;=\; 1 \tag{6}$$

where C_w is the concentration of the "aggregated water", which is obviously equal to unity. Then, the amount of water of type μ contained in the domain of interest may be estimated as

$$m_\mu(t) \;=\; \int_\Omega C_\mu(t,\mathbf{x}) \; d\mathbf{x} \;, \quad \mu = r,s,o \;. \tag{7}$$

As the original water is progressively replaced by other waters, its content in Ω must decrease:

$$\frac{d}{dt} m_o(t) \;\le\; 0 \;. \tag{8}$$

The concentration of every type of water is governed by an advection-diffusion equation:

$$\frac{\partial C_\mu}{\partial t} \;=\; - \;\nabla \bullet (\mathbf{u}C_\mu - \mathbf{K} \bullet \nabla C_\mu) \;, \quad \mu = r,s,o \;, \tag{9}$$

where **K** is the diffusivity, which is symmetric and positive definite (e.g. Deleersnijder et al. 2001). At the initial instant, common sense suggests that the following values of the water concentrations be prescribed:

$$C_o(0,\mathbf{x}) \;=\; 1 \;, \quad C_r(0,\mathbf{x}) \;=\; 0 \;=\; C_s(0,\mathbf{x}) \;. \tag{10}$$

Selecting the boundary conditions is less trivial, however, as they must lead to solutions satisfying relations (5), (6) and (8). Conceivable flux boundary conditions are listed in Table I. In the latter, use is made of operators selecting the positive or negative part of a function, which are defined as

$$\xi^+ \;=\; \frac{\xi + |\xi|}{2} \;, \quad \xi^- \;=\; \frac{\xi - |\xi|}{2} \;. \tag{11}$$

Table I. The value of the advection-diffusion flux $(\mathbf{u}C_\mu - \mathbf{K} \bullet \nabla C_\mu) \bullet \mathbf{n}$ for all types of water ($\mu = r,s,o$) on every boundary of the domain of interest.

	Γ^i	Γ^r	Γ^s
river water	0	$\mathbf{u} \bullet \mathbf{n}$	$(\mathbf{u} \bullet \mathbf{n})^+ C_r$
sea water	0	0	$(\mathbf{u} \bullet \mathbf{n})^+ C_s + (\mathbf{u} \bullet \mathbf{n})^-$
original water	0	0	$(\mathbf{u} \bullet \mathbf{n})^+ C_o$

Using the constraint (4), the evolution equation (9), the initial conditions (10), and the boundary conditions listed in Table I, the differential problem governing the concentration of the aggregated water, $C_w(t,\mathbf{x})$ is found to be:

$$\frac{\partial C_w}{\partial t} = - \nabla \bullet (\mathbf{u}C_w - \mathbf{K} \bullet \nabla C_w) , \tag{12}$$

$$C_w(0,\mathbf{x}) = 1 , \tag{13}$$

$$\text{on } \Gamma : \ (\mathbf{K} \bullet \nabla C_w) \bullet \mathbf{n} = 0 . \tag{14}$$

It is then readily seen that, as expected, the concentration $C_w(t,\mathbf{x})$ must be equal to unity at any time and position. Thus, condition (6) is met. Demonstrating that conditions (5) and (8) hold valid is less easy. First, a positivity lemma has to be established.

Positivity lemma: *Consider the function* $\xi(t,\mathbf{x})$ *which is the solution, in the domain Ω, of the following partial differential problem*

$$\frac{\partial \xi}{\partial t} = p - \nabla \bullet (\mathbf{u}\xi - \mathbf{K} \bullet \nabla \xi) , \tag{15}$$

$$\xi(0,\mathbf{x}) \geq 0 , \tag{16}$$

$$\text{on } \Gamma^i : \ (\mathbf{K} \bullet \nabla \xi) \bullet \mathbf{n} = 0 , \tag{17}$$

$$\text{on } \Gamma^r : \ (\mathbf{u}\xi - \mathbf{K} \bullet \nabla \xi) \bullet \mathbf{n} = \gamma^r \mathbf{u} \bullet \mathbf{n} , \tag{18}$$

$$\text{on } \Gamma^s : \ (\mathbf{u}\xi - \mathbf{K} \bullet \nabla \xi) \bullet \mathbf{n} = (\mathbf{u} \bullet \mathbf{n})^+ \xi + \gamma^s (\mathbf{u} \bullet \mathbf{n})^- , \tag{19}$$

where p, γ^r *and* γ^s *are non-negative functions. Then, at any time t and location* $\mathbf{x}$*, the function* $\xi(t,\mathbf{x})$ *is positive.*

To show that the lemma is valid, a method similar to that used in Appendix C of Deleersnijder et al. (2001) is resorted to. The negative part of $\xi(t,\mathbf{x})$, if any, obeys the integral equation

$$\begin{aligned} \frac{d}{dt} \int_\Omega \frac{(\xi^-)^2}{2} \, d\mathbf{x} = & \int_\Omega (p\xi^- - \nabla \xi^- \bullet \mathbf{K} \bullet \nabla \xi^-) \, d\mathbf{x} \\ & + \frac{1}{2} \int_{\Gamma^r} \xi^- (\xi^- - 2\gamma^r) \mathbf{u} \bullet \mathbf{n} \, d\mathbf{x}^\Gamma \\ & + \frac{1}{2} \int_{\Gamma^s} [\xi^- (\xi^- - 2\gamma^s)(\mathbf{u} \bullet \mathbf{n})^- - (\xi^-)^2 (\mathbf{u} \bullet \mathbf{n})^+] \, d\mathbf{x}^\Gamma \end{aligned} \tag{20}$$

which is obtained by manipulating (15) and (17)-(19). Clearly, in the right-hand side of the relation above, all terms are negative. As a consequence, the integral of the square of the negative part of $\xi(t,\mathbf{x})$ cannot increase. Taking into account the fact that $\xi^-(t,\mathbf{x})$ is zero at t=0 — since $\xi(0,\mathbf{x}) \geq 0$ —, $\xi^-(t,\mathbf{x})$ remains zero at any time and position. Having no negative part, the function $\xi(t,\mathbf{x})$ is thus positive. QED.

The differential problem governing the evolution of the concentration of every type of water may be cast into a form consistent with (15)-(16). Therefore, all water concentrations remain positive. As their sum is equal to unity, the water concentrations satisfy inequalities (5). Integrating over the domain of interest the equation governing the concentration of the original water, applying the divergence theorem and the relevant boundary conditions, one obtains

$$\frac{d}{dt} m_o(t) \;=\; - \int_{\Gamma^s} (\mathbf{u} \bullet \mathbf{n})^+ C_o \;\; d\mathbf{x}^s \; . \tag{21}$$

This derivative is negative since $(\mathbf{u} \bullet \mathbf{n})^+ \geq 0$ and $C_o \geq 0$, so that constraint (8) is satisfied.

Estimating timescales

As the amount of original water contained in the domain does not increase, it is possible to estimate the mean residence time of this water by means of the following formula (Bolin and Rodhe 1973):

$$\theta_o \;=\; - \frac{1}{m_o(0)} \int_{m_o(0)}^{0} t \;\; dm_o \;\; , \tag{22}$$

The latter expression may be transformed to a relation easier to deal with:

$$\theta_o \;=\; - \frac{1}{m_o(0)} \int_0^{\infty} t \; \frac{dm_o}{dt} \; dt \;=\; \frac{1}{m_o(0)} \int_0^{\infty} m_o(t) \;\; dt \; . \tag{23}$$

Table II. The value of the age concentration advection-diffusion flux $(\mathbf{u}\alpha_\mu - \mathbf{K} \bullet \nabla\alpha_\mu) \bullet \mathbf{n}$ for all types of water ($\mu = r,s,o$) on every boundary of the domain of interest.

	Γ^i	Γ^r	Γ^s
river water	0	0	$(\mathbf{u} \bullet \mathbf{n})^+ \alpha_r$
sea water	0	0	$(\mathbf{u} \bullet \mathbf{n})^+ \alpha_s$
original water	0	0	$(\mathbf{u} \bullet \mathbf{n})^+ \alpha_o$

To gain a deeper insight into the water renewal processes, it may be useful to calculate the age of every type of water at any time and location. According to the Constituent-oriented Age Theory (CAT) (Delhez et al. 1999, Deleersnijder et al. 2001), the age of every type water, $a_\mu(t,\mathbf{x})$, may be obtained as

$$a_\mu(t,\mathbf{x}) \;=\; \frac{\alpha_\mu(t,\mathbf{x})}{C_\mu(t,\mathbf{x})} \;, \quad \mu = r,s,o \;, \tag{24}$$

where $\alpha_\mu(t,\mathbf{x})$ is the age concentration, which satisfies the equation

$$\frac{\partial\alpha_\mu}{\partial t} \;=\; C_\mu \;-\; \nabla\bullet(\mathbf{u}C_\mu - \mathbf{K}\bullet\nabla C_\mu) \;, \quad \mu = r,s,o \;. \tag{25}$$

All ages are assumed to be zero at the initial instant, which requires

$$\alpha_\mu(0,\mathbf{x}) \;=\; 0 \;, \quad \mu = r,s,o \;. \tag{26}$$

For the original water, the age is meant to be a measure of the elapsed time, while for the other waters, the age is to estimate the amount of time elapsed since entering the domain through the riverine or the marine boundary. The relevant boundary conditions are listed in Table II.

Using the positivity lemma, it is possible to demonstrate that all ages meet the following common sense constraints:

$$0 \;\le\; a_\mu(t,\mathbf{x}) \;\le\; t \;, \quad \mu = r,s,o \;. \tag{27}$$

In addition, it may be seen that, at any time and location, the age of the original water is

$$a_o(t,\mathbf{x}) \;=\; t \;. \tag{28}$$

This is no useful diagnostic, but this result is reassuring as to the relevance of the problem from which ages are derived herein.

Discussion

The generic approach developed above allows to trace three types of water and estimate the associated ages so as to gain in-depth insight into the way the water is renewed in a semi-enclosed domain. The well-foundedness of the mathematical problems from which the concentration and age of the waters may be derived has been demonstrated.

It is desirable to extend the present theory to a variable-volume domain. This would help study the impact of tidal motions. On the other hand, no major modification is needed if several rivers or several marine boundaries are to be taken into account. Finally, it is necessary to illustrate the present theoretical development by means of numerical results from an estuary, a bay of a gulf. In this respect, collaboration is to be sought with colleagues using relevant models.

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Water renewal of a semi-enclosed domain: the relationship between the mean residence time and the domain-averaged age

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Herein the relationship is examined between the mean residence time of the original water and the domain-averaged age of the water renewing the original water assuming that *the velocity and the eddy diffusivity tensor are time-independent.*

The mean residence time may be obtained from

$$\theta \; = \; \frac{1}{m_o(0)} \int_0^\infty m_o(t) \;\; dt \; . \tag{1}$$

Since,

$$m_o(t) \; = \; \int_\Omega C_o(t,\mathbf{x}) \;\; d\mathbf{x} \; , \tag{2}$$

and

$$C_o(0,\mathbf{x}) \; = \; 1 \; , \tag{3}$$

it is readily seen that (1) may be transformed to

$$\theta \; = \; \frac{1}{\Omega} \int_0^\infty \int_\Omega C_o(t,\mathbf{x}) \;\; d\mathbf{x} \;\; dt \; . \tag{4}$$

Let the variable $D_o(\mathbf{x})$ be defined to be[2]

$$D_o(\mathbf{x}) \; = \; \int_0^\infty C_o(t,\mathbf{x}) \;\; dt \; . \tag{5}$$

Then, the mean residence time may be regarded as the domain-averaged value of $D_o(\mathbf{x})$,

$$\theta \; = \; \frac{1}{\Omega} \int_\Omega D_o(\mathbf{x}) \;\; d\mathbf{x} \; . \tag{6}$$

The following partial differential problem is obeyed by $D_o(\mathbf{x})$:

$$0 \; = \; 1 - \; \nabla \bullet (\mathbf{u}D_o - \mathbf{K} \bullet \nabla D_o) \; , \tag{7}$$

$$\text{on } \; \Gamma^i \; \text{ and } \Gamma^r : \;\; (\mathbf{u}D_o - \mathbf{K} \bullet \nabla D_o) \bullet \mathbf{n} \; = \; 0 \; , \tag{8}$$

$$\text{on } \; \Gamma^s : \;\; (\mathbf{u}D_o - \mathbf{K} \bullet \nabla D_o) \bullet \mathbf{n} \; = \; (\mathbf{u} \bullet \mathbf{n})^+ D_o \; . \tag{9}$$

The original water is progressively replaced in the domain of interest by river and sea waters. These two waters can be dealt with as an aggregate (e.g. Delhez et al. 1999, Deleersnijder et al. 2001) — hereafter identified by the subscript "r+s" —, of which the concentration, age concentration and age are

$$C_{r+s} = C_r + C_s \;\; , \;\;\; \alpha_{r+s} = \alpha_r + \alpha_s \;\; , \;\;\; a_{r+s} = \frac{\alpha_{r+s}}{C_{r+s}} \;\; . \tag{10}$$

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[2] The usefulness of this variable was shown by Jean-Marie Beckers in his latest Champagne-winning note, in which a different residence time problem was solved.

