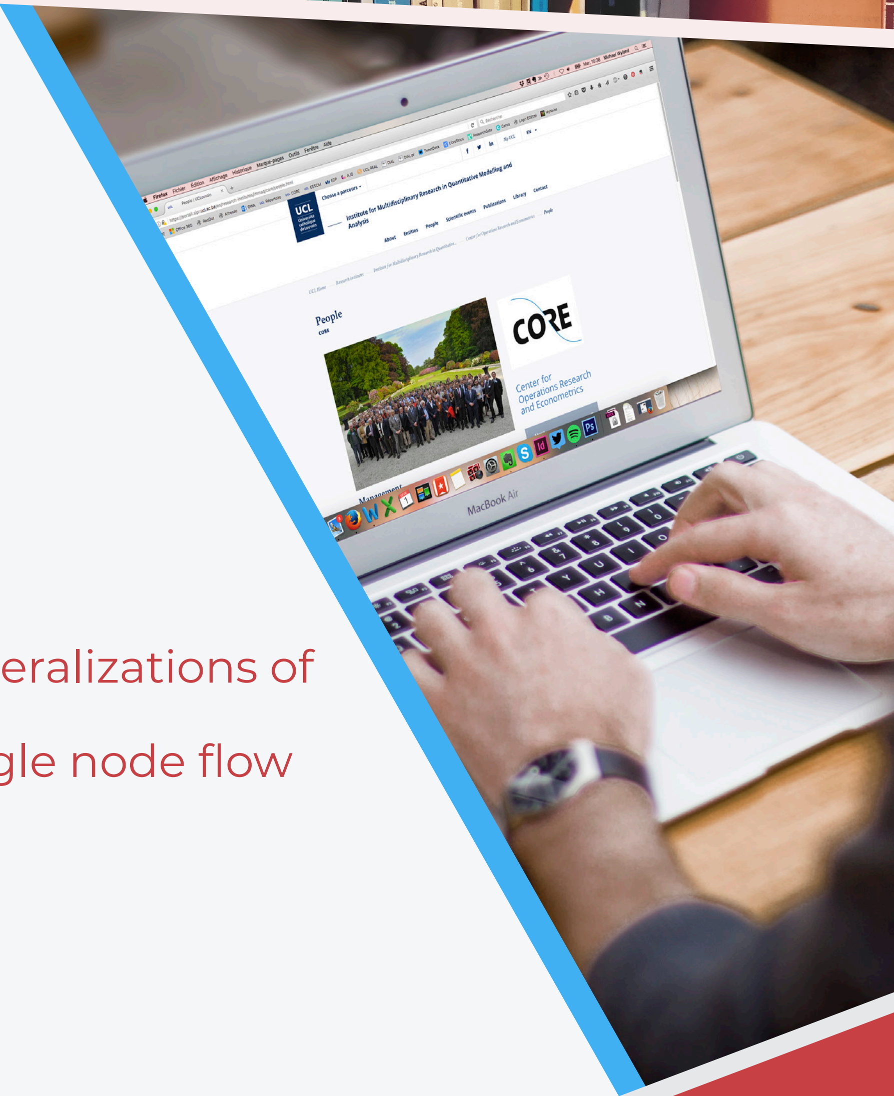
3100

REPRINT

Laurence A. Wolsey and Hande Yaman



Convex hull results of generalizations of the constant capacity single node flow set

Mathematical Programming A, online, 2020

CORE
Voie du Roman Pays 34, L1.03.01
B-1348 Louvain-la-Neuve
Tel (32 10) 47 43 04
Email:lidam-library@uclouvain.be
https://uclouvain.be/en/research-institutes/
lidam/core/reprints.html

Mathematical Programming
https://doi.org/10.1007/s10107-020-01481-6

FULL LENGTH PAPER

Series A

Convex hull results for generalizations of the constant capacity single node flow set

Laurence A. Wolsey[1] · Hande Yaman[2]

Received: 5 December 2018 / Accepted: 12 February 2020

Abstract

For single node flow sets with fixed costs and constant capacities on the inflow and outflow arcs, a family of constant capacity flow covers are known to provide the convex hull in different special cases and are conjectured to provide it in the general case. Here we study more general mixed integer sets for which such single node flow cover inequalities suffice to give the convex hull. In particular we consider the case of a path in which each node has one (or several) incoming and outgoing arcs with constant capacities and fixed costs. This can be seen as a lot-sizing set with production and sales decisions driven by costs and prices and by the lower and upper bounds on stocks instead of being driven by demands as in the standard lot-sizing model. The approach we take is classical: we characterize the extreme points, derive tight extended formulations and project out the additional variables. Specifically we show that Fourier–Motzkin elimination, though far from elegant, can be used to carry out the non-trivial projections. The validity of the conjecture for the single node flow set follows from our results.

Keywords Single node flow set · Flow cover inequalities · Lot-sizing with sales · Convex hull · Extended formulation · Fourier–Motzkin elimination

Mathematics Subject Classification 90C11 · 90B30

1 Introduction

Here we study generalizations of single node flow sets with constant capacities for which a simple class of flow cover inequalities provide the convex hull. Alternatively

✉ Hande Yaman
hande.yaman@kuleuven.be

Laurence A. Wolsey
laurence.wolsey@uclouvain.be

1 CORE, UCLouvain, 1348 Louvain-la-Neuve, Belgium

2 ORSTAT, Faculty of Economics and Business, KU Leuven, 3000 Leuven, Belgium

we are considering special cases of the constant capacity single-item lot-sizing problem with sales, lower and upper bounds on stocks and possibly also constant capacities and fixed costs on the sales.

The starting point is the single node flow set

$$\left\{(x,z): \sum_{j\in N^+} x_j - \sum_{j\in N^-} x_j \le b, 0 \le x_j \le c_j z_j \;\; j \in N, z \in \{0,1\}^{|N|}\right\},$$

with $N = N^+ \cup N^-$, where N^+ and N^- are disjoint sets.

A basic flow cover inequality for this set is the inequality

$$\sum_{j\in T}(x_j + (c_j - \lambda)^+(1 - z_j)) \le b + \lambda \sum_{j\in V} z_j + \sum_{j\in N^-\setminus V} x_j$$

where $T \subseteq N^+$ such that $\lambda = \sum_{j\in T} c_j - b > 0$ and $V \subseteq N^-$.

In the *constant capacity* case when $c_j = c$ for all $j \in N$ and b is not a multiple of c, the constant capacity flow cover inequality

$$\sum_{j\in T}(x_j - (c - \lambda)z_j)) \le \left\lfloor \frac{b}{c} \right\rfloor \lambda + \lambda \sum_{j\in V} z_j + \sum_{j\in N^-\setminus V} x_j, \tag{1}$$

is valid for $T \subseteq N^+$, $V \subseteq N^-$ and $\lambda = \lceil \frac{b}{c} \rceil c - b$.

It is known that the addition of the constant capacity flow cover inequalities gives the convex hull when $N^- = \emptyset$ [17] and it is conjectured/assumed to be true when $N^- \neq \emptyset$.

To simplify the notation, for two integers n_1 and $n_2 \ge n_1$, we let $[n_1, n_2] = \{n_1, \ldots, n_2\}$. For a vector $\alpha \in \mathbb{R}^n$ and for $T \subseteq [1, n]$, we let $\alpha(T) = \sum_{j\in T} \alpha_j$. When $T = [k, t]$, we use α_{kt} instead of $\alpha([k, t])$, to denote $\sum_{j=k}^{t} \alpha_j$.

We generalize the single node flow set by considering the set of flows with constant capacities and fixed costs along a directed path. Suppose that we have a digraph with n nodes. Let sets N_t^+ and N_t^- for $t \in [1, n]$ be such that $N_1^+ \subseteq \ldots \subseteq N_n^+$ and $N_1^- \subseteq \ldots \subseteq N_n^-$. We define $N_t = N_t^+ \cup N_t^-$ for $t = 1, \ldots, n$, $N^+ = N_n^+$, $N^- = N_n^-$ and $N = N^+ \cup N^-$. Let $S^+ \subseteq N^+$, $S^- \subseteq N^-$, $S = S^+ \cup S^-$, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be reals so that $a_t \le b_t$ for all $t \in [1, n]$.

We define set X to be the set of solutions to the system

$$\begin{aligned}
&a_t \le s_t \le b_t \quad t \in [1, n],\\
&s_t = x(N_t^+) - x(N_t^-) \;\; t \in [1, n],\\
&x_j \le c_j z_j \quad j \in S,\\
&x \in \mathbb{R}_+^{|N|}, z \in \{0,1\}^{|S|}.
\end{aligned}$$

Here variable x_j for $j \in N_t^+ \setminus N_{t-1}^+$ denotes an inflow at node t with fixed cost variable z_j if $j \in S^+$, x_j for $j \in N_t^- \setminus N_{t-1}^-$ is an outflow variable at node t with fixed cost

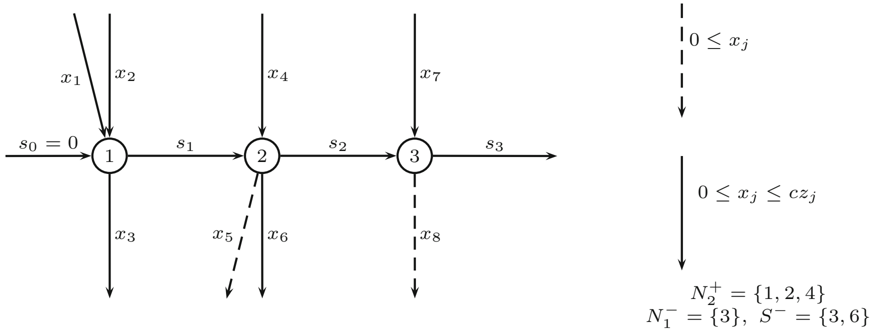


Fig. 1 Flows on a Dipath

variable z_j if $j \in S^-$ and s_t is the flow (positive or negative) from node t to node $t+1$, see Fig. 1.

Set X is also the feasible set of a lot-sizing problem with lower and upper bounds on stocks as well as multiple production and sales modes, possibly with capacities and setup costs (note that the constraints $s_t = x(N_t^+) - x(N_t^-)$ for $t \in [1, n]$ can equivalently be written as $s_{t-1} + x(N_t^+ \setminus N_{t-1}^+) = x(N_t^- \setminus N_{t-1}^-) + s_t$ for $t \in [1, n]$ with $s_0 = 0$ and $N_0^+ = N_0^- = \emptyset$). In particular, x_j for $j \in N_t^+ \setminus N_{t-1}^+$ is the amount produced using mode j in period t, with a setup variable z_j if $j \in S^+$, x_j for $j \in N_t^- \setminus N_{t-1}^-$ is the amount sold using mode j in period t, again with an associated setup variable z_j if $j \in S^-$ and s_t is the stock at the end of period t.

Such sets also arise in the constant capacity fixed charge network flow problem. A digraph $G = (V, A)$ is given with node-arc incidence matrix D, lower and upper bounds a_t, b_t on the net flow entering at node $t \in V$ and constraints $a \leq Dx \leq b$, $0 \leq x \leq cz, z \in \{0, 1\}^{|A|}$ where x_j is the flow on arc j and $z_j = 1$ if capacity is installed on arc j. Aggregating the constraints over a subset of nodes $V' \subset V$ gives

$$a(V') \leq x([V', V \setminus V']) - x([V \setminus V', V']) \leq b(V'),$$

where $[V', V \setminus V']$ is the set of arcs that start in set V' and end in set $V \setminus V'$ and $[V \setminus V', V']$ is the set of arcs in the opposite direction. That, along with the variable upper bound and integrality constraints produces a single node flow set with both lower and upper bounds on the net flow.

Some special cases and restrictions of set X have been studied previously:

1. The single node flow set is the special case of X with $n = 1$, $S^+ = N^+$, $S^- = N^-$ and $a_1 = -\infty$. The first results on this set can be found in Padberg et al. [17]. They describe the flow cover inequality when the capacities are varying for the case with $N^- = \emptyset$. The case with outflows is examined in Van Roy and Wolsey [22]. Further generalizations can be found in Gu et al. [13] and Louveaux and Wolsey [15]. Stallaert [20] introduced a complementary class of flow cover inequalities and Atamtürk [2] developed the class of flow pack facets. Recently Atamtürk et al. [6] present path inequalities for fixed charge network flow problems significantly

generalizing flow cover inequalities.

In the constant capacity case, it was shown in [17] that the addition of the flow cover inequalities suffices to give the convex hull when $N^- = \emptyset$. For $N^- \neq \emptyset$, Atamtürk [3] showed that adding the inequalities (1) suffices when the z variables are nonnegative integers. The same result with z variables in $\{0, 1\}^n$ is reported to hold in Atamtürk et al. [4], but we have not found any proof in the literature.

