From Kurzweil-Henstock integration to

Charges in Euclidean spaces

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Foreword

Before giving a brief survey of the results, we expose, quite informally, the general plan of the dissertation.

A natural context for working with objects of dimension $0 \leqslant m \leqslant n$ in the Euclidean space $\mathbb{R}^n$ is that of *currents*.

Inspired by the natural duality between manifolds and differential forms, an m-dimensional current is simply a linear form acting on compactly supported smooth m-differential forms in $\mathbb{R}^n$, continuous for a topology on the space of compactly supported smooth m-differential forms in $\mathbb{R}^n$, recalling Laurent Schwartz [Sch66] distributions. We get a boundary operator acting on m-currents as dual to the exterior differentiation of m-differential forms: $\langle \partial T, \omega \rangle = \langle T, d\omega \rangle$. The *mass* of any m-dimensional current T is a (possibly infinite) number $\mathbf{M}(T)$ coinciding with the Hausdorff m-dimensional measure of M whenever T_M is the current naturally associated to the m-dimensional compact, smooth, orientable submanifold M of $\mathbb{R}^n$ by integrating on M the action of ω on the tangent vectorfield $\vec{T}_M$ to M: $\langle T_M, \omega \rangle = \int_M \langle \vec{T}_M, \omega \rangle \, d\mathscr{H}^m$. According to Stokes' theorem, we get $\partial T_M = T_{\partial M}$ for each smooth, compact, orientable m-dimensional submanifdold M of $\mathbb{R}^n$.

We also define the *flat norm* of T as the greater lower bound for $\mathbf{M}(R) + \mathbf{M}(S)$ whenever $R + \partial S$ is a decomposition of T.

Among m-dimensional currents, we distinguish the so called m-dimensional *normal* currents, having compact support and such that $\mathbf{M}(T)$ and $\mathbf{M}(\partial T)$ are finite. We denote by $\mathbf{N}_m(\mathbb{R}^n)$ the space of all such currents. One interest of working with normal currents is the compactness of the set

$$\mathbf{A}(K,c) = \{T \in \mathbf{N}_m(\mathbb{R}^n) : T \text{ is supported in } K, \mathbf{M}(T) + \mathbf{M}(\partial T) \leqslant c\}$$

under the flat norm[1], whenever $K \subseteq \mathbb{R}^n$ is compact and $c > 0$ is a real number.

In particular (see [Pfe01, Proposition 1.2.2]) the strongest topology $\mathfrak{T}$ for which all inclusions

$$(\mathbf{A}(K,c), \mathbf{F}) \hookrightarrow (\mathbf{N}_m(\mathbb{R}^n), \mathfrak{T})$$

are continuous, turns $\mathbf{N}_m(\mathbb{R}^n)$ into a locally convex, sequential topological vector space. Any element of the (topological) dual

$$\mathbf{CH}_m(\mathbb{R}^n) = (\mathbf{N}_m(\mathbb{R}^n), \mathfrak{T})^*$$

is an *m-charge* in $\mathbb{R}^n$.

An equivalent condition for a linear functional $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ to be an m-charge is the following: for each $\varepsilon > 0$ and each compact set $K \subseteq \mathbb{R}^n$, there exists $\theta > 0$ such that

$$F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$$

holds for each $T \in \mathbf{N}_m(\mathbb{R}^n)$ supported in K.

Typical examples of m-charges are the linear functionals $T \mapsto \langle T, \omega \rangle$ and $T \mapsto \langle \partial T, \zeta \rangle$ where ω and ζ are continuous differential forms of order m and $m-1$ respectively, the duality between normal currents and *continuous* forms being defined using a straightforward regularization process.

To each couple (ω, ζ) whose first component is a continuous m-differential form and whose second component is a continuous $(m-1)$-differential form,

[1] Observe H. Federer localizes the flat norm. Given an m-current T and a compact set $K \subseteq \mathbb{R}^n$ the flat seminorm $\mathbf{F}_K(T)$ is the greatest lower bound for $\mathbf{M}(T - \partial S) + \mathbf{M}(S)$ where S ranges over the class of $(m+1)$-currents *supported in* K. In particular the $\mathbf{F}$-compactness of $\mathbf{A}(K,c)$ follows *not* from the classical compactness theorem [Fed69, Section 4.2.17].

we thus associate an m-charge $\Theta(\omega, \zeta)$ defined by

$$\Theta(\omega, \zeta)(T) = \langle T, \omega \rangle + \langle \partial T, \zeta \rangle.$$

Using the Banach-Grothendieck theorem we show that Θ has a dense image in the space $\mathbf{CH}_m(\mathbb{R}^n)$. Indeed, we show that the mapping $\Phi : \mathbf{CH}_m(\mathbb{R}^n)^* \to \mathbf{N}_m(\mathbb{R}^n)$ defined by

$$\langle \Phi(\alpha), \omega \rangle = \alpha(\Theta(\omega, 0))$$

is a linear bijection. In particular each $\alpha \in \mathbf{CH}_m(\mathbb{R}^n)^*$ vanishing on the space $\operatorname{im}\Theta(\cdot, 0)$ is zero, and $\operatorname{im}\Theta$ is dense in $\mathbf{CH}_m(\mathbb{R}^n)$. On the other hand, an application of the closed range theorem guarantees that Θ has a closed range in $\mathbf{CH}_m(\mathbb{R}^n)$. Consequently each m-charge in $\mathbb{R}^n$ is of the form

$$T \mapsto \langle T, \omega \rangle + \langle \partial T, \zeta \rangle$$

for some continuous m- and $(m-1)$-forms ω and ζ respectively.

In case F is an m-charge vanishing on boundaries, i.e. satisfying $F(\partial T) = 0$ for each $T \in \mathbf{N}_m(\mathbb{R}^n)$, we obtain that F is of the form

$$T \mapsto \langle \partial T, \zeta \rangle$$

for some continuous $(m-1)$-form ζ.

In [Whi57], H. Whitney introduces *m-flat cochains*. For our purposes, an m-flat cochain in $\mathbb{R}^n$ is a linear functional $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ such that there exists $c > 0$ for which $F(T) \leqslant c\mathbf{F}(T)$ holds for each $T \in \mathbf{N}_m(\mathbb{R}^n)$. In particular, any m-flat cochain F in $\mathbb{R}^n$ is an m-charge in $\mathbb{R}^n$. Wolfe's theorem [Whi57, Theorem 7C] states that there exists a *flat m-form* ω (merely an m-differential form having essentially bounded coefficients and an essentially bounded exterior derivative, which need not to be continuous) for which

$$F(T_M) = \int_M \langle \vec{T}_M(x), \omega(x) \rangle \, d\mathscr{H}^m(x)$$

holds for each m-dimensional, compact, smooth, orientable submanifold M of $\mathbb{R}^n$. The aforementioned representation theorem for m-charges hence furnishes a new representation theorem for m-flat cochains by a pair of *continuous* forms.

Following J. Kurzweil [Kur57] and R. Henstock [Hen61] which provided a constructive generalized Riemann integral integrating each derivative, J. Mawhin (see [Maw81]) (working with multidimensional intervals and using a different ratio) and W.F. Pfeffer [Pfe91] showed the necessity of controlling uniformly from below the isoperimetric ratios

$$\frac{\mathscr{H}^n(B^j)}{\mathscr{H}^{n-1}(\partial B^j)\,\mathrm{diam}(\{x^j\}\cup B^j)}, \quad 1 \leqslant j \leqslant m$$

of the sets $B^1, B^2, \ldots, B^m$ used to partition a smooth volume B so that a limiting process on the Riemann sums

$$\sum_{j=1}^{m} \mathrm{div}\, v(x^j)\, \mathscr{H}^n(B^j)$$

yields the attended value $\int_{\partial B} v \cdot \nu_{\mathrm{ext}}\, d\mathscr{H}^{n-1}$ (the flux of v through ∂B) in case $v : \mathbb{R}^n \to \mathbb{R}^n$ is a *differentiable* vectorfield.

Following W.F. Pfeffer, define the *regularity* of an m-dimensional normal current T at $x \in \mathbb{R}^n$ as the isoperimetric ratio

$$\mathrm{reg}(T, x) = \frac{\mathbf{M}(T)}{\mathbf{M}(\partial T)\,\mathrm{diam}(\mathrm{supp}\, T \cup \{x\})}.$$

We can define the derivative $DF(x, \xi)$ of an m-charge F at $x \in \mathbb{R}^n$ in the m-direction ξ as the limit (in case it exists)

$$\lim_k \frac{F(T_k)}{\mathbf{M}(T_k)}$$

where $T_0, T_1, T_2, \ldots$ is any sequence of currents "tending to x along ξ" and satisfying $\inf_k \mathrm{reg}(T_k, x) > 0$.

We could hope that given an m-charge F, the derivative $\xi \mapsto DF(x, \xi)$ defines at each $x \in \mathbb{R}^n$ a m-differential form ω_F for which

$$F(T) = \langle T, \omega_F \rangle$$

holds for each m-dimensional normal current T; unfortunately charges are in general not sufficiently regular for the previous relation to hold.

Yet, following N. Luzin [Lus12], we showed with W.F. Pfeffer [MP08] that given $1 \leqslant m \leqslant n$, any measurable m-differential form ω in $\mathbb{R}^n$ is almost everywhere the derivative of an m-charge F: for almost every $x \in \mathbb{R}^n$ and each m-direction ξ

$$\langle \xi, \omega(x) \rangle = DF(x, \xi).$$

A brief survey

Let $\mathscr{D}(\mathbb{R}^n)$ denote the space of all infinitely differentiable functions on $\mathbb{R}^n$ vanishing outside a compact set and call $C(\mathbb{R}^n, \mathbb{R}^n)$ the space of all continuous vectorfields in $\mathbb{R}^n$.

A *fluxing charge* is a linear functional $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ for which there exists $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\mathscr{F}(\varphi) = -\int_{\mathbb{R}} v \cdot \nabla\varphi \tag{1}$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

In particular, the formula $\mathscr{F}_v(\varphi) = \int_{\mathbb{R}^n} \varphi \operatorname{div} v$ defines a fluxing charge for each $v \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

A *strong charge* (see [DP04]) is a linear map $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ such that to each $\varepsilon > 0$ and each compact set $K \subseteq \mathbb{R}^n$ we can associate $\theta > 0$ for which

$$\mathscr{F}(\varphi) \leqslant \theta \int_{\mathbb{R}^n} |\varphi| + \varepsilon \int_{\mathbb{R}^n} |\nabla\varphi|$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\varphi(x) = 0$ for $x \in \mathbb{R}^n \setminus K$. Each fluxing charge is a strong charge, as we can show easily.

The intermediate dimensional case

Fix integers $0 \leqslant m \leqslant n$. Call $\mathscr{C}^m(\mathbb{R}^n)$ the set of all continuous m-differential forms in $\mathbb{R}^n$ and let $\mathscr{C}^{-1}(\mathbb{R}^n) = \{0\}$ by convention.

An m-dimensional *normal current* in $\mathbb{R}^n$ is the data of two Radon measures with compact support $T : \mathscr{C}^m(\mathbb{R}^n) \to \mathbb{R}$ and $\partial T : \mathscr{C}^{m-1}(\mathbb{R}^n) \to \mathbb{R}$, the second one vanishing in case $m = 0$, or satisfying

$$\langle T, d\omega \rangle = \langle \partial T, \omega \rangle$$

for each infinitely differentiable $(m-1)$-differential form ω having compact support, in case $m \geqslant 1$. In particular, each normal current is determined by the first Radon measure T and we will use it to specify any normal current.

Let $\mathbf{N}_m(\mathbb{R}^n)$ be the set of all m-dimensional normal currents in $\mathbb{R}^n$ and define the *normal mass* of $T \in \mathbf{N}_m(\mathbb{R}^n)$ as the sum of the variations of T and ∂T: $\mathbf{N}(T) = \|T\|(\mathbb{R}^n) + \|\partial T\|(\mathbb{R}^n)$. The *flat norm* of $T \in \mathbf{N}_m(\mathbb{R}^n)$ is the number

$$\mathbf{F}(T) = \inf\{\|T - \partial S\|(\mathbb{R}^n) + \|S\|(\mathbb{R}^n) : S \in \mathbf{N}_{m+1}(\mathbb{R}^n)\}$$

in case $m \leqslant n-1$ and $\mathbf{F}(T) = \|T\|(\mathbb{R}^n)$ in case $m = n$.

An m-fluxing charge in $\mathbb{R}^n$ is a linear map $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ such that there exists $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ for which $F(T) = \langle \partial T, \zeta \rangle$ holds for each $T \in \mathbf{N}_m(\mathbb{R}^n)$.

Given any m-fluxing charge $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$, choose $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ associated to F in such a way that $F(T) = \langle \partial T, \zeta \rangle$ holds for each $T \in \mathbf{N}_m(\mathbb{R}^n)$, fix $\varepsilon > 0$, a compact set $K \subseteq \mathbb{R}^n$ and ψ an infinitely differentiable $(m-1)$-differentiable form with compact support in $\mathbb{R}^n$ satisfying $|\zeta(x) - \psi(x)| \leqslant \varepsilon$ for each $x \in K$. In case $m \leqslant n-1$ compute for any

$S \in \mathbf{N}_{m+1}(\mathbb{R}^n)$

$$\begin{aligned}
\langle \partial T, \zeta \rangle &= \langle \partial T, \zeta - \psi \rangle + \langle \partial (T - \partial S), \psi \rangle, \\
&= \langle \partial T, \zeta - \psi \rangle + \langle T - \partial S, d\psi \rangle, \\
&\leqslant \varepsilon \|\partial T\|(\mathbb{R}^n) + \sup_{\mathbb{R}^n} |d\psi(x)| \|T - \partial S\|(\mathbb{R}^n), \\
&\leqslant \varepsilon \|\partial T\|(\mathbb{R}^n) + \sup_{\mathbb{R}^n} |d\psi(x)| [\|T - \partial S\|(\mathbb{R}^n) + \|S\|(\mathbb{R}^n)].
\end{aligned}$$

In particular we get

$$F(T) \leqslant \sup_{\mathbb{R}^n} |d\psi(x)| \mathbf{F}(T) + \varepsilon \mathbf{N}(T), \tag{2}$$

this inequality being also verified in case $m = n$.

An m-charge in $\mathbb{R}^n$ is a linear map $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ such that to each $\varepsilon > 0$ and each compact set $K \subseteq \mathbb{R}^n$ corresponds a $\theta > 0$ for which

$$F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$$

holds whenever $T \in \mathbf{N}_m(\mathbb{R}^n)$ is supported in K. In particular (2) guarantees each m-fluxing charge is an m-charge.

Remark. An n-differential form in $\mathbb{R}^n$ corresponds bijectively to a function $u : \mathbb{R}^n \to \mathbb{R}$, whereas to any $(n-1)$-differential form in $\mathbb{R}^n$ corresponds bijectively a vectorfield $v : \mathbb{R}^n \to \mathbb{R}^n$. Using these identifications, each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ defines a normal current $T_\varphi \in \mathbf{N}_n(\mathbb{R}^n)$ by the formula

$$\langle T_\varphi, u \rangle = \int_{\mathbb{R}^n} u\varphi,$$

for $u \in C(\mathbb{R}^n)$ while its boundary ∂T_φ acts on any continuous vectorfield $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ by the formula

$$\langle \partial T_\varphi, v \rangle = -\int_{\mathbb{R}^n} \tilde{v} \cdot \nabla \varphi,$$

if we let $\tilde{v}_i = (-1)^{i-1} v_i$, $1 \leqslant i \leqslant n$. In particular, we get $\mathbf{F}(T_\varphi) = \mathbf{M}(T_\varphi) = \int_{\mathbb{R}^n} |\varphi|$ and

$$\int_{\mathbb{R}^n} |\nabla \varphi| \leqslant \mathbf{N}(T_\varphi) \leqslant \sqrt{n} \int_{\mathbb{R}^n} |\nabla \varphi|.$$

Given an n-charge $F : \mathbf{N}_n(\mathbb{R}^n) \to \mathbb{R}$ we define a linear functional $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ by the formula $\mathscr{F}(\varphi) = F(T_\varphi)$. In particular we get for each $\varepsilon > 0$ and each compact set $K \subseteq \mathbb{R}^n$, a real number $\theta > 0$ for which

$$\mathscr{F}(\varphi) \leqslant \theta \mathbf{F}(T_\varphi) + \varepsilon \mathbf{N}(T_\varphi)$$

holds whenever $\varphi \in \mathscr{D}(\mathbb{R}^n)$ is supported in K. Consequently $\mathscr{F}$ is a strong charge, using the above estimations for $\mathbf{F}(T_\varphi)$ and $\mathbf{N}(T_\varphi)$. Conversely, one can show that each strong charge induces an n-charge in $\mathbb{R}^n$.

On the other hand, given an n-fluxing charge $F : \mathbf{N}_n(\mathbb{R}^n) \to \mathbb{R}$, there exists a continuous vectorfield $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ for which

$$\mathscr{F}(\varphi) = F(T_\varphi) = \langle \partial T_\varphi, v \rangle = -\int_{\mathbb{R}^n} \tilde{v} \cdot \nabla \varphi$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ where $\tilde{v}$ is defined as above. Hence $\mathscr{F}$ is a fluxing charge. Conversely, one also shows that each fluxing charge induces an n-fluxing charge in $\mathbb{R}^n$.

A representation theorem for m-charges

The objective of our work was to find conditions under which an m-charge in $\mathbb{R}^n$ is a m-fluxing charge.

In fact, we first obtain (Theorem 5.15) a representation theorem for m-charges: given any m-charge $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ we can find $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ and $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ for which

$$F(T) = \langle T, \omega \rangle + \langle \partial T, \zeta \rangle$$

holds for every $T \in \mathbf{N}_m(\mathbb{R}^n)$.

Unfortunately, we think there is no hope to obtain an (even weaker) representation theorem using derivating strategies, as we explain in Chapter 6.

Instead, we use functional analysis tools to prove the map associating to $(\omega, \zeta) \in \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)$ the m-charge F in $\mathbb{R}^n$ defined by $F(T) = \langle T, \omega \rangle + \langle \partial T, \zeta \rangle$ is onto. For that purpose we use a closed range theorem.

In case $m \geqslant 1$, we can define the exterior derivative of a m-charge F by the formula $dF(T) = F(\partial T)$. One readily verifies that dF is a $(m+1)$-charge.

Using a construction based on the homotopy formula for currents (or using Theorem 5.15 and a regularizing argument), we show that given $m \geqslant 1$ and an m-charge F in $\mathbb{R}^n$ for which $dF = 0$, there exists a $(m-1)$-charge G in $\mathbb{R}^n$ satisfying $dG = F$ (see Theorem 3.13 or Lemma 5.18 for that Poincaré-like lemma).

Combining the representation theorem for m-charges to the Poincaré-like lemma yields the desired result: an m-charge F in $\mathbb{R}^n$ is an m-fluxing charge in $\mathbb{R}^n$ if and only if $dF = 0$. As each n-charge F in $\mathbb{R}^n$ satisfies $dF = 0$, we get that each strong charge in $\mathbb{R}^n$ is a fluxing charge in $\mathbb{R}^n$, a result previously obtained by T. De Pauw and W. F. Pfeffer in [DP04].

A Luzin's type theorem for m-charges

In [Alb91], G. Alberti shows that given a measurable vectorfield v in an open set $U \subseteq \mathbb{R}^n$ of finite Lebesgue measure, there exists an arbitrary close to U (in measure) compact set $K \subseteq U$ and a function u of class C^1 in U for which $\nabla u = v$ on K.

Luzin's theorem [Lus12] states that each measurable function on the real line is, almost everywhere, the derivative of an almost everywhere differentiable continuous function. In a joint work with W.F. Pfeffer [MP08] we start from Alberti's result and show, following the original Luzin's argument, that each measurable vectorfield on $\mathbb{R}^n$ is, almost everywhere, the gradient of an almost everywhere differentiable function.

Given $0 \leqslant m \leqslant n-1$ and an $(m+1)$-differential form

$$\omega = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n} \omega_{i_1, i_2, \ldots, i_{m+1}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{m+1}}$$

with measurable coefficients $\omega_{i_1, i_2, \ldots, i_{m+1}}$, $1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n$, we thus know that for each sequence $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ there exists a continuous function $\Omega_{i_1,i_2,\dots,i_m}$ satisfying

$$\partial_i \Omega_{i_1,i_2,\dots,i_m}(x) = \begin{cases} \omega_{i,i_1,i_2,\dots,i_m} & \text{if} \quad i < i_1, \\ 0 & \text{if} \quad i \geqslant i_1 \end{cases}$$

for each $1 \leqslant i \leqslant n$. In particular we get a continuous m-differential form

$$\Omega = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \Omega_{i_1,i_2,\dots,i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

satisfying $d\Omega = \omega$ a.e. on $\mathbb{R}^n$.

We are then able to prove that given an $(m+1)$-differential form ω with measurable coefficients, its action on any m-direction ξ is, for almost every $x \in \mathbb{R}^n$, the derivative at x in the direction of ξ of the m-charge $F_\Omega : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$F_\Omega(T) = \langle T, \Omega \rangle,$$

where Ω is constructed from ω as above; this derivative being obtained by taking limits of quotients $F_\Omega(T)/\|T\|(\mathbb{R}^n)$ whenever T "tends to x along ξ".

In particular, each measurable function on $\mathbb{R}^n$ is almost everywhere the derivative of a strong charge, and we thereby generalize the result [HP04] by E.J. Howard and W.F. Pfeffer.

Removable singularities for $\operatorname{div} v = 0$

The problem of removing singularities for the equation $\operatorname{div} v = 0$ is the following: given a class $\mathscr{B}$ of locally bounded measurable vectorfields, give conditions on the compact set $S \subseteq \mathbb{R}^n$ for the following to hold: for all $v \in \mathscr{B}$, if

$$\int_{\mathbb{R}^n} v \cdot \nabla \varphi = 0 \tag{3}$$

is satisfied for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ supported outside S, then (3) holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$. In such a case we say that S is removable for the equation $\operatorname{div} v = 0$ with respect to $\mathscr{B}$.

Fix $\mathscr{B}$ any collection of locally bounded measurable functions on $\mathbb{R}^n$ for $n \geqslant 2$. Let us illustrate the previous definition by showing that $S = \{0\}$ is removable for $\operatorname{div} v = 0$ with respect to $\mathscr{B}$.

Define for each integer $k \geqslant 1$ an infinitely differentiable function $\beta_k \in \mathscr{D}(\mathbb{R}^n)$ by the formula

$$\beta_k(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant 1/k, \\ \exp\left(1 + \frac{1}{k^2|x|^2 - 2k|x|}\right) & \text{for} \quad 1/k < |x| < 2/k, \\ 0 & \text{for} \quad |x| \geqslant 2/k. \end{cases}$$

Pointwise on $\mathbb{R}^n$, we get

$$\beta_k \to \chi_{\{0\}}, k \to \infty,$$

while a simple computation yields

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\nabla \beta_k| = 0.$$

For $v \in \mathscr{B}$, $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and an integer $k \geqslant 1$, write

$$\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^n} v \cdot \nabla(\beta_k \varphi)\, dx + \int_{\mathbb{R}^n} v \cdot \nabla[(1-\beta_k)\varphi]\, dx.$$

Assuming $\int_{\mathbb{R}^n} v \cdot \nabla\psi \, dx = 0$ for each $\psi \in \mathscr{D}(\mathbb{R}^n)$ vanishing in a neighbourhood of 0, we see the second term vanishes. On the other hand, Lebesgue's dominated convergence theorem guarantees that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} v \cdot \nabla(\beta_k\varphi)\, dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \beta_k v \cdot \nabla\varphi \, dx + \lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi v \cdot \nabla\beta_k = 0,$$

and we get $\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx = 0$. Hence $\{0\}$ is removable for the equation $\operatorname{div} v = 0$ with respect to $\mathscr{B}$.

Refining this procedure, we show that a compact set $S \subseteq \mathbb{R}^n$ whose $(n-1)$-dimensional Hausdorff measure vanishes, is removable for the equation $\operatorname{div} v = 0$ with respect to any class of locally bounded measurable functions.

Moreover, we prove that a compact set $S \subseteq \mathbb{R}^n$ is removable for the equation $\operatorname{div} v = 0$ with respect to the class $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of bounded measurable vectorfields if and only if the $(n-1)$-dimensional Hausdorff measure of S vanishes. This result appeared first in [Moo06] but the simpler proof presented in this text is new and due to M. Torres and P.C. Phuc [PT07].

We also observe with a simple regularizing argument that compact sets with zero $(n-1)$-Hausdorff dimensional measure are also characterized by the following property: for all bounded measurable vectorfields $v : \mathbb{R}^n \to \mathbb{R}^n$, *of class C^∞ outside S*, if

$$\int_{\mathbb{R}^n} v \cdot \nabla\varphi = 0$$

is satisfied for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ supported outside S, then it holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

After that, we show that each compact set $S \subseteq \mathbb{R}^n$ with finite $(n-1)$-dimensional Hausdorff measure is removable for the equation $\operatorname{div} v = 0$ with respect to the class $C(\mathbb{R}^n, \mathbb{R}^n)$ of continuous vectorfields on $\mathbb{R}^n$.

Furthermore, following the ideas of M. Torres and P.C. Phuc [PT07], we use the representation theorem for n-charges together with Frostman's lemma to prove that each compact set $S \subseteq \mathbb{R}^n$ removable for the equation $\operatorname{div} v = 0$ with respect to the class $C(\mathbb{R}^n, \mathbb{R}^n)$ of continuous vectorfields on $\mathbb{R}^n$, has Hausdorff dimension at most $n-1$.

Contents

Chapter 1

Preliminaries and notations

Euclidean space

Given (extended) real numbers $-\infty \leqslant a \leqslant b \leqslant \infty$, we give the notations (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ their usual meanings.

Given an integer $n \geqslant 1$, the Euclidean space $\mathbb{R}^n$ endowed by the usual inner product $x \cdot y$, and we let $|x| = \sqrt{x \cdot x}$. For $x \in \mathbb{R}^n$ and $r > 0$ we let $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $B[x, r] = \{y \in \mathbb{R}^n : |x - y| \leqslant r\}$ and we often write $B(r)$ and $B[r]$ instead of $B(0, r)$ and $B[0, r]$ respectively.

Sometimes, we also employ the maximum norm defined by

$$|x|_\infty = \max\{|x_i| : 1 \leqslant i \leqslant n\}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$ we let $B_\infty(x, r) = \{y \in \mathbb{R}^n : |x - y|_\infty < r\}$ and $B_\infty[x, r] = \{y \in \mathbb{R}^n : |x - y|_\infty \leqslant r\}$; while $B_\infty(r)$ and $B_\infty[r]$ will abbreviate $B_\infty(0, r)$ and $B_\infty[0, r]$ respectively.

Given $E \subseteq \mathbb{R}^n$, we let $\bar{E}$ and $\operatorname{cl} E$ denote the closure of E, while $\mathring{E}$ and $\operatorname{int} E$ stand for the interior of E. One also let $\partial E = \bar{E} \setminus \mathring{E}$. Moreover the distance from $x \in \mathbb{R}^n$ to E is the number

$$\operatorname{dist}(x, E) = \inf\{|x - y| : y \in E\}.$$

For $F \subseteq \mathbb{R}^n$ we let

$$\operatorname{dist}(F, E) = \inf\{\operatorname{dist}(y, E) : y \in F\}.$$

Given $E \subseteq \mathbb{R}^n$, we let $\mathcal{K}(E)$ denote the set of all compact subsets of E.

The indicator of $E \subseteq \mathbb{R}^n$ will be denoted by χ_E. The outer Lebesgue measure of $E \subseteq \mathbb{R}^n$ is denoted $|E|$, while we let $\mathscr{H}^m(E)$ stand for the m-dimensional Hausdorff outer measure of E ($0 \leqslant m \leqslant n$).

For $x \in \mathbb{R}^n$ and $E \subseteq \mathbb{R}^n$ we will write $E - x = \{h \in \mathbb{R}^n : x + h \in E\}$.

Regular functions

Given $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $F \subseteq E$ we denote by $\operatorname{supp} f = \operatorname{cl}\{x \in \mathbb{R}^n : f(x) \neq 0\}$ the support of f and by $f \restriction F = \{(x, f(x)) : x \in F\}$ the restriction of f to F. The Lipschitz constant of f, denoted $\operatorname{Lip}(f)$, is defined by

$$\operatorname{Lip}(f) = \inf\{L > 0 : |f(x) - f(y)| \leqslant L|x - y| \text{ for all } x, y \in E\}.$$

For $1 \leqslant i \leqslant n$ and $x \in \mathbb{R}^n$ we let $\partial_i f(x)$ denote the i-th partial derivative of f at x. In case $m = 1$ we let $\nabla f(x) = (\partial_1 f(x), \partial_2 f(x), \dots, \partial_n f(x))$. In case $m = n$ we let $\operatorname{div} f(x) = \sum_{i=1}^n \partial_i f_i(x)$.

Fix $E \subseteq \mathbb{R}^n$. For an (extended) integer $0 \leqslant k \leqslant \infty$, we let $C^k(E, \mathbb{R}^m)$ denote the space of all functions of class C^k mapping E to $\mathbb{R}^m$ while $C_c^k(E, \mathbb{R}^m)$ denotes the subspace of $C^k(E, \mathbb{R}^m)$ consisting of functions having compact support in E. We abbreviate $C(E, \mathbb{R}^m)$ and $C(E)$ for $C^0(E, \mathbb{R}^m)$ and $C^0(E,$-$\mathbb{R})$ respectively; while $C^k(E)$ and $C_c^k(E)$ stands for $C^k(E, \mathbb{R})$ and $C_c^k(E, R)$ respectively. Finally we let $\mathscr{D}(E, \mathbb{R}^m) = C_c^\infty(E, \mathbb{R}^m)$ and $\mathscr{D}(E) = C_c^\infty(E)$.

Regularization

Fix a radial function $\rho \in \mathscr{D}(\mathbb{R}^n)$ satisfying $0 \leqslant \rho \leqslant 1$, $\operatorname{supp} \rho = B[1]$ and $\int_{\mathbb{R}^n} \rho \, dx = 1$. For each integer $k \geqslant 1$ define $\rho_k \in \mathscr{D}(\mathbb{R}^n)$ by the formula $\rho_k(x) = k^n \rho(kx)$.

Given an open set $U \subseteq \mathbb{R}^n$ and a map f defined and locally integrable on U we let for each integer $k \geqslant 1$ and each $x \in \mathbb{R}^n$ satisfying $\operatorname{dist}(x, \complement U) \leqslant \frac{1}{k}$

$$\rho_k * f(x) = \int_{\mathbb{R}^n} \rho_k(y) f(x-y)\, dy.$$

Lebesgue spaces

When there is no explicit mention of a particular measure, the expressions 'measurable', 'almost everywhere', 'negligible' etc. refer to the Lebesgue measure. Moreover all integrals are Lebesgue integrals whenever there is no explicit reference to another measure.

We denote by $M(\mathbb{R}^n, \mathbb{R}^m)$ the space of all measurable functions mapping $\mathbb{R}^n$ to $\mathbb{R}^m$, in which we identify two functions equal almost everywhere.

Given $1 \leqslant p < \infty$, let

$$L^p(\mathbb{R}^n, \mathbb{R}^m) = \left\{ u \in M(\mathbb{R}^n, \mathbb{R}^m) : \int_{\mathbb{R}^n} |u|^p\, dx < \infty \right\}$$

and define a norm $|\cdot|_p$ on $L^p(\mathbb{R}^n, \mathbb{R}^m)$ by the formula $|u|_p = \left(\int_{\mathbb{R}^n} |u|^p\, dx \right)^{1/p}$.

The *regularization theorem* ensures

$$\lim_{k \to \infty} |\rho_k * u - u|_p = 0$$

for each $u \in L^p(\mathbb{R}^n, \mathbb{R}^m)$.

We also define

$$\begin{aligned} L^\infty(\mathbb{R}^n, \mathbb{R}^m) = \{ u \in M(\mathbb{R}^n, \mathbb{R}^m) : &\text{ there exists } c > 0 \text{ such that} \\ & |u(x)| \leqslant c \text{ for almost every } x \in \mathbb{R}^n \}, \end{aligned}$$

and we define a norm $|\cdot|_\infty$ on $L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ by the formula

$$|u|_\infty = \inf \{ c > 0 : |u(x)| \leqslant c \text{ for almost every } x \in \mathbb{R}^n \}.$$

For $1 \leqslant p \leqslant \infty$, we use $L^p(\mathbb{R}^n)$ to abbreviate $L^p(\mathbb{R}^n, \mathbb{R})$ and we let

$$L^p_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m) = \{ u \in M(\mathbb{R}^n, \mathbb{R}^m) : u\chi_K \in L^p(\mathbb{R}^n, \mathbb{R}^m) \text{ for } K \in \mathcal{K}(\mathbb{R}^n) \};$$

and we use the abbreviation $L^p_{\text{loc}}(\mathbb{R}^n)$ for $L^p_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$.

A generalized integration by parts formula

The following result follows (for example) from [Pfe05, Corollary 2.10].

Theorem 1.1. *Assume $S \in \mathcal{K}(\mathbb{R}^n)$ has σ-finite $(n-1)$-dimensional Hausdorff measure and $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ is bounded and differentiable outside S. If $\operatorname{div} v \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for $\varphi \in \mathscr{D}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} \varphi \operatorname{div}\, v(x)\, dx = -\int_{\mathbb{R}^n} v(x) \cdot \nabla\varphi(x)\, dx.$$

Vitali's covering theorem

We will use Vitali's covering theorem.

Theorem 1.2 (Vitali)**.** *Assume $B \subseteq \mathbb{R}^n \times (0, \infty)$ is such that*

$$\sup_{(x,r)\in B} r < \infty.$$

Then there exists a sequence $((x_k, r_k)) \subseteq B$ for which the following are satisfied:

1. $B[x_k, r_k] \cap B[x_l, r_l] = \emptyset$ *for* $k \neq l$;
2. $\displaystyle\bigcup_{(x,r)\in B} B[x, r] \subseteq \bigcup_{k\in\mathbb{N}} B[x_k, 5r_k].$

Frostman's lemma

Recall Frostman's lemma, characterizing Borel sets with positive Hausdorff measure.

Lemma 1.3 (Frostman)**.** *Let $B \subseteq \mathbb{R}^n$ be a Borel set and $0 \leqslant m \leqslant n$ be a real number. Then $\mathscr{H}^m(B) > 0$ if and only if there exists a nonzero Radon measure with compact support in B satisfying $\mu(B[x, r]) \leqslant r^m$ for $x \in \mathbb{R}^n$ and $r > 0$.*

Vector-valued Radon measures

A $\mathbb{R}^m$-valued Radon measure in $\mathbb{R}^n$ is a map

$$\mu : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^m,$$

defined on the class $\mathcal{B}(\mathbb{R}^n)$ of Borel subsets of $\mathbb{R}^n$ satisfying $\mu(\emptyset) = 0$ and $\mu\left(\bigcup_{k\in\mathbb{N}} B_k\right) = \sum_{k=0}^{\infty} \mu(B_k)$ whenever $(B_k) \subseteq \mathcal{B}(\mathbb{R}^n)$ is a sequence of pairwise disjoint Borel subsets of $\mathbb{R}^n$. In such a case the formula

$$\|\mu\|(B) \quad = \quad \sup\left\{\sum_{k=0}^{\infty} |\mu(B_k)| : (B_k) \text{ pairwise disjoint}, B = \bigcup_{k\in\mathbb{N}} B_k\right\}$$

defines a finite, nonnegative Radon measure in $\mathbb{R}^n$. We call $\|\mu\|$ the *variation* of μ.

Functions with bounded variation

Call $BV(\mathbb{R}^n)$ the space of (Lebesgue) integrable functions with bounded variation in $\mathbb{R}^n$ (see [EG92, Chapter 5]). Each function having bounded variation in $\mathbb{R}^n$ induces a $\mathbb{R}^n$-valued Radon measure Du in $\mathbb{R}^n$ for which

$$\int_{\mathbb{R}^n} u \operatorname{div} v \, dx = -\int_{\mathbb{R}^n} v \cdot d[Du]$$

holds for each $v \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n)$. We let $\|u\| = \|Du\|(\mathbb{R}^n)$.

The *regularization theorem for BV functions* ensures that given a function $u \in BV(\mathbb{R}^n)$

$$\|u\| = \lim_{k\to\infty} \|\rho_k * u\|.$$

A vectorfield $v \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ is the weak gradient of $u \in L^1(\mathbb{R}^n)$ whenever

$$\int_{\mathbb{R}^n} u\nabla\varphi \, dx = -\int_{\mathbb{R}^n} \varphi v \, dx$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n)$. In such a case we write $v = \nabla u$.

In case $u \in L^1(\mathbb{R}^n)$ has a weak gradient $\nabla u \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, we have $u \in BV(\mathbb{R}^n)$ and $\|u\| = \int_{\mathbb{R}^n} |\nabla u| dx = |\nabla u|_1$.

A set $E \subseteq \mathbb{R}^n$ has *finite perimeter* in case $\chi_E \in BV(\mathbb{R}^n)$. We then let $\|\partial E\| = \|D\chi_E\|$ and call $\|E\| = \|\chi_E\|$ the *perimeter* of E.

The *coarea formula for BV functions* ensures

$$\|u\| = \int_{\mathbb{R}} \|\{u > t\}\| \, dt$$

holds for each $u \in BV(\mathbb{R}^n)$, if we let $\{u > t\} = \{x \in \mathbb{R}^n : u(x) > t\}$. In particular the level set $\{u > t\}$ has finite perimeter for a.e. $t \in \mathbb{R}$.