As the water velocity and diffusivity tensor are assumed to be time-independent, the three variables above tend to steady-state values at time increases. Clearly, $C_{r+s}(\infty,\mathbf{x}) = 1$, so that $\alpha_{r+s}(\infty,\mathbf{x}) = a_{r+s}(\infty,\mathbf{x})$. Then, from the previous note — Note1.pdf, dated 11 May 2003 — it is readily seen that, in the limit $t \rightarrow \infty$, the age $a_{r+s}(\infty,\mathbf{x}) = a^{\infty}_{r+s}(\mathbf{x})$ is the solution of the following partial differential problem:

$$\boxed{\begin{array}{c} 0 \ = \ 1 - \ \nabla \bullet (\mathbf{u} a^{\infty}_{r+s} - \mathbf{K} \bullet \nabla a^{\infty}_{r+s}) \ , \\ \text{on } \ \Gamma^i \ \text{ and } \Gamma^r : \ (\mathbf{u} a^{\infty}_{r+s} - \mathbf{K} \bullet \nabla a^{\infty}_{r+s}) \bullet \mathbf{n} \ = \ 0 \ , \\ \text{on } \ \Gamma^s : \ (\mathbf{u} a^{\infty}_{r+s} - \mathbf{K} \bullet \nabla a^{\infty}_{r+s}) \bullet \mathbf{n} \ = \ (\mathbf{u} \bullet \mathbf{n})^+ a^{\infty}_{r+s} \ . \end{array}} \qquad \begin{array}{r} (11) \\ (12) \\ (13) \end{array}$$

The problem (11)-(13) is equivalent to the problem (7)-(9), implying that

$$a_{r+s}(\infty,\mathbf{x}) \ = \ D_0(\mathbf{x}) \ . \qquad (14)$$

Hence

$$\boxed{\theta \ = \ \frac{1}{\Omega} \int_{\Omega} a_{r+s}(\infty,\mathbf{x}) \ d\mathbf{x} \ .} \qquad (15)$$

So, *the mean residence time is equal to the steady-state, domain-averaged age of the aggregate made up of river and sea waters*, i.e. the waters replacing the original water. This result is far from counterintuitive!

A definition of the mean residence time

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It is now common knowledge that the dynamics of a tracer may be studied either in the Lagrangian framework or in the Eulerian one, yielding identical results if the number of Lagrangian particles is arbitrarily large. So, for instance, the original water may be seen as a set of J ($\rightarrow \infty$) particles. Every particle is identified by subscript j ($j=1,2,\dots J$), and is characterised by its mass m_j, and the function of time $\omega_j(t)$, which is equal to unity if the particle under consideration is in the domain of interest, and is zero otherwise. At the initial instant, all particles are inside the domain of interest, so that $\omega_j(0)=1$ ($i=1,2...,J$). This approach does not rule out *a priori* the fact that every particle may be alternatively inside and outside the domain of interest.

The mass of the original water present in the domain of interest at time t obviously is

$$m_o(t) \; = \; \sum_{j=1}^{J} m_j \omega_j(t) \; . \tag{1}$$

The residence time of a particle may be defined as amount of time that this particle has spend in the domain of interest. Thus, the residence θ_j of the j-th particle is to be calculated by means of formula

$$\theta_j \; = \; \int_0^\infty \omega_j(t) \; dt \; . \tag{2}$$

The mean residence time of all the particles of the original water may be evaluated as the mass-weighted average of the residence times of these particles, i.e.

$$\theta \; = \; \frac{\sum_{j=1}^{J} m_j \theta_j}{\sum_{j=1}^{J} m_j} \; . \tag{3}$$

Then, it is readily seen that (3) may be transformed to

$$\theta \; = \; \frac{1}{m_o(0)} \int_0^\infty m_o(t) \; dt \; . \tag{4}$$

This expression of the mean residence is valid irrespective of the temporal behaviour of the function $m_o(t)$. In particular, with the approach developed herein, the mass of the original water present in the domain of interest must not necessarily be a non-increasing function.

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On the assumptions underlying the box model generally used to estimate the mean residence time

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The residence time of a semi-enclosed domain has often been estimated as the ratio of the volume of the domain to the flux entering or leaving it. It must be kept in mind, however, that this approach rests on a number of hypotheses, which are sometimes overlooked. These assumptions are:

- *Hypothesis H1:* The water flux entering the domain of interest and that leaving it are constant in time.
- *Hypothesis H2:* The water flux entering the domain of interest is equal to that leaving it.
- *Hypothesis H3:* The domain is well-mixed.
- *Hypothesis H4:* Every water parcel passes only once through the domain of interest.

In accordance with the previous note, it is assumed that the water present in the domain of interest is made up of original, river, and sea water, which are identified by subscripts "*o*", "*r*" and "*s*", respectively. Thus, at time *t*, the mass of the water in the domain of interest is

$$m(t) = m_o(t) + m_r(t) + m_s(t) , \tag{1}$$

Similarly, the volume fluxes entering and leaving the domain consist of three components, related to the three types of water, i.e.

$$\phi^{in}(t) = \phi_o^{in}(t) + \phi_r^{in}(t) + \phi_s^{in}(t) , \tag{2}$$

$$\phi^{out}(t) = \phi_o^{out}(t) + \phi_r^{out}(t) + \phi_s^{out}(t) . \tag{3}$$

Then, upon denoting ρ the density of the water — which can be considered as constant according to the Boussinesq approximation —, the mass of every type of water reads

$$\frac{d}{dt} m_\mu(t) = \rho\phi_\mu^{in}(t) - \rho\phi_\mu^{out}(t) \quad (\mu = o, r, s) , \tag{4}$$

which implies that the overall mass budget of the domains obeys

$$\frac{d}{dt} m(t) = \rho\phi^{in}(t) - \rho\phi^{out}(t) . \tag{5}$$

According to the the previous notes, the initial conditions are

$$m_o(0) = \rho\Omega , \quad m_r(0) = 0 , \quad m_s(0) = 0 , \tag{6}$$

where Ω is the volume of the domain of interest.

By virtue of assumptions *H1* and *H2*, $\phi^{in}(t)$ and $\phi^{out}(t)$ are equal and constant, so that

$$\phi^{in}(t) = \Phi = \phi^{out}(t) , \tag{7}$$

where Φ is a positive constant. Thus, according to (5), the mass of the water contained in the domain of interest is constant, i.e.

$$m(t) = \rho\Omega , \tag{8}$$

where Ω is also a positive constant.

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Hypothesis *H3* has it that the domain is well-mixed, implying that the concentration of every water component may be regarded as homogeneous over the domain of interest. Thus, the concentration of a given water component is proportional to the mass of this water component. Therefore, the associated outgoing water flux must be proportional to the mass of this water component, i.e.

$$\phi_\mu^{out}(t) \;=\; \frac{m_\mu(t)}{\tau} \quad (\mu = o, r, s) \ , \tag{9}$$

where the positive, constant timescale τ depends only on the hydrodynamics of the domain — and is independent of the water component under consideration.

By virtue of assumption *H4*, the incoming flux of the original water is zero:

$$\phi_o^{in}(t) \;=\; 0 \ . \tag{10}$$

Below, it is appropriate to consider only two types of water, the original water and the water progressively replacing it, i.e. the aggregate consisting of the river water and the sea water — which is identified by subscript “*r+s*”.

Combining all equations above, the differential problem to be solved is

$$\frac{d}{dt}m_o(t) \;=\; -\;\frac{m_o(t)}{\tau} \qquad\qquad \frac{d}{dt}m_{r+s}(t) \;=\; \rho\Phi \;-\; \frac{m_{r+s}(t)}{\tau} \tag{11}$$

$$m_o(0) \;=\; \rho\Omega \qquad\qquad m_{r+s}(0) \;=\; 0 \tag{12}$$

It is readily that the solution is

$$m_o(t) \;=\; \rho\Omega e^{-t/\tau} \ , \tag{13}$$

$$m_{r+s}(t) \;=\; \rho\tau\Phi(1 - e^{-t/\tau}) \ . \tag{14}$$

For the total water mass to be constant, it is necessary that

$$\Omega \;=\; \tau\Phi \ . \tag{15}$$

The mean residence time of the original water, θ, may be estimated by using the formula suggested in the previous note, i.e.

$$\theta \;=\; \frac{1}{m_o(t)} \int_0^\infty m_o(t) \;\; dt \ . \tag{16}$$

Substituting (13) into (16), and using (15) yields

$$\boxed{\theta \;=\; \frac{\Omega}{\Phi} \;=\; \tau} \tag{17}$$

This formula is well known and widely is used, and yet it is quite unfortunate that the hypotheses underlying it are often ignored.

Combining the solutions obtained above leads to the final form of the solutions of the well-mixed model dealt with herein:

$$\boxed{\begin{array}{cc} m_o(t) \;=\; \rho\theta\Phi e^{-t/\theta} & m_{r+s}(t) \;=\; \rho\theta\Phi(1 - e^{-t/\theta}) \\ \phi_o^{in}(t) \;=\; 0 & \phi_{r+s}^{in}(t) \;=\; \Phi \\ \phi_o^{out}(t) \;=\; \Phi e^{-t/\theta} & \phi_o^{out}(t) \;=\; \Phi(1 - e^{-t/\theta}) \end{array}}$$

On the impact of the return coefficient on the residence time in a well-mixed domain

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Herein, we revisit the developments of the previous note by relaxing hypothesis *H4*, i.e. some of the parcels of the orginal water that have left the domain of interest are now allowed to come back into it. So, the budget of the original water obeys equation

$$\frac{d}{dt} m_o(t) \;=\; \rho\phi_o^{in}(t) \;-\; \rho\phi_o^{out}(t) \;, \tag{1}$$

where the incoming flux $\phi_o^{in}(t)$, or *return flux*, is no longer assumed to be zero. As the domain is still assumed to be well-mixed, the outgoing flux is paramaterised as

$$\rho\phi_o^{out}(t) \;=\; \frac{m_o(t)}{\tau} \;, \tag{2}$$

where τ is the residence time that would prevail if the incoming flux of original water were assumed to be zero — as was the case in the previous note.

Integrating (1) over time yields

$$m_o(\infty) \;-\; m_o(0) \;=\; \rho \int_0^\infty \phi_o^{in}(t)\;dt \;-\; \frac{1}{\tau}\int_0^\infty m_o(t)\;dt \;. \tag{3}$$

It is assumed that the return flux does not prevent all parcels of the original water from eventually leaving the domain of interest. Hence, in the limit $t \to \infty$, the mass of the original water tends to zero:

$$m_o(\infty) \;=\; 0 \;. \tag{4}$$

Substituting (4) into (3), using the definition of the mean residence time established in Note # 3,

$$\theta \;=\; \frac{1}{m_o(0)} \int_0^\infty m_o(t)\;dt \;, \tag{5}$$

one obtains

$$-m_o(0) \;=\; \rho \int_0^\infty \phi_o^{in}(t)\;dt \;-\; \frac{\theta}{\tau}\; m_o(0) \;. \tag{6}$$

The magnitude of the return flux may be estimated by means of the dimensionless ratio

$$r \;=\; \frac{\rho \displaystyle\int_0^\infty \phi_o^{in}(t)\;dt}{m_o(0)} \;, \tag{7}$$

which is termed herein the *return coefficient.* The latter is the ratio of the mass of the original water that came back to the domain to the mass of the original water present in the domain at the initial instant. Clearly, this coefficient may be larger than unity.

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Combining (6) and (7), bearing in mind that the timescale τ is the mean residence time prevailing if r=0, the mean residence time may be expressed as a function of the return coefficient as follows:

$$\boxed{\theta(r) = (1+r)\ \theta(0)} \tag{8}$$

It is quite natural that the mean residence time increases as the return coefficient increases.

Applying the well-mixed box model to a domain influenced by tides

Eric Deleersnijder, 18 August 2003
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We revisit the developments of Note # 4 on well-mixed box models. Hypothesis H1 is relaxed, which now allows us to deal with a domain influenced by tides. Only one tidal component, with period *T*, is considered. The water fluxes entering and leaving the domain of interest, $\phi^{in}(t)$ and $\phi^{out}(t)$, are periodic — with period *T* — and exhibit the same mean value,

$$\frac{1}{T}\int_t^{t+T}\phi^{in}(t')\ dt' \ = \ \Phi \ = \ \frac{1}{T}\int_t^{t+T}\phi^{out}(t')\ dt' \ , \tag{1}$$

so that the volume of the domain of interest, $\Omega(t)$, is, on average, constant. The tidally-averaged volume is then denoted

$$\overline{\Omega} \ = \ \frac{1}{T}\int_t^{t+T}\Omega(t')\ dt' \ . \tag{2}$$

The mass of the original water satisfies the equation

$$\frac{dm}{dt} \ = \ -\ \frac{\phi^{out}(t)}{\Omega(t)}\ m(t)\ , \tag{3}$$

where the subscript "*o*" is omitted as we will only deal with the original water. The solution of (3) is readily seen to be

$$m(t) \ = \ m(0)\ \exp\left[-\int_0^t \frac{\phi^{out}(t')}{\Omega(t')}\ dt'\right] . \tag{4}$$

If

$$\frac{\overline{\Omega}}{\Phi} \ >> \ T \ , \tag{5}$$

then solution (4) is asymptotic to

$$m(t) \ \sim \ m(0)\ \exp\frac{-\Phi t}{\overline{\Omega}}\ , \quad \Phi T << \overline{\Omega}\ . \tag{6}$$

Therefore, the residence time is asymptotically equal to $\overline{\Omega}/\Phi$.