2. The capacitated lot-sizing set is the restriction obtained by setting $z_j = 1$ and $x_j = c_j$ (i.e., sales is equal to demand) for $j \in N^-$ in the case where N_t^+ and N_t^- are singletons for $t \in [1, n]$, $S^+ = N^+$, $S^- = N^-$, $a_t = 0$ and $b_t = \infty$ for $t \in [1, n]$. Single-item lot-sizing was first treated in the seminal paper of Wagner and Whitin [23]. Valid inequalities that are sufficient to describe the convex hull for single item lot-sizing sets are given for the uncapacitated case (c_j arbitrarily large for $j \in N^+$) in Barany et al. [7] and the constant capacity case in Pochet and Wolsey [18] ($c_j = c$ for $j \in N^+$). Optimizing over this set with time dependent capacities is NP-hard (see [8,11]). Lot-sizing with stock upper bounds was first treated in Love [16]. Valid inequalities are presented in Atamtürk and Küçükyavuz [3] both with and without fixed charges on the stock variables. See Wolsey [24,25] for a tight extended formulation and the equivalence to lot-sizing with production time windows. Other equivalent problems are presented in van den Heuvel and Wagelmans [21].

3. The lot-sizing set with sales is the restriction obtained by setting $z_j = 1$ for $j \in N^-$ in the special case where N_t^+ and N_t^- are singletons for $t \in [1, n]$, $S^+ = N^+$, $S^- = N^-$, $a_t = 0$ and $b_t = \infty$ for $t \in [1, n]$. In this setting the fixed demands are zero. If nonzero fixed demands exist, then this can be handled by defining $a_t = d_{1t}$ for $t \in [1, n]$, where d_{1t} is the cumulative fixed demand for periods 1 to t. Valid inequalities and convex hull results for the uncapacitated case where the fixed demands are zero but stock lower bounds are possibly nonzero (representing safety stocks, or equivalently stock lower bounds are zero and fixed demands are possibly nonzero) are given in Loparic et al. [14]. Note that lot sizing with sales models are equivalent to lot sizing with lost sales, studied among others in Aksen et al. [1]. Most of the above lot-sizing results are collected together in Pochet and Wolsey [19].

An important observation concerning the results on lot-sizing sets is that the valid inequalities depend explicitly on the demands in each period. Here in contrast we suppose throughout that fixed demands are zero and sales may possibly be capacitated. Thus feasibility in X is entirely determined by the inflow (production) and outflow (sales) capacities and the bounds on the stocks.

A similar set arises in the warehouse problem in which the aim is, given an initial stock and a capacitated warehouse, to decide when to sell and purchase a product to maximize profit. This problem does not impose capacities on production and sales; instead it includes an initial stock, a stock capacity and a constraint imposing that the sales amount in a given period cannot exceed the stock amount from the previous period. Wolsey and Yaman [26] present convex hull results for variants of this problem and show that the nontrivial facet defining inequalities are flow cover inequalities based on single node set relaxations. The approach taken there is to present tight extended formulations based on properties of the extreme points and then project out the additional variables using Fourier–Motzkin elimination. The same approach will be used here.

Throughout the paper, we consider constant capacities, i.e., $c_j = c$ for all $j \in N$. We provide the following results:

1. a description of $\text{conv}(X)$ when $S^+ = N^+, S^- = N^-, a_t = -\infty$ and $b_t \mod c = b$ for all $t \in [1, n]$ (this proves the conjecture about the convex hull of the constant capacity single node flow set),
2. a description of $\text{conv}(X)$ when $n = 1$, $S^+ = N^+$, $S^- = N^-$ and $a_1 \leq 0 < b_1$, (this provides a stronger result than that of the conjecture about the convex hull of the constant capacity single node flow set),
3. a description of $\text{conv}(X)$ when $S^+ = N^+$, $S^- = \emptyset$, N_t^+ and N_t^- are singletons, $a_t = 0$ and $b_t \mod c = b$ for all $t \in [1, n]$,
4. a tight extended formulation for $\text{conv}(X)$ in the "discrete lot-sizing case" when $S^+ = N^+$, $S^- = \emptyset$, N_t^+ and N_t^- are singletons, $x_t = cz_t$ for all $t \in [1, n]$ and the bounds $a_t < b_t$ are arbitrary,
5. a description of $\text{conv}(X)$ in the "discrete lot-sizing case" when $S^+ = N^+, S^- = \emptyset$, N_t^+ and N_t^- are singletons, $x_t = cz_t$ for all $t \in [1, n]$, $a_t = a$ and $b_t = b$ for all $t \in [1, n]$ with $a \mod c < b \mod c$.

Most of our results are based on the assumption that $b_t \mod c = b$ or $a_t \mod c = a$ or $a_t = a$ and $b_t = b$ for all $t \in [1, n]$. This is the case, for instance, for the lot sizing problem with sales and no fixed demands where the parameters a_t and b_t's represent lower and upper bounds on the stocks that are typically constant over the time horizon. However our assumption may be restrictive if these parameters are determined by time varying demands.

2 The extreme points of conv(X)

A standard approach for solving lot-sizing problems and optimizing over the set X is through the use of regeneration intervals. These results follow immediately from the work of Love [16] treating stock upper bounds and Florian and Klein [10] dealing with constant production capacities. We present briefly the main ideas.

- With stock bounds, the interval $[k, \ell]$ is a *regeneration interval* if $s_{k-1} \in \{a_{k-1}, b_{k-1}\}$, $s_\ell \in \{a_\ell, b_\ell\}$ and $a_t < s_t < b_t$ for $t \in [k, \ell - 1]$.
- Any extreme point solution can be decomposed into a series of regeneration intervals plus a (possibly empty) final interval $[k, n]$ with $s_{k-1} \in \{a_{k-1}, b_{k-1}\}$ and $a_t < s_t < b_t$ for $t \in [k, n]$. Note the net inflow in the interval is $s_\ell - s_{k-1}$. We also assume $s_0 = a_0 = b_0$ throughout.
- When the setup variables z_j's are given, optimizing over set X becomes a minimum cost network flow problem for which the basic variables do not form a cycle in an extreme point. This implies that there can be at most one inflow or outflow arc x_q whose flow is not at one of its bounds in each regeneration interval.

- Consider a regeneration interval $[k, \ell]$. Set $\delta_{k,\ell} = s_\ell - s_{k-1} \mod c$. When $\delta_{k,\ell} > 0$, all x_j are at their bounds with one exception x_q for some $q \in [k, \ell]$. Due to the constant capacities, either x_q is an inflow variable with $q \in S^+$ and $x_q = \delta_{k,\ell}$, or it is an outflow variable with $q \in S^-$ and $x_q = c - \delta_{k,\ell}$. If $\delta_{kl} = 0$ or it is a final interval, all x variables are at their bounds.
- A minimum cost solution for each regeneration interval can be found in polynomial time using dynamic programming. Finally optimizing over X is polynomial using a shortest path algorithm to find the optimal partition of the interval $[1, n]$ into regeneration intervals.

We now specialize to the cases treated below.

Proposition 1 *Suppose that* $b_t \mod c = b$ *for all* $t \in [1, n]$ *and let* $\lambda = c - b$.

(i) When $a_t = -\infty$ *for all* $t \in [1, n]$, *at an extreme point of* $\text{conv}(X)$, *there is at most one* $j \in S$ *with* $0 < x_j < c$. *If* $j \in S^+$, *then* $x_j = c - \lambda$ *and if* $j \in S^-$, *then* $x_j = \lambda$.
(ii) When $S^- = \emptyset$, $S^+ = N^+$ *and* $a_t = 0$ *for all* $t \in [1, n]$, *at an extreme point,* $x_j \in \{0, c - \lambda, c\}$ *for* $j \in N^+$ *and* $s_t \mod c \in \{0, c - \lambda\}$ *for* $t \in [1, n]$. *In addition, if* $x_j = c - \lambda$ *for some* $j \in N_t^+ \backslash N_{t-1}^+$ *then* $s_{t-1} \mod c = 0$ *and* $s_t \mod c = c - \lambda$.

Proof (i) The regeneration intervals with $s_0 = 0$ and $s_\ell = 0$, or $s_{k-1} = b_{k-1}$ and $s_\ell = b_\ell$ have $\delta_{k\ell} = 0$, so $x_j \in \{0, c\}$ for all $j \in N_\ell^+ \backslash N_{k-1}^+$. The only other regeneration intervals have $s_0 = 0$ and $s_\ell = b_\ell$. Then $\delta_{1\ell} = c - \lambda$ and the claim follows. If $s_n \neq b_n$ then $x_j \in \{0, c\}$ for all $j \in (N \backslash N_{k-1}) \cap S$ where $[k, n]$ is the last interval.

(ii) The regeneration intervals with $s_{k-1} = 0$ and $s_\ell = 0$, or $s_{k-1} = b_{k-1}$ and $s_\ell = b_\ell$ have $\delta_{k\ell} = 0$, so $x_j \in \{0, c\}$ for all $j \in (N_\ell \backslash N_{k-1}) \cap S$. The other possibilities are $s_{k-1} = 0$ and $s_\ell = b_\ell$ or $s_{k-1} = b_{k-1}$ and $s_\ell = 0$. In the first case $x_j = c - \lambda$ for some $j \in N_t^+ \backslash N_{t-1}^+$ provides a jump from $s_{t-1} \mod c = 0$ to $s_t \mod c = c - \lambda$. The second transition is only possible if $x_j \mod c = c - \lambda$ for some $j \in N_\ell^- \backslash N_{k-1}^-$ and $x_j \in \{0, c\}$ for $j \in N_\ell^+ \backslash N_{k-1}^+$. Finally, if $s_n \notin \{0, b_n\}$, then $x_j \in \{0, c\}$ for all $j \in N^+ \backslash N_{k-1}^+$ where $[k, n]$ is the last interval.

□

3 Convex hull results for the case of setups both for inflows and outflows

In this section, we consider the case with $S^+ = N^+$ and $S^- = N^-$. We are first interested in the convex hull of the mixed integer set $X_{\leq}$ defined as

$$x(N_t^+) - x(N_t^-) \leq b_t \quad t \in [1, n], \tag{2}$$
$$0 \leq x_j \leq cz_j \quad j \in N, \tag{3}$$
$$z_j \in \{0, 1\} \quad j \in N. \tag{4}$$

Set $X_{\leq}$ is a special case of set X with $S^+ = N^+$, $S^- = N^-$ and $a_t = -\infty$ for $t \in [1, n]$ and variables s_t projected out.

In the sequel, we prove that when $b_t \mod c = b$ for $t \in [1, n]$ and $b > 0$, $\text{conv}(X_{\leq})$ is described by the constraints (2), (3), $z_j \leq 1$ for $j \in N$ and inequalities

$$x(T) - (c - \lambda)z(T) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^- \setminus V) + \lambda z(V) \tag{5}$$

for $\emptyset \subset T \subseteq N_t^+$, $V \subseteq N_t^-$ and $t \in [1, n]$ where $\lambda = c - b$. If $b = 0$, then inequalities (5) are implied and $\text{conv}(X_{\leq})$ is described by (2), (3) and $z_j \leq 1$ for $j \in N$.

3.1 An extended formulation

We use the property of the extreme points derived in Proposition 1 (i). For $j \in N^+$, we define variable z_j^1 to be 1 if $x_j = c$ and 0 otherwise and variable z_j^2 to be 1 if $x_j = c - \lambda$ and 0 otherwise. Similarly, for $j \in N^-$, we define variable z_j^1 to be 1 if $x_j = c$ and 0 otherwise and variable z_j^2 to be 1 if $x_j = \lambda$ and 0 otherwise. We provide an extended formulation using these additional variables.

