

The Bessel kernel determinant on large intervals and Birkhoff's ergodic theorem

ELLIOT BLACKSTONE
KTH Royal Institute of Technology

CHRISTOPHE CHARLIER
KTH Royal Institute of Technology

AND

JONATAN LENELLS
KTH Royal Institute of Technology

Abstract

The Bessel process models the local eigenvalue statistics near 0 of certain large positive definite matrices. In this work, we consider the probability

$$\mathbb{P}\left(\exists \text{ no points in the Bessel process on } (0, x_1) \cup (x_2, x_3) \cup \cdots \cup (x_{2g}, x_{2g+1})\right),$$

where $0 < x_1 < \cdots < x_{2g+1}$ and $g \geq 0$ is any non-negative integer. We obtain asymptotics for this probability as the size of the intervals becomes large, up to and including the oscillations of order 1. In these asymptotics, the most intricate term is a one-dimensional integral along a linear flow on a g -dimensional torus, whose integrand involves ratios of Riemann θ -functions associated to a genus g Riemann surface. We simplify this integral in two generic cases: (a) If the flow is ergodic, we compute the leading term in the asymptotics of this integral explicitly using Birkhoff's ergodic theorem. (b) If the linear flow has certain "good Diophantine properties", we obtain improved estimates on the error term in the asymptotics of this integral. In the case when the flow is both ergodic and has "good Diophantine properties" (which is always the case for $g = 1$, and "almost always" the case for $g \geq 2$), these results can be combined, yielding particularly precise and explicit large gap asymptotics. © 2000 Wiley Periodicals, Inc.

1 Introduction

Let $g \geq 0$ be an integer, let $\vec{x} = (x_1, x_2, \dots, x_{2g+1})$ be such that $0 < x_1 < \cdots < x_{2g+1} < +\infty$, and consider the Fredholm determinant

$$(1.1) \quad F(\vec{x}) = \det(1 - \mathcal{K}|_{I_g}), \quad I_g = (0, x_1) \cup (x_2, x_3) \cup \cdots \cup (x_{2g}, x_{2g+1}),$$

where $\mathcal{K}|_{I_g}$ is the trace class operator acting on $L^2(I_g)$ whose kernel is given by

$$(1.2) \quad K(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}, \quad \alpha > -1,$$

and J_α is the Bessel function of the first kind of order α .

The determinantal point process on $(0, +\infty)$ associated to the kernel (1.2) is referred to as the Bessel point process. This is a universal point process in random matrix theory which models the behavior of the smallest eigenvalues for a wide class of large positive definite random matrices, see e.g. [37, 39]. As is well-known, $F(\vec{x})$ is the probability of finding no points on I_g in the Bessel process. In this paper, we obtain asymptotics for $F(r\vec{x}) = F(rx_1, \dots, rx_{2g+1})$ as $r \rightarrow +\infty$.

Known results for $g = 0$. Tracy and Widom in [51] have shown that $F(x_1)$ can be naturally expressed in terms of the solution of a Painlevé V equation. They also obtained large r asymptotics of $F(rx_1)$, namely,

$$(1.3) \quad F(rx_1) = \det(1 - \mathcal{K}|_{l_0}) = \exp\left(-\frac{rx_1}{4} + \alpha\sqrt{rx_1} - \frac{\alpha^2}{4}\log r + C + O(r^{-\frac{1}{2}})\right)$$

as $r \rightarrow +\infty$, where C is independent of r . Based on numerics, they conjectured that

$$(1.4) \quad C = G(1 + \alpha)(2\pi)^{-\frac{\alpha}{2}} - \frac{\alpha^2}{4}\log x_1,$$

where G is Barnes' G -function. The expression (1.4) for the constant C was first established rigorously by Ehrhardt in [34] for $\alpha \in (-1, 1)$, and then by Deift, Krasovsky, and Vasilevska in [27] for all values of $\alpha \in (-1, +\infty)$.

For $g \geq 1$, it is known from [19] that $F(\vec{x})$ is related to a solution of a system of $2g + 1$ coupled Painlevé V equations. However, to the best of our knowledge, there exist no results prior to this work on the large r asymptotics of $F(r\vec{x})$ for $g \geq 1$.

Statement of results. Let X be the Riemann surface of genus g associated to $\sqrt{\mathcal{R}(z)}$, where

$$(1.5) \quad \mathcal{R}(z) = \prod_{j=1}^{2g+1} (z + x_j).$$

We view X as two copies of $\mathbb{C}$ that are glued along $(-\infty, -x_{2g+1}) \cup \dots \cup (-x_4, -x_3) \cup (-x_2, -x_1)$. Given $z \in \mathbb{C}$, we write $z^+ \in X$ (resp. $z^- \in X$) for the points on the first (resp. second) sheet of X that project onto z . We choose the sheets such that $\sqrt{\mathcal{R}(z^+)} > 0$ for $z \in (-x_1, +\infty)$. Consider the cycles $A_1, \dots, A_g, B_1, \dots, B_g$ shown in Figure 1.1. The cycle B_j lies entirely in the first sheet and surrounds $(-x_{2j}, -x_{2j-1})$ in the positive direction, while A_j lies partly in the upper half-plane of the first sheet (the solid red curves in Figure 1.1) and partly in the lower half-plane of the second sheet (the dashed red curves in Figure 1.1). These cycles form a canonical homology basis of X . Define the $g \times g$ matrix $\mathbb{A}$ and the column vector $\vec{a}$ by

$$\mathbb{A} = (a_{i,j})_{i,j=1}^g, \quad \vec{a} = (a_{1,g+1} \ \dots \ a_{g,g+1})^t,$$

where

$$a_{i,j} = \oint_{A_i} \frac{s^{j-1}}{\sqrt{\mathcal{R}(s)}} ds, \quad i = 1, \dots, g, \quad j = 1, \dots, g+1,$$

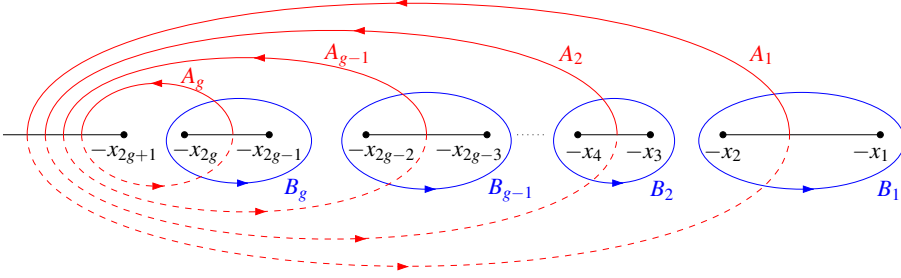


FIGURE 1.1. Canonical homology basis.

and t denotes the transpose operation. Because $x_1, \dots, x_{2g+1}$ are real, it follows that $a_{i,j} \in \mathbb{R}$ for all $i = 1, \dots, g$, $j = 1, \dots, g+1$. Let $\vec{\omega} = (\omega_1, \dots, \omega_g)$ be the row vector of holomorphic one-forms given by

$$(1.6) \quad \vec{\omega} = \frac{dz}{\sqrt{\mathcal{R}(z)}} (1 \ z \ \dots \ z^{g-1}) \mathbb{A}^{-1},$$

and let q be the following polynomial of degree g :

$$(1.7) \quad q(z) = \frac{z^g}{2} + \sum_{j=0}^{g-1} q_j z^j, \quad \text{where} \quad (q_0, \dots, q_{g-1})^t = -\frac{1}{2} \mathbb{A}^{-1} \vec{a}.$$

By definition, $\vec{\omega}$ and q satisfy

$$(1.8) \quad \oint_{A_k} \omega_j = \delta_{jk}, \quad j, k = 1, \dots, g,$$

$$\oint_{A_j} \frac{q(s)}{\sqrt{\mathcal{R}(s)}} ds = 0, \quad j = 1, \dots, g.$$

Since $a_{i,j} \in \mathbb{R}$, q has real coefficients, and we show in Section 3.1 that q has exactly one zero in each of the intervals $(-x_{2j+1}, -x_{2j})$, $j = 1, \dots, g$. The constant c defined by

$$(1.9) \quad c = q_{g-1} - \frac{1}{4} \sum_{j=1}^{2g+1} x_j,$$

and the constants Ω_j , $j = 1, \dots, g$, defined by

$$(1.10) \quad \Omega_j = -i \oint_{B_j} \frac{q(s) ds}{\sqrt{\mathcal{R}(s)}} = -2 \int_{-x_{2j-1}}^{-x_{2j}} \frac{q(s) ds}{|\mathcal{R}(s)|^{\frac{1}{2}}} \in (0, +\infty)$$

appear in our final results. Define

$$(1.11) \quad d_1 = \frac{1}{2} \sum_{j=0}^g \tilde{a}_j a_{j,g+1}, \quad a_{0,j} = -2 \operatorname{Re} \int_{-x_{2g+1}}^{0^+} \frac{s^{j-1} ds}{\sqrt{\mathcal{R}(s)}}, \quad j = 1, \dots, g+1,$$

where the path of integration lies in the upper half-plane of the first sheet and $\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_g \in \mathbb{R}$ are given by

$$(1.12) \quad \tilde{\alpha}_0 = \frac{\alpha}{2}, \quad (\tilde{\alpha}_1 \ \tilde{\alpha}_2 \ \dots \ \tilde{\alpha}_g) = -\frac{\alpha}{2}(a_{0,1} \ a_{0,2} \ \dots \ a_{0,g})\mathbb{A}^{-1}.$$

Given $r \geq 0$, we let $\vec{V}(r) = \vec{v} = (v_1, \dots, v_g)^t$ be the column vector with components

$$(1.13) \quad v_j = -\tilde{\alpha}_j - \Omega_j \frac{\sqrt{r}}{2\pi} \in \mathbb{R}, \quad j = 1, \dots, g.$$

It follows from the general theory of Riemann surfaces, see e.g. [36, p. 63], that the period matrix

$$(1.14) \quad \tau := \left(\oint_{B_i} \omega_j \right)_{i,j=1,\dots,g}$$

is symmetric and has positive definite imaginary part. The associated Riemann θ -function is defined by

$$(1.15) \quad \theta : \mathbb{C}^g \rightarrow \mathbb{C}, \quad \vec{z} \mapsto \theta(\vec{z}) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi(\vec{n}^t \tau \vec{n} + 2\vec{n}^t \vec{z})}.$$

This function is entire and satisfies

$$(1.16) \quad \theta(-\vec{z}) = \theta(\vec{z}), \quad \theta(\vec{z} + \vec{\lambda}' + \tau \vec{\lambda}) = e^{-i\pi(2\vec{\lambda}'^t \vec{z} + \vec{\lambda}'^t \tau \vec{\lambda})} \theta(\vec{z}),$$

for all $\vec{z} \in \mathbb{C}^g$ and $\vec{\lambda}, \vec{\lambda}' \in \mathbb{Z}^g$. The Abel map $\vec{\phi}$ is defined by

$$(1.17) \quad \vec{\phi} : X \rightarrow J(X), \quad P \mapsto \vec{\phi}(P) = \int_{-x_1}^P \vec{\omega}^t,$$

where $J(X)$ is the Jacobian variety of X :

$$(1.18) \quad J(X) = \mathbb{C}^g / L(X), \quad \text{where} \quad L(X) = \{ \vec{z} \in \mathbb{C}^g : \vec{z} = \vec{n} + \tau \vec{m} \text{ for some } \vec{n}, \vec{m} \in \mathbb{Z}^g \}.$$

Let $\vec{e}_1$ be the first column of the $g \times g$ identity matrix. We show in Proposition 7.7 that for each $r \geq 0$, the multi-valued function $P \mapsto \theta(\vec{\phi}(P) + \frac{\vec{e}_1}{2} + \vec{v}(r))$ has a well-defined zero set of cardinality g , and that the projections of these zeros on $\mathbb{C}$, which are denoted $b_1(\vec{v}(r)), \dots, b_g(\vec{v}(r))$, satisfy $b_k(\vec{v}(r)) \in [-x_{2k+1}, -x_{2k}]$, $k = 1, \dots, g$. Define the function $\mathcal{B} : \mathbb{C} \times \mathbb{R}^g / \mathbb{Z}^g \rightarrow \mathbb{C}$ by

$$(1.19) \quad \mathcal{B}(z, \vec{u}) = \frac{\prod_{k=1}^g (z - b_k(\vec{u}))}{q(z)}, \quad z \in \mathbb{C}, \vec{u} \in \mathbb{R}^g / \mathbb{Z}^g.$$

We mention that $\mathcal{B}$ admits another equivalent (but more complicated) expression involving certain ratios of θ -functions, see (7.16).

Our first theorem provides an asymptotic formula for $F(r\vec{x})$ for any choice of the points $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$.

Theorem 1.1 (The general case). *Let $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ be fixed. As $r \rightarrow +\infty$, we have*

$$(1.20) \quad F(r\vec{x}) = \exp \left(cr - d_1 \sqrt{r} + \frac{1-4\alpha^2}{16} \log r + \log \theta(\vec{v}(r)) \right. \\ \left. - \frac{1}{32} \sum_{j=1}^{2g+1} \int_M^r \mathcal{B}(-x_j, \vec{v}(t)) \frac{dt}{t} + \tilde{C} + O(r^{-\frac{1}{2}}) \right),$$

where F is the Fredholm determinant (1.1), $M > 0$ is arbitrary but independent of r and $\tilde{C} = \tilde{C}(M)$ is independent of r . Furthermore, for any $j = 1, \dots, 2g+1$, as $r \rightarrow +\infty$ we have

(1.21)

$$\int_M^r \mathcal{B}(-x_j, \vec{v}(t)) \frac{dt}{t} = \mathcal{B}_j \log r + o(\log r), \quad \mathcal{B}_j := \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{B}(-x_j, \vec{v}(t^2)) dt.$$

Remark 1.2. If $g = 0$, then the term $\log \theta(\vec{v}(r))$ in (1.20) should be interpreted as being equal to 0.

The asymptotic formula (1.21) is the best we can say for general choices of $x_1, \dots, x_{2g+1}$. However, if the linear flow

$$(1.22) \quad (0, +\infty) \ni r \mapsto (v_1(r^2) \bmod 1, v_2(r^2) \bmod 1, \dots, v_g(r^2) \bmod 1)$$

has either “good Diophantine properties” or is ergodic in the g -dimensional torus $\mathbb{R}^g/\mathbb{Z}^g$, then one can strengthen (1.21) (and thereby (1.20)). Let us define these notions:

- The flow (1.22) is said to have “good Diophantine properties” if there exist $\delta_1, \delta_2 > 0$ such that

$$(1.23) \quad |\vec{m}^t \vec{\Omega}| \geq \delta_1 \|\vec{m}\|^{-\delta_2} \quad \text{for all } \vec{m} \in \mathbb{Z}^{g \times 1} \text{ such that } \vec{m}^t \vec{\Omega} \neq 0,$$

where $\vec{\Omega} = (\Omega_1 \ \Omega_2 \ \dots \ \Omega_g)^t$ and $\|\vec{m}\|^2 = \vec{m}^t \vec{m}$.

- The flow (1.22) is said to be ergodic in $\mathbb{R}^g/\mathbb{Z}^g$ if the set $\{\vec{v}(r^2) \bmod \mathbb{Z}^g\}_{r \in (0, +\infty)}$ is dense in $\mathbb{R}^g/\mathbb{Z}^g$. Equivalently, the flow (1.22) is ergodic in $\mathbb{R}^g/\mathbb{Z}^g$ if $\Omega_1, \dots, \Omega_g$ are rationally independent, that is, if $\Omega_1, \dots, \Omega_g$ satisfy the following condition: If $(n_1, n_2, \dots, n_g) \in \mathbb{Z}^g$ is such that

$$(1.24) \quad \Omega_1 n_1 + \Omega_2 n_2 + \dots + \Omega_g n_g = 0,$$

then $n_1 = n_2 = \dots = n_g = 0$.

Let $\mathcal{D}, \mathcal{E} \subseteq (0, +\infty)^g$ be given by

$$\mathcal{D} = \{(\Omega_1, \dots, \Omega_g) : (1.23) \text{ holds true}\}, \quad \mathcal{E} = \{(\Omega_1, \dots, \Omega_g) : (1.24) \text{ holds true}\}.$$

The sets $\mathcal{D}$ and $\mathcal{E}$ are generic, in the sense that $(0, +\infty)^g \setminus \mathcal{D}$ and $(0, +\infty)^g \setminus \mathcal{E}$ have Lebesgue measure zero. In the case of $\mathcal{E}$, this follows easily from the definition; in the case of $\mathcal{D}$, this follows from the Khintchine–Groshev theorem, see e.g. [6,

Corollary 1.4 (b)]. If $g = 1$, then $\mathcal{D} = \mathcal{E} = (0, +\infty)$, but for $g \geq 2$, it is easy to see that $\mathcal{D} \subsetneq \mathcal{E}$, $\mathcal{E} \subsetneq \mathcal{D}$, and $\mathcal{D} \neq (0, +\infty)^g \neq \mathcal{E}$. For example,

- if $g = 2$ then $(1, \sqrt{2}) \in \mathcal{D} \cap \mathcal{E}$,
- if $g = 2$ then $(1, 1) \in \mathcal{D} \setminus \mathcal{E}$,
- if $g = 2$ then $(1, \mathcal{L}) \in \mathcal{E} \setminus \mathcal{D}$, where $\mathcal{L} = \sum_{n=1}^{+\infty} 10^{-n!}$ is Liouville's constant,
- if $g = 3$ then $(1, \mathcal{L}, 1) \notin \mathcal{D} \cup \mathcal{E}$,

where $(1, \sqrt{2}) \in \mathcal{D}$ is a consequence of Liouville's theorem on Diophantine approximation and the fact that $\sqrt{2}$ is a root of the degree 2 polynomial $x^2 - 2$. We show in Appendix A that the image of the mapping $(x_1, x_2, \dots, x_{2g+1}) \mapsto (\Omega_1, \Omega_2, \dots, \Omega_g)$ contains an open ball in $(0, +\infty)^g$, so that each of the four different cases

$$(1.25) \quad \begin{aligned} &(\Omega_1, \dots, \Omega_g) \notin \mathcal{D} \cup \mathcal{E}, \quad (\Omega_1, \dots, \Omega_g) \in \mathcal{D} \setminus \mathcal{E}, \\ &(\Omega_1, \dots, \Omega_g) \in \mathcal{E} \setminus \mathcal{D}, \quad (\Omega_1, \dots, \Omega_g) \in \mathcal{D} \cap \mathcal{E} \end{aligned}$$

can and do occur for certain choices of $(x_1, x_2, \dots, x_{2g+1})$. (In fact, for $g = 2$ there are only three cases because $\mathcal{D} \cup \mathcal{E} = (0, +\infty) \times (0, +\infty)$).

