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In this paper, we measure the market and the longevity risks borne by an insurer by computing their solvency capital for a given annuity and within an investment strategy. For this purpose, we propose the investment strategy in such a way as to mitigate the solvency capital of the insurer and improve the internal rate of return of a shareholder investing on a given annuity. Numerically, we study the sensitivity of both the solvency capital and the internal rate of return with respect to some significant parameters.

Keywords: Life annuity; solvency capital; equity risk; longevity risk; interest rate risk; sensitivity analysis.

1. Introduction

Since the adoption in 2009 and the implementation in 2016 of the new regulation framework for European Union insurance and reinsurance called the Solvency II (SII) framework. It is compulsory to insurance and reinsurance companies to put aside on a yearly basis a capital that will guarantee their solvency with a high confidence level called solvency capital (SC). Therefore, there is a real need for insurers to know the level of their SC at any time (from the contract inception until the end of the contract), in order to value their risk and their possible return or gain from such investment. Their decision on selling or not an annuity depends on both their return and their SC. The SII regulation was established in order to protect the policyholders against possible inflations on insurance products. The features of SII along with the methods of evaluation of the SC were verbally presented in the documents of the European Insurance and Occupational Pensions Authority EIOPA (2014a), EIOPA (2014b). The SII regulation has significantly impacted the European insurance industry and several authors have developed some specific analysis of it. Namely Eling *et al.* (2007) summarized the features of the SII in 2007; Dreyfuss (2013) presented the main principles of SII; Devolder (2011) valued the SC under different risk measures. Some critics have been made as well toward the SII, as shown in the article written by Doff (2008).

Our study focuses on the life insurance which is subject to the continuous increase of the population life expectancy. The increase of population life expectancy implies an increase of the longevity risk. This risk directly implies the increases of the insurer's SC and this might imply the increase of annuity prices. Therefore despite the multiple advantages of selling and buying annuity products, both the insurance companies and policyholders are reluctant to respectively sell and buy annuities. Hence, there is a need of making life annuity products attractive and competitive on the market for both insurers and policyholders. In order to do so, one could think of way to reduce the level of the SC for the insurer, reduce the prices of annuities for the policyholders as well as to increase the level of the annual benefit received by the policyholders. As the (annual or single) premiums paid by the policyholders to the insurance company (to purchase an annuity), the SC has to be invested as well in a financial market. The equity risk is then represented by the random outcome or return on a risky investment and the interest rate risk is represented by a stochastic short rate. In a conventional annuity, a bad investment outcome as well as mortality improvements are borne by the insurer and not by the policyholder, this implies a high value of SC. For the sake of mitigating these risks, insurers tend to increase annuity prices and/or reducing benefit payout of the annuitants. A way of coping with the high level of SC, high annuity prices and low annual benefits could be to build suitable investment strategies of both the premiums and the SC or to consider risk sharing annuities, or natural hedging.

We focus here on the building of investment strategies based on some specific assumptions. For instance, in this paper we assumed that the premium is invested on a mixed asset and for simplicity purpose, we assumed the SC to be invested on a stochastic discount bond. This will allow us to build an investment strategy for which we value the level of the SC with respect to some significant parameters. Several authors and insurance companies are paying attention to the computation of the SC within particular frameworks. Olivieri & Pitacco (2008) investigated rules for evaluating the SC by a portfolio of life annuities to meet longevity risk. The problem of valuation of the SC has been solved for the case of multiple premiums and single cash flow within Lévy driven market by Devolder & Tassa (2011); also the case of single premium and single cash flow within Brownian motion driven market have been studied by Devolder & Lebègue (2017). In the latter paper, they consider a contract in which the annuitants pay a single premium at inception in order to receive a single benefit at retirement. Our main goal in this paper is to extend the results presented in the latter article within a setting made of equity, interest rate and longevity risks. For instance, we consider a contract in which the annuitants pay a single premium at inception in order to receive a series of benefits as long as they are alive or during the contract duration.

The approach we use to cope with this issue of assessing the SC consists of computing it within our proposed strategy and by comparing the obtained results for a lifetime, a deferred and a term annuities. We assume the equity, the interest rate and the longevity risks to be driven by a Brownian motion or more precisely by a log-normal process. Moreover. the longevity risk is captured using the force of mortality process which we consider to follow a Hull-White (HW) model (see Lichters *et al.* (2015)) with a Gompertz mean reversion parameter (see Zeddouk & Devolder (2019)). This serves us to clearly develop a method of assessing the SC of an insurer with respect to his investment strategy and to the specific annuity he sells. This task is very helpful not only for annuity trading, but also for investment decision making of insurers. Meaning that in this work we provide the insurer with enough information regarding his risks while selling a given annuity, so as to facilitate his choice of the most solvency capital consuming product.

Our contribution in this paper is to measure the solvency of an insurer selling life annuity products. The approach consists of the valuation of the insurer's SC for the lifetime, the deferred and the term annuity for comparative purpose. To this end, we propose a profitable attractive investment strategy used to assess the SC of an insurer for a given annuity product as regards the equity, the interest rate and the longevity risks. Our investment strategy comes as a modified version of those proposed by Bauer & Weber (2008). In fact, in their paper they assess the risk linked to annuity using an investment strategy refers to as the fully liability hedging strategy. In this paper, we propose a partial (or temporal) liability hedging strategy refers to as the m guaranteed cash flows strategy. The attractiveness of our strategy goes on the one hand toward the policyholder as it guarantees with a hundred percent a given number (m) of benefits. Notice that the attractiveness (for the policyholder) in terms of consumption has been studied by Hanewald *et al.* (2013), where they used the optimal consumption to find the optimal retirement plan in a portfolio made of zero-coupon bond (ZCB), life annuity, longevity bond and group self-annuitization. On the other hand toward the insurer in the sense that it shows that the added SC could yield great return since it gives quite good internal rate of return. There exists alternative ways of managing the insurer's risks, for instance the natural hedging as shown by Luciano *et al.* (2017). Moreover, we compare the SC obtained with our strategy and the one obtained with the strategy proposed by Bauer & Weber (2008). This comparative study is made for the lifetime, the term and the deferred annuity so as to point out the most solvency capital consuming product. In line with Bauer & Weber (2008), we find that the more benefits we guarantee the low SC we have for both the lifetime and term annuities whereas for the deferred annuity, there exits at least one number of guaranteed benefit that gives a lower SC as compared to the no-guaranteed (i.e. $m = 0$) and fully guaranteed benefits (i.e. m is equal to the total number of payouts). We further find that our strategy gives a better internal rate of return (IRR) as compared to that of Bauer & Weber (2008) in the sense that ours gives a concave IRR with respect to the number of guaranteed benefits.

The structure of this paper is following. In Sec. 2, we present the features of the equity-(interest)rate-longevity model as well as the model's setting and hypothesis. In other words we define in detail the financial market, the investment strategy as well as the mortality model (i.e. the HW model), the interest rate model (as an Ornstein–Uhlenbeck process) and the liabilities of the insurer. In Sec. 3, our main theoretical results are presented. We compare the obtained numerical results and draw up some remarks in Sec. 4. The IRR is defined and computed in Sec. 5, where a comparative study is made on the IRR. Finally, the conclusion follows in the last section.

2. Model's Features

We base our work on the Black–Scholes setting (see Black & Scholes (1973) and Merton (1973)) with stochastic short rate within a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. In this paper, we assume the insurance company is exposed to the equity, interest rate and longevity risks.

In order to be consistent with the SII regulation, we consider the static risk measure VaR as the risk measurement tool. In this context, the safety level recommended by SII on a one year horizon is $\alpha = 99.5\%$.[a] Taking into account the long-term aspect of annuity products, here we will measure the risk directly at maturity and not on a one-year time. Therefore, we have also to adapt the safety level, based now on T years instead of one year. In order to define this new confidence level function, we start with the natural overall confidence $\alpha = (99.5\%)^T$ (see for instance Devolder & Lebègue (2017), Devolder & Tassa (2011), Devolder (2011)). Moreover, we avoid having too low confidence level (which arises when T is large) by defining a minimum confidence level α_0 such that the confidence level function never goes below α_0. Hence, the confidence level function is given by

$$\alpha(T, \alpha_0) = \max(\alpha^T, \alpha_0), \tag{2.1}$$

where $\alpha = 99.5\%$ and T is the length of the risky period. Note that if $\alpha_0 = \alpha$ then we obtained the SII safety level of $\alpha(T, \alpha_0) = 99.5\%$. We assume independence between the equity and the longevity risks; between the interest rate and the longevity risk but a dependence between the equity and the interest rate risks.