Let us illustrate the developments above. Assume that the outgoing water flux is the piecewise constant — crenelation-like — function defined as follows:

$$\begin{aligned} \phi^{out}(t) \ &= \ 2\Phi \ , \quad kT \le t < (k+1/2)T \\ \phi^{out}(t) \ &= \ 0 \ , \quad (k+1/2)T \le t < (k+1)T \end{aligned} \ . \tag{7}$$

If the variation of the domain volume is negligible — as suggested by (5) —, the mass of the original water is given by

$$\begin{aligned} m(t) \ &= \ m(0)\ \exp\left[-\frac{2\Phi}{\overline{\Omega}}(t-kT/2)\right], \quad kT \le t < (k+1/2)T \\ m(t) \ &= \ m(0)\ \exp\left[-\frac{\Phi}{\overline{\Omega}}(k+1)T\right], \quad (k+1/2)T \le t < (k+1)T \end{aligned} \ . \tag{8}$$

The mean residence time is then

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$$\theta \ = \ \left(\frac{1}{2} + \frac{\Phi T}{2\overline{\Omega}}\frac{1}{e^{\Phi T/\overline{\Omega}} - 1}\right)\ \frac{\overline{\Omega}}{\Phi}\ , \tag{9}$$

an expression which, not surprisingly, admits the following asymptotic expansion

$$\theta \ \sim \ \frac{\overline{\Omega}}{\Phi}\left(1 - \ \frac{\Phi T}{\overline{\Omega}}\right), \quad \frac{\Phi T}{\overline{\Omega}} \to 0\ . \tag{10}$$

It is also no wonder that

$$\int_0^\infty \frac{m(0)\exp\left(-\frac{\overline{\Omega}t}{\Phi}\right) - \ m(t)}{m(0)}\ \frac{dt}{\overline{\Omega}/\Phi} \ = \ \frac{1}{2} - \ \frac{\Phi T}{2\overline{\Omega}}\frac{1}{e^{\Phi T/\overline{\Omega}} - 1} \ \sim \ \frac{\Phi T}{4\overline{\Omega}}\ , \quad \frac{\Phi T}{\overline{\Omega}} \to 0\ . \tag{11}$$

Renouvellement de l'eau d'un domaine ouvert 1D

Eric Deleersnijder, le 2 février 2006

On considère l'écoulement 1D à vitesse constante U et diffusivité constante K dans un domaine infini. On s'intéresse au renouvellement de l'eau qui se trouve dans le segment défini par les inégalités $-L \le x \le L$. On rend les variables adimensionnelles de la manière habituelle; on travaille ici uniquement en variables adimensionnelles. Les équations font apparaître un seul paramètre adimensionnel, le nombre de Peclet,

$$Pe = \frac{UL}{K} . \tag{1}$$

Types d'eau

Il convient de distinguer 3 types d'eau:

- *eau de gauche*: eau qui se trouve au temps initial t=0 dans le domaine $x < -1$;
- *eau initiale*: eau qui se trouve au temps initial t=0 dans le segment $-1 \le x \le 1$;
- *eau de droite*: eau qui se trouve au temps initial t=0 dans le domaine $1 < x$.

On identifiera l'eau initiale, l'eau de droite et l'eau de gauche à l'aides des indices “*i*”, “*d*” et “*g*”. La nature d'une particule d'eau se modifie selon les deux règles suivantes:

- en entrant dans le sous-domaine $1 < x$, une particule d'eau devient de l'eau de droite;
- en entrant dans le sous-domaine $x < -1$, une particule d'eau devient de l'eau de gauche.

Les 3 definitions et les 2 règles ci-dessus suffisent pour établir les problèmes différentiels qui régissent les concentrations des 3 types d'eau dans le segment domaine d'intérêt.

En dehors du segment $-1 \le x \le 1$, les concentrations des 3 types d'eau sont:

$$\begin{cases} C_i(t,x) = 0 \ , \ \ C_g(t,x) = 1 \ , \ \ C_d(t,x) = 0 \ \ \text{pour} \ \ x < -1 \\ C_i(t,x) = 0 \ , \ \ C_g(t,x) = 0 \ , \ \ C_d(t,x) = 1 \ \ \text{pour} \ \ 1 < x \end{cases} . \tag{2}$$

Dans le segment $-1 \le x \le 1$, les concentrations des 3 types d'eau sont les solutions de l'équation générique

$$\frac{\partial C_s}{\partial t} + \frac{\partial C_s}{\partial x} = \frac{1}{Pe}\frac{\partial^2 C_s}{\partial x^2} \ , \ \ s = g,i,d \ . \tag{3}$$

Les valeurs initiales et conditions aux limites appropriées sont:

$$\begin{cases} C_g(0,x) = 0 \ , \ \ C_g(t,-1) = 1 \ , \ \ C_g(t,1) = 0 \\ C_i(0,x) = 1 \ , \ \ C_i(t,-1) = 0 \ , \ \ C_i(t,1) = 0 \\ C_d(0,x) = 0 \ , \ \ C_d(t,-1) = 0 \ , \ \ C_d(t,1) = 1 \end{cases} . \tag{4}$$

La solution de (3) est du type

$$C_s(t,x) = C_s(\infty,x) + e^{Pe\,x/2} \sum_{n=1}^{\infty} a_{s,n}\, e^{-\lambda_n t} \sin[k_n(1+x)] \ , \ \ s = g,i,d \ , \tag{5}$$

avec

$$k_n = \frac{n\pi}{2} \ , \tag{6}$$

$$\lambda_n = \frac{Pe}{4} + \frac{k_n^2}{Pe} , \tag{7}$$

$$a_{g,n} = \frac{-4e^{Pe/2}k_n}{Pe^2 + 4k_n^2} , \quad a_{i,n} = \frac{4\left[e^{Pe/2} - e^{-Pe/2}(-1)^n\right]k_n}{Pe^2 + 4k_n^2} , \quad a_{d,n} = \frac{4e^{-Pe/2}(-1)^n k_n}{Pe^2 + 4k_n^2} . \tag{8}$$

Bien entendu, on a recours à la définition

$$C_s(\infty,x) = \lim_{t\to\infty} C_s(t,x) . \tag{9}$$

Après quelques calculs, il vient

$$C_g(\infty,x) = \frac{e^{Pe} - e^{Pe\,x}}{e^{Pe} - e^{-Pe}} , \quad C_i(\infty,x) = 0 , \quad C_d(\infty,x) = \frac{e^{Pe\,x} - e^{-Pe}}{e^{Pe} - e^{-Pe}} . \tag{10}$$

On montre facilement que

$$C_g(t,x) + C_i(t,x) + C_d(t,x) = 1 . \tag{11}$$

Temps de résidence de l'eau initiale

Le temps de résidence $\theta_i(x)$ de l'eau initiale est la solution du problème différentiel

$$\begin{cases} -\dfrac{d\theta_i}{dx} = 1 + \dfrac{1}{Pe}\dfrac{d^2\theta_i}{dx^2} \\ \theta_i(-1) = 0 = \theta_i(1) \end{cases} . \tag{12}$$

On obtient alors aisément

$$\theta_i(x) = 1 - x - \frac{2(e^{-Pe\,x} - e^{-Pe})}{e^{Pe} - e^{-Pe}} . \tag{13}$$

Age des eaux

La concentration d'âge des différents types d'eau obéit à l'équation générique

$$\frac{\partial\alpha_s}{\partial t} + \frac{\partial\alpha_s}{\partial x} = C_s + \frac{1}{Pe}\frac{\partial^2\alpha_s}{\partial x^2} , \quad s = g,i,d . \tag{14}$$

Les conditions initiales et conditions aux limites sont

$$\alpha_s(0,x) = 0 , \quad \alpha_s(t,-1) = 0 , \quad \alpha_s(t,1) = 0 , \quad s = g,i,d . \tag{15}$$

La solution est du type

$$\alpha_s(t,x) = \alpha_s(\infty,x) + e^{Pe\,x/2}\sum_{n=1}^{\infty}(a_{s,n}t + b_{s,n})e^{-\lambda_n t}\sin[k_n(1+x)] , \quad s = g,i,d , \tag{16}$$

avec

$$b_{s,n} = \int_{-1}^{+1} e^{-Pe\,x/2}\alpha_s(\infty,x)\sin[k_n(1+x)]\,dx . \tag{17}$$

Pour l'eau initiale, on a $\alpha_i(\infty,x) = 0$, de sorte que $\alpha_i(t,x) = tC_i(t,x)$. Comme on pouvait s'y attendre, l'âge de l'eau initiale est égal au temps écoulé:

$$a_i(t,x) = \frac{\alpha_i(t,x)}{C_i(t,x)} = \frac{tC_i(t,x)}{C_i(t,x)} = t . \tag{18}$$

Pour obtenir l'âge des autres eaux, il faudrait effectuer des calculs très compliqués. Je suis pas persuadé qu'il faille faire cet effort. Je me bornerai à calculer les âges stationnaires des eaux de gauche et de droite.

Les concentrations d'âge stationnaires sont

$$\alpha_g(\infty,x) = \frac{e^{2Pe}+3-(3e^{Pe}+e^{-Pe})e^{Pe\,x}+(e^{Pe}-e^{-Pe})(e^{Pe}+e^{Pe\,x})x}{(e^{Pe}-e^{-Pe})^2} , \tag{19}$$

$$\alpha_d(\infty,x) = \frac{-e^{-2Pe}-3+(e^{Pe}+3e^{-Pe})e^{Pe\,x}-(e^{Pe}-e^{-Pe})(e^{-Pe}+e^{Pe\,x})x}{(e^{Pe}-e^{-Pe})^2} . \tag{20}$$

Aux limites du domaine d'intérêt, les âges correspondants se comportent comme suit:

$$a_g(\infty,-1) = 0 , \tag{21}$$

$$a_g(\infty,x) \sim \frac{2[1+Pe+(Pe-1)e^{2Pe}]}{Pe(e^{2Pe}-1)} - \frac{Pe}{6}(x-1)^2 , \quad x \to 1 , \tag{22}$$

$$a_d(\infty,x) \sim \frac{2[1+Pe+(Pe-1)e^{2Pe}]}{Pe(e^{2Pe}-1)} - \frac{Pe}{6}(x+1)^2 , \quad x \to -1 , \tag{23}$$

$$a_d(\infty,1) = 0 . \tag{24}$$

Eau de renouvellement

L'eau de renouvellement est l'aggrégat composé de l'eau de gauche et de l'eau de droite. Sa concentration et sa concentration d'âge valent

$$C_{g+d}(t,x) = C_g(t,x)+C_d(t,x) , \quad \alpha_{g+d}(t,x) = \alpha_g(t,x)+\alpha_d(t,x) . \tag{25}$$

A l'état stationnaire, on a

$$C_{g+d}(\infty,x) = 1 , \quad \alpha_{g+d}(\infty,x) = x - \frac{2e^{Pe\,x}}{e^{Pe}-e^{-Pe}} , \tag{26}$$

de sorte que l'âge de l'eau de renouvellement est

$$a_{g+d}(\infty,x) = x - \frac{2e^{Pe\,x}}{e^{Pe}-e^{-Pe}} . \tag{27}$$

On peut maintenant vérifier une propriété que j'avais établie de façon générale dans une autre note sur le renouvellement de l'eau. A l'état stationnaire, le temps de résidence moyen de l'eau initiale et l'âge moyen de l'eau de renouvellement,

$$(\bar{\theta}_i,\bar{a}_{g+d}) = \frac{1}{2}\int_{-1}^{+1}[\theta_i(\infty,x),a_{g+d}(\infty,x)]\,dx , \tag{28}$$

sont égaux,

$$\boxed{\bar{\theta}_i = \frac{e^{Pe}+e^{-Pe}}{e^{Pe}-e^{-Pe}} - \frac{1}{Pe} = \bar{a}_{g+d}} \tag{29}$$

On déduit facilement les comportements asymptotiques de cette expressions:

$$\bar{\theta}_i \sim \frac{Pe}{3} - \frac{Pe^3}{45} , \quad Pe \to 0 , \tag{30}$$

$$\bar{\theta}_i \sim 1 - \frac{1}{Pe}\,, \quad Pe \to \infty\,. \qquad (31)$$

Temps d'exposition dans un écoulement 1D

Eric Deleersnijder, les 4-5 février 2006

On reprend l'écoulement uni-dimensionnel de la note précédente. On va maintenant calculer les temps d'exposition — au lieu des temps de résidence — dans le segment $-1 \leq x \leq 1$[1]. Le temps d'exposition au segment concerné d'un traceur dont la concentration est $C(t,x)$ vaut

$$\Theta[C(0,x)] = \frac{\int_0^\infty \int_{-1}^{+1} C(t,x)\, dx\, dt}{\int_{-\infty}^{+\infty} C(0,x)\, dx} . \tag{1}$$

Cette formule est valable pour toute distribution initiale du traceur. Mais, on sait qu'il est avantageux de considérer que cette distribution initiale est une impulsion de Dirac, car le temps d'exposition correspondant pourra être utilisé pour évaluer le temps d'exposition de tout traceur. En effet, si $C(t,x,x_0)$ est la concentration du traceur dont la distribution initiale est l'impulsion de Dirac localisée en x_0, c'est-à-dire $C(0,x,x_0) = \delta(x - x_0)$, et si le temps d'exposition associé est

$$\Theta(x_0) = \frac{\int_0^\infty \int_{-1}^{+1} C(t,x,x_0)\, dx\, dt}{\int_{-\infty}^{+\infty} C(0,x,x_0)\, dx\, dt} = \int_0^\infty \int_{-1}^{+1} C(t,x,x_0)\, dx\, dt \ , \tag{2}$$

alors le temps d'exposition (1) devient

$$\Theta[C(0,x)] = \frac{\int_{-\infty}^{+\infty} C(0,\xi)\, \Theta(\xi)\, d\xi}{\int_{-\infty}^{+\infty} C(0,\xi)\, d\xi} . \tag{3}$$

Ci-dessous, on ne considérera que le temps d'exposition $\Theta(x_0)$ — puisque l'on peut en déduire tous les autres. On va calculer ce temps d'exposition par deux méthodes, l'approche directe et l'approche adjointe, et on va montrer que ces dernières fournissent — heureusement! — des résultats équivalents.

