On the (a)symmetry of the stress tensor in continuum mechanics

Eric Deleersnijder, 9 August 2021

Abstract. *It is erroneous to believe that, in continuum mechanics, there exists a first principle about the angular momentum budget of a material volume. The latter must be established from the momentum equation (i.e., Newton's second law) with the help of common vector calculus tools (divergence theorem, Reynolds' transport theorem, etc.). If there are no body moments (i.e., no local production/destruction of angular momentum), which is often but not always the case, then the stress tensor must be symmetric. Various forms of the angular momentum budget of a material volume are derived, including that in which the angular momentum is evaluated with respect to the centre of mass.*

Angular momentum budget of material volume

Let t and $\mathbf{x} = (x,y,z)$ represent the time and the position-vector. Consider a compressible fluid whose density and velocity are denoted $\rho(t,\mathbf{x})$ and $\mathbf{v}(t,\mathbf{x})$, respectively. These variables satisfy the momentum budget equation (i.e. Newton's second law)

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) \;=\; \rho\mathbf{f} \;-\; \nabla\bullet(\rho\mathbf{v}\mathbf{v}-\mathbf{T}) \tag{1}$$

where $\mathbf{f}$ is the acceleration associated with the body/volume forces acting at a distance (e.g. gravity, Lorentz force) whilst $\mathbf{T}$ is the stress tensor.

Let Ω, Γ and $\mathbf{n}$ denote a material control volume, the surface delimiting it and the boundary's outward unit normal vector, respectively. Using Reynolds' transport theorem, the divergence theorem and momentum equation (1), the angular momentum budget of the fluid present in the control volume may be written as follows:

$$\begin{aligned}\frac{d}{dt}\overbrace{\int_\Omega \mathbf{x}\times(\rho\mathbf{v})\,d\Omega}^{\text{angular momentum of the fluid in }\Omega} &= \int_\Omega \frac{\partial}{\partial t}[\mathbf{x}\times(\rho\mathbf{v})]\,d\Omega + \int_\Gamma [\mathbf{x}\times(\rho\mathbf{v})]\mathbf{v}\bullet\mathbf{n}\,d\Gamma \\ &= \int_\Omega \underbrace{\frac{\partial\mathbf{x}}{\partial t}}_{=0}\times(\rho\mathbf{v})\,d\Omega + \int_\Omega \mathbf{x}\times\frac{\partial}{\partial t}(\rho\mathbf{v})\,d\Omega + \int_\Gamma [\mathbf{x}\times(\rho\mathbf{v})]\mathbf{v}\bullet\mathbf{n}\,d\Gamma \\ &= \int_\Omega \mathbf{x}\times[\rho\mathbf{f}-\nabla\bullet(\rho\mathbf{v}\mathbf{v}-\mathbf{T})]\,d\Omega + \int_\Gamma [\mathbf{x}\times(\rho\mathbf{v})]\mathbf{v}\bullet\mathbf{n}\,d\Gamma\end{aligned} \tag{2}$$

Using identity (A.1), which is established in the Appendix, allows simplifying (2) to

$$\begin{aligned}\frac{d}{dt}\int_\Omega \mathbf{x}\times(\rho\mathbf{v})\,d\Omega &= \int_\Omega \mathbf{x}\times(\rho\mathbf{f}+\nabla\bullet\mathbf{T})\,d\Omega \\ &\quad \underbrace{-\int_\Omega \mathbf{x}\times[\nabla\bullet(\rho\mathbf{v}\mathbf{v})]\,d\Omega + \int_\Gamma [\mathbf{x}\times(\rho\mathbf{v})]\mathbf{v}\bullet\mathbf{n}\,d\Gamma}_{=0\ ,\ \text{see (A.1)}}\end{aligned} \tag{3a}$$

or, equivalently,

$$\frac{d}{dt}\int_{\Omega} \mathbf{x}\times(\rho\mathbf{v})\,d\Omega \;=\; \int_{\Omega} \mathbf{x}\times(\rho\mathbf{f}+\nabla\bullet\mathbf{T})\,d\Omega \quad . \tag{3b}$$

Using (A.2) again to manipulate the right-hand side member of (3b) eventually leads to

$$\begin{aligned} \frac{d}{dt}\int_{\Omega} \mathbf{x}\times(\rho\mathbf{v})\,d\Omega \;=\; & \int_{\Omega} \mathbf{x}\times(\rho\mathbf{f})\,d\Omega + \int_{\Gamma} \mathbf{x}\times(\mathbf{T}\bullet\mathbf{n})\,d\Gamma \\ & - \int_{\Omega} [(T_{zy}-T_{yz})\mathbf{e}_x + (T_{xz}-T_{zx})\mathbf{e}_y + (T_{yx}-T_{xy})\mathbf{e}_z]\,d\Omega \end{aligned} \tag{4}$$