In the following subsections, we present respectively in detail the financial market framework, the mortality model, the liability of the insurer as well as the investment strategy.

[a]Article 101 stipulates that the SCR corresponds to the Value-at-Risk with a confidence level of 99.5%.

2.1. *Financial market*

We assume that the financial market is made of a money market account B, a stock S and discount bonds P. We assume that there are no dividends, no transaction costs and no taxes. Following the Black–Scholes model, we define the stock by a constant drift ($\mu \in \mathbb{R}$), constant volatility ($\sigma \in \mathbb{R}_+^*$); regarding the money market account and the discount bonds we consider a stochastic interest rate $\{r_t\}$. As we are now facing negative interest rate and since this is a practical paper, we consider a famous interest rate model: the Vasicek model. In other words, we define the interest rate as an Ornstein–Uhlenbeck process with the following dynamic under the real measure $\mathbb{P}$

$$dr_t = b(c - r_t)dt + \sigma_r dW_t^r, \tag{2.2}$$

where $b, c, \sigma_r, r_t \in \mathbb{R}$ with $b, \sigma_r > 0$ and W_t^r is a Brownian motion on the physical probability space. Note that alternative models could be considered as well in this paper. The dynamic of the money market account is then given by

$$dB_t = r_t B_t \, dt, \tag{2.3}$$

for $t \geq 0$ and with $B_0 = 1$.

Denote by $\mathbb{F}^r = (\mathcal{F}_t^r)_{t \geq 0}$ the natural filtration of W_t^r and for a fix maturity T, we denote by $\mathbb{Q}_r$ the risk-neutral measure of $\mathcal{F}_T^r$. Following the Girsanov theorem (see Girsanov (1960)), we have

$$\frac{d\mathbb{Q}_r}{d\mathbb{P}} = \exp\left(\lambda_r W_T^r - \frac{1}{2}\lambda_r^2 T\right), \tag{2.4}$$

where $\lambda_r \in \mathbb{R}$ represents the market price of interest rate risk. One can show that the value at time $t \geq 0$ of a zero-coupon bond (ZCB) that pays a unit of currency at maturity time $s \geq 0$ with $s \geq t$ (i.e. the discount bond) is a random variable given by (see Vasicek & Fong (1982))

$$P(t, s) = \mathbb{E}^{Q_r}\left[\frac{B_t}{B_s} | \mathcal{F}_t^r\right] = A_r(t, s) e^{-B_r(t,s) r_t}, \tag{2.5}$$

where $\mathbb{E}^{Q_r}$ is the expectation under measure $\mathbb{Q}_r$, with

$$A_r(t, s) = \exp\left[\left(c - \frac{\lambda_r \sigma_r}{b} - \frac{\sigma_r^2}{2b^2}\right)(B_r(t, s) - (s - t)) - \frac{\sigma_r^2}{4b} B_r(t, s)^2\right] \tag{2.6}$$

and

$$B_r(t, s) = \frac{1}{b}(1 - e^{-b(s-t)}). \tag{2.7}$$

From the Ito formula, we obtain the following dynamic of the discount bond:

$$dP(t, s) = (r_t - \lambda_r \sigma_r B_r(t, s)) P(t, s) dt - \sigma_r B_r(t, s) P(t, s) dW_t^r. \tag{2.8}$$

As for the stock, let $\{W_t\}_{t\in\mathbb{N}}$ be a Brownian motion under the physical probability measure $\mathbb{P}$; the stock satisfies the following stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.9}$$

with $S_0 = 1$. One can show that the value at a given time $t \in \mathbb{R}_+$ of a unit invested on such stock is given by

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}. \tag{2.10}$$

We assume dependence between the stock and the interest rate, i.e. W_t is correlated with W_t^r with $\rho = corr(W_t, W_t^r)$ being the correlation parameter.

2.2. *Mortality model*

In this subsection we present the stochastic mortality model used to capture the policyholders' real survival probabilities. We assume the force of mortality of the cohort to follow a HW model with the following dynamic (see Lichters *et al.* (2015)):

$$d\mu_t^{x_0} = (\theta(t) - a\mu_t^{x_0})dt + \sigma_\mu \, dW_t^\mu, \quad \text{for all } t \geq 0, \tag{2.11}$$

where x_0 is the annuitant initial age; a is the mean reversion rate and σ_μ is the volatility (or standard deviation) of the force of mortality. $\theta(t)$ follows a Gompertz model (see Gompertz (1825)) and is given by $\theta(t) = Ae^{Bt}$, where A is the level of the mean reversion level at initial time (i.e. the baseline mortality) and B is the rate of the mean reversion level over time (i.e. the senescent component). Moreover, x_0, a, σ_μ, A and B are all positive constants. For more details on this mortality model, we refer the reader to the paper proposed by Zeddouk & Devolder (2019). We assume independence between equity and longevity risk, i.e. we assume both W_t and W_t^r are independent from W_t^μ. We define the survival index at time s of an annuitant initially aged x_0, alive at time t and surviving $s-t$ more years by

$$I_{s-t}^{x_0+t} = e^{-\int_t^s \mu_u^{x_0} du} = e^{X(t,s)} = e^{m^{x_0}(t,s) + \sigma^{x_0}(t,s)Z}, \tag{2.12}$$

where Z is normally distributed random variable with mean 0 and variance 1 (denoted by $Z \rightsquigarrow \mathcal{N}(0,1)$).

One can show that the negative integral of the force of mortality is given by

$$X(t,s) = -\int_t^s \mu_u^{x_0} du = m^{x_0}(t,s) - \frac{\sigma_\mu}{a}\int_t^s (1 - e^{-a(s-u)})dW_u^\mu. \tag{2.13}$$

$$X(t,s) \rightsquigarrow \mathcal{N}\left(m^{x_0}(t,s), (\sigma^{x_0}(t,s))^2\right) \Rightarrow \frac{X(t,s) - m^{x_0}(t,s)}{\sigma^{x_0}(t,s)} \rightsquigarrow \mathcal{N}(0,1), \tag{2.14}$$

where

$$m^{x_0}(t,s) = \frac{\mu_t^{x_0}(e^{-a(s-t)}-1)}{a} - \frac{Ae^{Bt}}{B(a+B)}(e^{B(s-t)}-1) - \frac{Ae^{Bt}}{a(a+B)}\left(e^{-a(s-t)}-1\right) \tag{2.15}$$

and

$$(\sigma^{x_0}(t,s))^2 = \frac{\sigma_\mu^2}{a^2}\left[s-t-\frac{1-e^{-a(s-t)}}{a} - \frac{(1-e^{-a(s-t)})^2}{2a}\right]. \tag{2.16}$$

In order to define the benefits of the annuitants, we assume an initially guaranteed life table given by the best estimate of the survival index at retirement age 65. We denote it by $p_{65}(0,t)$ and, we have

$$p_{65}(0,t) = \mathbb{E}[I_t^{65}] = A_{65}(0,t)e^{-B_{65}(0,t)\mu_0^{65}}, \tag{2.17}$$

with

$$B_{65}(0,t) = \frac{1}{a}(1-e^{-at}) \tag{2.18}$$

and

$$A_{65}(0,t) = \exp\left(\frac{A}{a+B}\left[\frac{1-e^{Bt}}{B} + B_{65}(0,t)\right] - \frac{\sigma_\mu^2}{2a^2}(B_{65}(0,t)-t) - \frac{\sigma_\mu^2}{4a}B_{65}(0,t)^2\right). \tag{2.19}$$