Theorem 1.3 (The “good Diophantine” case). *Let $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ be fixed, and assume that $(\Omega_1, \Omega_2, \dots, \Omega_g) \in \mathcal{D}$. In this case, (1.21) can be improved to*

$$(1.26) \quad \int_M^r \mathcal{B}(-x_j, \vec{v}(t)) \frac{dt}{t} = \mathcal{B}_j \log r + C_j + O(r^{-\frac{1}{2}}) \quad \text{as } r \rightarrow +\infty, \quad j = 1, \dots, 2g+1,$$

where $C_1, \dots, C_{2g+1}$ are independent of r . Therefore, (1.20) can be improved to

$$\begin{aligned} F(r\vec{x}) = \exp \left(cr - d_1 \sqrt{r} + \left(\frac{1-4\alpha^2}{16} - \frac{1}{32} \sum_{j=1}^{2g+1} \mathcal{B}_j \right) \log r \right. \\ \left. + \log \theta(\vec{v}(r)) + C + O(r^{-\frac{1}{2}}) \right), \end{aligned}$$

as $r \rightarrow +\infty$, where C is independent of r .

Remark 1.4. The constant C in Theorem 1.3 equals $\tilde{C} - \frac{1}{32}(C_1 + \dots + C_{2g+1})$, where $\tilde{C}$ is as in (1.20) and $C_1, \dots, C_{2g+1}$ are as in Theorem 1.3.

Theorem 1.5 (The ergodic case). *Let $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ be fixed, and assume that $(\Omega_1, \Omega_2, \dots, \Omega_g) \in \mathcal{E}$. In this case, (1.21) can be improved to*

$$(1.27) \quad \int_M^r \mathcal{B}(-x_j, \vec{v}(t)) \frac{dt}{t} = 2 \log r + o(\log r) \quad \text{as } r \rightarrow +\infty, \quad j = 1, \dots, 2g+1.$$

Therefore, (1.20) can be improved to

$$F(r\vec{x}) = \exp\left(cr - d_1\sqrt{r} - \frac{g+2\alpha^2}{8}\log r + o(\log r)\right) \quad \text{as } r \rightarrow +\infty.$$

By combining Theorems 1.3 and 1.5, we immediately obtain the following.

Corollary 1.6 (The “good Diophantine” and ergodic case). *Let $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ be fixed, and assume that $(\Omega_1, \Omega_2, \dots, \Omega_g) \in \mathcal{D} \cap \mathcal{E}$. Then*

$$F(r\vec{x}) = \exp\left(cr - d_1\sqrt{r} - \frac{g+2\alpha^2}{8}\log r + \log \theta(\vec{v}(r)) + C + O(r^{-\frac{1}{2}})\right) \quad \text{as } r \rightarrow +\infty,$$

where C is independent of r .

Remark 1.7. Corollary 1.6 has been verified numerically for $g = 1, 2, 3$ and several choices of $x_1, \dots, x_{2g+1}$ using [7].

The case $g = 1$ is simpler because both (1.23) and (1.24) are automatically satisfied for any choice of $0 < x_1 < x_2 < x_3 < +\infty$. Hence, Corollary 1.6 always applies when $g = 1$, and we get the following result.

Corollary 1.8 (The case $g = 1$). *Let $g = 1$ and fix $\alpha > -1$, $0 < x_1 < x_2 < x_3 < +\infty$. As $r \rightarrow +\infty$,*

$$F(r(x_1, x_2, x_3)) = \exp\left(cr - d_1\sqrt{r} - \frac{1+2\alpha^2}{8}\log r + \log \theta(v_1(r)) + C + O(r^{-\frac{1}{2}})\right),$$

where C is independent of r .

Remark 1.9. (The case $g = 0$). If $g = 0$, then (1.9) gives $c = -\frac{x_1}{4}$ and (1.11) gives

$$d_1 = \frac{\alpha}{4}a_{0,1} = -\frac{\alpha}{2} \int_{-x_1}^0 \frac{ds}{\sqrt{s+x_1}} = -\alpha\sqrt{x_1}.$$

Since both (1.23) and (1.24) are automatically satisfied for $g = 0$, using Corollary 1.6, we obtain

$$F(rx_1) = \exp\left(-\frac{rx_1}{4} + \alpha\sqrt{rx_1} - \frac{\alpha^2}{4}\log r + C + O(r^{-\frac{1}{2}})\right) \quad \text{as } r \rightarrow +\infty,$$

which is consistent with (1.3).

Related work. The determination of large gap asymptotics is a classical problem in random matrix theory with a long history. There exist various results on large gap asymptotics in the case of a gap on a *single* interval (the so-called “one-cut regime”), see [25, 30, 32, 33, 43, 53] for the sine process, [2, 24] for the Airy process, [27, 34] for the Bessel process, [14, 20, 21] for the Wright’s generalized Bessel and Meijer- G point processes, [22] for the Pearcey process, [3, 8–13, 15–18, 23] for thinned-deformations of these universal point processes, and [31, 50, 52] for the sine- β , Airy- β and Bessel- β point processes. We also refer to [44] and [38] for two overviews.

Large gap asymptotics in the case of *multiple* intervals has been much less explored. For the sine process, this problem was first investigated by Widom [54], who obtained an explicit expression for the leading term in the asymptotics, and characterized the oscillations in terms of the solution to a Jacobi inversion problem. These oscillations were later described more explicitly by Deift, Its, and Zhou [26] in terms of θ -functions. Actually, the results of [26] are, in many respects, similar to the results of our first two theorems (Theorem 1.1 and Theorem 1.3). In particular, the asymptotic formula of [26] for the probability to observe g gaps in the sine process involves (1) an explicit leading term, (2) some oscillations of order 1 described in terms of θ -functions, and (3) a rather complicated integral involving ratios of θ -functions. As in our case (see (1.21)), the latter integral grows logarithmically with the size of the intervals. It was also noted in [26] that, in the generic case where a certain vector has “good Diophantine properties”, this integral can be estimated with good control of the error term (this idea of [26] has been the main inspiration for our Theorem 1.3). However, the coefficient of the logarithmic term was written in [26] as a time average of ratios of θ -functions (cf. our formula (1.21)) and was not simplified. Recently, for the case of two intervals (this corresponds to a genus 1 situation), Fahs and Krasovsky in [35] showed that this coefficient is identically equal to $-1/2$. Large gap asymptotics for the Airy process in certain genus 1 situations were recently computed in [4, 5, 45], and there too simple expressions for the log coefficients appearing in the large gap asymptotics were obtained. We also mention that the multiplicative constants in the asymptotics were explicitly computed in [35, 45].

In Theorem 1.5, we provide a simplified formula for the coefficient of the logarithmic term appearing in the large r asymptotics of $F(r\vec{x})$ for an arbitrary $g \geq 0$, in the generic case when the flow (1.22) is ergodic in $\mathbb{R}^g/\mathbb{Z}^g$. The proof of Theorem 1.5 requires the novel use of Birkhoff’s ergodic theorem. More precisely, using the ergodicity of the flow (1.22), we rewrite the one-dimensional integrals $\mathcal{B}_j$ of (1.21) (which we call *time averages*) as g -dimensional integrals over $\mathbb{R}^g/\mathbb{Z}^g$ (which we call *space averages*). Quite remarkably, these space averages can in turn be evaluated explicitly after a non-trivial change of variables involving the Abel map, see Proposition 7.6 and Lemma 7.12. We expect that a similar simplification of the logarithmic term can be achieved by means of Birkhoff’s ergodic theorem for any genus $g \geq 1$ also for other point processes such as the sine and Airy processes. As noted above Corollary 1.8, the case $g = 1$ is somewhat simpler in several respects; in particular, for $g = 1$ every flow is both ergodic and has “good Diophantine properties”. Hence there is no need to invoke Birkhoff’s ergodic theorem when $g = 1$, see [4, 5, 35, 45]. For these reasons, we feel that Theorem 1.5 is an important novel contribution of the current paper.

Remark 1.10 (The nongeneric case). Even if the argument in Appendix A implies that each of the four cases in (1.25) occurs whenever $g \geq 3$ (and that three

of these cases occur if $g = 2$), it is still possible that the expansion in Corollary 1.6 is valid for all $0 < x_1 < x_2 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ for any $g \geq 1$. We know of no counterexample. If this is the case, then the time average $\mathcal{B}_j = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{B}(-x_j, \vec{v}(t^2)) dt$, where $v_j(t^2) = -\tilde{\alpha}_j - t \frac{\Omega_j}{2\pi}$, is always equal to the space average of $\mathcal{B}(-x_j, \cdot)$ (which is in turn always equal to 2 in accordance with (1.27)). In general, the time average of a smooth function on the torus $\mathbb{R}^g / \mathbb{Z}^g$ along a curve pointing in the direction $\vec{\Omega} = (\Omega_1, \dots, \Omega_g)$ is not equal to the space average if $\Omega_1, \dots, \Omega_g$ are rationally dependent (i.e., if $\vec{\Omega} \notin \mathcal{E}$). Moreover, numerical computations make it clear that the time average of $\mathcal{B}(-x_j, \cdot)$ along a curve pointing in the direction of one of the coordinate axes typically differs from its space average. However, the curve $\vec{v}(t^2)$ never points in the direction of one of the coordinate axes since all the Ω_j are strictly positive, and the functions $\mathcal{B}(-x_j, \cdot)$ are not independent of $\vec{\Omega}$, so it remains open whether the best-case scenario of Corollary 1.6 is in fact valid also in the nongeneric case of $\vec{\Omega} \notin \mathcal{D} \cap \mathcal{E}$.

Overview of the paper. In Section 2, we link the Fredholm determinant $F(r\vec{x})$ to the solution Φ of a certain Riemann–Hilbert (RH) problem. In Sections 3–6, we perform an asymptotic analysis of Φ by means of the Deift/Zhou steepest descent method. This analysis is split into several sections as follows. In Section 3, we normalize the RH problem for Φ at infinity using an appropriate g -function, and then we proceed with the opening of the lenses. In Section 4, we construct the global parametrix $P^{(\infty)}$ in terms of Riemann θ -functions. In Section 5, we construct local parametrices at $z = -x_j$, $j = 1, \dots, 2g+1$; these are built using the well-known solution of the Bessel model RH problem (see Appendix B). In Section 6, we complete our RH analysis with a small norm analysis. Section 7 relies on the results of Sections 3–6 and contains the proofs of Theorems 1.1–1.5. Birkhoff’s ergodic theorem is used in the proof of Theorem 1.5.

2 Differential identity for F

We begin by establishing a relationship between the Fredholm determinant $F(r\vec{x})$ and the solution to a particular Riemann–Hilbert problem. The goal of this section is to prove the following Proposition.

Proposition 2.1. *Let $0 < x_1 < \dots < x_{2g+1} < +\infty$ and $\alpha > -1$ be fixed. We have the identity*

$$(2.1) \quad \partial_r \log F(r\vec{x}) = \frac{1}{2ir} \Phi_{1,12}(r) + \frac{1-4\alpha^2}{16r},$$

where $F(\vec{x})$ is defined in (1.1), $\Phi_1(r) = \Phi_1(r; \vec{x}, \alpha)$ is defined by

$$(2.2) \quad \Phi_1(r) = \lim_{z \rightarrow \infty} rz \left(\Phi(z) \left((rz)^{-\frac{\sigma_3}{4}} M e^{\sqrt{rz} \sigma_3} \right)^{-1} - I \right),$$

where

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\Phi(\cdot) = \Phi(\cdot; r, \vec{x}, \alpha)$ is the unique solution of the following RH problem.

RH problem for $\Phi(\cdot) = \Phi(\cdot; r, \vec{x}, \alpha)$

- (a) $\Phi: \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where the contour $\Sigma_\Phi = (-\infty, 0] \cup \Sigma_+ \cup \Sigma_-$ is oriented as shown in Figure 2.1 with

$$\Sigma_+ := -x_{2g+1} + e^{\frac{2\pi i}{3}}(0, +\infty), \quad \Sigma_- := -x_{2g+1} + e^{-\frac{2\pi i}{3}}(0, +\infty).$$

- (b) The limits of $\Phi(z)$ as z approaches $\Sigma_\Phi \setminus \{-x_{2g+1}, -x_{2g}, \dots, -x_1, 0\}$ from the left (+ side) and from the right (− side) exist, are continuous on $\Sigma_\Phi \setminus \{-x_{2g+1}, -x_{2g}, \dots, -x_1, 0\}$ and are denoted by Φ_+ and Φ_- respectively. Furthermore, they are related by

$$\begin{aligned} \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pm \pi i \alpha} & 1 \end{pmatrix}, & z \in \Sigma_\pm, \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\infty, -x_{2g+1}), \\ \Phi_+(z) &= \Phi_-(z) \begin{pmatrix} e^{\pi i \alpha} & 1 \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, & z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g, \\ \Phi_+(z) &= \Phi_-(z) e^{\pi i \alpha \sigma_3}, & z \in (-x_{2j+1}, -x_{2j}), \quad j = 0, \dots, g, \end{aligned}$$

where $x_0 := 0$.

- (c) As $z \rightarrow \infty$, we have

$$(2.3) \quad \Phi(z) = \left(I + O(z^{-1}) \right) (rz)^{-\frac{\sigma_3}{4}} M e^{\sqrt{rz} \sigma_3},$$

where the principal branch is chosen for each fractional power.

- (d) As $z \rightarrow -x_j$, $j = 1, \dots, 2g+1$, we have

$$(2.4) \quad \Phi(z) = G_j(z) \begin{pmatrix} 1 & \frac{(-1)^{j+1}}{2\pi i} \log(rz + rx_j) \\ 0 & 1 \end{pmatrix} V_j(z) e^{\frac{\pi i \alpha}{2} \operatorname{sgn}(\operatorname{Im} z) \sigma_3} H(z),$$

where sgn is the signum function, G_j is analytic in a neighborhood of $-x_j$ and satisfies $\det G_j \equiv 1$, $V_j(z)$ is defined by

$$(2.5) \quad V_j(z) = I, \quad \text{if } j \text{ is odd}, \quad V_j(z) = \begin{cases} I, & \operatorname{Im} z > 0, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} z < 0, \end{cases} \quad \text{if } j \text{ is even},$$

and $H(z)$ is defined by

$$(2.6) \quad H(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z + x_{2g+1}) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix} & \text{for } \frac{2\pi}{3} < \arg(z + x_{2g+1}) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix} & \text{for } -\pi < \arg(z + x_{2g+1}) < -\frac{2\pi}{3}. \end{cases}$$

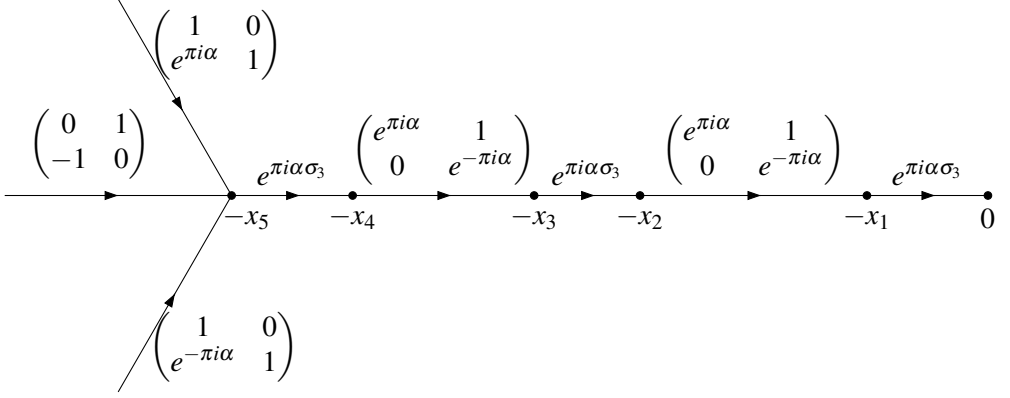


FIGURE 2.1. The jump contour Σ_Φ and the associated jump matrices in the case where $g = 2$.

(e) As z tends to 0, Φ takes the form

$$(2.7) \quad \Phi(z) = G_0(z)(rz)^{\frac{\alpha}{2}\sigma_3}, \quad \alpha > -1,$$

where G_0 is analytic in a neighborhood of 0 and satisfies $\det G_0 \equiv 1$.

Proof of Proposition 2.1. We follow the method developed by Its, Izergin, Korepin and Slavnov [41], and then further pursued in [26]. Note from (1.2) that K can be rewritten as

$$K(x, y) = \frac{\vec{f}^t(x)\vec{g}(y)}{x-y}, \quad \vec{f}(x) := \frac{1}{2} \begin{pmatrix} J_\alpha(\sqrt{x}) \\ \sqrt{x}J'_\alpha(\sqrt{x}) \end{pmatrix}, \quad \vec{g}(y) := \begin{pmatrix} \sqrt{y}J'_\alpha(\sqrt{y}) \\ -J_\alpha(\sqrt{y}) \end{pmatrix}.$$

Also, from (1.1) and a direct rescaling, we get

$$F(r\vec{x}) = \det(I - \mathcal{K}|_{rI_g}) = \det(I - \mathcal{K}_r|_{I_g}),$$

where $\mathcal{K}_r|_{I_g}$ is the trace-class integral operator acting on $L^2(I_g)$ whose kernel is given by

$$K_r(x, y) := rK(rx, ry) = \frac{\vec{f}^t(rx)\vec{g}(ry)}{x-y}.$$

Using well-known identities for trace-class integral operators, we get

$$(2.8) \quad \begin{aligned} \partial_r \log F(r\vec{x}) &= \partial_r \log \det(I - \mathcal{K}_r|_{I_g}) \\ &= \partial_r \text{Tr} \log(I - \mathcal{K}_r|_{I_g}) = -\text{Tr}((I - \mathcal{K}_r|_{I_g})^{-1} \partial_r \mathcal{K}_r|_{I_g}). \end{aligned}$$

A direct computation using [40, eq. (8.401)] gives

$$\partial_r K_r(x, y) = \frac{1}{4} J_\alpha(\sqrt{rx}) J_\alpha(\sqrt{ry}) = -\frac{1}{2} \vec{f}^t(rx) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{g}(ry),$$

and it follows that the kernel of the integral operator $(I - \mathcal{K}_r|_{I_g})^{-1} \partial_r \mathcal{K}_r|_{I_g}$ is given by

$$-\frac{1}{2} \vec{h}^t(rx) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{g}(ry) = -\frac{1}{2} h_1(rx) g_2(ry),$$

where $\vec{h}(\mu) := \left((I - \mathcal{K}|_{rI_g})^{-1} \vec{f}_1(\mu), (I - \mathcal{K}|_{rI_g})^{-1} \vec{f}_2(\mu) \right)^t$. Thus, by (2.8) we have

$$\begin{aligned} \partial_r \log \det (I - \mathcal{K}|_{rI_g}) &= \partial_r \log \det (I - \mathcal{K}_r|_{I_g}) \\ &= \frac{1}{2} \int_{I_g} h_1(rx) g_2(rx) dx = \frac{1}{2r} \int_{rI_g} h_1(x) g_2(x) dx. \end{aligned}$$

On the other hand,

$$Y(z) := I - \int_{rI_g} \frac{\vec{h}(x) \vec{g}^t(x)}{x - z} dx = I + \frac{Y_1(r)}{z} + O(z^{-2}) \quad \text{as } z \rightarrow \infty$$

where

$$Y_1(r) = \int_{rI_g} \begin{pmatrix} h_1(x) g_1(x) & h_1(x) g_2(x) \\ h_2(x) g_1(x) & h_2(x) g_2(x) \end{pmatrix} dx,$$

and so we conclude that

$$(2.9) \quad \partial_r \log F(r\vec{x}) = \partial_r \log \det (I - \mathcal{K}|_{rI_g}) = \frac{Y_{1,12}(r)}{2r}.$$

We will need the following result from [19]. For the reader's convenience, we also include a proof here.