Proposition 2 *When $b_t \mod c = b$ for all $t \in [1, n]$, the following is an extended formulation for set $X_{\leq}$:*

$$z^1(N_t^+) - z^1(N_t^-) - z^2(N_t^-) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1, n], \tag{6}$$

$$z^2(N^+) + z^2(N^-) \leq 1, \tag{7}$$

$$z_j^1 + z_j^2 - z_j \leq 0 \quad j \in N, \tag{8}$$

$$z_j^1, z_j^2 \geq 0 \quad j \in N, \tag{9}$$

$$z_j \leq 1 \quad j \in N, \tag{10}$$

$$z_j^1, z_j^2, z_j \in \{0, 1\} \quad j \in N, \tag{11}$$

$$x_j = cz_j^1 + (c - \lambda)z_j^2 \quad j \in N^+, \tag{12}$$

$$x_j = cz_j^1 + \lambda z_j^2 \quad j \in N^-. \tag{13}$$

Proof Suppose that $b_t \mod c = b$ for all $t \in [1, n]$. It is easy to see that for any extreme point (x, z) of $\text{conv}(X_{\leq})$, there exist z^1, z^2 such that (x, z, z^1, z^2) satisfies the system (7)–(13). Point (x, z, z^1, z^2) also satisfies (6) since this inequality is obtained by rewriting constraint (2) using the additional variables, dividing it by c and applying rounding. Now for any (x, z, z^1, z^2) that satisfies the system (6)–(13), (x, z) clearly satisfies (3)–(4). For $t \in [1, n]$,

$$\begin{aligned}x(N_t^+) - x(N_t^-) &= cz^1(N_t^+) + (c-\lambda)z^2(N_t^+) - cz^1(N_t^-) - \lambda z^2(N_t^-)\\ &= c(z^1(N_t^+) - z^1(N_t^-) - z^2(N_t^-)) + (c-\lambda)(z^2(N_t^+) + z^2(N_t^-))\\ &\leq c\left\lfloor \frac{b_t}{c} \right\rfloor + c - \lambda = b_t.\end{aligned}$$

So (x, z) also satisfies (2) and is in $X_{\leq}$. This proves that (6)–(13) is an extended formulation for set $X_{\leq}$. □

Next we prove that the extended formulation is tight. For this purpose, we first give a very simple lemma.

Lemma 1 *Let*

$$\Delta = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\}.$$

If $\delta^1, \delta^2 \in \Delta$, then either $\delta^1 + \delta^2 \in \Delta$ or $\delta^1 - \delta^2 \in \Delta$.

Proof It suffices to check case by case. □

Proposition 3 *At the extreme points of the polytope defined by* (6)–(10)*, z_j^1, z_j^2, z_j are 0–1 for all $j \in N$.*

Proof To prove this, we show that the matrix associated with constraints (6)–(8) is totally unimodular (TU).

Given an expression $\sum_{j \in C^+} w_j - \sum_{j \in C^-} w_j$ that is part of a $0, +1, -1$ constraint and a partition (J^+, J^-) of a subset J of the variables where we associate weights $+1$ and -1 to the members of J^+ and J^- respectively, the *weight* associated to the expression is $|J^+ \cap C^+| + |J^- \cap C^-| - |J^+ \cap C^-| - |J^- \cap C^+|$.

We use the characterization of Ghouila-Houri [12]: Given a $0, +1, -1$ matrix, the matrix is totally unimodular if, for any subset J of the columns, there exists a partition (J^+, J^-) of J such that the weight associated to each row is $0, +1$ or -1.

We observe that each variable z_j only occurs once with other variables, and can be ignored. Thus we only need to consider the z_j^1, z_j^2 variables. In addition if both z_j^1 and z_j^2 lie in J, the two variables cannot be assigned the same weight due to constraints (8).

Given a subset J of the variables, we order them selecting first from N_1, then $N_2 \backslash N_1$, etc.

Let $\chi_j(t)$ be the weight assigned to the left hand side of constraint (6) for t by taking just the variables $1, \ldots, j$ that are in J. If $j \in N_k$, we observe that $\chi_j(t) = \chi_j(k)$ for $t \geq k$ and $\chi_j(t) = \chi_{j'}(t)$ for $j' \geq j$ and $t < k$. In addition, if $j+1 \in N_\ell$, then $\chi_{j+1}(t) = \chi_j(t)$ for $t < \ell$. Similarly let η_j be the weight assigned to the left hand side of constraint (7) by taking the variables with indices $1, \ldots, j$ that are in J.

As induction hypothesis, suppose that there exists an assignment of the pairs of variables for $1, \ldots, j$ with $j \in N_k$ such that $\chi_j(t) \in \{0, +1, -1\}$ for $t < k$ and $(\chi_j(k), \eta_j) \in \Delta$. Consider the pair of variables (z_{j+1}^1, z_{j+1}^2) with $j+1 \in N_\ell$. Since $\chi_{j+1}(t) = \chi_j(t)$ for $t < \ell$, $\chi_j(t) = \chi_j(k)$ for $t \geq k$ and $\chi_j(t) \in \{0, +1, -1\}$ for

$t \leq k$, we have $\chi_{j+1}(t) \in \{0, +1, -1\}$ for $t < \ell$. Now to show that the induction hypothesis holds for $j+1$, it is sufficient to show that we can assign weights to z^1_{j+1} and z^2_{j+1}, whichever are in J, such that $(\chi_{j+1}(\ell), \eta_{j+1}) \in \Delta$. If z^1_{j+1} is in J, we assign it weight $+1$ and if z^2_{j+1} is in J, we assign it weight -1. It is easily checked that $(\chi_{j+1}(\ell) - \chi_j(\ell), \eta_{j+1} - \eta_j) \in \Delta$.

Since $\chi_j(\ell) = \chi_j(k)$ and $(\chi_j(k), \eta_j) \in \Delta$, $(\chi_{j+1}(\ell), \eta_{j+1})$ is the sum of two vectors in Δ and thus by the lemma their sum or difference is also in Δ. So by possibly switching the assignment of the variables (z^1_{j+1}, z^2_{j+1}) lying in J, we have extended the assignment to $j+1$. Finally observe that when $j = 0$, $(0, 0) \in \Delta$, so the result holds for $j = 0$. The claim follows by induction. □

3.2 Projection

Next we project out the additional variables to obtain a linear description in the original space.

Theorem 1 *When $b_t \mod c = b$ for all $t \in [1, n]$, $\mathrm{conv}(X_{\leq})$ is described by the constraints (2), (3), $z_j \leq 1$ for $j \in N$, plus the inequalities (5) for $T \subseteq N_t^+$, $V \subseteq N_t^-$ and $t \in [1, n]$.*

Proof Substituting $(c-\lambda)z_j^2 = x_j - cz_j^1$ for $j \in N^+$ and $\lambda z_j^2 = x_j - cz_j^1$ for $j \in N^-$ in (6)–(10) and setting $\alpha_j = z_j^1$ for $j \in N^+$ and $\beta_j = z_j^1$ for $j \in N^-$, we obtain

$$\lambda\alpha(N_t^+) + (c-\lambda)\beta(N_t^-) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, n], \tag{14}$$

$$\lambda(x(N^+) - c\alpha(N^+)) + (c-\lambda)(x(N^-) - c\beta(N^-))\big) \leq \lambda(c-\lambda), \tag{15}$$

$$\lambda\alpha_j \geq x_j - (c-\lambda)z_j \quad j \in N^+, \tag{16}$$

$$(c-\lambda)\beta_j \geq x_j - \lambda z_j \quad j \in N^-, \tag{17}$$

$$x_j \geq c\alpha_j \quad j \in N^+, \tag{18}$$

$$x_j \geq c\beta_j \quad j \in N^-, \tag{19}$$

$$\alpha_j \geq 0 \quad j \in N^+ \tag{20}$$

$$\beta_j \geq 0 \quad j \in N^-, \tag{21}$$

$$z_j \leq 1 \quad j \in N, \tag{22}$$

where (14) and (15) come from (6) and (7), (16) and (17) come from (8), and (18) and (19) come from the non-negativity of z_j^2.

We prove the theorem by projecting out the variables α and β from the system (14)–(22). We first show that after projecting out the variables α_j and β_j for $j \in N \backslash N_k$, the resulting polytope is given by

$$\lambda\alpha(N_t^+) + (c-\lambda)\beta(N_t^-) \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, k], \tag{23}$$

$$\lambda\alpha(N_k^+) + (c-\lambda)\beta(N_k^-) + \sum_{j\in N_t^+\setminus N_k^+}(x_j-(c-\lambda)z_j)^+ + \sum_{j\in N_t^-\setminus N_k^-}(x_j-\lambda z_j)^+$$
$$\leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t\in[k+1,n], \tag{24}$$
$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_k^+) - (c-\lambda)c\beta(N_k^-) \leq \lambda(c-\lambda), \tag{25}$$
$$x_j \geq c\alpha_j \quad j\in N_k^+, \tag{26}$$
$$x_j \geq c\beta_j \quad j\in N_k^-, \tag{27}$$
$$\lambda\alpha_j \geq (x_j-(c-\lambda)z_j)^+ \quad j\in N_k^+, \tag{28}$$
$$(c-\lambda)\beta_j \geq (x_j-\lambda z_j)^+ \quad j\in N_k^-, \tag{29}$$
$$x(N_t^+) - x(N_t^-) \leq b_t \quad t\in[k+1,n], \tag{30}$$
$$x_j \leq cz_j \quad j\in N\setminus N_k, \tag{31}$$
$$x_j \geq 0 \quad j\in N\setminus N_k, \tag{32}$$
$$z_j \leq 1 \quad j\in N. \tag{33}$$

We prove this result by induction. When $k=n$, we obtain the system (14)–(22). Now we first project out α_u for some $u\in N_k^+\setminus N_{k-1}^+$ using Fourier–Motzkin elimination.

Combining inequality (26) with (28) give $x_u \geq 0$ and $x_u \leq cz_u$. Combining inequality (26) with (25) gives

$$\lambda x(N_k^+\setminus\{u\}) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_k^+\setminus\{u\}) - (c-\lambda)c\beta(N_k^-) \leq \lambda(c-\lambda).$$

We treat (23) for $t=k$ and (24) for $t\in[k+1,n]$ together. Combining them with (28) gives

$$\lambda\alpha(N_k^+\setminus\{u\}) + (c-\lambda)\beta(N_k^-) + \sum_{j\in N_t^+\setminus N_k^+\cup\{u\}}(x_j-(c-\lambda)z_j)^+$$
$$+ \sum_{j\in N_t^-\setminus N_k^-}(x_j-\lambda z_j)^+ \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t\in[k,n].$$

Combining (23) for $t=k$ and (24) for $t\in[k+1,n]$ with (25) gives

$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) + c\sum_{j\in N_t^+\setminus N_k^+}(x_j-(c-\lambda)z_j)^+ + c\sum_{j\in N_t^-\setminus N_k^-}(x_j-\lambda z_j)^+$$
$$\leq \lambda b_t + cx(N_t^-) \quad t\in[k,n]. \tag{34}$$

For $t=k$, inequality (34) gives $x(N_k^+) - x(N_k^-) \leq b_k$. Now we prove that (34) is dominated when $t>k$. Inequality (34) is equivalent to the family of inequalities:

$$\lambda x(N_k^+) + (c-\lambda)x(N_k^-) + c\sum_{j\in T}(x_j - (c-\lambda)z_j)$$
$$+ c\sum_{j\in V}(x_j - \lambda z_j) \le \lambda b_t + cx(N_t^-) \tag{35}$$

for all $T \subseteq N_t^+\backslash N_k^+$ and $V \subseteq N_t^-\backslash N_k^-$. For given T and V, this inequality can be obtained by taking $x(N_t^+) - x(N_t^-) \le b_t$ with weight λ, $-x_j \le 0$ for $j \in N_t^+\backslash(N_k^+ \cup T)$ with weight λ, $x_j - cz_j \le 0$ for $j \in T$ with weight $c-\lambda$, $-x_j \le 0$ for $j \in N_t^-\backslash(N_k^- \cup V)$ with weight $c-\lambda$ and $x_j - cz_j \le 0$ for $j \in V$ with weight λ.

The elimination for all α_u's with $u \in N_k^+\backslash N_{k-1}^+$ is the same. After eliminating these variables, we obtain the polytope given by

$$\lambda\alpha(N_t^+) + (c-\lambda)\beta(N_t^-) \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, k-1], \tag{36}$$

$$\lambda\alpha(N_{k-1}^+) + (c-\lambda)\beta(N_k^-) + \sum_{j\in N_t^+\backslash N_{k-1}^+}(x_j - (c-\lambda)z_j)^+ + \sum_{j\in N_t^-\backslash N_k^-}(x_j - \lambda z_j)^+$$
$$\le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k, n], \tag{37}$$

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^-) - \lambda c\alpha(N_{k-1}^+) - (c-\lambda)c\beta(N_k^-) \le \lambda(c-\lambda), \tag{38}$$

$$x_j \ge c\alpha_j \quad j \in N_{k-1}^+, \tag{39}$$

$$x_j \ge c\beta_j \quad j \in N_k^-, \tag{40}$$

$$\lambda\alpha_j \ge (x_j - (c-\lambda)z_j)^+ \quad j \in N_{k-1}^+, \tag{41}$$

$$(c-\lambda)\beta_j \ge (x_j - \lambda z_j)^+ \quad j \in N_k^-, \tag{42}$$

$$x(N_t^+) - x(N_t^-) \le b_t \quad t \in [k, n], \tag{43}$$

$$x_j \le cz_j \quad j \in (N\backslash N_k) \cup N_k^+, \tag{44}$$

$$x_j \ge 0 \quad j \in (N\backslash N_k) \cup N_k^+, \tag{45}$$

$$z_j \le 1 \quad j \in N. \tag{46}$$

The next step is to eliminate β_u with $u \in N_k^-\backslash N_{k-1}^-$. Combining (42) and (40) gives $x_u \ge 0$ and $x_u \le cz_u$. Combining (38) and (40) gives

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^-\backslash\{u\}) - \lambda c\alpha(N_{k-1}^+) - (c-\lambda)c\beta(N_k^-\backslash\{u\}) \le \lambda(c-\lambda).$$

Combining (42) with (37) for $t \in [k, n]$ gives

$$\lambda\alpha(N_{k-1}^+) + (c-\lambda)\beta(N_k^-\backslash\{u\}) + \textstyle\sum_{j\in N_t^+\backslash N_{k-1}^+}(x_j - (c-\lambda)z_j)^+$$
$$+ \textstyle\sum_{j\in N_t^-\backslash N_k^-\cup\{u\}}(x_j - \lambda z_j)^+ \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [k, n].$$

Finally, we combine (38) with (37) for $t \in [k, n]$ to obtain

$$\lambda x(N_{k-1}^+) + (c-\lambda)x(N_k^-) + c \sum_{j \in N_t^+ \setminus N_{k-1}^+} (x_j - (c-\lambda)z_j)^+ + c \sum_{j \in N_t^- \setminus N_k^-} (x_j - \lambda z_j)^+$$
$$\leq \lambda b_t + cx(N_t^-) \quad t \in [k, n].$$

It can be proved as above that these inequalities are redundant. All variables β_u with $u \in N_k^- \setminus N_{k-1}^-$ are projected out in the same way.

