



Split Extensions and Actions of Bialgebras and Hopf Algebras

Florence Sterck^{1,2}

Received: 28 January 2021 / Accepted: 19 August 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

We introduce a notion of split extension of (non-associative) bialgebras which generalizes the notion of split extension of magmas introduced by M. Gran, G. Janelidze and M. Sobral. We show that this definition is equivalent to the notion of action of (non-associative) bialgebras. We particularize this equivalence to (non-associative) Hopf algebras by defining split extensions of (non-associative) Hopf algebras and proving that they are equivalent to actions of (non-associative) Hopf algebras. Moreover, we prove the validity of the Split Short Five Lemma for these kinds of split extensions, and we examine some examples.

Keywords (Non-associative) bialgebras · (Non-associative) Hopf algebras · Actions · Split extensions · Split short five lemma

Mathematics Subject Classification 16T10 · 16T05 · 18C40 · 18E99 · 18M05 · 17D99 · 16S40

1 Introduction

In the category of groups, split extensions have a lot of interesting properties. A split extension of groups is a diagram of the form

$$0 \longrightarrow K \xrightarrow{k} G \xrightleftharpoons[k]{s} H \longrightarrow 0 \quad (1.1)$$

where $f \cdot s = 1_H$ and k is the kernel of f . Then f is the cokernel of k , so that (k, f) is a short exact sequence, as the diagram indicates. One of the interesting properties of split extensions of groups is the fact that the category of split extensions is equivalent to the category of group actions. An action of a group $(G, 1)$ on a group $(X, 1)$ is a map $\rho: G \times X \rightarrow X: (g, x) \mapsto {}^g x$

Communicated by George Janelidze.

✉ Florence Sterck
florence.sterck@uclouvain.be

¹ Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

² Département de Mathématique, Université Libre de Bruxelles, Campus de la Plaine – CP 210 Boulevard du Triomphe, 1050 Bruxelles, Belgium

such that for any $g, g' \in G$ and $x, x' \in X$ the following identities hold:

$$gg'x = g(g'x), \quad (1.2)$$

$$1x = x, \quad (1.3)$$

$$g(xx') = g^g x g^g x'. \quad (1.4)$$

In any semi-abelian category [16] there is an equivalence between (internal) actions and split extensions [6], since there is a natural categorical notion of semidirect product introduced by D. Bourn and G. Janelidze in [5]. Unfortunately, one can not expect this correspondence to hold in general algebraic categories: for example, in the category of monoids, split extensions and monoid actions do not form equivalent categories. Nevertheless, D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral proved in [7] that, in this case of monoids, there is still a restricted equivalence for the so-called “Schreier split epimorphisms”. The terminology “Schreier split epimorphisms” came from a paper of A. Patchkoria [23], who worked on a notion of a Schreier internal category in the category of monoids and proved that the category of Schreier internal categories in the category of monoids is equivalent to the category of crossed semimodules. A further step of generalization, of importance for this work, was considered in a recent paper by M. Gran, G. Janelidze and M. Sobral, where a natural notion of split extension of *unitary magmas* was introduced and shown to correspond to suitable actions [12].

By a result of [14] (see also [13]) saying that the category of cocommutative Hopf K -algebras is semi-abelian, where K is a field, it is known that there is an equivalence between the actions of cocommutative Hopf algebras and the split extensions of cocommutative Hopf algebras (see [4, 14, 22, 25]). Moreover, when we consider cocommutative (non-)associative bialgebras over a symmetric monoidal category $\mathcal{C}$, they can be seen as internal monoids (or magmas) in the category of cocommutative coalgebras over $\mathcal{C}$, and then we can apply the results of [12].

But what happens in the non-cocommutative case? This paper will give an answer to this question. We define different split extensions that are equivalent to the actions of non-associative bialgebras, bialgebras, non-associative Hopf algebras and Hopf algebras in any symmetric monoidal category. This general context provides a wide variety of possible applications. In particular, the definition of split extensions of non-associative bialgebras generalizes the notion of *split extensions of unitary magmas* [12], which was a non-associative generalization of the concept of “Schreier split epimorphisms” of monoids [7].