Lemma 2.2. (Taken from [19, eq. (4.17)]) *The function Φ can be expressed in terms of Y as follows:*

$$(2.10) \quad \Phi(z) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} e^{\frac{\pi i}{4} \sigma_3} \sigma_3 Y(-rz) \sigma_3 \tilde{P}_{\text{Be}}(rz),$$

where $\tilde{P}_{\text{Be}}$ is defined by

$$(2.11) \quad \tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4} \sigma_3} \pi^{\frac{\sigma_3}{2}} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi} K_\alpha(z^{\frac{1}{2}}) \\ \pi i z^{\frac{1}{2}} I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}} K'_\alpha(z^{\frac{1}{2}}) \end{pmatrix} \mathcal{S}(z)$$

with

$$\mathcal{S}(z) := \begin{cases} I, & \arg(z + rx_m) \in (-\frac{2\pi}{3}, \frac{2\pi}{3}), \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \arg(z + rx_m) \in (\frac{2\pi}{3}, \pi), \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \arg(z + rx_m) \in (-\pi, -\frac{2\pi}{3}), \end{cases}$$

I_α and K_α are the modified Bessel functions of the first and second kind, and the principal branches are taken for $z^{\frac{1}{2}}$ and $\arg(z + rx_m)$. Moreover,

$$(2.12) \quad \tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} \left(I + \frac{\tilde{P}_{\text{Be},1}}{z} + O(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M e^{z^{\frac{1}{2}}\sigma_3} \quad \text{as } z \rightarrow \infty,$$

where

$$(2.13) \quad \tilde{P}_{\text{Be},1,12} = \frac{1 - 4\alpha^2}{8i}.$$

Proof. Let X be the right-hand side of (2.10). Clearly, X satisfies condition (a) of the RH problem for Φ . From the jumps

$$(2.14) \quad \tilde{P}_{\text{Be},+}(rz) = \tilde{P}_{\text{Be},-}(rz) \begin{pmatrix} e^{\pi i \alpha} & 1 \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_m, 0),$$

we infer that X has the same jumps as Φ on Σ_Φ . Also, using the small z asymptotics of $I_\alpha(z)$ and $K_\alpha(z)$ (given e.g. in [49, Section 10.40]), and using (2.12), we verify that $X(z)$ satisfies properties (d) and (e) of the RH problem for Φ . Hence, both Φ and X satisfy the RH problem for Φ ; by uniqueness of the solution they must be equal, which proves (2.10). The expansion (2.12) follows by using (2.11) and the large z asymptotics of $I_\alpha(z)$ and $K_\alpha(z)$ (given e.g. in [49, Section 10.40]). $\square$

Using (2.3), we infer that

$$\Phi_1(r) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3} \left(-\sigma_3 Y_1(r) \sigma_3 \right) e^{-\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{i}{8}(4\alpha^2 + 3) & 1 \end{pmatrix} + \tilde{P}_{\text{Be},1},$$

and thus

$$(2.15) \quad \Phi_{1,12}(r) = iY_{1,12}(r) + \tilde{P}_{\text{Be},1,12}.$$

The result now follows by using (2.13) and (2.15) in (2.9). $\square$

Our next goal is to obtain large r asymptotics of $\Phi_{1,12}(r)$. For this, we will first perform a Deift–Zhou [29] steepest descent analysis on the RH problem for Φ . This analysis is carried out in Sections 3–6.

3 Steepest descent for Φ : first steps

In this section, we proceed with the first steps of the steepest descent method. We start by finding a g -function that will be used to normalize the RH problem for Φ at ∞ .

3.1 g-function

The function $\sqrt{\mathcal{R}(z)}$, defined in (1.5), satisfies

$$(3.1) \quad \sqrt{\mathcal{R}(z)}_+ + \sqrt{\mathcal{R}(z)}_- = 0, \quad \text{for } z \in \bigcup_{j=1}^{g+1} (-x_{2j}, -x_{2j-1}),$$

where $\sqrt{\mathcal{R}(z)}_+$ and $\sqrt{\mathcal{R}(z)}_-$ denote the limits of $\sqrt{\mathcal{R}(s)}$ as $s \rightarrow z$ from the upper and lower half-plane of the first sheet of X , respectively. We define the $\mathfrak{g}$ -function by

$$(3.2) \quad \mathfrak{g}(z) = \int_{-x_1}^{z^+} \frac{q(s)}{\sqrt{\mathcal{R}(s)}} ds, \quad q(z) = \frac{z^g}{2} + \sum_{j=0}^{g-1} q_j z^j, \quad z \in \mathbb{C} \setminus (-\infty, -x_1),$$

where the path of integration lies on the first sheet of X and does not cross $(-\infty, -x_1]^+$ and coefficients q_j , $j = 0, \dots, g-1$ are given by (1.7). The following lemma extracts the necessary properties of the $\mathfrak{g}$ -function.

Lemma 3.1. *The $\mathfrak{g}$ -function enjoys the following properties:*

- (1) *The $\mathfrak{g}$ -function is analytic in $\mathbb{C} \setminus (-\infty, -x_1]$ and satisfies $\mathfrak{g}(z) = \overline{\mathfrak{g}(\bar{z})}$.*
- (2) *The $\mathfrak{g}$ -function satisfies the jump conditions*

$$(3.3) \quad \mathfrak{g}_+(z) + \mathfrak{g}_-(z) = 0, \quad z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1,$$

$$(3.4) \quad \mathfrak{g}_+(z) - \mathfrak{g}_-(z) = i(\Omega_1 + \dots + \Omega_j), \quad z \in (-x_{2j+1}, -x_{2j}), \quad j = 1, \dots, g,$$

where $\Omega_j > 0$ is defined in (1.10).

- (3) *As $z \rightarrow \infty$, $z \notin (-\infty, -x_{2g+1})$, we have*

$$(3.5) \quad \mathfrak{g}(z) = \sqrt{z} - \frac{2c}{\sqrt{z}} + O(z^{-3/2}),$$

where c is defined in (1.9).

- (4) *$\operatorname{Re} \mathfrak{g}(z) \geq 0$ for $z \in \mathbb{C}$ with equality only when $z \in [-x_{2j}, -x_{2j-1}]$, $j = 1, \dots, g+1$, where $x_{2g+2} := -\infty$.*

Proof. We begin with the observation that the second set of equations in (1.8) can be rewritten as

$$(3.6) \quad \int_{-x_{2j+1}}^{-x_{2j}} \frac{q(s)}{\sqrt{\mathcal{R}(s)}} ds = 0, \quad j = 1, \dots, g,$$

where the path lies on the first sheet of X . Because $\sqrt{\mathcal{R}(z)}$ is real and non-zero on $(-x_{2j+1}, -x_{2j})^+$, it follows from (3.6) that q has exactly one simple zero on each of the intervals $(-x_{2g+1}, -x_{2g}), \dots, (-x_5, -x_4), (-x_3, -x_2)$. Since $q(z) \rightarrow +\infty$ as $z \rightarrow +\infty$, it satisfies

$$(3.7) \quad \operatorname{sgn} q(z) = (-1)^{j+1}, \quad \text{for all } z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1.$$

- (1) The fact that $\mathfrak{g}$ is analytic in $\mathbb{C} \setminus (-\infty, -x_1]$ follows directly from (3.2), and $\overline{\mathfrak{g}(\bar{z})} = \mathfrak{g}(z)$ follows from the fact that the coefficients of $\mathcal{R}$ and q are real.
- (2) The jumps (3.3) follow from (3.1) and (3.6). For $z \in (-x_{2j+1}, -x_{2j})$, by (3.2) we have

$$\mathfrak{g}_+(z) - \mathfrak{g}_-(z) = \sum_{k=1}^j \int_{B_k} \mathfrak{g}'(s) ds = i(\Omega_1 + \cdots + \Omega_j),$$

and this is (3.4). Using (3.7) and the definition of Ω_j in (1.10), it follows that $\Omega_j > 0$.

- (3) A direct computation using (3.2) gives $\mathfrak{g}'(z) = \frac{1}{2}z^{-\frac{1}{2}} + cz^{-\frac{3}{2}} + O(z^{-\frac{5}{2}})$ as $z \rightarrow \infty$. Hence,

$$(3.8) \quad \mathfrak{g}(z) = \sqrt{z} + g_0 - \frac{2c}{\sqrt{z}} + O(z^{-\frac{3}{2}}) \quad \text{as } z \rightarrow \infty,$$

for a certain $g_0 \in \mathbb{C}$. We know from (3.3) that $\mathfrak{g}_+(z) + \mathfrak{g}_-(z) = 0$, which implies $g_0 = 0$.

- (4) It follows from the zeros of $q(z)$ and (3.7) that $\operatorname{Re} \mathfrak{g}(z) > 0$ for $z \in (-x_1, +\infty)$,

$$(3.9) \quad \begin{aligned} \operatorname{Re} \mathfrak{g}(z) &= 0 & \text{for } z \in \bigcup_{j=1}^{g+1} [-x_{2j}, -x_{2j-1}], \\ \operatorname{Re} \mathfrak{g}(z) &> 0 & \text{for } z \in \bigcup_{j=1}^g (-x_{2j+1}, -x_{2j}). \end{aligned}$$

It follows from the definition (3.2) that $\mathfrak{g}$ admits an expansion of the form $\mathfrak{g}(z) \sim \sqrt{z}(1 + \sum_{j=1}^{\infty} \frac{\mathfrak{g}_j}{z^j})$ as $z \rightarrow \infty$, and since $\overline{\mathfrak{g}(\bar{z})} = \mathfrak{g}(z)$, we infer that $\mathfrak{g}_j \in \mathbb{R}$ for all $j \geq 1$. Hence, as $z = |z|e^{i\theta} \rightarrow \infty$, $|\theta| \leq \pi$, we have

$$\operatorname{Re} \mathfrak{g}(z) \sim \sqrt{|z|} \cos \frac{\theta}{2} \left(1 + \sum_{j=1}^{\infty} \frac{\mathfrak{g}_j}{|z|^j} \frac{\cos((j - \frac{1}{2})\theta)}{\cos \frac{\theta}{2}} \right) \quad \text{as } z \rightarrow \infty.$$

Since

$$\left| \frac{\cos((j - \frac{1}{2})\theta)}{\cos \frac{\theta}{2}} \right| \leq 2j - 1, \quad \text{for all } |\theta| \leq \pi \text{ and } j = 1, 2, \dots$$

there exist a constant $\rho_* > 0$ such that

$$\left| \frac{\operatorname{Re} \mathfrak{g}(|z|e^{i\theta})}{\sqrt{|z|} \cos \frac{\theta}{2}} - 1 \right| \leq \frac{1}{2}, \quad \text{for all } |z| \geq \rho_* \text{ and } |\theta| \leq \pi.$$

This implies that

$$(3.10) \quad \frac{1}{2} \sqrt{|z|} \cos \frac{\theta}{2} \leq \operatorname{Re} \mathfrak{g}(|z|e^{i\theta}) \leq \frac{3}{2} \sqrt{|z|} \cos \frac{\theta}{2} \quad \text{for all } |z| \geq \rho_* \text{ and } |\theta| \leq \pi.$$

Let $\mathcal{U}_s$ be the open half disk in the upper half-plane centered at the origin with radius $s \geq \rho_*$. Since $\operatorname{Re} g(z)$ is harmonic on $\mathcal{U}_s$, its minimum value over the set $\mathcal{U}_s$ is attained only on $\partial\mathcal{U}_s$, which by (3.9) and (3.10) is equal to 0. Thus $\operatorname{Re} g(z) > 0$ for all $z \in \mathcal{U}_s$. Since $s \geq \rho_*$ was arbitrary, we conclude that $\operatorname{Re} g(z) > 0$ for all z in the upper half-plane. The symmetry $\overline{g(\bar{z})} = g(z)$ now implies that $\operatorname{Re} g(z) > 0$ for all z in the lower half-plane, which finishes the proof. $\square$

3.2 Rescaling of the RH problem

We now use the g -function (3.2) to define

$$(3.11) \quad T(z) := \begin{pmatrix} 1 & 0 \\ -2ic\sqrt{r} & 1 \end{pmatrix} r^{\frac{\sigma_3}{4}} \Phi(z) e^{-\sqrt{r}g(z)\sigma_3}.$$

It can now be verified, using Lemma 3.1 and the RH problem for Φ , that T is the unique solution to the following RH problem.

RH problem for $T(\cdot) = T(\cdot; r, \vec{x}, \alpha)$

- (a) $T : \mathbb{C} \setminus \Sigma_\Phi \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, and we recall that Σ_Φ is shown in Figure 2.1.
- (b) The jumps for T are given by

$$\begin{aligned} T_+(z) &= T_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\infty, -x_{2g+1}), \\ T_+(z) &= T_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi\alpha - 2\sqrt{r}g(z)} & 1 \end{pmatrix}, & z \in \Sigma_\pm, \\ T_+(z) &= T_-(z) e^{(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\sigma_3}, & z \in (-x_{2j+1}, -x_{2j}), \quad j = 0, \dots, g, \end{aligned}$$

where we define

$$\hat{\Omega}_0 = 0 \quad \text{and} \quad \hat{\Omega}_j = \Omega_1 + \dots + \Omega_j, \quad j = 1, \dots, g.$$

For $z \in \bigcup_{j=1}^g (-x_{2j}, -x_{2j-1})$ we have

$$(3.12) \quad T_+(z) = T_-(z) \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha - 2\sqrt{r}g_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha - 2\sqrt{r}g_+(z)} & 1 \end{pmatrix}.$$

- (c) As $z \rightarrow \infty$, we have

$$(3.13) \quad T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M,$$

where T_1 is independent of z and satisfies $T_{1,12} = \frac{\Phi_{1,12}}{\sqrt{r}} - 2ic\sqrt{r}$.

- (d) $T(z) = \mathcal{O}(\log(z + x_j))$ as $z \rightarrow -x_j$, $j = 1, \dots, 2g + 1$.
- (e) As $z \rightarrow 0$, T takes the form

$$(3.14) \quad T(z) = T_0(z) z^{\frac{\alpha}{2}\sigma_3},$$

where T_0 is analytic in a neighborhood of 0.

3.3 Opening of the lenses

For each $j \in \{1, \dots, g\}$, let $\Sigma_{j,+}$ and $\Sigma_{j,-}$ be two open curves starting at $-x_{2j}$, ending at $-x_{2j-1}$, and lying in the upper and lower half-plane respectively, see also Figure 3.1. The bounded lens-shaped region delimited by $\Sigma_{j,+} \cup \Sigma_{j,-}$ will be denoted by $\mathcal{L}_j$, $j = 1, \dots, g$. With the factorization (3.12) in mind, we define S in terms of T as follows

$$(3.15) \quad S(z) := T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha - 2\sqrt{r}g(z)} & 1 \end{pmatrix}, & z \in \bigcup_{j=1}^g \mathcal{L}_j \text{ and } \operatorname{Im} z > 0, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha - 2\sqrt{r}g(z)} & 1 \end{pmatrix}, & z \in \bigcup_{j=1}^g \mathcal{L}_j \text{ and } \operatorname{Im} z < 0, \\ I, & \text{otherwise.} \end{cases}$$

It is straightforward to verify that S satisfies the following RH problem.

RH problem for $S(\cdot) = S(\cdot; r, \vec{x}, \alpha)$

(a) $S: \mathbb{C} \setminus \Sigma_S \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, with

$$(3.16) \quad \Sigma_S = (-\infty, 0] \cup \gamma_+ \cup \gamma_-, \quad \gamma_{\pm} = \Sigma_{\pm} \cup \Sigma_{1,\pm} \cup \dots \cup \Sigma_{g,\pm},$$

and Σ_S is oriented as in Figure 3.1.

(b) The boundary values of S are related by

$$(3.17) \quad S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1,$$

$$(3.18) \quad S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pm \pi i \alpha - 2\sqrt{r}g(z)} & 1 \end{pmatrix}, \quad z \in \gamma_{\pm},$$

$$(3.19) \quad S_+(z) = S_-(z) e^{(\pi i \alpha - i \hat{\Omega}_{2j} \sqrt{r}) \sigma_3}, \quad z \in (-x_{2j+1}, -x_{2j}), \quad j = 0, \dots, g,$$

where we recall that $x_{2g+2} := -\infty$.

(c) As $z \rightarrow \infty$, we have

$$(3.20) \quad S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M.$$

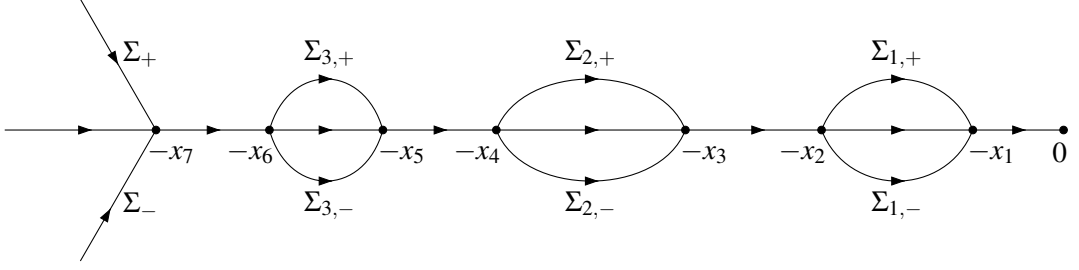
(d) As $z \rightarrow -x_j$, $j = 1, \dots, 2g+1$, we have

$$(3.21) \quad S(z) = \mathcal{O}(\log(z + x_j)).$$

(e) As $z \rightarrow 0$, S takes the form

$$(3.22) \quad S(z) = S_0(z) z^{\frac{\alpha}{2} \sigma_3},$$

where S_0 is analytic in a neighborhood of 0.

FIGURE 3.1. Jump contours Σ_S for S with $g = 3$.

Recall from Lemma 3.1 that

$$\begin{aligned} \operatorname{Re} g(z) &> 0 \text{ for all } z \in \mathbb{C} \setminus \bigcup_{j=1}^{g+1} [-x_{2j}, -x_{2j-1}], \\ \operatorname{Re} g(z) &= 0 \text{ for all } z \in \bigcup_{j=1}^{g+1} [-x_{2j}, -x_{2j-1}]. \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $c_1 > 0$ such that

$$S_-(z)^{-1} S_+(z) - I = O(e^{-c_1 r}) \text{ as } r \rightarrow +\infty$$

uniformly for

$$z \in (\gamma_+ \cup \gamma_-) \cap \left\{ z : \min_{j=1, \dots, 2g+1} \{|z + x_j|\} > \varepsilon \right\}.$$

However, $S_-(z)^{-1} S_+(z) - I$ is not exponentially small as $r \rightarrow +\infty$ when simultaneously $z \in \gamma_+ \cup \gamma_-$, $z \rightarrow -x_j$ for $j = 1, \dots, 2g+1$.