2.3. *Liability of the insurer*

In the remaining parts of this paper, the considered contract consists of a homogeneous group of retired policyholders buying either a lifetime, a deferred or a term annuity. As both the lifetime and term annuities could be seen as particular cases of the deferred annuity, hence our theoretical results will focus only on the deferred annuity. The liability of the insurer depends on the contract settled at inception, meaning it depends on the annuity purchased, the length of the payment period as well as the amount of benefit (initially) guaranteed by the insurer. The value of the initial liability of the insurer is then given by the sum of the discounted annual benefit of the cohort following the initially guaranteed mortality table. This is used as an assumption in order to define the cohort annual benefit ($\overline{R}$) as follows. We consider an initial cohort of N_0 annuitants paying a total single premium $\overline{A_0}$ for a d-year deferred annuity; using the guaranteed life table $p_{65}(0,t)$ (equals to the best estimate of the survival index) given by Formula (2.17). We consider a fixed guaranteed discount rate r describing the discount factor $P_f(t,s) = e^{-(s-t)r}$, hence the premium is computed using a fixed technical interest rate r and the fixed life table $p_{65}(0,t)$ corresponding to the initial best estimate of the mortality model. Note that r represents in fact the guaranteed annuity technical rate used by the insurer. Hence, it follows that $\overline{A_0} = P_f(0,d)\mathbb{E}[\overline{L_d}]$ where the expected value of the discounted benefits evaluated at the beginning of the payment period ($t = d$) is given by

$$\mathbb{E}[\overline{L_d}] = \mathbb{E}\left[\sum_{j=d}^{n} RN_j P_f(d,j)\right] = \sum_{j=d}^{n} RN_0 \; p_{65}(0,j) \; P_f(d,j), \tag{2.20}$$

with R representing individual annual benefit; N_j is the number of survivors at time $t = j$ and n is the length of the mortality table considered (or the duration of the contract). The total cohort's annual benefit $\overline{R} = RN_0$ is given by

$$\overline{R} = \frac{\frac{\overline{A_0}}{P_f(0,d)}}{\sum_{j=d}^{n} \; p_{65}(0,j) \; P_f(d,j)}. \tag{2.21}$$

$\overline{R}$ is in fact the cohort's annual benefit for a d-year deferred annuity and will be used to obtain the stochastic liability of the insurer. The formula of the annual benefit of a d'-year term annuity is obtained just by substituting n by $d' < n$ (which is the duration of the payment period) and by setting $d = 0$ in Formula (2.21). The annual benefit of a lifetime annuity is obtained by setting $d = 0$.

2.4. *Investment strategy*

We consider that the premium received by the insurer from the policyholders is invested in the financial market described above. Our model consists of different investment strategies depending on the level of hedging. We consider two investment strategies. These strategies have no effect on the level of the benefits, but rather have effect on the assets.

2.4.1. *Constant proportion strategy without guaranteed*

Denote by A_0 the single premium paid by a policyholder, then following our described financial market a constant proportion $x_s \in [0,1]$ of A_0 is assumed to be invested in the stock S_t, a proportion $x_p \in [0,1]$ is invested in the rolling bond $P(t,T)$ with maturity $T > 0$ and the remaining proportion $(1-x_s-x_p)$ is invested in the money market account B_t as defined in Sec. 2.1. The case of time-depending proportions and proportional rebalancing was partly studied by Devolder and Lebègue (see Devolder & Lebègue (2017)) and can be extended to our setting. Denote by A_t the value at time $t \geq 0$ of the portfolio made of x_s amount of the single premium A_0 in the stock, x_p amount in the $P(t,T)$ and $(1 - x_s - x_p)$ in the money market account. It follows that the dynamic of A_t is given by:

$$\begin{aligned} dA_t &= x_s \frac{dS_t}{S_t} A_t + x_p \frac{dP(t,t+T)}{P(t,t+T)} A_t + (1 - x_s - x_p) \frac{dB_t}{B_t} A_t \\ &= ((1 - x_s) r_t + x_s \mu + x_p \lambda_r \sigma_r B_r(t,t+T)) A_t dt \\ &\quad + x_s \sigma A_t dW_t - x_p \sigma_r B_r(t,t+T) A_t dW_t^r. \end{aligned}$$

For simplification purpose, let

$$B_r(T) = B_r(t, t+T) = \frac{1}{b}(1 - e^{-bT}). \tag{2.22}$$

One can show that the value at a given time v ($v \geq t \geq 0$) of an investment of the portfolio during $[t, v]$ is given by

$$\begin{aligned} A(t,v) = A_t \exp\Bigg[\Bigg(& x_s\mu + x_p\lambda_r\sigma_r B_r(T) - \frac{1}{2}(x_s^2\sigma^2 + x_p^2\sigma_r^2 B_r^2(T) \\ & -2\rho x_s x_p \sigma\sigma_r B_r(T))\Bigg)(v-t)\Bigg] \\ & \times \exp\left[(1-x_s)\int_t^v r_u du + x_s\sigma W_{v-t} - x_p\sigma_r B_r(T) W^r_{v-t}\right]. \end{aligned} \tag{2.23}$$

This constant proportion allocation strategy will be used in the deferred period and we will use the notation $A_t = A(0, t)$.

2.4.2. *Constant proportion strategy with guaranteed: the m guaranteed cash flows* (*mG-cf*) *strategy*

The strategy in this period is the combination of the constant proportion allocation strategy and a zero-coupon (ZC) strategy (i.e. the risk-free investment). Our motivation for this strategy comes from our desire to increase the attractiveness of annuity products for the insurer, so as to obtain a product that consumes less solvency capital for the insurer or is gainful for the shareholders as we will show in Sec. 4. Following the strategies proposed by Bauer & Weber (2008), who considered two strategies: the constant proportion allocation strategy and the liability hedging strategy, we based on the latter to define a periodic liability hedging strategy. In fact, instead of considering the liability hedging strategy on the contract duration as they did, we consider it only at the beginning of the contract and we refer to the obtained strategy as the m guaranteed cash flows. The term guaranteed here refers to the fact that the first m benefits are not subject to any risk as they are invested on a risk-free asset and depends on a deterministic life table. In order to make our strategy understandable, we propose the following detailed strategy for the lifetime annuity that we generalize later on to the deferred annuity.

(a) Lifetime annuity

Consider a homogeneous initial cohort of N_0 policyholders paying a total single premium $\overline{A_0}$ in order to purchase an immediate lifetime annuity for which the payment stream ends at most at time $t = n$ (in case the policyholder is alive). Our strategy on the payment period consists of investing the first m benefits (with $0 \leq m < n$) per survivors in a risk-free asset during the period $[0, m)$ and the remainder last $m_0 = n+1-m$ benefits are subject to a mixed investment during the whole contract duration $[0, n)$. This strategy is called the m guaranteed cash flows (mG-cf) strategy.

Notice that those m first benefits are assumed to be equity, interest rate guaranteed and conditionally longevity-guaranteed. In this case, we have two sub-periods: the guaranteed period $[0, m)$, where the insurer is hedged against the risks and the non-guaranteed period $[m, n)$, where he is exposed to the risks. The difference between these two sub-periods is that in the guaranteed period the benefit payout does not affect the risky investment whereas on the non-guaranteed period the benefit payout does affect the risky investment. By equity, interest rate-guaranteed benefits, we mean the benefits which are fully hedged against both the equity and the interest rate risks; whereas by conditionally longevity-guaranteed benefits we refer to the benefits fully hedged against longevity risk under some predefined conditions. This implies that, if the longevity condition is satisfied, then the insurance company will be 100% solvent for the first m benefits (i.e. from time $t = 0$ to time $t = m - 1$) and 0.5% uncertain for the remaining period $[m, n)$ according to the SII.

The mentioned (longevity) condition depends on both the policyholders and the insurer; it is stated at the contract inception and could be for example a given life table, a given trend such as the average trend of the mortality or the average trend of the survival and so on. In our case, we consider the best estimate of the survival index denoted by $S_{65}(0, t)$, where $x_0 = 65$ represents the retirement age of the policyholders. In other words, for all $t \in [0, m - 1)$ the number of survivors at time t is

$$N_t = N_0 S^{65}(0, t) = N_0\ \mathbb{E}[I_t^{65}], \tag{2.24}$$

where $S^{65}(0, t)$ is the survival probability of an individual initially aged 65, of living at least up to age $65 + t$ following the best estimate of the survival index. Note that $S^{65}(0, \cdot) = p_{65}(0, \cdot)$. Therefore, the m conditionally guaranteed benefits for a lifetime annuity evaluated at inception $t = 0$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=0}^{m-1} R\ \mathbb{E}[N_j]\ P(0, j) = \sum_{j=0}^{m-1} \overline{R}\ S^{65}(0, j)\ P(0, j). \tag{2.25}$$

This amount of benefits will then be deduced from the unique premium and the remaining asset will be

$$\overline{C_0} = \overline{A_0} - \overline{L_0^{(m)}}, \tag{2.26}$$

where $\overline{A_0} = \overline{A}(0, t)$ is defined by Formula (2.23).