Approche directe

La concentration $C(t,x,x_0)$ du traceur est la solution du problème différentiel suivant

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{\partial C}{\partial x} = \dfrac{1}{Pe} \dfrac{\partial^2 C}{\partial x^2} \\ C(0,x) = \delta(x - x_0) \end{cases} . \tag{4}$$

---

[1] D'emblée, on travaille en variables adimensionnelles.

La concentration vaut

$$C(t,x,x_0) = \left(\frac{Pe}{4\pi t}\right)^{1/2} \exp\left[\frac{-Pe(x-x_0-t)^2}{4t}\right] . \tag{5}$$

Le temps d'exposition dans le segment $-1 \le x \le 1$ est alors

$$\Theta(x_0) = \left(\frac{Pe}{4\pi}\right)^{1/2} \int_0^{\infty}\int_{-1}^{+1} t^{-1/2} \exp\left[\frac{-Pe(x-x_0-t)^2}{4t}\right] dx\,dt \ . \tag{6}$$

Cette expression peut être transformée comme suit

$$\Theta(x_0) = \left(\frac{Pe}{4\pi}\right)^{1/2} \int_{-1-x_0}^{1+x_0} \exp\left(\frac{Pe\xi}{2}\right) \int_0^{\infty} t^{-1/2} \exp\left[-\frac{Pe\xi^2}{4t} - \frac{Pe\,t}{4}\right] dt\,d\xi \ . \tag{7}$$

On sait que

$$\int_0^{\infty} t^{-1/2} \exp\left[-\frac{Pe\xi^2}{4t} - \frac{Pe\,t}{4}\right] dt = \left(\frac{4\pi}{Pe}\right)^{1/2} e^{-Pe|\xi|/2} \ , \tag{8}$$

la relation (7) devient

$$\Theta(x_0) = \int_{-1-x_0}^{1-x_0} \exp\left[\frac{Pe(\xi-|\xi|)}{2}\right] d\xi \ . \tag{9}$$

Le temps d'exposition vaut alors

$$\boxed{\begin{aligned} & x_0 \le -1: \quad \Theta(x_0) = \int_{-1-x_0}^{1-x_0} e^0\,d\xi = 2 \\ & -1 \le x_0 \le 1: \quad \Theta(x_0) = \int_{-1-x_0}^{0} e^{Pe\xi}\,d\xi + \int_0^{1-x_0} e^0\,d\xi \\ & \qquad\qquad = 1 + \frac{1}{Pe} - x_0 - \frac{e^{-Pe(x_0+1)}}{Pe} \\ & 1 \le x_0: \quad \Theta(x_0) = \int_{-1-x_0}^{1-x_0} e^{Pe\xi}\,d\xi = \frac{e^{Pe} - e^{-Pe}}{Pe} e^{-Pe\,x_0} \end{aligned}} \tag{10}$$

Approche adjointe

Le temps d'exposition est la solution de l'équation différentielle suivante:

$$\frac{1}{Pe}\frac{d^2\Theta}{dx_0^2} + \frac{d\Theta}{dx_0} = \begin{cases} 0 \ , \ \text{si } |x_0| \ge 1 \\ 1 \ , \ \text{si } |x_0| < 1 \end{cases} . \tag{11}$$

La solution $\Theta(x_0)$ doit être continue et dérivable partout, y compris en $x_0 = \pm 1$. A la limite $x_0 \to -\infty$, le temps d'exposition doit être fini. A la limite $x_0 \to +\infty$, le sens physique suggère que le temps d'exposition doit tendre vers zéro. Dans ces conditions, la solution de (11) est équivalente aux expressions (10).

Il n'est évidemment pas nécessaire d'estimer le temps de résidence au moyen de deux méthodes. Une seule suffit. Mais, il est toujours rassurant de constater qu'elles donnent le même résultat...

Remarque

Ici aussi, on peut identifier une eau de gauche, une eau initiale et une eau de droite. Mais les particules de ces eaux ne changent pas de nature quand elles traversent les frontières du segment $-1 \le x \le 1$. Les concentrations initiales sont

$$C_g(0,x) = \begin{cases} 1 \text{ , si } x < -1 \\ 0 \text{ , sinon} \end{cases} , \tag{12}$$

$$C_i(0,x) = \begin{cases} 1 \text{ , si } |x| < 1 \\ 0 \text{ , sinon} \end{cases} , \tag{13}$$

$$C_d(0,x) = \begin{cases} 1 \text{ , si } x > 1 \\ 0 \text{ , sinon} \end{cases} . \tag{14}$$

En vertu de la théorie de Green, la concentration de ces eaux vaut

$$C_g(t,x) = \left(\frac{Pe}{4\pi t}\right)^{1/2} \int_{-\infty}^{-1} \exp\left[\frac{-Pe(x-\xi-t)^2}{4t}\right] d\xi = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{Pe}{4t}}(x+1-t)\right] , \tag{15}$$

$$\begin{aligned} C_i(t,x) &= \left(\frac{Pe}{4\pi t}\right)^{1/2} \int_{-1}^{1} \exp\left[\frac{-Pe(x-\xi-t)^2}{4t}\right] d\xi \\ &= \frac{1}{2}\operatorname{erf}\left[\sqrt{\frac{Pe}{4t}}(x-1-t)\right] + \frac{1}{2}\operatorname{erf}\left[\sqrt{\frac{Pe}{4t}}(-x+1+t)\right] \end{aligned} , \tag{16}$$

$$C_d(t,x) = \left(\frac{Pe}{4\pi t}\right)^{1/2} \int_{1}^{\infty} \exp\left[\frac{-Pe(x-\xi-t)^2}{4t}\right] d\xi = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{Pe}{4t}}(-x+1+t)\right] , \tag{17}$$

où erf[2] et erfc[3] désignent la fonction d'erreur et la fonction d'erreur complémentaire. En utilisant (3), on obtient les temps d'exposition correspondants, c'est-à-dire

$$\Theta_g = \Theta[C_g(0,x)] = 2 \tag{18}$$

$$\Theta_i = \Theta[C_i(0,x)] = 1 + \frac{1}{Pe} - \frac{1-e^{-2Pe}}{2Pe^2} \tag{19}$$

$$\Theta_d = \Theta[C_d(0,x)] = 0 \tag{20}$$

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[2] La fonction d'erreur vaut $\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\upsilon^2} d\upsilon$ et est telle que $-1 \le \operatorname{erf}(\eta) \le 1$, avec $\operatorname{erf}(-\infty) = -1$, $\operatorname{erf}(0) = 0$ et $\operatorname{erf}(\infty) = 1$.

[3] La fonction d'erreur complémentaire vaut $\operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-\upsilon^2} d\upsilon = 1 - \operatorname{erf}(\eta)$ et est telle que $0 \le \operatorname{erfc}(\eta) \le 2$, avec $\operatorname{erfc}(-\infty) = 2$, $\operatorname{erfc}(0) = 1$ et $\operatorname{erfc}(\infty) = 0$.

Computing the residence time in a one-dimensional time-dependent flow

Eric Deleersnijder, September 3, 2008

The forward problem

Consider the one-dimensional domain of interest $0 \le x' \le L$, in which the flow is time-dependent. The velocity and the constant eddy diffusivity are denoted $u(t')$ and K (>0), respectively, where t' is the time. We study a passive tracer, whose concentration $C(t,x,t',x')$ obeys the following partial differential problem:

$$\frac{\partial C}{\partial t'} = -u\frac{\partial C}{\partial x'} + K\frac{\partial^2 C}{\partial x'^2} \tag{1}$$

$$C(t,x,t',x') = \begin{cases} 0 \ , & t' < t \\ \delta(x'-x) \ , & t' = t \end{cases} \tag{2}$$

$$C(t,x,t',0) = 0 = C(t,x,t',L) \tag{3}$$

where δ is the Dirac function and $0 < x < L$. Clearly, the amount of tracer present in the domain behaves as follows

$$\int_0^L C(t,x,t',x') \ dx' = \begin{cases} 0 \ , & t' < t \\ 1 \ , & t' = t \\ <1 \ , & t < t' \end{cases} \tag{4}$$

with

$$\lim_{\substack{t' \to \infty \\ t < t'}} \int_0^L C(t,x,t',x') \, dx' = 0 \ . \tag{5}$$

The residence time $\theta(t,x)$ is defined to be the mean time that the tracer particles released at time t and location x need to reach the boundary of the domain. This timescale may be estimated as follows:

$$\theta(t,x) = \frac{\int_t^\infty \int_0^L C(t,x,t',x') \, dx' \, dt'}{\int_0^L C(t,x,t,x') \, dx'} = \int_t^\infty \int_0^L C(t,x,t',x') \, dx' \, dt' \ . \tag{6}$$

One has to solve as many forward problems as the number of points where the residence time is to be determined. This is likely to be computationally too expensive, which why the backward — or inverse, or adjoint — approach may be preferred.

The adjoint problem

The residence time obeys the following equation

$$\frac{\partial \theta}{\partial t} = -1 - u\frac{\partial \theta}{\partial x} - K\frac{\partial^2 \theta}{\partial x^2} \ . \tag{7}$$

The boundary conditions to prescribe are

$$\theta(t,0) \ = \ 0 \ = \ \theta(t,L) \ . \tag{8}$$

Equation (7) is to be integrated backward from time T, so that it is convenient to introduce the new variable

$$\tau \ = \ T \ - \ t \ . \tag{9}$$

Using the latter, equation (7) transforms to

$$\frac{\partial \theta}{\partial \tau} \ = \ 1 \ + \ u\frac{\partial \theta}{\partial x} \ + \ K\frac{\partial^2 \theta}{\partial x^2} \ , \tag{10}$$

an equation that can be integrated forward with respect to the modified time τ.

The “initial” condition — to be prescribed at $\tau = 0$ — is usually unknown. Therefore, an arbitrary residence time distribution is to be set at $\tau = 0$, hoping that its influence will progressively vanish. It is believed that the residence time will be acceptable as soon as

$$\tau \ \gg \ \max(T_a, T_d) \ , \tag{11}$$

where T_a and T_d denote the timescales associated advection and diffusion, respectively. However, the residence time is itself a timescale characterizing transport processes, constraint (11) is probably equivalent to

$$\tau \ \gg \ \max_{t,x}[\theta(t,x)] \ . \tag{12}$$

The boundary layer of the steady-state solution

If the velocity is constant, then the residence time does not depend on time and is the solution of equation

$$K\frac{d^2\theta}{dx^2} \ + \ U\frac{d\theta}{dx} \ = \ -1 \ , \tag{13}$$

where U denotes the velocity. Solving this equation under boundary conditions (8) yields

$$\theta(x) \ = \ \frac{L}{U}\frac{1-e^{-Ux/K}}{1-e^{-UL/K}} \ - \ \frac{x}{U} \ . \tag{14}$$

If the Peclet number $Pe = |U|L/K$ is much larger than unity, the solution (14) exhibits a boundary layer whose thickness is of the order of $K/|U|$. The boundary layer is adjacent to the incoming boundary, which is at x=0 or x=L according to whether U>0 or U<0, respectively. This may be illustrated by rewriting (14) as follows:

$$\theta(x) \ = \ \underbrace{\frac{L-\xi}{|U|}}_{\substack{advective \\ res.\ time}} \ - \ \underbrace{\frac{L}{|U|}\frac{e^{-|U|\xi/K} - e^{-|U|L/K}}{1-e^{-|U|L/K}}}_{diffusive\ \ correction} \ , \tag{15}$$

where ξ ($\in [0,L]$) represents the distance to the boundary to which the boundary layer is adjacent, i.e.

$$\xi \ = \ \begin{cases} x \ , \ \ U > 0 \\ L - x \ , \ \ U < 0 \end{cases} \ . \tag{16}$$

Outside the boundary layer, the diffusive correction tends to zero as the Peclet number increases, so that the residence time tends to its purely advective component, $(L-\xi)/|U|$, which is the time needed to travel at speed $|U|$ the distance to the outgoing boundary, $L-\xi$.

Time-dependent solution

If the velocity depends on time, there is probably no closed-form solution to equation (10). Thus, the latter has to be solved numerically. *The mesh size must be much smaller than* $K/|U|$.

A tentative definition of the return coefficient

Eric Deleersnijder, September 9, 2008

Basic concepts

The residence time of a tracer particle is the time needed for this particle to reach an open boundary of the domain of interest. This concept does not take into account the fact that once a particle has hit an open boundary, it may either stay in the domain of interest for some more time or leave the domain and come back into it at a later time. This is why the exposure time has been introduced, which is the time spent in the domain of interest (Figure 1). Clearly, the exposure time is always larger than or equal to the residence time.

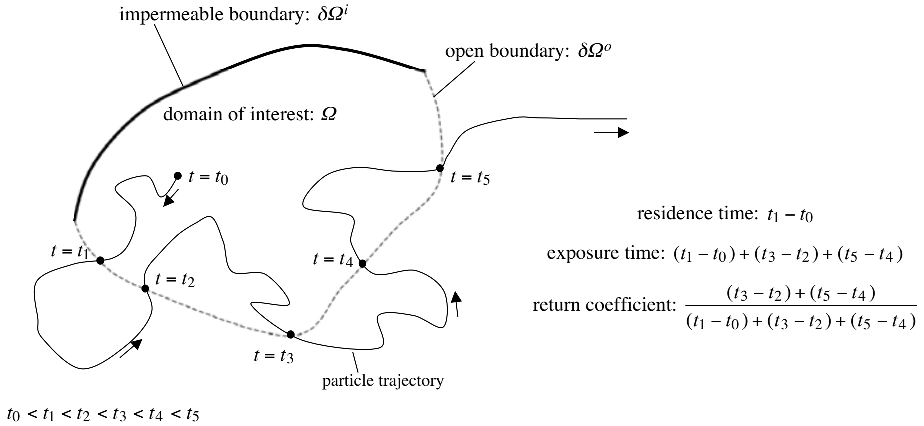


Figure 1. Illustration of the concepts of residence time, exposure time and return coefficient.