The first section of this paper is devoted to the preliminaries, where we recall the definition of bialgebras and Hopf algebras in a symmetric monoidal category.

In the second section, we define split extensions of non-associative bialgebras and show that they form a category that is equivalent to the category of actions of non-associative bialgebras. We show that these split extensions have the interesting property of being exact sequences. We prove a variation of the Split Short Five Lemma in the category of non-associative bialgebras, when one restricts it to the split extensions that we have introduced.

In the third section, we investigate the particular cases of cocommutative and associative bialgebras in a symmetric monoidal category. In particular, in the case of cocommutative bialgebras, the equivalence between actions and split extensions gives us the results in Section 4.6 in [12].

The last section describes the case of Hopf algebras and provides some examples of split extensions of Hopf algebras. In particular, we investigate the case where a split epimorphism $\alpha: A \rightarrow B$ of associative Hopf algebras satisfies the additional condition $H\text{Ker}(\alpha) =$

$LKer(\alpha)$, which is a condition given by N. Andruskiewitsch in [2] in order to define what he calls an *exact extension* of Hopf algebras.

2 Preliminaries

We recall [17] that a *monoidal category* is given by a triple $(\mathcal{C}, \otimes, I)$ where $\mathcal{C}$ is a category, $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ a bifunctor and I is the identity object (we omit the three natural isomorphisms, the associator, the right unit and the left unit).

A *braided monoidal category* is a 4-tuple $(\mathcal{C}, \otimes, I, \sigma)$ where $(\mathcal{C}, \otimes, I)$ is a monoidal category and σ is a *braiding*. A braiding consists of a natural isomorphism $\sigma = (\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X)_{X,Y \in \mathcal{C}}$ such that for any objects X, Y and Z in $\mathcal{C}$ the following equations are satisfied

$$\begin{aligned}\sigma_{X \otimes Y, Z} &= (\sigma_{X, Z} \otimes 1_Y) \cdot (1_X \otimes \sigma_{Y, Z}), \\ \sigma_{X, Y \otimes Z} &= (1_Y \otimes \sigma_{X, Z}) \cdot (\sigma_{X, Y} \otimes 1_Z).\end{aligned}$$

A braided monoidal category is called *symmetric* when

$$\sigma_{Y, X}^{-1} = \sigma_{X, Y}.$$

An algebra in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$ is given by an object $A \in \mathcal{C}$ endowed with a morphism $m: A \otimes A \rightarrow A$, called the multiplication. An algebra is unital when there is a morphism $u_A: I \rightarrow A$ called the unit, such that the following equalities are satisfied

$$m \cdot (u_A \otimes 1_A) = 1_A = m \cdot (1_A \otimes u_A), \quad (2.1)$$

$$\begin{array}{c} A \xrightarrow{u_A \otimes 1_A} A \otimes A \xleftarrow{1_A \otimes u_A} A \\ \searrow \quad \downarrow m \quad \swarrow \\ A \end{array}$$

All the algebras that we will consider in this paper are unital. However, we do not require any associativity condition on algebras. A morphism of algebras $f: A \rightarrow B$ is a morphism in $\mathcal{C}$ such that the following two diagrams commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ \downarrow m & & \downarrow m \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} I & \xrightarrow{u_A} & A \\ & \searrow u_B & \downarrow f \\ & & B \end{array}$$

A coalgebra is the dual notion of the notion of an algebra. In other words, a coalgebra over $(\mathcal{C}, \otimes, I, \sigma)$ is an object $C \in \mathcal{C}$ with a comultiplication $\Delta: C \rightarrow C \otimes C$. From now on, the coalgebras will always be coassociative, i.e. the following equality holds

$$(\Delta \otimes 1_C) \cdot \Delta = (1_C \otimes \Delta) \cdot \Delta \quad (2.2)$$