4 Global parametrix

In this section we construct the global parametrix $P^{(\infty)}(z)$, which will serve as an approximation to $S(z)$ when z is bounded away from the endpoints $-x_j$, $j = 1, \dots, 2g+1$.

RH problem for $P^{(\infty)}$

- (a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- (b) The jumps for $P^{(\infty)}$ are given by

$$\begin{aligned} P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1, \\ (4.1) \quad P_+^{(\infty)}(z) &= P_-^{(\infty)}(z) e^{(\pi i \alpha - i \hat{\Omega}_j \sqrt{r}) \sigma_3}, \quad z \in (-x_{2j+1}, -x_{2j}), \quad j = 0, \dots, g, \end{aligned}$$

(c) As $z \rightarrow \infty$, we have

$$(4.2) \quad P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M,$$

where $P_1^{(\infty)}$ is independent of z .

(d) As $z \rightarrow -x_j$, $j = 1, \dots, 2g+1$, we have

$$(4.3) \quad P^{(\infty)}(z) = O((z+x_j)^{-\frac{1}{4}}).$$

(e) As $z \rightarrow 0$, $P^{(\infty)}(z)$ takes the form

$$(4.4) \quad P^{(\infty)}(z) = P_0^{(\infty)}(z) z^{\frac{\sigma_3}{2}},$$

where $P_0^{(\infty)}$ is analytic in a neighborhood of 0.

We will construct $P^{(\infty)}$ in terms of Riemann θ -functions using some ideas from [4, 5, 26, 47]. However, because $P^{(\infty)}$ is not analytic on an unbounded cut, our construction is technically different from [26, 47]. As in [47], we proceed in two steps. First, we construct the solution $\tilde{P}^{(\infty)}$ of an auxiliary RH problem which has no root-type singularity at the origin. Next, we consider a function D with a root-type behavior at 0 and suitable jumps. We will see that $P^{(\infty)}$ is expressed in terms of $\tilde{P}^{(\infty)}$ and D .

4.1 Construction of $P^{(\infty)}$

For the moment we let $\vec{v} = (v_1, \dots, v_g) \in \mathbb{R}^g$ be arbitrary (we will require $\vec{v}$ to be as in (1.13) later). Consider the following RH problem:

RH problem for $\tilde{P}^{(\infty)}(\cdot) = \tilde{P}^{(\infty)}(\cdot; \vec{v})$

(a) $\tilde{P}^{(\infty)} : \mathbb{C} \setminus (-\infty, -x_1] \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $\tilde{P}^{(\infty)}$ are given by

$$(4.5) \quad \begin{aligned} \tilde{P}_+^{(\infty)}(z) &= \tilde{P}_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1, \\ \tilde{P}_+^{(\infty)}(z) &= \tilde{P}_-^{(\infty)}(z) e^{2\pi i(v_1 + \dots + v_j)\sigma_3}, & z \in (-x_{2j+1}, -x_{2j}), \quad j = 1, \dots, g. \end{aligned}$$

(c) As $z \rightarrow \infty$, we have

$$(4.6) \quad \tilde{P}^{(\infty)}(z) = \left(I + \frac{\tilde{P}_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M,$$

where $\tilde{P}_1^{(\infty)}$ is independent of z .

(d) As $z \rightarrow -x_j$, $j = 1, \dots, 2g+1$, we have

$$(4.7) \quad \tilde{P}^{(\infty)}(z) = O((z+x_j)^{-\frac{1}{4}}).$$

We express $\tilde{P}^{(\infty)}$ in terms of Riemann θ -functions, whose definition and relevant properties are given in (1.15) and (1.16). Recall from (1.17) that the Abel map $\vec{\varphi}$ is given by

$$(4.8) \quad \vec{\varphi} : X \rightarrow J(X), \quad P \mapsto \vec{\varphi}(P) = \int_{-x_1}^P \vec{\omega}^t,$$

and that $\vec{\omega}$ is defined in (1.6). It will be convenient to also consider the following function, which we denote by $\vec{\varphi}^{\mathbb{C}}$:

$$(4.9) \quad \vec{\varphi}^{\mathbb{C}} : \mathbb{C} \setminus (-\infty, -x_1] \rightarrow \mathbb{C}^g, \quad z \mapsto \vec{\varphi}^{\mathbb{C}}(z) = \int_{-x_1}^{z^+} \vec{\omega}^t,$$

where the path of integration lies on the first sheet. Recall that the period matrix τ is defined by (1.14). It is well-known, see e.g. [36, p. 63], that τ is symmetric and has positive definite imaginary part. The following properties of $\vec{\varphi}^{\mathbb{C}}$ follow readily from its definition.

Proposition 4.1. *For $z \in \mathbb{C} \setminus (-\infty, -x_1]$, we have*

$$(4.10) \quad \vec{\varphi}_+^{\mathbb{C}}(z) + \vec{\varphi}_-^{\mathbb{C}}(z) = \vec{e}_1 - \vec{e}_j, \quad z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1,$$

$$(4.11) \quad \vec{\varphi}_+^{\mathbb{C}}(z) - \vec{\varphi}_-^{\mathbb{C}}(z) = \sum_{k=1}^j \vec{\tau}_k, \quad z \in (-x_{2j+1}, -x_{2j}), \quad j = 1, \dots, g,$$

where $\vec{e}_j$ denotes the j th column of the $g \times g$ identity matrix, $\vec{e}_{g+1} := \vec{0}$, and $\vec{\tau}_j$ is the j th column of the matrix τ . Moreover, we have $\vec{\varphi}^{\mathbb{C}}(-x_1) = \vec{0}$, $\vec{\varphi}^{\mathbb{C}}(\infty) = \frac{1}{2}\vec{e}_1$, and, for $j = 1, \dots, g$,

$$(4.12) \quad \vec{\varphi}_{\pm}^{\mathbb{C}}(-x_{2j}) = \frac{1}{2}(\vec{e}_1 - \vec{e}_j) \pm \frac{1}{2} \sum_{k=1}^j \vec{\tau}_k, \quad \vec{\varphi}_{\pm}^{\mathbb{C}}(-x_{2j+1}) = \frac{1}{2}(\vec{e}_1 - \vec{e}_{j+1}) \pm \frac{1}{2} \sum_{k=1}^j \vec{\tau}_k.$$

Define

$$(4.13) \quad \beta(z) = \sqrt[4]{\frac{(z+x_2)(z+x_4) \cdots (z+x_{2g})}{(z+x_1)(z+x_3) \cdots (z+x_{2g+1})}}, \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^{g+1} [x_{2j}, x_{2j-1}],$$

where the branch is chosen such that $\beta(z) > 0$ for $z > -x_1$, and define $\mathcal{G} : \mathbb{C}^g \rightarrow \mathbb{C}$ by

$$(4.14) \quad \mathcal{G}(\vec{z}) := \frac{\theta(\vec{z} + \vec{v})}{\theta(\vec{z})}.$$

For any $\vec{\lambda}, \vec{\lambda}' \in \mathbb{Z}^g$, $\mathcal{G}$ satisfies

$$(4.15) \quad \mathcal{G}(\vec{z} + \vec{\lambda}' + \tau \vec{\lambda}) = e^{-2\pi i \vec{\lambda}' \cdot \vec{v}} \mathcal{G}(\vec{z}).$$

This periodicity property follows immediately from (1.16). Since $-i\tau$ is real and positive definite, $\theta(\vec{u}) > 0$ for all $\vec{u} \in \mathbb{R}^g$, see [28, Section 4]. In particular, $\mathcal{G}(\vec{u}) > 0$

for all $\vec{u} \in \mathbb{R}^g$. Recall from (1.6) that the holomorphic differentials $\omega_1, \dots, \omega_g$ are given by

$$(4.16) \quad \omega_j = (1 \ z \ \dots \ z^{g-1}) \mathbb{A}^{-1} \vec{e}_j \frac{dz}{\sqrt{\mathcal{R}(z)}} = \sum_{k=1}^g (\mathbb{A}^{-1})_{kj} z^{k-1} \frac{dz}{\sqrt{\mathcal{R}(z)}}, \quad j = 1, \dots, g.$$

Define

$$(4.17) \quad \tilde{P}^{(\infty)}(z) = \tilde{Q} \frac{\beta(z)^{\sigma_3}}{\sqrt{2}} \begin{pmatrix} \mathcal{G}(-\vec{\varphi}^{\mathbb{C}}(z) + \frac{\vec{e}_1}{2}) & i\mathcal{G}(\vec{\varphi}^{\mathbb{C}}(z) + \frac{\vec{e}_1}{2}) \\ i\mathcal{G}(-\vec{\varphi}^{\mathbb{C}}(z)) & \mathcal{G}(\vec{\varphi}^{\mathbb{C}}(z)) \end{pmatrix},$$

where

$$\tilde{Q} = \begin{pmatrix} \frac{1}{\mathcal{G}(\vec{0})} & 0 \\ -ic_{\mathcal{G}} & \frac{1}{\mathcal{G}(\frac{\vec{e}_1}{2})} \end{pmatrix}$$

and

$$(4.18) \quad c_{\mathcal{G}} := \frac{2(\mathbb{A}^{-1})_{g:} \nabla \mathcal{G}(\frac{\vec{e}_1}{2})}{\mathcal{G}(\frac{\vec{e}_1}{2})} = \frac{2}{\mathcal{G}(\frac{\vec{e}_1}{2})} \sum_{j=1}^g (\mathbb{A}^{-1})_{gj} \partial_j \mathcal{G}(\frac{\vec{e}_1}{2}).$$

Here $(\mathbb{A}^{-1})_{g:}$ denotes the g -th row of $\mathbb{A}^{-1}$, $\nabla \mathcal{G}$ is the gradient of $\mathcal{G}$ (of size $g \times 1$), and $\partial_j \mathcal{G}$ denotes the j -th partial derivative of $\mathcal{G}$. From Proposition 4.1 and (4.15), we infer that $\tilde{P}^{(\infty)}$ satisfies condition (b) of the RH problem for $\tilde{P}^{(\infty)}$. One also verifies using (4.16) that $\tilde{P}^{(\infty)}$ satisfies condition (c) of the RH problem for $\tilde{P}^{(\infty)}$ with

$$(4.19) \quad \tilde{P}_{1,12}^{(\infty)} = \frac{-2i(\mathbb{A}^{-1})_{g:} \nabla \mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})} = \frac{-2i}{\mathcal{G}(\vec{0})} \sum_{j=1}^g (\mathbb{A}^{-1})_{gj} \partial_j \mathcal{G}(\vec{0}).$$

By (7.10), (7.15) and Lemma 7.5 below, $z \mapsto \theta(\pm \vec{\varphi}(z))$ have simple zeros at $-x_3, -x_5, \dots, -x_{2g+1}$ and no other zeros, and $z \mapsto \theta(\frac{\vec{e}_1}{2} \pm \vec{\varphi}(z))$ have simple zeros at $-x_2, -x_4, \dots, -x_{2g}$ and no other zeros. This already shows that $\tilde{P}^{(\infty)}$ satisfies condition (a) of the RH problem for $\tilde{P}^{(\infty)}$, that $z \mapsto \mathcal{G}(\pm \vec{\varphi}^{\mathbb{C}}(z))$ blow up as inverse square roots near $-x_3, -x_5, \dots, -x_{2g+1}$, and that $z \mapsto \mathcal{G}(\pm \vec{\varphi}^{\mathbb{C}}(z) + \frac{\vec{e}_1}{2})$ blow up as inverse square roots near $-x_2, -x_4, \dots, -x_{2g}$. Using (4.13) and (4.17), it is now straightforward to verify that $\tilde{P}^{(\infty)}$ also satisfies condition (d) of the RH problem for $\tilde{P}^{(\infty)}$. We have thus shown that $\tilde{P}^{(\infty)}$ in (4.17) is the unique solution to the RH problem for $\tilde{P}^{(\infty)}$. Note that $P^{(\infty)}$ and $\tilde{P}^{(\infty)}$ have the same jumps on $(-x_{2j}, -x_{2j-1})$, $j = 1, \dots, g+1$ but not on $(-x_{2j+1}, -x_{2j})$, $j = 1, \dots, g$. Let us introduce the following function

$$(4.20) \quad D(z) = \exp \left(\sqrt{\mathcal{R}(z)} \sum_{j=0}^g \int_{-x_{2j+1}}^{-x_{2j}} \frac{(\tilde{\alpha}_0 + \dots + \tilde{\alpha}_j) ds}{\sqrt{\mathcal{R}(s)}(s-z)} \right),$$

where $\tilde{\alpha}_0 := \frac{\alpha}{2}$ and the constants $\tilde{\alpha}_j$ are defined by (1.12). The definition of the $\tilde{\alpha}_j$'s ensure that $D(z) \rightarrow 1$ as $z \rightarrow \infty$. It can be verified that D satisfies the following properties:

(a) $D : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ is analytic.

(b) D has the following jumps:

$$(4.21) \quad D_+(z)D_-(z) = 1, \quad z \in (-x_{2j}, -x_{2j-1}), \quad j = 1, \dots, g+1,$$

$$(4.22) \quad D_+(z) = D_-(z)e^{2\pi i(\tilde{\alpha}_0 + \dots + \tilde{\alpha}_j)}, \quad z \in (-x_{2j+1}, -x_{2j}), \quad j = 0, \dots, g.$$

(c) As $z \rightarrow \infty$, we have

$$(4.23) \quad D(z) = \exp\left(\frac{d_1}{\sqrt{z}} + \frac{d_2}{z^{\frac{3}{2}}} + O(z^{-\frac{5}{2}})\right) = 1 + \frac{d_1}{\sqrt{z}} + \frac{d_1^2}{2z} + \frac{\frac{d_1^3}{6} + d_2}{z^{\frac{3}{2}}} + O(z^{-2}),$$

where d_1 is given by (1.11).

(d) As $z \rightarrow 0$, we have

$$(4.24) \quad D(z) = z^{\tilde{\alpha}_0} O(1).$$

(e) As $z \rightarrow -x_j$, $\text{Im } z > 0$,

$$(4.25) \quad D(z) = e^{i\pi(\tilde{\alpha}_0 + \dots + \tilde{\alpha}_j)} + O(\sqrt{z + x_j}),$$

for $j = 1, \dots, 2g+1$, where $\tilde{j} = \left\lfloor \frac{j}{2} \right\rfloor$.

We are now ready to solve the RH problem for $P^{(\infty)}(z)$.

Proposition 4.2. *The solution to the RH problem for $P^{(\infty)}$ is given by*

$$(4.26) \quad P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix} \tilde{P}^{(\infty)}(z) D(z) \sigma_3,$$

where $v_1, \dots, v_g$ are defined in (1.13).

Proof. Using the properties of $D(z)$, $\tilde{P}^{(\infty)}(z)$, and the definition of $v_1, \dots, v_g$, we verify that the right-hand side of (4.26) solves the RH problem for $P^{(\infty)}$. $\square$

4.2 Various asymptotics of $P^{(\infty)}(z)$

For future references, we compute $P_{1,12}^{(\infty)}$ as well as the first terms in the expansions of $P^{(\infty)}(z)$ as $z \rightarrow -x_j$, $\text{Im } z > 0$, $j = 1, \dots, 2g+1$.

Computation of $P_{1,12}^{(\infty)}$

From (4.17), (4.19), (4.23) we obtain

$$P^{(\infty)}(z) = \left(I + \frac{1}{z} \left[\begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix} \tilde{P}_1^{(\infty)} \begin{pmatrix} 1 & 0 \\ id_1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{d_1^2}{2} & -id_1 \\ -\frac{i}{3}(d_1^3 - 3d_2) & -\frac{d_1^2}{2} \end{pmatrix} \right] + O(z^{-2}) \right) z^{-\frac{\sigma_3}{4}} M$$

as $z \rightarrow \infty$, and in particular, using (4.19)

$$(4.27) \quad P_{1,12}^{(\infty)} = \tilde{P}_{1,12}^{(\infty)} - id_1 = \frac{-2i(\mathbb{A}^{-1})_g \cdot \nabla \mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})} - id_1.$$

Asymptotics of $P^{(\infty)}(z)$ as $z \rightarrow -x_{2j}$, $\text{Im } z > 0$, $j = 1, \dots, g$

Associated to the Jacobian variety $J(X)$ are a total of 2^{2g} half-periods, $2^{g-1}(2^g - 1)$ of which are odd half-periods, see e.g. [36, p. 303]. It is well-known, see e.g. [36, p. 304], that the θ -function vanishes at each odd half-period. Proposition 4.1 allows us to express some of the odd half-periods in terms of the Abel map.