The *relative isoperimetric inequality* states that there exists a constant $c = c(n)$ such that for each set $E \subseteq \mathbb{R}^n$ having finite perimeter and each $r > 0$

$$\min\{|B(x,r) \setminus E|, |B(x,r) \cap E|\} \leqslant cr\|\partial E\|[B(x,r)].$$

Newtonian potential

Define for $z \in \mathbb{R}^n \setminus \{0\}$

$$\Gamma(z) = \begin{cases} \frac{1}{n(n-2)|B(0,1)|} \frac{1}{|z|^{n-2}} & \text{if} \quad n > 2, \\ -\frac{1}{2\pi} \ln |z| & \text{if} \quad n = 2. \end{cases}$$

Given $p > n$ and a function $u \in L^p(\mathbb{R}^n)$ having compact support, the *Newtonian potential* K_u of u defined by

$$K_u(x) = \int_{\mathbb{R}^n} \Gamma(x - y) u(y) \, dy$$

has a continuous weak gradient ∇K_u satisfying

$$\int_{\mathbb{R}^n} \nabla K_u \cdot \nabla \varphi \, dx = - \int_{\mathbb{R}^n} u\varphi \, dx$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ (see [LL01, Theorems 6.21 and 10.2]).

If moreover u is of class C^∞ on $B[x,r]$ for some $x \in \mathbb{R}^n$ and $r > 0$, choose $\tilde{u} \in \mathscr{D}(\mathbb{R}^n)$ satisfying $u = \tilde{u}$ on $B[x,r]$ and observe

$$\int_{\mathbb{R}^n} \nabla(K_u - K_{\tilde{u}}) \cdot \nabla \varphi \, dx = 0$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ supported in $B(x,r)$. In particular $h = K_u - K_{\tilde{u}}$ is harmonic in $B(x,r)$ (see [LL01, Theorem 9.3]) and so $K_u = K_{\tilde{u}} + h$ is of class C^∞ on $B(x,r)$ (see [LL01, Theorem 10.3]), and we get $\operatorname{div} \nabla K_u(x) = u(x)$.

Directed systems

In a topological space $(X, \mathcal{T})$, we denote by $\mathcal{V}_{\mathcal{T}}(x), \mathcal{V}_X(x)$ or $\mathcal{V}(x)$ the filter of all neighbourhoods of x in X. Given $A \subseteq X$, we let $\operatorname{cl} A, \operatorname{int} A, \partial A$ have the usual meanings.

Let $(I, \leqslant)$ be an ordered set such that for every $i, j \in I$ there exists $k \in A$ for which $i \leqslant k$ and $j \leqslant k$, and let X be a topological space. A *directed system indexed by* I in $A \subseteq X$ is a map $x : I \to A$; we shall write $(x_i)_{i \in I} \subseteq A$. A sequence is a directed system indexed by $\mathbb{N}$. The directed system $(x_i)_{i\in I}$ is said to *converge* to $x \in X$ whenever to each $V \in \mathcal{V}(x)$ corresponds one $i \in I$ such that $x_j \in V$ holds for every $j \in I$ satisfying $j \geqslant i$. One says that $x \in X$ is a *cluster point* for the directed system $(x_i)_{i\in I}$ if and only if to every $V \in \mathcal{V}(x)$ and $i \in I$ corresponds $j \in I$ satisfying $j \geqslant i$ and for which $x_j \in V$. In a Hausdorff space, each directed system converges to at most one point. Given a sequence $(x_k) \subseteq X$ and an increasing sequence of integers (k_j), one calls (x_{k_j}) a *subsequence* of (x_k), and one writes $(x_{k_j}) \subseteq (x_k)$.

Fix $A \subseteq X$. One has $x \in \operatorname{cl} A$ if and only there exists a directed system $(x_i)_{i\in I} \subseteq A$ converging to x.

A map $f : X \to Y$ between two topological spaces is continuous if and only if for each $x \in X$, the directed system $(f(x_i))_{i\in I}$ converges to $f(x)$ whenever $(x_i)_{i\in I}$ is a directed system converging to x.

Sequential spaces

In a topological space $(X, \mathcal{T})$, a set $A \subseteq X$ is called *sequentially closed* whenever for each sequence $(x_k) \subseteq A$ converging to $x \in X$, we get $x \in A$.

A topological space X is called *sequential* whenever it is necessary and sufficient for $A \subseteq X$ to be closed that A be sequentially closed.

Given two topological spaces X, Y, a map $f : X \to Y$ is called *sequentially continuous* whenever for each $x \in X$, the sequence $(f(x_k))$ converges to $f(x)$ whenever (x_k) is a sequence converging to X. If we assume X to be sequential, for a map $f : X \to Y$ to be continuous it is necessary and sufficient that f be sequentially continuous.

Metric spaces

A *metric* on a set X is a map $d : X \times X \to [0, \infty)$ satisfying the following properties for each $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leqslant d(x, z) + d(z, y)$.

The metric d on X is called *complete* whenever to each sequence $(x_k) \subseteq X$ satisfying $\lim_{j,k\to\infty} d(x_j, x_k) = 0$ corresponds $x \in X$ for which

$$\lim_k d(x_k, x) = 0.$$

Topological vector spaces

In the sequel, X is a (real) a vector space. We let X' denote the *algebraic dual* of X.

For $A, B \subseteq X$, $x \in X$ and $\lambda \in \mathbb{R}$ we shall use the following notations:

$$\begin{aligned} x + A &= \{x + a : a \in A\}, \\ x - A &= \{x - a : a \in A\}, \\ A + B &= \{a + b : a \in A, b \in B\}, \\ \alpha A &= \{\alpha a : a \in A\}. \end{aligned}$$

A set $B \subseteq X$ is called *balanced* whenever $\alpha B \subseteq B$ holds for each $\alpha \in \mathbb{R}$ veryfing $|\alpha| \leqslant 1$.

A set $C \subseteq X$ is called *convex* whenever $tC + (1-t)C \subseteq C$ holds for each $0 \leqslant t \leqslant 1$.

Given a vector space X and a topology $\mathfrak{T}$ on X, we say that $\mathfrak{T}$ is a *vector topology* on X whenever the following conditions hold:

1. for each $x \in X$, $\{x\}$ is closed,

2. the map $X \times X \to X, (x, y) \mapsto x + y$ is continuous,

3. the map $\mathbb{R} \times X \mapsto X, (\alpha, x) \mapsto \alpha x$ is continuous.

Whenever $\mathfrak{T}$ is a vector topology on X we shall say that $(X, \mathfrak{T})$ is a *topological vector space* (or shortly, a t.v.s.).

Given a t.v.s. X, we denote by X^* the vector space whose elements are continuous linear functionals on X. We call X^* the *topological dual* of X.

A subset $B \subseteq X$ of the t.v.s. X is called *bounded* whenever for each $V \in \mathcal{V}(0)$ one can find $s > 0$ such that $B \subseteq tV$ holds for every $t > s$.

The directed system $(x_i)_{i\in I}$ is called a *Cauchy system* whenever to each $V \in \mathcal{V}(0)$ corresponds one $i \in I$ such that $x_j - x_k \in V$ hold for every $j, k \in I$ satisfying $j \geqslant i$ and $k \geqslant i$.

A t.v.s. X is said to be *complete* if each Cauchy system in X converges to some $x \in X$. A t.v.s. X is called *sequentially complete* whenever each Cauchy sequence in X converges to some $x \in X$.

The metric d on the vector space X is called invariant if $d(x, y) = d(x + z, y + z)$ holds for every $x, y \in X$.

In case a t.v.s. X has its topology induced by an invariant metric d, X is complete if and only if d is complete.

A t.v.s. X is called *locally convex* whenever there exists a local base at 0 consisting of convex sets. The abreviation l.c.t.v.s. will mean 'locally convex topological vector space' in the sequel.

A *Fréchet space* is a l.c.t.v.s. whose topology is induced by a complete invariant metric. Equivalently, a Fréchet space is a complete l.c.t.v.s. whose topology is induced by an invariant metric.

Topology induced by a sequence of seminorms

A *seminorm* on a vector space X is a function $p : X \to [0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $\alpha \in \mathbb{R}$:

1. $p(x+y) \leqslant p(x)+p(y)$,

2. $p(\alpha x)=|\alpha| p(x)$.

Remark 1.4. A *norm* on a vector space X is a seminorm $\|\cdot\|$ on X satisfying $\|x\|=0$ if and only if $x=0$. Given a norm $\|\cdot\|$ on a vector space X, we call the pair $(X,\|\cdot\|)$ a *normed space*.

A family $\mathscr{P}$ of seminorms on X is said to be *separating* if to each $x \in X \setminus\{0\}$ corresponds at least one $p \in \mathscr{P}$ for which $p(x) \neq 0$.

Theorem 1.5. *Assume $\mathscr{P}=(p_k)_{k \in \mathbb{N}}$ is a separating sequence of seminorms on the vector space X. Then the formula*

$$d(x, y)=\max_k \frac{p_k(x-y)}{2^k[1+p_k(x-y)]}$$

defines an invariant metric on X inducing a locally convex vector topology $\mathfrak{T}_{\mathscr{P}}$ on X.

We report to the next section the study of further properties of the topology $\mathfrak{T}_{\mathscr{P}}$. All the same, we illustrate the previous construction by an example.

Example 1.6 (The Fréchet space $C(\mathbb{R}^n, E)$)**.** Let $(E,\|\cdot\|)$ be a finite-dimensional normed space. Given $K \in \mathfrak{K}(\mathbb{R}^n)$, we define a seminorm ν_K^0 on the space $C(\mathbb{R}^n, E)$ of continuous functions mapping $\mathbb{R}^n$ to E by the formula

$$\nu_K^0(f)=\sup \{\|f(x)\|: x \in K\}.$$

It is clear that $\mathscr{P}=\{\nu_{B[k]}^0: k \in \mathbb{N}\}$ is a separating family of seminorms. Let d be the metric induced on $C(\mathbb{R}^n, E)$ by $\mathscr{P}$ according to (1.5). To show d is complete, fix a sequence $(f_j) \subseteq C(\mathbb{R}^n, E)$ satisfying $d(f_i, f_j) \rightarrow 0$; $i, j \rightarrow \infty$. In particular, we obtain for each $k \in \mathbb{N}$

$$\sup_{x \in B[k]}\|f_j(x)-f_i(x)\| \rightarrow 0 ; i, j \rightarrow \infty,$$

and we find $f \in C(\mathbb{R}^n, E)$ such that $f_j \rightarrow f, j \rightarrow \infty$ uniformly on each compact set $K \in \mathfrak{K}(\mathbb{R}^n)$. Fix $\varepsilon>0$ and choose $l \in \mathbb{N}$ such that $d(f_j, f_i) \leqslant \varepsilon$

holds for $j \geqslant i \geqslant l$, and fix $j \geqslant i \geqslant l$. Let k_0 denote the first nonnegative integer for which $2^{-k_0} < \varepsilon$. Compute for every $k \in \mathbb{N}, k < k_0$ and every $x \in B[k]$:

$$\|f_j(x) - f_i(x)\| \leqslant \frac{2^k \varepsilon}{1 - 2^k \varepsilon}$$

so that we get

$$\nu^0_{B[k]}(f_i - f) = \sup\{\|f_i(x) - f(x)\| : x \in B[k]\} \leqslant \frac{2^k \varepsilon}{1 - 2^k \varepsilon}$$

for $i \geqslant l$. On the other hand we get

$$\frac{\nu^0_{B[k]}(f_i - f)}{2^k[1 + \nu^0_{B[k]}(f_i - f)]} \leqslant 2^{-k} \leqslant \varepsilon$$

in case $k \geqslant k_0$. Consequently we have for $i \geqslant l$:

$$d(f_i, f) = \max_{k \in \mathbb{N}} \frac{\nu^0_{B[k]}(f_i - f)}{2^k[1 + \nu^0_{B[k]}(f_i - f)]} \leqslant \varepsilon,$$

and d is complete.

Another example will be useful in the context of *currents*.

Example 1.7 (The Fréchet spaces $C^\infty(\mathbb{R}^n, E)$ and $\mathscr{D}_K(\mathbb{R}^n, E)$)**.** Let $(E, \|\cdot\|)$ be a finite dimensional normed space. A multi-index is a n-tuple $\alpha \in \mathbb{N}^n$, and we shall write $|\alpha| = \sum_{i=1}^n |\alpha_i|$. To each multi-index α is associated a differential operator of order $|\alpha|$

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$$

defined on $C^\infty(\mathbb{R}^n, E)$ (with the convention that ∂_i^0 is the identity operator on $C(\mathbb{R}^n, E)$). For $K \in \boldsymbol{\mathcal{K}}(\mathbb{R}^n)$, we let

$$\mathscr{D}_K(\mathbb{R}^n, E) = \{f \in C^\infty(\mathbb{R}^n, E) : \operatorname{supp} f \subseteq K\}.$$

For each $k \in \mathbb{N}$ and each compact set $K \in \boldsymbol{\mathcal{K}}(\mathbb{R}^n)$, define a seminorm ν^k_K on $C^\infty(\mathbb{R}^n, E)$ by the formula

$$\nu^k_K(f) = \sup\{\|\partial^\alpha f(x)\| : x \in K, \alpha \in \mathbb{N}^n, |\alpha| \leqslant k\}.$$

Let d be the metric associated to $\mathscr{P} = (\nu^k_{B[k]})_{k\in\mathbb{N}}$ according to Theorem 1.5 and assume $(f_j) \subseteq C^\infty(\mathbb{R}^n, E)$ is such that $d(f_j, f_i) \to 0$; $i, j \to \infty$. For each $\alpha \in \mathbb{N}^n$ and each $k \in \mathbb{N}$, we obtain

$$\sup_{B[k]} \|\partial^\alpha f_j - \partial^\alpha f_i\| \to 0; i, j \to \infty,$$

and consequently for each $\alpha \in \mathbb{N}^n$ we get a function $g_\alpha \in C(\mathbb{R}^n, E)$ such that $\partial^\alpha f_j \to g_\alpha, j \to \infty$ uniformly on each compact set $K \in \mathfrak{K}(\mathbb{R}^n)$. In particular $g_\alpha = \partial^\alpha g_0$ holds for each $\alpha \in \mathbb{N}^n$, so that $g_0 \in C^\infty(\mathbb{R}^n, E)$. We conclude that $C^\infty(\mathbb{R}^n, E)$ is a Fréchet space by showing $d(f_i, g_0) \to 0$;$i \to \infty$ as in Example 1.6.

As $\mathscr{D}_K(\mathbb{R}^n, E)$ is a closed subspace of $C^\infty(\mathbb{R}^n, E)$, it is a Fréchet space.

Topology induced by a family of seminorms

We can generalize the construction of the previous section to an arbitrary separating family of seminorms.

Theorem 1.8. *Let $\mathscr{P}$ be a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and each $k \in \mathbb{N}$ the set*

$$V(p, k) = \left\{x \in X : p(x) < 2^{-k}\right\},$$

and let $\mathfrak{B}$ be the collection of all finite intersections $\bigcap_{i=1}^m V(p_i, k_i)$ where $p_i \in \mathscr{P}$, $k_i \in \mathbb{N}$, $1 \leqslant i \leqslant m$. Then $\mathfrak{B}$ is a convex balanced local base for a topology $\mathfrak{T}_{\mathscr{P}}$ on X, which turns X into a locally convex space for which:

1. *every $p \in \mathscr{P}$ is continuous,*
2. *a set $B \subseteq X$ is bounded if and only if each $p \in \mathscr{P}$ is bounded on B,*
3. *a directed system $(x_i)_{i\in I}$ converges to $x \in X$ for the toplogy $\mathfrak{T}_{\mathscr{P}}$ if and only if the directed system of real numbers $(p(x_i - x))_{i\in I}$ converges to 0 for each $p \in \mathscr{P}$,*

4. *for a seminorm q on X to be $\mathfrak{T}_{\mathscr{P}}$-continuous on X, it is necessary and sufficient that there exists $p_j \in \mathscr{P}$, $1 \leqslant j \leqslant m$ and $C > 0$ such that $q(x) \leqslant C \max_{1 \leqslant j \leqslant m} p_j(x)$ be satisfied for each $x \in X$,*

5. *for linear map $u \in X'$ to be $\mathfrak{T}_{\mathscr{P}}$-continuous on X, it is necessary and sufficient that there exists $p_j \in \mathscr{P}$, $1 \leqslant j \leqslant m$ and $C > 0$ such that $|\langle u, x\rangle| \leqslant C \max_{1 \leqslant j \leqslant m} p_j(x)$ be satisfied for each $x \in X$.*

Moreover, the topology $\mathfrak{T}$ is induced by the metric d defined in Theorem 1.5 if furthermore $\mathscr{P}$ is countable.

The following example will be of importance in the context of currents.

Example 1.9. Let $(E, \|\cdot\|)$ be a finite dimensional normed space and let $\mathscr{D}(\mathbb{R}^n, E)$ be the vector space of all infinitely differentiable functions $f : \mathbb{R}^n \to E$ having compact support. Denote $\mathfrak{M}$ the set of all increasing sequences of integers and $\mathfrak{D}$ the set of all sequences of real numbers decreasing to 0. For $\boldsymbol{\mu} = (\mu_k) \in \mathfrak{M}$ and $\boldsymbol{\varepsilon} = (\varepsilon_k) \in \mathfrak{D}$ define a seminorm $N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})$ on $\mathscr{D}(\mathbb{R}^n, E)$ by the formula

$$N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})(\varphi) = \sup_k \left(\sup \left\{ \frac{\|\partial^\alpha \varphi(x)\|}{\varepsilon_k} : |x| \geqslant k, |\alpha| \leqslant \mu_k \right\} \right).$$

The collection $\mathscr{P} = \{N(\boldsymbol{\mu}, \boldsymbol{\varepsilon}) : \mu \in \mathfrak{M}, \varepsilon \in \mathfrak{D}\}$ is a separating family of seminorms on $\mathscr{D}(\mathbb{R}^n, E)$. According to Theorem 1.8 it endows $\mathscr{D}(\mathbb{R}^n, E)$ with a locally convex vector topology $\mathfrak{T}_{\mathscr{P}}$.

Given $K \in \mathfrak{K}(\mathbb{R}^n)$, observe that $\mathfrak{T}_{\mathscr{P}}$ turns the closed subspace $\mathscr{D}_K(\mathbb{R}^n,$-$E)$ of $\mathscr{D}(\mathbb{R}^n, E)$ into a l.c.t.v.s. Moreover, the subspace topology $\mathfrak{T}_{\mathscr{P},K}$ is compatible with the distance defined on $\mathscr{D}_K(\mathbb{R}^n, E)$ in Example 1.7. Indeed, given a compact set $K \in \mathfrak{K}(\mathbb{R}^n)$ there is one $m \in \mathbb{N}$ such that $K \subseteq B[m]$ and for $\boldsymbol{\mu} \in \mathfrak{M}$, $\boldsymbol{\varepsilon} \in \mathfrak{D}$ and $\varphi \in \mathscr{D}_K(\mathbb{R}^n, E)$ we get $m \leqslant \mu_m$ and

$$N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})(\varphi) \leqslant \frac{1}{\varepsilon_m} \sup\{\|\partial^\alpha \varphi(x)\| : x \in K, |\alpha| \leqslant \mu_m\} \leqslant \frac{1}{\varepsilon_m} \nu^{\mu_m}_{B[\mu_m]}(\varphi).$$

Conversely, given $k \in \mathbb{N}$ and $\varphi \in \mathscr{D}_K(\mathbb{R}^n, E)$, choose $\boldsymbol{\mu} \in \mathfrak{M}$ and $\boldsymbol{\varepsilon} \in \mathfrak{D}$ satisfying $\mu_0 = k$, $\varepsilon_0 = 1$ and observe

$$\nu^k_{B[k]}(\varphi) \leqslant \sup \left\{ \frac{\|\partial^\alpha \varphi(x)\|}{\varepsilon_0} : x \geqslant 0, |\alpha| \leqslant \mu_0 \right\} \leqslant N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})(\varphi).$$

In any l.c.t.v.s., a continuous linear functional is identically zero provided it vanishes on a dense subspace.

Theorem 1.10. *Let X be a l.c.t.v.s., $Y \subseteq X$ be a vector space endowed by the subspace topology and $x_0 \in X \setminus \operatorname{cl} Y$. There exists a linear continuous functional u on X satisfying $\langle u, x_0 \rangle \neq 0$ yet $u \restriction Y = 0$.*

The following result is an easy corollary of Theorem 1.10.

Corollary 1.11. *Let X be a l.c.t.v.s. and fix $x \neq y$ in X. There exists $u \in X^*$ satisfying $\langle u, x \rangle \neq \langle u, y \rangle$.*

Locally convex topology induced by a sequence of compact convex sets

Assume $(X, \mathfrak{S})$ is a l.c.t.v.s. and $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ are compact convex subsets of X satisfying the following conditions:

1. $0 \in K_0$ and $X = \bigcup_{k \in \mathbb{N}} K_k$;
2. for each $k \in \mathbb{N}$, each $x \in X$ and each $t \in \mathbb{R}$ there exists $l \in \mathbb{N}$ for which $x + K_k \subseteq K_l$ and $tK_k \subseteq K_l$.

Call $\mathfrak{T}$ the strongest topology on X such that for each $k \in \mathbb{N}$ the inclusion

$$(K_k, \mathfrak{S}) \hookrightarrow (X, \mathfrak{T})$$

is continuous. The topology $\mathfrak{T}$ is locally convex, sequential in case $\mathfrak{S}$ is sequential, and satisfies the following conditions according to [Pfe01, Proposition 1.2.2]:

1. a set $A \subseteq X$ is $\mathfrak{T}$-closed if and only if $A \cap K_k$ is $\mathfrak{S}$-closed for each $k \in \mathbb{N}$;

2. given a topological space Y, it is necessary and sufficient for a map $f : (X, \mathfrak{T}) \to Y$ to be continuous that $f \upharpoonright K_k : (K_k, \mathfrak{S}) \to Y$ be continuous for each $k \in \mathbb{N}$;

3. a sequence $(x_j) \subseteq X$ $\mathfrak{T}$-converges to an $x \in X$ if and only if there exists $k \in \mathbb{N}$ for which $(x_j) \subseteq K_k$ and (x_j) $\mathfrak{S}$-converges to x.

The weak* topology on a dual space

In the sequel, X stands for a t.v.s. and $\operatorname{Fin}_* X$ denotes the collection of nonempty finite subsets of X.

For every $F \in \operatorname{Fin}_* X$, we define a seminorm on X^* by the formula

$$p_F(u) = \sup\{|\langle u, x\rangle| : x \in F\}.$$

Let $\sigma(X^*, X)$ be the topology induced by the separating family of seminorms $\mathscr{P} = \{p_F : F \in \operatorname{Fin}_* X\}$.

When working in (a subspace of) a dual space X^*, the adverb '*weakly**' will mean 'relatively to the (topolgy induced on that subspace by the) $\sigma(X^*,$-$X)$-topology' in the sequel.

The following result characterizes the weakly*-continuous linear functionals on X^*. For $x \in X$, define $j(x) \in (X^*)'$ by the fomula $\langle j(x), u\rangle = \langle u, x\rangle$.

Theorem 1.12. *A linear functional α on X^* is weakly*-continuous if and only if there exists $x \in X$ for which $\alpha = j(x)$.*

Given $A \subseteq X$, define the *polar set A° of A* by

$$A^\circ = \{u \in X^* : |\langle u, x\rangle| \leqslant 1 \text{ for each } x \in A\}.$$

In a locally convex, complete topological vector space, the Banach-Grothendieck theorem ensures weak continuity is implied by weak continuity on every polar set.

Theorem 1.13 (Banach-Grothendieck)**.** *Let X be a complete l.c.t.v.s. A linear functional α on X^* is weakly*-continuous if and only if α is weakly*-continuous on V° for each $V \in \mathcal{V}(0)$.*

We state the important Banach-Alaoglu theorem.

Theorem 1.14 (Banach-Alaoglu)**.** *If V is a neighbourhood of 0 in X, then V° is weakly*-compact.*

The strong topology associated to a t.v.s.

In the sequel, $\mathfrak{B}$ stands for the collection of bounded subsets of the t.v.s. X. For $S \in \mathfrak{B}$ define a seminorm p_S on X^* by the formula

$$p_S(u) = \sup\{|\langle u, x\rangle| : x \in S\},$$

and note that $\mathscr{P}_{\mathfrak{S}} = \{p_S : S \in \mathfrak{B}\}$ is a separating family of seminorms. According to Theorem 1.8 it generates on X^* a locally convex topology $\mathfrak{B}(X^*,$-$X)$ called the strong topology on X^*.

When working in a dual space X^*, the adverb '*strongly*' will mean 'relatively to the $\mathfrak{B}(X^*, X)$-topology' in the sequel.

Adjoint maps

Let X, Y be l.c.t.v.s. and fix a linear continuous map $T : X \to Y$. The adjoint map $T^* : Y^* \to X^*$ (exists and) is defined by the formula

$$\langle T^*(u), x\rangle = \langle u, T(x)\rangle.$$

The following fact about adjoint maps is classical (see [Edw95, Section 8.6]).

Proposition 1.15. *Assume X and Y are l.c.t.v.s. and $T : X \to Y$ is continuous. Then $\operatorname{im} T$ is dense if and only if T^* is one-to-one.*

In [Edw95, Theorem 8.6.13] we find the following version of the *closed range theorem*.

Theorem 1.16 (Closed Range)**.** *Let X, Y be Fréchet spaces and assume $T : X \to Y$ is linear and continuous. The following are equivalent:*

1. $\operatorname{im} T$ *is closed in* Y*;*

2. $\operatorname{im} T^*$ *is weakly***-closed in* X^**;*

3. $\operatorname{im} T^*$ *is strongly closed in* X^*.

Remark 1.17. Looking to the proof of Theorem 8.6.13 in [Edw95] we see that condition 3 in Theorem 1.16 can be replaced by the following:

4. $\operatorname{im} T^*$ is strongly sequentially closed in X^*.

Combining Proposition 1.15 and Theorem 1.16 we get the following.

Theorem 1.18. *Assume X, Y are Fréchet spaces and $T : X \to Y$ is linear, continuous and one-to-one. If moreover* $\operatorname{im} T$ *is closed, then* $\operatorname{im} T^* = X^*$.

Chapter 2

Currents

We introduce the geometrical objects we will use.

2.1 Differential forms and currents

2.1.1 Differential forms in $\mathbb{R}^n$

In the sequel, integers $0 \leqslant m \leqslant n$ are given. Let $\Lambda_m\mathbb{R}^n$ and $\Lambda^m\mathbb{R}^n$ denote the sets of all m-vectors and m-forms in $\mathbb{R}^n$ respectively (with the usual convention that $\Lambda_0\mathbb{R}^n = \Lambda^0\mathbb{R}^n = \mathbb{R}$). We have

$$\dim \Lambda_m\mathbb{R}^n = \dim \Lambda^m\mathbb{R}^n = C_n^m = \frac{n!}{m!(n-m)!}.$$

The duality between $\Lambda_m\mathbb{R}^n$ and $\Lambda^m\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. In particular, for

$$\eta = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \eta_{i_1,\dots,i_m} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \quad \text{and}$$

$$\phi = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \phi_{i_1,\dots,i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

we get

$$\langle \eta, \phi \rangle = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \eta_{i_1,\dots,i_m} \phi_{i_1,\dots,i_m}.$$

The *Euclidean norm* of the m-vector

$$\eta = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \eta_{i_1,\dots,i_m} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$$

is defined by

$$|\eta| = \left(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \eta^2_{i_1,i_2,\dots,i_m} \right)^{1/2}.$$

Call an m-vector $\eta \in \Lambda_m \mathbb{R}^n$ *simple* whenever there exists $v_1, v_2, \dots, v_m \in \mathbb{R}^n$ for which $\eta = v_1 \wedge v_2 \wedge \cdots \wedge v_m$. We will denote by $\mathbf{G}_0(n, m)$ the set of all simple m-vectors $\eta \in \Lambda_m \mathbb{R}^n$ satisfying $|\eta| = 1$. Given $\eta \in \mathbf{G}_0(n, m)$ we define a linear subspace

$$\operatorname{span}\langle \eta \rangle = \{ v \in \mathbb{R}^n : \eta \wedge v = 0 \},$$

and one checks $\dim \operatorname{span}\langle \eta \rangle = m$.

The *comass* of $\phi \in \Lambda^m \mathbb{R}^n$ is the number

$$\|\phi\| = \sup\{ \langle \eta, \phi \rangle : \eta \in \Lambda_m \mathbb{R}^n \text{ is simple}, |\eta| \leqslant 1 \}.$$

One easily verifies $|\phi| \geqslant \|\phi\| \geqslant (C_n^m)^{-1/2} |\phi|$. For $\psi \in \Lambda^l \mathbb{R}^n$ we have $\|\phi \wedge \psi\| \leqslant C^l_{l+m} \|\psi\| \|\psi\|$. The *mass* of $\eta \in \Lambda_m \mathbb{R}^n$ is the number

$$\|\eta\| = \sup\{ \langle \eta, \phi \rangle : \phi \in \Lambda^m \mathbb{R}^n, \|\phi\| \leqslant 1 \}.$$

One easily verifies $|\eta| \leqslant \|\eta\| \leqslant (C_n^m)^{1/2} |\eta|$. For $\xi \in \Lambda_l \mathbb{R}^n$ we have $\|\eta \wedge \xi\| \leqslant \|\xi\| \|\eta\|$. One also checks the comass (resp. the mass) defines a norm on $\Lambda^m \mathbb{R}^n$ (resp. $\Lambda_m \mathbb{R}^n$). See [Fed69, Section 1.8] for more detail.

An m-differential form on $\mathbb{R}^n$ is a map $\omega : \mathbb{R}^n \to \Lambda^m \mathbb{R}^n$. Its *comass* $\mathbf{M}(\omega)$ is the extended real number $\mathbf{M}(\omega)$ defined by the formula

$$\mathbf{M}(\omega) = \sup\{ \|\omega(x)\| : x \in \mathbb{R}^n \}.$$

Dually, an m-vectorfield on $\mathbb{R}^n$ is a map $\xi : \mathbb{R}^n \to \Lambda_m \mathbb{R}^n$. The *mass* of ξ is the extended real number $\mathbf{M}(\xi)$ defined by the formula

$$\mathbf{M}(\xi) = \sup\{ \|\xi(x)\| : x \in \mathbb{R}^n \}.$$

For $0 \leqslant m \leqslant m$ and a differentiable m-differential form $\omega : \mathbb{R}^n \to \Lambda^m \mathbb{R}^n$ with

$$\omega(x) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \omega_{i_1, i_2, \ldots, i_m}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

for $x \in \mathbb{R}^n$, define the $(m+1)$-differential form $d\omega : \mathbb{R}^n \to \Lambda^{m+1}\mathbb{R}^n$ by the formula

$$d\omega(x) = \sum_{j=1}^{n} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \partial_j \omega_{i_1, i_2, \ldots, i_m}(x)\, dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}.$$

If $\omega : \mathbb{R}^n \to \Lambda^n \mathbb{R}^n$ is a differentiable n-differential form, we get in particular $d\omega(x) = 0$ for each $x \in \mathbb{R}^n$. For a differentiable 0-differential form (i.e. a differentiable function) $\omega : \mathbb{R}^n \to \mathbb{R}$, we define the 1-differential form $d\omega : \mathbb{R}^n \to \Lambda^1 \mathbb{R}^n$ by

$$d\omega(x) = \sum_{j=1}^{m} \partial_j \omega(x)\, dx_j.$$

Given a differentiable map $f : \mathbb{R}^p \to \mathbb{R}^n$ and an m-differential form ω in $\mathbb{R}^n$, we define the m-differential form $f^{\#}\omega$ in $\mathbb{R}^p$ by the formula $\omega \circ f$ in case $m = 0$ and by

$$f^{\#}\omega(y) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \omega_{i_1, i_2, \ldots, i_m}[f(y)]\, df_{i_1}(y) \wedge df_{i_2}(y) \wedge \cdots \wedge df_{i_m}(y)$$

in case $m \geqslant 1$ and

$$\omega(x) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \omega_{i_1, i_2, \ldots, i_m}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}.$$

In case f is twice differentiable and ω is differentiable, we get $d(f^{\#}\omega) = f^{\#}(d\omega)$.

2.1.2 Currents in $\mathbb{R}^n$

We abbreviate $\mathscr{C}^m(\mathbb{R}^n) = C(\mathbb{R}^n, \Lambda^m \mathbb{R}^n)$ the space of all continuous m-differential forms in $\mathbb{R}^n$ and define by convention $\mathscr{C}^{-1}(\mathbb{R}^n) = \{0\}$. For

$K \in \mathcal{K}(\mathbb{R}^n)$ define a seminorm ν_K^0 on $\mathscr{C}^m(\mathbb{R}^n)$ by the formula

$$\nu_K^0(\omega) = \sup\{\|\omega(x)\| : x \in K\}.$$

According to Example 1.6, these seminorms induce on $\mathscr{C}^m(\mathbb{R}^n)$ a structure of Fréchet space.

Given $K \in \mathcal{K}(\mathbb{R}^n)$ and $k \in \mathbb{N}$, we define on $C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$ a seminorm ν_K^k by the formula

$$\nu_K^k(\omega) = \sup\{\|\partial^\alpha \omega(x)\| : x \in K, \alpha \in \mathbb{N}^n, |\alpha| \leqslant k\}.$$

When K and k range over $\mathcal{K}(\mathbb{R}^n)$ and $\mathbb{N}$ respectively, the seminorms ν_K^k induce on $C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$ a metrizable, locally convex, complete topology. Hence given $K \in \mathcal{K}(\mathbb{R}^n)$, the subspace $\mathscr{D}_K^m(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$ whose elements are those of $C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$ supported in K, is endowed with a structure of Fréchet space (see Example 1.7 for more detail).

Finally, let $\mathscr{D}^m(\mathbb{R}^n)$ be the subspace of $C^\infty(\mathbb{R}^n, \Lambda^m\mathbb{R}^n)$ whose elements have compact support, and define by convention $\mathscr{D}^{-1}(\mathbb{R}^n) = \{0\}$. For $\zeta \in \mathscr{D}^{-1}(\mathbb{R}^n)$ write $d\zeta = 0$.

Denote $\mathfrak{M}$ the set of all increasing sequences of integers and $\mathfrak{D}$ the set of all sequences of real numbers decreasing to 0. For $\boldsymbol{\mu} = (\mu_k) \in \mathfrak{M}$ and $\boldsymbol{\varepsilon} = (\varepsilon_k) \in \mathfrak{D}$ define a seminorm $N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})$ on $\mathscr{D}^m(\mathbb{R}^n)$ by the formula

$$N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})(\omega) = \sup_k \left(\sup \left\{ \frac{\|\partial^\alpha \omega(x)\|}{\varepsilon_k} : |x| \geqslant k, |\alpha| \leqslant \mu_k \right\} \right).$$

When $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ range over $\mathfrak{M}$ and $\mathfrak{D}$ respectively, the seminorms $N(\boldsymbol{\mu}, \boldsymbol{\varepsilon})$ endow $\mathscr{D}^m(\mathbb{R}^n)$ with a structure of complete, sequential, locally convex topological vector space according to Example 1.9. An element of the topological dual $\mathscr{D}_m(\mathbb{R}^n) = [\mathscr{D}^m(\mathbb{R}^n)]^*$ is called an *m-current*.

The support $\operatorname{supp} T$ of $T \in \mathscr{D}_m(\mathbb{R}^n)$ is defined in the usual way by

$$\begin{aligned} \operatorname{supp} T = \mathbb{R}^n \setminus \cup\{U \subseteq \mathbb{R}^n : U \text{ is open}, \langle T, \omega \rangle = 0 \\ \text{for each } \omega \in \mathscr{D}^m(\mathbb{R}^n) \text{ satisfying } \operatorname{supp} \omega \subseteq U\}. \end{aligned}$$

For $m \geqslant 1$, the boundary ∂T of $T \in \mathscr{D}_m(\mathbb{R}^n)$ is the $(m-1)$-dimensional current $\partial T \in \mathscr{D}_{m-1}(\mathbb{R}^n)$ defined by the formula $\langle \partial T, \omega \rangle = \langle T, d\omega \rangle$. For $T \in \mathscr{D}_0(\mathbb{R}^n)$ we let $\partial T = 0$.

The *mass* of $T \in \mathscr{D}_m(\mathbb{R}^n)$ is the extended real number

$$\mathbf{M}(T) = \sup\{\langle T, \omega \rangle : \omega \in \mathscr{D}^m(\mathbb{R}^n), \mathbf{M}(\omega) \leqslant 1\},$$

and its *flat norm* is the extended real number

$$\mathbf{F}(T) = \sup\{\langle T, \omega \rangle : \omega \in \mathscr{D}^m(\mathbb{R}^n), \mathbf{M}(\omega) \leqslant 1, \mathbf{M}(d\omega) \leqslant 1\}. \tag{2.1}$$

The following result gives another expression for the flat norm $\mathbf{F}(T)$ of $T \in \mathscr{D}_m(\mathbb{R}^n)$.

Proposition 2.1. *For $T \in \mathscr{D}_m(\mathbb{R}^n)$, we have*

$$\mathbf{F}(T) = \begin{cases} \mathbf{M}(T) & \textit{in case } m = n; \\ \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathscr{D}_{m+1}(\mathbb{R}^n)\} & \textit{otherwise}. \end{cases} \tag{2.2}$$

Proof. Assume $m \leqslant n-1$, fix $T \in \mathbf{N}_m(\mathbb{R}^n)$ and compute for any $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\|\omega\| \leqslant 1$, $\|d\omega\| \leqslant 1$ and any $S \in \mathscr{D}_{m+1}(\mathbb{R}^n)$

$$\langle T, \omega \rangle = \langle T - \partial S, \omega \rangle + \langle S, d\omega \rangle \leqslant \mathbf{M}(T - \partial S) + \mathbf{M}(S).$$

Consequently we get $\mathbf{F}(T) \leqslant \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathscr{D}_{m+1}(\mathbb{R}^n)\}$.