We obtain the result of the theorem for $k = 0$ with $N_0^+ = N_0^- = N_0 = \emptyset$. In this case, inequalities (24), which are the only nontrivial inequalities, become

$$\sum_{j \in N_t^+} (x_j - (c-\lambda)z_j)^+ + \sum_{j \in N_t^-} (x_j - \lambda z_j)^+ \leq \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + x(N_t^-) \quad t \in [1, n],$$

which are equivalent to the flow cover inequalities (5). □

With $n = 1$, observe that the single node flow set conjecture from the Introduction is now proven.

3.3 Extensions

As we did not assume that b_t's are positive, there is an immediate corollary:

Corollary 1 *Let $X_{\geq}$ be the set of solutions to the system*

$$x(N_t^+) - x(N_t^-) \geq a_t \quad t \in [1, n],$$
$$0 \leq x_j \leq cz_j \quad j \in N,$$
$$z_j \in \{0, 1\} \quad j \in N.$$

If $a_t \mod c = a$ for $t = 1, \ldots, n$, then $\mathrm{conv}(X_{\geq})$ is described by the original constraints and the inequalities

$$x(N_t^+ \setminus T) + az(T) \geq a \left\lceil \frac{a_t}{c} \right\rceil + x(V) - (c-a)z(V) \tag{47}$$

for all $t \in [1, n]$, $T \subseteq N_t^+$ and $V \subseteq N_t^-$.

Again if $a = 0$, then the inequalities (47) are implied and the original constraints are sufficient to describe the convex hull.

Set X is the intersection of sets $X_{\geq}$ and $X_{\leq}$. As shown by an example in Sect. 6, $\mathrm{conv}(X) \neq \mathrm{conv}(X_{\geq}) \cap \mathrm{conv}(X_{\leq})$ in general. However, this holds in some special cases, in particular when $n = 1$.

Theorem 2 *If* $n = 1$, $a_1 \leq 0 < b_1$, *then* $\text{conv}(X) = \text{conv}(X_{\geq}) \cap \text{conv}(X_{\leq})$.

Proof Suppose that $n = 1$, $a_1 \leq 0 < b_1$ and we are minimizing $\sum_{j \in N}(f_j z_j + g_j x_j)$ over sets X, $X_{\leq}$ and $X_{\geq}$ with (f, g) not equal to zero. Let $X(f, g)$, $X_{\leq}(f, g)$ and $X_{\geq}(f, g)$ be the respective sets of all optimal solutions. We would like to show that $X(f, g) \subseteq X_{\geq}(f, g)$ or $X(f, g) \subseteq X_{\leq}(f, g)$ and hence all optimal solutions lie on a face defined by one of the facet defining inequalities of $\text{conv}(X_{\geq})$ or $\text{conv}(X_{\leq})$.

Suppose to the contrary that $X(f, g) \nsubseteq X_{\geq}(f, g)$ and $X(f, g) \nsubseteq X_{\leq}(f, g)$. There exists $(x', z') \in X(f, g) \backslash X_{\geq}(f, g)$. Since $(x', z') \notin X_{\geq}(f, g)$, we have $\min_{(x,z) \in X} \sum_{j \in N}(f_j z_j + g_j x_j) > \min_{(x,z) \in X_{\geq}} \sum_{j \in N}(f_j z_j + g_j x_j)$. Then $X(f, g) \cap X_{\geq}(f, g) = \emptyset$ and consequently $x(N^+) - x(N^-) > b_1$ for all $(x, z) \in X_{\geq}(f, g)$. In this case, at extreme points of $\text{conv}(X_{\geq})$ that are in $X_{\geq}(f, g)$, $x_j \in \{0, c\}$ for all $j \in N$.

Let $N^+_> = \{j \in N^+ : f_j + cg_j > 0\}$, $N^+_< = \{j \in N^+ : f_j + cg_j < 0\}$ and $N^+_0 = \{j \in N^+ : f_j + cg_j = 0\}$. Similarly, $N^-_> = \{j \in N^- : f_j + cg_j > 0\}$, $N^-_< = \{j \in N^- : f_j + cg_j < 0\}$ and $N^-_0 = \{j \in N^- : f_j + cg_j = 0\}$. If $(x, z) \in X_{\geq}(f, g)$ and it is an extreme point of $\text{conv}(X_{\geq})$, then we should have $x_j = c$ for all $j \in N^+_< \cup N^-_<$ and $x_j = 0$ for all $j \in N^+_> \cup N^-_>$ since if there exists $j \in N^+_< \cup N^-_<$ with $x_j = 0$ (respectively, $j \in N^+_> \cup N^-_>$ with $x_j = c$) then setting $z_j = 1$ and $x_j = c$ (respectively, setting $z_j = 0$ and $x_j = 0$ (this produces a feasible solution since $x(N^+) - x(N^-) \geq \left\lceil \frac{b_1}{c} \right\rceil c$ and $\left\lceil \frac{b_1}{c} \right\rceil c - c \geq 0 \geq a_1$)) gives a feasible solution with a better objective function value. Hence $|N^+_<| - |N^-_<| \geq \left\lceil \frac{b_1}{c} \right\rceil$.

There exists also $(x', z') \in X(f, g) \backslash X_{\leq}(f, g)$. Following the same arguments, we get that $|N^+_<| - |N^-_<| \leq \left\lfloor \frac{a_1}{c} \right\rfloor$. We obtain $\left\lceil \frac{b_1}{c} \right\rceil \leq \left\lfloor \frac{a_1}{c} \right\rfloor$, which is a contradiction. So $X(f, g) \subseteq X_{\geq}(f, g)$ or $X(f, g) \subseteq X_{\leq}(f, g)$. □

4 Convex hull results for constant capacity lot-sizing with sales and stock upper bounds

Here we consider the case where $S^+ = N^+$, $S^- = \emptyset$, N^+_t and N^-_t's are singletons, $a_t = 0$ and $b_t \mod c = b > 0$ for $t \in [1, n]$. To simplify the presentation, we use x_t to represent the inflow (production) at node t and y_t to represent the outflow (sales) at node t. We define set Y to be the feasible set of the system:

$$s_t = s_{t-1} + x_t - y_t \quad t \in [1, n], \tag{48}$$

$$s_0 = 0, \tag{49}$$

$$0 \leq s_t \leq b_t \quad t \in [1, n], \tag{50}$$

$$0 \leq x_t \leq cz_t \quad t \in [1, n], \tag{51}$$

$$y_t \geq 0 \quad t \in [1, n], \tag{52}$$

$$z_t \in \{0, 1\} \quad t \in [1, n]. \tag{53}$$

This is the feasible set of the constant capacity lot-sizing problem with sales and stock upper bounds. Again we define $\lambda = c - b$.

4.1 An extended formulation

Using the property of the extreme points derived in Proposition 1 (ii), we introduce the following variables to obtain an extended formulation for set Y:

$\alpha_t^1 = 1$ if $x_t = c$ and 0 otherwise,
$\alpha_t^2 = 1$ if $x_t = c - \lambda$ and 0 otherwise,
$\sigma_t^1 = \lfloor \frac{s_t}{c} \rfloor$: the integer multiples of c of s_t,
$\sigma_t^2 = 1$ if $s_t \mod c = c - \lambda$ and 0 otherwise.

With these new variables, we have $x_t = c\alpha_t^1 + (c - \lambda)\alpha_t^2$ and $s_t = c\sigma_t^1 + (c - \lambda)\sigma_t^2$ for $t \in [1, n]$.

Proposition 4 *The following is an extended formulation for set Y when* $b_t \mod c = b$ *for all* $t \in [1, n]$*:*

$$x_t = c\alpha_t^1 + (c - \lambda)\alpha_t^2 \quad t \in [1, n], \tag{54}$$
$$s_t = c\sigma_t^1 + (c - \lambda)\sigma_t^2 \quad t \in [1, n], \tag{55}$$
$$y_t = s_{t-1} + x_t - s_t \quad t \in [1, n], \tag{56}$$
$$s_0 = \sigma_0^1 = \sigma_0^2 = 0, \tag{57}$$
$$\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \ge 0 \quad t \in [1, n], \tag{58}$$
$$\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2 \ge 0 \quad t \in [1, n], \tag{59}$$
$$\alpha_t^1 + \alpha_t^2 - z_t \le 0 \quad t \in [1, n], \tag{60}$$
$$\alpha_t^2 - \sigma_t^2 \le 0 \quad t \in [1, n], \tag{61}$$
$$\sigma_t^1 \le \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1, n], \tag{62}$$
$$\sigma_t^1 \ge 0 \quad t \in [1, n], \tag{63}$$
$$0 \le \alpha_t^1, \alpha_t^2, \sigma_t^2, z_t \le 1 \quad t \in [1, n], \tag{64}$$
$$\alpha_t^1, \alpha_t^2, \sigma_t^1, \sigma_t^2, z_t \quad \textit{integer} \quad t \in [1, n]. \tag{65}$$

In addition the polytope defined by (57)–(64) *is integral.*

Proof Suppose that $b_t \mod c = b$ for all $t \in [1, n]$. Let (x, s, y, z) be an extreme point of $conv(Y)$. From Proposition 1 (ii), we have that there exist $\alpha^1, \alpha^2, \sigma^1$ and σ^2 such that $x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2$ and $s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2$ with $\sigma_t^1 \ge 0$ and integer and $\alpha_t^1, \alpha_t^2, \sigma_t^2 \in \{0, 1\}$ for $t \in [1, n]$. The fact that if $x_t = c - \lambda$ then $s_t \mod c = c - \lambda$ implies $\alpha_t^2 \le \sigma_t^2$ for $t \in [1, n]$.

Let $t \in [1, n]$. If $\alpha_t^1 = 1$, then we have $s_{t-1} + c \ge s_t$ implying that $\sigma_{t-1}^1 + 1 \ge \sigma_t^1$. If $\alpha_t^1 = 0$ and $x_t = 0$, then $s_{t-1} + 0 \ge s_t$ implying that $\sigma_{t-1}^1 \ge \sigma_t^1$. If $\alpha_t^1 = 0$ and $x_t = c - \lambda$, then $s_{t-1} \mod c = 0$. Therefore $s_{t-1} = c\sigma_{t-1}^1$, $s_t \le c\sigma_{t-1}^1 + (c - \lambda)$ and thus $\sigma_t \le \sigma_{t-1}$. Thus in every case $\sigma_{t-1}^1 + \alpha_t^1 \ge \sigma_t^1$.

The inequality $c\sigma^1_{t-1} + (c-\lambda)\sigma^2_{t-1} + c\alpha^1_t + (c-\lambda)\alpha^2_t - c\sigma^1_t - (c-\lambda)\sigma^2_t \geq 0$ can be rewritten as $(c-\lambda)(\sigma^1_{t-1} + \sigma^2_{t-1} + \alpha^1_t + \alpha^2_t - \sigma^1_t - \sigma^2_t) + \lambda(\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t) \geq 0$. We know that $\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t \geq 0$. If $\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t = 0$, the above inequality implies $\sigma^1_{t-1} + \sigma^2_{t-1} + \alpha^1_t + \alpha^2_t \geq \sigma^1_t + \sigma^2_t$. On the other hand if $\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t \geq 1$, then as $\sigma^2_{t-1} + \alpha^2_t - \sigma^2_t \geq -1$, we again have $\sigma^1_{t-1} + \sigma^2_{t-1} + \alpha^1_t + \alpha^2_t \geq \sigma^1_t + \sigma^2_t$.