(b) Generalization to the deferred annuity

Here we generalize the mG-cf strategy to a d-year deferred annuity as follows. Considering the same initial cohort as previously, paying the same single premium for a d-year deferred annuity, we have three sub-periods:

(a) the deferred period $[0, d)$ where no benefit payout is made but the mortality is considered;

(b) the guaranteed period $[d, d+m)$ where the mortality is considered and the benefit payout does not affect the risky investment: the insurer is hedged against the financial risks in this period.
(c) and the non-guaranteed period $[d+m, n)$ where the mortality is considered and the benefit payout affects the risky investment: the insurers are exposed to both the financial and mortality risks in this period.

The strategy consists of assessing at time $t = 0$ the first m benefits (with $0 \leq m < n + 1 - d$) per survivor invested in a risk-free asset during the guaranteed period $[d, d+m)$. Let us denote it by $\overline{L_0^{(m)}}$. Note that the remaining $m_0 = n + 1 - d - m$ benefits will be subject to a mixed investment (*cf.* 2.4.1) during the contract duration $[0, n)$. $\overline{L_0^{(m)}}$ will then be deduced from the unique premium $\overline{A_0}$ and the remaining premium denoted by $\overline{C_0}$ will be invested on the mixed asset during $[0, n]$ in order to pay the non-guaranteed benefits. It follows that the m conditionally guaranteed benefits for a d-year deferred annuity evaluated at time $t = 0$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=d}^{d+m-1} R\, \mathbb{E}[N_j\ P(0,j)] = \sum_{j=d}^{d+m-1} \overline{R}\ S^{65}(0,j)\ P(0,j). \tag{2.27}$$

The remaining asset at time $t = 0$ will be

$$\overline{C_0} = \overline{A_0} - \overline{L_0^{(m)}}. \tag{2.28}$$

Figure 1 represents the mG-cf strategy for a d-year deferred annuity

We observe that for $m = 0$, the mG-cf strategy is equivalent to the constant allocation strategy (*cf.* 2.4.1) since there is no benefit guaranteed. Whereas for $m = n$ we obtain the liability hedging strategy proposed by Bauer & Weber (2008).

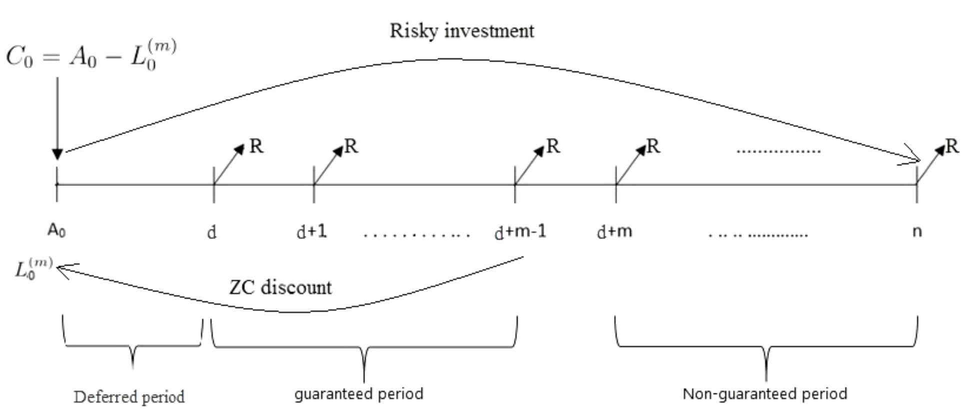

Fig. 1. Illustration mG-cf for a d-year deferred annuity with a constant annual payment R.

This formulation of the mG-cf strategy leads to the following question: what to do if the cohort's survival does not follow the average trend of the survival during the guaranteed period? In other words, how would the insurer pay the benefits of the gap of survivors during the guaranteed period? To answer this question, we need to take into account the risk generated by a possible gap of survivors during the guaranteed period. This risk is evaluated yearly and given by the difference between the survivors under the real mortality trend (i.e. using the survival index) and the survivors under the best estimate of the survival index (i.e. using the guaranteed mortality table). In the next section, we present the formulae used to measure the risk borne by an insurer selling a d-year deferred annuity within the mG-cf strategy by computing the insurer's SC.

3. Insurer's Solvency Capital

This section shows the main formulae of our model, i.e. the formulae of the SC. Recall that the first m guaranteed cash flows are evaluated at inception and deduced from the single premium, the remaining premium is invested on a mixed asset and the SC is invested on a rolling bond from the current time t till the supposed end of the contract n. Moreover, we value the SC by the use of the static risk measure VaR with respect to the final surplus for a given annuity. The theoretical analysis below corresponds to the computation of the SC for a d-year deferred annuity of length n. The corresponding analysis for a lifetime annuity will be obtained by setting $d = 0$ and that of a d'-year term annuity by setting both $d = 0$ and $n = d'$.

It is important to stress that the parameters of the HW mortality model of a given cohort are supposed to be known at inception, as they are calibrated from the cohort life table and these parameters are valid only for the cohort from which they have been calibrated. We consider a d-year deferred annuity, with ($d < n$) for which we have three computational intervals as described in Sec. 2.4, i.e. the deferred period $[0, d)$, the guaranteed period $[d, d+m)$ and non-guaranteed $[d+m, n)$ period. Note that the following conditions should be satisfied

$$0 < m < n + 1 - d; \tag{3.1}$$

meaning the number of guaranteed benefits m should be less or equal to the maximum number of benefits $n + 1 - d$. Below is given the formula of the SC of an insurer selling a deferred annuity in each period.

3.1. *SC during the deferred period*

In the deferred period $[0, d)$, the value at inception $t = 0$ of the m guaranteed benefits $\overline{L_0^{(m)}}$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=d}^{d+m-1} R\mathbb{E}[N_j]P(0,j) = \sum_{j=d}^{d+m-1} \overline{R}S^{65}(0,j)P(0,j), \tag{3.2}$$

where $\overline{R}$ is the cohort annual benefit given by Formula (2.21) and $S^{65}(0,j)$ is the best estimate of the survival index. For simplification purpose, let us denote the portfolio return by

$$\begin{aligned}\text{Asset}(t,v) = \exp\Bigg[\Bigg(& x_s\mu + x_p\lambda_r\sigma_r B_r(T) - \frac{1}{2}(x_s^2\sigma^2 + x_p^2\sigma_r^2 B_r^2(T) \\ & -2\rho x_s x_p \sigma\sigma_r B_r(T))\Bigg)(v-t)\Bigg] \\ & \times \exp\left[(1-x_s)\int_t^v r_u du + x_s\sigma W_{v-t} - x_p\sigma_r B_r(T) W_{v-t}^r\right]. \end{aligned} \tag{3.3}$$

The value at the computational time $t \in [0,d)$ of the remaining asset available for the non-guaranteed benefits is given by

$$\overline{C_t} = \overline{C_0}\ Asset(0,t) \tag{3.4}$$

which is known at time t and where $\overline{C_0}$ is given by Formula (2.28) and $\text{Asset}(t,d)$ is given by Formula (3.3).