Lemma 4.3. *The points $\vec{\varphi}(-x_{2j+1}) \in J(X)$, $j = 1, \dots, g$, are odd half-periods. The points $\vec{\varphi}(-x_{2j}) + \frac{\vec{e}_1}{2} \in J(X)$, $j = 1, \dots, g$, are also odd half-periods.*

Proof. It follows from (4.12) that $\vec{\varphi}(-x_k)$ can be written in the form

$$\vec{\varphi}(-x_k) = \frac{1}{2}\vec{\alpha}_k + \frac{1}{2}\tau\vec{\beta}_k \pmod{L(X)} \quad \text{for certain } \vec{\alpha}_k, \vec{\beta}_k \in \mathbb{Z}^g, \quad k = 1, \dots, 2g+1.$$

Since $\vec{\alpha}_k$ is odd if $k = 2j+1$ for $j = 1, \dots, g$, we conclude that $\vec{\varphi}(-x_{2j+1}) \in J(X)$, $j = 1, \dots, g$, are odd half-periods. It follows similarly from (4.12) that $\vec{\varphi}(-x_{2j}) + \frac{\vec{e}_1}{2}$, $j = 1, \dots, g$, are odd half-periods. $\square$

Lemma 4.3 implies that

$$(4.28) \quad \theta(\pm \vec{\varphi}_+^{\mathbb{C}}(-x_{2j}) + \frac{\vec{e}_1}{2}) = \theta(\vec{\varphi}(-x_{2j}) + \frac{\vec{e}_1}{2}) = 0, \quad j = 1, \dots, g.$$

Using (4.28), we infer that the asymptotics of $P^{(\infty)}(z)$ as $z \rightarrow -x_{2j}$, $\text{Im } z > 0$, $j = 1, \dots, g$ are given by

$$(4.29) \quad P^{(\infty)}(z) = (P^{(\infty)})_{-x_{2j}}^{(-\frac{1}{4})}(z+x_{2j})^{-\frac{1}{4}} + O((z+x_{2j})^{\frac{1}{4}}),$$

$$(4.30) \quad (P^{(\infty)})_{-x_{2j}}^{(-\frac{1}{4})} = \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix} \tilde{Q} \frac{(\beta_{2j}^{(\frac{1}{4})})^{\sigma_3}}{\sqrt{2}} \\ \times \begin{pmatrix} \tilde{\mathcal{G}}_{2j}^- & i\tilde{\mathcal{G}}_{2j}^+ \\ i\mathcal{G}(-\vec{\varphi}_+^{\mathbb{C}}(-x_{2j})) & \mathcal{G}(\vec{\varphi}_+^{\mathbb{C}}(-x_{2j})) \end{pmatrix} e^{i\pi(\tilde{\alpha}_j + \dots + \tilde{\alpha}_j)\sigma_3},$$

where

(4.31)

$$\beta(z) = \beta_{2j}^{(\frac{1}{4})}(z+x_{2j})^{\frac{1}{4}} + O((z+x_{2j-1})^{\frac{5}{4}}), \quad \beta_{2j}^{(\frac{1}{4})} = e^{-\frac{\pi i}{4}} \frac{\prod_{\substack{k=1 \\ k \neq j}}^g |x_{2k} - x_{2j}|^{\frac{1}{4}}}{\prod_{k=0}^g |x_{2k+1} - x_{2j}|^{\frac{1}{4}}},$$

$$\begin{aligned}\tilde{\mathcal{G}}_{2j}^{\pm} &:= \lim_{\substack{z \rightarrow -x_{2j} \\ \text{Im } z > 0}} \sqrt{z + x_{2j}} \frac{\theta(\pm \vec{\varphi}^{\mathbb{C}}(z) + \frac{\vec{e}_1}{2} + \vec{v})}{\theta(\pm \vec{\varphi}^{\mathbb{C}}(z) + \frac{\vec{e}_1}{2})} \\ &= \frac{\theta(\pm \vec{\varphi}_+^{\mathbb{C}}(-x_{2j}) + \frac{\vec{e}_1}{2} + \vec{v}) \prod_{\substack{k=1 \\ k \neq 2j}}^{2g+1} \sqrt{x_k - x_{2j} +}}{\pm 2 \sum_{k=1}^g p_k(-x_{2j}) \partial_k \theta(\pm \vec{\varphi}_+^{\mathbb{C}}(-x_{2j}) + \frac{\vec{e}_1}{2})}.\end{aligned}$$

Asymptotics of $P^{(\infty)}(z)$ as $z \rightarrow -x_1$, $\text{Im } z > 0$

The asymptotics for $P^{(\infty)}(z)$ as $z \rightarrow -x_1$, $\text{Im } z > 0$, are given by

$$(4.32) \quad P^{(\infty)}(z) = (P^{(\infty)})_{-x_1}^{(-\frac{1}{4})}(z + x_1)^{-\frac{1}{4}} + O((z + x_1)^{\frac{1}{4}}),$$

$$(4.33) \quad (P^{(\infty)})_{-x_1}^{(-\frac{1}{4})} = \frac{\beta_1^{(-\frac{1}{4})} \mathcal{G}(\frac{\vec{e}_1}{2})}{\sqrt{2} \mathcal{G}(\vec{0})} \begin{pmatrix} 1 & i \\ -i(c_{\mathcal{G}} + d_1) & c_{\mathcal{G}} + d_1 \end{pmatrix} e^{\frac{i\pi\alpha}{2} \sigma_3},$$

where

$$\beta(z) = \frac{\beta_1^{(-\frac{1}{4})}}{(z + x_1)^{\frac{1}{4}}} + O((z + x_1)^{\frac{3}{4}}), \quad \beta_1^{(-\frac{1}{4})} = \prod_{k=1}^g \frac{|x_{2k} - x_1|^{\frac{1}{4}}}{|x_{2k+1} - x_1|^{\frac{1}{4}}}.$$

Asymptotics of $P^{(\infty)}(z)$ as $z \rightarrow -x_{2j+1}$, $\text{Im } z > 0$, $j = 1, \dots, g$

The asymptotics for $P^{(\infty)}(z)$ as $z \rightarrow -x_{2j+1}$, $\text{Im } z > 0$, $j = 1, \dots, g$, are given by

$$(4.34) \quad P^{(\infty)}(z) = (P^{(\infty)})_{-x_{2j+1}}^{(-\frac{1}{4})}(z + x_{2j+1})^{-\frac{1}{4}} + O((z + x_{2j+1})^{\frac{1}{4}}),$$

$$\begin{aligned}(P^{(\infty)})_{-x_{2j+1}}^{(-\frac{1}{4})} &= \begin{pmatrix} 1 & 0 \\ -id_1 & 1 \end{pmatrix} \tilde{Q} \frac{(\beta_{2j+1}^{(-\frac{1}{4})})^{\sigma_3}}{\sqrt{2}} \\ &\times \begin{pmatrix} \mathcal{G}(-\vec{\varphi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{\vec{e}_1}{2}) & i \mathcal{G}(\vec{\varphi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{\vec{e}_1}{2}) \\ i \mathcal{G}_{2j+1}^- & \mathcal{G}_{2j+1}^+ \end{pmatrix} e^{i\pi(\tilde{\alpha}_0 + \dots + \tilde{\alpha}_j) \sigma_3},\end{aligned}$$

where

(4.35)

$$\beta(z) = \frac{\beta_{2j+1}^{(-\frac{1}{4})}}{(z + x_{2j+1})^{\frac{1}{4}}} + O((z + x_{2j+1})^{\frac{3}{4}}), \quad \beta_{2j+1}^{(-\frac{1}{4})} = \frac{\prod_{k=1}^g |x_{2k} - x_{2j+1}|^{\frac{1}{4}}}{\prod_{\substack{k=0 \\ k \neq j}}^g |x_{2k+1} - x_{2j+1}|^{\frac{1}{4}}},$$

$$\begin{aligned}\tilde{\mathcal{G}}_{2j+1}^{\pm} &:= \lim_{\substack{z \rightarrow -x_{2j+1} \\ \text{Im } z > 0}} \sqrt{z + x_{2j+1}} \frac{\theta(\pm \vec{\varphi}^{\mathbb{C}}(z) + \vec{v})}{\theta(\pm \vec{\varphi}^{\mathbb{C}}(z))} \\ &= \frac{\theta(\pm \vec{\varphi}_+^{\mathbb{C}}(-x_{2j+1}) + \vec{v}) \prod_{\substack{k=1 \\ k \neq 2j+1}}^{2g+1} \sqrt{x_k - x_{2j+1} +}}{\pm 2 \sum_{k=1}^g p_k(-x_{2j+1}) \partial_k \theta(\pm \vec{\varphi}_+^{\mathbb{C}}(-x_{2j+1}))}.\end{aligned}$$

5 Local parametrices

The local parametrix $P^{(-x_j)}$ is defined in a small open disk $\mathbb{D}_{-x_j}$ centered at $-x_j$, $j = 1, \dots, 2g + 1$. The radii of the disks are chosen sufficiently small, but independent of r , such that they do not intersect each other. Inside $\mathbb{D}_{-x_j}$, $P^{(-x_j)}$ has the same jumps as S , and is such that $S(z)P^{(-x_j)}(z)^{-1} = O(1)$ as $z \rightarrow -x_j$. Furthermore, on the boundary of $\mathbb{D}_{-x_j}$, we impose that $P^{(-x_j)}$ “mathches” with $P^{(\infty)}$, in the sense that

$$(5.1) \quad P^{(x_j)}(z) = (I + o(1))P^{(\infty)}(z) \quad \text{as } r \rightarrow +\infty,$$

uniformly for $z \in \partial\mathbb{D}_{-x_j}$. We will follow the pioneering work [26] and build these parametrices in terms of Bessel functions. We will show in Section 6 that $P^{(-x_j)}(z)$ approximates $S(z)$ for $z \in \mathbb{D}_{-x_j}$.

5.1 Parametrix at $-x_{2j+1}$, $j = 0, \dots, g$

Let f_{2j+1} be the conformal map from $\mathbb{D}_{-x_{2j+1}}$ to a neighborhood of 0 defined by

$$(5.2) \quad f_{2j+1}(z) = \frac{1}{4} \left(\mathfrak{g}(z) \mp \frac{i\hat{\Omega}_j}{2} \right)^2, \quad \pm \text{Im } z > 0.$$

The expansion of $f_{2j+1}(z)$ as $z \rightarrow -x_{2j+1}$ is given by

$$(5.3) \quad f_{2j+1}(z) = c_{2j+1}(z + x_{2j+1})(1 + O(z + x_{2j+1})), \quad c_{2j+1} = \frac{q(-x_{2j+1})^2}{\prod_{\substack{k=1 \\ k \neq 2j+1}}^{2g+1} (x_k - x_{2j+1})} > 0.$$

We deform the lenses such that $f_{2j+1}(\gamma_{\pm} \cap \mathbb{D}_{-x_{2j+1}}) \subseteq e^{\pm \frac{2\pi i}{3}}(0, +\infty)$, and define

$$(5.4) \quad P^{(-x_{2j+1})}(z) := E_{2j+1}(z) \Phi_{\text{Be}}(rf_{2j+1}(z); 0) e^{-\sqrt{r}\mathfrak{g}(z)\sigma_3} e^{\pm \frac{i\pi\alpha}{2}\sigma_3}, \quad \pm \text{Im } z > 0,$$

$$(5.5) \quad E_{2j+1}(z) = P^{(\infty)}(z) e^{\mp(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} M^{-1} \left(2\pi\sqrt{r}f_{2j+1}(z)^{\frac{1}{2}} \right)^{\frac{\sigma_3}{2}}, \quad \pm \text{Im } z > 0,$$

where Φ_{Be} is the solution to the RH problem presented in Section B. Using the jumps of $P^{(\infty)}$, we verify that $E_{2j+1}(z)$ is analytic for $z \in \mathbb{D}_{-x_{2j+1}}$, and using the jumps (3.3)–(3.4) of $\mathfrak{g}$, that $P^{(-x_{2j+1})}$ has the same jumps as S inside $\mathbb{D}_{-x_{2j+1}}$ and satisfies $S(z)P^{(-x_j)}(z)^{-1} = O(1)$ as $z \rightarrow -x_{2j+1}$. A direct computation using (B.2) shows that

$$(5.6) \quad P^{(-x_{2j+1})}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)e^{\mp(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} \Phi_{\text{Be},1} e^{\pm(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}}{\sqrt{r}f_{2j+1}(z)^{\frac{1}{2}}} + O(r^{-1})$$

as $r \rightarrow +\infty$ uniformly for $z \in \partial\mathbb{D}_{-x_{2j+1}}$, where the $\pm$ and $\mp$ signs correspond to $\pm\text{Im}z > 0$.

5.2 Parametrix at $-x_{2j}$, $j = 1, \dots, g$

Let f_{2j} be the conformal map from $\mathbb{D}_{-x_{2j}}$ to a neighborhood of 0 defined by

$$(5.7) \quad f_{2j}(z) = -\frac{1}{4} \left(\mathfrak{g}(z) \mp \frac{i\hat{\Omega}_j}{2} \right)^2, \quad \pm\text{Im}z > 0.$$

The expansion of $f_{2j}(z)$ as $z \rightarrow -x_{2j}$ is given by

$$(5.8) \quad f_{2j}(z) = c_{2j}(z + x_{2j})(1 + O(z + x_{2j})), \quad c_{2j} = \frac{-q^2(-x_{2j})}{\prod_{\substack{k=1 \\ k \neq 2j}}^{2g+1} (x_k - x_{2j})} > 0.$$

We choose the lenses such that $-f_{2j}(\gamma_{\pm} \cap \mathbb{D}_{-x_{2j}}) \subseteq e^{\pm \frac{2\pi i}{3}}(0, +\infty)$. Let us define $P^{(-x_{2j})}$ by

$$(5.9) \quad P^{(-x_{2j})}(z) := E_{2j}(z) \Phi_{\text{Be}}(-r f_{2j}(z)) \sigma_3 e^{-\sqrt{r} \mathfrak{g}(z) \sigma_3} e^{\pm \frac{i\pi \alpha}{2} \sigma_3}, \quad \pm\text{Im}z > 0,$$

$$(5.10) \quad E_{2j}(z) = P^{(\infty)}(z) e^{\mp(i\pi \alpha - i\hat{\Omega}_j \sqrt{r}) \frac{\sigma_3}{2}} \sigma_3 M^{-1} \left(2\pi \sqrt{r} (-f_{2j}(z))^{\frac{1}{2}} \right)^{\frac{\sigma_3}{2}}, \quad \pm\text{Im}z > 0,$$

where we verify that $E_{2j}(z)$ is analytic for $z \in \mathbb{D}_{-x_{2j}}$. Also, a computation using (B.2) shows that

$$(5.11) \quad P^{(-x_{2j})}(z) P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z) e^{\mp(i\pi \alpha - i\hat{\Omega}_j \sqrt{r}) \frac{\sigma_3}{2}} \sigma_3 \Phi_{\text{Be},1} \sigma_3 e^{\pm(i\pi \alpha - i\hat{\Omega}_j \sqrt{r}) \frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}}{\sqrt{r} (-f_{2j}(z))^{\frac{1}{2}}} + O(r^{-1})$$

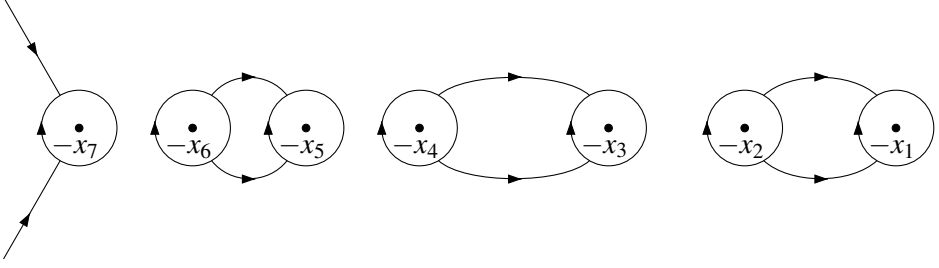
as $r \rightarrow +\infty$ uniformly for $z \in \partial\mathbb{D}_{-x_{2j}}$, where the $\pm$ and $\mp$ signs correspond to $\pm\text{Im}z > 0$.

6 Small norm problem

Consider

$$(6.1) \quad R(z) := \begin{cases} S(z) P^{(-x_j)}(z)^{-1}, & z \in \mathbb{D}_{-x_j}, j = 1, \dots, 2g+1, \\ S(z) P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \bigcup_{j=1}^{2g+1} \mathbb{D}_{-x_j}. \end{cases}$$

We have verified in Section 5 that $P^{(-x_j)}$ and S have the same jumps inside $\mathbb{D}_{-x_j}$, and that $R(z) = S(z) P^{(-x_j)}(z)^{-1}$ remains bounded as $z \rightarrow -x_j$, $j = 1, \dots, 2g+1$.

FIGURE 6.1. The contour Σ_R for R with $g = 3$.

Thus, using the jumps of $P^{(\infty)}$, we infer that R is analytic in $\mathbb{C} \setminus \Sigma_R$, where Σ_R is given by

$$\Sigma_R = \left(\gamma_+ \cup \gamma_- \cup \bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j} \right) \setminus \bigcup_{j=1}^{2g+1} \mathbb{D}_{-x_j}.$$

For convenience, we orient the circles $\partial \mathbb{D}_{-x_j}$ in the clockwise direction, see Figure 6.1. The jump matrix for R , namely

$$J_R : \Sigma_R \rightarrow \mathbb{C}^{2 \times 2}, \quad z \mapsto J_R(z) := R_-(z)^{-1} R_+(z),$$

is given by

$$(6.2) \quad J_R(z) = \begin{cases} P^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi\alpha - 2\sqrt{r}g(z)} & 1 \end{pmatrix} P^{(\infty)}(z)^{-1}, & z \in \gamma_{\pm} \setminus \bigcup_{j=1}^{2g+1} \mathbb{D}_{-x_j} \\ P^{(-x_j)}(z) P^{(\infty)}(z)^{-1}, & z \in \partial \mathbb{D}_{-x_j}, j = 1, \dots, 2g+1. \end{cases}$$

Since $\mathcal{G}(\vec{u}) \neq 0$ for all $\vec{u} \in \mathbb{R}^g$, it follows from (4.17) and (4.26) that $P^{(\infty)}(z)$ is bounded as $r \rightarrow +\infty$ uniformly for z bounded away from $0, -x_1, \dots, -x_{2g+1}$. Hence, using Lemma 3.1 and (5.6), (5.11), we obtain

$$(6.3) \quad J_R(z) = \begin{cases} I + O(e^{-\tilde{c}|rz|^{\frac{1}{2}}}) & \text{as } r \rightarrow +\infty \text{ uniformly for } z \in \Sigma_R \setminus \left(\bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j} \right), \\ I + \frac{J_R^{(1)}(z)}{\sqrt{r}} + O(r^{-1}) & \text{as } r \rightarrow +\infty \text{ uniformly for } z \in \bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j}, \end{cases}$$

where $\tilde{c} > 0$ is a sufficiently small constant, and $J_R^{(1)}(z) = O(1)$ as $r \rightarrow +\infty$ uniformly for $z \in \bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j}$. The matrix $J_R^{(1)}(z)$ has been computed in (5.6), (5.11), and is given by

$$(6.4) \quad J_R^{(1)}(z) = \begin{cases} \frac{P^{(\infty)}(z) e^{\mp(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} \Phi_{\text{Be},1} e^{\pm(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}}{f_{2j+1}(z)^{\frac{1}{2}}}, & z \in \partial \mathbb{D}_{-x_{2j+1}}, j = 0, \dots, g, \\ \frac{P^{(\infty)}(z) e^{\mp(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} \sigma_3 \Phi_{\text{Be},1} \sigma_3 e^{\pm(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1}}{(-f_{2j}(z))^{\frac{1}{2}}}, & z \in \partial \mathbb{D}_{-x_{2j}}, j = 1, \dots, g. \end{cases}$$

By the small-norm theory for RH problems [28, 29], it follows that R exists for sufficiently large r and

$$(6.5) \quad R(z) = I + \frac{R^{(1)}(z)}{\sqrt{r}} + O(r^{-1}) \quad \text{as } r \rightarrow +\infty,$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. Using (6.3) and the relation

$$(6.6) \quad R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(\xi)(J_R(\xi) - I)}{\xi - z} d\xi, \quad z \in \mathbb{C} \setminus \Sigma_R,$$

we infer that

$$(6.7) \quad R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j}} \frac{J_R^{(1)}(\xi)}{\xi - z} d\xi, \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j}.$$

It is easy to see from (6.4) that $J_R^{(1)}$ admits an analytic continuation from $\bigcup_{j=1}^{2g+1} \partial \mathbb{D}_{-x_j}$ to $\bigcup_{j=1}^{2g+1} \overline{\mathbb{D}}_{-x_j} \setminus \{-x_j\}$, and furthermore

$$J_R^{(1)}(z) = (J_R^{(1)})_{-x_j}^{(-1)}(z + x_j)^{-1} + O(1) \quad \text{as } z \rightarrow -x_j, \quad j = 1, \dots, 2g+1.$$

Recalling that $\partial \mathbb{D}_{-x_j}$ is oriented in the clockwise direction, a simple residue calculation shows that, for z outside the disks, (6.7) can be rewritten as

$$(6.8) \quad R^{(1)}(z) = \sum_{j=1}^{2g+1} \frac{1}{z + x_j} (J_R^{(1)})_{-x_j}^{(-1)}, \quad \text{for } z \in \mathbb{C} \setminus \bigcup_{j=1}^{2g+1} \mathbb{D}_{-x_j},$$

and in particular, we have

$$(6.9) \quad R^{(1)}(z) = \frac{1}{z} \sum_{j=1}^{2g+1} (J_R^{(1)})_{-x_j}^{(-1)} + O(z^{-2}) \quad \text{as } z \rightarrow \infty.$$