Symmetry of the stress tensor

The first two integrals in the right-hand side member of (4) have the form of the cross product of a lever arm vector and a force, i.e. they represent the moment of the body forces and that of the surface forces, respectively. Such terms are expected to be found in an angular momentum budget equation. However, the last integral in (4) is of a completely different nature. It represents the local rate of production of angular momentum and is, therefore, independent of the origin of the reference frame. Such a term is associated with none of the physical phenomena commonly considered in continuum mechanics. This is why it has to be identically zero, implying that the stress tensor must be symmetric, i.e.

$$T_{xy} = T_{yx} \;\; , \quad T_{xz} = T_{zx} \;\; , \quad T_{yz} = T_{zy} \quad . \tag{5}$$

As a consequence, the final form of the angular momentum budget reads

$$\frac{d}{dt}\overbrace{\int_{\Omega} \mathbf{x}\times(\rho\mathbf{v})\,d\Omega}^{\text{angular momentum of the fluid in } \Omega} \;=\; \overbrace{\int_{\Omega} \mathbf{x}\times(\rho\mathbf{f})\,d\Omega}^{\text{moment of the body forces}} + \overbrace{\int_{\Gamma} \mathbf{x}\times(\mathbf{T}\bullet\mathbf{n})\,d\Gamma}^{\text{moment of the surface forces}} \tag{6}$$

This expression holds valid irrespective of the nature of the stress tensor **T**, be it associated with viscous (i.e. molecular-scale) or turbulent momentum transfer.

Angular momentum budget (6) is not a first principle of continuum mechanics. Instead, it was derived from the momentum equation (Newton's second law) with the help of a physically-grounded assumption about the production of angular momentum (zero local production/destruction rate of angular momentum). It must be stressed, however, that (5) is not universally valid: if there are body moments, the stress tensor cannot be assumed to be identically symmetric (e.g. Gasparoux et Prost 1971, Tseng et al. 1972, Lin and Segel 1974[1], Straub and Lauster 1994)[2].

Budget of the angular momentum with respect to the centre of mass

The momentum budget is often studied in a moving reference frame having its origin at the centre of mass of the material control volume under consideration. To derive the relevant equation, the continuity equation

[1] Section 14.3: *Balance of Angular Momentum*
[2] H.M. Schuttelaars drew my attention to some of these references.

$$\frac{\partial\rho}{\partial t} + \nabla \bullet (\mathbf{v}\rho) = 0 \qquad (7)$$

need be introduced. Then, by definition of the concept of a material volume, the mass of fluid present in Ω,

$$m = \int_\Omega \rho(t,\mathbf{x})\, d\Omega \ , \qquad (8)$$

is constant. This is easily seen by integrating (7) over the control volume and using Reynolds' transport theorem and the divergence theorem. The position of the centre of mass of the control volume is

$$\mathbf{r}_c(t) = \frac{1}{m}\int_\Omega \rho(t,\mathbf{x})\mathbf{x}\, d\Omega \ . \qquad (9)$$

The velocity of this point is

$$\mathbf{v}_c(t) = \frac{d}{dt}\mathbf{r}_c(t) \ . \qquad (10)$$

Then, combining (7), (9)-(10), and using Reynolds' transport theorem, the following manipulations may be performed:

$$\begin{aligned}
\mathbf{v}_c &= \frac{d}{dt}\left[\frac{1}{m}\int_\Omega \rho\,\mathbf{x}\, d\Omega\right] = \frac{1}{m}\frac{d}{dt}\int_\Omega \rho\,\mathbf{x}\, d\Omega \\
&= \frac{1}{m}\int_\Omega \frac{\partial}{\partial t}(\rho\mathbf{x})\, d\Omega + \frac{1}{m}\int_\Gamma (\rho\mathbf{x})\mathbf{v}\bullet\mathbf{n}\, d\Omega = \frac{1}{m}\int_\Omega\left[\frac{\partial}{\partial t}(\rho\mathbf{x}) + \nabla\bullet(\rho\mathbf{x}\mathbf{v})\right] d\Omega \\
&= \frac{1}{m}\int_\Omega \underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{v})\right]}_{=0,\ \text{see (7)}}\mathbf{x}\, d\Omega + \frac{1}{m}\int_\Omega \rho\underbrace{\frac{\partial\mathbf{x}}{\partial t}}_{=0} d\Omega + \frac{1}{m}\int_\Omega \underbrace{(\nabla\mathbf{x})}_{=\mathbf{I}}\bullet(\rho\mathbf{v})\, d\Omega \\
&= \frac{1}{m}\int_\Omega \mathbf{I}\bullet(\rho\mathbf{v})\, d\Omega = \frac{1}{m}\int_\Omega \rho\mathbf{v}\, d\Omega
\end{aligned} \qquad (11)$$

where $\mathbf{I}$ is the identity tensor, i.e. $\mathbf{I} = diag(1,1,1)$. This leads to

$$\mathbf{v}_c = \frac{1}{m}\int_\Omega \rho\mathbf{v}\, d\Omega \ , \qquad (12)$$

which means that the velocity of the centre of mass of the control volume is the density-weighted average of the velocity of the fluid parcels contained in this volume. This is far from unexpected.