The value at time n of the remaining $m_0 = n+1-d-m$ non-guaranteed benefits deduced yearly from the asset value between time $d+m$ and time n is given by

$$\begin{aligned}\overline{L'}(t,n,d,m) &= \sum_{j=m+d}^{n} R\ N_j \text{Asset}(j,n) \\ &= \sum_{j=m+d}^{n} R\ N_{d+m}\ I_{j-d-m}^{65+d+m}\text{Asset}(j,n) \\ &= \sum_{j=m+d}^{n} R\ N_d\ I_m^{65+d}\ I_{j-d-m}^{65+d+m}\text{Asset}(j,n) \\ &= \sum_{j=m+d}^{n} R\ N_t\ I_{d-t}^{65+t}\ I_m^{65+d}\ I_{j-d-m}^{65+d+m}\text{Asset}(j,n).\end{aligned}$$

Note that N_t is the observed number of survivors at time t, in other words N_j is known for $j \leq t$ and random for $j > t$. It follows that the risk of possible gap on survival during the guaranteed period is captured by

$$\overline{L'_r}(d,m) = \sum_{j=d}^{d+m-1} R(N_j - N_0\ S^{65}(0,j))\text{Asset}(j,d+m), \tag{3.5}$$

where $N_j = N_t\ I_{d-t}^{65+t}\ I_{j-d}^{65+d}$. Hence at any time $t \in [0,d)$, the SC satisfies the following solvency condition at maturity

$$\mathbb{P}_t[\overline{C}(t,n) - \overline{\text{Bene}f2} + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L'}(t,n,d,m)] \leq 1-\alpha(n-t),$$

where

(a) $\overline{C}(t,n)$ is the value at n of $\overline{C_t}$ (i.e. Formula (3.4)) and it is given by

$$\overline{C}(t,n) = \overline{C_t}\text{Asset}(t,n); \tag{3.6}$$

(b) $\overline{\text{Bene}f2}$ is the value at n of the benefit gap occurred on the guaranteed period and added or deduced on the remaining asset. It is computed as

$$\overline{\text{Bene}f2} = \overline{L'_r}(d,m)\text{Asset}(d+m,n), \tag{3.7}$$

where $\overline{L'_r}(d,m)$ is given by Eq. (3.5);

(c) $\overline{L'}(t,n,d,m)$ is the value at n of the non-guaranteed benefits computed above.

3.2. *SC during the guaranteed period*

The difference between this period $[d,d+m)$ and the previous period is that this is a payment guaranteed period as the payments are made but not from the risky investment; whereas the previous period is just the deferred period, where no payments are made. Hence, for a given computational time $t \in [d,d+m)$ the value at n of the non-guaranteed benefits denoted by $\overline{L''}(t,n,d,m)$ has to be adjusted based on the information obtained at time t

$$\begin{aligned}\overline{L''}(t,n,d,m) &= \sum_{j=d+m}^{n} R\ N_j\text{Asset}(j,n)\\ &= \sum_{j=d+m}^{n} R\ N_{d+m}\ I_{j-d-m}^{65+d+m}\text{Asset}(j,n)\\ &= \sum_{j=d+m}^{n} R\ N_t\ I_{d+m-t}^{65+t-T}\ I_{j-d-m}^{65+d+m}\text{Asset}(j,n),\end{aligned}$$

where $\text{Asset}(j,n)$ is given by Formula (3.3). In this period, the possible benefits gap that might occur during the remaining guaranteed period $[t,d+m)$, is obtained according to the actual number of survivors. Therefore, the future benefits gap is

$$\overline{L''_r}(t,d,m) = \sum_{j=t+1}^{d+m-1} R(N_j - N_0\ S^{65}(0,j))\text{Asset}(j,d+m), \tag{3.8}$$

where $N_j = N_t\ I_{j-t}^{65+t}$ with N_t observed. The asset value $\overline{C_t}$ at time t is obtained based on both the past gaps of survival and the past investment experience and it is equal to

$$\overline{C_t} = \overline{C_0}\ \text{Asset}(0,t) - \sum_{j=d}^{t}\overline{R}(I_j^{65} -\ S^{65}(0,j))\text{Asset}(j,t), \tag{3.9}$$

where the second term of the right hand side (RHS) is observed and represents the benefit gap observed before and up to the computational time t.

Hence, the SC satisfies the solvency condition

$$\mathbb{P}_t[\overline{C}(t,n) - \overline{\text{Bene}f3} + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L''}(t,n,d,m)] \leq 1 - \alpha(n-t), \tag{3.10}$$

where

(a) $\overline{C}(t,n)$ is the value at n of $\overline{C_t}$ (given by Eq. (3.9)), i.e.

$$\overline{C}(t,n) = \overline{C_t}\text{Asset}(t,n), \tag{3.11}$$

(b) $\overline{\text{Bene}f3}$ is the value at n of the future gap occurring on the remaining guaranteed period. It is given by

$$\overline{\text{Bene}f3} = \overline{L''_r}(t,d,m)\text{Asset}(d+m,n);$$

(c) $\overline{L''}(t,n,d,m)$ is the value at n of the non-guaranteed benefits computed above.

3.3. *SC during the non-guaranteed period*

In the non-guaranteed period $[d+m,n)$, for a given computational time $t \in [d+m,n)$, the SC satisfies the following solvency condition:

$$\mathbb{P}_t[\overline{C}(t,n) + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L'''}(t,n)] \leq 1 - \alpha(n-t), \tag{3.12}$$

where

(a) $\overline{C}(t,n)$ is the value of $\overline{C_t}$ at time n. The latter (i.e. $\overline{C_t}$) is obtained based on the gap of survivors during the guaranteed period, the past investment experience as well as the non-guaranteed benefits paid from time $d+m$ up to time t. More explicitly, since the information up to time t is known, it follows that $\overline{C_t}$ and N_t are known. The value at time $d+m$ of the asset used for the $n-d-m+1$ non-guaranteed benefits is known and given by

$$\overline{C'_{d+m}} = \overline{C_0}\text{Asset}(0,d+m) - \sum_{j=d}^{d+m-1} \overline{R}(I_j^{65} - \ S^{65}(0,j))\text{Asset}(j,d+m), \tag{3.13}$$

where the second term of the RHS represents the gap of benefits observed and deduced from the asset during the guaranteed period. It follows that the asset at time t is given by

$$\overline{C_t} = \overline{C'_{d+m}}\text{Asset}(d+m,t) - \sum_{j=d+m}^{t} \overline{R}\ I_j^{65}\ \text{Asset}(j,t), \tag{3.14}$$

where the second term of the RHS represents the benefits paid from time $d+m$ up to the computational time t. Therefore, we have

$$\overline{C}(t,n) = \overline{C_t}\text{Asset}(t,n). \tag{3.15}$$

(b) $\overline{L'''}(t,n)$ is the value at n of the remaining non-guaranteed benefits yearly deduced from the asset and it is given by

$$\overline{L'''}(t,n) = \sum_{j=t+1}^{n} R\ N_j \mathrm{Asset}(j,n)$$

$$= \sum_{j=t+1}^{n} R\ N_t\ I_{j-t}^{65+t} \mathrm{Asset}(j,n).$$

Note that for $m = 0$, the mG-cf strategy defined in 2.4.2 becomes the constant proportion strategy defined in 2.4.1 and in this case Sec. 3.2 does not exist any more.

4. Comparative Discussions from Numerical Results

In this section, we present some numerical results of the SC by the use of Monte Carlo simulation for two different cohorts, with initial ages $x_0 = 65$ and $x_0 = 75$, respectively. We further draw conclusions and remarks from the obtained results.

4.1. *Simulation framework*

To obtain the numerical results of the SC we first calibrate both the HW and the Vasicek models given by Formula (2.11) and (2.2). For the HW model, we consider two generations, i.e. an unisex projected generational life table of individual aged $x_0 = 65$ and $x_0 = 75$ with an ultimate age of 110 (available on the IA|BE life table proposed by Antonio *et al.* (2015)). From this we calibrate, the model using the mean square error (MSE); and we obtain the following parameters Table 1:

Concerning the Vasicek model, we estimate the parameters from the European Central Bank overnight rates[b] using the MLE and we obtain Table 2.

The market price of interest rate risk is obtained from the 30 years maturity AAA-rated Euro area central government bonds in 30 September 2019 using the MSE and, we have $\lambda_r = 0.0897$.

Secondly, we use the Monte Carlo (MC) simulations of the VaR (see Bauer *et al.* (2012)). We used 100000 simulations. For this purpose, we consider two cohorts of

Table 1. Calibration parameters of the HW model using MSE. Second and third rows represent the parameters for cohorts with initial age $x_0 = 65$ and $x_0 = 75$, respectively. This has been calibrated using the IA|BE mortality table.

μ_0^{65} and μ_0^{75}	A	B	a	σ^μ	MSE
0.0105677	0.0005749505	0.1304207503	0.0014965354	0.0083530153	0.000303644
0.02633591	0.009698091	0.115573842	0.310552836	0.051888992	0.000145998

[b] https://www.ecb.europa.eu/stats/policy_and_exchange_rates/key_ecb_interest_rates/html/index.en.html, downloaded on the 02/09/2019.