The return coefficient is introduced to evaluate the propensity of a particle to return into the domain of interest after hitting for the first time an open boundary. It is suggested that this coefficient be defined as the ratio of the time spent in the domain of interest after hitting for the first an open boundary to the total time spent in the domain, i.e.

$$\boxed{\text{return coefficient} = \frac{(\text{exposure time}) - (\text{residence time})}{\text{exposure time}}} \tag{1}$$

Clearly, the return coefficient lies between zero and unity.

Computing the residence and exposure times

Let $\mathbf{u}$ and $\mathbf{K}$ denote the velocity vector, which is divergence-free ($\nabla \bullet \mathbf{u} = 0$), and the diffusivity tensor, which is symmetric and positive definite. The residence time $\theta(t,\mathbf{x})$ is the solution of the differential equation

$$\frac{\partial\theta}{\partial t} + \nabla\bullet(\mathbf{u}\theta + \mathbf{K}\bullet\nabla\theta) = -1 , \tag{2}$$

which is to be integrated backward in time from instant T. At this instant, the distribution of the residence probably is unknown. However, the residence time must be specified for the backward integration process to begin. It is easy to prescribe a constant value, say zero, i.e.

$$\theta(T,\mathbf{x}) = 0 . \tag{3}$$

The boundary conditions are

$$\left[\theta(t,\mathbf{x})\right]_{\mathbf{x}\in\delta\Omega^o} = 0 \tag{4}$$

and

$$\left[(\mathbf{K}\bullet\nabla\theta)\bullet\mathbf{n}\right]_{\mathbf{x}\in\delta\Omega^i} = 0 . \tag{5}$$

The value of the residence time will be realistic for

$$t << T - O(\theta) . \tag{6}$$

To obtain the exposure time $\Theta(t,\mathbf{x})$, the computational domain must be enlarged, so that calculations can be carried out in the environment Γ of the domain of interest Ω. Therefore, to obtain the exposure time, the computational domain is $\Omega \cup \Gamma$, and the boundary conditions are to be prescribed on $\delta\Gamma^i$ and $\delta\Gamma^o$, which are the impermeable and open parts of the boundary of Γ, respectively. The exposure time obeys the governing equation

$$\frac{\partial\Theta}{\partial t} + \nabla\bullet(\mathbf{u}\Theta + \mathbf{K}\bullet\nabla\Theta) = \begin{cases} -1 , & \text{if } \mathbf{x}\in\Omega \\ 0 , & \text{if } \mathbf{x}\in\Gamma \end{cases} . \tag{7}$$

The “final” condition is

$$\Theta(T,\mathbf{x}) = 0 . \tag{8}$$

and the impermeable boundary condition reads

$$\left[(\mathbf{K}\bullet\nabla\Theta)\bullet\mathbf{n}\right]_{\mathbf{x}\in\delta\Gamma^i} = 0 . \tag{9}$$

The condition to be prescribed on the open boundaries is still an open question that will be addressed in another working note.

Computing the return coefficient

At any time and position in the domain of interest Ω the return coefficient is

$$r(t,\mathbf{x}) = \frac{\Theta(t,\mathbf{x}) - \theta(t,\mathbf{x})}{\Theta(t,\mathbf{x})} . \tag{10}$$

As $\theta(t,\mathbf{x}) \le \Theta(t,\mathbf{x})$, the return coefficient satisfies inequalities

$$0 \le r(t,\mathbf{x}) \le 1 . \tag{11}$$

The larger the return coefficient, the more likely it is for a particle to reenter the domain of interest after hitting its boundary. Notice also that the return coefficient is equal to unity on an open boundary, as the residence time is prescribed to be zero on such a boundary — and the exposure time is unlikely to be zero there.

The relation (10) defines the pointwise return coefficient. It may also be useful to define a return coefficient that would apply to a subdomain of Ω. The latter would measure the propensity of tracer particles uniformly distributed at the initial instant to reenter the domain of interest. Let $\overline{\psi}_{\Omega_s}(t)$ denote the average of function $\psi(t,\mathbf{x})$ over the subdomain Ω_s, i.e.

$$\overline{\psi}_{\Omega_s}(t) \;=\; \frac{1}{\Omega_s} \int_{\Omega_s} \psi(t,\mathbf{x})\, d\mathbf{x} \;. \tag{12}$$

Using this notation the return coefficient applicable to subdomain Ω_s is then

$$r_{\Omega_s}(t) \;=\; \frac{\overline{\Theta}_{\Omega_s}(t) - \overline{\theta}_{\Omega_s}(t)}{\overline{\Theta}_{\Omega_s}(t)} \;. \tag{13}$$

One may object that $\overline{r}_{\Omega_s}(t)$ would also do. However, there is no simple direct problem that naturally leads to this measure of the propensity of particles to reenter the domain of interest. Therefore, I would not recommend having recourse to this alternative definition of the return coefficient.

A one-dimensional, steady-state illustration

Consider a one-dimensional flow characterized by velocity U and diffusivity K, where U and K are positive constants. The domain of interest is the interval $0 \le x \le 1$. It is convenient to use dimensionless variables. The latter are defined as follows:

$$\tilde{x} = \frac{x}{L} \;,\; \tilde{\theta} = \frac{\theta}{L/U} \;,\; \tilde{\Theta} = \frac{\Theta}{L/U} \;. \tag{14}$$

From here on, only dimensionless variables will be used, which is why the tildes will be dropped.

The flow being stationary, the residence and exposure times are time-independent. They are the solution of the ordinary differential problems

$$\begin{cases} \dfrac{1}{Pe}\dfrac{d^2\theta}{dx^2} + \dfrac{d\theta}{dx} = -1 \;,\; x \in [0,1] \\ \theta(0) = 0 = \theta(1) \end{cases} \tag{15}$$

and

$$\begin{cases} \dfrac{1}{Pe}\dfrac{d^2\Theta}{dx^2} + \dfrac{d\Theta}{dx} = \begin{cases} -1 \;,\; x \in [0,1] \\ 0 \;,\; x \in]-\infty,0[\cup]1,+\infty[\end{cases} \\ \Theta(-\infty) < \infty \;,\; \Theta(+\infty) = 0 \end{cases} \tag{16}$$

where $Pe = UL/K$ denotes the Peclet number. The residence and exposure times are

$$\theta(x) \;=\; 1 - x - \frac{e^{-Pe\,x} - e^{-Pe}}{1 - e^{-Pe}} \;,\; 0 \le x \le 1 \tag{17}$$

and

$$\Theta(x) \;=\; 1 \;,\; -\infty < x < 1 \;, \tag{18a}$$

$$\Theta(x) \;=\; 1 - x + \frac{1 - e^{-Pe\,x}}{Pe} \;,\; 0 \le x \le 1 \;, \tag{18b}$$

$$\Theta(x) \;=\; \frac{e^{Pe} - 1}{Pe} e^{-Pe\,x} \;,\; 1 < x < +\infty \;. \tag{18c}$$

These timescale and the corresponding return coefficient are illustrated in Figures 2 and 3.

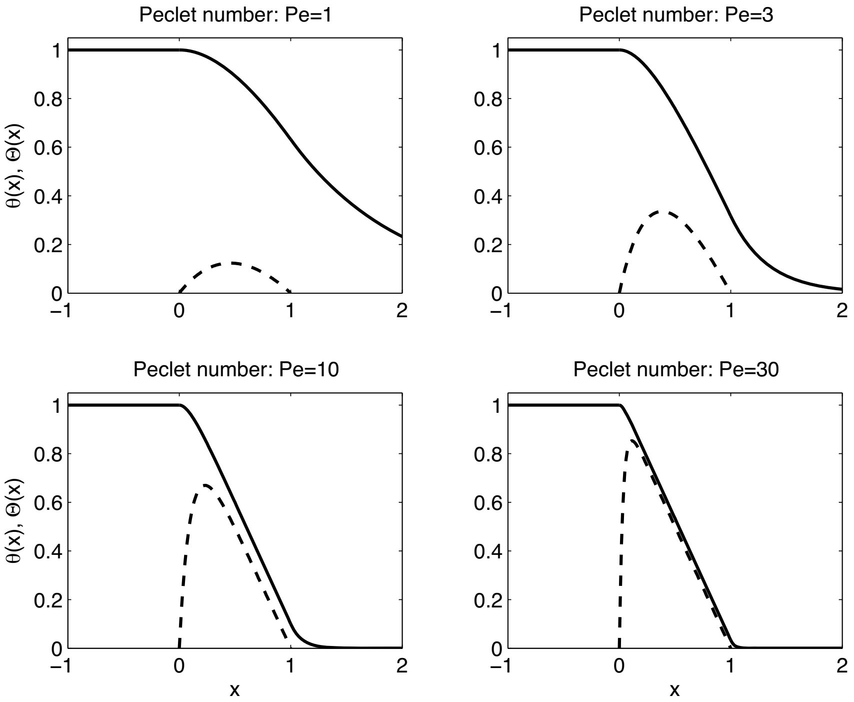


Figure 2. The residence time (dashed curve) and the exposure time (solid curve) for several values of the Peclet number.

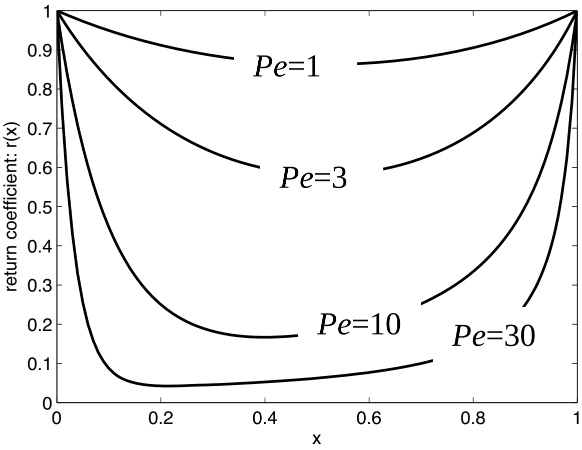


Figure 3. The pointwise return coefficient $r(x)$ for several values of the Peclet number Pe.

A simple illustration of the concept of return coefficient

Eric Deleersnijder, September 12, 2008

In a previous working note, the return coefficient at time t and location $\mathbf{x}$ was defined to be

$$r(t,\mathbf{x}) = \frac{\Theta(t,\mathbf{x}) - \theta(t,\mathbf{x})}{\Theta(t,\mathbf{x})} , \tag{1}$$

where $\theta(t,\mathbf{x})$ and $\Theta(t,\mathbf{x})$ denote the residence time and the exposure time, respectively (see www.climate.be/CART for relevant definitions and references). In a well-mixed domain with a steady-state flow, the residence and exposure times and, hence, the return coefficient should be time- and position-independent. This will be illustrated herein.

We will study the fate of marked particles of water. Let $m(t)$ denote the mass of the particles that are present in the domain of interest at time t. As is well known, the time the particles under consideration spend in the domain is

$$T = \frac{1}{m(0)} \int_0^\infty m(t)\, dt . \tag{2}$$

This timescale may represent either the residence time or the exposure time according to whether the particles are discarded or not once they hit the boundary of the domain.

The mass budget of the water under study is given by differential equation

$$\frac{dm}{dt} = \phi_{in} - \phi_{out} , \tag{3}$$

where ϕ_{in} and ϕ_{out} denote the incoming and outgoing fluxes. As the domain is assumed to be well mixed with a steady-state flow, the outgoing flux may be parameterised as follows

$$\phi_{out} = \frac{m}{\tau} , \tag{4}$$

where the constant τ denotes a relevant timescale.