Next, the acceleration of the centre of mass,

$$\mathbf{a}_c(t) \equiv \frac{d}{dt}\mathbf{v}_c(t) \ , \qquad (13)$$

obeys

$$\begin{aligned}
\mathbf{a}_c(t) &= \frac{d}{dt}\left[\frac{1}{m}\int_\Omega \rho\mathbf{v}\, d\Omega\right] = \frac{1}{m}\frac{d}{dt}\int_\Omega \rho\mathbf{v}\, d\Omega \\
&= \frac{1}{m}\int_\Omega \frac{\partial}{\partial t}(\rho\mathbf{v})\, d\Omega + \frac{1}{m}\int_\Gamma (\rho\mathbf{v})\mathbf{v}\bullet\mathbf{n}\, d\Gamma = \frac{1}{m}\int_\Omega \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\bullet(\rho\mathbf{v}\mathbf{v})\right] d\Omega
\end{aligned}$$

$$= \frac{1}{m}\int_\Omega \underbrace{\left[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\mathbf{v})\right]}_{=0,\text{ see (7)}}\mathbf{v}\,d\Omega + \frac{1}{m}\int_\Omega \rho\underbrace{\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\bullet\nabla\mathbf{v}\right]}_{=D_t\mathbf{v}}d\Omega = \frac{1}{m}\int_\Omega \rho\, D_t\mathbf{v}\,d\Omega \qquad (14)$$

where $D_t = \partial/\partial t + \mathbf{v}\bullet\nabla$ denotes the material derivative operator. Combining this expression with (1) and using the divergence theorem yields

$$m\mathbf{a}_c = \underbrace{\int_\Omega \rho\mathbf{f}\,d\Omega}_{\substack{\text{resultant of}\\ \text{body forces}}} + \underbrace{\int_\Gamma \mathbf{T}\bullet\mathbf{n}\,d\Gamma}_{\substack{\text{resultant of}\\ \text{surface forces}}} \qquad (15)$$

Thus, the dynamics of the centre of mass is that of a particle whose mass is that of the fluid in the control volume on which the resultant of all forces is applied. This is a well-known result of classical mechanics.

Since

$$\frac{d}{dt}[\mathbf{r}_c\times(m\mathbf{v}_c)] = \underbrace{\frac{d\mathbf{r}_c}{dt}}_{=\mathbf{v}_c}\times(m\mathbf{v}_c) + \mathbf{r}_c\times\underbrace{\left(m\frac{d\mathbf{v}_c}{dt}\right)}_{=m\mathbf{a}_c} = \underbrace{\mathbf{v}_c\times(m\mathbf{v}_c)}_{=0} + \mathbf{r}_c\times(m\mathbf{a}_c) \qquad (16)$$

the angular momentum budget of the centre of mass of the control volume reads

$$\frac{d}{dt}[\mathbf{r}_c\times(m\mathbf{v}_c)] = \mathbf{r}_c\times(m\mathbf{a}_c) = \mathbf{r}_c\times\int_\Omega \rho\mathbf{f}\,d\Omega + \mathbf{r}_c\times\int_\Gamma \mathbf{T}\bullet\mathbf{n}\,d\Gamma \ . \qquad (17)$$

This is a straightforward consequence of momentum budget (15) and is not an additional first principle of Newtonian mechanics.

It is now convenient to introduce the vectors $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{r}_c$ and $\hat{\mathbf{v}} = \mathbf{v} - \mathbf{v}_c$. Their density-weighted average over the control volume is readily seen to be zero,

$$\int_\Omega \rho\hat{\mathbf{x}}\,d\Omega = 0\ , \quad \int_\Omega \rho\hat{\mathbf{v}}\,d\Omega = 0 \qquad (18)$$

leading to

$$\int_\Omega \mathbf{x}\times(\rho\mathbf{v})\,d\Omega = \mathbf{r}_c\times(m\mathbf{v}_c) + \int_\Omega \hat{\mathbf{x}}\times(\rho\hat{\mathbf{v}})\,d\Omega \qquad (19)$$

Then, subtracting (17) from (6), the angular momentum budget evaluated in a reference frame whose origin is at the centre of mass of the control volume is obtained, i.e.

$$\frac{d}{dt}\int_\Omega \hat{\mathbf{x}}\times(\rho\hat{\mathbf{v}})\,d\Omega = \int_\Omega \hat{\mathbf{x}}\times(\rho\mathbf{f})\,d\Omega + \int_\Gamma \hat{\mathbf{x}}\times(\mathbf{T}\bullet\mathbf{n})\,d\Gamma \qquad (20)$$

Since this expression of the angular momentum budget focuses on the rotation of the control volume around its centre of mass, it is likely to prove easier to handle and more useful for practical purposes than relation (6).