Let $R_1 = \lim_{z \rightarrow \infty} z(R(z) - I)$. From (6.9), we infer that

$$(6.10) \quad R_1(r) = \frac{R_1^{(\frac{1}{2})}}{\sqrt{r}} + O(r^{-1}), \quad R_1^{(\frac{1}{2})} = \sum_{j=1}^{2g+1} (J_R^{(1)})_{-x_j}^{(-1)}.$$

The matrices $(J_R^{(1)})_{-x_j}^{(-1)}$ can be easily computed using (4.29), (4.32), (4.34), (5.3), (5.8), (6.4) and the fact that $\det P^{(\infty)}(z) \equiv 1$. Thus, we have

$$(J_R^{(1)})_{-x_{2j+1}}^{(-1)} = c_{2j+1}^{-\frac{1}{2}} (P^{(\infty)})_{-x_{2j+1}}^{(-\frac{1}{4})} e^{-(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} \Phi_{\text{Be},1} e^{(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} (P_{\text{inv}}^{(\infty)})_{-x_{2j+1}}^{(-\frac{1}{4})},$$

for $j = 0, \dots, g$,

$$(J_R^{(1)})_{-x_{2j}}^{(-1)} = i(c_{2j})^{-\frac{1}{2}} (P^{(\infty)})_{-x_{2j}}^{(-\frac{1}{4})} e^{-(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} \sigma_3 \Phi_{\text{Be},1} \sigma_3 e^{(i\pi\alpha - i\hat{\Omega}_j\sqrt{r})\frac{\sigma_3}{2}} (P_{\text{inv}}^{(\infty)})_{-x_{2j}}^{(-\frac{1}{4})},$$

for $j = 1, \dots, g$, where

$$(P_{\text{inv}}^{(\infty)})_{-x_j}^{(-\frac{1}{4})} = \begin{pmatrix} (P^{(\infty)})_{-x_j, 22}^{(-\frac{1}{4})} & -(P^{(\infty)})_{-x_j, 12}^{(-\frac{1}{4})} \\ -(P^{(\infty)})_{-x_j, 21}^{(-\frac{1}{4})} & (P^{(\infty)})_{-x_j, 11}^{(-\frac{1}{4})} \end{pmatrix}, \quad j = 1, \dots, 2g+1.$$

In particular, using (1.16), we find

$$(6.11) \quad \left[(J_R^{(1)})_{-x_{2j+1}}^{(-1)} \right]_{12} = \frac{(\beta_{2j+1}^{(-\frac{1}{4})})^2 \mathcal{G}(\vec{\phi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{\vec{e}_1}{2}) \mathcal{G}(-\vec{\phi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{\vec{e}_1}{2})}{16i\sqrt{c_{2j+1}} \mathcal{G}(\vec{0})^2},$$

for $j = 0, \dots, g$ and

$$(6.12) \quad \left[(J_R^{(1)})_{-x_{2j}}^{(-1)} \right]_{12} = \frac{(\beta_{2j}^{(\frac{1}{4})})^2 \tilde{\mathcal{G}}_{2j}^+ \tilde{\mathcal{G}}_{2j}^-}{16\sqrt{c_{2j}} \mathcal{G}(\vec{0})^2}, \quad j = 1, \dots, g.$$

More precisely, the right-hand side of (6.11) for $j = 1, \dots, g$ has been obtained after some simplifications using $\theta(\vec{\phi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{e_1}{2} + \vec{v}) = e^{-2\pi i(v_1 + \dots + v_j)} \theta(-\vec{\phi}_+^{\mathbb{C}}(-x_{2j+1}) + \frac{e_1}{2} + \vec{v})$, and the right-hand side of (6.12) has been obtained using the relation $\tilde{\mathcal{G}}_{2j}^+ = -e^{-2\pi i(v_1 + \dots + v_j)} \tilde{\mathcal{G}}_{2j}^-$.

We note from (6.11) and (6.12) that the expressions for $\left[(J_R^{(1)})_{-x_j}^{(-1)} \right]_{12}$ appear unrelated for odd and even j . In Proposition 7.8 below, we show that they can all (both for j even and odd) be rewritten in an unified way.

7 Proofs of Theorems 1.1-1.5

Recall from Proposition 2.1 that

$$(7.1) \quad \partial_r \log F(r\vec{x}) = \frac{1}{2ir} \Phi_{1,12}(r) + \frac{1-4\alpha^2}{16r}.$$

7.1 Proof of Theorem 1.1

The proof relies on equation (7.1). We first compute the large r asymptotics of $\Phi_{1,12}(r)$.

Proposition 7.1. *We have*

$$\frac{\Phi_{1,12}(r)}{2ir} = c + \frac{P_{1,12}^{(\infty)} + R_{1,12}}{2i\sqrt{r}} = c + \frac{P_{1,12}^{(\infty)}}{2i\sqrt{r}} + \frac{R_{1,12}^{(\frac{1}{2})}}{2ir} + O(r^{-\frac{3}{2}}) \quad \text{as } r \rightarrow +\infty.$$

Proof. From (3.11), (3.15) and (6.1), we have, for all sufficiently large z ,

$$\Phi(z) = r^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ 2ic\sqrt{r} & 1 \end{pmatrix} R(z) P^{(\infty)}(z) e^{\sqrt{r}g(z)\sigma_3}.$$

Taking $z \rightarrow \infty$ and using (2.2), (3.5), (4.2) and (6.10), we obtain the claim. $\square$

Our next goal is to rewrite $\frac{P_{1,12}^{(\infty)}}{2i\sqrt{r}}$ as an r -derivative. To prepare ourselves for that matter, we first prove the following lemma.

Lemma 7.2. *We have $(\mathbb{A}^{-1})_{gj} = \frac{\Omega_j}{4\pi}$, $j = 1, \dots, g$.*

Proof. Let us define

$$(7.2) \quad w_r := \frac{z^{r-1}}{\sqrt{\mathcal{R}(z)}} dz, \quad r = 1, \dots, g+1,$$

so that

$$\omega_r = \sum_{k=1}^g w_k (\mathbb{A}^{-1})_{kr}, \quad r = 1, \dots, g.$$

Using the definition (1.10) of Ω_j , and noting that $a_{r,\ell} = \int_{A_r} w_\ell$, we get

$$(7.3) \quad \begin{aligned} \frac{\Omega_j}{4\pi} &= \frac{1}{4\pi i} \int_{B_j} \frac{q(s)}{\sqrt{\mathcal{R}(s)}} ds = \frac{1}{8\pi i} \int_{B_j} w_{g+1} - \frac{1}{8\pi i} \sum_{k=1}^g \sum_{r=1}^g \left(\int_{B_j} w_k \right) (\mathbb{A}^{-1})_{kr} a_{r,g+1} \\ &= \frac{1}{8\pi i} \left(\int_{B_j} w_{g+1} - \sum_{r=1}^g \int_{B_j} \omega_r \int_{A_r} w_{g+1} \right). \end{aligned}$$

The lemma will follow from (7.3) if we can show that

$$(7.4) \quad \int_{B_j} w_{g+1} - \sum_{r=1}^g \int_{A_r} w_{g+1} \int_{B_j} \omega_r = 8\pi i (\mathbb{A}^{-1})_{gj}.$$

To prove (7.4), first note that the differential w_{g+1} is holomorphic except at $z = \infty$. As $z \rightarrow \infty$, we have $\sqrt{\mathcal{R}(z)} = \sqrt{z}(z^g + O(z^{g-1}))$, and hence $w_{g+1} = \frac{dz}{\sqrt{z}}(1 + O(z^{-1}))$. In terms of the local analytic coordinate $u := z^{-\frac{1}{2}}$ at $z = \infty$, this becomes

$$w_{g+1} = -2 \frac{du}{u^2} + O(1) \quad \text{as } u \rightarrow 0.$$

It follows that the differential $\tilde{\eta}$, defined by

$$(7.5) \quad \tilde{\eta} = -\frac{1}{2} \left[w_{g+1} - \sum_{r=1}^g \left(\int_{A_r} w_{g+1} \right) \omega_r \right],$$

is an Abelian differential of the second kind with vanishing A -periods whose only pole is a double pole at $z = \infty$ (i.e. at $u = 0$); moreover, at this pole $\tilde{\eta}$ has the singular part du/u^2 . Applying Riemann's bilinear identity to $\tilde{\eta}$ and a holomorphic differential η , we find (see e.g. [36, p. 64, eq. (3.0.2)])

$$(7.6) \quad \sum_{l=1}^g \left(\int_{A_l} \eta \int_{B_l} \tilde{\eta} - \int_{B_l} \eta \int_{A_l} \tilde{\eta} \right) = 2\pi i \operatorname{Res}_{u=0} (f \tilde{\eta}),$$

where f is a function such that $df = \eta$ near $u = 0$. Let us fix $j \in \{1, \dots, g\}$ and choose $\eta = \omega_j$. Then (7.6) reduces to

$$(7.7) \quad \int_{B_j} \tilde{\eta} = 2\pi i \operatorname{Res}_{u=0} (f_j \tilde{\eta}),$$

where $df_j = \omega_j$ near $u = 0$. Using the definition (4.16) of ω_j , we obtain

$$\omega_j = z^{-3/2}(\mathbb{A}^{-1})_{gj}(1 + O(z^{-1}))dz = -2(\mathbb{A}^{-1})_{gj}(1 + O(u^2))du \quad \text{as } u \rightarrow 0,$$

which implies $f_j(u) = f_j(0) - 2(\mathbb{A}^{-1})_{gj}u + O(u^2)$ as $u \rightarrow 0$, and thus

$$(7.8) \quad 2\pi i \operatorname{Res}_{u=0}(f_j \tilde{\eta}) = -4\pi i (\mathbb{A}^{-1})_{gj}.$$

The identity (7.4) follows by substituting (7.5) and (7.8) into (7.7). This completes the proof. $\square$

Proposition 7.3. *We have the identity*

$$\frac{P_{1,12}^{(\infty)}}{2i\sqrt{r}} = -\frac{d_1}{2\sqrt{r}} - \frac{1}{\sqrt{r}} \sum_{j=1}^g (\mathbb{A}^{-1})_{gj} \frac{\partial_j \mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})} = \frac{d}{dr} [\log \theta(\vec{v}) - d_1 \sqrt{r}].$$

Proof. The first equality follows from (4.27). Note that $\mathcal{G}(\vec{0})$ depends on r , whereas d_1 and $(\mathbb{A}^{-1})_{gj}$ do not. Let $\vec{z} = (z_1, \dots, z_g) \in \mathbb{C}^g$. Using the definition (4.14) of $\mathcal{G}$, we get

$$\frac{\partial_j \mathcal{G}(\vec{z})}{\mathcal{G}(\vec{z})} = \frac{\partial_j \theta(\vec{z} + \vec{v})}{\theta(\vec{z} + \vec{v})} - \frac{\partial_j \theta(\vec{z})}{\theta(\vec{z})}, \quad j = 1, \dots, g.$$

Since $\theta(\vec{z}) = \theta(-\vec{z})$ for all $\vec{z} \in \mathbb{C}^g$, we have $\partial_j \theta(\vec{0}) = 0$ for all $j = 1, \dots, g$, and thus

$$\frac{\partial_j \mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})} = \frac{\partial_j \theta(\vec{v})}{\theta(\vec{v})}, \quad j = 1, \dots, g.$$

Lemma 7.2 gives the relation $\frac{dv_j}{dr} = -\frac{(\mathbb{A}^{-1})_{gj}}{\sqrt{r}}$, so we conclude that

$$-\frac{1}{\sqrt{r}} \sum_{j=1}^g (\mathbb{A}^{-1})_{gj} \frac{\partial_j \mathcal{G}(\vec{0})}{\mathcal{G}(\vec{0})} = \sum_{j=1}^g \frac{dv_j}{dr} \frac{\partial_j \theta(\vec{v})}{\theta(\vec{v})} = \frac{d}{dr} \log \theta(\vec{v}).$$

$\square$

Finding a simplified expression for $R_{1,12}^{(\frac{1}{2})}$ is more challenging and requires some preparation. A divisor on X is a formal product of points of the form

$$P_1^{\alpha_1} \cdots P_k^{\alpha_k}, \quad \text{where } P_1, \dots, P_k \in X, \quad \alpha_1, \dots, \alpha_k \in \mathbb{Z}.$$

Let X_g denote the n -fold symmetric product of the Riemann surface X , that is, X_g is the set of all divisors of the form $P_1 \cdots P_g$ where $P_1, \dots, P_g \in X$. It is known [36, section III.11.9] that X_g can be given a complex structure, so that X_g is a manifold of complex dimension g . The Abel map $\vec{\varphi}$ was defined in (1.17) as a map from X to $J(X)$. Following [36, p.92], we extend the map $\vec{\varphi}$ to X_g as follows:

$$(7.9) \quad \vec{\varphi} : X_g \rightarrow J(X), \quad D = P_1 \cdots P_g \mapsto \vec{\varphi}(D) = \sum_{j=1}^g \int_{-x_1}^{P_j} \vec{\omega}^t.$$

Let $\pi : X \rightarrow \mathbb{C}$ denote the projection of X onto $\mathbb{C}$, and let $\mathcal{A}$ be the subset of X_g consisting of all divisors $D = P_1 \cdots P_g$ such that $P_j \in \pi^{-1}([-x_{2j+1}, -x_{2j}])$, $j = 1, \dots, g$. One can equip $\mathcal{A}$ with an atlas of smooth local charts inherited from the atlas of analytic local charts of X_g , so that $\mathcal{A}$ is a smooth submanifold of X_g of real dimension g .

We recall from [36, p. 110 (a)] that a divisor $P_1 \cdots P_g$ is special if and only if there exists a non-zero holomorphic differential $\tilde{\omega}$ which vanishes at each of the points $P_1, \dots, P_g$.

Lemma 7.4. *Each divisor in $\mathcal{A}$ is non-special.*

Proof. Since $\omega_1, \dots, \omega_g$ form a basis of the space of holomorphic differentials, it follows that any holomorphic differential can be written in the form $\tilde{\omega} = \tilde{p}(z)dz/\sqrt{\mathcal{R}(z)}$ where $\tilde{p}$ is a polynomial of degree at most $g-1$. If $P_1 \cdots P_g \in \mathcal{A}$, then by definition of $\mathcal{A}$ we have $P_{j_1} \neq P_{j_2}$ for $1 \leq j_1 \neq j_2 \leq g$. Note that $dz/\sqrt{\mathcal{R}(z)}$ vanishes only at ∞ . Hence, $\tilde{\omega}$ vanishes at P_j if and only if $\tilde{p}$ vanishes at P_j . Since there exists no non-zero polynomial of degree $\leq g-1$ vanishing at g distinct points, the proof is complete. $\square$

According to Lemma 4.3, the g points $\vec{\varphi}(-x_3), \vec{\varphi}(-x_5), \dots, \vec{\varphi}(-x_{2g+1})$ are odd half-periods in $J(X)$. Following [36, p. 325, eq. (1.2.1)], we define the vector of Riemann constants $\mathcal{K} \in J(X)$ by

$$\begin{aligned} \mathcal{K} &:= \vec{\varphi}((-x_3)(-x_5) \cdots (-x_{2g+1})) \\ (7.10) \quad &= \sum_{j=1}^g \vec{\varphi}(-x_{2j+1}) = \frac{g}{2} \vec{e}_1 - \frac{1}{2} \sum_{j=2}^g \vec{e}_j + \frac{1}{2} \sum_{j=1}^g (g-j+1) \vec{\tau}_j \pmod{L(X)}, \end{aligned}$$

where we have used (4.12) for the last equality.

We recall the following result from [36, p. 312].

Lemma 7.5. *If $e = \vec{\varphi}(D) + \mathcal{K}$ and $D \in X_g$, then the multi-valued function $X \ni P \mapsto \theta(\vec{\varphi}(P) - e)$ vanishes identically if and only if D is special. Furthermore, this function has a well-defined zero set, and if D is not special, then D is the divisor of zeros of $X \ni P \mapsto \theta(\vec{\varphi}(P) - e)$.*

Clearly, the set

$$\mathcal{T} := \{ \mathcal{K} + \vec{x} + \tau \vec{m} : \vec{x} \in \mathbb{R}^g, \vec{m} \in \mathbb{Z}^g \} / L(X) = \mathcal{K} + \mathbb{R}^g / \mathbb{Z}^g$$

is a nonempty connected smooth submanifold of $J(X)$ of real dimension g . Let $\vec{\varphi}|_{\mathcal{A}}$ denote the restriction of $\vec{\varphi} : X_g \rightarrow J(X)$ to $\mathcal{A}$.

Proposition 7.6. *The map $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ is a diffeomorphism, that is, $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ is a smooth bijection, and $(\vec{\varphi}|_{\mathcal{A}})^{-1} : \mathcal{T} \rightarrow \mathcal{A}$ is smooth.*

Proof. Let $D = P_1 \cdots P_g \in \mathcal{A}$. Since $\mathcal{K} = \vec{\varphi}((-x_3)(-x_5) \cdots (-x_{2g+1}))$, we have

$$\vec{\varphi}(D) = \mathcal{K} + \sum_{j=1}^g \int_{-x_{2j+1}}^{P_j} \vec{\omega} \in \mathcal{T},$$

where we have used that the restriction of $\vec{\omega}$ to $\pi^{-1}([-x_{2j+1}, -x_{2j}])$, $j = 1, \dots, g$, takes values in $\mathbb{R}^g$. This shows that $\vec{\varphi}(\mathcal{A}) \subseteq \mathcal{T}$.

It readily follows from its definition that the map $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ is smooth. We have shown in Lemma 7.4 that all divisors in $\mathcal{A}$ are non-special. Let $D = P_1 \cdots P_g$ be an arbitrary divisor in $\mathcal{A}$. Since D is non-special, the tangent map of $\vec{\varphi} : X_g \rightarrow J(X)$ at D has full rank (see e.g. [36, section VI.2.5]); hence the tangent map of $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ at D also has full rank. Therefore, by the inverse function theorem, $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ is a local diffeomorphism.

Since $\vec{\varphi}|_{\mathcal{A}}$ is continuous, $\mathcal{A}$ is compact, and $\mathcal{T}$ is Hausdorff, it follows that $\vec{\varphi}|_{\mathcal{A}}$ is a proper map [48, Proposition A.53]. Therefore, we have shown that $\vec{\varphi}|_{\mathcal{A}}$ is a proper local diffeomorphism. Since $\mathcal{A}$ and $\mathcal{T}$ are nonempty and connected, this implies that $\vec{\varphi}|_{\mathcal{A}}$ is a smooth covering map [48, Proposition 4.46], that is, $\vec{\varphi}|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{T}$ is a smooth surjective map, and for each $\vec{t} \in \mathcal{T}$, there exists a neighborhood $U \subseteq \mathcal{T}$ of $\vec{t}$ such that each component of $(\vec{\varphi}|_{\mathcal{A}})^{-1}(U)$ is mapped diffeomorphically onto U by $\vec{\varphi}|_{\mathcal{A}}$.