For $t \in [1, n]$, constraint $s_t \leq b_t$ is the same as $c\sigma^1_t + (c-\lambda)\sigma^2_t \leq b_t$ which implies $\sigma^1_t \leq \left\lfloor \frac{b_t}{c} \right\rfloor$. Also constraint $x_t \leq cz_t$, which is the same as $c\alpha^1_t + (c-\lambda)\alpha^2_t \leq cz_t$, gives $\alpha^1_t + \alpha^2_t \leq z_t$.

Hence for each extreme point (x, s, y, z) of $conv(Y)$, there exist $\alpha^1, \alpha^2, \sigma^1$ and σ^2 such that $(x, s, y, z, \alpha^1, \alpha^2, \sigma^1, \sigma^2)$ is feasible in the extended formulation.

Next, we show that for any $(x, s, y, z, \alpha^1, \alpha^2, \sigma^1, \sigma^2)$ that is feasible in the extended formulation, $(x, s, y, z) \in Y$. It is easy to see that (x, s, y, z) satisfies (48)–(51) and (53). To show that $y_t \geq 0$, observe that $y_t = (c-\lambda)(\sigma^1_{t-1} + \sigma^2_{t-1} + \alpha^1_t + \alpha^2_t - \sigma^1_t - \sigma^2_t) + \lambda(\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t)$. As both $\sigma^1_{t-1} + \sigma^2_{t-1} + \alpha^1_t + \alpha^2_t - \sigma^1_t - \sigma^2_t$ and $\sigma^1_{t-1} + \alpha^1_t - \sigma^1_t$ are nonnegative, the result follows.

To prove that the polytope defined by (57)–(64) is integral, we show that the constraint matrix associated with (58)–(61) is totally unimodular. We again use the characterization of Ghouila-Houri [12]. Let Δ^* be the same as the set Δ of Lemma 1, except that the last two vectors are replaced by $\begin{pmatrix} 1 & 1 \end{pmatrix}^T, \begin{pmatrix} -1 & -1 \end{pmatrix}^T$. Lemma 1 also holds for Δ^*.

First we observe that the variables z_t just occur once and can thus be ignored.

Consider a subset J of the remaining variables and let J_{k-1} be those variables in J with subscript at most $k-1$. The induction hypothesis is that there exists a partition (J^+_{k-1}, J^-_{k-1}) of J_{k-1} with associated weights $(+1, -1)$ such that

(i) the weights associated to the constraints (58)–(61) for $t \leq k-1$ lie in $\{0, +1, -1\}$
(ii) the weights δ^1 associated to the terms $(\sigma^1_{k-1}, \sigma^1_{k-1} + \sigma^2_{k-1})$ from (58)–(59) for $t = k$ are such that $\delta^1 \in \Delta^*$.

Now we show that we can associate weights to J_k such that the induction hypothesis holds for k. If present in $J_k \backslash J_{k-1}$, we assign the variables with σ^2_k and α^2_k to the same set so that the weight associated to (61) lies in $\{0, 1, -1\}$. Also α^1_k and α^2_k are assigned to different sets of the partition so that the weight associated to (60) lies in $\{0, 1, -1\}$. Now let δ^2 be the weight associated to the terms $(\alpha^1_k - \sigma^1_k, \alpha^1_k + \alpha^2_k - \sigma^1_k - \sigma^2_k)$ from (58)–(59) for $t = k$. First we observe that if less than two of the subscript k variables lie in J_k, then $\delta^2 \in \Delta^*$. Below we propose 11 possible assignments when two or more of the subscript k variables lie in J_k.

We have thus verified case by case that $\delta^2 \in \Delta^*$. Now δ^1 and δ^2 can be combined to give a vector in Δ^* showing that there is a partition of J_k in which the weight assigned to rows (58)–(59) for $t = k$ lies in $\{0, 1, -1\}$. Also none of the assignments assign σ^1_k and σ^2_k to the same set, and thus the weight δ^1 associated to $(\sigma^1_k, \sigma^1_k + \sigma^2_k)$ lies in Δ^*.

Finally the induction hypothesis holds for $k = 1$ taking $\sigma^1_0 = \sigma^2_0 = 0$ and thus the claim follows by induction. □

α_k^1	α_k^2	σ_k^1	σ_k^2	δ_1^2	δ_2^2
J^+	J^-	J^+	J^-	0	0
J^+	J^-	J^+	.	0	− 1
J^-	J^+	.	J^+	− 1	− 1
J^+	.	J^+	J^-	0	+ 1
.	J^+	J^-	J^+	+ 1	+ 1
J^+	J^-	.	.	+ 1	0
J^+	.	J^+	.	0	0
J^+	.	.	J^+	+ 1	0
.	J^+	J^+	.	− 1	0
.	J^+	.	J^+	0	0
.	.	J^+	J^-	− 1	0

4.2 Projection

Theorem 3 *If* $b_t \mod c = b$ *for all* $t \in [1, n]$, $\text{conv}(Y)$ *is described by constraints* (48)–(51), $z_t \leq 1$ *for* $t \in [1, n]$, *plus the inequalities*

$$s_t - s_j \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=j+1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad j \in [1, n], t \in [j+1, n],$$

$$s_t \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad t \in [1, n].$$

Proof We start with the extended formulation (54)–(64) and we project out the variables $\alpha_t^1, \alpha_t^2, \sigma_t^1, \sigma_t^2$ variables working down from $t = n, n-1, \ldots, 1$.

Consider the sets R^k

$$x_t \leq cz_t \quad t \in [k+1, n], \tag{66}$$

$$x_t \geq 0 \quad t \in [k+1, n], \tag{67}$$

$$s_t \geq 0 \quad t \in [k+1, n], \tag{68}$$

$$s_t \leq b_t \quad t \in [k+1, n], \tag{69}$$

$$s_t \leq s_{t-1} + x_t \quad t \in [k+1, n], \tag{70}$$

$$s_t - s_j \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=j+1}^{t} \min\{x_u, (c-\lambda)z_u\} \quad j \in [k+1, n], t \in [j+1, n], \tag{71}$$

$$s_t \leq \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \sum_{u=k+1}^{t} \min\{x_u, (c-\lambda)z_u\} + (c-\lambda)(\sigma_k^1 + \sigma_k^2) \quad t \in [k+1, n], \tag{72}$$

and Q^k

$$x_t = c\alpha_t^1 + (c-\lambda)\alpha_t^2 \quad t \in [1, k], \tag{73}$$

$$s_t = c\sigma_t^1 + (c-\lambda)\sigma_t^2 \quad t \in [1,k], \tag{74}$$
$$s_0 = \sigma_0^1 = \sigma_0^2 = 0, \tag{75}$$
$$\sigma_{t-1}^1 + \alpha_t^1 - \sigma_t^1 \geq 0 \quad t \in [1,k], \tag{76}$$
$$\sigma_{t-1}^1 + \sigma_{t-1}^2 + \alpha_t^1 + \alpha_t^2 - \sigma_t^1 - \sigma_t^2 \geq 0 \quad t \in [1,k], \tag{77}$$
$$\alpha_t^1 + \alpha_t^2 - z_t \leq 0 \quad t \in [1,k], \tag{78}$$
$$\alpha_t^2 - \sigma_t^2 \leq 0 \quad t \in [1,k], \tag{79}$$
$$\sigma_t^1 \leq \left\lfloor \frac{b_t}{c} \right\rfloor \quad t \in [1,k], \tag{80}$$
$$\sigma_t^2 \leq 1 \quad t \in [1,k], \tag{81}$$
$$\alpha_t^2, \sigma_t^1 \geq 0 \quad t \in [1,k], \tag{82}$$
$$\alpha_t^1 \geq 0 \quad t \in [1,k], \tag{83}$$
$$z_t \leq 1 \quad t \in [1,n]. \tag{84}$$

Note that $Q^n \cap R^n$ is the original system and $Q^0 \cap R^0$ is the final system with variables y_t projected out.

We prove by induction that after eliminating the variables with subscripts $t = n, n-1, \ldots, k+1$, the resulting projection is $Q^k \cap R^k$. The proof is given in "Appendix 1". □

5 Convex hull results for constant capacity discrete lot-sizing with sales and stock bounds

Here we consider the discrete lot-sizing version in which $x_t = cz_t$ for all $t \in [1,n]$. Let set Z be the set of solutions to

$$s_{t-1} + cz_t \geq s_t \quad t \in [1,n], \tag{85}$$
$$s_0 = 0, \tag{86}$$
$$s_t \leq b_t \quad t \in [1,n], \tag{87}$$
$$s_t \geq a_t \quad t \in [1,n], \tag{88}$$
$$z_t \in \{0,1\} \quad t \in [1,n]. \tag{89}$$

We assume that $a_t < b_t$, $b_t = \min\{b_t, b_{t-1} + c\}$ and $a_t = \max\{a_t, a_{t+1} - c\}$ for all $t \in [1,n]$ with $b_0 = 0$ and $a_{n+1} = -\infty$.

5.1 Extreme points and extended formulation

Let $F = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, 0\}$, where $\alpha_j = a_j \mod c$ and $\beta_j = b_j \mod c$ for all $j \in [1,n]$.

Observation 1 *In an extreme point of* $\text{conv}(Z)$, $s_t \mod c \in F$ *for all* $t \in [1,n]$.

Observation 2 *With the change of variable* $\zeta_t = -z_{1t}$ *and* $\sigma_t = s_t - cz_{1t} = s_t + c\zeta_t$, *we obtain the equivalent model:*

$$\sigma_t - \sigma_{t-1} \leq 0 \quad t \in [1, n], \tag{90}$$

$$\sigma_t - c\zeta_t \leq b_t \quad t \in [1, n], \tag{91}$$

$$\sigma_t - c\zeta_t \geq a_t \quad t \in [1, n], \tag{92}$$

$$\zeta_t - \zeta_{t-1} \leq 0 \quad t \in [1, n], \tag{93}$$

$$\zeta_t - \zeta_{t-1} \geq -1 \quad t \in [1, n], \tag{94}$$

$$\sigma_0 = \zeta_0 = 0, \tag{95}$$

$$\zeta_t \in \mathbb{Z} \quad t \in [1, n].$$

This is a dual network MIP, see [9]. From Observation 1, σ_t/c can only take at most the $K+1$ distinct fractional values in F with $K \leq 2n$ and thus from Section 4 in [9] one obtains the tight polynomial size extended formulation of the convex hull that we now describe.

We order the elements of $F\backslash\{0\}$ in decreasing order, i.e., $f_0 = c > f_1 > \cdots f_K > f_{K+1} = 0$ Let ℓ_t and k_t be the positions of α_t and β_t, respectively, in this ordering.

The extended formulation is

$$\begin{aligned}
&\sigma_t = \sum_{j=0}^{K} (f_j - f_{j+1})\mu_t^j \quad t \in [1, n],\\
&\mu_{t-1}^j - \mu_t^j \geq 0 \quad j \in [0, K], t \in [1, n]\\
&\mu_t^j - \mu_t^{j+1} \leq 0 \; j \in [0, K-1], \; t \in [1, n],\\
&\mu_t^K - \mu_t^0 \leq 1 \; t \in [1, n],\\
&\mu_t^{(k_t-1)} - \zeta_t \leq \left\lfloor \frac{b_t}{c} \right\rfloor \; t \in [1, n], \beta_t = f_{k_t},\\
&\mu_t^{\ell_t} - \zeta_t \geq \left\lfloor \frac{a_t}{c} \right\rfloor + 1 \; t \in [1, n] : \alpha_t = f_{\ell_t} > 0,\\
&\mu_t^0 - \zeta_t \geq \left\lfloor \frac{a_t}{c} \right\rfloor \; t \in [1, n] : \alpha_t = 0,\\
&\zeta_t - \zeta_{t-1} \leq 0 \quad t \in [1, n],\\
&\zeta_t - \zeta_{t-1} \geq -1 \quad t \in [1, n],\\
&\zeta_0 = 0,\\
&\mu_0^i = 0 \quad i \in [0, K],
\end{aligned}$$

where the interpretation of the μ variables is as follows: $\mu_t^0 = \lfloor \frac{\sigma_t}{c} \rfloor$, $\mu_t^j = \mu_t^0$ if $\sigma_t - c\lfloor \frac{\sigma_t}{c} \rfloor < f_j$ and $\mu_t^j = \mu_t^0 + 1$ otherwise.