To obtain the residence time, every particle is disregarded once it hits the boundary of the domain of interest. Therefore, the incoming flux must be zero ($\phi_{in} = 0$) and $m(t)$ is readily seen to be

$$m(t) = m(0) e^{-t/\tau} . \tag{5}$$

Substituting (5) into (2) yields the residence time

$$\theta = \tau . \tag{6}$$

To estimate the exposure time, the particles under study are no longer discarded upon leaving the domain of interest. Therefore, there is the possibility for some of them to re-enter the domain of interest at a later time, implying that the incoming flux is not zero anymore. If this flux may be assumed to be proportional to the outgoing one, i.e.

$$\phi_{in} = \alpha \phi_{out} , \tag{7}$$

with $0 < \alpha < 1$, the mass of the marked water particles decays as

$$m(t) = m(0) e^{-(1-\alpha)t/\tau} . \tag{8}$$

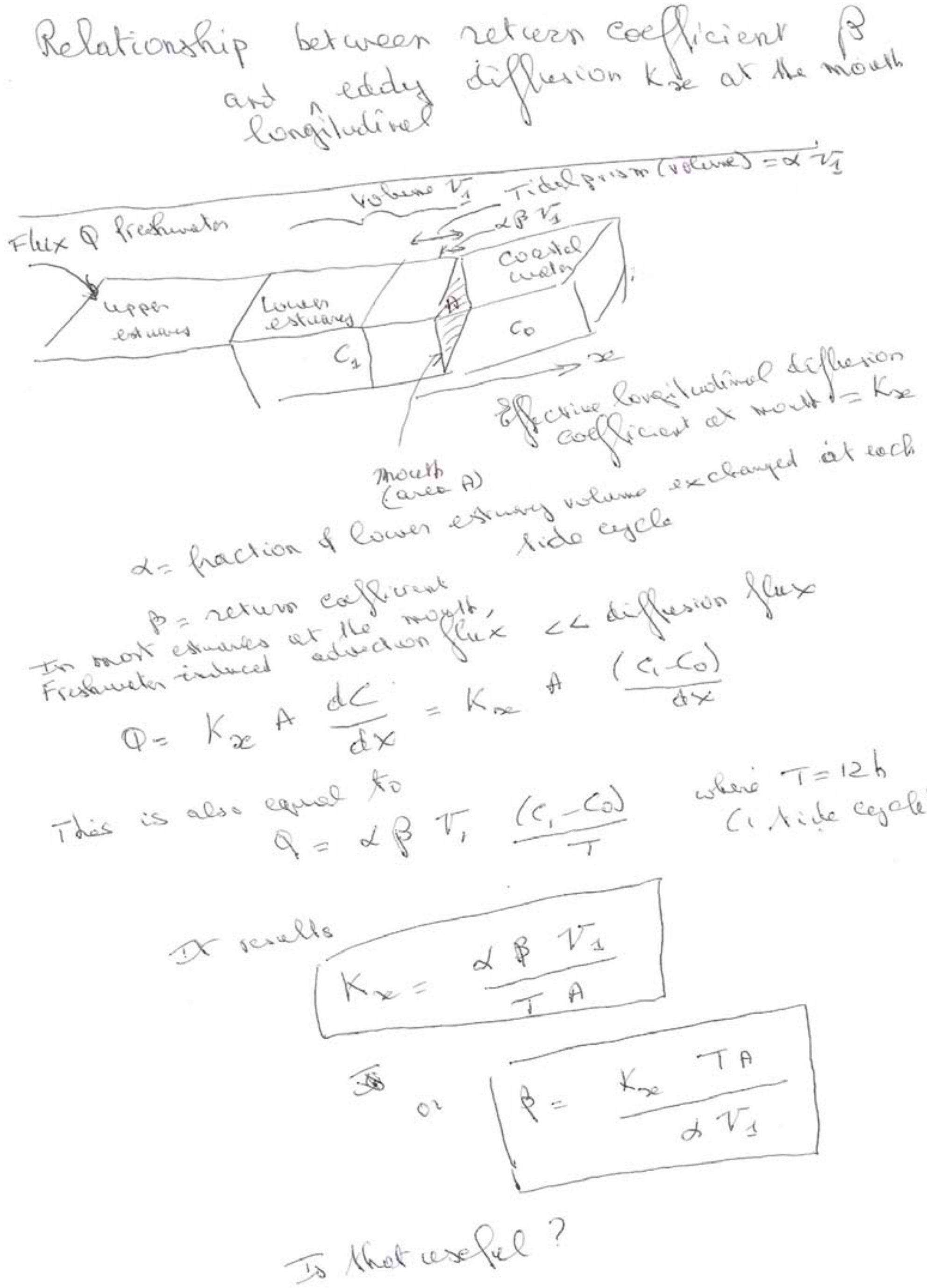
Combining (2) and (8) yields the exposure time:

$$\Theta = \frac{\tau}{1-\alpha} . \qquad (9)$$

Substituting (6) and (9) into (1) leads to the return coefficient

$$r = \alpha . \qquad (10)$$

This remarkably simple result bears some similarity with the apparently different return coefficient used by Eric Wolanski in the working note reproduced below — that was forwarded to me on September 11, 2008.



Relationship between return coefficient β and longitudinal eddy diffusion K_x at the mouth

α = fraction of lower estuary volume exchanged at each tide cycle

β = return coefficient

In most estuaries at the mouth, Freshwater induced advection flux << diffusion flux

$$Q = K_x A \frac{dC}{dx} = K_x A \frac{(C_1 - C_0)}{dx}$$

This is also equal to

$$Q = \alpha \beta V_1 \frac{(C_1 - C_0)}{T}$$

where $T = 12h$ (1 tide cycle)

It results

$$\boxed{K_x = \frac{\alpha \beta V_1}{T A}}$$

or

$$\boxed{\beta = \frac{K_x T A}{\alpha V_1}}$$

Is that useful ?

The %$#@&?ϟ£❖ boundary layer of the exposure time

Eric Deleersnijder, October 17, 2008

The residence time is the time needed for a tracer or water particle to hit for the first time the boundary of the domain of interest. On the other hand, the exposure time is the time spent in the domain of interest. Clearly, the residence time is defined in the domain of interest only, while the exposure time is also defined in the environment of the domain of interest.

To calculate the residence time, the boundary conditions to be prescribed on the boundaries of the domain of interest were established in Delhez et al. (2004) and Delhez and Deleersnijder (2006). However, to evaluate the exposure time, calculations have to be carried out both in the domain of interest and in the environment of the latter. How to define the environment of the domain of interest and the boundary conditions to prescribe on its limits has not been discussed so far. Contributing to filling this gap is the objective of the present note.

One-dimensional solutions in an infinite environment

Consider a steady-state, one-dimensional flow in the domain $-\infty < x < \infty$. The velocity U and the diffusivity K are positive constants. The domain of interest is $0 \le x \le L$. Then, dimensionless variables are introduced in the common way, with L/U being the timescale used to scale the residence and exposure times. From now, only dimensionless variables will be used.

The flow being stationary, the residence time θ and the exposure time Θ are time-independent. They are the solution of the ordinary differential problems

$$\begin{cases} \dfrac{1}{Pe}\dfrac{d^2\theta}{dx^2} + \dfrac{d\theta}{dx} = -1 \ , \quad x \in [0,1] \\ \theta(0) = 0 = \theta(1) \end{cases} \tag{1}$$

and

$$\begin{cases} \dfrac{1}{Pe}\dfrac{d^2\Theta}{dx^2} + \dfrac{d\Theta}{dx} = \begin{cases} -1 \ , & x \in [0,1] \\ 0 \ , & x \notin [0,1] \end{cases} \\ \Theta(-\infty) < \infty \ , \quad \Theta(+\infty) = 0 \end{cases} \tag{2}$$

where $Pe = UL/K$ denotes the Peclet number. The residence and exposure times are

$$\theta(x) = 1 - x - \frac{e^{-Pe\,x} - e^{-Pe}}{1 - e^{-Pe}} \ , \quad 0 \le x \le 1 \tag{3}$$

and

$$\Theta(x) = \begin{cases} 1 \ , & -\infty < x < 0 \\ 1 - x + Pe^{-1}(1 - e^{-Pe\,x}) \ , & 0 \le x \le 1 \ , \\ Pe^{-1}(e^{Pe} - 1)\, e^{-Pe\,x} \ , & 1 < x < +\infty \end{cases} \tag{4}$$

One-dimensional solutions in a finite environment

In the example above, the environment of the domain of interest is infinite. However, for a numerical approximation of the solutions to be obtained, it is necessary that the computational domain be finite. This is why we will imagine that the upstream and downstream ends of the environment of the domain of interest are located at $x = -\xi$ and $x = 1+\xi$, respectively. In other words, a segment of length ξ is added on the upstream and downstream sides of the domain of interest. This has no impact on the estimation of the residence time. This is not so, however, for the exposure time, which will be denoted Θ_ξ to distinguish it from the "true" residence time Θ that was derived above. If the modified exposure time Θ_ξ is prescribed to be zero on the newly-defined boundaries, i.e.

$$\Theta_\xi(-\xi) \;=\; 0 \;=\; \Theta_\xi(1+\xi) \;, \tag{5}$$

then we have

$$\Theta_\xi(x) \;=\; \begin{cases} A[1-e^{-Pe(x+\xi)}] \;, & -\xi < x < 0 \\ B \;-\; x \;+\; C e^{-Pe\,x} \;, & 0 \le x \le 1 \\ D[e^{Pe[(1+\xi)-x]}-1] \;, & 1 < x < 1+\xi \end{cases} \tag{6}$$

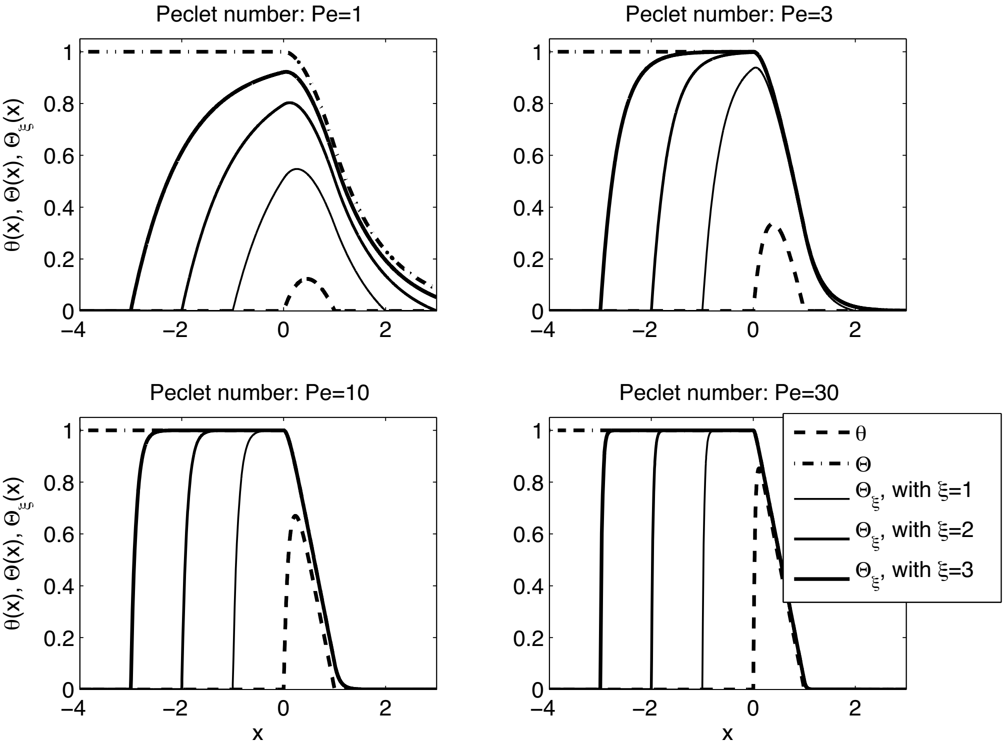
with

$$\begin{cases} A \;=\; \dfrac{e^{Pe\xi} - e^{Pe(1+\xi)} + Pe\, e^{Pe(1+2\xi)}}{Pe[e^{Pe(1+2\xi)}-1]} \\ B \;=\; \dfrac{-1 + e^{Pe\xi} - e^{Pe(1+\xi)} + (1+Pe)e^{Pe(1+2\xi)}}{Pe[e^{Pe(1+2\xi)}-1]} \\ C \;=\; \dfrac{e^{Pe} - e^{Pe(1+2\xi)} - Pe\, e^{Pe(1+\xi)}}{Pe[e^{Pe(1+2\xi)}-1]} \\ D \;=\; \dfrac{-Pe - e^{Pe\xi} + e^{Pe(1+\xi)}}{Pe[e^{Pe(1+2\xi)}-1]} \end{cases} \tag{7}$$

This solution is displayed in the figure below along with the residence time and the "true" exposure time $\Theta(x)$. Next to the upstream boundary, the modified exposure time Θ_ξ exhibits a boundary layer whose width is of the order of Pe^{-1}.

One could object that it would be possible to avoid the boundary layer at the incoming boundary by applying a zero normal gradient condition on this boundary. This solution is worth investigating. However, a weakness of this approach is easy to identify: it corresponds to a zero flux of tracer through the boundary (Delhez et al. 2004), which is rather awkward since the boundary actually is open.

One might also object that, theoretically, all problems could be avoided by considering the whole World Ocean — so that there would be no open boundary anymore. Doing so would unfortunately lead to an infinite value of the exposure time both in the domain of interest and in its environment. Therefore, there needs to be an open boundary on which the particles are discarded as soon as they hit this boundary. This is crucial for preventing the exposure time from becoming infinite.



References

Delhez E.J.M. and E. Deleersnijder, 2006, The boundary layer of the residence time field, *Ocean Dynamics*, 56, 139-150

Delhez E.J.M., A.W. Heemink and E. Deleersnijder, 2004, Residence time in a semi-enclosed domain from the solution of an adjoint problem, *Estuarine, Coastal and Shelf Science*, 61, 691-702

The dependency of the exposure time on the size of the environment of the domain of interest

Eric Deleersnijder, October 23, 2008

To illustrate how the exposure times increases with the size of the environment of the domain of interest, consider a purely diffusive problem. The boundary of the domain of interest and that of its environment are concentric circles (Figure 1), whose radiuses are R_i and R_e, respectively.

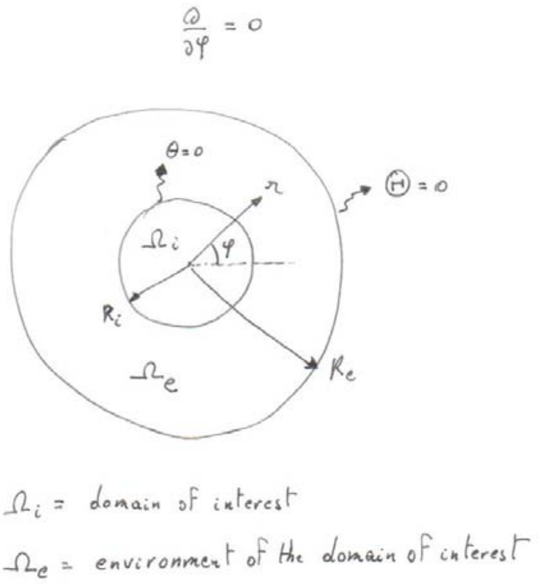


Figure 1. The domain of interest and its environment.