Let $D_1, D_2 \in \mathcal{A}$. Lemma 7.4 implies that D_1 and D_2 are non-special. Hence, Lemma 7.5 implies that D_j is the divisor of zeros of $P \mapsto \theta(\vec{\varphi}(P) - \vec{\varphi}(D_j) - \mathcal{K})$, $j = 1, 2$. In particular, if $\vec{\varphi}(D_1) = \vec{\varphi}(D_2)$, then necessarily $D_1 = D_2$. This shows that $\vec{\varphi}|_{\mathcal{A}}$ is injective. Since an injective smooth covering map is a diffeomorphism, see [48, Proposition 4.33 (b)], the proof is complete. $\square$

Let $\vec{\mu} \in \mathbb{R}^g / \mathbb{Z}^g$. In view of Proposition 7.6, there exist $\tilde{D} = \tilde{P}_1 \cdots \tilde{P}_g \in \mathcal{A}$ and $\hat{D} = \hat{P}_1 \cdots \hat{P}_g \in \mathcal{A}$ such that

$$(7.11) \quad \tilde{D} = (\vec{\varphi}|_{\mathcal{A}})^{-1}(-\vec{\mu} - \frac{\vec{e}_1}{2} - \mathcal{K}), \quad \hat{D} = (\vec{\varphi}|_{\mathcal{A}})^{-1}(\vec{\mu} + \frac{\vec{e}_1}{2} - \mathcal{K}),$$

or equivalently

$$(7.12) \quad \vec{\mu} + \frac{\vec{e}_1}{2} = -\vec{\varphi}(\tilde{D}) - \mathcal{K} \pmod{L(X)}, \quad -\vec{\mu} - \frac{\vec{e}_1}{2} = -\vec{\varphi}(\hat{D}) - \mathcal{K} \pmod{L(X)}.$$

It follows from Lemma 7.4 that $\tilde{D}$ and $\hat{D}$ are non-special, and from Lemma 7.5 that

$$P \mapsto \theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{\mu}) = \theta(\vec{\varphi}(P) - \vec{\varphi}(\tilde{D}) - \mathcal{K})$$

and

$$P \mapsto \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{\mu}) = \theta(\vec{\varphi}(P) - \vec{\varphi}(\hat{D}) - \mathcal{K})$$

do not vanish identically and that their zero divisors are given by $\tilde{D}$ and $\hat{D}$, respectively. Let $\iota : X \rightarrow X$ be the sheet-changing involution. Since $\vec{\varphi}(P) = -\vec{\varphi}(\iota(P))$

and $\mathcal{K} = -\mathcal{K} \bmod L(X)$, it follows from (7.12) that $\tilde{P}_k = \iota(\hat{P}_k)$ (and in particular $\pi(\tilde{P}_k) = \pi(\hat{P}_k)$) for $k = 1, \dots, g$. Let us define

$$(7.13) \quad b_k(\vec{\mu}) := \pi(\tilde{P}_k) = \pi(\hat{P}_k) \in [-x_{2k+1}, -x_{2k}], \quad k = 1, \dots, g.$$

Proposition 7.7. *For each $P \in X$ and $\vec{\mu} \in \mathbb{R}^g / \mathbb{Z}^g$, we have*

$$(7.14) \quad \frac{\theta^2(\vec{0})}{\theta^2(\vec{\mu})} \frac{\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{\mu}) \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{\mu})}{\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2}) \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2})} = \prod_{k=1}^g \frac{z - b_k(\vec{\mu})}{z + x_{2k}}, \quad z = \pi(P).$$

Proof. Let $h(P)$ denote the left-hand side of (7.14). Using (1.16), it is easily verified that the function $\hat{h}$ defined by

$$\vec{u} \mapsto \hat{h}(\vec{u}) := \frac{\theta^2(\vec{0})}{\theta^2(\vec{\mu})} \frac{\theta(\vec{u} + \frac{\vec{e}_1}{2} + \vec{\mu}) \theta(-\vec{u} + \frac{\vec{e}_1}{2} + \vec{\mu})}{\theta(\vec{u} + \frac{\vec{e}_1}{2}) \theta(-\vec{u} + \frac{\vec{e}_1}{2})}, \quad \vec{u} \in \mathbb{C}^g,$$

satisfies

$$\hat{h}(\vec{u}) = \hat{h}(\vec{u} + \vec{\lambda}' + \tau \vec{\lambda}), \quad \vec{\lambda}, \vec{\lambda}' \in \mathbb{Z}^g.$$

This shows that $h(P) = \hat{h}(\varphi(P))$ is a well-defined function of $P \in X$. From the discussion above (7.13), we conclude that h has exactly $2g$ zeros (counting multiplicities) at $\tilde{P}_1, \dots, \tilde{P}_g, \hat{P}_1, \dots, \hat{P}_g$. Also, it follows from Lemma 4.3 that $\vec{\varphi}(-x_{2k}) + \frac{\vec{e}_1}{2}$ are odd half-periods, and therefore the function $\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2}) = \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2})$ has g simple zeros at $-x_2, -x_4, \dots, -x_{2g}$. In fact, $\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2})$ has no other zeros, because from (4.12) and (7.10), we deduce that

$$(7.15) \quad \vec{\varphi}((-x_2)(-x_4) \cdots (-x_{2g})) = \mathcal{K} + \frac{\vec{e}_1}{2},$$

and thus Lemma 7.5 ensures that $(-x_2)(-x_4) \cdots (-x_{2g})$ is the divisor of zeros of $\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2})$.

Since $\vec{\varphi}(P) = -\vec{\varphi}(\iota(P))$, h satisfies $h(P) = h(\iota(P))$ for all $P \in X$. In other words, $h(P)$ only depends on $\pi(P)$ and therefore h can be viewed as a meromorphic function on the Riemann sphere. It directly follows from the above discussion and from (7.13) that h , viewed as a function on the Riemann sphere, has g simple zeros at $b_1(\vec{\mu}), \dots, b_g(\vec{\mu})$, and g simple poles at $-x_2, -x_4, \dots, -x_{2g}$. Hence the ratio of the left- and right-hand sides of (7.14) is equal to a constant κ . By Proposition 4.1, $\vec{\varphi}(\infty) = \frac{\vec{e}_1}{2} \bmod L(X)$. This shows that $\kappa = 1$, which finishes the proof. $\square$

Define the function $\mathcal{B} : \mathbb{C} \times \mathbb{R}^g / \mathbb{Z}^g \rightarrow \mathbb{C} \cup \{\infty\}$ by

$$(7.16) \quad \mathcal{B}(z, \vec{u}) := \frac{\prod_{k=1}^g (z - b_k(\vec{u}))}{q(z)} = \frac{\theta^2(\vec{0})}{\theta^2(\vec{u})} \frac{\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{u}) \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2} + \vec{u})}{\theta(\vec{\varphi}(P) + \frac{\vec{e}_1}{2}) \theta(-\vec{\varphi}(P) + \frac{\vec{e}_1}{2})} \frac{\prod_{k=1}^g (z + x_{2k})}{q(z)},$$

where $P \in X$ is such that $z = \pi(P)$. The second equality in (7.16) follows from Proposition 7.7.

Proposition 7.8. *Recall that $\vec{v}(r) \in \mathbb{R}^g$ is defined in (1.13). We have*

$$R_{1,12}^{(\frac{1}{2})} = \frac{1}{16i} \sum_{j=1}^{2g+1} \mathcal{B}(-x_j, \vec{v}(r)).$$

Proof. After a direct computation using (4.31), (4.35), (5.3), (5.8), (6.11), (6.12), the definition of $\mathcal{G}$ (4.14), and the right-most expression for $\mathcal{B}$ in (7.16), we obtain

$$\frac{1}{16i} \mathcal{B}(-x_j, \vec{v}(r)) = \left[(J_R^{(1)})_{-x_j}^{(-1)} \right]_{12}, \quad j = 1, 2, \dots, 2g+1.$$

By (6.10), $R_{1,12}^{(\frac{1}{2})} = \sum_{j=1}^{2g+1} \left[(J_R^{(1)})_{-x_j}^{(-1)} \right]_{12}$, and thus the claim follows. $\square$

We are now ready to compute the large r asymptotics of $\frac{\Phi_{1,12}(r)}{2ir}$.

Proposition 7.9. *As $r \rightarrow +\infty$, we have*

$$(7.17) \quad \frac{\Phi_{1,12}(r)}{2ir} = \frac{d}{dr} \left[cr - d_1 \sqrt{r} + \log \theta(\vec{v}) - \frac{1}{32} \int_M^r \sum_{j=1}^{2g+1} \mathcal{B}(-x_j, \vec{v}(t)) \frac{dt}{t} \right] + O(r^{-\frac{3}{2}}),$$

where $M > 0$ is independent of r .

Proof. It follows from Propositions 7.1, 7.3 and 7.8 that

$$\begin{aligned} \frac{\Phi_{1,12}(r)}{2ir} &= c + \frac{P_{1,12}^{(\infty)}}{2i\sqrt{r}} + \frac{R_{1,12}^{(\frac{1}{2})}}{2ir} + O(r^{-\frac{3}{2}}) \\ &= c + \frac{d}{dr} [\log \theta(\vec{v}) - d_1 \sqrt{r}] - \frac{1}{32r} \sum_{j=1}^{2g+1} \mathcal{B}(-x_j, \vec{v}(r)) + O(r^{-\frac{3}{2}}) \end{aligned}$$

as $r \rightarrow +\infty$, which is equivalent to (7.17). $\square$

We obtain (1.20) after substituting the asymptotics (7.17) into (7.1), integrating in r , and then exponentiating both sides. To finish the proof of Theorem 1.1, it remains to prove (1.21).

Proposition 7.10. *Let $M > 0$ and let $\mathcal{H} : \mathbb{R}^g / \mathbb{Z}^g \rightarrow \mathbb{R}$ be continuous. Then $\widehat{\mathcal{H}} \in \mathbb{R}$ is well-defined by*

$$\widehat{\mathcal{H}} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{H}(\vec{v}(t^2)) dt$$

and, for all $r > 0$,

$$(7.18) \quad \int_M^r \mathcal{H}(\vec{v}(r')) \frac{dr'}{r'} = \frac{2}{\sqrt{r}} \int_{\sqrt{M}}^{\sqrt{r}} \mathcal{H}(\vec{v}(t^2)) dt \\ + 2 \int_{\sqrt{M}}^{\sqrt{r}} \frac{1}{t} \left(\frac{\int_{\sqrt{M}}^t \mathcal{H}(\vec{v}(t'^2)) dt'}{t} - \widehat{\mathcal{H}} \right) dt + \widehat{\mathcal{H}} \log \frac{r}{M}.$$

In particular,

$$(7.19) \quad \int_M^r \mathcal{H}(\vec{v}(r')) \frac{dr'}{r'} = \widehat{\mathcal{H}} \log r + o(\log r) \quad \text{as } r \rightarrow +\infty.$$

Proof. Because $\mathbb{R}^g/\mathbb{Z}^g$ is compact and $\mathcal{H}$ is continuous, $t \mapsto \mathcal{H}(\vec{v}(t^2))$ is an almost-periodic function (see [42, Definition 5.1]), and thus $\widehat{\mathcal{H}} \in \mathbb{R}$ is well-defined (see [42, p. 176]). The identity (7.18) can be verified by differentiating both sides. Since the first and second terms on the right-hand side of (7.18) are $O(1)$ and $o(\log r)$ as $r \rightarrow +\infty$, respectively, (7.19) follows. $\square$

Applying Proposition 7.10 to $\mathcal{H} = \mathcal{B}(-x_j, \cdot)$, we get

$$\int_M^r \frac{\mathcal{B}(-x_j, \vec{v}(r'))}{r'} dr' = \mathcal{B}_j \log r + o(\log r) \quad \text{as } r \rightarrow +\infty,$$

which is (1.21). This finishes the proof of Theorem 1.1.

7.2 Proof of Theorem 1.3

In this subsection, whose content is inspired by [26, p. 156], we assume that there exist $\delta_1, \delta_2 > 0$ such that (1.23) holds.

Proposition 7.11. *Let $\mathcal{H} : \mathbb{R}^g/\mathbb{Z}^g \rightarrow \mathbb{R}$ be analytic and assume that (1.23) holds. For any $M > 0$, we have*

$$(7.20) \quad \frac{1}{t} \int_{\sqrt{M}}^t \mathcal{H}(\vec{v}(t'^2)) dt' = \widehat{\mathcal{H}} + O(t^{-1}) \quad \text{as } t \rightarrow +\infty,$$

where $\widehat{\mathcal{H}} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{H}(\vec{v}(t^2)) dt$.

Proof. We already know from Proposition 7.10 that $\frac{1}{t} \int_{\sqrt{M}}^t \mathcal{H}(\vec{v}(t'^2)) dt' = \widehat{\mathcal{H}} + o(1)$ as $t \rightarrow +\infty$, so it only remains to show that the error term can be reduced from $o(1)$ to $O(t^{-1})$. Since $\mathcal{H}$ is analytic on $\mathbb{R}^g/\mathbb{Z}^g$, its Fourier series

$$\sum_{\vec{m} \in \mathbb{Z}^g} \ell_{\vec{m}} e^{2\pi i \vec{m} \cdot \vec{u}}$$

converges to $\mathcal{H}(\vec{u})$ uniformly for $\vec{u} \in \mathbb{R}^g/\mathbb{Z}^g$, and the Fourier coefficients $\ell_{\vec{m}}$ decay exponentially fast, i.e., there exist $c_1, c_2 > 0$ such that

$$(7.21) \quad |\ell_{\vec{m}}| \leq c_1 e^{-c_2 \|\vec{m}\|} \quad \text{for all } \vec{m} \in \mathbb{Z}^g.$$

Hence,

$$\begin{aligned} \frac{1}{t} \int_{\sqrt{M}}^t \mathcal{H}(\vec{v}(t'^2)) dt' &= \frac{1}{t} \int_{\sqrt{M}}^t \sum_{\vec{m} \in \mathbb{Z}^g} \ell_{\vec{m}} e^{-i\vec{m}^t (\vec{\Omega}t + 2\pi\vec{\alpha})} dt = \frac{1}{t} \sum_{\vec{m} \in \mathbb{Z}^g} \int_{\sqrt{M}}^t \ell_{\vec{m}} e^{-i\vec{m}^t (\vec{\Omega}t + 2\pi\vec{\alpha})} dt \\ &= \frac{1}{t} \sum_{\substack{\vec{m} \in \mathbb{Z}^g \\ \vec{m}^t \vec{\Omega} \neq 0}} \ell_{\vec{m}} e^{-2\pi i \vec{m}^t \vec{\alpha}} \frac{e^{-i\vec{m}^t \vec{\Omega}t} - e^{-i\vec{m}^t \vec{\Omega}\sqrt{M}}}{-i\vec{m}^t \vec{\Omega}} + \frac{1}{t} \sum_{\substack{\vec{m} \in \mathbb{Z}^g \\ \vec{m}^t \vec{\Omega} = 0}} \ell_{\vec{m}} e^{-2\pi i \vec{m}^t \vec{\alpha}} (t - \sqrt{M}). \end{aligned}$$

The claim now follows from (1.23) and (7.21). $\square$

Applying Proposition 7.11 to $\mathcal{H} = \mathcal{B}(-x_j, \cdot)$ gives

$$(7.22) \quad \frac{1}{t} \int_{\sqrt{M}}^t \mathcal{B}(-x_j, \vec{v}(t'^2)) dt' = \mathcal{B}_j + O(t^{-1}) \quad \text{as } t \rightarrow +\infty,$$

and substituting these asymptotics into (7.18) (again with $\mathcal{H} = \mathcal{B}(-x_j, \cdot)$), we get

$$(7.23) \quad \int_M^r \mathcal{B}(-x_j, \vec{v}(r')) \frac{dr'}{r'} = \mathcal{B}_j \log r + C_j + O(r^{-\frac{1}{2}}) \quad \text{as } r \rightarrow +\infty,$$

for some C_j independent of r . This completes the proof of (1.26), and hence also of Theorem 1.3.

7.3 Proof of Theorem 1.5

In this subsection we assume that $\Omega_1, \dots, \Omega_g$ are rationally independent. This condition implies that the linear flow

$$(1, +\infty) \ni t \mapsto (v_1(t^2) \bmod 1, v_2(t^2) \bmod 1, \dots, v_g(t^2) \bmod 1)$$

is ergodic in $\mathbb{R}^g/\mathbb{Z}^g$, where we recall that $\vec{v}(r)$ is given by (1.13). Given a function $\mathcal{H} : \mathbb{R}^g/\mathbb{Z}^g \rightarrow \mathbb{R}$, the Birkhoff ergodic theorem states that the average of $\mathcal{H}$ along the above flow (often referred to as the *time average* in the literature) is equal to the average of $\mathcal{H}$ over the torus $\mathbb{R}^g/\mathbb{Z}^g$ (the *space average*). More precisely, the time average of $\mathcal{H}$ is defined by

$$(7.24) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_M^T \mathcal{H}(\vec{v}(t^2) + \vec{x}) dt, \quad \vec{x} \in \mathbb{R}^g, \quad M > 0,$$

(this limit is clearly independent of M) and the space average of $\mathcal{H}$, which we denote by $\langle \mathcal{H} \rangle$, is defined by

$$(7.25) \quad \langle \mathcal{H} \rangle = \int_{[0,1]^g} \mathcal{H}(\vec{x}) d\vec{x} = \int_0^1 \int_0^1 \cdots \int_0^1 \mathcal{H}(x_1, x_2, \dots, x_g) dx_1 dx_2 \cdots dx_g.$$

Because $\Omega_1, \dots, \Omega_g$ are rationally independent, Birkhoff's ergodic theorem states that (see e.g. [1, p. 286])

$$(7.26) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_M^T \mathcal{H}(\vec{v}(t^2) + \vec{x}) dt = \langle \mathcal{H} \rangle \quad \text{for every } \vec{x} \in \mathbb{R}^g.$$

In particular, the time average is independent of $\vec{x}$. To complete the proof of Theorem 1.5, it remains to evaluate $\langle \mathcal{B}(-x_j, \cdot) \rangle$ explicitly for $j = 1, 2, \dots, 2g+1$.

Let us orient the closed loop $\pi^{-1}([-x_{2j+1}, -x_{2j}])$ so that $\pi^{-1}([-x_{2j+1}, -x_{2j}])$ is homotopic to $A_j - A_{j+1}$, where $A_1, \dots, A_g$ are the cycles shown in Figure 1.1. This induces an orientation on $\mathcal{A}$ via the identification

$$\mathcal{A} \cong \pi^{-1}([-x_3, -x_2]) \times \pi^{-1}([-x_5, -x_4]) \times \cdots \times \pi^{-1}([-x_{2g+1}, -x_{2g}]).$$

Our next lemma shows that with this orientation of $\mathcal{A}$, the map $\vec{\phi}|_{\mathcal{A}} - \mathcal{K} - \frac{\vec{e}_1}{2}$ is an orientation-preserving diffeomorphism from $\mathcal{A}$ to the subset $\mathbb{R}^g/\mathbb{Z}^g$ of the Jacobian variety $J(X)$.