To prove the converse inequality, assume $\mathbf{F}(T) < \infty$. On the space $\mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^{m+1}(\mathbb{R}^n)$, define a seminorm p by the formula

$$p(\omega, \phi) = \max\{\mathbf{M}(\omega), \mathbf{M}(\phi)\},$$

introduce the linear injection

$$u : \mathscr{D}^m(\mathbb{R}^n) \to \mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^{m+1}(\mathbb{R}^n), \omega \mapsto (\omega, d\omega)$$

and observe that $p[u(\omega)] = \max\{\mathbf{M}(\omega), \mathbf{M}(d\omega)\}$. In particular, we get

$$T \circ u^{-1} \restriction \operatorname{im} u \leqslant \mathbf{F}(T)\, p \restriction \operatorname{im} u.$$

Using Hahn-Banach' theorem, we find a linear map

$$L : \mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^{m+1}(\mathbb{R}^n) \to \mathbb{R}$$

satisfying $L \restriction \operatorname{im} u = T \circ u^{-1}$ and $L \leqslant \mathbf{F}(T)p$ on $\mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^m(\mathbb{R}^n)$. Define $R : \mathscr{D}^m(\mathbb{R}^n) \to \mathbb{R}$ and $S : \mathscr{D}^{m+1}(\mathbb{R}^n) \to \mathbb{R}$ by the formulas $\langle R, \omega \rangle = L(\omega, 0)$ and $\langle S, \phi \rangle = L(0, \phi)$. For $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and $\phi \in \mathscr{D}^{m+1}(\mathbb{R}^n)$ we get $\langle R, \omega \rangle = L(\omega, 0) \leqslant \mathbf{F}(T)\mathbf{M}(\omega)$ and $\langle S, \phi \rangle = L(0, \phi) \leqslant \mathbf{F}(T)\mathbf{M}(\phi)$. In particular we get $R \in \mathscr{D}_m(\mathbb{R}^n)$, $S \in \mathscr{D}_{m+1}(\mathbb{R}^n)$ as well as $\mathbf{M}(R) \leqslant \mathbf{F}(T)$ and $\mathbf{M}(S) \leqslant \mathbf{F}(T)$. On the other hand, compute for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$,

$$\langle T, \omega \rangle = L(\omega, d\omega) = \langle R, \omega \rangle + \langle S, d\omega \rangle = \langle R + \partial S, \omega \rangle.$$

Moreover we get $\mathbf{M}(\partial S) \leqslant \mathbf{M}(T) + \mathbf{M}(R)$. Finally, for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and $\phi \in \mathscr{D}^{m+1}(\mathbb{R}^n)$ satisfying $\mathbf{M}(\omega) \leqslant 1$ and $\mathbf{M}(\phi) \leqslant 1$ we compute

$$\langle T - \partial S, \omega \rangle + \langle S, \phi \rangle = \langle R, \omega \rangle + \langle S, \phi \rangle = L(\omega, \phi) \leqslant \mathbf{F}(T),$$

so that $\mathbf{M}(T - \partial S) + \mathbf{M}(S) \leqslant \mathbf{F}(T)$. $\square$

Given $u, v \in \mathbb{R}^n$ we define $[\![u, v]\!] \in \mathscr{D}_1(\mathbb{R}^n)$ by the formula

$$[\![u, v]\!](\omega) = \int_0^1 \langle v - u, \omega[(1-t)u + tv] \rangle \, dt.$$

A function f mapping a subset of $\mathbb{R}^n$ to $\mathbb{R}^p$ is called *proper* whenever $f^{-1}(K) \in \mathcal{K}(\mathbb{R}^n)$ holds for all $K \in \mathcal{K}(\mathbb{R}^p)$.

Assume $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ and $T \in \mathscr{D}_m(\mathbb{R}^n)$ are such that $f \restriction \operatorname{supp} T$ is proper. Then we define $f_\# T \in \mathscr{D}_m(\mathbb{R}^p)$ by the formula

$$\langle f_\# T, \omega \rangle = \langle T, f^\# \omega \rangle,$$

for each $\omega \in \mathscr{D}^m(\mathbb{R}^p)$. We obtain $\operatorname{supp}(f_\# T) \subseteq f(\operatorname{supp} T)$, $\partial(f_\# T) = f_\#(\partial T)$ as well as

$$\mathbf{M}(f_\# T) \leqslant [\operatorname{Lip}(f \restriction \operatorname{supp} T)]^m \mathbf{M}(T) \tag{2.3}$$

and

$$\mathbf{F}(f_{\#}T) \leqslant \max\{[\mathrm{Lip}(f \restriction \operatorname{supp} T)]^m, [\mathrm{Lip}(f \restriction \operatorname{supp} T)]^{m+1}\}\mathbf{F}(T). \quad (2.4)$$

If moreover $g \in C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ is given, we get $(g \circ f)_{\#}T = g_{\#}(f_{\#}T)$.

Given $S \in \mathscr{D}_l(\mathbb{R}^n)$ and $T \in \mathscr{D}_m(\mathbb{R}^p)$, there exists a unique $S \times T \in \mathscr{D}_{l+m}(\mathbb{R}^{n+p})$ satisfying the following conditions:

1. $\langle S \times T, \omega_1 \wedge \omega_2 \rangle = \langle S, \omega_1 \rangle \cdot \langle T, \omega_2 \rangle$ for every $\omega_1 \in \mathscr{D}^l(\mathbb{R}^n)$ and $\omega_2 \in \mathscr{D}^m(\mathbb{R}^p)$;

2. $\langle S \times T, \omega_1 \wedge \omega_2 \rangle = 0$ for every $\omega_1 \in \mathscr{D}^{l'}(\mathbb{R}^n)$ and $\omega_2 \in \mathscr{D}^{l+m-l'}(\mathbb{R}^p)$ such that $l' \neq l$.

We have $\operatorname{supp}(S \times T) = \operatorname{supp} S \times \operatorname{supp} T$ as well as

$$\partial(S \times T) = \begin{cases} (\partial S) \times T + (-1)^l S \times (\partial T) & \text{if} \quad l > 0, m > 0; \\ (\partial S) \times T & \text{if} \quad l > m = 0; \\ S \times (\partial T) & \text{if} \quad m > l = 0. \end{cases}$$

Assume f, g map $\mathbb{R}^n$ to $\mathbb{R}^p$. We call *homotopy mapping f to g* each map $h \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^p)$ satisfying the following conditions:

1. $h(0, x) = f(x)$ for each $x \in \mathbb{R}^n$;

2. $h(1, x) = g(x)$ for each $x \in \mathbb{R}^n$.

Recall the homotopy formula for currents.

Proposition 2.2 (Homotopy formula for currents). *Let $T \in \mathscr{D}_m(\mathbb{R}^n)$ and assume f, g map $\mathbb{R}^n$ to $\mathbb{R}^p$ and h is a homotopy mapping f to g such that $h \restriction [0,1] \times \operatorname{supp} T$ is proper. Then,*

$$g_{\#}T - f_{\#}T = \begin{cases} \partial[h_{\#}([\![0,1]\!] \times T)] + h_{\#}([\![0,1]\!] \times \partial T) & \textit{if} \quad m > 0; \\ \partial[h_{\#}([\![0,1]\!] \times T)] & \textit{if} \quad m = 0. \end{cases}$$

2.1.3 Normal currents in $\mathbb{R}^n$

In case $m > 0$, an m-dimensional current $T \in \mathscr{D}_m(\mathbb{R}^n)$ is called a locally normal m-dimensional current in $\mathbb{R}^n$ if there exist Radon measures $\|T\|$ and $\|\partial T\|$ in $\mathbb{R}^n$ together with a $\|T\|$-measurable map $\vec{T} : \mathbb{R}^n \to \Lambda_m \mathbb{R}^n$ and a $\|\partial T\|$-measurable map $\vec{\partial T} : \mathbb{R}^n \to \Lambda_{m-1}\mathbb{R}^n$ for which one has

$$\langle T, \omega \rangle = \int_{\mathbb{R}^n} \langle \vec{T}, \omega \rangle \, d\|T\| \quad \text{and} \quad \langle \partial T, \zeta \rangle = \int_{\mathbb{R}^n} \langle \vec{\partial T}, \zeta \rangle \, d\|\partial T\| \tag{2.5}$$

whenever $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and $\zeta \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ are given, $\|\vec{T}\| = 1$ $\|T\|$-a.e. in $\mathbb{R}^n$ and $\|\vec{\partial T}\| = 1$ $\|\partial T\|$-a.e. in $\mathbb{R}^n$. If moreover $\operatorname{supp} T$ is compact, then T is called an m-dimensional normal current in $\mathbb{R}^n$.

Still in case $m \geqslant 1$ we can, using a smoothing argument, extend in a unique way any locally normal m-dimensional current in $\mathbb{R}^n$ and its boundary ∂T as linear maps $T : \mathscr{C}^m(\mathbb{R}^n) \to \mathbb{R}$ and $\partial T : \mathscr{C}^{m-1}(\mathbb{R}^n) \to \mathbb{R}$ satisfying the previous conditions (2.5) for $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ and $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$.

If T is a locally normal m-dimensional current for $m \geqslant 1$ one let $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$, and one observes $\mathbf{M}(T) = \|T\|(\mathbb{R}^n)$, $\mathbf{M}(\partial T) = \|\partial T\|(\mathbb{R}^n)$.

A 0-current $T \in \mathscr{D}_0(\mathbb{R}^n)$ is called a locally normal 0-dimensional current in $\mathbb{R}^n$ if there exists a Radon measure $\|T\|$ in $\mathbb{R}^n$ together with a $\|T\|$-measurable function $\vec{T} : \mathbb{R}^n \to \mathbb{R}$ for which one has

$$\langle T, \omega \rangle = \int_{\mathbb{R}^n} \langle \vec{T}, \omega \rangle \, d\|T\|, \tag{2.6}$$

for each $\omega \in \mathscr{D}(\mathbb{R}^n)$ and $\|\vec{T}\| = 1$ a.e. in $\mathbb{R}^n$. If moreover $\operatorname{supp} T$ is compact, then T is called a 0-dimensional normal current in $\mathbb{R}^n$.

Any locally normal 0-dimensional current T in $\mathbb{R}^n$ can be extended in a unique way to a linear map $T : \mathscr{C}^0(\mathbb{R}^n) \to \mathbb{R}$ satisfying (2.6) for each $\omega \in \mathscr{C}^0(\mathbb{R}^n)$.

If T is a locally normal 0-dimensional current, one lets $\mathbf{N}(T) = \mathbf{M}(T)$ and one observes $\mathbf{M}(T) = \|T\|(\mathbb{R}^n)$.

If $K \in \mathcal{K}(\mathbb{R}^n)$ is compact we let $\mathbf{N}_{m,K}(\mathbb{R}^n)$ be the set of all m-dimensional normal currents in $\mathbb{R}^n$ supported in K and we let

$$\mathbf{N}_m(\mathbb{R}^n) = \cup\{\mathbf{N}_{m,K}(\mathbb{R}^n) : K \in \mathcal{K}(\mathbb{R}^n)\}.$$

The notation $\mathbf{N}_m^{\mathrm{loc}}(\mathbb{R}^n)$ stands for the set of all locally normal m-dimensional currents in $\mathbb{R}^n$. For $T \in \mathbf{N}_m(\mathbb{R}^n)$ we obviously have $\mathbf{N}(T) < \infty$. Conversely, if $T \in \mathscr{D}_m(\mathbb{R}^n)$ verifies $\mathbf{M}(T) + \mathbf{N}(T) < \infty$, then T is locally normal. In particular, one can reformulate (2.2) in the following way:

$$\mathbf{F}(T) = \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathbf{N}_{m+1}^{\mathrm{loc}}(\mathbb{R}^n)\}. \tag{2.7}$$

Given $S \in \mathbf{N}_l(\mathbb{R}^n)$ and $T \in \mathbf{N}_m(\mathbb{R}^p)$, a simple application of (2.7) yields

$$\mathbf{F}(S \times T) \leqslant \mathbf{N}(S)\mathbf{F}(T).$$

Given a normal current $T \in \mathbf{N}_m(\mathbb{R}^n)$ and $k \geqslant 1$ an integer, the formula

$$\langle T_k, \omega\rangle = \langle T, \rho_k * \omega\rangle \tag{2.8}$$

defines a normal current $T_k \in \mathbf{N}_m(\mathbb{R}^n)$ (see [Fed69, Section 4.1.2]) for which there exists $\xi_k \in \mathscr{D}(\mathbb{R}^n, \Lambda_m\mathbb{R}^n)$ satisfying

$$\langle T_k, \omega\rangle = \int_{\mathbb{R}^n} \langle \xi_k, \omega\rangle \, dx$$

for each $\omega \in \mathscr{D}^m(\mathbb{R}^n)$. Moreover, using the smoothing homotopy formula [Fed69, Section 4.1.18] we get

$$\mathbf{N}(T_k) \leqslant \mathbf{N}(T) \quad \text{and} \quad \mathbf{F}(T_k - T) \leqslant \frac{1}{k}\mathbf{N}(T).$$

Remark 2.3. Using an approximation by smooth functions [Fed69, Section 4.1.14], given a locally lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^p$ and a normal current $T \in \mathbf{N}_m(\mathbb{R}^n)$ we can define $f_\# T \in \mathbf{N}_m(\mathbb{R}^p)$. The usual properties are conserved, for example $\operatorname{supp}(f_\# T) \subseteq f(\operatorname{supp} T)$ and $\partial(f_\# T) = f_\#(\partial T)$. Given $g : \mathbb{R}^p \to \mathbb{R}^q$ we also have $(g \circ f)_\# T = g_\#(f_\# T)$. Moreover, the homotopy formula (Proposition 2.2) generalizes to any locally lipschitz homotopy in case T is normal.

We will need the following restriction lemma.

Lemma 2.4. *For* $\chi \in \mathscr{D}(\mathbb{R}^n)$ *and* $T \in \mathbf{N}_m(\mathbb{R}^n)$, *define* $T \llcorner \chi : \mathscr{D}^m(\mathbb{R}^n) \to \mathbb{R}$ *by the formula* $\langle T \llcorner \chi, \omega \rangle = \langle T, \chi\omega \rangle$. *Then the following conditions hold:*

1. $T = T \llcorner \chi + T \llcorner (1 - \chi)$,
2. $T \llcorner \chi \in \mathbf{N}_m(\mathbb{R}^n)$,
3. $\operatorname{supp} T \llcorner \chi \subseteq \operatorname{supp} T \cap \operatorname{supp} \chi$,
4. $\mathbf{N}(T \llcorner \chi) \leqslant 2m \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)] \mathbf{N}(T)$,
5. $\mathbf{F}(T \llcorner \chi) \leqslant 2(m+1) \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)] \mathbf{F}(T)$.

Proof. We can of course assume $\chi \neq 0$. It is obvious that $\operatorname{supp} T \llcorner \chi \subseteq \operatorname{supp} T \cap \operatorname{supp} \chi$. Moreover, compute for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\mathbf{M}(\omega) \leqslant 1$

$$\langle T \llcorner \chi, \omega \rangle \leqslant \mathbf{M}(\chi)\mathbf{M}(T).$$

On the other hand, for $\zeta \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ satisfying $\mathbf{M}(\zeta) \leqslant 1$ we get

$$\langle T \llcorner \chi, d\zeta \rangle = \langle \partial T, \chi\zeta \rangle - \langle T, d\chi \wedge \zeta \rangle \leqslant m \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)] \mathbf{N}(T),$$

so that $\mathbf{N}(T \llcorner \chi) \leqslant 2m \max[\mathbf{M}(\chi), \mathbf{M}(d\zeta)] \mathbf{N}(T)$. On the other hand fix $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\max[\mathbf{M}(\omega), \mathbf{M}(d\omega)] \leqslant 1$, observe $\mathbf{M}(\chi\omega) \leqslant \mathbf{M}(\chi)$ and compute

$$\begin{aligned} \mathbf{M}[d(\chi\omega)] &\leqslant \mathbf{M}(\chi d\omega) + \mathbf{M}(d\chi \wedge \omega) \\ &\leqslant \mathbf{M}(\chi) + (m+1)\mathbf{M}(d\chi) \leqslant 2(m+1) \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)]. \end{aligned}$$

So we get $\max\{\mathbf{M}(\chi\omega), \mathbf{M}[d(\chi\omega)]\} \leqslant 2(m+1) \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)]$ and

$$\begin{aligned} \langle T \llcorner \chi, \omega \rangle &\leqslant \mathbf{F}(T) \max\{\mathbf{M}(\chi\omega), \mathbf{M}[d(\chi\omega)]\} \\ &\leqslant 2(m+1) \max[\mathbf{M}(\chi), \mathbf{M}(d\chi)] \mathbf{F}(T). \end{aligned}$$

The proof is complete. ☐

2.2 A compactness theorem

Let $C = [0,1]^n$ be the unit-cube in $\mathbb{R}^n$ and call m-face of C each set of the form

$$F = [0,1]^{\varepsilon_1} \times [0,1]^{\varepsilon_2} \times \cdots \times [0,1]^{\varepsilon_n}$$

for $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{+,-,1\}^n$ satisfying $\#\{1 \leqslant i \leqslant n : \varepsilon_i = 1\} = m$, if we let

$$[0,1]^+ = \{1\},\ [0,1]^- = \{0\} \text{ and } [0,1]^1 = [0,1];$$

moreover we define $\vec{T}_F = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$ if $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ denote the elements of $\{1 \leqslant i \leqslant n : \varepsilon_i = 1\}$. We also define for $\rho > 0$

$$L_m(\rho) = \{\rho(z+F) : z \in \mathbb{Z}^n, F \text{ is a } m-\text{face of } C\};$$

and for $F' = \rho(z+F) \in L_m(\rho)$ we define $\vec{T}_{F'} = \vec{T}_F$. To each $F \in L_m(\rho)$ we associate $[\![F]\!] \in \mathscr{D}_m(\mathbb{R}^n)$ by defining

$$[\![F]\!](\omega) = \int_{\mathbb{R}^n} \langle \vec{T}_F, \omega(x) \rangle \, d\mathscr{H}^m \llcorner F(x)$$

for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$.

Recall the *deformation theorem* ([Sim83, Theorem 29.1]).

Theorem 2.5 (Deformation theorem)**.** *There exists a constant $\gamma = \gamma(n,m)$ such that given $T \in \mathbf{N}_m(\mathbb{R}^n)$ and $\rho > 0$ we can write $T - P = \partial S + R$ where:*

1. $P = \sum_{F \in L_m(\rho)} \alpha_F [\![F]\!]$ *for some real numbers* $\alpha_F \in \mathbb{R}$*,* $F \in L_m(\rho)$*;*
2. $\mathbf{M}(P) \leqslant \gamma \mathbf{M}(T)$*,* $\mathbf{M}(\partial P) \leqslant \gamma \mathbf{M}(\partial T)$*;*
3. $\mathbf{M}(S) \leqslant \gamma\rho \mathbf{M}(T)$*,* $\mathbf{M}(R) \leqslant \gamma\rho \mathbf{M}(\partial T)$*;*
4. $\operatorname{supp} P \cup \operatorname{supp} S \subseteq \{x : \operatorname{dist}(x, \operatorname{supp} T) < 2\rho\sqrt{n}\}$*;*
5. $\operatorname{supp} \partial P \cup \operatorname{supp} R \subseteq \{x : \operatorname{dist}(x, \operatorname{supp} \partial T) < 2\rho\sqrt{n}\}$.

As a corollary of the deformation theorem, we get the following.

Proposition 2.6. *Fix $K \in \mathcal{K}(\mathbb{R}^n)$. For any $c > 0$, the set*

$$\mathbf{A}(K,c) = \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\} \tag{2.9}$$

is totally bounded under the flat norm $\mathbf{F}$.

Proof. Fix $\varepsilon > 0$, define $\rho = \varepsilon/4c\gamma$ (where γ is the constant appearing in Theorem 2.5) and consider the set $\Phi \subseteq \mathbf{N}_m(\mathbb{R}^n)$ whose elements are all currents

$$P = \sum_{F \in L_m(\rho), F \cap \{x \in \mathbb{R}^n : \operatorname{dist}(x,K) \leqslant \rho\sqrt{2n}\} \neq \emptyset} \alpha_F [\![F]\!].$$

for which $\mathbf{N}(T) \leqslant 2c\gamma$. Call F^j,$1 \leqslant j \leqslant p$ the elements of $L_m(\rho)$ meeting the set $\{x \in \mathbb{R}^n : \operatorname{dist}(x,K) \leqslant \rho\sqrt{2n}\}$ and observe one can write

$$P = \sum_{j=1}^{p} \alpha_j(P) [\![F^j]\!]$$

for each $P \in \Phi$. Of course, we get for $P \in \Phi$

$$\rho^m \max_{1 \leqslant j \leqslant p} |\alpha_j(P)| \leqslant \mathbf{M}(P) \leqslant 2c\gamma \tag{2.10}$$

and

$$\mathbf{M}(P) \leqslant \rho^m \sum_{j=1}^{p} |\alpha_j(P)|. \tag{2.11}$$

Given a sequence $(P_k) \subseteq \Phi$, choose a subsequence $(P_{k_l}) \subseteq (P_k)$ for which $(\alpha_j(P_{k_l}))$ converges to $\alpha_j^* \in \mathbb{R}$ for each $1 \leqslant j \leqslant p$. Defining $P^* \in \Phi$ by $P^* = \sum_{i=1}^{p} \alpha_j^* [\![F^j]\!]$ we get

$$\mathbf{F}(P_{k_l} - P) \leqslant \mathbf{M}(P_{k_l} - P) \leqslant \rho^m \sum_{j=1}^{p} |\alpha_j(P_{k_l}) - \alpha_j^*|$$

from which it follows that (P_{k_l}) $\mathbf{F}$-converges to T. Consequently Φ is $\mathbf{F}$-compact. In particular there exists $P_i \in \Phi$, $1 \leqslant i \leqslant r$ for which

$$\Phi \subseteq \bigcup_{i=1}^{r} \left\{ T \in \mathbf{N}_m(\mathbb{R}^n) : \mathbf{F}(T - P_i) \leqslant \frac{\varepsilon}{2} \right\}.$$

On the other hand given $T \in \mathbf{A}(K,c)$, Theorem 2.5 ensures the existence of $P \in \Phi$ and of R, S satisfying the conclusions 1.-5. of Theorem 2.5. In particular

$$\mathbf{F}(T-P) \leqslant \mathbf{M}(T-P-\partial S) + \mathbf{M}(S) = \mathbf{M}(R) + \mathbf{M}(S) \leqslant 2c\gamma\rho = \frac{\varepsilon}{2}$$

and we obtain

$$\mathbf{A}(K,c) \subseteq \bigcup_{i=1}^{r} \{T \in \mathbf{N}_m(\mathbb{R}^n) : \mathbf{F}(T-P_i) \leqslant \varepsilon\}.$$

The proof is complete. □

Each Cauchy sequence in $\mathbf{A}(K,c)$ converges to a current in $\mathbf{A}(K,c)$.

Proposition 2.7. *Fix $K \in \mathcal{K}(\mathbb{R}^n)$. For each $c > 0$, any $\mathbf{F}$-Cauchy sequence $(T_k) \subseteq \mathbf{A}(K,c)$, $\mathbf{F}$-converges to $T \in \mathbf{A}(K,c)$.*

Proof. Fix a $\mathbf{F}$-Cauchy sequence $(T_k) \subseteq \mathbf{A}(K,c)$. In particular $(\langle T_k, \omega\rangle)$ is a Cauchy sequence of real numbers for each $\omega \in \mathscr{D}(\mathbb{R}^n)$, and the formula $\langle T, \omega\rangle = \lim_{k\to\infty}\langle T_k, \omega\rangle$ defines a normal current $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ satisfying

$$\mathbf{N}(T) \leqslant \varliminf_{k\to\infty} \mathbf{N}(T_k) \leqslant c.$$

Moreover given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\mathbf{F}(T_k - T_l) \leqslant \varepsilon$ holds for $k, l \geqslant m$. Fix $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\mathbf{M}(\omega) \leqslant 1$ and $\mathbf{M}(d\omega) \leqslant 1$. We get for $k \geqslant m$

$$\langle T_k - T, \omega\rangle = \lim_{l\to\infty}\langle T_k - T_l, \omega\rangle \leqslant \varlimsup_{l\to\infty} \mathbf{F}(T_k - T_l) \leqslant \varepsilon;$$

and we obtain $\mathbf{F}(T_k - T) \leqslant \varepsilon$ for $k \geqslant m$. So (T_k) $\mathbf{F}$-converges to T. □

It is now easy to conclude the set in (2.9) is $\mathbf{F}$-compact.

Theorem 2.8. *Fix $K \in \mathcal{K}(\mathbb{R}^n)$. For any $c > 0$, the set*

$$\mathbf{A}(K,c) = \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\}$$

is $\mathbf{F}$-compact.

Proof. According to Fréchet's theorem, $\mathbf{A}(K,c)$ being complete and totally bounded under the flat norm $\mathbf{F}$, is also $\mathbf{F}$-compact. □

Chapter 3

Charges in Euclidean spaces

Using Theorem 2.8, we define a topology on $\mathbf{N}_m(X)$ which will be our interest in the sequel.

3.1 The m-charges in $\mathbb{R}^n$

First, observe the flat norm $\mathbf{F}$ endows $\mathbf{N}_m(\mathbb{R}^n)$ with a locally convex topology $\mathfrak{S}$.

For $K \in \mathfrak{K}(\mathbb{R}^n)$ and $c > 0$, observe

$$\mathbf{A}(K, c) = \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\} \subseteq \mathbf{N}_m(\mathbb{R}^n)$$

is a compact, convex subset of $(\mathbf{N}_m(\mathbb{R}^n), \mathfrak{S})$ following Theorem 2.8.

Call $\mathfrak{T}$ the strongest topology on $\mathbf{N}_m(\mathbb{R}^n)$ such that for each $K \in \mathfrak{K}(\mathbb{R}^n)$ and each $c > 0$ the inclusion

$$[\mathbf{A}(K, c), \mathfrak{S}] \hookrightarrow [\mathbf{N}_m(\mathbb{R}^n), \mathfrak{T}]$$

is continuous. Then $\mathfrak{T}$ is locally convex, sequential and the following conditions hold:

1. a set $A \subseteq \mathbf{N}_m(\mathbb{R}^n)$ is $\mathfrak{T}$-closed if and only if $\mathbf{A}(K, c) \cap A$ is $\mathfrak{S}$-closed for every $K \in \mathfrak{K}(\mathbb{R}^n)$ and $c > 0$;

2. given any topological space Y and a map $f : (\mathbf{N}_m(\mathbb{R}^n), \mathfrak{T}) \to Y$, it is necessary and sufficient for f to be continuous that
$$f \restriction \mathbf{A}(K, c) : [\mathbf{A}(K, c), \mathfrak{S}] \to Y$$
be continuous for every $K \in \mathfrak{K}(\mathbb{R}^n)$ and $c > 0$;

3. a sequence $(T_k) \subseteq \mathbf{N}_m(\mathbb{R}^n)$ $\mathfrak{T}$-converges to an $T \in \mathbf{N}_m(\mathbb{R}^n)$ if and only if there exists $K \in \mathfrak{K}(\mathbb{R}^n)$ and $c > 0$ for which $(T_k) \subseteq \mathbf{A}(K, c)$ and (T_k) $\mathfrak{S}$-converges to T.

We introduce m-*charges* in $\mathbb{R}^n$.

Definition 3.1. A linear functional $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ is called an m-*charge* in case it is $\mathfrak{T}$-continuous.

The following conditions give characterizations of m-charges.

Proposition 3.2. *Given a linear functional $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$, the following conditions are equivalent:*

1. *F is an m-charge in $\mathbb{R}^n$;*

2. *for each $\varepsilon > 0$ and any $K \in \mathfrak{K}(\mathbb{R}^n)$, there exists $\delta > 0$ such that one has $F(T) \leqslant \varepsilon$ whenever $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ satisfies $\mathbf{F}(T) \leqslant \delta$ and $\mathbf{N}(T) \leqslant \frac{1}{\varepsilon}$;*

3. *for each $\varepsilon > 0$ and any $K \in \mathfrak{K}(\mathbb{R}^n)$, there exists $\theta > 0$ such that one has $F(T) \leqslant \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T)$ for every $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$;*

4. *one has $\lim_k F(T_k) = 0$ whenever $(T_k) \subseteq \mathbf{N}_m(\mathbb{R}^n)$ satisfies $\operatorname{supp} T_k \subseteq K$ for some $K \in \mathfrak{K}(\mathbb{R}^n)$, $\sup_k \mathbf{N}(T_k) < \infty$ and $\mathbf{F}(T_k) \to 0, k \to \infty$.*

Proof. To infer 2. from 1. assume F is an m-charge, fix $K \in \mathfrak{K}(\mathbb{R}^n)$ and $\varepsilon > 0$. As $F \restriction \mathbf{A}\left(K, \frac{1}{\varepsilon}\right)$ is $\mathfrak{S}$-continuous we find $\delta > 0$ such that $F(T) \leqslant \varepsilon$ holds for each $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ satisfying $\mathbf{N}(T) \leqslant \frac{1}{\varepsilon}$ and $\mathbf{F}(T) \leqslant \delta$.

We prove condition 2. implies condition 3. Given $0 < \varepsilon < 1$ and a compact set $K \in \mathfrak{K}(\mathbb{R}^n)$, fix $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$. Find $\delta > 0$ given by condition 2. Assume first that $\mathbf{N}(T) = 1$. If moreover $\mathbf{F}(T) \leqslant \delta$ then one gets $F(T) \leqslant \varepsilon$. In case $\mathbf{F}(T) > \delta$ observe $F(\delta T/\mathbf{F}(T)) \leqslant \varepsilon$ that is $F(T) \leqslant (\varepsilon/\delta)\mathbf{F}(T)$. In either case writing $\theta = \varepsilon/\delta$ one gets

$$F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon. \tag{3.1}$$

In case $\mathbf{N}(T) \neq 1$ either $T = 0$ or $T \neq 0$, in which case applying (3.1) to $T/\mathbf{N}(T)$ yields

$$F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T).$$

Is is obvious that condition 3. implies condition 4.

To prove that any linear functional $F : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$ satisfying 4. is an m-charge, recall F is $\mathfrak{T}$-continuous if and only if F is $\mathfrak{S}$-continuous on $\mathbf{A}(K, c)$ for each $K \in \mathfrak{K}(\mathbb{R}^n)$ and each $c > 0$. So fix $K \in \mathfrak{K}(\mathbb{R}^n)$, $c > 0$ and let $(T_k) \subseteq \mathbf{A}(K, c)$ $\mathfrak{S}$-converge to $T \in \mathbf{A}(K, c)$ as $k \to \infty$. In particular $\mathbf{F}(T_k - T) \to 0, k \to \infty$. Using 4., we obtain $F(T_k) \to F(T), k \to \infty$ so that F is sequentially continuous on $(\mathbf{A}(K, c), \mathfrak{S})$. As $\mathfrak{S}$ is sequential, F is continuous on $(\mathbf{A}(K, c), \mathfrak{S})$ as was to be proved. $\square$

We give an interesting property of m-charges.

Proposition 3.3. *Assume the directed system $(T_i)_{i \in I} \subseteq \mathbf{N}_m(\mathbb{R}^n)$ satisfies the following conditions:*

1. $\sup_{i \in I} \mathbf{N}(T_i) \leqslant c < \infty$;
2. *there exists $K \in \mathfrak{K}(\mathbb{R}^n)$ such that one has* $\operatorname{supp} T_i \subseteq K$, $i \in I$;
3. *for each $\omega \in \mathscr{D}^m(\mathbb{R}^n)$, $(\langle T_i, \omega \rangle)_{i \in I}$ converges to* 0.

Then the directed system $(\mathbf{F}(T_i))_{i \in I}$ converges to 0.

Proof. Proceed by contradiction and assume that one can find an $\varepsilon > 0$ such that given $i \in I$, there exists $j \in I$ satisfying $j \geqslant i$ for which $\mathbf{F}(T_j) \geqslant \varepsilon$. Let

$C = \{j \in I : \mathbf{F}(T_j) \geqslant \varepsilon\}$ and notice that for any $j_1, j_2 \in C$ one can find $j_3 \in C$ so that $j_1 \leqslant j_3$ and $j_2 \leqslant j_3$. The directed system

$$(T_j)_{j \in C} \subseteq \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\} \tag{3.2}$$

has a cluster point in the flat norm topology, using Theorem 2.8. So there exist $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ and $j \in C$ for which $\mathbf{N}(T) \leqslant c$ and $\mathbf{F}(T_j - T) \leqslant \frac{\varepsilon}{2}$. It yields

$$\mathbf{F}(T) \geqslant \mathbf{F}(T_j) - \mathbf{F}(T_j - T) \geqslant \frac{\varepsilon}{2} > 0.$$

Yet $(T_j)_{j \in C}$ weakly*-converges to 0 so that 0 is the only cluster point of $(T_j)_{j \in C}$ in the weak*-topology. As T is also a weak*-cluster point of $(T_j)_{j \in C}$, (3.2) yields a contradiction. □

The following result follows readily from Propositions 3.2 and 3.3.

Proposition 3.4. *Suppose F is an m-charge and the directed system $(T_i)_{i \in I} \subseteq \mathbf{N}_m(\mathbb{R}^n)$ satisfies the following conditions:*

1. $\sup_{i \in I} \mathbf{N}(T_i) \leqslant c < \infty$;
2. *there exists $K \in \mathcal{K}(\mathbb{R}^n)$ such that one has* $\operatorname{supp} T_i \subseteq K$, $i \in I$;
3. *for each $\omega \in \mathscr{D}^m(\mathbb{R}^n)$, $(\langle T_i, \omega \rangle)_{i \in I}$ converges to* 0.

Then the directed system $(F(T_i))_{i \in I}$ converges to 0.

We give an important class of m-charges in $\mathbb{R}^n$.

Example 3.5. Let $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ and $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$. The linear map

$$\Theta(\omega, \zeta) : \mathbf{N}_m(\mathbb{R}^n) \to \mathbb{R}$$

is defined by the formula

$$\Theta(\omega, \zeta)(T) = \langle T, \omega \rangle + \langle \partial T, \zeta \rangle.$$

We show that $\Theta(\omega, \zeta)$ is an m-charge in $\mathbb{R}^n$. To that purpose, fix $\varepsilon > 0$, $K \in \mathcal{K}(\mathbb{R}^n)$ and choose $\phi \in \mathscr{D}^m(\mathbb{R}^n)$ and $\psi \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ for which the inequalities $\|\omega(x) - \phi(x)\| \leqslant \varepsilon$ and $\|\zeta(x) - \psi(x)\| \leqslant \varepsilon$ hold for $x \in K$ and let $\theta = \max[\mathbf{M}(\phi), \mathbf{M}(d\phi)] + \mathbf{M}(d\psi)$. For $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ and $S \in \mathbf{N}^{\mathrm{loc}}_{m+1}(\mathbb{R}^n)$, compute

$$\begin{aligned}
\langle T, \omega \rangle &= \langle T, \omega - \phi \rangle + \langle T, \phi \rangle, \\
&= \langle T, \omega - \phi \rangle + \langle T - \partial S, \phi \rangle + \langle \partial S, \phi \rangle, \\
&= \langle T, \omega - \phi \rangle + \langle T - \partial S, \phi \rangle + \langle S, d\phi \rangle, \\
&\leqslant \varepsilon \mathbf{M}(T) + \mathbf{M}(T - \partial S)\mathbf{M}(\phi) + \mathbf{M}(S)\mathbf{M}(d\phi), \\
&\leqslant \varepsilon \mathbf{M}(T) + \max[\mathbf{M}(\phi), \mathbf{M}(d\phi)][\mathbf{M}(T - \partial S) + \mathbf{M}(S)],
\end{aligned}$$

and

$$\begin{aligned}
\langle \partial T, \zeta \rangle &= \langle \partial T, \zeta - \psi \rangle + \langle \partial (T - \partial S), \psi \rangle, \\
&= \langle \partial T, \zeta - \psi \rangle + \langle T - \partial S, d\psi \rangle, \\
&\leqslant \varepsilon \mathbf{M}(\partial T) + \mathbf{M}(T - \partial S)\mathbf{M}(d\psi), \\
&\leqslant \varepsilon \mathbf{M}(\partial T) + \mathbf{M}(d\psi)[\mathbf{M}(T - \partial S) + \mathbf{M}(S)],
\end{aligned}$$

Adding the two previous estimates yields

$$\Theta(\omega, \zeta) \leqslant \theta[\mathbf{M}(T - \partial S) + \mathbf{M}(S)] + \varepsilon \mathbf{N}(T).$$

As S is arbitrary, formula (2.7) allows us to conclude $\Theta(\omega, \zeta)$ is an m-charge.

We will show in chapter 5 that the m-charges given in example 3.5 are generic. More precisely, calling $\mathbf{CH}_m(\mathbb{R}^n)$ the space of all m-charges in $\mathbb{R}^n$, we will show that the map

$$\Theta : \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n), (\omega, \zeta) \mapsto \Theta(\omega, \zeta)$$

is surjective.

3.2 Charges vanishing on boundaries

One defines on $\mathbf{CH}_m(\mathbb{R}^n)$ a differential operator by duality.

Definition 3.6. Assume F is an m-charge in $\mathbb{R}^n$. For $m < n$, the exterior derivative of F denoted $\mathbf{d}F$ is the $(m+1)$-charge defined by the formula $\mathbf{d}F(T) = F(\partial T)$ for $T \in \mathbf{N}_{m+1}(\mathbb{R}^n)$. In case $m = n$ one let $\mathbf{d}F = 0$.