At a steady state, assuming cyclindrical symmetry, i.e. no azimutal variations, the residence time $\theta(r)$ is the solution of the following differential problem:

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(rK\dfrac{d\theta}{dr}\right) = -1 \ , & 0 \le r \le R_i \\ \left[rK\dfrac{d\theta}{dr}\right]_{r=0} = 0 \ , & \theta(R_i) = 0 \end{cases} \tag{1}$$

where r represents the distance to the centre of the domain of interest. The exposure time $\Theta(r)$ obeys the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(rK\frac{d\Theta}{dr}\right) = \begin{cases} -1 \ , & 0 \le r \le R_i \\ 0 \ , & R_i \le r \le R_e \end{cases} \tag{2}$$

and the boundary conditions

$$\left[rK\frac{d\Theta}{dr}\right]_{r=0} = 0 \ , \quad \Theta(R_e) = 0 \ . \tag{3}$$

It is readily seen that the residence time and the exposure time are

$$\theta(r) = \frac{R_i^2 - r^2}{4K}, \quad 0 \le r \le R_i \tag{4}$$

and

$$\Theta(r) = \begin{cases} \dfrac{R_i^2 - r^2}{4K} + \dfrac{R_i^2}{2K}\log\dfrac{R_e}{R_i}, & 0 \le r \le R_i \\ \dfrac{R_i^2}{2K}\log\dfrac{R_e}{r}, & R_i \le r \le R_e \end{cases} \tag{5}$$

The solutions above are displayed in Figure 2.

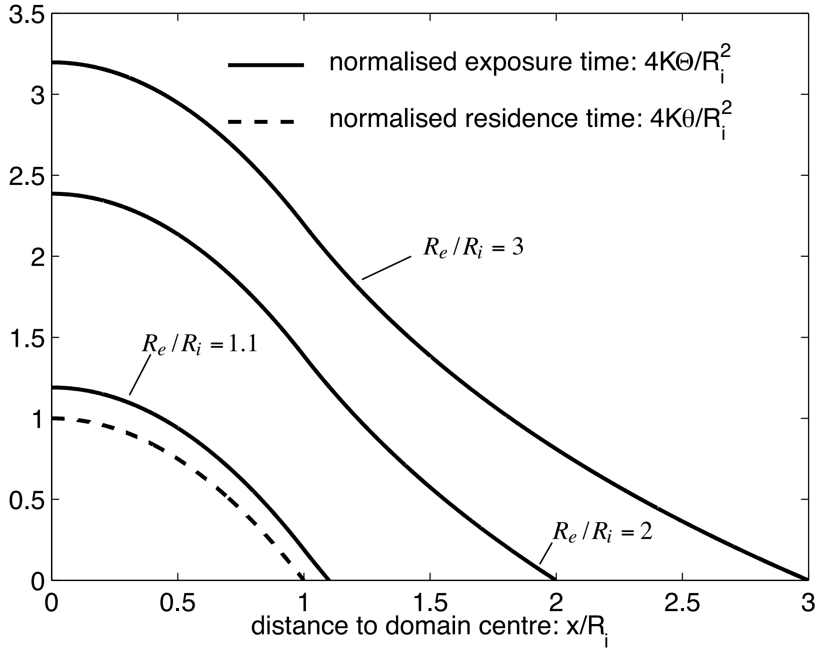


Figure 2. The residence time and the exposure time for various values of R_e.

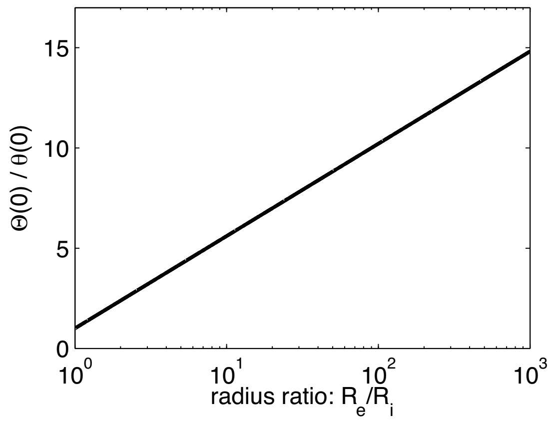


Figure 3. Graphical representation of formula (6).

It is also instructive to look at the ratio (Figure 3)

$$\frac{\max\limits_{0\le r\le R_e} \Theta(r)}{\max\limits_{0\le r\le R_i} \theta(r)} = \frac{\Theta(0)}{\theta(0)} = 1 + 2\log\frac{R_e}{R_i} , \tag{6}$$

which admits the following asymptotic expansion

$$\frac{\Theta(0)}{\theta(0)} \sim 1 + 2\frac{R_e - R_i}{R_i} - \frac{(R_e - R_i)^2}{R_i^2} , \quad R_e \to R_i . \tag{7}$$

Formula (6) may be rewritten as

$$\boxed{\frac{\Theta(0)}{\theta(0)} = 1 + \log(1 + \Omega_e / \Omega_i)} \tag{8}$$

which admits the asymptotic axpansion

$$\frac{\Theta(0)}{\theta(0)} \sim 1 + \frac{\Omega_e}{\Omega_i} - \frac{\Omega_e^2}{2\Omega_i^2} , \quad \Omega_e / \Omega_i \to 0 . \tag{9}$$

Formula (8) holds valid in the present case, in which the boundaries of the domain of interest and its environment are concentric circles. *Would it hold true in general?*

Mathematical properties water renewal-related variables in the Scheldt Estuary

Eric Deleersnijder, March 19 and March 24, 2011

Water renewal refers to the processes by which water that is initially in the domain of interest, the *original water*, is progressively replaced by water originating from its environment, the *renewing water*. To assess the rate at which this water renewal occurs, it is convenient to calculate timescales pertaining to the aforementioned water types. Accordingly, the *residence time* of an original water parcel is the time needed for this water parcel to hit an open boundary of the domain and the *age* of a renewing water parcel is the time elapsed since this water parcel entered the domain. Using CART (www.climate.be/cart), the concentration of the water types and the related timescales may be obtaiend at every time and position as the solutions of partial differential problems.

At the initial instant, it is natural to prescribe that the concentration of the original water and that of the renewing water be equal to unity and zero, respectively. Then, common sense has it that the following properties must hold valid:

1. the sum of the water type concentrations must be equal to unity at any time and location;
2. at any time and location, every water type type concentration must be greater than or equal to zero, and smaller than or equal to unity;
3. at any location, as time progresses, the concentration of the original water must tend to zero,while that of the renewing water must approach unity.

Making sure that this properties are satisfied for the problem under consideration is not always trivial (e.g. Gourgue et al. 2007).

If it is appropriate to split the original water or the renewing water into sub-categories, their concentrations must be seen to obey similar properties. Doing so for the Scheldt Estuary is the primary objective of the present note, which also aims at establishing the differential problems to be solved to obtain the relevant water renewal timescales.

Hydrodynamics

Let $\mathbf{x}$, Ω and Γ denote the position vector, the domain of interest and the boundary of the latter, respectively. The domain boundary consists of several contributions:

$$(1) \qquad \Gamma = \Gamma_c \cup \Gamma_u \cup \Gamma_d \cup \Gamma_s \ ,$$

where Γ_c, Γ_u, Γ_d and Γ_s respectively denote the coastline, the upstream open boundary, the downstream open boundary and semi-open boundaries that will be treated as spillways (Delhez 2010). The coastline being impermeable, the depth-averaged horizontal velocity $\mathbf{u}(t,\mathbf{x})$ satisfies at any time t the following boundary condition

(2) $$\left[\mathbf{u}\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_c} = 0 \ ,$$

where $\mathbf{n}$ is the outward unit normal vector to the domain boundary. On the spillway boundary, water is entering the domain according to the following formula:

(3) $$\left[H\mathbf{u}\bullet\mathbf{n}+U\right]_{\mathbf{x}\in\Gamma_s} = 0 \ ,$$

where $H(t,\mathbf{x})$ is height of the water column while $U(t,\mathbf{x})$ is a non-negative function representing the transport entering the domain through the spillways. On the upstream and downstream boundaries, Γ_u and Γ_d, water alternatively enters and leaves the domain due to the tidal variability, Needless to say, the usual depth-integrated continuity equation,

(4) $$\frac{\partial H}{\partial t} + \nabla\bullet(H\mathbf{u}) = 0 \ ,$$

is satisfied at any time and $\mathbf{x}\in\Omega$.

Water types

Let $C_o(t,\mathbf{x})$ and $C_r(t,\mathbf{x})$ denote the concentration of the *original* and *renewing water*, respectively. The latter is made up of the water originating from the upstream, downstream and spillway boundaries, whose concentrations read $C_u(t,\mathbf{x})$, $C_d(t,\mathbf{x})$ and $C_s(t,\mathbf{x})$, respectively; these water types may be termed *river*, *sea* and *spillway water*, respectively. Therefore, the concentration of the renewing water is

(5) $$C_r = C_u + C_d + C_s \ .$$

The *total water* concentration, $C_w(t,\mathbf{x})$, is defined to be the sum of all the aforementioned concentrations:

(6) $$C_w = C_o + \underbrace{C_u + C_d + C_s}_{C_r} \ .$$

All of the water (sub-)types are regarded as passive tracers, implying that the related concentrations satisfy an equation of the form

(7) $$\frac{\partial(HC_\chi)}{\partial t} = -\nabla\bullet\boldsymbol{\phi}_\chi \ , \quad \chi = o,u,d,s \ .$$

where

$$(8) \qquad \boldsymbol{\phi}_\chi = H\mathbf{u}C_\chi - H\mathbf{K}\bullet\nabla C_\chi$$

is total flux of the relevant water type, consisting of an advective part, $H\mathbf{u}C_\chi$ and a diffusive component, $-H\mathbf{K}\bullet\nabla C_\chi$. The latter is parameterised with the help of the 2×2 diffusivity tensor **K**, which is symmetric and positive-definite. Then, by performing the appropriate sums, it is readily seen that the renewing water concentration and the water concentration, C_r and C_w, satisfy generic equation (7) too. The relevant initial and boundary conditions are listed in Table 1.

Table 1. Initial and boundary conditions for the concentration of the various water (sub-)types.

	original water	river water	sea water	spillway water	renewing water	total water
$t=0$	$C_o=1$	$C_u=0$	$C_d=0$	$C_s=0$	$C_r=0$	$C_w=1$
$\mathbf{x}\in\Gamma_c$	$\boldsymbol{\phi}_o\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_u\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_u\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_s\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_r\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_w\bullet\mathbf{n}=0$
$\mathbf{x}\in\Gamma_u$	$C_o=0$	$C_u=1$	$C_d=0$	$C_s=0$	$C_r=1$	$C_w=1$
$\mathbf{x}\in\Gamma_d$	$C_o=0$	$C_u=0$	$C_d=1$	$C_s=0$	$C_r=1$	$C_w=1$
$\mathbf{x}\in\Gamma_s$	$\boldsymbol{\phi}_o\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_u\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_u\bullet\mathbf{n}=0$	$\boldsymbol{\phi}_s\bullet\mathbf{n}=-U$	$\boldsymbol{\phi}_r\bullet\mathbf{n}=-U$	$\boldsymbol{\phi}_w\bullet\mathbf{n}=-U$

Most of the relations in Table 1 are trivial. Nonetheless, the impermeability conditions along the coastline (Γ_c) need some explanations: combining the zero flux conditions $\boldsymbol{\phi}_\chi\bullet\mathbf{n}=0$ with the impermeability condition (2) yields $(-H\mathbf{K}\bullet\nabla C_\chi)\bullet\mathbf{n}=0$. In other words, the advective and the diffusive components of the flux normal to the coastline are both zero. As suggested by Delhez (2010), the spillway condition is of a special nature: the total flux of each water type crossing the boundary is zero, except for the flux of the spillway water, which has to be prescribed in such a way that the desired amount of water enters the domain through Γ_s.