Appendix

Let Ω represent the domain of interest. The latter is delimited by surface Γ, whose outward unit normal is denoted $\mathbf{n}$. The following relation

$$\int_{\Omega} \mathbf{x}\times(\nabla\bullet\mathbf{A})\,d\Omega \;=\; \int_{\Gamma} \mathbf{x}\times(\mathbf{A}\bullet\mathbf{n})\,d\Gamma - \int_{\Omega} [(A_{zy}-A_{yz})\mathbf{e}_x + (A_{xz}-A_{zx})\mathbf{e}_y + (A_{yx}-A_{xy})\mathbf{e}_z]\,d\Omega \tag{A.1}$$

holds valid for any tensor $\mathbf{A}(t,\mathbf{x})$ (E. J.M. Delhez, personal communication, 2012).

First, it is convenient to introduce the symbol of Kronecker

$$\delta_{ij} \;=\; \begin{cases} 1\,, & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

as well as that of Levi-Civita

$$\varepsilon_{ijk} \;=\; \begin{cases} 1\,, & \text{if } (i,j,k)=(1,2,3) \text{ or } (3,1,2) \text{ or } (2,3,1) \\ -1\,, & \text{if } (i,j,k)=(1,3,2) \text{ or } (3,2,1) \text{ or } (2,1,3) \\ 0 & \text{otherwise} \end{cases} \tag{A.3}$$

Next, using the index notation and Einstein's summation convention, the left- and right-hand sides of (A.1) may be rewritten as follows

$$\int_{\Omega} \mathbf{x}\times(\nabla\bullet\mathbf{A})\,d\Omega \;=\; \int_{\Omega} \varepsilon_{ijk}\,x_j\,\partial_l A_{kl}\,d\Omega \tag{A.4}$$

and

$$\int_{\Gamma} \mathbf{x}\times(\mathbf{A}\bullet\mathbf{n})\,d\Gamma - \int_{\Omega} [(A_{zy}-A_{yz})\mathbf{e}_x + (A_{xz}-A_{zx})\mathbf{e}_y + (A_{yx}-A_{xy})\mathbf{e}_z]\,d\Omega = \int_{\Gamma} \varepsilon_{ijk}\,x_j\,A_{kl}\,n_l\,d\Gamma - \int_{\Omega} \varepsilon_{ijk}\,A_{kj}\,d\Omega \tag{A.5}$$

where n_l denotes (in the index notation) the outward unit normal vector to the boundary of the domain of interest. Clearly, demonstrating that (A.1) holds valid amounts to proving that

$$\int_{\Omega} \varepsilon_{ijk}\,x_j\,\partial_l A_{kl}\,d\Omega \;=\; \int_{\Gamma} \varepsilon_{ijk}\,x_j\,A_{kl}\,n_l\,d\Gamma - \int_{\Omega} \varepsilon_{ijk}\,A_{kj}\,d\Omega \tag{A.6}$$

is correct.

The integrand appearing in the left-hand side of (A.6) may be manipulated as follows:

$$\varepsilon_{ijk}\,x_j\,\partial_l A_{kl} \;=\; \partial_l(\varepsilon_{ijk}\,x_j\,A_{kl}) - \varepsilon_{ijk}\overbrace{\partial_l x_j}^{=\delta_{lj}}A_{kl} \;=\; \partial_l(\varepsilon_{ijk}\,x_j\,A_{kl}) - \varepsilon_{ijk}A_{kj} \tag{A.7}$$

Then, using (A.7) and the divergence theorem, the left-hand side of (A.6) transforms to

$$\int_{\Omega} \varepsilon_{ijk}\,x_j\,\partial_l A_{kl}\,d\Omega \;=\; \int_{\Omega} [\partial_l(\varepsilon_{ijk}\,x_j\,A_{kl}) - \varepsilon_{ijk}\,A_{kj}]\,d\Omega \;=\; \int_{\Gamma} \varepsilon_{ijk}\,x_j\,A_{kl}\,n_l\,d\Gamma - \int_{\Omega} \varepsilon_{ijk}\,A_{kj}\,d\Omega \tag{A.8}$$

Obviously, this relation is equivalent to (A.6). QED.

References

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