

Algebraically coherent categories: definition, examples and basic properties

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joint work with Alan S Cigoli and James RA Gray

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Quick definition, motivation

A regular category with pushouts is called **algebraically coherent** when the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs.

Why this condition?

- ▶ A variation on it appears in [S Mantovani & G Metere, 2010].
- ▶ Needed in recent work of AS Cigoli–S Mantovani–G Metere, AS Cigoli–A Montoli and M Hartl–TVdL.
- ▶ JRA Gray: (LACC) too strong to include all *categories of interest*. Need for a weaker condition which does, while still implying others such as (SH), (NH), strong protomodularity.
- ▶ Leads to non-trivial results such as the *Three Subobjects Lemma*.
- ▶ A situation where a concept in Categorical Algebra corresponds to a concept in Topos Theory. [G Janelidze, 2012]

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Full definition, basic properties I

In a finitely complete category $\mathbb{X}$, a pair (f, g) is **jointly strongly epimorphic** if and only if it is **jointly extremal epimorphic**: for each commutative diagram

$$\begin{array}{ccc} & M & \\ f' \nearrow & \downarrow m & \nwarrow g' \\ X & \xrightarrow{f} Z \xleftarrow{g} & Y, \end{array}$$

if m is a monomorphism, then m is an isomorphism.

When $\mathbb{X}$ is regular and admits binary sums, this happens if and only if $\begin{pmatrix} f \\ g \end{pmatrix}: X + Y \rightarrow Z$ is a regular epimorphism.

A functor between regular categories with binary sums is called **coherent** when it preserves finite limits and jointly strongly epimorphic pairs.

A regular category with pushouts is called **algebraically coherent** ($\mathbf{C}_{\text{Alg}}$) when the change-of-base functors in the fibration of points are coherent.

Unless mentioned otherwise, we restrict ourselves to semi-abelian $\mathbb{X}$.

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Full definition, basic properties II

Theorem

The following conditions are equivalent:

- i $\mathbb{X}$ is algebraically coherent;*
- ii all kernel functors $\text{Ker } \text{!}_B^* : \mathbf{Pt}_B(\mathbb{X}) \rightarrow \mathbb{X}$ are coherent;*
- iii all $Bb(-) : \mathbb{X} \rightarrow \mathbb{X}$ preserve jointly strongly epimorphic pairs.*

Theorem

All categories of interest [G Orzech, 1976] are algebraically coherent.

Proposition

Algebraic coherence is stable under: taking slices and coslices; taking internal actions; taking presheaves; taking a subcategory, closed under subobjects and products—in particular, a (regular epi)-reflective subcategory; taking internal categories, groups or (pre)crossed modules.

- ▶ Groups; associative algebras; Lie, Leibniz, Poisson algebras; n -nilpotent or n -solvable groups, rings, algebras; torsion-free groups, reduced rings; crossed modules of such; n -cat-groups.

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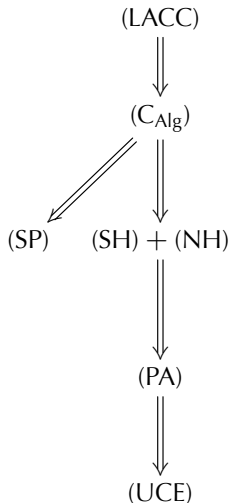
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Relation with some other categorical-algebraic conditions



Gp, Lie_R, HopfAlg_{K,coc} are examples

all categories of interest are examples

internal crossed modules simplify
cohomology classifies higher central extensions
 $[X, X] \triangleleft X$ is characteristic

extension is abelianised
by abelianising its kernel

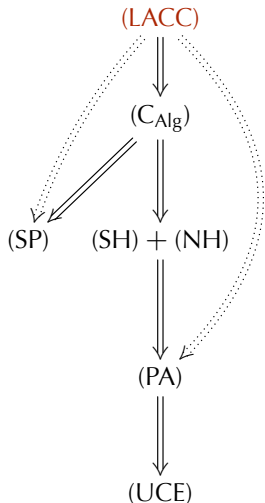
good theory of universal central extensions
Grün's Lemma

(LACC) loc. algebraically cartesian closed

[D Bourn & JRA Gray, 2012]

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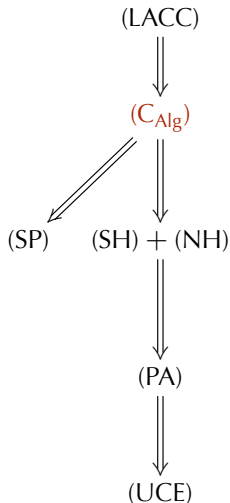
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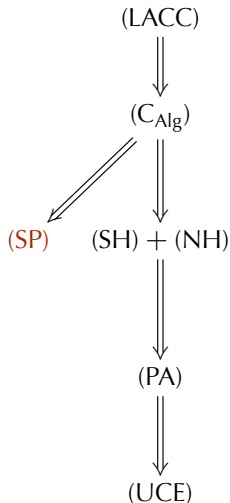
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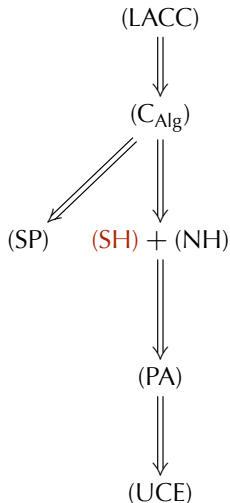
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(SP) strongly protomodular [D Bourn, 2004]