Remark 3.7. To justify $\mathbf{d}F$ is a m-charge in case $m \leqslant n$, fix $\varepsilon > 0$, a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ and choose $\theta > 0$ such that the inequality $F(S) \leqslant \theta\mathbf{F}(S) + \varepsilon\mathbf{N}(S)$ holds for each $S \in \mathbf{N}_{m,K}(\mathbb{R}^n)$. As one easily observes $\mathbf{F}(\partial T) \leqslant \mathbf{F}(T)$ and $\mathbf{N}(\partial T) \leqslant \mathbf{N}(T)$ we obtain for $T \in \mathbf{N}_{m+1,K}(\mathbb{R}^n)$

$$\mathbf{d}F(T) = F(\partial T) \leqslant \theta\mathbf{F}(\partial T) + \varepsilon\mathbf{N}(\partial T) \leqslant \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T),$$

and $\mathbf{d}F$ is an m-charge.

We get an easy description of the exterior derivative of $\Theta(\omega, \zeta)$.

Proposition 3.8. *Let* $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ *and* $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$. *Then* $\mathbf{d}\Lambda(\omega) = \Gamma(\omega)$, $\mathbf{d}\Gamma(\zeta) = 0$ *and* $\mathbf{d}\Theta(\omega, \zeta) = \Gamma(\omega)$. *In particular,*

$$\Theta(\omega, \zeta) = \Lambda(\omega) + \mathbf{d}\Lambda(\zeta).$$

Given $m \geqslant 1$ and $F \in \mathbf{CH}_m(\mathbb{R}^n)$, we construct $P(F) : \mathbf{N}_{m-1}(\mathbb{R}^n) \to \mathbb{R}$ in the following way. Define $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by the formula $h(t, x) = tx$ and let for $T \in \mathbf{N}_{m-1}(\mathbb{R}^n)$

$$P(F)(T) = F(\boldsymbol{\delta}_0 \divideontimes T)$$

where $\boldsymbol{\delta}_0 \divideontimes T = h_{\#}([\![0,1]\!] \times T)$.

Proposition 3.9. *For each* $F \in \mathbf{CH}_m(\mathbb{R}^n)$, $P(F)$ *is an* $(m-1)$*-charge.*

Claim 3.10. Given a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ there exists $c_1 > 0$ such that for every $T \in \mathbf{N}_{m-1,K}(\mathbb{R}^n)$ one has $\mathbf{F}(\boldsymbol{\delta}_0 \divideontimes T) \leqslant c_1\mathbf{F}(T)$.

Proof of the claim. Using (2.3) and (2.4) compute for $T \in \mathbf{N}_{m-1,K}(\mathbb{R}^n)$

$$\begin{aligned}\mathbf{F}(\boldsymbol{\delta}_0 \divideontimes T) = \mathbf{F}(h_{\#}([\![0,1]\!] \times T)) \leqslant \max\{&[\mathrm{Lip}(h \restriction [0,1] \times K)]^m,\\ &[\mathrm{Lip}(h \restriction [0,1] \times K)]^{m+1}\}\, \mathbf{F}([\![0,1]\!] \times T),\end{aligned}$$

and

$$\mathbf{F}([\![0,1]\!] \times T) \leqslant \mathbf{N}([\![0,1]\!])\mathbf{F}(T) \leqslant 3\mathbf{F}(T).$$

The claim is proved. $\square$

Claim 3.11. Given a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ there exists $c_2 > 0$ such that for every $T \in \mathbf{N}_{m-1,K}(\mathbb{R}^n)$ one has $\mathbf{N}(\boldsymbol{\delta}_0 \mathbin{⋇} T) \leqslant c_2\mathbf{N}(T)$.

Proof of the claim. We get for $T \in \mathbf{N}_{m-1,K}(\mathbb{R}^n)$

$$\mathbf{M}(h_{\#}([\![0,1]\!] \times T)) \leqslant [\mathrm{Lip}(h \restriction [0,1] \times K)]^m \mathbf{M}([\![0,1]\!] \times T)$$

while $\mathbf{M}([\![0,1]\!] \times T) \leqslant \mathbf{M}(T)$. On the other hand the homotopy formula for currents (Proposition 2.2) provides

$$\partial h_{\#}([\![0,1]\!] \times T) = T - h_{\#}([\![0,1]\!] \times \partial T)$$

so that

$$\mathbf{M}(\partial h_{\#}([\![0,1]\!] \times T)) \leqslant \mathbf{M}(T) + [\mathrm{Lip}(h \restriction [0,1] \times K)]^m \mathbf{M}(\partial T).$$

The claim is proved. $\square$

Proof of Proposition 3.9. Fix $\varepsilon > 0$, a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ and let $\eta > 0$ satisfy

$$F(S) \leqslant \eta \mathbf{F}(S) + \varepsilon c_2^{-1} \mathbf{N}(S)$$

for each $S \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ and let $\theta = c_1\eta$. For $T \in \mathbf{N}_{m-1}(\mathbb{R}^n)$ we get

$$P(F)(T) = F(\boldsymbol{\delta}_0 \mathbin{⋇} T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$$

and $P(F)$ is an $(m-1)$-charge. $\square$

Proposition 3.12. *Assume that $F \in \mathbf{CH}_m(\mathbb{R}^n)$ is such that $\mathbf{d}F = 0$. Then $F = \mathbf{d}P(F)$.*

Proof. Compute for $T \in \mathbf{N}_m(\mathbb{R}^n)$, using the homotopy formula (Proposition 2.2)

$$P(F)(\partial T) = F(h_\#([\![0,1]\!] \times \partial T)) = F(T) - F(\partial h_\#([\![0,1]\!] \times T))$$

which equals $F(T)$ as $\mathbf{d}F = 0$. □

We state now a Poincaré-like lemma in the context of m-charges.

Theorem 3.13. *Let $F \in \mathbf{CH}_m(\mathbb{R}^n)$ satisfy $\mathbf{d}F = 0$. There exists $\Phi \in \mathbf{CH}_{m-1}(\mathbb{R}^n)$ satisfying $\mathbf{d}\Phi = F$.*

Proof. Theorem 3.13 follows readily from Proposition 3.12. □

Using the representation theorem (Theorem 5.15) we can prove Theorem 3.13 in a way interesting for itself.

3.3 Top-dimensional charges in $\mathbb{R}^n$

To $\varphi \in \mathscr{D}(\mathbb{R}^n)$ we associate the current $T_\varphi \in \mathbf{N}_n(\mathbb{R}^n)$ defined by the formula

$$\langle T_\varphi, \psi \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \rangle = \int_{\mathbb{R}^n} \varphi\psi \, dx,$$

in such a way that $\mathbf{F}(T_\varphi) = \mathbf{M}(T_\varphi) = |\varphi|_1$. Moreover the integration by parts formula yields the equality

$$-\int_{\mathbb{R}^n} w \cdot \nabla\varphi \, dx = \left\langle \partial T_\varphi, \sum_{i=1}^{n} (-1)^{i-1} w_i dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right\rangle \quad (3.3)$$

for $w = (w_1, \ldots, w_n) \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^n)$. In particular we have

$$|\nabla\varphi|_1 \leqslant \mathbf{M}(\partial T_\varphi) \leqslant \sqrt{n}|\nabla\varphi|_1.$$

Each n-charge $F \in \mathbf{CH}_n(\mathbb{R}^n)$ induces on $\mathscr{D}(\mathbb{R}^n)$ a linear functional $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ by defining $\mathscr{F}(\varphi) = F(T_\varphi)$. Given $\varepsilon > 0$ and $K \in \mathcal{K}(\mathbb{R}^n)$, there exists $\theta > 0$ such that

$$\mathscr{F}(\varphi) \leqslant \theta \mathbf{F}(T_\varphi) + \varepsilon \mathbf{N}(T_\varphi) \leqslant (\theta + \varepsilon)|\varphi|_1 + \varepsilon\sqrt{n}|\nabla\varphi|_1$$

holds for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subseteq K$. In particular $\mathscr{F}$ is a distribution.

Conversely, fix a linear functional $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ for which to every $\varepsilon > 0$ and $K \in \mathcal{K}(\mathbb{R}^n)$ corresponds $\theta > 0$ so that

$$\mathscr{F}(\varphi) \leqslant \theta|\varphi|_1 + \varepsilon|\nabla\varphi|_1 \tag{3.4}$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subseteq K$.

In particular, letting $\mathbf{D}(\mathbb{R}^n) = \{T_\varphi : \varphi \in \mathscr{D}(\mathbb{R}^n)\}$ we define a functional $\tilde{F} : \mathbf{D}(\mathbb{R}^n) \to \mathbb{R}$ by the formula $\tilde{F}(T_\varphi) = \mathscr{F}(\varphi)$. Hence to every $\varepsilon > 0$ and $K \in \mathcal{K}(\mathbb{R}^n)$ corresponds $\theta > 0$ so that

$$\tilde{F}(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T) \tag{3.5}$$

holds for each $T \in \mathbf{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp} T \subseteq K$.

Given any $T \in \mathbf{N}_n(\mathbb{R}^n)$, the current T_k defined for any integer $k \geqslant 1$ by smoothing T as in (2.8), belongs to $\mathbf{D}(\mathbb{R}^n)$ as there exists $\xi_k \in \mathscr{D}(\mathbb{R}^n, \Lambda^n \mathbb{R}^n)$ together with $\varphi_k \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\xi_k = \varphi_k e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and

$$\langle T_k, \psi dx_1 \wedge dx_2 \wedge \ldots dx_n \rangle = \int_{\mathbb{R}^n} \langle \xi_k, \psi dx_1 \wedge dx_2 \wedge \ldots dx_n \rangle \, dx = \int_{\mathbb{R}^n} \varphi_k \psi \, dx$$

for each $\psi \in \mathscr{D}(\mathbb{R}^n)$; that is $T_k \in \mathbf{D}(\mathbb{R}^n)$.

Claim 3.14. The formula $F(T) = \lim_{k\to\infty} \tilde{F}(T_k)$ defines an n-charge in $\mathbb{R}^n$.

Proof of the claim. Fix $T \in \mathbf{N}_n(\mathbb{R}^n)$ and $\varepsilon > 0$. Let

$$K = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp} T) \leqslant 1\}$$

and choose $\theta > 0$ for which

$$\tilde{F}(S) \leqslant \theta \mathbf{F}(S) + \frac{\varepsilon}{4[\mathbf{N}(T)+1]}\mathbf{N}(S)$$

holds whenever $S \in \mathbf{D}_n(\mathbb{R}^n)$ satisfy $\operatorname{supp} S \subseteq K$. Let $j \geqslant \frac{4\theta\mathbf{N}(T)}{\varepsilon}$ be an integer. For integers $k, l \geqslant j$, compute

$$\begin{aligned}\tilde{F}(T_k - T_l) &\leqslant \theta\mathbf{F}(T_k - T_l) + \frac{\varepsilon[\mathbf{N}(T_k)+\mathbf{N}(T_l)]}{4[\mathbf{N}(T)+1]} \\ &\leqslant \theta[\mathbf{F}(T_k - T) + \mathbf{F}(T_l - T)] + \frac{\varepsilon}{2} \leqslant \frac{2\theta\mathbf{N}(T)}{j} + \frac{\varepsilon}{2} \leqslant \varepsilon.\end{aligned}$$

Consequently the formula $F(T) = \lim_{k\to\infty} \tilde{F}(T_k)$ defines a linear functional F on $\mathbf{N}_n(\mathbb{R}^n)$.

To show F is an n-charge in $\mathbb{R}^n$, fix $\varepsilon > 0$, $K \in \mathcal{K}(\mathbb{R}^n)$, choose $T \in \mathbf{N}_n(\mathbb{R}^n)$ satisfying $\operatorname{supp} T \subseteq K$ and let

$$K' = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp} T) \leqslant 1\}.$$

Choose $\theta > 0$ for which $\tilde{F}(T) \leqslant \theta\mathbf{F}(T) + \frac{\varepsilon}{3}\mathbf{N}(T)$ holds for each $T \in \mathbf{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp} T \subseteq K'$. Assume first $T \neq 0$ and choose an integer $k \geqslant 1$ for which

$$|F(T) - \tilde{F}(T_k)| \leqslant \frac{\varepsilon}{3}\mathbf{N}(T) \quad \text{and} \quad \frac{\theta}{k} \leqslant \frac{\varepsilon}{3}.$$

Compute then

$$\begin{aligned}F(T) = [F(T) - \tilde{F}(T_k)] + \tilde{F}(T_k) &\leqslant \frac{\varepsilon}{3}\mathbf{N}(T) + \theta\mathbf{F}(T_k) + \frac{\varepsilon}{3}\mathbf{N}(T_k) \\ &\leqslant \theta\mathbf{F}(T) + \theta\mathbf{F}(T_k - T) + \frac{2\varepsilon}{3}\mathbf{N}(T) \leqslant \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T).\end{aligned}$$

The proof is complete as the last inequality is clear in case $T = 0$. □

We have associated to each linear function $\mathscr{F}$ on $\mathscr{D}(\mathbb{R}^n)$ satisfying (3.4) an n-charge F in $\mathbb{R}^n$. Moreover, we compute for $\psi \in \mathscr{D}(\mathbb{R}^n)$

$$\begin{aligned}&\langle (T_\varphi)_k, \psi dx_1 \wedge dx_2 \wedge \ldots dx_n\rangle = \langle T_\varphi, \rho_k * \psi\, dx_1 \wedge dx_2 \wedge \ldots dx_n\rangle = \\ &\int_{\mathbb{R}^n} \varphi(\rho_k * \psi)\, dx = \int_{\mathbb{R}^n} (\rho_k * \varphi)\psi\, dx = \langle T_{\rho_k * \varphi}, \psi dx_1 \wedge dx_2 \wedge \ldots dx_n\rangle,\end{aligned}$$

so that we get

$$F(T_\varphi) = \lim_{k\to\infty} \tilde{F}[(T_\varphi)_k] = \lim_{k\to\infty} \mathscr{F}(\rho_k * \varphi) = \mathscr{F}(\varphi), \tag{3.6}$$

using (3.4).

In [DP04], Thierry De Pauw and Washek F. Pfeffer call *strong charges* the linear functionals on $\mathscr{D}(\mathbb{R}^n)$ satisfying (3.4).

Definition 3.15. A linear functional F on $\mathscr{D}(\mathbb{R}^n)$ is called a *strong charge* whenever to every $\varepsilon > 0$ and $K \in \mathcal{K}(\mathbb{R}^n)$ corresponds $\theta > 0$ such that $\mathscr{F}(\varphi) \leqslant \theta|\varphi|_1 + \varepsilon|\nabla\varphi|_1$ holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subseteq K$.

Equation (3.6) leads us to call *strong extension* of $\mathscr{F}$ the n-charge F associated to the strong charge $\mathscr{F}$ by the previous procedure, as it yields an n-charge F "extending" $\mathscr{F}$ in a natural way.

Chapter 4

The charge-cohomology of S^{n-1}

Fix $X \subseteq \mathbb{R}^n$ a closed set in the whole chapter.

4.1 Charges in a closed set

Fix $0 \leqslant m \leqslant n$ an integer in the whole section.

An m-dimensional normal current in X is an element of the space

$$\mathbf{N}_m(X) = \{T \in \mathbf{N}_m(\mathbb{R}^n) : \operatorname{supp} T \subseteq X\}.$$

For $K \in \mathcal{K}(X)$ we also define

$$\mathbf{N}_{m,K}(X) = \{T \in \mathbf{N}_m(X) : \operatorname{supp} T \subseteq K\}.$$

The space $\mathbf{N}_m(X)$ (and all its subspaces) is endowed by the flat norm topology $\mathcal{S}$.

Call $\mathcal{T}_X$ the strongest topology on $\mathbf{N}_m(X)$ such that for each $K \in \mathcal{K}(X)$ and each $c > 0$ the inclusion

$$[\mathbf{A}(K,c), \mathcal{S}] \hookrightarrow [\mathbf{N}_m(X), \mathcal{T}_X]$$

is continuous. Then $\mathcal{T}_X$ is locally convex, sequential and the following conditions hold:

1. a set $A \subseteq \mathbf{N}_m(X)$ is $\mathfrak{T}_X$-closed if and only if $\mathbf{A}(K,c) \cap A$ is $\mathfrak{S}$-closed for every $K \in \mathfrak{K}(X)$ and $c > 0$;

2. given any topological space Y and a map $f : (\mathbf{N}_m(X), \mathfrak{T}) \to Y$, it is necessary and sufficient for f to be continuous that

$$f \restriction \mathbf{A}(K,c) : [\mathbf{A}(K,c), \mathfrak{S}] \to Y$$

   be continuous for every $K \in \mathfrak{K}(X)$ and $c > 0$;

3. a sequence $(T_k) \subseteq \mathbf{N}_m(X)$ $\mathfrak{T}_X$-converges to an $T \in \mathbf{N}_m(X)$ if and only if there exists $K \in \mathfrak{K}(X)$ and $c > 0$ for which $(T_k) \subseteq \mathbf{A}(K,c)$ and (T_k) $\mathfrak{S}$-converges to T.

An m-charge in X is a $\mathfrak{T}_X$-continuous linear functional on $\mathbf{N}_m(X)$.

Definition 4.1. An m-charge in X is an element of the set $\mathbf{CH}_m(X) = [\mathbf{N}_m(X), \mathfrak{T}_X]^*$.

The following conditions give characterizations of m-charges in X.

Proposition 4.2. *Given a linear functional $F : \mathbf{N}_m(X) \to \mathbb{R}$, the following conditions are equivalent:*

1. *F is an m-charge in X;*

2. *for each $\varepsilon > 0$ and any $K \in \mathfrak{K}(X)$, there exists $\delta > 0$ such that one has $F(T) \leqslant \varepsilon$ whenever $T \in \mathbf{N}_{m,K}(X)$ satisfies $\mathbf{F}(T) \leqslant \delta$ and $\mathbf{N}(T) \leqslant \frac{1}{\varepsilon}$;*

3. *for each $\varepsilon > 0$ and any $K \in \mathfrak{K}(X)$, there exists $\theta > 0$ such that one has $F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$ for every $T \in \mathbf{N}_{m,K}(X)$;*

4. *one has $\lim_k F(T_k) = 0$ whenever $(T_k) \subseteq \mathbf{N}_m(X)$ satisfies $\operatorname{supp} T_k \subseteq K$ for some $K \in \mathfrak{K}(X)$, $\sup_k \mathbf{N}(T_k) < \infty$ and $\mathbf{F}(T_k) \to 0, k \to \infty$.*

Proof. The proof is virtually indentical to that of Proposition 3.2. $\square$

Any m-charge in X extends to an m-charge in $\mathbb{R}^n$.

Proposition 4.3. *Given $F \in \mathbf{CH}_m(X)$, there exists $\bar{F} \in \mathbf{CH}_m(\mathbb{R}^n)$ satisfying $\bar{F} \restriction \mathbf{N}_m(X) = F$.*

Proof. Fix $F \in \mathbf{CH}_m(X)$ and call $\mathfrak{T}'_X$ the locally convex topology induced on $\mathbf{N}_m(X)$ by the locally convex topology $\mathfrak{T}$ defined on $\mathbf{N}_m(\mathbb{R}^n)$ in Section 3.1. Observe $\mathfrak{T}'_X$ is sequential as $\mathbf{N}_m(X)$ is $\mathfrak{T}$-closed in $\mathbf{N}_m(\mathbb{R}^n)$. Fix $(T_k) \subseteq \mathbf{N}_m(X)$ and assume its $\mathfrak{T}'_X$-converges to $T \in \mathbf{N}_m(X)$. In particular there exist $K \in \mathfrak{K}(\mathbb{R}^n)$ and $c > 0$ such that $(T_k) \subseteq \mathbf{A}(K, c)$ and (T_k) $\mathfrak{S}$-converges to T. Hence we have $(T_k) \subseteq \mathbf{A}(K \cap X, c)$, (T_k) $\mathfrak{S}$-converges to T and it follows that (T_k) $\mathfrak{T}_X$-converges to T. In particular $F(T) = \lim_k F(T_k)$ and F is $\mathfrak{T}'_X$-sequentially continuous on $\mathbf{N}_m(X)$. As $\mathfrak{T}'_X$ is sequential, F is $\mathfrak{T}'_X$-continuous on $\mathbf{N}_m(X)$. On the other hand $\mathfrak{T}$ is locally convex and so there exists an m-charge $\bar{F}$ in $\mathbb{R}^n$ extending F to $\mathbf{N}_m(\mathbb{R}^n)$. □

Conversely, any m-charge in $\mathbb{R}^n$ induces an m-charge in X.

Proposition 4.4. *Given $F \in \mathbf{CH}_m(\mathbb{R}^n)$, we have $F \restriction \mathbf{N}_m(X) \in \mathbf{CH}_m(X)$.*

Proof. Given $\varepsilon > 0$ and $K \in \mathfrak{K}(X)$ we get $\theta > 0$ for which $F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T)$ and Proposition 4.2 allows us to conclude. □

4.2 A poincaré-like lemma

Fix an m-charge $F \in \mathbf{N}_m(X)$. In case $m = n$ we let $\mathbf{d}F = 0$. In case $0 \leqslant m \leqslant n-1$ we define the exterior derivative $\mathbf{d}F$ of F as the linear functional $\mathbf{d}F : \mathbf{N}_{m+1}(X) \to \mathbb{R}$ defined by $\mathbf{d}F(T) = F(\partial T)$ for each $T \in \mathbf{N}_{m+1}(X)$. One checks $\mathbf{d}F$ is an $(m+1)$-charge in X as in Remark 3.7.

Definition 4.5. We say that X is *L-contractible* if there exists a Lipschitz map $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions:

1. there exists $x_0 \in X$ such that $h(0, x) = x_0$ for each $x \in \mathbb{R}^n$;

2. $h(1, x) = x$ for each $x \in \mathbb{R}^n$;

3. for every $x \in X$ and $0 \leqslant t \leqslant 1$, $h(t,x) \in X$.

Given $1 \leqslant m \leqslant n$, the following construction associates to each m-charge in X an $(m-1)$-charge in X.

Proposition 4.6. *Given $1 \leqslant m \leqslant n$ and a Lipschitz map $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $h(t,x) \in X$ for each $x \in X$ and $0 \leqslant t \leqslant 1$, the formula*

$$P_h F(T) = F[h_\#([\![0,1]\!] \times T)]$$

defines an $(m-1)$-charge in X.

Proof. Follow the proof of Proposition 3.9, use Remark 2.3 and observe that $h_\#([\![0,1]\!] \times T) \in \mathbf{N}_m(X)$ for $T \in \mathbf{N}_{m-1}(X)$. □

Given $1 \leqslant m \leqslant n$, any m-charge in the L-contractible closed set X is the exterior derivative of an $(m-1)$-charge in X.

Theorem 4.7. *Assume X is L-contractible and $h : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map associated to X according to Definition 4.5. For an integer $1 \leqslant m \leqslant n$ and $F \in \mathbf{CH}_m(X)$, we construct $P(F) : \mathbf{N}_{m-1}(X) \to \mathbb{R}$ in the following way:*

$$P(F)(T) = F[h_\#([\![0,1]\!] \times T)].$$

Then, $P(F)$ is an $(m-1)$-charge in X satisfying $\mathbf{d}P(F) = F$.

Proof. Use Proposition 4.6 and observe that $h(0,\cdot)_\# T = 0$, $h(1,\cdot)_\# T = T$. The homotopy formula (see Remark 2.3) yields for $T \in \mathbf{N}_m(X)$

$$P(F)(\partial T) = F(h_\#([\![0,1]\!] \times \partial T)) = F(T) - F(\partial h_\#([\![0,1]\!] \times T))$$

which equals $F(T)$ as $\mathbf{d}F = 0$. □

4.3 Computing the charge-cohomology of S^{n-1}

We naturally get a cochain complex as follows:

$$0 \longrightarrow \mathbf{CH}_0(X) \xrightarrow{\mathbf{d}} \mathbf{CH}_1(X) \xrightarrow{\mathbf{d}} \cdots \xrightarrow{\mathbf{d}} \mathbf{CH}_n(X) \xrightarrow{\mathbf{d}} 0.$$

We define for $0 \leqslant m \leqslant n$

$$Z^m_{\text{ch}}(X) = \{F \in \mathbf{CH}_m(X) : \mathbf{d}F = 0\}$$

and

$$B^m_{\text{ch}}(X) = \begin{cases} 0 & \text{in case } m = 0, \\ \{\mathbf{d}F : F \in \mathbf{CH}_{m-1}(X)\} & \text{in case } 1 \leqslant m \leqslant n. \end{cases}$$

For $0 \leqslant m \leqslant n$ we naturally define the *charge-cohomology vector space*

$$H^m_{\text{ch}}(X) = Z^m_{\text{ch}}(X)/B^m_{\text{ch}}(X).$$

In case X is L-contractible, Theorem 4.7 ensures $H^m_{\text{ch}}(X) = \{0\}$ for each $0 \leqslant m \leqslant n$.

Our objective is to describe the vector spaces $H^m_{\text{ch}}(S^{n-1})$ associated to the unit-sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

The case $n = 1$

We describe

$$H^0_{\text{ch}}(S^0) = \{F \in \mathbf{CH}_0(S^0) : \mathbf{d}F = 0\}.$$

For that purpose, we will need a description of the space $\mathbf{N}_0(S^0)$.

Given $x \in X$, we let $\boldsymbol{\delta}_x \in \mathbf{N}_0(X)$ be defined by $\langle \boldsymbol{\delta}_x, \varphi \rangle = \varphi(x)$ for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

Claim 4.8. Assume $x \in \mathbb{R}^n$ and $T \in \mathbf{N}_0(\mathbb{R}^n)$ satisfy $\operatorname{supp} T \subseteq \{x\}$. Then there exists $\alpha \in \mathbb{R}$ such that $T = \alpha \boldsymbol{\delta}_x$.

Proof. First, observe that the equality $\varphi(x) = 0$ implies $\langle T, \varphi \rangle = 0$. Indeed, choose a sequence $(\chi_k) \subseteq \mathscr{D}(\mathbb{R}^n)$ so that

$$\chi_{[x-2^{-k-1}, x+2^{-k-1}]} \leqslant \chi_k \leqslant \chi_{[x-2^{-k}, x+2^{-k}]}$$

holds for each $k \in \mathbb{N}$ and observe

$$\langle T, \varphi \rangle = \lim_{k\to\infty} \langle T, (1 - \chi_k)\varphi \rangle = 0.$$

Fix $\varphi_0 \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\varphi_0(x) = 1$ and let $\alpha = \langle T, \varphi_0 \rangle$. We get for $\varphi \in \mathscr{D}(\mathbb{R}^n)$

$$\langle T, \varphi \rangle = \langle T, \varphi(x)\varphi_0 \rangle = \alpha\varphi(x).$$

□

Claim 4.9. Given $T \in \mathbf{N}_0(S^0)$, there exist real numbers α_-, α_+ so that $T = \alpha_-\boldsymbol{\delta}_{-1} + \alpha_+\boldsymbol{\delta}_1$.

Proof. Fix $\chi_0 \in C^\infty(\mathbb{R})$ satisfying $\chi_{(\infty,-1/2]} \leqslant \chi_0 \leqslant \chi_{(-\infty,1/2)}$ and choose $\chi \in \mathscr{D}(\mathbb{R})$ satisfying $\chi = \chi_0$ on a neighbourhood of $\operatorname{supp} T$. Define $T^\pm \in \mathbf{N}_0(\mathbb{R}^n)$ by $T^- = T \llcorner \chi$ and $T^+ = T \llcorner (1-\chi)$. As we get $\operatorname{supp} T^- \subseteq \{-1\}$, $\operatorname{supp} T^+ \subseteq \{1\}$ and $T = T^- + T^+$, we conclude using Claim 4.8. □

Claim 4.10. We have $\mathbf{N}_1(S^0) = \{0\}$.

Proof. Fix $T \in \mathbf{N}_1(S^0)$. Proceeding as in the proof of Claim 4.9, we find real numbers β_-, β_+ such that $\langle T, \varphi\, dx \rangle = \beta_-\varphi(-1) + \beta_+\varphi(1)$ for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$. For $\varphi \in \mathscr{D}(\mathbb{R})$ we get

$$\langle \partial T, \varphi \rangle = \langle T, \varphi' dx \rangle = \beta_-\varphi'(-1) + \beta_+\varphi'(1)$$

which is a Radon measure if and only if $\beta_- = \beta_+ = 0$, that is $T = 0$. □

Proposition 4.11. *We have* $H^0_{\text{ch}}(S^0) = \mathbb{R} \oplus \mathbb{R}$ *and* $H^1_{\text{ch}}(S^0) = \{0\}$.

Proof. Observe $H^0_{\text{ch}}(S^0) = \{F \in \mathbf{CH}_0(S^0) : \mathbf{d}F = 0\}$. As $\mathbf{N}_1(S^0) = \{0\}$ we get $\mathbf{d}F = 0$ for each $F \in \mathbf{CH}_1(S^0)$. Given real numbers F_-, F_+, the formula

$$F(\alpha_-\boldsymbol{\delta}_{-1} + \alpha_+\boldsymbol{\delta}_1) = \alpha_-F_- + \alpha_+F_+$$

defines an 0-charge in S^0. Conversely, given $F \in \mathbf{CH}_0(S^0)$ we get

$$F(\alpha_-\boldsymbol{\delta}_{-1} + \alpha_+\boldsymbol{\delta}_1) = \alpha_-F(\boldsymbol{\delta}_{-1}) + \alpha_+F(\boldsymbol{\delta}_1).$$

As a consequence, we have $H^0_{\text{ch}}(S^0) = \mathbf{CH}_0(S^0) \simeq \mathbb{R} \oplus \mathbb{R}$. On the other hand it is obvious that $H^1_{\text{ch}}(S^0) = \{0\}$. □

The case $n > 1$: a Mayer-Vietoris exact sequence

Let

$$D_+^{n-1} = \left\{ x \in S^{n-1} : x_n \geqslant -\frac{1}{2} \right\}, \quad D_-^{n-1} = \left\{ x \in S^{n-1} : x_n \leqslant \frac{1}{2} \right\}$$

and

$$C^{n-1} = \left\{ x \in S^{n-1} : -\frac{1}{2} \leqslant x_n \leqslant \frac{1}{2} \right\}.$$

Observe $D_+^{n-1} \cup D_-^{n-1} = S^{n-1}$ and $D_+^{n-1} \cap D_-^{n-1} = C^{n-1}$.

For $0 \leqslant m \leqslant n$, we construct mappings

$$\mathbf{CH}_m(S^{n-1}) \xrightarrow{\Phi_m} \mathbf{CH}_m(D_+^{n-1}) \oplus \mathbf{CH}_m(D_-^{n-1}) \xrightarrow{\Psi_m} \mathbf{CH}_m(C^{n-1})$$

in the following way: let

$$\Phi_m(F) = F \restriction \mathbf{N}_m(D_+^{n-1}) \oplus F \restriction \mathbf{N}_m(D_-^{n-1})$$

and

$$\Psi_m(F \oplus G) = F \restriction \mathbf{N}_m(C^{n-1}) - G \restriction \mathbf{N}_m(C^{n-1}).$$

Claim 4.12. The map $\Phi_m : \mathbf{CH}_m(S^{n-1}) \to \mathbf{CH}_m(D_+^{n-1}) \oplus \mathbf{CH}_m(D_-^{n-1})$ is injective.

Proof. Fix $\chi_0 \in C^\infty(\mathbb{R})$ satisfying $\chi_{(-\infty,-1/3]} \leqslant \chi_0 \leqslant \chi_{(-\infty,1/3]}$ and choose $\chi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\chi(x) = \chi_0(x_n)$ in a neighbourhood of S^{n-1}. Assuming $F \in \mathbf{CH}_m(S^{n-1})$ is nonzero, there exists $T \in \mathbf{N}_m(X)$ for which

$$0 \neq F(T) = F(T \llcorner \chi) + F(T \llcorner (1 - \chi)).$$

In particular one of the two real numbers $F(T \llcorner \chi)$ and $F(T \llcorner (1-\chi))$ is nonzero. Observing $T \llcorner \chi \in \mathbf{N}_m(D_-^{n-1})$ and $T \llcorner (1-\chi) \in \mathbf{N}_m(D_+^{n-1})$ we see that $F \restriction \mathbf{N}_m(D_-^{n-1})$ and $F \restriction \mathbf{N}_m(D_-^{n-1})$ cannot both vanish. $\square$

Claim 4.13. We have $\operatorname{im} \Phi_m = \ker \Psi_m$.

Proof. It is obvious that $\operatorname{im}\Phi_m \subseteq \ker \Psi_m$. On the other hand, fix $F_+ \oplus F_- \in \mathbf{CH}_m(D_+^{n-1}) \oplus \mathbf{CH}_m(D_-^{n-1})$ such that $F_+ \restriction C^{n-1} = F_- \restriction C^{n-1}$. Fix $\chi \in \mathscr{D}(\mathbb{R}^n)$ as in the proof of Claim 4.12 and define a linear functional F on $\mathbf{N}_m(S^{n-1})$ by the formula $F(T) = F_+(T \llcorner (1-\chi)) + \mathbf{F}_-(T \llcorner \chi)$. Use Lemma 2.4 to infer F is an m-charge in S^{n-1}. For $T \in \mathbf{N}_m(D_+^{n-1})$ observe $\operatorname{supp}(T \llcorner \chi) \subseteq C^{n-1}$. Consequently $F_-(T \llcorner \chi) = F_+(T \llcorner \chi)$ and

$$\begin{aligned} F(T) &= F_+(T \llcorner (1-\chi)) + \mathbf{F}_-(T \llcorner \chi) \\ &= F_+(T \llcorner (1-\chi)) + \mathbf{F}_+(T \llcorner \chi) = F_+(T). \end{aligned}$$

Similarly, we get $F(T) = F_-(T)$ for each $T \in \mathbf{N}_m(D_-^{n-1})$. Shortly,

$$\Phi_m(F) = F \restriction \mathbf{N}_m(D_+^{n-1}) \oplus F \restriction \mathbf{N}_m(D_-^{n-1}) = F_+ \oplus F_-.$$

The proof is complete. $\square$

Claim 4.14. The map $\Psi_m : \mathbf{CH}_m(D_+^{n-1}) \oplus \mathbf{CH}_m(D_-^{n-1}) \to \mathbf{CH}_m(C^{n-1})$ is surjective.

Proof. Fix $F \in \mathbf{CH}_m(C^{n-1})$ and choose $G \in \mathbf{CH}_m(\mathbb{R}^n)$ satisfying $G \restriction \mathbf{N}_m(C^{n-1}) = F$ according to Proposition 4.3. Let $G_+ = G \restriction \mathbf{N}_m(D_+^{n-1})$, observe $G_+ \in \mathbf{CH}_m(D_+^{n-1})$ according to Proposition 4.4 and compute

$$\Psi_m(G_+ \oplus 0) = F.$$

The proof is complete. $\square$

We can summarize the previous informations by saying the following sequence is exact:

$$0 \longrightarrow \mathbf{CH}_m(S^{n-1}) \xrightarrow{\Phi_m} \mathbf{CH}_m(D_+^{n-1}) \oplus \mathbf{CH}_m(D_-^{n-1}) \xrightarrow{\Psi_m} \mathbf{CH}_m(C^{n-1}) \longrightarrow 0.$$

Moreover, observe the following diagram is commutative:

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\mathbf{CH}_{m+1}(S^{n-1}) \arrow[r, "\Phi_{m+1}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m+1}(D_-^{n-1}) \oplus \mathbf{CH}_{m+1}(D_+^{n-1}) \arrow[r, "\Psi_{m+1}"] \arrow[u, "\mathbf{d}\oplus\mathbf{d}"] & \mathbf{CH}_{m+1}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\mathbf{CH}_{m}(S^{n-1}) \arrow[r, "\Phi_{m}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m}(D_-^{n-1}) \oplus \mathbf{CH}_{m}(D_+^{n-1}) \arrow[r, "\Psi_{m}"] \arrow[u, "\mathbf{d}\oplus\mathbf{d}"] & \mathbf{CH}_{m}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\mathbf{CH}_{m-1}(S^{n-1}) \arrow[r, "\Phi_{m-1}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m-1}(D_-^{n-1}) \oplus \mathbf{CH}_{m-1}(D_+^{n-1}) \arrow[r, "\Psi_{m-1}"] \arrow[u, "\mathbf{d}\oplus\mathbf{d}"] & \mathbf{CH}_{m-1}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\vdots \arrow[u, "\mathbf{d}"] & \vdots \arrow[u, "\mathbf{d}"] & \vdots \arrow[u, "\mathbf{d}"]
\end{tikzcd}$$

Consequently we get a long exact sequence in cohomology

$$\cdots \xrightarrow{\Delta^{m-1}} H^m_{\mathrm{ch}}(S^{n-1}) \xrightarrow{H^m(\Psi)} H^m_{\mathrm{ch}}(D_+^{n-1}) \oplus H^m_{\mathrm{ch}}(D_-^{n-1}) \xrightarrow{H^m(\Psi)}$$
$$H^m_{\mathrm{ch}}(C^{n-1}) \xrightarrow{\Delta^m} H^{m+1}_{\mathrm{ch}}(S^{n-1}) \longrightarrow \cdots \qquad (4.1)$$

Computing $H^*_{\mathrm{ch}}(D_\pm^{n-1})$ Choose a map $q \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ defined in a neighbourhood of D_-^{n-1} by the formula

$$q_i(x) = \frac{2x_i}{1+x_n}.$$

Conversely, define a map $p \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^n)$ by

$$p_i(y) = \begin{cases} \frac{4y_i}{|y|^2+4} & \text{for } 1 \leqslant i \leqslant n-1, \\ \frac{|y|^2-4}{|y|^2+4} & \text{for } i = n. \end{cases}$$

Define a map $h \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ using p and q by the formula

$$h(t, x) = p(tq(x)).$$

We get $h(1,x) = x$ and $h(0,x) = (0,\ldots,0,-1) \in D_-^{n-1}$ for $x \in D_-^{n-1}$. For $0 \leqslant t \leqslant 1$ and $x \in D_-^{n-1}$ one readily checks $h(t,x) \in D_-^{n-1}$. According to Theorem 4.7 we get $H^m(D_-^{n-1}) = \{0\}$ for each $1 \leqslant m \leqslant n$.