Table 2. Parameters of the Vasicek interest rate model using European Central Bank overnight rates.

b	c	σ_r
0.6587	0.01039	0.0122

$N_0 = 1000$ annuitants, respectively, aged $x_0 = 65$ and $x_0 = 75$ at the affiliation time $t = 0$. Each participant pays $A_0 = 1000\$$ to the insurer at inception in order to receive an amount R during the payment period, then $\overline{A_0} = N_0 A_0 = 1000 * 1000\$$. Moreover, we assume $n = 45$, $r = 1\%$ and a 15-year term annuity i.e. $d' = 15$. For the deferred annuity, we assume that the payment starts once the annuitant is aged $x_d = 81$ (called the deferred age). In other words for the cohort initially aged $x_0 = 65$ we consider a $d = 16$ years deferred annuity and for the cohort with $x_0 = 75$ we consider a $d = 6$ years deferred annuity.

We study the sensitivity of the SC with respect to the confidence level defined in Sec. 2 and for that purpose we consider the following values of the minimal confidence level $\alpha_0 \in \{85\%, 90\%, 95\%, 99\%, 99.5\%\}$. For these values of α_0, Fig. 2 shows the confidence level function (2.1) with respect to the time t.

Concerning the parameters of the stock, we consider those of the geometric Brownian motion calibrated with the MLE (see Phillips & Yu (2009)) using the S&P500 indexes[c] from 1871 to 2018. Hence, we obtained that the drift equals

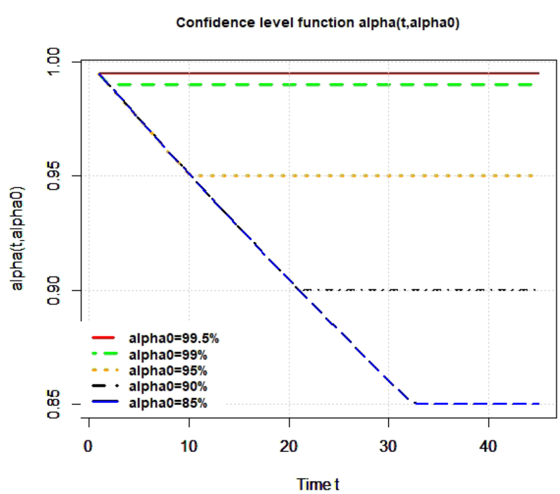

Fig. 2. Confidence level function with respect to the time t for different values of the minimum safety level α_0.

[c]http://www.multpl.com/s-p-500-historical-prices/table/by-year [Accessed on 7 March 2019].

$\mu = 0.058702$, the volatility equals $\sigma = 0.204172$ and the correlation between the stock and the discount bond is $\rho = -60\%$.

4.2. *Results*

In this section, some numerical results for each annuity are given in order to provide sufficient information to an insurer to guide his trading decision making.

4.2.1. *Basic case: lifetime annuity*

Table 3 shows the sensitivity of SC for a lifetime annuity with respect to ageing, m and α_0. We observe a decreasing trend with the number of guaranteed benefits m. In other words, for a lifetime annuity, the more benefits we guarantee the less SC we have. As expected, the less minimum safety level we use, the less SC we have. Furthermore, unlike the fully guaranteed strategy (i.e. $m = n$), the younger cohort of annuitants leads to a higher SC for $m < n$. Note that when $m = n$, the obtained SC reflects both longevity and interest rate risk borne by the insurer.

For comparison purposes, we consider the standard case where $m = 0$ and $\alpha_0 = 90\%$. We choose the non-guaranteed cash flow strategy (i.e. $m = 0$) because it is a common strategy in the literature and in practice. The choice of $\alpha_0 = 90\%$ comes from the fact that the time t for which $(99.5\%)^t = \alpha_0$ is closest to $n/2$ for both lifetime and deferred annuities as shown in Fig. 2.

4.2.2. *Deferred analysis*

Figure 3 illustrates the effect of the deferred time d on the SC for both cohorts with five benefits guaranteed. We observe for both cohorts that, the SC has a convex behaviour with respect to d; meaning that in this case there exists an optimal value of d that minimised the SC. Moreover, ageing effect increases with d and we have higher SC for the older cohort than for the younger for large values of d. This comes from the fact that, the longevity risk is more important at advanced ages as also shown in Table 4 where we can observe that the mG-cf strategy yields lower SC compared to both non-guaranteed ($m = 0$) and fully guaranteed

Table 3. Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ for a lifetime annuity, with $x_s = 15\%$ and $x_p = 25\%$. The values were obtained for different values of the minimum safety level α_0 and number of guaranteed benefits m as well as for different cohorts (or initial ages).

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n$	$m = 0$	$m = 5$	$m = n$
$\alpha_0 = 85\%$	0.267	0.132	0.0282	0.261	0.115	0.0638
$\alpha_0 = 90\%$	**0.361**	0.198	0.0443	**0.344**	0.167	0.08624
$\alpha_0 = 95\%$	0.5067	0.2987	0.067	0.471	0.241	0.121
$\alpha_0 = 99.5\%$	0.893	0.571	0.1384	0.818	0.453	0.2235

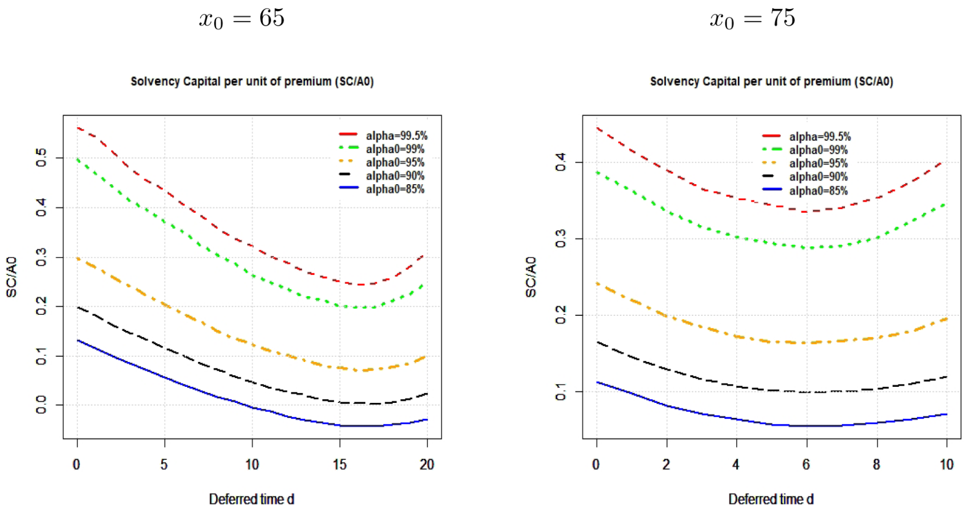

Fig. 3. $\overline{SC}(d)/\overline{A_0}$ at $t = 0$ for a d-year deferred annuity with respect to the deferred time d for $m = 5$ guaranteed cash flows for both cohorts and different values of the safety level α_0.

Table 4. Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ for a $d = 16$ years deferred annuity, with $x_s = 15\%$ and $x_p = 25\%$. The values were obtained for different values of the minimum safety level α_0 and number of guaranteed benefits m as well as for different cohorts (or initial ages).

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n - d$	$m = 0$	$m = 5$	$m = n - d$
$\alpha_0 = 85\%$	0.0282	−0.0432	0.09373	0.1674	0.0551	0.106
$\alpha_0 = 90\%$	**0.1071**	0.00553	0.1344	**0.2456**	0.0983	0.1425
$\alpha_0 = 95\%$	0.228	0.0747	0.2035	0.365	0.1632	0.2011
$\alpha_0 = 99.5\%$	0.544	0.2435	0.4015	0.681	0.337	0.3683

$(m = n - d)$ strategies. The increasing effect with the safety level is also observed in Table 4 and Fig. 3. Note that the negative SC obtained in Table 4 means that the insurer needs no extra capital to guarantee his solvency with the used confidence level.

From the standard case (i.e. values in bold), we observe that lifetime annuity yields higher SC compared to term and deferred annuities. Deferred annuity yields higher SC than term annuity for the older cohort, unlike for the younger cohort.