Properties of the water type concentrations

The total water concentration is equal to unity at the initial instant. Assuming that it remains equal to unity at any time time and location, then the evolution equation (7) transforms to the continuity equation (4) and the incoming flux of water through the spillway boundary is purely advective, the diffusive component being zero, which is consistent with boundary condition (3). Finally, the impermeability of the coast is indentically satisfied. So, the evolution equation, the initial and boundary conditions are consistent with the total water concentration being equal to unity at any time and location. Therefore, one has

$$\boxed{\forall t \geq 0,\ \forall \mathbf{x} \in \Omega :\quad C_w(t,\mathbf{x}) \;=\; 1} \tag{9}$$

Demonstrating that every water type concentration remains positive is equivalent to proving that the negative part of every concentration,

$$C_\chi^- \;=\; \frac{C_\chi - \left|C_\chi\right|}{2} \;, \tag{10}$$

is zero at any time and location. Multiplying evolution (7) by C_χ^-, using (2)-(4), (8) and Table 1, one obtains

$$\frac{d}{dt}\int_\Omega H\left(C_\chi^-\right)^2 d\Omega \;=\; -\int_{\Gamma_s} U\omega_\chi \, d\Gamma_s \;-\; 2\int_\Omega H\nabla C_\chi^- \bullet \mathbf{K} \bullet \nabla C_\chi^- \, d\Omega \;, \quad \chi = o,u,d,s \;, \tag{11}$$

with

$$\omega_o = \left(C_o^-\right)^2 \;,\quad \omega_u = \left(C_u^-\right)^2 \;,\quad \omega_d = \left(C_d^-\right)^2 \;,\quad \omega_s = \left(C_s^-\right)^2 - 2C_s^- \;. \tag{12}$$

The lengthy manipulations needed to derive these relations are inspired by Lewandowski (1997), Deleersnijder et al. (2001) and Gourgue et al. (2007). As $\omega_\chi \geq 0$ and $\nabla C_\chi^- \bullet \mathbf{K} \bullet \nabla C_\chi^- \geq 0$, the integral over the domain of interest of $H(C_\chi^-)^2$ cannot increase. As every water type concentration is non-negative at the initial instant, this integral is zero at t=0. Therefore, this integral will remain zero as time progresses, implying that C_χ^- is zero at any time and location, so that

$$\boxed{\forall t \geq 0,\ \forall \mathbf{x} \in \Omega :\quad C_\chi(t,\mathbf{x}) \;\geq\; 0 \;, \quad \chi = o,u,d,s} \tag{13}$$

Demonstrating that every water type concentration does not exceed unity is equivalent to proving that the positive part of $\hat{C}_\chi = C_\chi - 1$,

$$\hat{C}_\chi^+ \;=\; \frac{\hat{C}_\chi + \left|\hat{C}_\chi\right|}{2} \;, \tag{14}$$

is zero at any time and location. Then, one has

(15) $$\frac{d}{dt}\int_{\Omega} H\left(\hat{C}_{\chi}^{+}\right)^{2} d\Omega \;=\; -\int_{\Gamma_s} U\hat{\omega}_{\chi}\, d\Gamma_s \;-\; 2\int_{\Omega} H\nabla\hat{C}_{\chi}^{+}\bullet\mathbf{K}\bullet\nabla\hat{C}_{\chi}^{+}\, d\Omega\ , \quad \chi = o,u,d,s\ ,$$

with

(16) $$\hat{\omega}_{o} = \hat{C}_{o}^{+}(2+\hat{C}_{o}^{+})\ ,\quad \hat{\omega}_{u} = \hat{C}_{u}^{+}(2+\hat{C}_{u}^{+})\ ,\quad \hat{\omega}_{d} = \hat{C}_{d}^{+}(2+\hat{C}_{d}^{+})\ ,\quad \hat{\omega}_{s} = \left(\hat{C}_{s}^{+}\right)^{2}\ .$$

As $\hat{\omega}_{\chi} \geq 0$ and $\nabla\hat{C}_{\chi}^{-}\bullet\mathbf{K}\bullet\nabla\hat{C}_{\chi}^{-} \geq 0$, the integral over the domain of interest of $H(\hat{C}_{\chi}^{+})^{2}$ cannot increase. As every water type concentration is smaller than or equal to unity at the initial instant, this integral is zero at t=0. Therefore, this integral will remain zero as time progresses, implying that $\hat{C}_{\chi}^{+}$ is zero at any time and location, so that

(17) $$\boxed{\forall t \geq 0,\ \forall \mathbf{x} \in \Omega:\quad C_{\chi}(t,\mathbf{x}) \;\leq\; 1\ ,\quad \chi = o,u,d,s}$$

By multiplying the equation governing the concentration of original water by the concentration of the latter and taking into account the boundary conditions listed in Table 1, one obtains after some calculations

(18) $$\frac{d}{dt}\int_{\Omega} H C_{o}^{2}\, d\Omega \;=\; -\int_{\Gamma_s} U C_{o}^{2}\, d\Gamma_s \;-\; 2\int_{\Omega} H\nabla C_{o}\bullet\mathbf{K}\bullet\nabla C_{o}\, d\Omega\ .$$

Thus, the integral over the domain of interest of $H C_{o}^{2}$ decreases monotonically until the original water concentration is zero at any location, i.e.

(19) $$\boxed{\forall \mathbf{x} \in \Omega:\quad \lim_{t\to\infty} C_{o}(t,\mathbf{x}) \;=\; 0}$$

As the renewing water concentration is $C_r = 1 - C_o$, (19) implies that, as expected, the concentration of the renewing water tends to unity as time progresses:

(20) $$\boxed{\forall \mathbf{x} \in \Omega:\quad \lim_{t\to\infty} C_{r}(t,\mathbf{x}) \;=\; \lim_{t\to\infty}[C_{u}(t,\mathbf{x}) + C_{d}(t,\mathbf{x}) + C_{s}(t,\mathbf{x})] \;=\; 1}$$

Residence time

The residence time at time t_0 and position $\mathbf{x}_0$, $\theta(t_0,\mathbf{x}_0)$, is (e.g. Delhez et al. 2004, de Brauwere et al. 2011)

(21) $$\theta(t_0,\mathbf{x}_0) \;=\; \int_{0}^{\infty}\int_{\Omega} H(t_0+t',\mathbf{x})\,C(t_0+t',\mathbf{x})\, d\Omega\, dt'\ ,$$

where the pseudo-concentration C is the solution of the following partial differential problem:

$$(22)\qquad \begin{cases} \dfrac{\partial(Hc)}{\partial t} = -\nabla \bullet (H\mathbf{u}C - H\mathbf{K} \bullet \nabla C) \\ C(t_0,\mathbf{x}) = \dfrac{\delta(\mathbf{x}-\mathbf{x}_0)}{H(t_0,\mathbf{x}_0)} \ , \ \left[C(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma_u\cup\Gamma_d} = 0 \ , \ \left[(H\mathbf{u}C - H\mathbf{K}\bullet\nabla C)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_c\cup\Gamma_s} = 0 \end{cases}$$

It is readily seen that the domain-averaged residence time at t=0,

$$(23)\qquad \bar{\theta}(0,\mathbf{x}) = \frac{\int_{\Omega} H(0,\mathbf{x})\,\theta(0,\mathbf{x})\, d\Omega}{\int_{\Omega} H(0,\mathbf{x})\, d\Omega} \ ,$$

is equal to the residence time of the original water:

$$(24)\qquad \bar{\theta}(0,\mathbf{x}) = \frac{\int_0^{\infty}\int_{\Omega} H(t,\mathbf{x})\,C_o(t,\mathbf{x})\, d\Omega\, dt}{\int_{\Omega} H(0,\mathbf{x})\,C_o(0,\mathbf{x})\, d\Omega} \ .$$

Obtaining the residence time by numerically solving the forward problem above may entail excessively high computer costs. A more affordable approach probably consists in solving the adjoint problem (Delhez et al. 2004, Blaise et al. 2010):

$$(25)\qquad \begin{cases} \dfrac{\partial(H\theta)}{\partial t} = -\nabla \bullet (H\mathbf{u}\theta + H\mathbf{K}\bullet\nabla\theta) - H \\ \left[\theta(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma_u\cup\Gamma_d} = 0 \ , \ \left[(H\mathbf{K}\bullet\nabla\theta)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma_c\cup\Gamma_s} = 0 \end{cases}$$

The integration has to be performed backward in time from the instant $t = T$. The value of the residence at this moment being unknown, one may prescribe $\theta(T,\theta) = 0$ and keep in mind that the value of the residence time will be reliable only for $0 \le t << T - \Theta$, where Θ is the order of magnitude of the residence time.

Clearly, $\theta(t,\mathbf{x})$ is the mean time needed for water particles located at position $\mathbf{x}$ at time t to hit for the first time the open boundary of the domain of interest, i.e. $\Gamma_u \cup \Gamma_d$. In particular, $\theta(0,\mathbf{x})$ is the average of the time needed for original water particles located at t=0 at point $\mathbf{x}$ to hit for the first time the open boundary of the domain of interest.

Water ages

According to Delhez et al. (1999) or Deleersnijder et al. (2001), the age of a water type is the ratio of the age concentration, $\alpha_\chi(t,\mathbf{x})$, to the concentration, $C_\chi(t,\mathbf{x})$, i.e.

$$a_\chi = \frac{\alpha_\chi}{C_\chi} \; , \quad \chi = o,u,d,s \; . \tag{26}$$

The age concentration obeys the partial differential equation

$$\frac{\partial(H\alpha_\chi)}{\partial t} = -\nabla \bullet \boldsymbol{\varphi}_\chi + HC_\chi \; , \quad \chi = o,u,d,s \; , \tag{27}$$

with

$$\boldsymbol{\varphi}_\chi = H\mathbf{u}\alpha_\chi - H\mathbf{K} \bullet \nabla\alpha_\chi \; . \tag{28}$$

The initial condition and boundary conditions are listed in Table 2.

Table 2. Initial and boundary conditions for the age concentration of the various water (sub-)types.

	original water	river water	sea water	spillway water	renewing water
$t=0$	$\alpha_o = 0$	$\alpha_u = 0$	$\alpha_d = 0$	$\alpha_s = 0$	$\alpha_r = 0$
$\mathbf{x} \in \Gamma_c$	$\boldsymbol{\varphi}_o \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_u \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_d \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_s \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_s \bullet \mathbf{n} = 0$
$\mathbf{x} \in \Gamma_u$	$\alpha_o = 0$	$\alpha_u = 0$	$\alpha_d = 0$	$\alpha_s = 0$	$\alpha_r = 0$
$\mathbf{x} \in \Gamma_d$	$\alpha_o = 0$	$\alpha_u = 0$	$\alpha_d = 0$	$\alpha_s = 0$	$\alpha_r = 0$
$\mathbf{x} \in \Gamma_s$	$\boldsymbol{\varphi}_o \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_u \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_d \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_s \bullet \mathbf{n} = 0$	$\boldsymbol{\varphi}_r \bullet \mathbf{n} = 0$

With these initial and boundary conditions, the age concentration of the original water is $\alpha_o(t,\mathbf{x}) = tC_o(t,\mathbf{x})$, implying that the age of this water type simply is equal to the elapsed time, i.e. $a_o(t,\mathbf{x}) = t$. The age of the river, sea and spillway water is a measure of the time elapsed since entering the domain. Then, in accordance with the age-averaging hypothesis (Deleersnijder et al. 2001), the age concentration and the age of the renewing water are $\alpha_r = \alpha_u + \alpha_d + \alpha_s$ and $a_r = \alpha_r / C_r$. Finally, it may be seen that the age of the water is

$$\text{(29)} \qquad a_w = C_o t + (1 - C_o) a_r = (1 - C_r) t + C_r a_r \ .$$

Obviously, all ages should be non-negative and no larger than the elapsed time *t*. These constraints are satisfied by the age of the original water, for the latter is equal to *t*. If they are also satisfied by the age of river, sea and spillway waters, then all of the ages considered herein will be non-negative and no larger than the elapsed time *t*. The appropriate demonstrations are achieved below.

Let α_χ^- represent the negative part of the age concentration, i.e.

$$\text{(30)} \qquad \alpha_\chi^- = \frac{\alpha_\chi - |\alpha_\chi|}{2} \ .$$

It may be seen that this variable obeys the following formula:

$$\text{(31)} \qquad \frac{d}{dt} \int_\Omega H\left(\alpha_\chi^-\right)^2 d\Omega = -\int_{\Gamma_s} U\left(\alpha_\chi^-\right)^2 d\Gamma_s - 2\int_\Omega H\left(\nabla\alpha_\chi^- \bullet \mathbf{K} \bullet \nabla\alpha_\chi^- - C_\chi \alpha_\chi^-\right) d\Omega \ , \quad \chi = u,d,s$$

As $U(\alpha_\chi^-)^2$, $\nabla\alpha_\chi^- \bullet \mathbf{K} \bullet \nabla\alpha_\chi^-$ and $-C_\chi \alpha_\chi^-$ are non-negative, the right-hand side of (31) is smaller than or equal to zero. As α_χ^- is zero at the initial instant, it will remain zero, implying that the ages are non-negative at any time and position.

The variable $\hat{\alpha}_\chi = \alpha_\chi - C_\chi t = C_\chi (a_\chi - t)$ is zero at the initial instant and evolves in accordance with the following equation

$$\text{(32)} \qquad \frac{\partial (H \hat{\alpha}_\chi)}{\partial t} = -\nabla \bullet (\varphi_\chi - \phi_\chi t) \ , \quad \chi = u,d,s \ .$$

Its positive part,

$$\text{(33)} \qquad \hat{\alpha}_\chi^+ = \frac{\hat{\alpha}_\chi + |\hat{\alpha}_\chi|}{2} \ ,$$

obeys the integral relation

$$\text{(34)} \qquad \frac{d}{dt} \int_\Omega H\left(\hat{\alpha}_\chi^+\right)^2 d\Omega = -\int_{\Gamma_s} U \hat{\eta}_\chi \, d\Gamma_s - 2\int_\Omega H \nabla\hat{\alpha}_\chi^+ \bullet \mathbf{K} \bullet \nabla\hat{\alpha}_\chi^+ \, d\Omega \ , \quad \chi = u,d,s \ ,$$

with $\hat{\eta}_u = (\hat{\alpha}_u^+)^2$, $\hat{\eta}_d = (\hat{\alpha}_d^+)^2$ and $\hat{\eta}_s = (\hat{\alpha}_s^+)^2 + 2t\hat{\alpha}_s^+$. All the integrals in the right-hand side of (34) are non-positive. Therefore, $\hat{\alpha}_\chi$ is zero at any time and location, implying that the ages cannot exceed the elapsed time.

The results of the above developments can be summarised by the following inequalities:

$$(35) \qquad \boxed{\forall t \geq 0,\ \forall \mathbf{x} \in \Omega:\quad 0 \leq\ a_\chi(t,\mathbf{x})\ \leq\ t\ ,\quad \chi = o,u,d,s,r,w}$$

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