To compute $H^0(D_-^{n-1})$ fix $F \in \mathbf{CH}_0(D_-^{n-1})$, assume $\mathbf{d}F = 0$ and choose $\varphi \in C(X)$ such that $F(T) = \langle T, \varphi\rangle$ holds for each $T \in \mathbf{N}_0(D_-^{n-1})$. Given $x \neq y$ in D_-^{n-1}, let $T = p_\# [\![q(x), q(y)]\!]$ and observe $\partial T = \boldsymbol{\delta}_y - \boldsymbol{\delta}_x$. Compute

$$0 = F(\partial T) = \langle \boldsymbol{\delta}_y, \varphi\rangle - \langle \boldsymbol{\delta}_x, \varphi\rangle = \varphi(y) - \varphi(x),$$

and observe φ is constant on D_-^{n-1}. Conversely, each constant function φ on D_-^{n-1} defines a 0-charge in D_-^{n-1} with zero exterior derivative. Consequently, $H^0_{\text{ch}}(D_-^{n-1}) \simeq \mathbb{R}$.

Similarly, we compute $H^0(D_+^{n-1}) \simeq \mathbb{R}$ and $H^m(D_+^{n-1}) = \{0\}$ for $1 \leqslant m \leqslant n$.

Computing $H^*_{\text{ch}}(C^{n-1})$ For $x \in \mathbb{R}^n$, let $\hat{x} = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Choose $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^{n-1})$ satisfying $f(x) = \frac{\hat{x}}{|\hat{x}|}$ in a neighbourhood of C^{n-1} and define $g \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R}^n)$ by the formula $g(y) = (y, 0)$. There exists a map $h \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$h(t,x) = \left(\sqrt{1 - t^2 x_n^2} \cdot \frac{\hat{x}}{|\hat{x}|}, t x_n \right) \tag{4.2}$$

in a neighbourhood of $[0,1] \times C^{n-1}$. In particular we get for $x \in C^{n-1}$

$$h(0,x) = g(f(x)) \quad \text{and} \quad h(1,x) = x.$$

On the other hand we get $f(g(y)) = y$ for $y \in S^{n-2}$.

Given $F \in \mathbf{CH}_m(C^{n-1})$ we define $g^\# F \in \mathbf{CH}_m(S^{n-2})$ by $g^\# F(T) = F(g_\# T)$ for each $T \in \mathbf{N}_m(S^{n-2})$. Given $G \in \mathbf{CH}_m(S^{n-2})$ we define $f^\# G \in \mathbf{CH}_m(C^{n-1})$ in the same way. Observe

$$g^\#(f^\# G) = (f \circ g)^\# G \quad \text{and} \quad f^\#(g^\# F) = (g \circ f)^\# F.$$

Observe the following diagram is commutative:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
\mathbf{CH}_{m+1}(C^{n-1}) \arrow[r, "g^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m+1}(S^{n-2}) \arrow[r, "f^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m+1}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\mathbf{CH}_{m}(C^{n-1}) \arrow[r, "g^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m}(S^{n-2}) \arrow[r, "f^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\mathbf{CH}_{m-1}(C^{n-1}) \arrow[r, "g^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m-1}(S^{n-2}) \arrow[r, "f^{\#}"] \arrow[u, "\mathbf{d}"] & \mathbf{CH}_{m-1}(C^{n-1}) \arrow[u, "\mathbf{d}"] \\
\vdots \arrow[u, "\mathbf{d}"] & \vdots \arrow[u, "\mathbf{d}"] & \vdots \arrow[u, "\mathbf{d}"]
\end{tikzcd}
$$

As a consequence we get linear maps

$$H^*_{\mathrm{ch}}(C^{n-1}) \xrightarrow{H(g^{\#})} H^*_{\mathrm{ch}}(S^{n-2}) \xrightarrow{H(f^{\#})} H^*_{\mathrm{ch}}(C^{n-1}).$$

Let $0 \leqslant m \leqslant n$, fix $F \in \mathbf{CH}_m(C^{n-1})$ satisfying $\mathbf{d}F = 0$ and compute

$$H^m(f^{\#})[H^m(g^{\#})([F])] = H^m((g \circ f)^{\#})([F]) = [(g \circ f)^{\#} F].$$

The homotopy formula (Proposition 2.2) applied to h defined in (4.2) yields for $T \in \mathbf{N}_m(C^{n-1})$

$$(g \circ f)_{\#} T - T = \partial h_{\#}([\![0,1]\!] \times T) + h_{\#}([\![0,1]\!] \times \partial T)$$

in case $1 \leqslant m \leqslant n$ and

$$(g \circ f)_{\#} T - T = \partial h_{\#}([\![0,1]\!] \times T)$$

in case $m = 0$. As $\mathbf{d}F = 0$ we get for $T \in \mathbf{N}_m(C^{n-1})$ and $1 \leqslant m \leqslant n$

$$F((g \circ f)_{\#} T) - F(T) = F(h_{\#}([\![0,1]\!] \times \partial T)) = P_h F(\partial T) = \mathbf{d} P_h F(T),$$

that is $[(g \circ f)^{\#} F] = [F]$. In case $m = 0$ we obtain $F((g \circ f)_{\#} T) = F(T)$ for $T \in \mathbf{N}_0(C^{n-1})$ that is $[(g \circ f)^{\#} F] = [F]$. In Both cases we conclude $H^m(f^{\#}) \circ H^m(g^{\#}) = \mathrm{id}_{H^m_{\mathrm{ch}}(C^{n-1})}$.

On the other hand we get for $0 \leqslant m \leqslant n$ and $G \in \mathbf{CH}_m(S^{n-2})$ satisfying $\mathbf{d}F = 0$

$$H^m(g^{\#})[H^m(f^{\#})([G])] = [(f \circ g)^{\#} G] = [G],$$

as $f(g(y)) = y$ for $y \in S^{n-2}$.

Finally, we conclude $H^*_{\mathrm{ch}}(C^{n-1}) \simeq H^*_{\mathrm{ch}}(S^{n-2})$.

Computing $H^*_{\mathrm{ch}}(S^{n-1})$ Using the long exact sequence (4.1) and the previous computations we obtain, proceeding as for the singular cohomology

$$H^m_{\mathrm{ch}}(S^0) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{in case } m = 0, \\ 0 & \text{otherwise;} \end{cases}$$

and for $n > 1$

$$H^m_{\mathrm{ch}}(S^{n-1}) = \begin{cases} \mathbb{R} & \text{in case } m \in \{0, n-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 5

Representing m-charges

5.1 The space of m-charges

Denote by $\mathbf{CH}_m(\mathbb{R}^n)$ the set of all m-charges in $\mathbb{R}^n$. One defines a seminorm on $\mathbf{CH}_m(\mathbb{R}^n)$ for each $K \in \mathcal{K}(\mathbb{R}^n)$ by the formula

$$\|F\|_K = \sup\{F(T) : T \in \mathbf{N}_{m,K}(\mathbb{R}^n), \mathbf{N}(T) \leqslant 1\}.$$

As the seminorms $\|\cdot\|_{B[k]}$ form a separating family when k ranges over $\mathbb{N}$, they turn $\mathbf{CH}_m(\mathbb{R}^n)$ into a metrizable l.c.t.v.s. according to Theorem 1.5.

Proposition 5.1. *The topology defined above on the space* $\mathbf{CH}_m(\mathbb{R}^n)$ *turns* $\mathbf{CH}_m(\mathbb{R}^n)$ *into a Fréchet space.*

Proof. Call d the invariant metric induced on $\mathbf{CH}_m(\mathbb{R}^n)$ by the family $\{\|\cdot\|_{B[k]} : k \in \mathbb{N}\}$ according to Theorem 1.5. It remains to be shown that d is complete. For, assume $(F_k) \subseteq \mathbf{CH}_m(\mathbb{R}^n)$ is a Cauchy sequence, i.e. $d(F_j, F_k) \to 0$; $j, k \to \infty$. In particular, $\|F_j - F_k\|_{B[l]} \to 0$;$j, k \to \infty$, for each $l \in \mathbb{N}$ large enough for $\operatorname{supp} T \subseteq B[l]$ to hold. For $T \in \mathbf{N}_m(\mathbb{R}^n)$ one obtains $|F_j(T) - F_k(T)| \leqslant \mathbf{N}(T)\|F_j - F_k\|_{B[l]}$ for each $l \in \mathbb{N}$, so that $(F_k(T))$ is a Cauchy sequence. Define a linear functional F on $\mathbf{N}_m(\mathbb{R}^n)$ by $F(T) = \lim_k F_k(T)$. To show F is an m-charge, fix $\varepsilon > 0$, $K \in \mathcal{K}(\mathbb{R}^n)$ and choose $l \in \mathbb{N}$ for which $K \subseteq B[l]$. Let $k \in \mathbb{N}$ be such that $\|F - F_k\|_{B[l]} \leqslant \frac{\varepsilon}{2}$. For $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ compute

$$F(T) \leqslant F_k(T) + \mathbf{N}(T)\|F - F_k\|_{B[l]} \leqslant F_k(T) + \frac{\varepsilon}{2}\mathbf{N}(T). \tag{5.1}$$

As F_k is an m-charge, there is a $\theta > 0$ such that $F_k(S) \leqslant \theta\mathbf{F}(S) + \frac{\varepsilon}{2}\mathbf{N}(S)$ occurs for $S \in \mathbf{N}_{m,K}(\mathbb{R}^n)$. Combining this to (5.1) we get

$$F(T) \leqslant \theta\mathbf{F}(T) + \frac{\varepsilon}{2}\mathbf{N}(T) + \frac{\varepsilon}{2}\mathbf{N}(T) = \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T),$$

and $F \in \mathbf{CH}_m(\mathbb{R}^n)$. We conclude (F_k) d-converges to F as in Example 1.6. $\square$

5.2 A continuous map

Recall that Example 3.5 yields a map

$$\Theta : \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n).$$

The product topology turns the space $\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)$ into a Fréchet space whose topology is induced by the seminorms $\|\cdot\|_K$ defined by

$$\|(\omega, \zeta)\|_K = \max[\nu_K^0(\omega), \nu_K^0(\zeta)]$$

where K ranges over $\mathcal{K}(\mathbb{R}^n)$ and ν_K^0 is defined as in Example 1.6.

Fix $K \in \mathcal{K}(\mathbb{R}^n)$ and let $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ satisfy $\mathbf{N}(T) \leqslant 1$. Compute

$$\begin{aligned} \Theta(\omega,\zeta)(T) &= \langle T, \omega\rangle + \langle \partial T, \zeta\rangle, \\ &\leqslant \mathbf{M}(T)\nu_K^0(\omega) + \mathbf{M}(\partial T)\nu_K^0(\zeta), \\ &\leqslant \max[\nu_K^0(\omega), \nu_K^0(\zeta)]. \end{aligned}$$

In particular this yields $\|\Theta(\omega,\zeta)\|_K \leqslant \max[\nu_K^0(\omega), \nu_K^0(\zeta)]$, and Θ is continuous.

Define $\Lambda : \mathscr{C}^m(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n)$ and $\Gamma : \mathscr{C}^{m-1}(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n)$ by the formulas

$$\Lambda(\omega)(T) = \langle T, \omega\rangle \quad \text{and} \quad \Gamma(\zeta)(T) = \langle \partial T, \zeta\rangle,$$

so that $\Theta(\omega, \zeta) = \Lambda(\omega) + \Gamma(\zeta)$ for $\omega \in \mathscr{C}^m(\mathbb{R}^n)$, $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$. Obviously, Λ and Γ are continuous.

Remark 5.2. For $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and $K \in \mathfrak{K}(\mathbb{R}^n)$ compute

$$\|\Lambda(\omega)\|_K = \sup\{\langle T, \omega\rangle : T \in \mathbf{N}_{m,K}(\mathbb{R}^n), \mathbf{N}(T) \leqslant 1\} \leqslant \mathbf{M}(\omega) \tag{5.2}$$

and, for $\zeta \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ and $K \in \mathfrak{K}(\mathbb{R}^n)$ notice that

$$\|\Gamma(\zeta)\|_K = \sup\{\langle \partial T, \zeta\rangle : T \in \mathbf{N}_{m,K}(\mathbb{R}^n), \mathbf{N}(T) \leqslant 1\} \leqslant \mathbf{M}(\zeta). \tag{5.3}$$

Our goal is to show the continuous map

$$\Theta : \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n)$$

is onto. For that purpose, we will show that $\operatorname{im}\Theta$ is a dense, closed subset of $\mathbf{CH}_m(\mathbb{R}^n)$.

5.3 The dual space $\mathrm{CH}_m(\mathbb{R}^n)^*$

We describe the dual space $\mathbf{CH}_m(\mathbb{R}^n)^*$.

Proposition 5.3. *The map*

$$\Phi : \mathbf{CH}_m(\mathbb{R}^n)^* \to \mathbf{N}_m(\mathbb{R}^n)$$

defined by $\Phi(\alpha)(\omega) = \alpha(\Lambda(\omega))$ *for* $\omega \in \mathscr{D}^m(\mathbb{R}^n)$*, is a linear bijection.*

Claim 5.4. Assume $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and $K \in \mathfrak{K}(\mathbb{R}^n)$ are such that $K \cap \operatorname{supp}\omega = \emptyset$. Then, $\|\Lambda(\omega)\|_K = 0$.

Proof. For $T \in \mathbf{N}_{m,K}(\mathbb{R}^n)$ one gets $\langle T, \omega\rangle = 0$ and thus $\|\Lambda(\omega)\|_K = 0$. $\square$

Claim 5.5. For $\alpha \in \mathbf{CH}_m(\mathbb{R}^n)^*$ one has $\Phi(\alpha) \in \mathbf{N}_m(\mathbb{R}^n)$.

Proof. As α is continuous on $\mathbf{CH}_m(\mathbb{R}^n)$, there exist $c > 0$ and a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ such that for $F \in \mathbf{CH}_m(\mathbb{R}^n)$ we can write

$$\alpha(F) \leqslant c\|F\|_K.$$

In particular, it yields for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$

$$\langle \Phi(\alpha), \omega \rangle = \alpha[\Lambda(\omega)] \leqslant c\|\Lambda(\omega)\|_K \leqslant c\mathbf{M}(\omega),$$

and for $\zeta \in \mathscr{D}^{m-1}(\mathbb{R}^n)$,

$$\langle \partial\Phi(\alpha), \zeta \rangle = \alpha[\Gamma(\zeta)] \leqslant c\|\Gamma(\zeta)\|_K \leqslant c\mathbf{M}(\zeta).$$

It follows immediately that $\Phi(\alpha)$ is a normal current. $\square$

Define a linear map

$$\Upsilon : \mathbf{N}_m(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n)'$$

by the formula $\Upsilon(T)(F) = F(T)$.

Claim 5.6. For each $T \in \mathbf{N}_m(\mathbb{R}^n)$ we have $\Upsilon(T) \in \mathbf{CH}_m(\mathbb{R}^n)^*$.

Proof. Fix $T \in \mathbf{N}_m(\mathbb{R}^n)$ and choose a compact set $K \in \mathcal{K}(\mathbb{R}^n)$ for which $\operatorname{supp} T \subseteq K$. As

$$|\Upsilon(T)(F)| = |F(T)| \leqslant \mathbf{N}(T)\|F\|_K,$$

we conclude that $\Upsilon(T)$ is continuous on $\mathbf{CH}_m(\mathbb{R}^n)$. $\square$

So we have a linear map

$$\Upsilon : \mathbf{N}_m(\mathbb{R}^n) \to \mathbf{CH}_m(\mathbb{R}^n)^*$$

satisfying $\Upsilon(T)(F) = F(T)$ for each $F \in \mathbf{CH}_m(\mathbb{R}^n)$.

Claim 5.7. For each $T \in \mathbf{N}_m(\mathbb{R}^n)$, we have $\Phi[\Upsilon(T)] = T$.

Proof. For $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ compute

$$\langle \Phi[\Upsilon(T)], \omega \rangle = \Upsilon(T)[\Lambda(\omega)] = \Lambda(\omega)(T) = \langle T, \omega \rangle.$$

The proof is complete since ω is arbitrary. $\square$

For $K \in \mathfrak{K}(\mathbb{R}^n)$ and $\varepsilon > 0$ we will use the notation

$$V(K, \varepsilon) = \{F \in \mathbf{CH}_m(\mathbb{R}^n) : \|F\|_K \leqslant \varepsilon\}.$$

The sets $V(K, \varepsilon)$ constitue a neighbourhood base at 0 in $\mathbf{CH}_m(\mathbb{R}^n)$ for ε and K ranging over $(0, \infty)$ and $\mathfrak{K}(\mathbb{R}^n)$ respectively.

The following claims will help us to prove Φ maps bijectively $\mathbf{CH}_m(\mathbb{R}^n)$ onto $\mathbf{N}_m(\mathbb{R}^n)$.

Claim 5.8. One has $\operatorname{supp} \Phi(\alpha) \subseteq K$ for $\alpha \in V(K, \varepsilon)^\circ$.

Proof. Indeed, fix $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ and suppose $K \cap \operatorname{supp} \omega = \emptyset$. For each $\lambda > 0$ one computes then $\|\lambda \Lambda(\omega)\|_K = 0$. This yields $\lambda \Lambda(\omega) \in V(K, \varepsilon)$ for each $\lambda > 0$. In particular, one has $\lambda \alpha[\Lambda(\omega)] = \alpha[\lambda \Lambda(\omega)] \leqslant 1$ for every $\lambda > 0$ so that $\Phi(\alpha)(\omega) = \alpha[\Lambda(\omega)] = 0$. $\square$

Claim 5.9. The equation $\mathbf{N}[\Phi(\alpha)] \leqslant \frac{2}{\varepsilon}$ holds for $\alpha \in V(K, \varepsilon)^\circ$.

Proof. Indeed, fix $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\mathbf{M}(\omega) \leqslant 1$ and note that

$$\|\varepsilon \Lambda(\omega)\|_K = \varepsilon \|\Lambda(\omega)\|_K \leqslant \varepsilon \mathbf{M}(\omega) \leqslant \varepsilon.$$

So one has $\alpha[\varepsilon \Lambda(\omega)] \leqslant 1$, that is

$$\langle \Phi(\alpha), \omega \rangle = \alpha[\Lambda(\omega)] \leqslant \frac{1}{\varepsilon}.$$

On the other hand for $\zeta \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ satisfying $\mathbf{M}(\zeta) \leqslant 1$ one computes

$$\|\varepsilon \Gamma(\zeta)\|_K = \varepsilon \|\Gamma(\zeta)\|_K \leqslant \varepsilon \mathbf{M}(\zeta) \leqslant \varepsilon.$$

It yields $\alpha[\varepsilon \Gamma(\zeta)] \leqslant 1$ or

$$\langle \partial \Phi(\alpha), \zeta \rangle = \alpha[\Gamma(\zeta)] \leqslant \frac{1}{\varepsilon}.$$

Finally we obtain $\mathbf{M}[\partial \Phi(\alpha)] \leqslant \frac{1}{\varepsilon}$, so that $\mathbf{N}(T) \leqslant \frac{2}{\varepsilon}$. $\square$

The map Υ is a left-inverse for Φ.

Claim 5.10. For each each $\alpha \in \mathbf{CH}_m^*(\mathbb{R}^n)$ one has $\Upsilon[\Phi(\alpha)] = \alpha$.

Proof. We have to show that for $F \in \mathbf{CH}_m(\mathbb{R}^n)$, $\Upsilon[\Phi(\alpha)](F) = \alpha(F)$, that is $F[\Phi(\alpha)] = \alpha(F)$. Fix $F \in \mathbf{CH}_m(\mathbb{R}^n)$ and let

$$\Delta_F : \mathbf{CH}_m(\mathbb{R}^n)^* \to \mathbb{R}, \alpha \mapsto F[\Phi(\alpha)].$$

We show that the map Δ_F is weakly* continuous on the set

$$V(K, \varepsilon)^\circ = \{\alpha \in \mathbf{CH}_m^* : |\alpha(F)| \leqslant 1 \text{ for each } F \in V(K, \varepsilon)\},$$

given $K \in \mathfrak{K}(\mathbb{R}^n)$ and $\varepsilon > 0$.

Let $(\alpha_i)_{i \in I} \subseteq V(K, \varepsilon)^\circ$ be a directed system weakly*-converging to 0. In particular:

1. $\operatorname{supp}[\Phi(\alpha_i)] \subseteq K$ for each $i \in I$ according to Claim 5.8;
2. $\sup_{i \in I} \mathbf{N}[\Phi(\alpha_i)] \leqslant \frac{2}{\varepsilon} < \infty$ according to Claim 5.9;
3. $(\Phi(\alpha_i)(\omega))_{i \in I} = (\alpha_i(\Lambda(\omega)))_{i \in I}$ converge to 0 for each $\omega \in \mathscr{D}^m(\mathbb{R}^n)$ as $\Lambda(\omega)$ is an m-charge and $(\alpha_i(F))_{i \in I}$ converges to 0 for each m-charge F.

As $\Delta_F(\alpha_i) = F[\Phi(\alpha_i)]$ holds for each $i \in I$, Proposition 3.4 implies that $(\Delta_F(\alpha_i))_{i \in I}$ converges to 0, and Δ_F is weakly*-continuous on $V(K, \varepsilon)^\circ$.

According to the Banach-Grothendieck theorem (Theorem 1.13) we find $F^* \in \mathbf{CH}_m(\mathbb{R}^n)$ such that $\Delta_F(\alpha) = \alpha(F^*)$ for all $\alpha \in \mathbf{CH}_m(\mathbb{R}^n)^*$. For $T \in \mathbf{N}_m(\mathbb{R}^n)$ we have, according to Claim 5.7,

$$F(T) = F(\Phi(\Upsilon(T))) = \Upsilon(T)(F^*) = F^*(T),$$

so that $F = F^*$. The result follows as $\Delta_F(\alpha) = \alpha(F)$ holds for every $\alpha \in \mathbf{CH}_m(\mathbb{R}^n)^*$. $\square$

We can now conclude this section with the following corollaries.

Corollary 5.11. *The map* $\Phi : \mathbf{CH}_m(\mathbb{R}^n)^* \to \mathbf{N}_m(\mathbb{R}^n)$ *defined by* $\Phi(\alpha)(\omega) = \alpha[\Lambda(\omega)]$ *is a linear isomorphism, whose inverse* Φ^{-1} *satisfies* $\Phi^{-1}(T)(F) = F(T)$ *for each* $T \in \mathbf{N}_m(\mathbb{R}^n)$.

Call *smooth* m*-charges* the elements of

$$\mathbf{CH}^s_m(\mathbb{R}^n) = \{\Lambda(\omega) : \omega \in \mathscr{D}^m(\mathbb{R}^n)\} \subseteq \mathbf{CH}_m(\mathbb{R}^n)$$

Corollary 5.12. *The space* $\mathbf{CH}^s_m(\mathbb{R}^n)$ *is dense in* $\mathbf{CH}_m(\mathbb{R}^n)$.

Proof. Assume the continuous linear functional α on $\mathbf{CH}_m(\mathbb{R}^n)$ is such that $\alpha[\Lambda(\omega)] = 0$ for each $\omega \in \mathscr{D}^m(\mathbb{R}^n)$. Then one has $\Phi(\alpha) = 0$ and $\alpha = \Upsilon(\Phi(\alpha)) = 0$. If $\mathbf{CH}^s_m(\mathbb{R}^n)$ weren't dense in $\mathbf{CH}_m(\mathbb{R}^n)$ there would exist $F \in \mathbf{CH}_m(\mathbb{R}^n) \setminus \mathrm{cl}(\mathbf{CH}^s_m(\mathbb{R}^n))$. Theorem 1.10 would then imply the existence of $\alpha_* \in \mathbf{CH}_m(\mathbb{R}^n)^*$ such that $\alpha_* \restriction \mathbf{CH}^s_m(\mathbb{R}^n) = 0$ yet $\alpha_*(F) \neq 0$, which is a contradiction. □

To show that the image of Θ is closed in $\mathbf{CH}_m(\mathbb{R}^n)$, one will use the closed range theorem.

5.4 An adjoint map with closed range

The adjoint map $\Theta^* : \mathbf{CH}_m(\mathbb{R}^n)^* \to [\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$ is defined in such a way that

$$\Theta^*(\alpha)(\omega, \zeta) = \alpha[\Theta(\omega, \zeta)].$$

Observe that $L \in [\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$ belongs to $\operatorname{im}\Theta^*$ if and only if L belongs to the image of the map

$$\Xi^* : \mathbf{N}_m(\mathbb{R}^n) \to [\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$$

defined by the formula

$$\Xi^*(T)(\omega, \zeta) = \Theta(\omega, \zeta)(T).$$

To see this, fix $L \in [\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$ and note that if $\alpha \in \mathbf{CH}_m(\mathbb{R}^n)^*$ is such that $\Theta^*(\alpha) = L$, we obtain

$$\begin{aligned}\Xi^*[\Phi(\alpha)](\omega,\zeta) = \Theta(\omega,\zeta)[\Phi(\alpha)] = \{\Phi^{-1}[\Phi(\alpha)]\}[\Theta(\omega,\zeta)] \\ = \alpha[\Theta(\omega,\zeta)] = \Theta^*(\alpha)(\omega,\zeta) = L(\omega,\zeta).\end{aligned}$$

Conversely, fix $L \in [\mathscr{C}^m(\mathbb{R}^n)\times\mathscr{C}^{m-1}(\mathbb{R}^n)]^*$. Assuming $T \in \mathbf{N}_m(\mathbb{R}^n)$ satisfy $\Xi(T) = L$, we compute

$$\begin{aligned}\Theta^*[\Phi^{-1}(T)](\omega,\zeta) = \Phi^{-1}(T)[\Theta(\omega,\zeta)] = \Theta(\omega,\zeta)(T) \\ = \Xi^*(T)(\omega,\zeta) = L(\omega,\zeta).\end{aligned}$$

We show that $\operatorname{im}\Xi^*$ is strongly sequentially closed in $[\mathscr{C}^m(\mathbb{R}^n)\times\mathscr{C}^{m-1}(\mathbb{R}^n)]^*$.

Proposition 5.13. *The map* $\Xi^* : \mathbf{N}_m(\mathbb{R}^n) \to [\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$ *is injective. Moreover* $\operatorname{im}\Xi^*$ *is strongly sequentially closed in* $[\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$.

Proof. Note first that the map Ξ^* is injective. Indeed, suppose $\Xi^*(T) = 0$ for $T \in \mathbf{N}_m(\mathbb{R}^n)$ and compute for $\omega \in \mathscr{D}^m(\mathbb{R}^n)$

$$\langle T,\omega\rangle = \Theta(\omega,0)(T) = \Xi^*(T)(\omega,0) = 0.$$

For a set $A \subseteq \mathscr{C}^m(\mathbb{R}^n)$, compute

$$\begin{aligned}&\sup_{(\omega,\zeta)\in A\times\{0\}} |\Xi^*(T)(\omega,\zeta)| \\ &\qquad = \sup\{|\Theta(\omega,0)(T)| : \omega \in A\} = \sup\{|\langle T,\omega\rangle| : \omega \in A\},\end{aligned}$$

and, similarly for $A' \subseteq \mathscr{C}^{m-1}(\mathbb{R}^n)$

$$\begin{aligned}&\sup_{(\omega,\zeta)\in\{0\}\times A'} |\Xi^*(T)(\omega,\zeta)| \\ &\qquad = \sup\{|\Theta(0,\zeta)(T)| : \zeta \in A'\} = \sup\{|\langle \partial T,\zeta\rangle| : \omega \in A'\}.\end{aligned}$$

Define sets

$$\begin{aligned} B &= \{(\omega,0) \in \mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^{m-1}(\mathbb{R}^n) : \mathbf{M}(\omega) \leqslant 1\}, \\ C &= \{(0,\zeta) \in \mathscr{D}^m(\mathbb{R}^n) \times \mathscr{D}^{m-1}(\mathbb{R}^n) : \mathbf{M}(\zeta) \leqslant 1\}, \end{aligned}$$

and observe B and C are bounded since given $K \in \mathcal{K}(\mathbb{R}^n)$

$$\sup_{(\omega,\zeta)\in B} \max[\nu_K^0(\omega), \nu_K^0(\zeta)] \leqslant 1 \quad \text{and} \quad \sup_{(\omega,\zeta)\in C} \max[\nu_K^0(\omega), \nu_K^0(\zeta)] \leqslant 1.$$

In particular the seminorm $q : \operatorname{im} \Xi^* \to [0,\infty)$ defined by

$$q(L) = \sup_{(\omega,\zeta)\in B} |L(\omega,\zeta)| + \sup_{(\omega,\zeta)\in C} |L(\omega,\zeta)|$$

is strongly continuous on $\operatorname{im}\Xi^*$. On the other hand, compute for $T \in \mathbf{N}_m(\mathbb{R}^n)$

$$\sup_{(\omega,\zeta)\in B} |\Xi^*(T)(\omega,\zeta)| = \sup\{|\langle T,\omega\rangle| : \omega \in \mathscr{D}^m(\mathbb{R}^n), \mathbf{M}(\omega) \leqslant 1\} = \mathbf{M}(T).$$

Similarly, one obtains for $T \in \mathbf{N}_m(\mathbb{R}^n)$

$$\sup_{(\omega,\zeta)\in C} |\Xi^*(T)(\omega,\zeta)| = \mathbf{M}(\partial T)$$

and in particular $q(\Xi^*(T)) = \mathbf{N}(T)$.

To show that $\operatorname{im}\Xi^*$ is strongly sequentially closed, assume that the sequence $(\Xi^*(T_k)) \subseteq \operatorname{im}\Xi^*$ strongly converges to $L \in [\mathscr{C}^m(\mathbb{R}^n)\times\mathscr{C}^{m-1}(\mathbb{R}^n)]^*$ for some $(T_k) \subseteq \mathbf{N}_m(\mathbb{R}^n)$.

As q is strongly continuous and because of the previous computations, one obtains

$$\sup_k \mathbf{N}(T_k) = \sup_k q(\Xi^*(T_k)) = c < \infty.$$

Claim 5.14. There exists $K \in \mathcal{K}(\mathbb{R}^n)$ containing $\operatorname{supp} T_k$ for each $k \in \mathbb{N}$.

Proof of the claim. Proceding toward a contradiction, assume there exists a sequence of open sets (U_k) satisfying the following conditions:

1. $\bar{U}_k \subseteq U_{k+1}$ for each $k \in \mathbb{N}$,

2. for each $k \in \mathbb{N}$, $\operatorname{supp} T_k \cap U'_k \neq \emptyset$ if we let $U'_k = U_k \setminus \bar{U}_{k-1}$ for $k \geqslant 1$ and $U'_0 = U_0$.

In view of this, choose $\omega_k \in \mathscr{D}^m(\mathbb{R}^n)$ satisfying $\operatorname{supp} \omega_k \subseteq U'_k$, $\mathbf{M}(\omega_k) = 1$ and $a_k = \langle T_k, \omega_k \rangle > 0$. Define for $k \in \mathbb{N}$, $b_k = \max_{1 \leqslant l \leqslant k} a_l^{-1}$. Define a bounded subset $D \subseteq \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)$ by

$$D = \{(\omega, 0) \in \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n) : \nu^0_{\bar{U}_k}(\omega) \leqslant k b_k \text{ for every } k \in \mathbb{N}\}$$

and observe $(\omega'_k, 0) \in D$ for each $k \in \mathbb{N}$, where $\omega'_k = k b_k \omega_k$. Let q_D be the seminorm defined on $\operatorname{im} \Xi^*$ by the formula $q_D(L) = \sup_{(\omega,\zeta) \in D} |L(\omega, \zeta)|$. We have a contradiction since

$$k \leqslant k b_k a_k = k b_k \langle T_k, \omega_k \rangle = \langle T_k, \omega'_k \rangle$$

would imply $\sup_{k \in \mathbb{N}} q_D[\Xi^*(T_k)] = +\infty$. □

To conclude the proof, observe that

$$(T_k) \subseteq \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\} \tag{5.4}$$

and use the compactness of the right side of (5.4) in the flat norm topology to find a subsequence $(T_{k_l}) \subseteq (T_k)$ and $T \in \{T \in \mathbf{N}_{m,K}(\mathbb{R}^n) : \mathbf{N}(T) \leqslant c\}$ for which $T_{k_l} \to T$, $l \to \infty$ in the flat norm topology $\mathcal{S}$. In particular, we obtain for $(\omega, \zeta) \in \mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)$,

$$\begin{aligned} L(\omega, \zeta) = \lim_{l \to \infty} \Xi^*(T_{k_l})(\omega, \zeta) = \lim_{l \to \infty} \Theta(\omega, \zeta)(T_{k_l}) \\ = \Theta(\omega, \zeta)(T) = \Xi^*(T)(\omega, \zeta), \end{aligned}$$

that is $L \in \operatorname{im} \Xi^*$. □

So we can prove our fundamental result.

Theorem 5.15. *Let $F \in \mathbf{CH}_m(\mathbb{R}^n)$. There exist $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ and $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ for which the equality*

$$F(T) = \langle T, \omega \rangle + \langle \partial T, \zeta \rangle$$

holds whenever $T \in \mathbf{N}_m(\mathbb{R}^n)$ is given.

Proof. As Θ has a dense image in $\mathbf{CH}_m(\mathbb{R}^n)$, and because $\operatorname{im}\Theta^*$ is strongly sequentially closed in $[\mathscr{C}^m(\mathbb{R}^n) \times \mathscr{C}^{m-1}(\mathbb{R}^n)]^*$, the closed range theorem (Theorem 1.16) and Remark 1.17 guarantee that $\operatorname{im}\Theta = \mathbf{CH}_m(\mathbb{R}^n)$. $\square$

Theorem 5.15 yields without effort a representation for m-charges in a closed set.

Fix $X \subseteq \mathbb{R}^n$ a closed set. In case $m \geqslant 0$ is an integer, define $\mathscr{C}^m(X)$ the linear space of all continuous m-forms in X, and let $\mathscr{C}^{-1}(X) = \{0\}$.

We get the following using Theorem 5.15, Proposition 4.3 and Proposition 4.4.

Corollary 5.16. *A linear functional $F : \mathbf{N}_m(X) \to \mathbb{R}$ is an m-charge in X if and only if there exists $\omega \in \mathscr{C}^m(X)$ and $\zeta \in \mathscr{C}^{m-1}(X)$ such that*

$$F(T) = \Theta(\omega,\zeta)(T)$$

holds for each $T \in \mathbf{N}_m(X)$, where $\Theta : \mathscr{C}^m(X) \times \mathscr{C}^{m-1}(X) \to \mathbf{CH}_m(X)$ is defined by $\Theta(\omega,\zeta)(T) = \langle T,\omega\rangle + \langle \partial T,\zeta\rangle$.

5.5 A representation theorem for strong charges

As a corollary of Theorem 5.15, we obtain the following result, due to T. De Pauw and W.F. Pfeffer [DP04].

Theorem 5.17 (De Pauw-Pfeffer)**.** *Given a strong charge $\mathscr{F} : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ there exists $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying*

$$\mathscr{F}(\varphi) = -\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

Proof. Call $F : \mathbf{N}_n(\mathbb{R}^n) \to \mathbb{R}$ the strong extension of $\mathscr{F}$. Observing $\mathbf{d}F = 0$, we get $\zeta \in \mathscr{C}^{n-1}(\mathbb{R}^n)$ satisfying $F = \mathbf{d}\boldsymbol{\zeta}$ using Theorem 5.19. Write

$$\zeta = \sum_{i=1}^{n} \zeta_i \, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

define $v = (v_1, v_2, \dots, v_n) \in C(\mathbb{R}^n, \mathbb{R}^n)$ by $v_i = (-1)^{i-1}\zeta_i$, $1 \leqslant i \leqslant n$ and use 3.3 to obtain

$$\mathscr{F}(\varphi) = \mathbf{d}\boldsymbol{\zeta}(T_\varphi) = \int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$. $\square$

5.6 Another proof of Theorem 3.13

Proposition 3.8 allows us to use the notation $\boldsymbol{\omega}$ for $\Lambda(\omega)$. In particular, one will write $\boldsymbol{\omega} + \mathbf{d}\boldsymbol{\zeta}$ for $\Theta(\omega, \zeta)$ and the Poincaré formula

$$\mathbf{d}^2\boldsymbol{\omega} = 0$$

holds for each $\omega \in \mathscr{C}^m(\mathbb{R}^n)$. We state and prove our version of Poincaré's lemma.