5.2 Projection

We provide the description of conv(Z) when $a_t = a$ and $b_t = b$ for all $t \in [1, n]$. Here $\beta = b \mod c$ and $\alpha = a \mod c$. We present the case in which $\beta > \alpha > 0$. One has $K = 2$, $(f_0, f_1, f_2, f_3) = (c, \beta, \alpha, 0)$ and the extended formulation becomes:

$$\sigma_t = (c-\beta)\mu_t^0 + (\beta-\alpha)\mu_t^1 + \alpha\mu_t^2 \;\; t \in [1, n], \tag{96}$$
$$\mu_{t-1}^i - \mu_t^i \geq 0 \;\; i \in [0, 2], t \in [1, n], \tag{97}$$
$$\mu_t^i - \mu_t^{i+1} \leq 0 \;\; i \in [0, 1], \;\; t \in [1, n], \tag{98}$$
$$\mu_t^2 - \mu_t^0 \leq 1, \;\; t \in [1, n], \tag{99}$$
$$\mu_t^0 - \zeta_t \leq \left\lfloor \frac{b}{c} \right\rfloor \;\; t \in [1, n], \tag{100}$$
$$\mu_t^2 - \zeta_t \geq \left\lfloor \frac{a}{c} \right\rfloor + 1 \;\; t \in [1, n], \tag{101}$$
$$\zeta_t - \zeta_{t-1} \leq 0 \;\; t \in [1, n], \tag{102}$$
$$\zeta_t - \zeta_{t-1} \geq -1 \;\; t \in [1, n], \tag{103}$$
$$\zeta_0 = 0, \tag{104}$$
$$\mu_0^i = 0 \;\; i \in [0, 2]. \tag{105}$$

We will now show that the projection on the σ, ζ space gives just the inequalities:

$$\sigma_t \leq \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + \lambda_b \zeta_t \;\; t \in [1, n]$$

and

$$\sigma_t - \sigma_k \leq \lambda_{b-a} \left\lfloor \frac{b-a}{c} \right\rfloor + \lambda_{b-a}(\zeta_t - \zeta_k) \;\; k \in [1, n], t \in [k+1, n],$$

where $\lambda_b = c - \beta$ and $\lambda_{b-a} = c - \beta + \alpha$. In other words,

Theorem 4 *If $a_t = a$ and $b_t = b$ for all $t \in [1, n]$ and $\beta = b \mod c > \alpha = a \mod c > 0$, then* conv$(Z)$ *is described by the constraints* (85)–(88)*,* $z \in [0, 1]^n$ *and inequalities*

$$s_t \leq \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + (c - \lambda_b) z_{1t} \;\; t \in [1, n], \textit{and}$$
$$s_t - s_{k-1} \leq \lambda_{b-a} \left\lfloor \frac{b-a}{c} \right\rfloor + (c - \lambda_{b-a}) z_{kt} \;\; k \in [2, n], t \in [k, n].$$

Proof We suppose that $\beta > \alpha > 0$. For $k \in [1, n]$, let P_k be the set of vectors that satisfy (96)–(105) for $t \in [1, k]$ and R_k be the set of vectors that satisfy (90)–(95) for $t \in [k+1, n]$ and

$$\sigma_t - \sigma_q \leq (c - \beta + \alpha) \left\lfloor \frac{b-a}{c} \right\rfloor + (c - \beta + \alpha)(\zeta_t - \zeta_q) \;\; q, t \in [k+1, n] : q < t.$$

After elimination of variables for $t \in [k+1, n]$, we obtain $P_k \cap R_k$ plus

$$-(\beta - \alpha)\mu_k^1 - \alpha\mu_k^2 + \sigma_t - (c - \beta)\zeta_t \leq (c - \beta) \left\lfloor \frac{b}{c} \right\rfloor \;\; t \in [k+1, n], \tag{106}$$

$$-(\beta - \alpha)\mu_k^1 + \sigma_t - (c - \beta + \alpha)\zeta_t \leq (c - \beta + \alpha) \left\lfloor \frac{b}{c} \right\rfloor + \alpha \;\; t \in [k+1, n]. \tag{107}$$

For $k = n$, we have P_n, which is the original system. When $k = 0$, we obtain R_0 and inequalities (106) and (107). We now show that the inequalities (107) are redundant. For $k = 0$ and $t \in [1, n]$, inequality (107) is

$$\sigma_t - (c - \beta + \alpha)\zeta_t \leq (c - \beta + \alpha) \left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This can be obtained by taking $\sigma_t - (c - \beta)\zeta_t \leq \left\lfloor \frac{b}{c} \right\rfloor (c - \beta)$ with weight $(\beta - \alpha)/\beta$ and $\sigma_t - c\zeta_t \leq b$ with weight α/β. We give the induction proof for the intermediate iterations in Appendix 2. □

The other cases in which $\alpha = 0$ or $\alpha \geq \beta \geq 0$ lead to the same result. However σ_t, k_t and ℓ_t are modified in the extended formulation and thus the projection also needs to be modified.

Observation 3 *If one modifies the set Z by replacing the single entering arc by multiple entering arcs with associated 0-1 variables z_t^j and sets $z_t = \sum_{j=1}^{J} z_t^j$, the new variables again give rise to dual network constraints, so that to obtain the projection of this new set, it suffices to remove the bounds on z_t, substitute for z_t and add the bound constraints $0 \leq z_t^j \leq 1$ for $j \in [1, J], t \in [1, T]$.*

6 Final remarks

Most mixed integer programming solvers are nowadays very good at generating flow cover inequalities so it is doubtful whether the inequalities and extended formulations presented here can have much computational impact. In particular it turns out that the lot-sizing problem with constraint set Y that is not demand driven is easily solved computationally. We see this work rather as part of a continuing attempt to analyze and understand simple mixed integer programs. It also confirms that, for certain tight extended formulations, Fourier–Motzkin elimination, though tedious and far from elegant, can be used to carry out non-trivial projections.

It is of interest to develop other valid inequalities for the case in which the constant capacity flow cover inequalities do not suffice to give the convex hull.

For example, consider an instance of the set X:

$$\begin{aligned}
&-4 \le x_1 - x_4 \le 7,\\
&-4 \le x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \le 7,\\
&0 \le x_j \le 10z_j,\ z_j \in \{0,1\} \quad j \in [1,6].
\end{aligned}$$

A facet-defining inequality for conv(X) is

$$-x_3 + x_5 + x_6 - 6z_1 - 9z_2 - z_5 - z_6 \le 3.$$

This inequality is not a flow cover inequality. However it can be obtained by lifting a flow cover inequality as follows. Combining inequalities $-4 \le x_1 + x_2 + x_3 - x_4 - x_5 - x_6$ and $x_1 - x_4 \le 7$ gives $x_5 + x_6 - x_2 - x_3 \le 11$. A flow cover inequality based on this relaxation is $-x_3 + x_5 + x_6 - 9z_2 - z_5 - z_6 \le 9$. This inequality is valid when $z_1 = 1$. We can lift it with z_1 by computing an α for which the inequality $-x_3 + x_5 + x_6 - 9z_2 - z_5 - z_6 \le 9 - \alpha(1 - z_1)$ is valid for X. The best $\alpha = \min_{(x,z)\in X:z_1=0}(9 + x_3 - x_5 - x_6 + 9z_2 + z_5 + z_6)$. One optimal solution to this minimization problem is $x_5 = 4$, $z_5 = 1$ and other components are zero. Then $\alpha = 6$ and we obtain the above facet-defining inequality. conv(X) also has more complicated facet-defining inequalities such as

$$9x_1 + 7x_2 + 9x_3 - 6x_5 - 6x_6 - 63z_1 - 7z_2 - 21z_3 - 27z_4 - 9z_5 - 9z_6 \le 42.$$

Finally, based on computational tests and an examination of small instances with PORTA, we think that the following more general result holds:
Conjecture: If $a_t = a < 0 < b_t = b$ for all $t \in [1,n]$, conv(Y) is described by the constraints (48)–(52), $z_t \le 1$ for $t \in [1,n]$ plus inequalities

$$s_t - s_j \le \lambda_{b-a}\left\lfloor \frac{b-a}{c} \right\rfloor + x(T) + (c - \lambda_{b-a})z([j+1,t]\backslash T)$$

for all $j \in [1, n-1]$, $t \in [j+1, n]$ and $T \subseteq [j+1, t]$ where $\lambda_{b-a} = \left\lceil \frac{b-a}{c} \right\rceil c - (b-a)$, and

$$s_t \le \lambda_b \left\lfloor \frac{b}{c} \right\rfloor + x(T) + (c - \lambda_b)z([1,t]\backslash T)$$

for all $t \in [1,n]$ and $T \subseteq [1,t]$ where $\lambda_b = \left\lceil \frac{b}{c} \right\rceil c - b$.

Acknowledgements Part of the research of the second author was carried out in the Department of Industrial Engineering, Bilkent University, Turkey.

Appendix 1: Proof of Theorem 3

The proof is by induction. The result holds for $k = n$. We show that if it holds for k, then it holds for $k-1$.

For ease of presentation, we introduce $\pi_u = \min\{x_u, (c-\lambda)z_u\}$ for $u \in [1, n]$.

Elimination of α_k^2 and σ_k^2

The first step is to use the equations (73) and (74) for $t = k$ to eliminate α_k^2 and σ_k^2 by substitution.

Inequality (77) for $t = k$ becomes

$$s_k \le (c-\lambda)(\sigma_{k-1}^1 + \sigma_{k-1}^2) + x_k - \lambda\alpha_k^1 + \lambda\sigma_k^1. \tag{108}$$

Inequality (78) for $t = k$ becomes

$$x_k \le (c-\lambda)z_k + \lambda\alpha_k^1. \tag{109}$$

Inequality (79) for $t = k$ becomes

$$x_k - c\alpha_k^1 \le s_k - c\sigma_k^1. \tag{110}$$

Inequality (81) for $t = k$ becomes

$$s_k \le c\sigma_k^1 + (c-\lambda). \tag{111}$$

Inequality (82) for $t = k$ becomes

$$x_k \ge c\alpha_k^1. \tag{112}$$

Inequalities (72) for $t \in [k+1, n]$ become

$$s_t \le \lambda\left\lfloor \frac{b_t}{c} \right\rfloor + \pi_{k+1,t} - \lambda\sigma_k^1 + s_k. \tag{113}$$

Elimination of α_k^1

Now we use Fourier–Motzkin to eliminate α_k^1. We have to combine the inequalities (108) and (112) with the inequalities (76), (83), (109) and (110).
Inequality (76) with (108) gives

$$s_k \le x_k + s_{k-1}.$$

Inequality (76) with (112) gives

$$x_k \ge c(\sigma_k^1 - \sigma_{k-1}^1). \tag{114}$$

Inequality (83) with (112) gives

$$x_k \geq 0.$$

Inequality (83) with (108) gives

$$s_k \leq (c-\lambda)(\sigma^1_{k-1}+\sigma^2_{k-1}) + x_k + \lambda\sigma^1_k. \tag{115}$$

Inequality (109) with (108) gives

$$s_k \leq (c-\lambda)z_k + (c-\lambda)(\sigma^1_{k-1}+\sigma^2_{k-1}) + \lambda\sigma^1_k. \tag{116}$$

These inequalities (115) and (116) together give

$$s_k \leq \pi_k + (c-\lambda)(\sigma^1_{k-1}+\sigma^2_{k-1}) + \lambda\sigma^1_k. \tag{117}$$

Inequality (109) with (112) gives

$$x_k \leq cz_k.$$

Inequality (110) with (108) gives

$$s_k \leq x_k + c(\sigma^1_{k-1}+\sigma^2_{k-1}). \tag{118}$$

Inequality (110) with (112) gives

$$s_k \geq c\sigma^1_k. \tag{119}$$

Note that inequality (118) is redundant as it is dominated by (114).

Elimination of σ^1_k

σ^1_k occurs in the inequalities (80), (113), (114), (119) and with the opposite sign in (82), (111), (117). We treat inequalities (80) and (113) together as (80) is the same as (113) for $t = k$.