4.2.3. *Term analysis*

For a d'-year term annuity, let us assume $m = [3 + \frac{d'-5}{8}]$, where $[x]$ represents the integer part of x. Figure 4 shows that long maturity yields high SC and ageing increases the SC as shown in Table 5. The non-smooth curves reflect the impact

$x_0 = 65$ $\qquad\qquad$ $x_0 = 75$

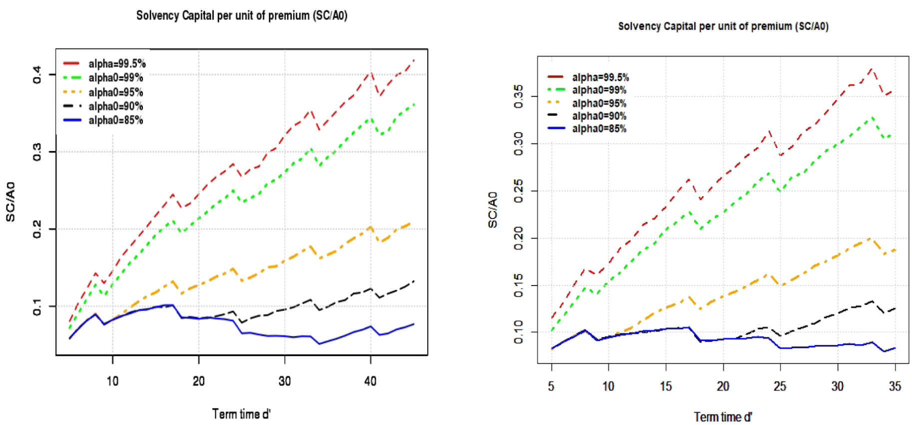

Fig. 4. $\overline{SC}(d')/\overline{A_0}$ at $t = 0$ of a d'-year term annuity with respect to the term time d', with $m = [3 + \frac{d'-5}{8}]$ for both cohorts and different values of the safety level α_0.

Table 5. Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ of a $d' = 15$ years term annuity, with $x_s = 15\%$ and $x_p = 25\%$. The values were obtained for different values of the minimum safety level α_0 and number of guaranteed benefits m as well as for different cohorts (initial ages).

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = d'$	$m = 0$	$m = 5$	$m = d'$
$\alpha_0 = 85\%$	0.1844	0.0825	0.031	0.2133	0.0899	0.0923
$\alpha_0 = 90\%$	0.185	0.0822	0.0313	0.2132	0.08914	0.0924
$\alpha_0 = 95\%$	0.216	0.1008	0.0374	0.2496	0.1097	0.1083
$\alpha_0 = 99.5\%$	0.3697	0.1876	0.0731	0.4302	0.211	0.1937

of the parameter m which changes with d'. Unlike the deferred annuity, we see from Table 5 that the fully guaranteed strategy yields a lower SC compared to non-guaranteed and mG-cf strategies. Note that SC obtained for the 15-year term annuity with the older cohort in Table 5 are close to that of the lifetime annuity, since in that case, we have an ultimate age of 90 years old.

4.2.4. *Guarantee effect*

Below, we present changes of the SC with respect to number of guaranteed benefits m for lifetime and $d = 16$ years deferred annuities. First row of Fig. 5 shows that for a 16-year deferred annuity, minimum SC depends on the minimum confidence level α_0. For instance, small α_0 and $m = 1$ produce the lower SC whereas for high α_0, we can find values of $m > 1$ that minimizes the SC for both cohorts. Furthermore,

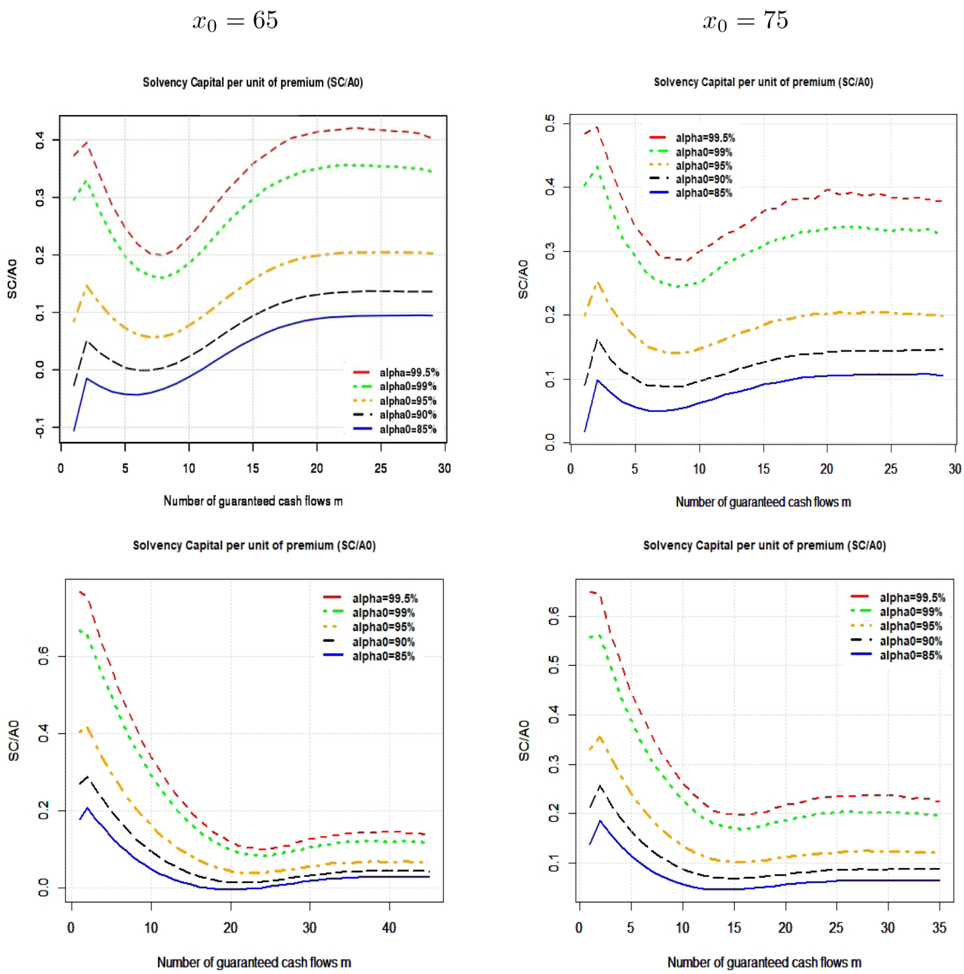

Fig. 5. $\overline{SC}(m)/\overline{A_0}$ at $t = 0$ of a 16-year deferred annuity (first raw) and of a lifetime annuity (second raw) with respect to the number of guaranteed benefits m, for both cohort and for different values of the safety level α_0.

the ageing and safety level do increase the SC. We have similar observations for the lifetime annuity which gives higher SC compared to the 16-year deferred annuity.

5. The Internal Rate of Return of the Shareholders

In this section, we discuss the possible profitability of the annuity's trading from the shareholder point of view. In other words we answer the question related to the choice of a shareholder to invest in an annuity or not. On the one hand, at inception an insurer could base his choice of the product to sell on the less SC consumer product (i.e. $SC = 0$) so as to minimise his SC. On the other hand, adding a SC could be seen as an investment, and assessing the returns of such an investment could help shareholders decide whether or not to invest in a given product, i.e. based on the profitability of the said product. The mentioned profitability can be measured based on the internal rate of return (IRR) which represents the ability of the invested SC to generate benefits (see Gronchi (1986)). It follows that the shareholder will lose all his initial capital (i.e. the SC) if IRR $= -1$, and when IRR $= 0$, then the exact amount of the SC will be recovered at the end of the contract. For a strictly positive IRR, the shareholder will make a profit upon the IRR. Moreover, a shareholder can define his minimum rate of return, says π, in such a way that he will invest on the annuity with an expected IRR strictly greater than π. The minimum rate π can be for example the risk-free rate available on the market at the contract inception or an expected rate of return of a given risky asset, etc.