Lemma 7.12. *The map $D \mapsto f(D) := \vec{\phi}(D) - \mathcal{K} - \frac{\vec{e}_1}{2}$ is an orientation-preserving diffeomorphism from $\mathcal{A}$ to $\mathbb{R}^g/\mathbb{Z}^g$. Moreover, the inverse of f satisfies $\pi \circ f^{-1} = \vec{b}$, where $\vec{b} : \mathbb{R}^g/\mathbb{Z}^g \rightarrow \mathbb{R}^g$ is defined by*

$$(7.27) \quad \vec{b}(\vec{u}) = (b_1(\vec{u}), \dots, b_g(\vec{u})), \quad \vec{u} \in \mathbb{R}^g/\mathbb{Z}^g$$

and $\pi : \mathcal{A} \rightarrow \mathbb{R}^g$ is defined by $\pi(D) = (\pi(P_1), \dots, \pi(P_g))$ for $D = P_1 \cdots P_g \in \mathcal{A}$.

Proof. By Proposition 7.6, the map $\vec{\phi}|_{\mathcal{A}} - \mathcal{K} - \frac{\vec{e}_1}{2}$ is a diffeomorphism from $\mathcal{A}$ to $\mathbb{R}^g/\mathbb{Z}^g$. To see that it is orientation-preserving, note that the multi-valued function

$$\vec{\phi} : \mathcal{A} \rightarrow \mathbb{C}^g, \quad D = P_1 \cdots P_g \mapsto \vec{\phi}(D) = \sum_{j=1}^g \int_{-x_1}^{P_j} \vec{\omega}^t$$

increases by $\vec{f}_j = \vec{e}_j - \vec{e}_{j+1}$ as P_j goes once around $\pi^{-1}([-x_{2j+1}, -x_{2j}])$. Since the matrix relating the two bases $\{\vec{e}_j\}_{j=1}^g$ and $\{\vec{f}_j\}_{j=1}^g$ has determinant $+1$, the map $\vec{\phi}|_{\mathcal{A}} - \mathcal{K} : \mathcal{A} \rightarrow \mathbb{R}^g/\mathbb{Z}^g$ is orientation-preserving.

Suppose $f(D) = \vec{u}$ where $D \in \mathcal{A}$ and $\vec{u} \in \mathbb{R}^g/\mathbb{Z}^g$. Then $\vec{u} + \frac{\vec{e}_1}{2} = \vec{\phi}(D) + \mathcal{K}$ in $J(X)$, so by Lemma 7.4 and Lemma 7.5, D is the divisor of zeros of $\theta(\vec{\phi}(\cdot) - \vec{u} - \frac{\vec{e}_1}{2})$. The definition (7.13) of b_k then implies that $\pi(D) = \vec{b}(\vec{u})$. This shows that $\pi(f^{-1}(\vec{u})) = \vec{b}(\vec{u})$ as desired. $\square$

Proposition 7.13. *For all $z \in \mathbb{C}$, we have*

$$(7.28) \quad \left\langle \prod_{k=1}^g (z - b_k(\cdot)) \right\rangle = \det(zI - T),$$

where $b_1, \dots, b_g$ are defined in (7.13), and the $g \times g$ matrix T is defined by

$$(7.29) \quad T = (T_{ij})_{i,j=1}^g, \quad T_{ij} := \int_{A_i} z \omega_j = \int_{A_i} \frac{dz}{\sqrt{\mathcal{R}(z)}} (z^2 \cdots z^g)(\mathbb{A}^{-1}) \vec{e}_j.$$

Proof. Fix $z_0 \in \mathbb{C}$. Defining the function F by

$$F : \pi(\mathcal{A}) \cong [-x_3, -x_2] \times [-x_5, -x_4] \times \cdots \times [-x_{2g+1}, -x_{2g}] \rightarrow \mathbb{C},$$

$$F(\vec{b}) = \prod_{k=1}^g (z_0 - b_k),$$

and the function $\vec{b} : \mathbb{R}^g / \mathbb{Z}^g \rightarrow \mathbb{R}^g$ by (7.27), we can write the space average of $F \circ \vec{b}$ as

$$(7.30) \quad \langle F \circ \vec{b} \rangle = \int_{[0,1]^g} F(\vec{b}(\vec{u})) d\vec{u}$$

By Lemma 7.12, the map $f := \bar{\varphi}|_{\mathcal{A}} - \mathcal{K} - \frac{\vec{e}_1}{2}$ is an orientation-preserving diffeomorphism from $\mathcal{A}$ to $\mathbb{R}^g / \mathbb{Z}^g \cong [0,1]^g$. Hence

$$(7.31) \quad \langle F \circ \vec{b} \rangle = \int_{f(\mathcal{A})} F \circ \vec{b} d\vec{u} = \int_{\mathcal{A}} f^*(F \circ \vec{b} d\vec{u}),$$

where $f^*(F \circ \vec{b} d\vec{u})$ denotes the pull-back by f of the g -form $F \circ \vec{b} d\vec{u}$. (The last step in (7.31) amounts to changing variables from $\vec{u} \in \mathbb{R}^g / \mathbb{Z}^g$ to $D \in \mathcal{A}$ according to $u_k = \varphi_k(D) - \mathcal{K}_k - \frac{(\vec{e}_1)_k}{2}$.) In the same way that we use the analytic coordinate z to perform calculations on X (viewing X as a two-sheeted cover of the complex z -plane), we can use $(z_1, \dots, z_g)$ as coordinates on

$$\mathcal{A} \cong \pi^{-1}([-x_3, -x_2]) \times \pi^{-1}([-x_5, -x_4]) \times \cdots \times \pi^{-1}([-x_{2g+1}, -x_{2g}]) \subset X \times \cdots \times X,$$

where z_j denotes the coordinate on the j -th factor of X . Evaluating the integral on the right-hand side of (7.31) in these coordinates and using that $f^*(F \circ \vec{b}) = F \circ \vec{b} \circ f = F \circ \pi$ by Lemma 7.12, we obtain

$$\begin{aligned} \langle F \circ \vec{b} \rangle &= \prod_{r=1}^g \int_{A_r - A_{r+1}} dz_r \det(\varphi'_i(z_j))_{i,j=1}^g F(z_1, \dots, z_g) \\ &= \prod_{r=1}^g \int_{A_r - A_{r+1}} dz_r \begin{vmatrix} \varphi'_1(z_1) & \varphi'_1(z_2) & \cdots & \varphi'_1(z_g) \\ \varphi'_2(z_1) & \varphi'_2(z_2) & \cdots & \varphi'_2(z_g) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_g(z_1) & \varphi'_g(z_2) & \cdots & \varphi'_g(z_g) \end{vmatrix} \prod_{k=1}^g (z_0 - z_k) \\ &= \prod_{r=1}^g \int_{A_r - A_{r+1}} dz_r \begin{vmatrix} \varphi'_1(z_1)(z_0 - z_1) & \varphi'_1(z_2)(z_0 - z_2) & \cdots & \varphi'_1(z_g)(z_0 - z_g) \\ \varphi'_2(z_1)(z_0 - z_1) & \varphi'_2(z_2)(z_0 - z_2) & \cdots & \varphi'_2(z_g)(z_0 - z_g) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_g(z_1)(z_0 - z_1) & \varphi'_g(z_2)(z_0 - z_2) & \cdots & \varphi'_g(z_g)(z_0 - z_g) \end{vmatrix}. \end{aligned}$$

Using that $\varphi'_j(z) dz = \omega_j$ and that $\oint_{A_k} \omega_j = \delta_{jk}$ for $j, k = 1, \dots, g$, we conclude that $\langle F \circ \vec{b} \rangle$ is given by the determinant

$$\begin{vmatrix} z_0 - \int_{A_1 - A_2} z \omega_1 & - \int_{A_2 - A_3} z \omega_1 & \cdots & - \int_{A_{g-1} - A_g} z \omega_1 & - \int_{A_g} z \omega_1 \\ -z_0 - \int_{A_1 - A_2} z \omega_2 & z_0 - \int_{A_2 - A_3} z \omega_2 & \cdots & - \int_{A_{g-1} - A_g} z \omega_2 & - \int_{A_g} z \omega_2 \\ - \int_{A_1 - A_2} z \omega_3 & -z_0 - \int_{A_2 - A_3} z \omega_3 & \cdots & - \int_{A_{g-1} - A_g} z \omega_3 & - \int_{A_g} z \omega_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - \int_{A_1 - A_2} z \omega_{g-1} & - \int_{A_2 - A_3} z \omega_{g-1} & \cdots & z_0 - \int_{A_{g-1} - A_g} z \omega_{g-1} & - \int_{A_g} z \omega_{g-1} \\ - \int_{A_1 - A_2} z \omega_g & - \int_{A_2 - A_3} z \omega_g & \cdots & -z_0 - \int_{A_{g-1} - A_g} z \omega_g & z_0 - \int_{A_g} z \omega_g \end{vmatrix}.$$

By first adding the g -th column to the $(g-1)$ -th column, then the $(g-1)$ -th column to the $(g-2)$ -th column etc., we see that $\langle F \circ \vec{b} \rangle$ is equal to

$$\begin{aligned}
 & \begin{vmatrix} z_0 - \int_{A_1-A_2} z \omega_1 & -\int_{A_2-A_3} z \omega_1 & \cdots & -\int_{A_{g-1}} z \omega_1 & -\int_{A_g} z \omega_1 \\ -z_0 - \int_{A_1-A_2} z \omega_2 & z_0 - \int_{A_2-A_3} z \omega_2 & \cdots & -\int_{A_{g-1}} z \omega_2 & -\int_{A_g} z \omega_2 \\ -\int_{A_1-A_2} z \omega_3 & -z_0 - \int_{A_2-A_3} z \omega_3 & \cdots & -\int_{A_{g-1}} z \omega_3 & -\int_{A_g} z \omega_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\int_{A_1-A_2} z \omega_{g-1} & -\int_{A_2-A_3} z \omega_{g-1} & \cdots & z_0 - \int_{A_{g-1}} z \omega_{g-1} & -\int_{A_g} z \omega_{g-1} \\ -\int_{A_1-A_2} z \omega_g & -\int_{A_2-A_3} z \omega_g & \cdots & -\int_{A_{g-1}} z \omega_g & z_0 - \int_{A_g} z \omega_g \end{vmatrix} \\
 &= \cdots = \begin{vmatrix} z_0 - \int_{A_1} z \omega_1 & -\int_{A_2} z \omega_1 & \cdots & -\int_{A_{g-1}} z \omega_1 & -\int_{A_g} z \omega_1 \\ -\int_{A_1} z \omega_2 & z_0 - \int_{A_2} z \omega_2 & \cdots & -\int_{A_{g-1}} z \omega_2 & -\int_{A_g} z \omega_2 \\ -\int_{A_1} z \omega_3 & -\int_{A_2} z \omega_3 & \cdots & -\int_{A_{g-1}} z \omega_3 & -\int_{A_g} z \omega_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\int_{A_1} z \omega_{g-1} & -\int_{A_2} z \omega_{g-1} & \cdots & z_0 - \int_{A_{g-1}} z \omega_{g-1} & -\int_{A_g} z \omega_{g-1} \\ -\int_{A_1} z \omega_g & -\int_{A_2} z \omega_g & \cdots & -\int_{A_{g-1}} z \omega_g & z_0 - \int_{A_g} z \omega_g \end{vmatrix} \\
 &= \det(z_0 I - T).
 \end{aligned}$$

Since $z_0 \in \mathbb{C}$ was arbitrary, this finishes the proof. $\square$

Proposition 7.14. *For all $z \in \mathbb{C}$, we have the identity*

$$(7.32) \quad q(z) = \frac{1}{2} \det(zI - T),$$

where $q(z)$ is defined in (1.7).

Proof. Let $\hat{\mathbb{A}}$ be the $g \times g$ matrix defined by

$$(7.33) \quad \hat{\mathbb{A}} := (a_{i,j+1})_{i,j=1}^g, \quad a_{i,j+1} = \int_{A_i} w_{j+1} = \int_{A_i} \frac{z^j dz}{\sqrt{\mathcal{R}(z)}},$$

where we recall that w_{j+1} is defined in (7.2). We can write the definition (7.29) of T_{ij} as

$$T_{ij} = \int_{A_i} z \omega_j = \sum_{k=1}^g \hat{\mathbb{A}}_{ik} (\mathbb{A}^{-1})_{kj}, \quad i, j = 1, \dots, g,$$

and hence

$$(7.34) \quad T = \hat{\mathbb{A}} \mathbb{A}^{-1}.$$

On the other hand, since the j -th column of $\hat{\mathbb{A}}$ equals the $(j+1)$ -th column of $\mathbb{A}$ for $j = 1, \dots, g-1$, the matrix $\mathbb{A}^{-1}\hat{\mathbb{A}}$ has the form

$$(7.35) \quad \mathbb{A}^{-1}\hat{\mathbb{A}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix},$$

and by (1.7) the g -th column of $\mathbb{A}^{-1}\hat{\mathbb{A}}$ is given by

$$\mathbb{A}^{-1}\hat{\mathbb{A}}\vec{e}_g = \mathbb{A}^{-1}\vec{a} = -2(q_0, \dots, q_{g-1})^t.$$

Hence, we find

$$\begin{aligned} \frac{1}{2} \det(zI - T) &= \frac{1}{2} \det(zI - \hat{\mathbb{A}}\mathbb{A}^{-1}) = \frac{1}{2} \det(zI - \mathbb{A}^{-1}\hat{\mathbb{A}}) \\ &= \frac{z^g}{2} - \frac{1}{2} \sum_{j=1}^g (\mathbb{A}^{-1}\hat{\mathbb{A}})_{jg} z^{j-1} = q(z), \end{aligned}$$

where we have expanded the determinant along the last column to obtain the third equality. $\square$

We conclude from (7.19), (7.26), and the definition (1.19) of $\mathcal{B}$ that

$$\begin{aligned} \int_M^r \frac{\mathcal{B}(-x_j, \vec{V}(t))}{t} dt &= \langle \mathcal{B}(-x_j, \cdot) \rangle \log r + o(\log r) \\ &= \left\langle \frac{\prod_{k=1}^g (-x_j - b_k(\cdot))}{q(-x_j)} \right\rangle \log r + o(\log r), \quad r \rightarrow +\infty, \end{aligned}$$

for $j = 1, \dots, 2g+1$. Propositions 7.13 and 7.14 imply that

$$\left\langle \prod_{k=1}^g (-x_j - b_k(\cdot)) \right\rangle = 2q(-x_j), \quad j = 1, \dots, 2g+1.$$

Consequently, as $r \rightarrow +\infty$,

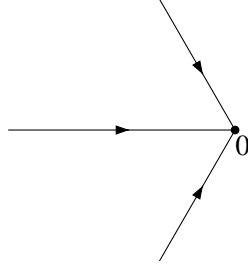
$$\int_M^r \frac{\mathcal{B}(-x_j, \vec{V}(t))}{t} dt = 2\log r + o(\log r), \quad j = 1, \dots, 2g+1,$$

which proves (1.27) and completes the proof of Theorem 1.5.

Appendix: On the mapping $(x_1, x_2, \dots, x_{2g+1}) \rightarrow (\Omega_1, \dots, \Omega_g)$

Since q has exactly one zero in each of the intervals $(-x_{2g+1}, -x_{2g}), \dots, (-x_5, -x_4), (-x_3, -x_2)$, we have

$$q(z) \rightarrow \frac{1}{2} \prod_{j=1}^g (z + x_{2j}) \quad \text{as } (x_2, x_4, \dots, x_{2g}) \rightarrow (x_3, x_5, \dots, x_{2g+1})$$

FIGURE B.1. The jump contour Σ_{Be} for Φ_{Be} .

uniformly for z in compact subsets of $\mathbb{C}$. Hence, by (1.10), for each $j \in \{1, 2, \dots, g\}$, $\Omega_j \rightarrow 2(\sqrt{x_{2j+1} - x_1} - \sqrt{x_{2j-1} - x_1})$ as $(x_2, x_4, \dots, x_{2g}) \rightarrow (x_3, x_5, \dots, x_{2g+1})$, and thus for each $\vec{x}_* := (x_1, x_3, x_3, x_5, x_5, \dots, x_{2g+1}, x_{2g+1})$ satisfying $0 < x_1 < x_3 < x_5 < \dots < x_{2g+1}$, $\Omega(\vec{x}_*)$ is analytic in each of the variables $x_1, x_3, x_5, \dots, x_{2g+1}$ and we have

$$\begin{aligned} \det \left(\frac{d(\Omega_j(\vec{x}_*))}{dx_{2k+1}} \right)_{j,k=1}^g &= \det \left((\partial_{2k} \Omega_j)(\vec{x}_*) + (\partial_{2k+1} \Omega_j)(\vec{x}_*) \right)_{j,k=1}^g \\ &= \prod_{j=1}^g \frac{1}{\sqrt{x_{2j+1} - x_1}} \neq 0. \end{aligned}$$

This shows that the image of the mapping

$$\{(x_1, x_2, \dots, x_{2g+1}) : x_1 < x_2 < \dots < x_{2g+1}\} \rightarrow (\Omega_1, \dots, \Omega_g)$$

contains an open ball in $(0, +\infty)^g$, so that for $g \geq 2$ each of the four cases in (1.25) does happen for certain choices of $(x_1, x_2, \dots, x_{2g+1})$ (for $g = 2$ there are only three cases because $\mathcal{D} \cup \mathcal{E} = (0, +\infty) \times (0, +\infty)$).

Appendix: Bessel model RH problem

- (a) $\Phi_{\text{Be}} : \mathbb{C} \setminus \Sigma_{\text{Be}} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic, where $\Sigma_{\text{Be}} = (-\infty, 0) \cup e^{\frac{2\pi i}{3}}(0, +\infty) \cup e^{-\frac{2\pi i}{3}}(0, +\infty)$ is oriented as shown in Figure B.1.
- (b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned} \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\infty, 0), \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in e^{\frac{2\pi i}{3}}(0, +\infty), \\ \Phi_{\text{Be},+}(z) &= \Phi_{\text{Be},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in e^{-\frac{2\pi i}{3}}(0, +\infty). \end{aligned} \tag{B.1}$$

- (c) As $z \rightarrow \infty$, $z \notin \Sigma_{\text{Be}}$, we have

$$\Phi_{\text{Be}}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} M \left(I + \frac{\Phi_{\text{Be},1}}{z^{\frac{1}{2}}} + O(z^{-1}) \right) e^{2z^{\frac{1}{2}} \sigma_3}, \tag{B.2}$$

where $\Phi_{\text{Be},1} = \frac{1}{16} \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}$ and $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

(d) As z tends to 0, the behavior of $\Phi_{\text{Be}}(z)$ is

$$(B.3) \quad \Phi_{\text{Be}}(z) = \begin{cases} \begin{pmatrix} O(1) & O(\log z) \\ O(1) & O(\log z) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} O(\log z) & O(\log z) \\ O(\log z) & O(\log z) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi. \end{cases}$$

The unique solution of this RH problem exists and can be explicitly constructed in terms of Bessel functions [26, 46].

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