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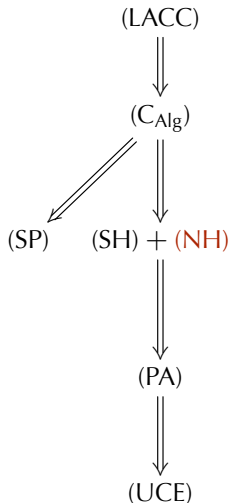
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(SH) *Smith is Huq* [N Martins-Ferreira & TVdL, 2012][D Rodelo & TVdL, 2012]

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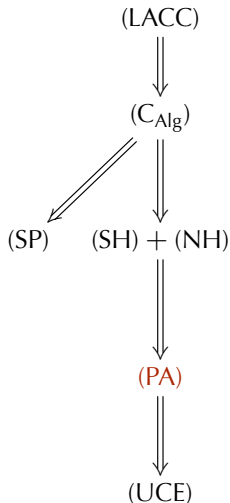
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(NH) Higgins commutators of normal subobjects are normal

[AS Cigoli, JRA Gray & TVdL, 2014] [AS Cigoli & A Montoli, 2014]

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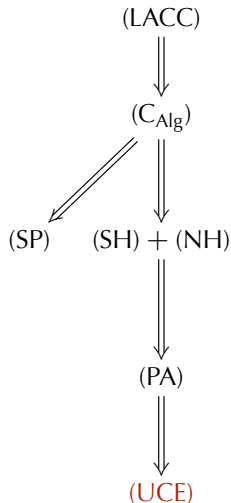
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(PA) peri-abelian [D Bourn, 2010] [JRA Gray & TVdL, 2014]

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(UCE) universal central extension condition [J-M Casas & TVdL, 2014]

Comparing conditions in terms of the fibration of points

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$\mathbb{X}$ finitely complete

$\mathbf{P}\mathbb{X} = \text{cod}: \mathbf{Pt}(\mathbb{X}) \rightarrow \mathbb{X}: ((p, s): E \rightrightarrows B) \mapsto B$ is the **fibration of points**, of which the change-of-base functors are determined by pulling back.

$$\begin{array}{ccc} & E & \\ & \uparrow s \quad \downarrow p & \\ A & \xrightarrow{f} & B \end{array} \quad (p, s)$$

$$\begin{aligned} f^*: \mathbf{Pt}_B(\mathbb{X}) &\rightarrow \mathbf{Pt}_A(\mathbb{X}) \\ (p, s) &\mapsto f^*(p, s) \end{aligned}$$

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The change-of-base functors...
reflect isos

The category $\mathbb{X}$ is...
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Comparing conditions in terms of the fibration of points

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Topos Theory vs. Categorical Algebra

[G Janelidze, 2012]

[PT Johnstone, 2002]

$\mathbb{X}$ regular with pushouts

Topos Theory

basic fibration $\mathbf{Arr}(\mathbb{X}) \rightarrow \mathbb{X}$

fibre: slice category $(\mathbb{X} \downarrow B)$

(loc.) cartesian closed

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There are no coherent categories in algebra: unital + coherent $\Rightarrow$ trivial.

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A result, some non-examples, some examples

Theorem

Any pointed algebraically coherent Mal'tsev category is protomodular.

Are semi-abelian but not algebraically coherent:

- ▶ (commutative) loops and digroups, which do not satisfy (SH);
- ▶ non-associative algebras and Jordan algebras, which need not satisfy (NH);
- ▶ Heyting semilattices, which form an arithmetical strongly protomodular category that does not satisfy (SSH).

Are algebraically coherent:

- ▶ naturally Mal'tsev categories: all abelian, additive, affine categories;
- ▶ all coherent categories—though they cannot be semi-abelian;
- ▶ $\mathbf{Haus}^{\mathbb{T}}$, $\mathbf{HComp}^{\mathbb{T}}$ for $\mathbb{T}$ a semi-abelian algebraically coherent theory;
- ▶ $\mathbf{Ext}(\mathbb{C}) \simeq \mathbf{NMono}(\mathbb{C})$, for $\mathbb{C}$ semi-abelian algebraically coherent, and $\mathbf{CExt}_{\mathbb{B}}(\mathbb{C})$ for $\mathbb{B} \subseteq \mathbb{C}$ Birkhoff;
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A result, some non-examples, some examples

Theorem

Any pointed algebraically coherent Mal'tsev category is protomodular.

Are semi-abelian but not algebraically coherent:

- ▶ (commutative) loops and digroups, which do not satisfy (SH);
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Theorem (Three Subobjects Lemma for normal subobjects)

If $K, L, M \triangleleft X$ in an algebraically coherent semi-abelian category, then

$$[K, L, M] = [[K, L], M] \vee [[M, K], L].$$

In particular, $[[L, M], K] \leqslant [[K, L], M] \vee [[M, K], L]$.

Note that in $\mathbf{Gp}$ this is also true for non-normal subobjects.

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Conclusion

A semi-abelian category is called **algebraically coherent** when the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs.

- ▶ $(\mathbf{C}_{\text{Alg}})$ captures consequences of (LACC) while still having all *categories of interest* as examples.
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Current questions:

- ▶ How to separate $(\mathbf{C}_{\text{Alg}})$ from action accessibility?
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More in our paper [arXiv:1409.4219v1](#). Comments welcome!

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