Lemma 5.18. *Assume $m \geqslant 1$ and $\omega \in \mathscr{C}^m(\mathbb{R}^n)$ is such that $\mathbf{d}\boldsymbol{\omega} = 0$. There exists $\lambda \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ such that $\mathbf{d}\boldsymbol{\lambda} = \boldsymbol{\omega}$.*

Proof. If $\phi = \sum_{1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n} \varphi_{i_1, i_2, \dots, i_m} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_m}$ we define $\operatorname{Prim}(\phi) \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ by

$$\operatorname{Prim}(\phi)(x) = \sum_{1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n} \sum_{j=1}^{m} (-1)^{j-1} \int_0^1 t^{m-1} \varphi_{i_1, \dots, i_m}(tx) \, dt$$
$$x_{i_j} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_m},$$

where a hat above an expression means it is omitted. It follows from a simple computation (see [Maw97, Théorème, pp. 630-632]) that $d[\operatorname{Prim}(\phi)] = \phi$ holds whenever ϕ is C^1. For $k \geqslant 1$, define $\omega_k = \rho_k * \omega$, $\lambda_k = \operatorname{Prim}(\omega_k)$ and $\lambda = \operatorname{Prim}(\omega)$. In case $m \leqslant n-1$ compute for $T \in \mathbf{N}_{m+1}(\mathbb{R}^n)$

$$\begin{aligned}
\langle T, d\omega_k \rangle &= \langle \partial T, \omega_k \rangle, \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(y) \langle \vec{\partial T}(x), \omega(x-y) \rangle \, dy \, d\|\partial T\|(x), \\
&= \int_{\mathbb{R}^n} \rho_k(y) \int_{\mathbb{R}^n} \langle \partial \vec{S}_y(x), \omega(x) \rangle \, d\|\partial S_y\|(x) \, dy, \qquad (5.5) \\
&= 0,
\end{aligned}$$

where we used Fubini's theorem to justify (5.5) and defined a current $S_y \in \mathbf{N}_m(\mathbb{R}^n)$ by the formula $\langle S_y, \psi \rangle = \langle T, \psi(\cdot - y) \rangle$ for $\psi \in \mathscr{D}^{m+1}(\mathbb{R}^n)$. The previous computations ensure $d\omega_k = 0$ (the case $m = n$ is obvious), so that $d\lambda_k = \omega_k$ for $k \geqslant 1$. Of course (ω_k) converges uniformly to ω on each $K \in \mathcal{K}(\mathbb{R}^n)$ and an easy computation reveals (λ_k) converges uniformly to λ on each $K \in \mathcal{K}(\mathbb{R}^n)$. Fixing $T \in \mathbf{N}_m(\mathbb{R}^n)$ and observing $\operatorname{supp} T$ and $\operatorname{supp} \partial T$ are compact subsets of $\mathbb{R}^n$, one computes

$$\mathbf{d}\boldsymbol{\lambda}(T) = \langle \partial T, \lambda \rangle = \lim_k \langle \partial T, \lambda_k \rangle = \lim_k \langle T, \omega_k \rangle = \langle T, \omega \rangle = \boldsymbol{\omega}(T),$$

and the proof is complete. $\square$

We deduce the following result from Theorem 5.15 and Lemma 5.18.

Theorem 5.19. *Given $m \geqslant 1$ and $F \in \mathbf{CH}_m(\mathbb{R}^n)$ such that $\mathbf{d}F = 0$, there exists $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ such that $F = \mathbf{d}\boldsymbol{\zeta}$.*

Chapter 6

Why an abstract proof?

First, let us study how the fundamental theorem of calculus can be improved on curves.

6.1 Derivating continuous functions along curves

Let $f \in C^1([0,1], \mathbb{R}^n)$ be injective, regular (i.e. $f'(t) \neq 0$ whenever $0 \leqslant t \leqslant 1$) and let U be an open set containing $\operatorname{im} f$.

Definition 6.1. For $\phi : U \to \mathbb{R}$ and $0 \leqslant t \leqslant 1$, the *upper directional derivative* of ϕ at $f(t)$ along f is the extended real number

$$\overline{D}_f \phi[f(t)] = \frac{1}{|f'(t)|} \overline{\lim_{h \to 0}} \frac{\phi[f(t) + hf'(t)] - \phi[f(t)]}{h}$$

whereas the *lower directional derivative* of ϕ at $f(t)$ along f is defined by

$$\underline{D}_f \phi[f(t)] = \frac{1}{|f'(t)|} \underline{\lim_{h \to 0}} \frac{\phi[f(t) + hf'(t)] - \phi[f(t)]}{h}.$$

In case $\overline{D}_f \phi[f(t)] = \underline{D}_f \phi[f(t)]$ is a real number we call its value the *directional derivative* of ϕ at $f(t)$ along f.

6.1.1 An intriguing example

Let $f \in C^\infty([0,1],\mathbb{R}^3)$ be the injective and regular function defined by $f(t) = (\cos t, \sin t, t)$. For $t \in [0,1]$ define $\Lambda(t) = \{f(t) + \lambda f'(t) : \lambda \in \mathbb{R}\}$ the tangent line to f at t.

Claim 6.2. Given $s \neq t$ in $[0,1]$, we have $\Lambda(s) \cap \Lambda(t) = \emptyset$.

Proof. Fix $s \neq t$ in $[0,1]$. We have to show that the system

$$\begin{aligned} \cos s - \lambda \sin s = \cos t - \mu \sin t \quad , \\ \sin s + \lambda \cos s = \sin t + \mu \cos t \quad , \\ s + \lambda = t + \mu \quad ; \end{aligned}$$

doesn't admit any solution $(\lambda,\mu) \in \mathbb{R}^2$. Proceeding toward a contradiction, and assuming (λ,μ) is a solution of our system, we obtain from the two first equations

$$\mu - \lambda = \frac{2 - 2\cos(s-t)}{\sin(s-t)} \quad ,$$

which, combined to the third one, leads to the equality

$$\frac{2 - 2\cos(s-t)}{\sin(s-t)} = s - t \quad .$$

Writing $\vartheta = s - t \neq 0$, we obtain

$$2 - 2\cos\vartheta = \vartheta \sin\vartheta \quad ,$$

that is

$$\tan\frac{\vartheta}{2} = \frac{\vartheta}{2} \quad .$$

We obtain a contradiction because $\vartheta \mapsto \tan\vartheta$ doesn't have any nonzero fixed point in the interval $[-1/2, 1/2]$. □

Let $\mathscr{F} = \bigcup_{t\in[0,1]} \Lambda(t)$ and $p : \mathscr{F} \to [0,1]$ the map which associates to $x \in \mathscr{F}$ the unique $t \in [0,1]$ for which $x \in \Lambda(t)$.

Claim 6.3. The map p is continuous in a neighbourhood of each point $x_0 \in \mathscr{F}$.

Proof of the claim. Let $x_0 \in \mathscr{F}$ and choose $(x_k) \subseteq \mathscr{F}$ be such that $x_k \to x_0, k \to \infty$. For each $k \in \mathbb{N}$, let $t_k \in [0,1]$ and $\lambda_k \in \mathbb{R}$ be such that one has $x_k = f(t_k) + \lambda_k f'(t_k)$ and let $t_0 \in [0,1]$ and $\lambda_0 \in \mathbb{R}$ satisfy $x_0 = f(t_0) + \lambda_0 f'(t_0)$. We have to show $t_k \to t_0, k \to \infty$. Proceeding toward a contradiction, assume (working if necessary with a subsequence) one has $t_k \to t$,$k \to \infty$ with $t \in [0,1]$, $t \neq t_0$, and $\lambda_k \to \lambda$, $k \to \infty$ for some $\lambda \in \mathbb{R}$. We write, for $k \in \mathbb{N}$,

$$x_k = (\cos t_k - \lambda_k \sin t_k, \sin t_k + \lambda_k \cos t_k, t_k + \lambda_k).$$

This yields in particular

$$\begin{pmatrix} \cos t - \lambda \sin t \\ \sin t + \lambda \cos t \\ t + \lambda \end{pmatrix} = \lim_{k\to\infty} x_k = \begin{pmatrix} \cos t_0 - \lambda \sin t_0 \\ \sin t_0 + \lambda_0 \cos t_0 \\ t_0 + \lambda_0 \end{pmatrix}.$$

As before, these equations imply $t = t_0$. We get a contradiction. □

Claim 6.4. The set $\mathscr{F}$ is closed.

Proof of the claim. Let $(x_k) \subseteq \mathscr{F}$ be such that $x_k \to x_0, k \to \infty$ for some $x_0 \in \mathbb{R}^3$. For each $k \in \mathbb{N}$, choose $t_k \in [0,1]$ and $\lambda_k \in \mathbb{R}$ for which $x_k = f(t_k) + \lambda_k f'(t_k)$. As (λ_k) is bounded, choose $(t_{k_l}, \lambda_{k_l}) \subseteq (t_k, \lambda_k)$, $t_0 \in [0,1]$, $\lambda_0 \in \mathbb{R}$ for which $t_{k_l} \to t_0$,$l \to \infty$ and $\lambda_{k_l} \to \lambda_0$,$l \to \infty$. We get

$$x_0 = \lim_{l\to\infty} [f(t_{k_l}) + \lambda_{k_l} f'(t_{k_l})] = f(t_0) + \lambda_0 f'(t_0) \in \mathscr{F}.$$

The claim is proved. □

For $x \in \mathscr{F}$ define $\phi(x) = p(x)$ and extend ϕ to $\mathbb{R}^3$ in a continuous way using the Tietze extension theorem. As ϕ is constant on $\Lambda[f^{-1}(x)]$ for each $x \in \operatorname{im} f$, one has $D_f\phi(x) = 0$ for each $x \in \operatorname{im} f$. Yet

$$\phi[f(1)] - \phi[f(0)] = 1 - 0 = 1 > 0.$$

So we get the following.

Example 6.5. There exists $f \in C^\infty([0,1],\mathbb{R}^3)$ injective and regular together with $\phi \in C(\mathbb{R}^3,\mathbb{R})$ for which the following conditions are satisfied:

1. for each $x \in \operatorname{im} f$, $D_f\phi(x) = 0$;
2. $\phi[f(t)] = t$ for each $0 \leqslant t \leqslant 1$.

The *oscillation* $\operatorname{osc}(\phi, r)$ of $\phi : U \to \mathbb{R}$ at scale $r > 0$ is the extended real number

$$\operatorname{osc}(\phi, r) = \sup\{|\phi(x) - \phi(y)| : x, y \in U, |x - y| \leqslant r\}.$$

The oscillation of a function ϕ provided by Example 6.5 satisfies

$$\varlimsup_{r\to 0} r^{-1/2} \operatorname{osc}(\phi, r) > 0,$$

it follows from the following result.

Proposition 6.6. *Assume that*

1. $f : [0,1] \to U$ *is of class* $C^{1,1}$*, injective and regular;*
2. $\phi \in C(U,\mathbb{R})$ *is such that for every* $x \in \operatorname{im} f$*,* $D_f\phi(x) = 0$*;*
3. $\phi[f(t)] = t$ *for* $0 \leqslant t \leqslant 1$*;*

then $\varlimsup_{r\to 0} r^{-1/2} \operatorname{osc}(\phi, r) > 0$.

Proof. Let $L = \operatorname{Lip}(f') > 0$. Given $t \in [0,1]$ compute

$$|h|^{-2} \left|f(t+h) - f(t) - hf'(t)\right| = |h|^{-2} \left| \int_t^{t+h} [f'(s) - f'(t)]\, ds \right| \leqslant L$$

for each $h \in [0,1] - t$. Moreover, for $t \in [0,1]$ choose $\delta(t) > 0$ so that

$$|\phi[f(t) + hf'(t)] - \phi[f(t)]| \leqslant \frac{1}{2}|h|$$

holds for $h \in [0,1] - t$ satisfying $f(t) + hf'(t) \in U$ and $|h| \leqslant \delta(t)$. For $h \in [0,1] - t$ with $|h| \leqslant \delta(t)$, compute

$$\frac{|\phi[f(t+h)] - \phi[f(t) + hf'(t)]|}{\sqrt{|f(t+h) - f(t) - hf'(t)|}} \geqslant \frac{|h| - |\phi[f(t) + hf'(t)] - \phi[f(t)]|}{\sqrt{|f(t+h) - f(t) - hf'(t)|}} \geqslant \frac{1}{2L}$$

and observe the result follows since h can be choosen as small as we want. $\square$

6.1.2 Some sufficient conditions for a mean value formula

We can avoid the situation appearing in Example 6.5 by imposing some Lipschitz condition on ϕ.

Theorem 6.7. *Assume that*

1. $f : [0,1] \to U$ *is of class* C^1*, injective and regular;*
2. $\phi \in C(U, \mathbb{R})$ *is such that for every* $x \in \operatorname{im} f$, $\underline{D}_f \phi(x) \geqslant 0$*;*
3. ϕ *is Lipschitz;*

then $\phi[f(1)] - \phi[f(0)] \geqslant 0$.

Proof. Write $L = \operatorname{Lip}(f)$ and introduce $M = \max_{0 \leqslant t \leqslant 1} |f'(t)| > 0$. Fix $\varepsilon > 0$. For each $t \in [0,1]$, choose $\delta_1(t) > 0$ be such that one has

$$\begin{aligned} \phi[f(t) + hf'(t)] - \phi[f(t)] &\geqslant -\frac{\varepsilon}{2M}|f'(t)|h \geqslant -\frac{\varepsilon}{2}h, && (6.1)\\ \phi[f(t) - hf'(t)] - \phi[f(t)] &\leqslant \frac{\varepsilon}{2M}|f'(t)|h \leqslant \frac{\varepsilon}{2}h && (6.2)\end{aligned}$$

whenever $h \geqslant 0$ is such that $f(t) \pm hf'(t) \in U$ and satisfies $h \leqslant \delta_1(t)$. Choose also a real number $\delta_2(t) > 0$ for which one has

$$|f(t+\eta) - f(t) - \eta f'(t)| \leqslant \frac{\varepsilon}{2L+1}|\eta|, \tag{6.3}$$

whenever $\eta \in [0,1] - t$ satisfy $0 \leqslant |\eta| \leqslant \delta_2(t)$. For $0 \leqslant t \leqslant 1$ define $\delta(t) = \min[\delta_1(t), \delta_2(t)]$ and find $0 = \alpha^0 < \alpha^1 < \alpha^2 < \cdots < \alpha^m = 1$ together with tags $t^j \in [\alpha^{j-1}, \alpha^j]$, $1 \leqslant j \leqslant m$ for which $[\alpha^{j-1}, \alpha^j] \subseteq [t^j - \delta(t^j), t^j + \delta(t^j)]$ holds for each $1 \leqslant j \leqslant m$. As ϕ is Lipschitz we compute for $1 \leqslant j \leqslant m$

$$\begin{aligned} &\phi[f(\alpha^j)] - \phi[f(t^j) + (\alpha^j - t^j)f'(t^j)] \\ &\qquad \geqslant -\frac{\varepsilon L}{2L+1}(\alpha^j - t^j) \geqslant -\frac{1}{2}\varepsilon(\alpha^j - t^j), \end{aligned} \tag{6.4}$$

and similarly

$$\phi[f(t^j) - (t^j - \alpha^{j-1})f'(t^j)] - \phi[f(\alpha^{j-1})] \geqslant -\frac{1}{2}\varepsilon(t^j - \alpha^j). \tag{6.5}$$

Moreover (6.1) and (6.2) yield

$$\phi[f(t^j)+(\alpha^j-t^j)f'(t^j)]-\phi[f(t^j)] \geqslant -\frac{\varepsilon M}{2M}(\alpha^j-t^j) = -\frac{1}{2}\varepsilon(\alpha^j-t^j), \quad (6.6)$$

and similarly

$$\phi[f(t^j)] - \phi[f(t^j) - (t^j - \alpha^{j-1})f'(t^j)] \geqslant -\frac{1}{2}\varepsilon(t^j - \alpha^{j-1}). \quad (6.7)$$

Fitting (6.4), (6.5), (6.6) and (6.7) together we get

$$\phi[f(\alpha^j)] - \phi[f(\alpha^{j-1})] \geqslant -\varepsilon(\alpha^j - \alpha^{j-1})$$

and the proof is complete. □

Whenever f' is Lipschitz, one can weaken the last condition in Theorem 6.7.

Theorem 6.8. *Assume that*

1. $f : [0,1] \to U$ *is of class* $C^{1,1}$*, injective and regular;*
2. $\phi \in C(U,\mathbb{R})$ *is such that for every* $x \in \operatorname{im} f$, $\underline{D}_f\phi(x) \geqslant 0$*;*
3. $\overline{\lim}_{r\to 0}\left[r^{-1/2}\operatorname{osc}(\phi,r)\right] = 0$*;*

then $\phi[f(1)] - \phi[f(0)] \geqslant 0$*.*

Proof. Let $L = \operatorname{Lip}(f') > 0$. As is the proof of Proposition 6.6, one observes

$$|h|^{-2}|f(t+h) - f(t) - hf'(t)| \leqslant L$$

holds for each $h \in [0,1] - t$. Let $M = \max_{0\leqslant t\leqslant 1}|f'(t)| > 0$, fix $0 < \varepsilon \leqslant 1$ and choose $r > 0$ such that

$$|\phi(x) - \phi(y)| \leqslant \frac{\varepsilon}{2\sqrt{L}}|x-y|^{1/2}, \quad (6.8)$$

holds whenever $x, y \in U$ satisfy $|x - y| \leqslant r$. For each $0 \leqslant t \leqslant 1$, let $\delta_1(t) > 0$ be such that (6.1) and (6.2) hold whenever $h \geqslant 0$ is such that

$f(t) \pm hf'(t) \in U$ and satisfies $h \leqslant \delta_1(t)$. For $0 \leqslant t \leqslant 1$ define $\delta(t) = \min[r^{1/2}L^{-1/2}, \delta_1(t), \delta_2(t)]$ and find points α^j,$0 \leqslant j \leqslant m$ and tags t^j as in the proof of Theorem 6.7. Decomposing $\phi[f(\alpha^j)] - \phi[f(\alpha^{j-1})]$ as in the above proof, one gets the estimates (6.6) and (6.7) in the same way. On the other hand, observe that

$$|f(\alpha^j) - f(t^j) - (\alpha^j - t^j)f'(t^j)| \leqslant L(\alpha^j - t^j)^2 \leqslant r,$$

so that we have the estimate

$$\begin{aligned} \phi[f(\alpha^j)] - \phi[f(t^j) + (\alpha^j - t^j)f'(t^j)] &\geqslant -\frac{\varepsilon}{2\sqrt{L}}[L(\alpha^j - t^j)^2]^{1/2} \\ &= -\frac{1}{2}\varepsilon(\alpha^j - t^j). \end{aligned} \tag{6.9}$$

Similarly one shows

$$\phi[f(t^j) - (t^j - \alpha^{j-1})f'(t^j)] - \phi[f(\alpha^{j-1})] \geqslant -\frac{1}{2}\varepsilon(t^j - \alpha^j). \tag{6.10}$$

One concludes as in the proof of Theorem 6.7. □

6.1.3 The case $n = 2$

In the plane, the situation appearing in Example 6.5 cannot hold.

Given $f : [0, 1] \to \mathbb{R}^2$ of class 2 one says that f has nonzero curvature at $0 \leqslant t \leqslant 1$, if $|f''(t) \cdot f'(t)| < |f'(t)||f''(t)|$.

We will use the following geometrical fact.

Lemma 6.9. *Suppose $U \subseteq \mathbb{R}^2$ is an open set and $f : [0, 1] \to U$ is of class C^2, injective and regular with nonzero curvature everywhere. For $t \in [0, 1]$,*

$$\lim_{h\to 0} \frac{|x_f(t, t+h) - f(t)|}{|hf'(t)|} = \lim_{h\to 0} \frac{|x_f(t, t+h) - f(t+h)|}{|hf'(t)|} = \frac{1}{2},$$

where $x_f(t, t+h)$ denotes the intersection of the tangent lines to f at t and $t+h$ respectively.

Proof. Without loss of generality, assume $f(t) = (0,0)$, $f'(t) = (f_1'(t), 0) \neq 0$ and $f_2''(t) \neq 0$. Choose $\eta > 0$ such that

$$\min_{\substack{h\in[0,1]-t \\ |h|\leqslant\eta}} |f_2''(t+h)| > 0,$$

and assume $h \in [0,1] - t$ satisfies $|h| \leqslant \eta$. The coordinates of the point $x_f(t, t+h) = (x_1(t, t+h), x_2(t, t+h))$ are given by

$$x_1(t, t+h) = f_1(t+h) - f_1'(t+h)\frac{f_2(t+h)}{f_2'(t+h)} \quad \text{and} \quad x_2(t, t+h) = 0.$$

Choose also $\delta > 0$ such that such that for each $h \in [0,1] - t$ verifying $|h| \leqslant \delta$ one can find points θ_h and ϑ_h in $[0,1]$ for which

$$f_2'(t+h) = hf_2''(t+\theta_h h) \quad \text{and} \quad f_2(t+h) = \frac{1}{2}h^2 f_2''(t+\vartheta_h h).$$

For $h \in [0,1] - t$ satisfying $|h| \leqslant \min\{\eta, \delta\}$, compute

$$x_1(t, t+h) = f_1(t+h) - \frac{1}{2}hf_1'(t+h)\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)}, \quad x_2(s,t) = 0$$

and

$$\frac{x_1(t,t+h) - f_1(t)}{hf_1'(t)} = \frac{1}{f_1'(t)}\frac{f_1(t+h) - f_1(t)}{h} - \frac{1}{2}\frac{f_1'(t+h)}{f_1'(t)}\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)}.$$

We thus get

$$\lim_{h\to 0}\frac{|x_f(t,t+h) - f(t)|}{|hf'(t)|} = \left|\lim_{h\to 0}\frac{x_1(t,t+h) - f_1(t)}{hf_1'(t)}\right| = \frac{1}{2}.$$

On the other hand, compute for $h \in [0,1] - t$ satisfying $|h| \leqslant \min\{\eta, \delta\}$

$$\lim_{h\to 0}\frac{x_1(t,t+h) - f_1(t+h)}{h} = -\lim_{h\to 0}\frac{1}{2}f_1'(t+h)\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)} = -\frac{1}{2}f_1'(t),$$

and from $f_2(t) = 0$ we get

$$\lim_{h\to 0}\frac{x_2(t,t+h) - f_2(t+h)}{h} = -\lim_{h\to 0}\frac{f_2(t+h) - f_2(t)}{h} = -f_2'(t) = 0.$$

Consequently we obtain

$$\lim_{h\to 0}\frac{|x_f(t,t+h)-f(t+h)|}{|hf'(t)|}=\frac{1}{2}$$

and the proof is complete. □

Using the previous lemma, we get the following.

Proposition 6.10. *Assume that $U\subseteq\mathbb{R}^2$ is an open set and*

1. *$f:[0,1]\to U$ is of class C^2, injective and regular with nonzero curvature at each $t\in[0,1]$,*
2. *$\phi:U\to\mathbb{R}$ is such that for every $0\leqslant t\leqslant 1$*

$$\lim_{h\to 0}\sup_{0\leqslant t\leqslant 1}\frac{\phi[f(t+h)]-\phi[x_f(t,t+h)]}{|f(t+h)-x_f(t,t+h)|}=0,$$

using the same notations as in Lemma 6.9.

Then $\phi[f(1)]=\phi[f(0)]$.

Proof. Fix $t\in[0,1]$ and $h\in[0,1]-t$ sufficiently small and compute, using the same notations as in Lemma 6.9,

$$\begin{aligned}\frac{\phi[f(t+h)]-\phi[f(t)]}{h}&=\\ &\frac{\phi[f(t+h)]-\phi[x_f(t,t+h)]}{|f(t+h)-x_f(t,t+h)|}\cdot\frac{|f(t+h)-x_f(t,t+h)|}{h}\\ &\quad+\frac{\phi[x_f(t,t+h)]-\phi[f(t)]}{|f(t)-x_f(t,t+h)|}\cdot\frac{|f(t)-x_f(t,t+h)|}{h}.\end{aligned}$$

In particular we get, using Lemma 6.9,

$$(\phi\circ f)'(t)=\lim_{h\to 0}\frac{\phi[f(t+h)]-\phi[f(t)]}{h}=0$$

and ϕ is constant on $\operatorname{im} f$. □

Hence the situation appearing in Example 6.5 cannot happen in the plane.

Theorem 6.11. *Assume that $U \subseteq \mathbb{R}^2$ is an open set and*

1. *$f : [0,1] \to U$ is of class C^2, injective and regular with nonzero curvature at each $t \in [0,1]$,*

2. *$\phi : U \to \mathbb{R}$ is constant on $f(t) + \operatorname{span}\langle f'(t)\rangle$ for each $0 \leqslant t \leqslant 1$.*

Then $\phi[f(1)] = \phi[f(0)]$.

6.2 No Wolfe-like representation theorem

Throughout this section, $1 \leqslant m \leqslant n$ will denote an integer.

For $x \in \mathbb{R}^n$ and $T \in \mathbf{N}_m(\mathbb{R}^n)$ we let $\operatorname{diam}(T,x) = \operatorname{diam}(\{x\} \cup \operatorname{supp} T)$ and define the *regularity* $\operatorname{reg}(T,x)$ of T at x by

$$\operatorname{reg}(T,x) = \begin{cases} \frac{\mathbf{M}(T)}{\mathbf{M}(\partial T)\operatorname{diam}(T,x)} & \text{if } T \neq 0 \text{ and } \partial T \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given $x \in \mathbb{R}^n$ and $\xi \in \mathbf{G}_0(n,m)$ we can define the *derivate* of an m-charge F in the direction of ξ as follows.

Definition 6.12. Let $F \in \mathbf{CH}_m(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and fix $\xi \in \mathbf{G}_0(n,m)$. For $\eta > 0$, we let

$$\underline{D}_\eta F(x,\xi) = \sup_{\delta>0} \inf \frac{F(T)}{\mathbf{M}(T)} \quad \text{and} \quad \overline{D}_\eta F(x,\xi) = \inf_{\delta>0} \sup \frac{F(T)}{\mathbf{M}(T)},$$

where the infimum and the supremum are taken over all $T \in \mathbf{N}_m(\mathbb{R}^n)$ satisfying the following conditions:

1. $\operatorname{diam}(T,x) \leqslant \delta$;

2. $\operatorname{supp} T \subseteq x + \operatorname{span}\langle \xi \rangle$;

3. $\operatorname{reg} T \geqslant \eta$.

One defines in the same way the *lower and upper derivates* of F along ξ by the formulas

$$\underline{D}F(x,\xi) = \inf_{\eta>0} \underline{D}_\eta F(x,\xi) \quad \text{and} \quad \overline{D}F(x,\xi) = \sup_{\eta>0} \overline{D}_\eta F(x,\xi) \quad .$$

In case the lower and upper directional derivates of F at (x,ξ) coincide and are real we denote by $DF(x,\xi)$ their common value, and call it the *derivate* of F at x along ξ.

Remark 6.13. In some sense, the previous definition of the derivate $DF(x,\xi)$ is the weakest we can think of. It gives *a priori* some hope to obtain the *existence* of $DF(x,\xi)$ for "a lot of" $x \in \mathbb{R}^n$ so that the mapping $\xi \mapsto DF(x,\xi)$ could be extended to $\phi(x) \in (\Lambda_m \mathbb{R}^n)^* \simeq \Lambda^m \mathbb{R}^n$. We would then obtain a Wolfe-like representation theorem if we can show

$$F(T) = \int_{\mathbb{R}^n} \langle \vec{T}(x), \phi(x) \rangle \, d\|T\|(x) = \int_{\mathbb{R}^n} DF(x, \vec{T}(x)) \, d\|T\|(x). \quad (6.11)$$

Unfortunately, (6.11) doesn't hold in general.

Example 6.14. Let f and ϕ be as in Example 6.5 and define $T_f \in \mathbf{N}_1(\mathbb{R}^3)$ by the formula

$$\langle T_f, \omega \rangle = \sum_{i=1}^{3} \int_0^1 f_i'(t) \omega_i[f(t)] \, dt,$$

for $\omega = \sum_{i=1}^3 \omega_i dx_i \in \mathscr{D}^1(\mathbb{R}^3)$. Observe for $\zeta \in \mathscr{D}(\mathbb{R}^3)$ that

$$\langle \partial T_f, \zeta \rangle = \int_0^1 \sum_{i=1}^{3} f_i'(t) \partial_i \zeta[f(t)] \, dt = \int_0^1 (\zeta \circ f)'(t) \, dt = \zeta[f(1)] - \zeta[f(0)].$$

Furthermore we have $\operatorname{supp} T_f \subseteq \operatorname{im} f$. Notice that

$$\Gamma(\phi)(T_f) = \phi[f(1)] - \phi[f(0)] = 1 > 0. \quad (6.12)$$

Yet if $T \in \mathbf{N}_1(\mathbb{R}^3)$ satisfies $\operatorname{supp} T \subseteq f(t) + \operatorname{span}\langle f'(t) \rangle$ for some $0 \leqslant t \leqslant 1$ we get

$$\Gamma(\phi)(T) = t \int_{\mathbb{R}^3} \partial \vec{T}(x) \, d\|\partial T\|(x) = \langle \partial T, \zeta \rangle = \langle T, d\zeta \rangle = 0$$

where $\zeta \in \mathscr{D}(\mathbb{R}^3)$ is any 0-differential form satisfying $\zeta = 1$ on a neighbourhood of $\operatorname{supp} T$. In particular we get

$$D\Gamma(\phi)(f(t), \vec{T}_f[f(t)]) = 0,$$

hence (6.11) cannot hold without contradicting (6.12).

As a consequence, there is no hope to obtain a representation theorem (even weaker than Theorem 5.15) using a derivation process as in the proof of Wolfe's theorem (see [Whi57]).

6.3 Derivating m-charges

In this section, $1 \leqslant m \leqslant n$ is an integer.

We introduce another way of derivating m-charges, that we will use in the sequel.

Definition 6.15. Let $F \in \mathbf{CH}_m(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and fix $\xi \in \mathbf{G}_0(n,m)$. For $\eta > 0$, we let

$$\underline{\mathfrak{D}}_\eta F(x,\xi) = \sup_{\delta>0} \inf \frac{F(T)}{\mathbf{M}(T)} \quad \text{and} \quad \overline{\mathfrak{D}}_\eta F(x,\xi) = \inf_{\delta>0} \sup \frac{F(T)}{\mathbf{M}(T)},$$

where the infimum and the supremum are taken over all $T \in \mathbf{N}_m(\mathbb{R}^n)$ satisfying the following conditions:

1. $\operatorname{diam}(T,x) \leqslant \delta$;
2. $\int_{\mathbb{R}^n} \|\vec{T}(y) - \xi\| \, d\|T\|(y) \leqslant \delta \mathbf{M}(T)$;
3. $\operatorname{reg} T \geqslant \eta$.

One defines in the same way the *lower and upper derivatives* of F along ξ by the formulas

$$\underline{\mathfrak{D}} F(x,\xi) = \inf_{\eta>0} \underline{D}_\eta F(x,\xi) \quad \text{and} \quad \overline{\mathfrak{D}} F(x,\xi) = \sup_{\eta>0} \overline{D}_\eta F(x,\xi) \quad .$$

In case the lower and upper directional derivatives of F at (x, ξ) coincide and are real we denote by $\mathfrak{D}F(x,\xi)$ their common value, call it the *derivative* of F at x along ξ and we say that F is *differentiable at x along ξ*.

We will prove that $\Gamma(\zeta)$ is differentiable at $x \in \mathbb{R}^n$ along all directions whenever ζ is differentiable at x.

Proposition 6.16. *Assume $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$ is differentiable at $x \in \mathbb{R}^n$. Then for each $\xi \in \mathbf{G}_0(n,m)$, $\Gamma(\zeta)$ is differentiable at x along ξ and*

$$\mathfrak{D}\Gamma(\zeta)(x,\xi) = \langle \xi, d\zeta(x) \rangle.$$

Proof. Fix $\xi \in \mathbf{G}_0(n,m)$, $\varepsilon > 0$ and $\eta > 0$. There exists a linear map $\lambda : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ and $0 < \delta \leqslant \varepsilon$ for which

$$\|\zeta(x+h) - \zeta(x) - \lambda(h)\| \leqslant \varepsilon\eta|h|$$

for each $h \in \mathbb{R}^n$ satisfying $|h| \leqslant \delta$. Define for $y \in \mathbb{R}^n$, $\phi(y) = \zeta(x) + \lambda(y-x)$ and observe $d\phi(y) = d\zeta(x)$ for all $y \in \mathbb{R}^n$. Let $T \in \mathbf{N}_m(\mathbb{R}^n)$ satisfy

1. $\operatorname{diam}(T,x) \leqslant \delta$;
2. $\int_{\mathbb{R}^n} \|\vec{T}(y) - \xi\| \, d\|T\|(y) \leqslant \delta\mathbf{M}(T)$;
3. $\operatorname{reg} T \geqslant \eta$.

and compute

$$|\Gamma(\zeta)(T) - \langle \xi, d\zeta(x)\rangle \mathbf{M}(T)| \leqslant |\langle \partial T, \zeta - \phi\rangle| + |\langle \partial T, \phi \rangle - \langle \xi, d\zeta(x)\rangle\mathbf{M}(T)|.$$

Yet

$$\begin{aligned} |\langle \partial T, \zeta - \phi\rangle| &\leqslant \int_{\mathbb{R}^n} \|\zeta(y) - \phi(y)\| \, d\|\partial T\|(y) \\ &\leqslant \varepsilon\eta \int_{\operatorname{supp} T} |y-x| \, d\|\partial T\|(y) \leqslant \varepsilon\eta \operatorname{diam}(T,x)\mathbf{M}(\partial T) \leqslant \varepsilon \mathbf{M}(T) \end{aligned}$$

and

$$\begin{aligned}
|\langle \partial T, \phi\rangle - \langle \xi, d\zeta(x)\rangle \mathbf{M}(T)| &= \left| \langle T, d\phi\rangle - \int_{\mathbb{R}^n} \langle \xi, d\zeta(x)\rangle \, d\|T\|(y) \right|, \\
&\leqslant \int_{\mathbb{R}^n} |\langle \vec{T}(x) - \xi, d\zeta(x)\rangle| \, d\|T\|(y), \\
&\leqslant \varepsilon \|d\zeta(x)\| \int_{\mathbb{R}^n} \|\vec{T}(x) - \xi\| \, d\|T\|(y), \\
&\leqslant \varepsilon \|d\zeta(x)\| \mathbf{M}(T). \qquad (6.13)
\end{aligned}$$

We get finally

$$|\Gamma(\zeta)(T) - \langle \xi, d\zeta(x)\rangle \mathbf{M}(T)| \leqslant \varepsilon(1 + \|d\zeta(x)\|)\mathbf{M}(T)$$

and $\mathfrak{D}\Gamma(\zeta)(x, \xi) = \langle \xi, d\zeta(x)\rangle$. □

Remark 6.17. The equality $\langle \partial T, \phi\rangle = \langle T, d\phi\rangle$ used to justify the first line in (6.13) holds as there exists $\psi \in \mathscr{D}^{m-1}(\mathbb{R}^n)$ coinciding with ϕ on a neighbourhood of $\operatorname{supp} T$.

Chapter 7

Luzin's theorem for m-charges

First, recall a result of G. Alberti in [Alb91].

7.1 A Luzin's theorem for gradients

The following lemma will be useful in the sequel.

Lemma 7.1. *Suppose $U \subseteq \mathbb{R}^n$ is a nonempty open set satisfying $|U| < \infty$. Given $0 < \varepsilon \leqslant 1$, $\eta > 0$ and $v \in C(U, \mathbb{R}^n)$, there exists $K \in \mathcal{K}(U)$ and a function $u \in C_c^1(U)$ for which the following hold:*

1. $|U \setminus K| \leqslant \varepsilon |U|$;
2. *for each* $x \in K$, $|v(x) - \nabla u(x)| \leqslant \eta$;
3. $|\nabla u|_\infty \leqslant C_1 \varepsilon^{-1} |v|_\infty$;

where $C_1 = C_1(n) > 0$ is a constant depending only on n.

Proof. Choose $K' \in \mathcal{K}(U)$ for which $|U \setminus K'| < \varepsilon |U|/2$. There exists $\delta > 0$ such that $B_\infty[x, 2\delta] \subseteq U$ holds for each $x \in K'$, and for which given $y \in U$ satisfying $|y - x|_\infty \leqslant \delta$ we have $|v(x) - v(y)| \leqslant \eta$.

Enumerate $x_1, x_2, \ldots, x_m$ the elements of the set

$$\left\{x \in \mathbb{R}^n : x \in \delta\mathbb{Z}^n, B_\infty\left[x, \frac{\delta}{2}\right] \cap K' \neq \emptyset\right\}$$

and let $Q_j = B_\infty[x_j, \delta/2] \subseteq U$, $1 \leqslant j \leqslant m$. For $1 \leqslant j \leqslant m$, define

$$a_j = \frac{1}{|Q_j|}\int_{Q_j} v\,dx \quad \text{and} \quad Q_j' = B_\infty\left[x_j, \left(1 - \frac{\varepsilon}{2n}\right)\frac{\delta}{2}\right].$$

Fix any $1 \leqslant j \leqslant m$ and from $\operatorname{dist}(Q_j', \mathbb{R}^n \setminus Q_j) = \frac{\varepsilon\delta}{4n} > 0$ deduce the existence of $\psi_j \in \mathscr{D}(\mathbb{R}^n)$ satisfying the following conditions:

1. $\chi_{Q_j'} \leqslant \psi_j \leqslant \chi_{Q_j}$,
2. $|\nabla\psi_j|_\infty \leqslant 2 \cdot \frac{4n}{\varepsilon\delta}$.

Define $u \in C_c^1(\mathbb{R}^n)$ by the formula

$$u(x) = \sum_{j=1}^m \psi_j(x)\, a_j \cdot (x - x_j).$$

Obviously $\operatorname{supp} u \subseteq \bigcup_{j=1}^m Q_j$ and for $x \in Q_j'$ we get

$$\nabla u(x) = \sum_{j=1}^m [\nabla\psi_j(x)\, a_j \cdot (x - x_j) + \psi_j(x) a_j] = a_j.$$

Fix $1 \leqslant j \leqslant m$. By the choice of Q_j' we obtain

$$|Q_j \setminus Q_j'| \leqslant \left[1 - \left(1 - \frac{\varepsilon}{2n}\right)^n\right] |Q_j|.$$

Yet

$$1 - \left(1 - \frac{\varepsilon}{2n}\right)^n = \left[1 - \left(1 - \frac{\varepsilon}{2n}\right)\right]\left[1 + \left(1 - \frac{\varepsilon}{2n}\right) + \cdots + \left(1 - \frac{\varepsilon}{2n}\right)^{n-1}\right] \leqslant \frac{\varepsilon}{2n} \cdot n = \frac{\varepsilon}{2}$$

and we get $|Q_j \setminus Q_j'| \leqslant \frac{\varepsilon}{2}|Q_j|$ from which it follows that the compact set $K = \bigcup_{j=1}^m Q_j'$ satisfies

$$|U \setminus K| \leqslant |U \setminus K'| + \sum_{j=1}^m |Q_j \setminus Q_j'| \leqslant \frac{\varepsilon}{2}|U| + \frac{\varepsilon}{2}|U| = \varepsilon|U|,$$

as it is obvious that $x \in U \setminus K$ belongs to $\bigcup_{j=1}^m Q_j \setminus Q_j'$ whenever it also belongs to K'.