Combining (82) with (119) gives

$$s_k \geq 0.$$

Combining (111) with (113) for $t \in [k, n]$ gives

$$cs_t - (c-\lambda)s_k \leq \lambda b_t + c\pi_{k+1,t}.$$

For $t = k$, we obtain

$$s_k \leq b_k.$$

For $t \in [k+1, n]$, these inequalities are redundant. To see this, we take inequality $s_u - s_{u-1} \le x_u$ for $[u \in k+1, t]$ with weight $(c-\lambda)$ and $s_t \le b_t$ with weight λ. We obtain $cs_t - (c-\lambda)s_k \le \lambda b_t + (c-\lambda)x_{k+1,t}$. Now $(c-\lambda)x_u \le cx_u$ and as $x_u \le cz_u$, $(c-\lambda)x_u \le c(c-\lambda)z_u$. Thus $(c-\lambda)x_u \le c\pi_u$ for $u \in [k+1, t]$ and the inequality is dominated. Combining (117) with (113) for $t \in [k, n]$ gives

$$s_t \le \lambda \left\lfloor \frac{b_t}{c} \right\rfloor + \pi_{kt} + (c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}) \quad t \in [k, n].$$

Note that these are inequalities (72) of $R^{k-1} \backslash R^k$.
Combining (82) with (113) for $t \in [k, n]$ gives

$$s_t - s_k \le \left\lfloor \frac{b_t}{c} \right\rfloor \lambda + \pi_{k+1,t}.$$

The inequality is redundant for $t = k$. For $t \in [k+1, n]$, we obtain the inequalities (71) for $j = k$ of $R^{k-1} \backslash R^k$.
Note that we have obtained all the inequalities describing R^{k-1}.

It now suffices to show that the remaining inequalities are redundant for $Q^{k-1} \cap R^{k-1}$.
Combining (82) with (80) and (114) give $\left\lfloor \frac{b_k}{c} \right\rfloor \ge 0$ and $x_k + c\sigma^1_{k-1} \ge 0$, respectively. These are both redundant.

Combining (111) with (119) gives $c - \lambda \ge 0$, which is redundant.

Combining (111) with (114) gives

$$s_k \le x_k + c\sigma^1_{k-1} + (c-\lambda).$$

Combining (117) with (119) gives

$$(c-\lambda)s_k \le c\pi_k + c(c-\lambda)(\sigma^1_{k-1} + \sigma^2_{k-1}).$$

Combining (117) with (114) gives

$$cs_k \le (c+\lambda)\pi_k + cs_{k-1}.$$

The last three inequalities are all dominated using (114), $s_k \le x_k + s_{k-1}$ and (118), $x_k \le cz_k$ together with $\pi_k = \min\{x_k, (c-\lambda)z_k\}$ and $s_{k-1} = c\sigma^1_{k-1} + (c-\lambda)\sigma^2_{k-1}$.

Now the result follows by induction as $R^0 \cap Q^0$ is as claimed in the Theorem. □

Appendix 2: Proof of Theorem 4

Suppose that after the elimination of variables for $t \in [k+1, n]$, we obtain $P_k \cap R_k$ plus inequalities (106) and (107) for $t \in [k+1, n]$. We will show that after eliminating μ^0_k, μ^1_k and μ^2_k, we obtain $P_{k-1} \cap R_{k-1}$ plus inequalities (106) and (107) for $t \in [k, n]$.

Elimination of μ_k^0

The first step is the elimination of μ_k^0 by substitution. The inequalities $\mu_k^0 \le \mu_{k-1}^0$, $\mu_k^0 \le \mu_k^1$, $\mu_k^0 - \zeta_k \le \left\lfloor \frac{b}{c} \right\rfloor$ and $\mu_k^2 - \mu_k^0 \le 1$ give

$$\sigma_k \le (c-\beta)\mu_{k-1}^0 + (\beta-\alpha)\mu_k^1 + \alpha\mu_k^2, \tag{120}$$
$$\sigma_k \le (c-\alpha)\mu_k^1 + \alpha\mu_k^2, \tag{121}$$
$$\sigma_k - (c-\beta)\zeta_k \le (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor + (\beta-\alpha)\mu_k^1 + \alpha\mu_k^2, \tag{122}$$
$$-\sigma_k + (\beta-\alpha)\mu_k^1 + (c-\beta+\alpha)\mu_k^2 \le (c-\beta), \tag{123}$$

respectively.

Elimination of μ_k^1

Now we eliminate μ_k^1 using Fourier–Motzkin elimination. Overall, after eliminating μ_k^0, we are left with $P_{k-1} \cap R_k$ plus (97), (98), (101), (106), (107), (120), (121), (122) and (123).

The constraints involving μ_k^1 are all these inequalities except (97) for $i = 2$ and (101).
As (122) is the same as (106) for $t = k$, we treat them together.

The resulting inequalities are: (97) and (106) for $t \in [k, n]$ give

$$(\beta-\alpha)\mu_{k-1}^1 \ge \sigma_t - (c-\beta)\left(\zeta_t + \left\lfloor \frac{b}{c} \right\rfloor\right) - \alpha\mu_k^2 \quad t \in [k, n]. \tag{124}$$

(97) and (107) for $t \in [k+1, n]$ give

$$(\beta-\alpha)\mu_{k-1}^1 \ge \sigma_t - (c-\beta+\alpha)\left(\zeta_t + \left\lfloor \frac{b}{c} \right\rfloor\right) - \alpha \quad t \in [k+1, n]. \tag{125}$$

(97) and (120) give

$$(\beta-\alpha)\mu_{k-1}^1 \ge \sigma_k - \alpha\mu_k^2 - (c-\beta)\mu_{k-1}^0. \tag{126}$$

(97) and (121) give $(c-\alpha)\mu_{k-1}^1 \ge \sigma_k - \alpha\mu_k^2$. This is redundant since adding (126) and $(c-\beta)$ times $\mu_{k-1}^1 \ge \mu_{k-1}^0$ gives this inequality.

(98) and (106) for $t \in [k, n]$ give

$$\beta\mu_k^2 \ge \sigma_t - (c-\beta)\left(\left\lfloor \frac{b}{c} \right\rfloor + \zeta_t\right) \quad t \in [k, n]. \tag{127}$$

(98) and (107) give $(\beta-\alpha)\mu_k^2 \geq \sigma_t - (c-\beta+\alpha)(\zeta_t + \lfloor \frac{b}{c} \rfloor) - \alpha \;\; t \in [k+1, n]$. This is redundant as it can be obtained by taking $b \geq \sigma_t - c\zeta_t$ with weight $\frac{\alpha}{\beta}$ and (127) for t with weight $\frac{\beta-\alpha}{\beta}$.

(98) and (120) give

$$\beta\mu_k^2 \geq \sigma_k - (c-\beta)\mu_{k-1}^0. \tag{128}$$

(98) and (121) give

$$c\mu_k^2 \geq \sigma_k. \tag{129}$$

(123) and (106) for $t \in [k, n]$ give

$$\sigma_k - (c-\beta)\mu_k^2 + (c-\beta)\left(\left\lfloor \frac{b}{c} \right\rfloor + 1 + \zeta_t\right) \geq \sigma_t \;\; t \in [k, n].$$

For $t \in [k+1, n]$, the inequality is redundant since adding $\frac{c-\beta}{c-\beta+\alpha}$ times (131) (obtained from (123) and (107) below) and $\frac{\alpha}{c-\beta+\alpha}$ times $\sigma_k \geq \sigma_t$ gives this inequality. For $t = k$, we obtain

$$\mu_k^2 - \zeta_k \leq 1 + \left\lfloor \frac{b}{c} \right\rfloor. \tag{130}$$

(123) and (107) give

$$\sigma_k - (c-\beta+\alpha)\mu_k^2 + (c-\beta+\alpha)\left(\left\lfloor \frac{b}{c} \right\rfloor + 1 + \zeta_t\right) \geq \sigma_t \;\; t \in [k+1, n]. \tag{131}$$

(123) and (120) give $\mu_k^2 \leq \mu_{k-1}^0 + 1$. This is redundant since $\mu_k^2 \leq \mu_{k-1}^2 \leq \mu_{k-1}^0 + 1$.

(123) and (121) give

$$-\sigma_k + c\mu_k^2 \leq c - \alpha. \tag{132}$$

The resulting formulation is $P_{k-1} \cap R_k$ plus inequalities

(125) $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu_{k-1}^1 \leq (c-\beta+\alpha)\lfloor \frac{b}{c} \rfloor + \alpha \;\; t \in [k+1, n]$

that do not involve μ_k^2 and the following families of inequalities that involve μ_k^2: (97) for $i = 2$, (130), (132), (131), (129), (101), (128), (126), (127) and (124).

Elimination of μ_k^2

Here we need to combine the four inequalities (97) for $i = 2$, (130), (132) and (131) with the six inequalities (129), (101), (128), (126), (127) and (124) to eliminate μ_k^2.

Combining (97) and (126) gives (90) for $t = k$:

$$\sigma_k - \sigma_{k-1} \leq 0.$$

Combining (97) and (124) gives for $t \in [k, n]$:

$$\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor.$$

These are inequalities (106) for $t \in [k, n]$.

Combining (130)and (127) for $t = k$ gives:

$$\sigma_k - c\zeta_k \leq b,$$

which is (91) for $t = k$.

Combining (130) and (124) for $t = k$ gives

$$\sigma_k - (c-\beta+\alpha)\zeta_k - (\beta-\alpha)\mu^1_{k-1} \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This is (107) for $t = k$. Together with (125), we have (107) for $t \in [k, n]$.

Combining (132) and (101) gives (92) for $t = k$:

$$-\sigma_k + c\zeta_k \leq -a.$$

Combining (131) and (101) gives for $t \in [k+1, n]$

$$\sigma_t - \sigma_k - (c-\beta+\alpha)(\zeta_t - \zeta_k) \leq (c-\beta+\alpha)\left(\left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor\right).$$

These are inequalities (106) since $\left\lfloor \frac{b-a}{c} \right\rfloor = \left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor$.

The remaining combinations are redundant:

Combining (97) and (129) gives

$$\sigma_k - c\mu^2_{k-1} \leq 0.$$

It is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight 1, $\mu^0_{k-1} - \mu^1_{k-1} \leq 0$ with weight $(c-\beta)$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(c-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight 1.

Combining (97) and (101) gives

$$-\mu^2_{k-1} + \zeta_k \leq -\left\lfloor \frac{a}{c} \right\rfloor - 1.$$

It is redundant taking $-\zeta_{k-1} + \zeta_k \leq 0$ with weight 1 and $\zeta_{k-1} - \mu^2_{k-1} \leq -\left\lfloor \frac{a}{c} \right\rfloor - 1$ with weight 1.

Combining (97) and (128) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - \beta\mu^2_{k-1} \leq 0.$$

It is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight 1, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(\beta-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight 1.

Combining (97) and (127) for $t \in [k, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - \beta\mu^2_{k-1} \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor.$$

It is redundant taking (106) $\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} \leq (c-\beta)\left\lfloor \frac{b}{c} \right\rfloor$ with weight 1 and $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(\beta-\alpha)$.

Combining (130) and (129) gives

$$\sigma_k - c\zeta_k \leq c\left(1 + \left\lfloor \frac{b}{c} \right\rfloor\right).$$

This is redundant taking $\sigma_k - c\zeta_k \leq b$.

Combining (130) and (101) gives

$$0 \leq \left\lfloor \frac{b}{c} \right\rfloor - \left\lfloor \frac{a}{c} \right\rfloor.$$

This is redundant as $a \leq b$.

Combining (130) and (128) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - \beta\zeta_k \leq \beta\left(1 + \left\lfloor \frac{b}{c} \right\rfloor\right).$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{c-\beta}{c}$, $\sigma_k - c\zeta_k \leq b$ with weight $\frac{\beta}{c}$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $\frac{c-\beta}{c}(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\frac{\beta}{c}(c-\beta)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $\frac{c-\beta}{c}$.

Combining (130) and (126) gives

$$\sigma_k - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\zeta_k \leq \alpha\left(1 + \left\lfloor \frac{b}{c} \right\rfloor\right).$$

This is redundant taking $-(\beta-\alpha)\mu^1_{k-1} + \sigma_k - (c-\beta+\alpha)\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{\alpha}{c-\beta+\alpha}$, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{c-\beta}{c-\beta+\alpha}$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\frac{\alpha(c-\beta)}{c-\beta+\alpha}$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $\frac{c-\beta}{c-\beta+\alpha}$.

Combining (130) and (127) for $t \in [k+1, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - \beta\zeta_k \leq b.$$

This is redundant taking $\sigma_k - c\zeta_k \leq b$ with weight $\frac{\beta}{c}$, $\sigma_t - c\zeta_t \leq b$ with weight $\frac{c-\beta}{c}$ and $-\sigma_{j-1} + \sigma_j \leq 0$ with weight $\frac{\beta}{c}$ for $j \in [k+1, t]$.

Combining (130) and (124) for $t \in [k+1, n]$ gives

$$\sigma_t - (c-\beta)\zeta_t - (\beta-\alpha)\mu^1_{k-1} - \alpha\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha.$$

This is redundant taking ((107) for k) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_k - (c-\beta+\alpha)\zeta_k \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{\alpha}{c-\beta+\alpha}$, ((107) for t) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_t - (c-\beta+\alpha)\zeta_t \leq (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $1 - \frac{\alpha}{c-\beta+\alpha}$ and $-\sigma_{j-1} + \sigma_j \leq 0$ with weight $\frac{\alpha}{c-\beta+\alpha}$ for $j \in [k+1, t]$.