As in the previous section, we have the following definition of the SC using our investment strategy

$$\mathbb{P}_t[\overline{C}(t,n) - \overline{\text{Bene}f} + \overline{SC}(t+n)P^{-1}(t,n) < \overline{L'}(t,n,d)] \leq 1 - \alpha(n-t),$$

from this we define the total final surplus of the contract. By total final surplus of a contract, we refer to the amount of both the premium and the SC that exceeds the liability of the insurer at the end of the contract. Hence, in our setting the total final surplus is a random variable given by

$$\chi(x,d,m;w) := \overline{C}(t,n) - \overline{\text{Bene}f} + \frac{\overline{SC}(t,n)}{P(t,n)} - \overline{L'}(t,n,d), \tag{5.1}$$

where $\overline{\text{Bene}f}$ represents the benefit's gap as defined in the previous section.

Let us denote the IRR by $\tau(x,d,m;w)$, which is a random variable satisfying the relation

$$\overline{SC}(t,n)(1+\tau(w))^{n-t} = (\chi(x,d,m;w))^+, \tag{5.2}$$

where $(x^+) = \max(x,0)$. It follows that the IRR is given by

$$\tau(x,d,m;w) = \left(\frac{(\chi(x,d,m;w))^+}{\overline{SC}(t,n)}\right)^{\frac{1}{n-t}} - 1. \tag{5.3}$$

Note that the IRR is computed if and only if the SC is strictly positive.

Tables 6 and 7 present the mean and variance of the IRR for both cohorts with a sensitive analysis on parameters m and α_0.

It comes out from Tables 6 and 7 that the expected IRR is higher for the younger cohort than for the older cohort, and it decreases when α_0 increases. Furthermore, the obtained values have a concave behavior with respect to m in the sense that the expected IRR increases with smaller values of m and decreases with larger values of m. This implies that there exists optimal value of m that maximises the expected IRR. The $d = 16$ years deferred annuity gives the better values of the expected IRR for the younger cohort with small values of m, whereas the $d' = 15$

Table 6. Mean (top values) and variance (bottom values in brackets) of the IRR at $t = 0$ for lifetime, $d' = 15$-year term and $d = 16$-year deferred annuities. Values obtained for $xs = 15\%$, $xp = 25\%$ and for the younger cohort with initial ages $x_0 = 65$.

	Lifetime annuity			Deferred annuity			Term annuity		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.7262%	1.5327%	1.327%	6.1262%	3.793%	2.086%	2.3311%	2.3209%	1.703%
	(0.04253%)	(0.03636%)	(0.01206%)	(0.11%)	(0.0536%)	(0.014%)	(0.2745%)	(0.2739%)	(0.11285%)
$m = 5$	2.48504%	2.0794%	1.5596%	/	9.454%	2.61%	4.1294%	4.1521%	2.6331%
	(0.04968%)	(0.04%)	(0.01274%)		(0.1425%)	(0.0152%)	(0.3031%)	(0.3018%)	(0.1227%)
$m = n$	2.299%	1.932%	1.4501%	1.888%	1.63%	1.33%	3.382%	3.3382%	2.1655%
	(0.04357%)	(0.0349%)	($9.557e^{-5}$)	(0.0392%)	(0.0323%)	(0.009%)	(0.261%)	(0.261%)	(0.09755%)

Table 7. Mean (top values) and variance (bottom values in brackets) of the IRR at $t = 0$ for lifetime, $d' = 15$-year term and $d = 16$-year deferred annuities. Values obtained for $xs = 15\%$, $xp = 25\%$ and for the younger cohort with initial ages $x_0 = 75$.

	Lifetime annuity			Deferred annuity			Term annuity		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.642%	1.407%	1.245%	2.065%	2.191%	1.582%	2.02%	2.003%	1.5333%
	(0.064%)	(0.0574%)	(0.01952%)	(0.074%)	(0.06135%)	(0.0199%)	(0.2632%)	(0.26495%)	(0.1071%)
$m = 5$	2.521%	2.103%	1.45%	4.08%	3.126%	1.942%	4.0997%	4.117%	2.5613%
	(0.07183%)	(0.061%)	(0.02025%)	(0.0945%)	(0.0697%)	(0.0216%)	(0.29023%)	(0.2896%)	(0.11204%)
$m = n$	2.502%	2.0971%	1.552%	1.557%	1.336%	1.1977%	1.5311%	1.5311%	1.234%
	(0.0734%)	(0.06084%)	(0.02011%)	(0.05825%)	(0.04998%)	(0.01496%)	(0.2287%)	(0.2287%)	(0.08425%)

years term annuity performs better for the older cohort as well as for the younger cohort with large values of m. It is important to stress that in some cases, we obtain high expected IRR and high variance as well; meaning that the IRR is subject to an important volatility (or risk).

Note that in Table 6, the mean and variance of the deferred annuity for $m = 5$ and $\alpha_0 = 85\%$ is not reported because in that case the SC is less than zero.

Overall, the mG-cf strategy generalizes the liability hedging strategy proposed by Bauer & Weber (2008) which is obtained when the number of guaranteed benefits is equal to the total number of benefits (i.e. $m = n + 1$ for the lifetime annuity, $m = d' + 1$ for the term annuity and $m = n - d + 1$ for the deferred annuity) within the mG-cf strategy. Therefore, alternative cases could improve or not the liability hedging strategy (in terms of both the value of the SC and the IRR) depending on the product sold or on the characteristics of the product. Thus, this highlights our contribution which is to measure the solvency of an insurer selling classical annuity products by assessing insurer's SC and IRR for each product. To do so, we have proposed a profitable and attractive investment strategy (for both the insurer and policyholders) generalizing the one proposed by Bauer & Weber (2008).

6. Conclusion

In this work we have proposed a triple-risk model, used to measure the equity, interest rate and longevity risks borne by an insurer selling a given annuity. From our model, we can deduce the equity risk-free model obtained by not investing on the risky asset. We have proposed a profitable investment strategy called the mG-cf strategy following the strategy proposed by Bauer & Weber (2008) called the fully liability hedging strategy. Our proposed strategy moves from the non-guaranteed liability strategy to the fully guaranteed liability strategy, depending on the number of cash flows guaranteed. In other words, the mG-cf strategy contains (i) the constant proportion allocation strategy, obtained when no benefits are guaranteed; (ii) the fully guaranteed cash flows (or liability hedging) strategy, obtained when all the benefits are guaranteed and (iii) the alternative strategies, obtained when the number of guaranteed benefits is between one and the maximum number of benefits. The risk measurement approach used consists of the valuation of both SC and IRR of an insurer using the mG-cf strategy for different annuity products and within the SII framework. The generalized formulae have been provided for a d-year deferred annuity since both the lifetime and term annuities are particular cases of the deferred annuity. Furthermore, numerical results for two different cohort and comparative observations have been provided with an in deep sensitivity analysis of the results with respect to some significant parameters.

We have found a significant increase of the SC from the younger to the older cohorts as well as with respect to the minimal confidence level. Furthermore, the mG-cf reduces the SC of the insurer as compared to the constant proportion strategy where no benefit is guaranteed. The concave behaviours of the IRR as well as convex behaviors of the SC (for a deferred annuity) show that the mG-cf strategy improves the strategy proposed by Bauer & Weber (2008). For a 15-year term annuity, the obtained SC is less volatile with respect to ageing as compared to both the lifetime and the deferred annuities. Moreover, we have found that the mG-cf strategy we have proposed yields higher expected IRR as compared to the fully liability hedging strategy proposed by Bauer & Weber (2008). Even though the ageing effect affects less the expected IRR of a term annuity, we notice higher expected IRR for the younger cohort for each of the annuities. Overall, we have found that the more benefits we guarantee the lower SC we have for both the lifetime and term annuities (which is in line with Bauer & Weber (2008)) whereas for the deferred annuity, there exits at least one number of guaranteed benefit that gives a lower SC as compared to the no-guaranteed (i.e. $m = 0$) and fully guaranteed benefits (i.e. m is equal to the total number of payouts). We have further found that our strategy gives a better IRR as compared to that of Bauer & Weber (2008) in the sense that ours gives a concave IRR with respect to the number of guaranteed benefits.

This paper has aimed to mitigate the risks from the insurer point of view. Therefore, future research could consist of focusing on the policyholder side by considering variable annual benefits so as to find the optimal parameters that maximize the benefit payouts of the policyholders. A more general model could consist of considering a dynamic risk measurement such as the iterated VaR or the iterated CTE for a heterogeneous cohort of policyholders (i.e. with different initial age, different premium and purchasing different annuity).

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