On the other hand, choose for each $1 \leqslant j \leqslant m$ a $x_j' \in Q_j$ for which $a_j = f(x_j')$ and observe for $x \in Q_j'$ that

$$|x - x_j'|_\infty \leqslant |x - x_j|_\infty + |x_j - x_j'|_\infty \leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

In particular given $x \in K$ we get a $1 \leqslant j \leqslant m$ for which $x \in Q_j'$ so that

$$|v(x) - \nabla u(x)| = |v(x) - v(x_j')| \leqslant \eta.$$

Finally, compute for $x \in U$

$$\nabla u(x) = \sum_{j=1}^m [\nabla\psi_j(x)\, a_j \cdot (x - x_j) + \psi_j(x) a_j]$$

and observe that given $x \in U$ for all but one $1 \leqslant j \leqslant m$ the numbers $\nabla\psi_j(x)$ and $\psi_j(x)$ are zero; in case one of those two numbers is nonzero, we get $x \in Q_j$. In particular

$$|\nabla u|_\infty \leqslant \frac{8n}{\delta\varepsilon}|v|_\infty \frac{\delta\sqrt{n}}{2} + |v|_\infty \leqslant (4n^{3/2} + 1)\varepsilon^{-1}|v|_\infty$$

as $0 < \varepsilon \leqslant 1$. □

We get the following as a corollary.

Corollary 7.2. *Assume $U \subseteq \mathbb{R}^n$ is a nonempty open set, $|U| < \infty$ and $v : U \to \mathbb{R}^n$ is a continuous, bounded vectorfield. For every $0 < \varepsilon \leqslant 1$ there exists an open set $U_0 \subseteq U$ and a function $u \in C_0^1(U)$ satisfying the following conditions:*

1. $|U_0| \leqslant \varepsilon |U|$*;*

2. for each $x \in U \setminus U_0$*,* $v(x) = \nabla u(x)$*;*

3. $|\nabla u|_\infty \leqslant C_2(n)\varepsilon^{-1}|v|_\infty$ *where* $C_2(n)$ *is a constant depending only on* n*.*

Proof. Let $(\eta_k) \subseteq (0, \infty)$ be a sequence satisfying

$$\sum_{k=0}^{\infty} 2^k \eta_k \leqslant |v|_\infty,$$

and construct inductively $(u_k) \subseteq C_c^1(U)$, $(K_k) \subseteq \mathcal{K}(U)$ and $(v_k) \subseteq C(U)$ as follows. According to Lemma 7.1, choose $u_0 \in C_c^1(U)$ and $K_0 \in \mathcal{K}(U)$ such that

1. $|U \setminus K_0| \leqslant \frac{\varepsilon}{2}|U|$;

2. for each $x \in K_0$, $|\nabla u_0(x) - v(x)| \leqslant \eta_0$;

3. $|\nabla u_0|_\infty \leqslant 2C_1\varepsilon^{-1}|v|_\infty$.

Assuming u_k, K_k and v_k have been constructed for $k \geqslant 0$, choose $u_{k+1} \in C_c^1(U)$ and $K_{k+1} \in \mathcal{K}(U)$ satisfying

1. $|U \setminus K_{k+1}| \leqslant \frac{\varepsilon}{2^{k+2}}|U|$;

2. for each $x \in K_{k+1}$, $|\nabla u_{k+1}(x) - v_k(x)| \leqslant \eta_{k+1}$;

3. $|\nabla u_{k+1}|_\infty \leqslant 2^{k+2}C_1\varepsilon^{-1}|v_k|_\infty$;

according to Lemma 7.1; choose $v_{k+1} \in C(U, \mathbb{R}^n)$ an extension of $(\nabla u_{k+1} - v_k) \restriction K_{k+1}$ to U satisfying

$$|v_{k+1}|_\infty = \sup_{x \in K_{k+1}} |\nabla u_{k+1}(x) - v_k(x)| \leqslant \eta_{k+1},$$

according to Tietze's extension theorem.

Observe

$$\sum_{k=0}^{\infty} |\nabla u_k|_\infty \leqslant 2C_1\varepsilon^{-1}\left[|v|_\infty + \sum_{k=0}^{\infty} 2^k |v_k|_\infty\right]$$
$$\leqslant 2C_1\varepsilon^{-1}\left[|v|_\infty + \sum_{k=0}^{\infty} 2^k \eta_k\right] \leqslant 4C_1\varepsilon^{-1}|v|_\infty. \quad (7.1)$$

As u_k together with its gradient can be extended in a contiuous way to $\bar{U}$ by 0, the series $\sum_{k\in\mathbb{N}} u_k(x_0)$ converges to 0 for any $x_0 \in \partial U$. On the other hand, (7.1) ensures the existence of $u \in C_0^1(U)$ for which

1. $\sum_{k\in\mathbb{N}} u_k$ uniformly converges to u on U;

2. $\sum_{k\in\mathbb{N}} \nabla u_k$ uniformly converges to ∇u on U.

Moreover we infer from (7.1)

$$|\nabla u|_\infty \leqslant C_2\varepsilon^{-1}|v|_\infty$$

if we let $C_2(n) = 4C_1(n)$.

Let $U_0 = U \setminus \bigcap_{k\in\mathbb{N}} K_k$ and observe

$$|U_0| \leqslant \sum_{k=0}^{\infty} |U \setminus K_k| \leqslant \sum_{k=0}^{\infty} \frac{\varepsilon|U|}{2^{k+1}} = \varepsilon|U|.$$

For each $x \in U \setminus U_0 = \bigcap_{k\in\mathbb{N}} K_k$ and $m \in \mathbb{N}$ we get $v_m(x) = v(x) - \sum_{k=0}^{m} \nabla u_k(x)$ and

$$|\nabla u(x) - v(x)| = \left|\sum_{k=0}^{\infty} \nabla u_k(x) - v(x)\right|$$
$$\leqslant |v_m(x)| + \sum_{k=m+1}^{\infty} |\nabla u_k(x)| \leqslant \eta_m + \sum_{k=m+1}^{\infty} |\nabla u_k|_\infty.$$

As $\lim_{m\to\infty} \eta_m = \lim_{m\to\infty} \sum_{k=m+1}^{\infty} |\nabla u_k|_\infty = 0$ the result follows immediately. □

Finally, we obtain the following result, due to G. Alberti [Alb91].

Theorem 7.3 (Alberti)**.** *Assume $U \subseteq \mathbb{R}^n$ is a nonempty open set satisfying $|U| < \infty$ and $v : U \to \mathbb{R}^n$ is a measurable vectorfield. Assume v is not a.e. 0 as the result is trivial otherwise. For every $0 < \varepsilon \leqslant 1$ there exists an open set $U_0 \subseteq U$ and a function $u \in C_0^1(U)$ satisfying the following conditions:*

1. *$|U_0| \leqslant \varepsilon |U|$;*

2. *for each $x \in U \setminus U_0$, $v(x) = \nabla u(x)$;*

3. *$|\nabla u|_\infty \leqslant C(n)\varepsilon^{-1}|v|_\infty$ where $C(n)$ is a constant depending only on n.*

Proof. Fix $0 < \varepsilon \leqslant 1$. There exists $r > 0$ such that the set $B = \{x \in U : |v(x)| > r\}$ satisfy $0 < |B| < \frac{\varepsilon|U|}{4}$. According to Luzin's theorem there exists $v_1 \in C(U, \mathbb{R}^n)$ satisfying $v_1 = v$ outside a Borel set $C \subseteq U$ satisfying $|C| \leqslant |B|$. Define $v_2 \in C(U, \mathbb{R}^n)$ by the formula

$$v_2(x) = \begin{cases} v_1(x) & \text{if} \quad |v_1(x)| \leqslant r, \\ \frac{rv_1(x)}{|v_1(x)|} & \text{if} \quad |v_1(x)| > r. \end{cases}$$

The vectorfield v_2 is bounded, continuous and agrees with v outside $C \cup B$, moreover $|C \cup B| < \frac{\varepsilon|U|}{2}$ and so there exists an open set $U_1 \supseteq C \cup B$ satisfying $|U_1| < \frac{\varepsilon|U|}{2}$ and for which $v_2 = v$ outside U_1. We distinguish two cases of figure:

1. if $|B| = 0$, then we get $|C \cup B| = 0$ and so

$$|v_2|_\infty = |v_2 \restriction [U \setminus (C \cup B)]|_\infty \leqslant |v|_\infty$$

   as v_2 and v agree outside $C \cup B$;

2. if $|B| > 0$ then notice that $|v_2|_\infty \leqslant r \leqslant |v|_\infty$.

As a consequence, $|v_2|_\infty \leqslant |v|_\infty$.

As v_2 is bounded and continuous, Corollary 7.2 yields an open set $U_2 \subseteq U$ satisfying $|U_2| \leqslant \frac{\varepsilon|U|}{2}$ and a function $u \in C_0^1(U)$ satisfying $\nabla u = v_2$ outside U_2 as well as

$$|\nabla u|_\infty \leqslant C_2 \left(\frac{\varepsilon}{2}\right)^{-1} |v_2|_\infty,$$

where $C_2 = C_2(n)$ is the constant appearing in Corollary 7.2. Consequently we obtain $\nabla u = v$ outside $U_0 = U_1 \cap U_2$ and we get

$$|\nabla u|_\infty \leqslant 2C_2\varepsilon^{-1}|v|_\infty.$$

The proof is complete if we let $C(n) = 2C_2(n)$. □

A simple corollary is the following.

Theorem 7.4. *Assume $U \subseteq \mathbb{R}^n$ is a nonempty open set, $|U| < \infty$ and $v : U \to \mathbb{R}^n$ is a measurable vectorfield. For every $0 < \varepsilon \leqslant 1$ there exists $K \in \mathcal{K}(U)$ and a function $u \in C_0^1(U)$ satisfying the following conditions:*

1. $|U \setminus K| \leqslant \varepsilon$;
2. *for each $x \in K$, $v(x) = \nabla u(x)$;*
3. $|\nabla u|_\infty \leqslant \beta(n)\varepsilon^{-1}|v|_\infty$ *where $\beta(n)$ is a constant depending only on n.*

Proof. Fix $\varepsilon > 0$ and let u and U_0 be associated to U, v and $\varepsilon/(2|U|)$ according to Theorem 7.3. Choose $K \in \mathcal{K}(U_0)$ satisfying $|U_0 \setminus K| \leqslant \frac{\varepsilon}{2|U|}$. Observe

$$|U \setminus K| \leqslant |U \setminus U_0| + |U_0 \setminus K| \leqslant \frac{\varepsilon}{2|U|}|U| + \frac{\varepsilon}{2|U_0|}|U| \leqslant \varepsilon.$$

Defining $\beta = 2C$ where $C = C(n)$ is the constant appearing in Theorem 7.3, we get condition 3. □

We will make use of the following consequence of Theorem 7.4.

Proposition 7.5 (M.-Pfeffer (2007))**.** *Assume $U \subseteq \mathbb{R}^n$ is a nonempty open set satisfying $|U| < \infty$ and $v : U \to \mathbb{R}^n$ is a measurable vectorfield. For every $0 < \varepsilon \leqslant 1$ there exists $K \in \mathcal{K}(U)$ and a function $u \in C_c^1(\mathbb{R}^n)$ supported in U and satisfying the following conditions:*

1. *$|U \setminus K| \leqslant \varepsilon$;*
2. *for each $x \in K$, $v(x) = \nabla u(x)$;*
3. *for every $x \in \mathbb{R}^n$, $|u(x)| \leqslant \varepsilon \min\{1, \operatorname{dist}^2(x, \mathbb{R}^n \setminus U)\}$.*

Proof. Choose an open set $\emptyset \neq V \subseteq U$ for which $\bar{V} \in \mathfrak{K}(U)$ and $|U \setminus V| \leqslant \frac{\varepsilon}{2}$. Choose $C \in \mathfrak{K}(V)$ satisfying $|V \setminus C| \leqslant \frac{\varepsilon}{4}$ and such that $u \restriction C$ is bounded. The formula

$$w(x) = \begin{cases} v(x) & \text{if} \quad x \in C, \\ 0 & \text{if} \quad x \in V \setminus C, \end{cases}$$

defines a bounded measurable vectorfield $w : V \to \mathbb{R}^n$. Define

$$\delta = \min\{1, \operatorname{dist}^2(V, \mathbb{R}^n \setminus U)\} \quad \text{and} \quad d = \frac{\varepsilon^2 \delta}{1 + 8\beta |V| |w|_\infty}$$

where β is the constant appearing in Theorem 7.4, and find nonoverlapping closed cubes $Q_1, Q_2, \ldots, Q_m$ satisfying $\operatorname{diam} Q_j \leqslant d$ and $Q_j \subseteq V$ for each $1 \leqslant j \leqslant m$, as well as

$$\left| V \setminus \bigcup_{j=1}^{m} Q_j \right| \leqslant \frac{\varepsilon}{8}.$$

Fix $1 \leqslant j \leqslant m$ and apply Theorem 7.4 to the vectorfield $w_j = w \restriction \mathring{Q}_j$ on the open set $\mathring{Q}_j$ and find a compact set $K_j \subseteq \mathring{Q}_j$ and a function $u_j \in C_c^1(\mathbb{R}^n)$ supported in Q_j satisfying the following conditions:

1. $|Q_j \setminus K_j| \leqslant \frac{\varepsilon}{8|V|} |Q_j|$;
2. for each $x \in K_j$, $\nabla u_j(x) = w(x)$;
3. $|\nabla u_j|_\infty \leqslant \frac{8\beta|V|}{\varepsilon} |w_j|_\infty$;

so that the mean value theorem implies

$$\begin{aligned} |u_j|_\infty \leqslant \operatorname{diam}(Q_j) |\nabla u_j|_\infty \leqslant d \cdot \frac{8\beta |V| |w_j|_\infty}{\varepsilon} \\ \leqslant \frac{\varepsilon^2 \delta}{1 + 8\beta |V| |w|_\infty} \frac{8\beta |V| |w|_\infty}{\varepsilon} \leqslant \varepsilon \delta. \end{aligned}$$

Let $K_0 = \bigcup_{j=1}^m K_j$ and observe K_0 is a compact subset of V with

$$|V \setminus K_0| \leqslant \left| V \setminus \bigcup_{j=1}^{m} Q_j \right| + \sum_{j=1}^{m} |Q_j \setminus K_j| \leqslant \frac{\varepsilon}{8} + \frac{\varepsilon}{8|V|} \sum_{j=1}^{m} |Q_j| \leqslant \frac{\varepsilon}{4}.$$

Moreover $u = \sum_{j=1}^m u_j$ belongs to $C_c^1(\mathbb{R}^n)$, is supported in $V \subseteq U$ and satisfies $|u|_\infty \leqslant \varepsilon\delta$ and for $x \in K_0$ we get $\nabla u(x) = w(x)$. In particular we have $\nabla u(x) = v(x)$ for each $x \in K$ if we let $K = K_0 \cap C$. Clearly, $K \in \mathcal{K}(U)$ and

$$|U \setminus K| \leqslant |U \setminus V| + |V \setminus K_0| + |V \setminus C| \leqslant \varepsilon.$$

Moreover $u(x) = 0$ for each $x \in \mathbb{R}^n \setminus V$, on the other hand for $x \in V$ we get

$$|u(x)| \leqslant |u|_\infty \leqslant \varepsilon\delta \leqslant \varepsilon \min\{1, \operatorname{dist}^2(x, \mathbb{R}^n \setminus U)\}.$$

The proof is complete. $\square$

We are now able to give the following generalization of Luzin's theorem.

Theorem 7.6 (M.-Pfeffer (2007)). *Assume $U \subseteq \mathbb{R}^n$ is an open set and $v : U \to \mathbb{R}^n$ is a measurable vectorfield. Given $0 < \varepsilon \leqslant 1$ there exists $u \in C(\mathbb{R}^n)$ differentiable almost everywhere in $\mathbb{R}^n$ and satisfying the following conditions:*

1. $|u|_\infty \leqslant \varepsilon$ *and* $\operatorname{supp} u \subseteq U$*;*
2. *for almost all $x \in U$, $v(x) = \nabla u(x)$;*
3. *for every $x \in \mathbb{R}^n \setminus U$, $\nabla u(x) = 0$.*

Proof. For $k \in \mathbb{N}$ write $B_k = B(0, 2^k)$. Let $U_0 = U \cap B_0$ and choose $u_0 \in C_c^1(\mathbb{R}^n)$ supported in U_0 and satisfying the following conditions, according to Proposition 7.5:

1. $|U_0 \setminus K_0| \leqslant 1/2$;
2. for each $x \in K_0$, $v(x) = \nabla u_0(x)$;

3. for every $x \in \mathbb{R}^n$, $|u_0(x)| \leqslant \frac{\varepsilon}{2} \min\{1, \operatorname{dist}^2(x, \mathbb{R}^n \setminus U_0)\}$.

Assuming $U_0, U_1, \dots, U_k$ and $u_0, u_1, \dots u_k$ have been constructed for some $k \in \mathbb{N}$, we define $U_{k+1} = U \cap B_{k+1} \setminus \bigcup_{j=0}^{k} K_j$ and choose $u_{k+1} \in C_c^1(\mathbb{R}^n)$ supported in U_{k+1} satisfying the following conditions, according to Proposition 7.5:

1. $|U_{k+1} \setminus K_{k+1}| \leqslant \frac{1}{2^{k+2}}$;
2. for each $x \in K_{k+1}$, $\nabla u_{k+1}(x) = v(x) - \sum_{j=0}^{k} \nabla u_j(x)$;
3. for every $x \in \mathbb{R}^n$, $|u_{k+1}(x)| \leqslant \frac{\varepsilon}{2^{k+2}} \min\{1, \operatorname{dist}^2(x, \mathbb{R}^n \setminus U_{k+1})\}$.

Let $K = \bigcup_{k \in \mathbb{N}} K_k$, define $u = \sum_{k=0}^{\infty} u_k$ and observe

$$\sum_{k=0}^{\infty} |u_k|_\infty \leqslant \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \varepsilon.$$

As a consequence the Weierstrass convergence criterion guarantees u is continuous in $\mathbb{R}^n$ and $|u|_\infty \leqslant \varepsilon$. It is clear that $\operatorname{supp} u \subseteq U$. For $k \in \mathbb{N}$ compute for any $l > k$

$$|U \cap B_k \setminus K| \leqslant \left| \left(U \cap B_l \setminus \bigcup_{j=0}^{l-1} K_j \right) \setminus K_l \right| = |U_l \setminus K_l| \leqslant \frac{1}{2^{l+1}}$$

and conclude $|U \cap B_k \setminus K| = 0$ for each $k \in \mathbb{N}$. Consequently

$$|U \setminus K| = \left| \bigcup_{k \in \mathbb{N}} U \cap B_k \setminus K \right| = 0.$$

For $x \in K$, choose $k \in \mathbb{N}$ for which $x \in K_k$ and observe u_l is supported in U_l for $l > k$ while $K_k \cap U_l = \emptyset$. In particular $u_l(x) = 0$ for $l > k$. If $h \in \mathbb{R}^n$ is such that $x + h \in \mathbb{R}^n \setminus U_l$ we get $u_l(x+h) = 0$, in case $x + h \in U_l$ we obtain

$$|u_l(x+h)| \leqslant \frac{1}{2^{l+1}} \min\{1, \operatorname{dist}^2(x+h, \mathbb{R}^n \setminus U_l)\} \leqslant \frac{|h|^2}{2^{l+1}}$$

In either case we get $|u_l(x+h)| \leqslant \frac{|h|^2}{2^{l+1}}$ for $l > k$ and

$$\left| \sum_{l=k+1}^{\infty} u_l(x+h) - \sum_{l=k+1}^{\infty} u_l(x) \right| \leqslant \sum_{l=k+1}^{\infty} |u_l(x+h)| \leqslant |h|^2 \sum_{l=k+1}^{\infty} \frac{1}{2^{l+1}}.$$

Consequently $\nabla u_l(x) = 0$ for $l > k$ and $\nabla \left[\sum_{l=k+1}^{\infty} u_l\right](x) = 0$ so that

$$\nabla u(x) = \sum_{j=0}^{k} \nabla u_j(x) = v(x) - \nabla u_{k+1}(x) = v(x).$$

Finally for $x \in \mathbb{R}^n \setminus U$ we get $u_k(x) = 0$. In case $h \in \mathbb{R}^n$ is such that $x + h \notin U_k$ we get $u_k(x+h)$, while we obtain

$$|u_k(x+h)| \leqslant \frac{1}{2^{k+1}} \min\{1, \operatorname{dist}^2(x+h, \mathbb{R}^n \setminus U_k)\} \leqslant \frac{|h|^2}{2^{k+1}}$$

in case $x + h \in U_k$. In either case

$$|u(x+h) - u(x)| \leqslant \sum_{k=0}^{\infty} |u_k(x+h)| \leqslant |h|^2$$

from which it follows that $\nabla u(x) = 0$. $\square$

7.2 A Luzin's theorem for differential forms

Let $0 \leqslant m \leqslant n$ denote an integer. An m-differential form

$$\omega = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \omega_{i_1, i_2, \ldots, i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

is called measurable (resp. a.e. differentiable) whenever $\omega_{i_1, i_2, \ldots, i_m}$ is a measurable (resp. a.e. differentiable) function for each m-tuple $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$.

Theorem 7.7 (M.-Pfeffer (2007)). *Let $0 \leqslant m \leqslant n-1$ be an integer and let ω be a measurable $(m+1)$-differential form in $\mathbb{R}^n$. For each $\varepsilon > 0$, there is an almost everywhere differentiable $\Omega \in \mathscr{C}^m(\mathbb{R}^n)$ satisfying the following conditions:*

1. $\mathbf{M}(\Omega) \leqslant \varepsilon$*;*

2. $d\Omega(x) = \omega(x)$ *for a.e.* $x \in \mathbb{R}^n$*.*

Proof. Write $\omega = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n} \omega_{i_1, i_2, \ldots, i_{m+1}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{m+1}}$ where $\omega_{i_1, i_2, \ldots, i_{m+1}}$ is measurable for each $1 \leqslant i_1 < i_2 < \cdots < i_{m+1} \leqslant n$. Using Theorem 7.6 find for every $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ an almost everywhere differentiable function $\Omega_{i_1, i_2, \ldots, i_m} \in C(\mathbb{R}^n)$ satisfying the following condition:

1. $|\Omega_{i_1, i_2, \ldots, i_m}|_\infty \leqslant \varepsilon (C_n^m)^{-1}$;

2. for almost every $x \in \mathbb{R}^n$ and each $1 \leqslant i \leqslant n$,

$$\partial_i \Omega_{i_1, i_2, \ldots, i_m}(x) = \begin{cases} \omega_{i, i_1, i_2, \ldots, i_m} & \text{if} \quad i < i_1, \\ 0 & \text{if} \quad i \geqslant i_1. \end{cases}$$

Let $\Omega = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \Omega_{i_1, i_2, \ldots, i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$ and observe a.e. on $\mathbb{R}^n$

$$\begin{aligned} d\Omega &= \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n \\ 1 \leqslant i \leqslant n}} \partial_i \Omega_{i_1, i_2, \ldots, i_m} dx_i \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}, \\ &= \sum_{1 \leqslant i < i_1 < i_2 < \cdots < i_m \leqslant n} \omega_{i, i_1, i_2, \ldots, i_m} dx_i \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}, \\ &= \omega. \end{aligned}$$

The proof is complete. □

7.3 A Luzin's theorem for m-charges

Given an integer $0 \leqslant m \leqslant n$, the *oscillation* of $F \in \mathbf{CH}_m(\mathbb{R}^n)$ is the number

$$\operatorname{osc}(F) = \inf \{ \varepsilon > 0 : F(T) \leqslant \varepsilon \text{ for } T \in \mathbf{N}_m(T) \text{ satisfying } \mathbf{N}(T) \leqslant 1 \}.$$

We estimate the oscillation of a flux.

Proposition 7.8. *For $\zeta \in \mathscr{C}^{m-1}(\mathbb{R}^n)$, we have* $\operatorname{osc}[\Gamma(\zeta)] \leqslant \mathbf{M}(\zeta)$.

Proof. Without loss of generality we can assume $0 < \mathbf{M}(\zeta) < \infty$. For $T \in \mathbf{N}_m(\mathbb{R}^n)$ satisfying $\mathbf{N}(T) \leqslant 1$ we get

$$\Gamma(\zeta)(T) = \langle \partial T, \zeta \rangle \leqslant \mathbf{M}(\partial T)\mathbf{M}(\zeta) \leqslant \mathbf{M}(\zeta)\mathbf{N}(T) \leqslant \mathbf{M}(\zeta),$$

so that $\operatorname{osc}[\Gamma(\zeta)] \leqslant \sqrt{\mathbf{M}(\zeta)}$. □

We conclude this chapter by a *Luzin's theorem* for m-charges.

Theorem 7.9 (M.-Pfeffer (2007))**.** *Let $1 \leqslant m \leqslant n$ be an integer and assume ω is a measurable m-differential form in $\mathbb{R}^n$. Given $\varepsilon > 0$ there is an almost everywhere differentiable m-charge $F \in \mathbf{CH}_m(\mathbb{R}^n)$ satisfying the following conditions:*

1. $\operatorname{osc}(F) \leqslant \varepsilon$*;*
2. *for a.e. $x \in \mathbb{R}^n$, we have $\mathfrak{D}F(x,\xi) = \langle \xi, \omega(x) \rangle$ for each $\xi \in \mathbf{G}_0(n,m)$.*

Proof. Let Ω be the $(m-1)$-differential form associated to ω and ε according to Theorem 7.7 and define $F = \Gamma(\Omega)$. Proposition 7.8 allows us to conclude F is a desired m-charge. □

Chapter 8

Removability for $\operatorname{div} v = 0$

In this chapter we assume $n \geqslant 2$ and we present a new proof, due to N.C. Phuc and M. Torres [PT07], of our result about removable singularities for the equation $\operatorname{div} v = 0$ in $\mathbb{R}^n$ [Moo06].

8.1 Removability for bounded vectorfields

The flux of a vectorfield $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ is the distributional divergence of v,

$$\mathfrak{div}\, v : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}, \varphi \mapsto -\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx. \tag{8.1}$$

Introduce removable sets.

Definition 8.1. Fix $\mathscr{B} \subseteq L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ a class of locally bounded measurable vectorfields and let $S \in \mathcal{K}(\mathbb{R}^n)$. We say that S is *removable* for the equation $\operatorname{div} v = 0$ with respect to $\mathscr{B}$ (or simply $\mathscr{B}$-removable) if the following condition holds: for every $v \in \mathscr{B}$,

$$\operatorname{supp}(\mathfrak{div}\, v) \subseteq S \quad \text{implies} \quad \mathfrak{div}\ v \equiv 0.$$

We will say that S is *removable* (for the equation $\operatorname{div} v = 0$) if it is removable with respect to $L^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

In other words there is no vectorfield $v \in \mathscr{B}$ whose nonzero distributional divergence vanishes outside a $\mathscr{B}$-removable set.

8.1.1 A sufficient condition for a compact set to be removable

Following Th. De Pauw [De 03] we get the following.

Lemma 8.2. *Let $S \in \mathcal{K}(\mathbb{R}^n)$ satisfy $0 \leqslant \mathscr{H}^{n-1}(S) < \infty$. There exists a sequence $(\tilde{\chi}_k) \subseteq BV(\mathbb{R}^n)$ satisfying the following conditions:*

1. *$\tilde{\chi}_k = 1$ in a neighbourhood of S, for each integer k;*
2. *$|\operatorname{supp} \tilde{\chi}_k| \to 0$ as $k \to \infty$;*
3. *$\|D\tilde{\chi}_k\| \leqslant C(n)\mathscr{H}^{n-1}(S) + \frac{1}{2^k}$ for $k \in \mathbb{N}$,*

where $C(n)$ is a constant depending only on n.

Proof. Call c_{n-1} the volume of the unit-ball in $\mathbb{R}^{n-1}$. Let k be a nonnegative integer. Choose a finite collection of open cubes $Q_1^k, \dots, Q_m^k$ with $\operatorname{diam} Q_j^k \leqslant 2^{-k}, 1 \leqslant j \leqslant m$, for which the following are satisfied:

$$S \subseteq \bigcup_{j=1}^m Q_j^k \quad \text{and} \quad \sum_{j=1}^m c_{n-1} \left(\frac{\operatorname{diam} Q_j^k}{2} \right)^{n-1} \leqslant \mathscr{H}^{n-1}(S) + \frac{c_{n-1}}{n2^{k+n}}.$$

Let $U_k = \bigcup_{j=1}^m Q_j^k$ and $\tilde{\chi}_k = \chi_{U_k}$. One computes

$$\|D\tilde{\chi}_k\| = \mathscr{H}^{n-1}(\partial U_k) \leqslant \sum_{j=1}^m \mathscr{H}^{n-1}(\partial Q_j^k),$$

and

$$\sum_{j=1}^m \mathscr{H}^{n-1}(\partial Q_j^k) \leqslant 2n \sum_{j=1}^m (\operatorname{diam} Q_j^k)^{n-1} \leqslant \frac{n2^n \mathscr{H}^{n-1}(S)}{c_{n-1}} + \frac{1}{2^k}.$$

On the other hand, one has $|\operatorname{supp} \tilde{\chi}_k| = |U_k| \to 0$ as $k \to \infty$. The result follows. $\square$

A smoothing argument yields the following result.

Corollary 8.3. *Suppose $S \in \mathcal{K}(\mathbb{R}^n)$ satisfies $0 \leqslant \mathscr{H}^{n-1}(S) < \infty$. There exists a sequence $(\chi_k) \subseteq \mathscr{D}(\mathbb{R}^n)$ satisfying the following conditions:*

1. *$\chi_k = 1$ in a neighbourhood of S, for each k;*
2. *$|\operatorname{supp} \chi_k| \to 0$ as $k \to \infty$;*
3. *$\int_{\mathbb{R}^n} |\nabla \chi_k| \, dx \leqslant C(n)\mathscr{H}^{n-1}(S) + \frac{1}{2^{k-1}}$ for each $k \in \mathbb{N}$,*

where $C(n)$ is a constant depending only on n.

Proof. We use the same notations as in the proof of Lemma 8.2, define

$$\delta_k = \operatorname{dist}(S, \partial U_k) > 0,$$

choose an integer $j(k) \geqslant 1$ satisfying $j(k) > \frac{1}{\delta_k}$ and for which

$$\left| \int_{\mathbb{R}^n} |\nabla(\rho_{j(k)} * \tilde{\chi}_k)| \, dx - \|D\tilde{\chi}_k\| \right| \leqslant \frac{1}{2^k}. \tag{8.2}$$

We can assume $j(k+1) > j(k)$ for each $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $\chi_k = \rho_{j(k)} * \tilde{\chi}_k$. We find $\chi_k = 1$ on the open set

$$V_k = \left\{ x \in U_k : \operatorname{dist}(x, \partial U_k) < \frac{1}{j(k)} \right\}$$

containing S. On the other hand for $k \in \mathbb{N}$ we have

$$\operatorname{supp} \chi_k = \left\{ x \in \mathbb{R}^n : \operatorname{dist}(x, U_k) \leqslant \frac{1}{j(k)} \right\}.$$

Observe $S = \bigcap_{k \in \mathbb{N}} U_k$ and $|S| = 0$ to infer $|\operatorname{supp} \chi_k| \to 0, k \to \infty$ using Lebesgue's dominated convergence theorem. From Lemma 8.2 and (8.2) we get conclusion 3. □

The following result follows readily.

Proposition 8.4. *Let $v \in L^\infty_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and assume $S \in \mathcal{K}(\mathbb{R}^n)$ satisfies $\mathscr{H}^{n-1}(S) = 0$. Then $\operatorname{supp}(\mathfrak{div}\, v) \subseteq S$ implies $\mathfrak{div}\, v \equiv 0$.*

Proof. Assume that $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies $\operatorname{supp}(\mathfrak{div}\, v) \subseteq S$. We have to show that $\mathfrak{div}\, v$ is the zero distribution. For that purpose let $(\chi_n) \subseteq \mathscr{D}(\mathbb{R}^n)$ be a sequence associated to S by Corollary 8.3. For $\varphi \in \mathscr{D}(\mathbb{R}^n)$, write

$$\langle \mathfrak{div}\, v, \varphi \rangle = \langle \mathfrak{div}\, v, \chi_k \varphi \rangle + \langle \mathfrak{div}\, v, (1 - \chi_k)\varphi \rangle \quad ,$$

whenever k is a nonnegative integer. It is clear that one has

$$\langle \mathfrak{div}\ v, (1 - \chi_k)\varphi \rangle = 0$$

using the fact that $(1 - \chi_k)\varphi$ is supported outside S. On the other hand, Lebesgue's dominated convergence theorem guarantees that

$$\lim_{k \to \infty} \langle \mathfrak{div}\, v, \chi_k \varphi \rangle = -\lim_{k \to \infty} \int_{\mathbb{R}^n} \chi_k v \cdot \nabla \varphi \, dx - \lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi v \cdot \nabla \chi_k = 0,$$

and we get $\langle \mathfrak{div}\, v, \varphi \rangle = 0$ as was to be proved. $\square$

It is now easy to infer the removability of compact sets whose $(n-1)$-dimensional Hausdorff measure vanishes.

Corollary 8.5. *Let $\mathscr{B} \subseteq L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and assume $S \in \mathcal{K}(\mathbb{R}^n)$ satisfies $\mathscr{H}^{n-1}(S) = 0$. Then, S is $\mathscr{B}$-removable for the equation* $\operatorname{div} v = 0$.

8.1.2 A necessary condition for a set to be removable

We will use the following boxing inequality (we use the techniques developed in [KKST07]).

Proposition 8.6 (Boxing inequality)**.** *There exists a constant $C = C(n) > 0$ such that given a nonempty open set $U \subseteq \mathbb{R}^n$ satisfying $|U| < \infty$ and having finite perimeter, one can find $x_k \in U$ and $R_k > 0$, $k \in \mathbb{N}$ for which the following conditions are satisfied:*

1. $U \subseteq \bigcup_{k \in \mathbb{N}} B(x_k, 6R_k)$,
2. $\sum_{k=0}^{\infty} R_k^{n-1} \leqslant C\|U\|$,

3. *for each* $k \in \mathbb{N}$*,*

$$\frac{1}{2^{n+1}} \leqslant \frac{|U \cap B(x_k, R_k)|}{|B(x_k, R_k)|} \leqslant \frac{1}{2}. \tag{8.3}$$

Proof. Fix $x \in U$. Since U is open, there exists $r_x > 0$ such that

$$\frac{|U \cap B(x, r_x)|}{|B(x, r_x)|} = 1.$$

As $\lim_{r\to\infty} |B(x,r)| = \infty$ and $|U| < \infty$ we get

$$\lim_{r\to\infty} \frac{|U \cap B(x, r)|}{|B(x, r)|} = 0.$$

Let k_x be the least integer for which

$$\frac{|U \cap B(x, 2^{k_x} r_x)|}{|B(x, 2^{k_x} r_x)|} \leqslant \frac{1}{2}. \tag{8.4}$$

By construction

$$\frac{|U \cap B(x, 2^{k_x} r_x)|}{|B(x, 2^{k_x} r_x)|} \geqslant \frac{|U \cap B(x, 2^{k_x-1} r_x)|}{2^n |B(x, 2^{k_x-1} r_x)|} > \frac{1}{2^{n+1}}. \tag{8.5}$$

Define $R_x = 2^{k_x} r_x$ and observe, since U is measurable,

$$|B(x, R_x) \setminus U| = |B(x, R_x)| - |B(x, R_x) \cap U| \geqslant |B(x, R_x) \cap U|$$

according to (8.5). Using (8.5) we get

$$\min\{|B(x,R_x)\setminus U|, |B(x,R_x)\cap U|\} = |B(x,R_x)\cap U| \geqslant \frac{|B(x, R_x)|}{2^{n+1}}. \tag{8.6}$$

Using the relative isoperimetric inequality for BV sets, we know there exists a constant $c = c(n)$ for which

$$\min\{|B(x, R_x) \setminus U|, |B(x, R_x) \cap U|\} \leqslant c R_x \|\partial U\|[B(x, R_x)].$$

Combining this to (8.6) we get

$$R_x^{n-1} |B(0,1)| = \frac{|B(x, R_x)|}{R_x} \leqslant 2^{n+1} c \|\partial U\|(B(x, R_x)). \tag{8.7}$$

In particular we get $\sup\{R_x : x \in U\} < \infty$ and the Vitali covering theorem yields a sequence of disjoint balls $\{B(x_k, R_{x_k}) : k \in \mathbb{N}\}$ for which

$$U \subseteq \bigcup_{x \in U} B[x, R_x] \subseteq \bigcup_{k \in \mathbb{N}} B[x_k, 5R_{x_k}] \subseteq \bigcup_{k \in \mathbb{N}} B(x_k, 6R_{x_k}).$$

Let $R_k = R_{x_k}$ for $k \in \mathbb{N}$ and compute

$$\begin{aligned}
\sum_{k=0}^{\infty} R_{x_k}^{n-1} &= \frac{1}{|B(0,1)|} \sum_{k=0}^{\infty} \frac{|B(x_k, R_{x_k})|}{R_{x_k}}, \\
&\leqslant C \sum_{k=0}^{\infty} \|\partial U\|[B(x_k, R_{x_k})], \\
&= C\|\partial U\| \left(\bigcup_{k \in \mathbb{N}} B(x_k, R_{x_k}) \right) \\
&\leqslant C\|U\|.
\end{aligned}$$

if we let $C = C(n) = 2^{n+1} c(n) |B(0,1)|^{-1}$. The proof is complete. $\square$

The following lemma can be proved using the Boxing inequality.