Combining (132) and (129) gives

$$c - \alpha \geq 0$$

which is redundant.

Combining (132) and (128) gives

$$(c-\beta)\sigma_k - (c-\beta)c\mu^0_{k-1} \leq (c-\alpha)\beta.$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\beta)$, $\mu^1_{k-1} - \mu^2_{k-1} \leq 0$ with weight $(c-\beta)(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $(c-\beta)\beta$, $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $(c-\beta)$and $(c-\beta)\beta \leq (c-\alpha)\beta$ with weight 1.

Combining (132) and (126) gives

$$(c-\alpha)\sigma_k - c(c-\beta)\mu^0_{k-1} - c(\beta-\alpha)\mu^1_{k-1} \leq \alpha(c-\alpha).$$

This is redundant taking $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\alpha)$, $\mu^0_{k-1} - \mu^1_{k-1} \leq 0$ with weight $\alpha(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \leq 1$ with weight $\alpha(c-\alpha)$ and $-\sigma_{k-1} + \sigma_k \leq 0$ with weight $(c-\alpha)$.

Combining (132) and (127) for $t \in [k, n]$ gives

$$c\sigma_t - c(c-\beta)\zeta_t - \beta\sigma_k \leq \beta(c-\alpha) + (c-\beta)c\left\lfloor \frac{b}{c} \right\rfloor.$$

This is redundant taking $\sigma_t - c\zeta_t \leq b$ with weight $(c-\beta)$, $-\sigma_{j-1} + \sigma_j \leq 0$ with weight β for $j \in [k+1, t]$, $b(c-\beta) = \beta(c-\beta) + (c-\beta)c\left\lfloor \frac{b}{c} \right\rfloor$ with weight 1 and $\beta(c-\beta) \leq \beta(c-\alpha)$ with weight 1.

Combining (132) and (124) for $t \in [k, n]$ gives

$$c\sigma_t - c(c-\beta)\zeta_t - \alpha\sigma_k - c(\beta-\alpha)\mu^1_{k-1} \leq c(c-\beta)\left\lfloor \frac{b}{c} \right\rfloor + (c-\alpha)\alpha.$$

This is redundant taking ((107) for t) $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{c(c-\beta)}{c-\beta+\alpha}$, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $\frac{\alpha(\beta-\alpha)}{c-\beta+\alpha}$, $\mu^0_{k-1} - \mu^1_{k-1} \le 0$ with weight $\alpha(\beta-\alpha)$, $\mu^2_{k-1} - \mu^0_{k-1} \le 1$ with weight $\frac{\alpha^2(\beta-\alpha)}{c-\beta+\alpha}$, $-\sigma_{k-1} + \sigma_k \le 0$ with weight $\frac{\alpha(\beta-\alpha)}{c-\beta+\alpha}$ and $-\sigma_{j-1} + \sigma_j \le 0$ with weight $\frac{\alpha c}{c-\beta+\alpha}$ for $j \in [k+1, t]$.

Combining (131) for $t \in [k+1, n]$ and (129) gives

$$c\sigma_t - c(c-\beta+\alpha)\zeta_t - (\beta-\alpha)\sigma_k \le c(c-\beta+\alpha)\left(\left\lfloor \frac{b}{c} \right\rfloor + 1\right).$$

This is redundant taking $-\sigma_{j-1} + \sigma_j \le 0$ with weight $(\beta-\alpha)$ for $j \in [k+1, t]$, $\sigma_t - c\zeta_t \le b$ with weight $(c-\beta+\alpha)$ and $\beta \le c$ with weight $c-\beta+\alpha$.

Combining (131) for $t \in [k+1, n]$ and (128) gives

$$\beta\sigma_t - \beta(c-\beta+\alpha)\zeta_t + (c-2\beta+\alpha)\sigma_k - (c-\beta+\alpha)(c-\beta)\mu^0_{k-1} \le \beta(c-\beta+\alpha)\left(\left\lfloor \frac{b}{c} \right\rfloor + 1\right).$$

Case 1. $c - 2\beta + \alpha \ge 0$

This is redundant taking ((107) for t) $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight β, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $c-2\beta+\alpha$, $\mu^1_{k-1} - \mu^2_{k-1} \le 0$ with weight $(\beta-\alpha)(c-\beta+\alpha)$, $-\mu^0_{k-1} + \mu^2_{k-1} \le 1$ with weight $\beta(c-\beta)$ and $-\sigma_{k-1} + \sigma_k \le 0$ with weight $c-2\beta+\alpha$.

Case 2. $c - 2\beta + \alpha < 0$

This is redundant taking $\sigma_t - c\zeta_t \le b$ with weight $\frac{(c-\beta+\alpha)(2\beta-c-\alpha)}{\beta-\alpha}$, $\sigma_t - (c-\beta+\alpha)\zeta_t - (\beta-\alpha)\mu^1_{k-1} \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $\frac{(c-\beta+\alpha)(c-\beta)}{\beta-\alpha}$, $\mu^1_{k-1} - \mu^2_{k-1} \le 0$ with weight $(c-\beta)(c-\beta+\alpha)$, $-\mu^0_{k-1} + \mu^2_{k-1} \le 1$ with weight $(c-\beta)(c-\beta+\alpha)$and $-\sigma_{j-1} + \sigma_j \le 0$ for $j \in [k+1, t]$ with weight $2\beta - c - \alpha$.

Combining (131) for $t \in [k+1, n]$ and (126) gives

$$\alpha\sigma_t + (c-\beta)\sigma_k - (c-\beta+\alpha)\left(\alpha\zeta_t + (c-\beta)\mu^0_{k-1} + (\beta-\alpha)\mu^1_{k-1}\right) \le \alpha(c-\beta+\alpha)\left(\left\lfloor \frac{b}{c} \right\rfloor + 1\right).$$

This is redundant taking ((107) for t) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_t - (c-\beta+\alpha)\zeta_t \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight α, $\sigma_{k-1} - (c-\beta)\mu^0_{k-1} - (\beta-\alpha)\mu^1_{k-1} - \alpha\mu^2_{k-1} = 0$ with weight $(c-\beta)$, $\mu^2_{k-1} - \mu^0_{k-1} \le 1$ with weight $\alpha(c-\beta)$ and $-\sigma_{k-1} + \sigma_k \le 0$ with weight $(c-\beta)$.

Combining (131) for $t \in [k+1, n]$ and (127) for $q \in [k, n]$ gives

$$\beta[\sigma_t - (c-\beta+\alpha)\zeta_t] + (c-\beta+\alpha)[\sigma_q - (c-\beta)\zeta_q] - \beta\sigma_k \le (c-\beta+\alpha)b.$$

Case 1. $t > q$
This is redundant taking $\sigma_t - c\zeta_t \le b$ with weight $(c-\beta+\alpha)\beta/c$, $\sigma_q - c\zeta_q \le b$ with weight $(c-\beta+\alpha)(c-\beta)/c$, $-\sigma_{j-1}+\sigma_j \le 0$ with weight β for $j \in [k+1, q]$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight $\beta(\beta-\alpha)/c$ for $j \in [q+1, t]$.
Case 2. $t \le q$
This is redundant taking $\sigma_t - c\zeta_t \le b$ with weight $(c-\beta+\alpha)\beta/c$, $\sigma_q - c\zeta_q \le b$ with weight $(c-\beta+\alpha)(c-\beta)/c$, $-\sigma_{j-1}+\sigma_j \le 0$ with weight β for $j \in [k+1, t]$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight $\beta(c-\beta+\alpha)/c$ for $j \in [t+1, q]$.

Combining (131) for $t \in [k+1, n]$ and (124) for $q \in [k, n]$ gives

$$\alpha[\sigma_t - (c-\beta+\alpha)\zeta_t] + (c-\beta+\alpha)[\sigma_q - (c-\beta)\zeta_q] \\ -\alpha\sigma_k - (c-\beta+\alpha)(\beta-\alpha)\mu^1_{k-1} \le (c-\beta+\alpha)[(c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha].$$

This is redundant taking ((107) for t) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_t - (c-\beta+\alpha)\zeta_t \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight α, ((107) for q) $-(\beta-\alpha)\mu^1_{k-1} + \sigma_q - (c-\beta+\alpha)\zeta_q \le (c-\beta+\alpha)\left\lfloor \frac{b}{c} \right\rfloor + \alpha$ with weight $(c-\beta)$ and $-\sigma_{j-1}+\sigma_j \le 0$ with weight α for $j \in [k+1, q]$.

The iteration is complete and we obtain the formulation with k replaced by $k-1$. □

References

1. Aksen, D., Altinkemer, K., Chand, S.: The single-item lot-sizing problem with immediate lost sales. Eur. J. Oper. Res. **147**, 558–566 (2003)
2. Atamtürk, A.: Flow pack facets of the single node fixed-charge flow polytope. Oper. Res. Lett. **29**, 107–114 (2001)
3. Atamtürk, A.: On capacitated network design cut-set polyhedra. Math. Program. **92**, 425–437 (2002)
4. Atamtürk, A., Gomez, A., Küçükyavuz, S.: Three-partition inequalities for constant capacity fixed-charge network flow problems. Networks **67**, 299–315 (2016)
5. Atamtürk, A., Küçükyavuz, S.: Lot-sizing with inventory bounds and fixed costs: polyhedral study and computation. Oper. Res. **53**, 711–730 (2005)
6. Atamtürk, A., Küçükyavuz, S., Tezel, B.: Path cover and path pack inequalities for the capacitated fixed-charge network flow problem. SIAM J. Optim. **27**, 194–1976 (2017)
7. Barany, I., Van Roy, T.J., Wolsey, L.A.: Uncapacitated lot-sizing: the convex hull of solutions. Math. Program. **22**, 32–43 (1984)
8. Bitran, G.R., Yanasse, H.H.: Computational complexity of the capacitated lot size problem. Manag. Sci. **28**, 1174–1186 (1982)
9. Conforti, M., Di Summa, M., Eisenbrand, F., Wolsey, L.A.: Network formulations of mixed-integer programs. Math. Oper. Res. **34**, 194–209 (2009)
10. Florian, M., Klein, M.: Deterministic production planning with concave costs and capacity constraints. Manag. Sci. **18**, 12–20 (1971)
11. Florian, M., Lenstra, J.K., Rinnooy Kan, H.G.: Deterministic production planning: algorithms and complexity. Manag. Sci. **26**, 669–679 (1980)
12. Ghouila-Houri, A.: Caractérisation des matrices totalement unimodulaire. C.R. Acad. Sci. Paris **254**, 1192–1194 (1962)

13. Gu, Z., Nemhauser, G.L., Savelsbergh, M.W.P.: Lifted flow cover inequalities for mixed 0–1 integer programs. Math. Program. **85**, 439–467 (1999)
14. Loparic, M., Pochet, Y., Wolsey, L.A.: The uncapacitated lot-sizing problem with sales and safety stocks. Math. Program. **89**, 487–504 (2001)
15. Louveaux, Q., Wolsey, L.A.: Lifting, superadditivity, mixed integer rounding and single node flow sets revisited. 4OR **1**, 173–207 (2003)
16. Love, S.F.: Bounded production and inventory models with piece- wise concave costs. Manag. Sci. **20**, 313–318 (1973)
17. Padberg, M.W., Van Roy, T.J., Wolsey, L.A.: Valid inequalities for fixed charge problems. Math. Program. **33**, 842–861 (1985)
18. Pochet, Y., Wolsey, L.A.: Lot-sizing with constant batches: formulation and valid inequalities. Math. Oper. Res. **18**, 767–785 (1993)
19. Pochet, Y., Wolsey, L.A.: Production Planning by Mixed Integer Programming. Springer, New York (2006)
20. Stallaert, J.I.A.: The complementary class of generalized flow cover inequalities. Discrete Appl. Math. **77**, 73–80 (1997)
21. van den Heuvel, W., Wagelmans, A.P.M.: Four equivalent lot-sizing models. Oper. Res. Lett. **36**, 465–470 (2008)
22. Van Roy, T.J., Wolsey, L.A.: Valid inequalities for mixed 0–1 Programs. Discrete Appl. Math. **14**, 199–213 (1986)
23. Wagner, H.M., Whitin, T.M.: Dynamic version of the economic lot size model. Manag. Sci. **5**, 89–96 (1958)
24. Wolsey, L.A.: Lot-sizing with production and delivery time windows. Math. Program. **107**, 471–489 (2006)
25. Wolsey, L.A.: Erratum: A tight formulation for uncapacitated lot-sizing with stock upper bounds. Math. Program. **161**, 603–609 (2017)
26. Wolsey, L.A., Yaman, H.: Convex hull results for the warehouse problem. Discrete Optim. **30**, 108–120 (2018)