Lemma 8.7. *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$ and assume there exists $M > 0$ such that $\mu(B(x,r)) \leqslant M r^{n-1}$ for each $x \in \mathbb{R}^n$ and each $r > 0$. Then*

$$\left| \int_{\mathbb{R}^n} \varphi \, d\mu \right| \leqslant C' M \int_{\mathbb{R}^n} |\nabla \varphi| \, dx,$$

holds for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ where $C' = C'(n)$ is a constant depending only on n.

Proof. Fix $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Recall that the weak gradient $\nabla|\varphi|$ satisfies $|\nabla|\varphi|| = |\nabla \varphi|$ a.e. on $\mathbb{R}^n$. The open set $\{|\varphi| > t\} = \mathbb{R}^n$ has empty boundary for $t < 0$. For a.e. $t > 0$, the coarea formula guarantees the open set $\{|\varphi| > t\}$ has finite perimeter, in which case we can choose $x_k \in \mathbb{R}^n$ and $R_k > 0$, $k \in \mathbb{N}$ according to Proposition 8.6 and compute

$$\mu(\{|\varphi| > t\}) \leqslant \sum_{k=0}^{\infty} \mu(B(x_k, 6R_k)) \leqslant C' M \sum_{k=0}^{\infty} R_k^{n-1} \leqslant C' M \|\{|\varphi| > t\}\|$$

if we let $C' = C'(n) = 6^{n-1}C(n)$ where $C(n)$ is the constant appearing in Proposition 8.6. Using the Cavalieri principle and the coarea formula for BV functions, we get

$$\int_{\mathbb{R}^n} |\varphi|\, d\mu = \int_0^\infty \mu(\{|\varphi| > t\})\, dt \leqslant C'M \int_0^\infty \|\{|\varphi| > t\}\|\, dt$$
$$= C'M \int_{\mathbb{R}^n} |\nabla|\varphi||\, dx = C'M \int_{\mathbb{R}^n} |\nabla\varphi|\, dx.$$

The proof is complete. □

Introduce the space

$$\mathscr{D}^{1,1}(\mathbb{R}^n) = \{u \in L^{\frac{n}{n-1}}(\mathbb{R}^n) : \nabla u \in L^1(\mathbb{R}^n, \mathbb{R}^n)\}$$

which, endowed by the norm $|u|_{1,1} = |\nabla u|_1$, becomes a Banach space (see [Wil08]). Observe $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{D}^{1,1}(\mathbb{R}^n)$.

Theorem 8.8 (Phuc-Torres). *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$ and assume there exists $M > 0$ such that $\mu(B(x,r)) \leqslant Mr^{n-1}$ for each $x \in \mathbb{R}^n$ and each $r > 0$. Then there exists $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying*

$$\langle \mathfrak{div}\, v, \varphi \rangle = \int_{\mathbb{R}^n} \varphi\, d\mu$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

Proof. Define an operator $T : \mathscr{D}^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n, \mathbb{R}^n)$ by $T(u) = -\nabla u$. Obviously $|T(u)|_1 = |u|_{1,1}$ for $u \in \mathscr{D}^{1,1}(\mathbb{R}^n)$. In particular T is injective. Moreover given $(u_k) \subseteq \mathscr{D}^{1,1}(\mathbb{R}^n)$ and $v \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, the assumption $T(u_k) \to v$, $k \to \infty$ implies

$$|u_j - u_k|_{1,1} = |T(u_j) - T(u_k)|_1 \to 0; j, k \to \infty.$$

The completeness of $\mathscr{D}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ then ensures the existence of a function $u \in \mathscr{D}^{1,1}(\mathbb{R}^n)$ for which $|u_k - u|_{1,1} \to 0, k \to \infty$. In particular we get

$$T(u) = \lim_{k\to\infty} T(u_k) = v,$$

and $\operatorname{im} T$ is closed. Theorem 1.18 implies the adjoint operator

$$T^* : L^\infty(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{D}^{1,1}(\mathbb{R}^n)^*$$

is surjective. On the other hand compute for $\varphi \in \mathscr{D}(\mathbb{R}^n)$

$$\langle T^*(v), \varphi \rangle = -\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx = \langle \mathfrak{div}\, v, \varphi \rangle.$$

So the result is proved as Lemma 8.7 ensures $\mu \upharpoonright \mathscr{D}^{1,1}(\mathbb{R}^n)$ belongs to the dual space $\mathscr{D}^{1,1}(\mathbb{R}^n)^*$. $\square$

We conclude this section by showing any removable set has zero $(n-1)$-dimensional Hausdorff measure. The result appeared in [Moo06] for the first time, while the proof was completely different in spirit. This new proof is due to N.C. Phuc and M. Torres in [PT07].

Theorem 8.9 (M. (2006) — Phuc & Torres (2007))**.** *Suppose $S \in \mathcal{K}(\mathbb{R}^n)$ is removable for the equation* $\operatorname{div} v = 0$*. Then $\mathscr{H}^{n-1}(S) = 0$.*

Proof. Fix $S \in \mathcal{K}(\mathbb{R}^n)$ satisfying $\mathscr{H}^{n-1}(S) > 0$ and use Frostman's lemma to find a nonzero Radon measure μ supported in a subset $S' \subseteq S$ satisfying $0 < \mathscr{H}^{n-1}(S') < \infty$ and for which there exists $M > 0$ such that $\mu(B(x,r)) \leqslant Mr^{n-1}$ holds for every $x \in \mathbb{R}^n$, $r > 0$. According to Theorem 8.8 there exists $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\operatorname{div} v = \mu$ on $\mathbb{R}^n$, in the distributional sense. As μ is nonzero, S is not removable for the equation $\operatorname{div} v = 0$. $\square$

8.2 A better removability result

We follow the smoothing argument of T. De Pauw [De 03] to obtain the following lemma.

Lemma 8.10. *Assume $S \in \mathcal{K}(\mathbb{R}^n)$ satisfies $0 < \mathscr{H}^{n-1}(S) < \infty$ and fix real numbers $\eta > 0$, $1 \leqslant p < \infty$. There exists $w \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the following conditions:*

1. *$w \restriction \mathbb{R}^n \setminus S$ is infinitely differentiable;*

2. *the function $x \mapsto \operatorname{div} w(x)$ belongs to $L^p(\mathbb{R}^n)$ and $|\operatorname{div} w|_p \leqslant \eta$;*

3. *there exists $\varphi \in \mathscr{D}(\mathbb{R}^n)$ for which*

$$\int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx \neq -\int_{\mathbb{R}^n} w \cdot \nabla\varphi \, dx. \tag{8.8}$$

Proof. As in the proof of Theorem 8.9, use Frostman's lemma to find a nonzero Radon measure μ supported in S and for which there exists $M > 0$ such that

$$\mu(B(x,r)) \leqslant Mr^{n-1}$$

holds for every $x \in \mathbb{R}^n$, $r > 0$. According to Theorem 8.8 there exists $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$-\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu.$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$. As μ is supported in S, we get in particular $\langle \mathfrak{div}\, v, \varphi \rangle > 0$ for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\varphi \geqslant \chi_S$.

Fix $\varphi_* \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\varphi_* \geqslant \chi_S$ and chosse $0 < \varepsilon < \eta$ so that

$$\varepsilon \left(|\varphi_*|_{p'} + |\nabla\varphi_*|_\infty \right) < \mu(S) \tag{8.9}$$

holds, where we called p' the conjugate exponent to p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. From the open covering

$$\mathfrak{U} = \left\{ B\left(x, \frac{\operatorname{dist}(x,S)}{2}\right) : x \in \mathbb{R}^n \setminus S \right\}$$

of $\mathbb{R}^n \setminus S$, extract a locally finite, countable open covering $\mathfrak{U}_0 = (U_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^n \setminus S$ subordinate to $\mathfrak{U}$ (with $U_k \subseteq B(x_k, \operatorname{dist}(x_k, S)/2)$, $k \in \mathbb{N}$); on the other hand choose $(\psi_k) \subseteq \mathscr{D}(\mathbb{R}^n)$ a partition of unity subordinate to $\mathfrak{U}$, so that $0 \leqslant \psi_k \leqslant \chi_{U_k}$, $k \in \mathbb{N}$ and $\sum_{k=0}^\infty \psi_k = \chi_{\mathbb{R}^n \setminus S}$.

Observe for each $k \in \mathbb{N}$

$$\lim_{l\to\infty} \left(\int_{\mathbb{R}^n} |\nabla \psi_k|^p |\rho_l * v - v|^p \, dx \right)^{1/p} \leqslant |\nabla \psi_k|_\infty \lim_{l\to\infty} \int_{U_k} |\rho_l * v - v|^p \, dx = 0,$$

and

$$0 \leqslant \lim_{l\to\infty} \int_{\mathbb{R}^n} \psi_k |\rho_l * v - v| \, dx \leqslant |\psi_k|_\infty \lim_{l\to\infty} \int_{U_k} |\rho_l * v - v| \, dx = 0.$$

So for $k \in \mathbb{N}$ choose $l_k \in \mathbb{N}^*$ satisfying $\frac{1}{l_k} < \frac{\operatorname{dist}(\bar{U}_k, S)}{2}$ as well as

$$\max \left\{ \left(\int_{\mathbb{R}^n} |\nabla \psi_k|^p |\rho_{l_k} * v - v|^p \, dx \right)^{1/p}, \int_{\mathbb{R}^n} \psi_k |\rho_{l_k} * v - v| \, dx \right\} \leqslant 2^{-k-1} \varepsilon. \quad (8.10)$$

Define then $w : \mathbb{R}^n \to \mathbb{R}^n$ by $w(x) = 0$ if $x \in S$ and

$$w(x) = \sum_{k=0}^{\infty} \psi_k(x) \rho_{l_k} * v(x)$$

for $x \in \mathbb{R}^n \setminus S$. Conclusion 1. follows readily.

Claim 8.11. For each $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp} \varphi \cap S = \emptyset$,

$$\left| \int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx \right| \leqslant \varepsilon |\varphi|_{p'}.$$

Proof of the claim. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfy $\operatorname{supp} \varphi \cap S = \emptyset$ and observe

$$-\int_{\mathbb{R}^n} v \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu = 0.$$

On the other hand observe the integration by parts formula yields

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx &= -\int_{\mathbb{R}^n} w \cdot \nabla \varphi \, dx, \\
&= -\int_{\mathbb{R}^n} (w - v) \cdot \nabla \varphi \, dx, \\
&= -\int_{\mathbb{R}^n} \sum_{k=0}^{\infty} \psi_k (\rho_{l_k} * v - v) \cdot \nabla \varphi \, dx.
\end{aligned}$$

On the other hand, compute for $N \in \mathbb{N}$

$$\sum_{k=0}^{N} |\psi_k(\rho_{l_k} * v - v) \cdot \nabla\varphi| \leqslant 2|v|_\infty |\nabla\varphi| \sum_{k=0}^{N} \psi_k \leqslant 2|v|_\infty |\nabla\varphi|$$

and notice that the Lebesgue dominated convergence theorem yields

$$\int_{\mathbb{R}^n} \sum_{k=0}^{\infty} \psi_k(\rho_{l_k} * v - v) \cdot \nabla\varphi \, dx = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} (\rho_{l_k} * v - v) \cdot (\psi_k \nabla\varphi) \, dx.$$

For $k \in \mathbb{N}$ note $\operatorname{supp}(\psi_k\varphi) \cap S = \emptyset$ and infer

$$\int_{\mathbb{R}^n} v \cdot \nabla(\psi_k\varphi) \, dx = \int_{\mathbb{R}^n} \psi_k\varphi \, d\mu = 0.$$

As ρ is radial, compute

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_{k_l} * v \cdot \nabla(\psi_k\varphi) \, dx &= \int_{\mathbb{R}^n} v \cdot [\rho_{k_l} * \nabla(\psi_k\varphi)] \, dx \\ &= \int_{\mathbb{R}^n} v \cdot \nabla[\rho_{k_l} * (\psi_k\varphi)] \, dx = \int_{\mathbb{R}^n} \rho_{k_l} * (\psi_k\varphi) \, d\mu = 0, \end{aligned}$$

as the choice of l_k ensures that $\rho_{k_l} * (\psi_k\varphi)$ is supported outside S. Hence

$$\int_{\mathbb{R}^n} (\rho_{k_l} * v - v) \cdot \nabla(\psi_k\varphi) \, dx = 0. \tag{8.11}$$

Continue by observing that (8.10) and the Hölder inequality give

$$\left| \int_{\mathbb{R}^n} (\rho_{k_l} * v - v) \cdot (\varphi \nabla\psi_k) \, dx \right| \leqslant \int_{\mathbb{R}^n} |\rho_{k_l} * v - v||\nabla\psi_k||\varphi| \, dx \leqslant \frac{\varepsilon}{2^{k+1}} |\varphi|_{p'},$$

from which we infer

$$\sum_{k=0}^{\infty} \left| \int_{\mathbb{R}^n} (\rho_{k_l} * v - v) \cdot (\varphi \nabla\psi_k) \, dx \right| \leqslant \varepsilon |\varphi|_{p'}. \tag{8.12}$$

Finally (8.11) and (8.12) and the equality $\psi_k \nabla\varphi = \nabla(\varphi\psi_k) - \varphi\nabla\psi_k$ provide

$$\left| \int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx \right| \leqslant \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}^n} (\rho_{l_k} * v - v) \cdot (\psi_k \nabla\varphi) \, dx \right| \leqslant \varepsilon |\varphi|_{p'},$$

as we wanted to prove. □

Claim 8.12. The map $x \mapsto \operatorname{div} w(x)$ belongs to $L^p(\mathbb{R}^n)$ and $|\operatorname{div} w|_p \leqslant \eta$.

Proof of the claim. Let $X = \{\varphi \in \mathscr{D}(\mathbb{R}^n) : \operatorname{supp} \varphi \cap S = \emptyset\}$. According to the Hahn-Banach theorem we see that the linear functional $f : X \subseteq L^{p'}(\mathbb{R}^n) \to \mathbb{R}$ defined by $\langle f, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx$ extends to $\bar{f} \in L^{p'}(\mathbb{R}^n)^*$ satisfying $|\langle \bar{f}, u \rangle| \leqslant \varepsilon |u|_{p'}$ for each $u \in L^{p'}(\mathbb{R}^n)$. So there exists $g \in L^p(\mathbb{R}^n)$ satisfying $|g|_p \leqslant \varepsilon$ and such that $\langle \bar{f}, u \rangle = \int_{\mathbb{R}^n} gu \, dx$ holds for each $u \in L^{p'}(\mathbb{R}^n)$. Since $|S| = 0$ we get

$$\int_{\mathbb{R}^n} (g - \operatorname{div} w) \varphi \, dx = 0$$

for each $\varphi \in \mathscr{D}(\mathbb{R}^n)$, so $\operatorname{div} w = g$ a.e. Consequently $\operatorname{div} w \in L^p(\mathbb{R}^n)$ and $|\operatorname{div} w|_p = |g|_p \leqslant \varepsilon \leqslant \eta$. □

As a consequence of Hölder's inequality, we get

$$\left| \int_{\mathbb{R}^n} \varphi_* \operatorname{div} w \, dx \right| \leqslant \varepsilon |\varphi_*|_{p'}, \tag{8.13}$$

where $\varphi_* \in \mathscr{D}(\mathbb{R}^n)$ is the function introduced at the beginning of the proof. On the other hand observe using (8.10) that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (w - v) \cdot \nabla \varphi_* \, dx \right| &= \left| \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} \psi_k (\rho_{l_k} * v - v) \cdot \nabla \varphi_* \, dx \right| \\ &\leqslant |\nabla \varphi_*|_\infty \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \psi_k |\rho_{l_k} * v - v| \, dx \leqslant \varepsilon |\nabla \varphi_*|_\infty . \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^n} w \cdot \nabla \varphi_* \, dx &= -\int_{\mathbb{R}^n} v \cdot \nabla \varphi^* \, dx + \int_{\mathbb{R}^n} (v - w) \cdot \nabla \varphi_* \, dx \\ &= \int_{\mathbb{R}^n} \varphi_* \, d\mu + \int_{\mathbb{R}^n} (v - w) \cdot \nabla \varphi_* \, dx \geqslant \mu(S) - \varepsilon |\nabla \varphi_*|_\infty . \end{aligned}$$

Yet (8.9) and (8.13) imply

$$\mu(S) - \varepsilon |\nabla \varphi_*|_\infty > \varepsilon |\varphi_*|_{p'} \geqslant \left| \int_{\mathbb{R}^n} \varphi_* \operatorname{div} w \, dx \right|,$$

and we conclude

$$-\int_{\mathbb{R}^n} \varphi_* \operatorname{div} w \, dx \neq \int_{\mathbb{R}^n} w \cdot \nabla \varphi_* \, dx.$$

The Lemma is proved. □

Remark 8.13. Multiplying if necessary the vectorfield w obtained in Lemma 8.10 by $\chi \in \mathscr{D}(\mathbb{R}^n)$ verifying $0 \leqslant \chi \leqslant 1$ on $\mathbb{R}^n$ and satisfying $\chi = 1$ on a neighbourhood of $\operatorname{supp} \varphi$, we get a vectorfield $w \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ having compact support and satisfying the following conditions:

1. $w \restriction \mathbb{R}^n \setminus S$ is infinitely differentiable;
2. the function $x \mapsto \operatorname{div} w(x)$ belongs to $L^p(\mathbb{R}^n)$;
3. there exists $\varphi \in \mathscr{D}(\mathbb{R}^n)$ for which
$$\int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx \neq -\int_{\mathbb{R}^n} w \cdot \nabla \varphi \, dx. \tag{8.14}$$

of Lemma 8.10.

We can improve Theorem 8.9 following the ideas developed in [Moo06, Section 5].

Definition 8.14. A compact set $S \subseteq \mathbb{R}^n$ is said to be *s-removable* for the equation $\operatorname{div} v = 0$ if the following condition holds: for every $v \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of class C^∞ outside S:

$$\operatorname{supp}(\mathfrak{div}\, v) \subseteq S \quad \text{implies} \quad \mathfrak{div}\, v \equiv 0 \quad .$$

Theorem 8.15 (M. (2006))**.** *A set $K \in \mathcal{K}(\mathbb{R}^n)$ is s-removable for the equation* $\operatorname{div} v = 0$ *if and only if* $\mathscr{H}^{n-1}(S) = 0$.

Proof. The sufficient part follows readily from Proposition 8.4. To prove it is also necessary, assume $0 < \mathscr{H}^{n-1}(S) < \infty$, choose a real number $n < p < \infty$ and let w and φ be associated with S and p by Remark 8.13. Choose $\chi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\chi = 1$ on a neighbourhood of $\operatorname{supp} \varphi$ and let $\tilde{w} =$

$\chi\nabla K_{\operatorname{div} w}$, where $K_{\operatorname{div} w}$ is the Newtonian potential of $\operatorname{div} w$. Observe $\tilde{w}$ is a bounded vectorfield of class C^∞ outside S and continuous in $\mathbb{R}^n$. As S is a set with finite $(n-1)$-dimensional Hausdorff measure, the generalised integration by parts formula guarantees the equalities

$$\int_{\mathbb{R}^n} \varphi \operatorname{div} w \, dx = \int_{\mathbb{R}^n} \varphi \operatorname{div} \tilde{w} \, dx = -\int_{\mathbb{R}^n} \tilde{w} \cdot \nabla\varphi \, dx \quad ,$$

are satisfied. Defining $v = w - \tilde{w}$ and using (8.8) we get

$$\int_{\mathbb{R}^n} v \cdot \nabla\varphi \, dx \neq 0 \quad ,$$

and so $\mathfrak{div}\, v$ is not the zero distribution. But using the integration by parts formula one sees $\langle \mathfrak{div}\, v, \psi \rangle = 0$ for each $\psi \in \mathscr{D}(\mathbb{R}^n)$ supported in $\mathbb{R}^n \setminus S$. $\square$

8.3 Charging measures

We first examine when a measure defines a strong charge.

Theorem 8.16 (Phuc-Torres)**.** *Let μ be a Radon measure in $\mathbb{R}^n$, and assume for each $K \in \mathcal{K}(\mathbb{R}^n)$ that*

$$\lim_{\delta\to 0^+} \sup_{x\in K} \sup \left\{ \int_{\mathbb{R}^n} \varphi \, d\mu : \varphi \in \mathscr{D}(\mathbb{R}^n), \operatorname{supp}\varphi \subseteq B(x,\delta), |\nabla\varphi|_1 \leqslant 1 \right\} = 0.$$

Then the formula $\mathscr{F}(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu$ defines a strong charge in $\mathbb{R}^n$.

Proof. Fix $\varepsilon > 0$, $K \in \mathcal{K}(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\varphi \subseteq K$. Let $K_1 = \{x \in \mathbb{R}^n : \operatorname{dist}(x,K) \leqslant 1\}$. By hypothesis, there exists $0 < \delta < 1$ such that for every $x \in K_1$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\psi \subseteq B(x,\delta)$, we have

$$\int_{\mathbb{R}^n} \psi \, d\mu \leqslant \varepsilon |\nabla\psi|_1. \tag{8.15}$$

Choose $\eta \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\chi_{B(0,1/2)} \leqslant \eta \leqslant \chi_{B(0,1)}$ and let $C = |\nabla\eta|_\infty$. For each n-tuple $I = (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ let $x_I = \frac{1}{3\sqrt{n}}\delta I$ and observe for each $x \in \mathbb{R}^n$ the number

$$1 \leqslant \# \left\{ I \in \mathbb{N}^n : x \in B\left(x_I, \frac{\delta}{2}\right) \right\}$$

is less than

$$N = \max \left[\max_{x\in\mathbb{R}^n} \# \left\{ I \in \mathbb{N}^n : x \in B\left(x_I, \frac{\delta}{2}\right) \right\}, \right.$$
$$\left. \# \left\{ I \in \mathbb{N}^n : K_1 \cap B\left(x_I, \frac{\delta}{2}\right) \neq \emptyset \right\} \right] < \infty.$$

For $I \in \mathbb{N}^n$ let $\eta_I \in \mathscr{D}(\mathbb{R}^n)$ be defined by $\eta_I(x) = \eta\left(\frac{x-x_I}{\delta}\right)$. Note that

$$\chi_{B(x_I,\delta/2)} \leqslant \eta \leqslant \chi_{B(x_I,\delta)} \quad \text{and} \quad |\nabla \eta_I| \leqslant \frac{C}{\delta}.$$

Choose $I_1, I_2, \ldots, I_m \in \mathbb{N}^n$ for which

$$K_1 \subseteq \bigcup_{j=1}^m B\left(x_{I_j}, \frac{\delta}{2}\right)$$

and define $\psi \in \mathscr{D}(\mathbb{R}^n)$ by the formula $\psi(x) = \sum_{j=1}^m \eta_{I_j}(x)$. In particular we get $1 \leqslant \psi(x) \leqslant N$ for each $x \in K_1$. Furthermore we get for $x \in \mathbb{R}^n$

$$|\nabla\psi(x)| \leqslant \sum_{j=1}^m |\nabla \eta_{I_j}(x)| \leqslant \frac{CN}{\delta}.$$

For $1 \leqslant j \leqslant m$ define $\xi_j(x) = \frac{\eta_{I_j}(x)}{\psi(x)}$ in case $\psi(x) \neq 0$. Otherwise let $\xi_j(x) = 0$ and observe $\sum_{j=1}^m \xi_j = 1$ on K_1. Moreover we find $\xi_j\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\xi_j\varphi) \subseteq B(x_{I_j}, \delta)$.

Compute using (8.15)

$$\int_{\mathbb{R}^n} \varphi\, d\mu = \int_{\mathbb{R}^n} \sum_{j=1}^m \xi_j\varphi\, d\mu \leqslant \varepsilon \sum_{j=1}^m \int_{\mathbb{R}^n} |\nabla(\xi_j\varphi)|\, dx,$$

and

$$\sum_{j=1}^m \int_{\mathbb{R}^n} |\nabla(\xi_j\varphi)|\, dx \leqslant \int_{\mathbb{R}^n} \sum_{j=1}^m |\nabla\xi_j||\varphi|\, dx + \int_{\mathbb{R}^n} \sum_{j=1}^m \xi_j |\nabla\varphi|\, dx$$
$$\leqslant \int_{\mathbb{R}^n} \sum_{j=1}^m |\nabla\xi_j||\varphi|\, dx + \int_{\mathbb{R}^n} |\nabla\varphi|\, dx.$$

Yet for $1 \leqslant j \leqslant m$ and $x \in K$

$$|\nabla \xi_j(x)| \leqslant \frac{|\nabla \eta_{I_j}(x)|}{\psi(x)} + \frac{\eta_{I_j}(x)}{\psi(x)} \cdot \frac{|\nabla \psi(x)|}{\psi(x)} \\ \leqslant \frac{|\nabla \eta_{I_j}(x)|}{\psi(x)} + \frac{|\nabla \psi(x)|}{\psi(x)} \leqslant \frac{C}{\delta} + \frac{CN}{\delta}$$

as $\psi(x) \geqslant 1$. Hence

$$\sum_{j=1}^{m} |\nabla \xi_j| \leqslant \frac{CN(N+1)}{\delta}$$

and we get

$$\int_{\mathbb{R}^n} \varphi \, d\mu \leqslant \theta \int_{\mathbb{R}^n} |\varphi| \, dx + \varepsilon \int_{\mathbb{R}^n} |\nabla \varphi| \, dx.$$

if we choose $\theta = \frac{CN(N+1)\varepsilon}{\delta}$. The proof is complete. □

We give a simple sufficient condition for a Radon measure to be a strong charge.

Theorem 8.17 (Phuc-Torres). *Let μ be a Radon measure in $\mathbb{R}^n$ and assume for each $K \in \mathcal{K}(\mathbb{R}^n)$ that*

$$\lim_{\delta \to 0^+} \sup_{x \in K} \sup_{0<r<\delta} \frac{\mu(B(x,r))}{r^{n-1}} = 0.$$

Then the formula $\mathscr{F}(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu$ defines a strong charge in $\mathbb{R}^n$.

Proof. Fix $K \in \mathcal{K}(\mathbb{R}^n)$, $\varepsilon > 0$ and define K_1 as in the proof of Theorem 8.16. By hypothesis there exists $0 < \delta_1 < 1$ such that $\mu(B(x,r)) \leqslant \varepsilon r^{n-1}$ holds for each $x \in K_1$ and every $0 < r < \delta_1$. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ satisfy $\operatorname{supp} \varphi \subseteq B(x_0, \delta)$ for some $0 < \delta < \frac{\delta_1}{12}$ and some $x_0 \in K$. For a.e. $t > 0$, the open set $\{|\varphi| > t\}$ has finite perimeter, in which case call $(x_k) \subseteq \{|\varphi| > t\}$ and $(R_k) \subseteq (0, \infty)$ two sequences associated to $\{|\varphi| > t\}$ according to Proposition 8.6. Observe (8.3) combined to the inclusion $\{|\varphi| > t\} \subseteq B(x_0, \delta)$ yields $R_k \leqslant 2\delta$ and $6R_k \leqslant 12\delta < \delta_1$ for each $k \in \mathbb{N}$. In particular we get $x_k \in K_1$ and

$$\mu(B(x_k, 6R_k)) \leqslant 6^{n-1} \varepsilon R_k^{n-1}$$

for each $k \in \mathbb{N}$. We get

$$\mu(\{|\varphi| > t\}) \leqslant \sum_{k=0}^{\infty} \mu(B(x_k, 6R_k)) \leqslant 6^{n-1}\varepsilon \sum_{k=0}^{\infty} R_k^{n-1} \leqslant C'\varepsilon \|\{|\varphi| > t\}\|$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi \, d\mu &\leqslant \int_{\mathbb{R}^n} |\varphi| \, d\mu = \int_0^{\infty} \mu\{|\varphi| > t\} \, dt \\ &\leqslant C'\varepsilon \int_0^{\infty} \|\{|\varphi\| > t\}\| = C'\varepsilon \int_{\mathbb{R}^n} |\nabla|\varphi|| \, dx = C'\varepsilon \int_{\mathbb{R}^n} |\nabla\varphi| \, dx.\end{aligned}$$

As it yields the vanishing of

$$\lim_{\delta \to 0^+} \sup_{x_0 \in K} \sup \left\{ \int_{\mathbb{R}^n} \varphi \, d\mu : \varphi \in \mathscr{D}(\mathbb{R}^n), \operatorname{supp} \varphi \subseteq B(x_0, \delta), |\nabla\varphi|_1 \leqslant 1 \right\},$$

Theorem 8.16 allows us to conclude. □

8.4 Removability for continuous vectorfields

First, we state a sufficient condition for a compact set to be $C(\mathbb{R}^n, \mathbb{R}^n)$-removable.

Lemma 8.18. *Suppose* $S \in \mathcal{K}(\mathbb{R}^n)$ *satisfy* $\mathscr{H}^{n-1}(S) < \infty$. *Then* S *is* $C(\mathbb{R}^n, \mathbb{R}^n)$*-removable for the equation* $\operatorname{div} v = 0$.

Proof. Assume $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ is such that $\operatorname{supp}(\mathfrak{div}\, v) \subseteq S$. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be nonzero and define $K = \operatorname{supp} \varphi$. Our goal is to show $\langle \mathfrak{div}\, v, \varphi \rangle = 0$.

To that purpose fix $\varepsilon > 0$. As $\mathfrak{div}\, v$ is a strong charge, choose $\theta > 0$ such that

$$\langle \mathfrak{div}\, v, \psi \rangle \leqslant \theta |\psi|_1 + \varepsilon \min\left\{1, \frac{1}{C(n)|\varphi|_\infty}\right\} |\nabla\psi|_1$$

holds for each $\psi \in \mathscr{D}(\mathbb{R}^n)$ satisfying $\operatorname{supp} \psi \subseteq K$, where $C(n)$ is the constant appearing in Corollary 8.3, and choose (χ_k) a sequence associated to S by Corollary 8.3. In particular we get for any $k \in \mathbb{N}$

$$\langle \mathfrak{div}\, v, \varphi \rangle = \langle \mathfrak{div}\, v, (1 - \chi_k)\varphi \rangle + \langle \mathfrak{div}\, v, \chi_k \varphi \rangle.$$

As $(1-\chi_k)\varphi$ is supported outside S, we get $\langle \mathfrak{div}\, v, (1-\chi_k)\varphi\rangle = 0$. On the other hand,

$$\langle \mathfrak{div}\, v, \chi_k \varphi\rangle \leqslant (\theta|\varphi|_\infty + \varepsilon|\nabla\varphi|_\infty)|\chi_k|_1 + \frac{\varepsilon}{C(n)}|\nabla\chi_k|_1.$$

Using Corollary 8.3 and taking the limit when $k \to \infty$ in the previous inequality we obtain $\langle \mathfrak{div}\, v, \varphi\rangle \leqslant \varepsilon\mathscr{H}^{n-1}(S)$. □

An interesting application of Theorem 5.15 yields a necessary condition for a set $S \in \mathcal{K}(\mathbb{R}^n)$ to be $C(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation $\operatorname{div} v = 0$.

Theorem 8.19 (Phuc-Torres). *Assume $S \in \mathcal{K}(\mathbb{R}^n)$ is $C(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation* $\operatorname{div} v = 0$. *Then $\mathscr{H}^{n-1+\varepsilon}(S) = 0$ for each $\varepsilon > 0$.*

Proof. Fix $\varepsilon > 0$ and suppose $S \in \mathcal{K}(\mathbb{R}^n)$ is $C(\mathbb{R}^n, \mathbb{R}^n)$-removable for the equation $\operatorname{div} v = 0$ yet satisfies $\mathscr{H}^{n-1+\varepsilon}(S) > 0$. According to Frostman's lemma there exists a nontrivial Radon measure μ supported in S and satisfying

$$\mu(B(x,r)) \leqslant r^{n-1+\varepsilon}$$

for all $x \in \mathbb{R}^n$, $r > 0$. In particular we find for $K \in \mathcal{K}(\mathbb{R}^n)$

$$0 \leqslant \lim_{\delta\to 0^+} \sup_{x\in K} \sup_{0<r<\delta} \frac{\mu(B(x,r))}{r^{n-1}} \leqslant \lim_{\delta\to 0^+} \delta^\varepsilon = 0$$

and the formula $\mathscr{F}(\varphi) = \int_{\mathbb{R}^n} \varphi\, d\mu$ defines a strong charge in $\mathbb{R}^n$ according to Theorem 8.17. Theorem 5.15 yields a vectorfield $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} v \cdot \nabla\varphi\, dx = \int_{\mathbb{R}^n} \varphi\, d\mu$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$. In particular $\mathfrak{div}\, v$ is nonzero yet $\operatorname{supp}(\mathfrak{div}\, v) \cap S = \emptyset$, hence S is not $C(\mathbb{R}^n, \mathbb{R}^n)$-removable. □

A perspective
Charges in Banach spaces

With Marie Snipes [MS07], we proposed a definition of m-charges in a Banach space.

Charges in Banach spaces

Let X be a Banach space and for any integer $m \in \mathbb{N}$ let $\Lambda_m X$ denote the space of all m-vectors in X (see [Fed69, Chapter 1]).

For each $k \in \mathbb{N}$, $\mathbf{G}(X, k)$ will stand for the collection of all k-dimensional linear subspaces of X.

Let us introduce *continuous partial forms* on X, following M. Snipes [Sni].

Definition 8.20. A *continuous m-partial form* on X is a map

$$\omega : X \times \Lambda_m X \to \mathbb{R}$$

satisfying the following condition:

1. for all $k \in \mathbb{N}$ and each $Y \in \mathbf{G}(X, k)$, the map $\omega \restriction X \times \Lambda_m Y$ is continuous;

2. for each $x \in X$, the map $\omega_x : \Lambda_m X \to \mathbb{R}$ is linear.

The notation $\mathscr{C}^m(X)$ stands for the set of all continuous partial differential forms in X.

To each m-dimensional oriented, convex polyhedron P in X we associate a tangent m-vector $\vec{T}_P$.

T. Adams [Ada04] introduces a space $\mathbf{F}_m(X)$ of m-dimensional flat chains in a Banach space X by completing the space $\mathbf{P}_m(X)$ of formal linear combinations of m-dimensional polyhedra under a *flat norm.* That flat norm extends to a flat norm on the space $\mathbf{F}_m(X)$, and we also have a (lower semicontinuous in the flat norm topology) mass functional $\mathbf{M}$ on $\mathbf{F}_m(X)$. We let

$$\mathbf{N}_m(X) = \{T \in \mathbf{F}_m(X) : \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty\}.$$

Definition 8.21. A linear functional $F : \mathbf{N}_m(X) \to \mathbb{R}$ is called an m-charge in X (or simply an m-charge) if for each $\varepsilon > 0$, each $k \in \mathbb{N}$, each $Y \in \mathbf{G}(X, k)$ and each $K \in \mathcal{K}(Y)$ there exists $\theta > 0$ such that for each $T \in \mathbf{N}_m(X)$ satisfying $\operatorname{supp} T \subseteq K$ one has

$$F(T) \leqslant \theta \mathbf{F}(T) + \varepsilon \mathbf{N}(T).$$

The space of all m-charges in X will be denoted by $\mathbf{CH}_m(X)$.

Remark 8.22. When $X = \mathbb{R}^n$ the previous definition is equivalent to Definition 3.1 according to Proposition 3.2.

For each $\omega \in \mathscr{C}^m(X)$, one defines a map

$$\Lambda(\omega) : \mathbf{P}_m(X) \to \mathbb{R}$$

by the formula $\Lambda(\omega)(P) = \sum_{i=1}^{l} \alpha_i \int_{P_i} \omega(x, \vec{T}_{P_i}) d\mathscr{H}^m(x)$, where

$$P = \sum_{i=1}^{l} \alpha_i [P_i]$$

for m-dimensional convex polyhedra P_i in X and $\alpha_i \in \mathbb{R}$, $1 \leqslant i \leqslant l$.

Using an approximation property by Lipschitz continuous m-forms (the use of such a generalization of the Stone-Weierstrass approximation theorem indicating the necessity of working with *finite dimensional* subspaces of X in the definition of m-charge), one should be able to prove $\Lambda(\omega)$ satisfies

$$\Lambda(\omega)(P) \leqslant \theta\mathbf{F}(P) + \varepsilon\mathbf{N}(P)$$

whenever $P \in \mathbf{P}_m(X)$ satisfies $\operatorname{supp} P \subseteq K$.

For $T \in \mathbf{N}_m(X)$ choose a sequence $(P_k) \subseteq \mathbf{P}_m(X)$ satisfying the following conditions (see [Ada04]):

1. $\lim_{k\to\infty} \mathbf{N}(P_k) = \mathbf{N}(T)$,
2. $\lim_{k\to\infty} \mathbf{F}(T - P_k) = 0$.

Writing $F = \Lambda(\omega)$ it yields

$$\lim_{k,l\to\infty} |F(P_k) - F(P_l)| = 0.$$

If $(\tilde{P}_k) \subseteq \mathbf{P}_m(X)$ is another sequence satisfying 1. and 2. then

$$\lim_{k\to\infty} |F(P_k) - F(\tilde{P}_k)| = 0$$

so that we can let

$$F(T) = \lim_{k\to\infty} F(T_k).$$

From the inequalities

$$F(P_k) \leqslant \theta\mathbf{F}(P_k) + \varepsilon\mathbf{N}(P_k)$$

one would infer

$$F(T) \leqslant \theta\mathbf{F}(T) + \varepsilon\mathbf{N}(T)$$

and $F = \Lambda(\omega)$ would be an m-charge in X.

With M. Snipes, we would like to investigate which type of representation theorem we can get for m-charges in Banach spaces.

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