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow \Delta & & \downarrow 1_C \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes 1_C} & C \otimes C \otimes C \end{array}$$

We will also assume that the coalgebras are counital, meaning that there exists a morphism $\epsilon_C : C \rightarrow I$, called counit, satisfying the condition:

$$(\epsilon_C \otimes 1_C) \cdot \Delta = 1_C = (1_C \otimes \epsilon_C) \cdot \Delta, \quad (2.3)$$

as expressed by the commutativity of the following diagram

$$\begin{array}{ccccc} & & C & \xleftarrow{\epsilon_C \otimes 1_C} & C \otimes C & \xrightarrow{1_C \otimes \epsilon_C} & C \\ & & \uparrow \Delta & & \uparrow \Delta & & \\ & & C & & C & & \end{array}$$

Similarly, a morphism of coalgebras $g : C \rightarrow D$ is a morphism in $\mathcal{C}$ such that the following two diagrams commute

$$\begin{array}{ccc} D \otimes D & \xleftarrow{g \otimes g} & C \otimes C \\ \uparrow \Delta & & \uparrow \Delta \\ D & \xleftarrow{g} & C \end{array} \quad \begin{array}{ccc} I & \xleftarrow{\epsilon_D} & D \\ \swarrow \epsilon_C & & \uparrow g \\ & & C \end{array}$$

We also recall that a bialgebra is a 5-tuple $(B, m, u_B, \Delta, \epsilon_B)$ where (B, m, u_B) is an algebra, (B, Δ, ϵ_B) is a coalgebra and Δ, ϵ_B are algebra morphisms (which is equivalent to asking that m and u_B are coalgebra morphisms) i.e. the following conditions hold

$$\Delta \cdot m = (m \otimes m) \cdot (1_B \otimes \sigma_{B,B} \otimes 1_B) \cdot (\Delta \otimes \Delta), \quad (2.4)$$

$$\begin{array}{ccccc} B \otimes B & \xrightarrow{\quad m \quad} & B \\ \downarrow \Delta \otimes \Delta & & \downarrow \Delta \\ B \otimes B \otimes B \otimes B & \xrightarrow{1_B \otimes \sigma_{B,B} \otimes 1_B} B \otimes B \otimes B \otimes B & \xrightarrow{m \otimes m} & B \otimes B \end{array}$$

$$\Delta \cdot u_B = u_B \otimes u_B, \quad (2.5)$$

$$\begin{array}{ccc} I & \xrightarrow{u_B} & B \\ \searrow u_B \otimes u_B & & \downarrow \Delta \\ & & B \otimes B \end{array}$$

$$\epsilon_B \cdot m = \epsilon_B \otimes \epsilon_B, \quad (2.6)$$

$$\begin{array}{ccc} B \otimes B & \xrightarrow{m} & B \\ \searrow \epsilon_B \otimes \epsilon_B & & \downarrow \epsilon_B \\ & & I \end{array}$$

$$\epsilon_B \cdot u_B = 1_I. \quad (2.7)$$

$$\begin{array}{ccc}
 I & \xrightarrow{u_B} & B \\
 & \searrow & \downarrow \epsilon_B \\
 & & I
 \end{array}$$

Moreover, a morphism in $\mathcal{C}$ is a morphism of bialgebras if it is a morphism of algebras and coalgebras.

A non-associative Hopf algebra is a 7-tuple $(A, m, u_A, \Delta, \epsilon_A, S_L, S_R)$ where $(A, m, u_A, \Delta, \epsilon_A)$ is a bialgebra and S_L and S_R are antihomomorphisms of coalgebras and algebras, called the left and the right antipode, such that the following diagram commutes

$$\begin{array}{ccccc}
 & & A \otimes A & \xrightleftharpoons[S_L \otimes 1_A]{1_A \otimes S_R} & A \otimes A \\
 & \nearrow \Delta & & & \searrow m \\
 A & \xrightarrow{\epsilon_A} & I & \xrightarrow{u_A} & A.
 \end{array} \quad (2.8)$$

A morphism of Hopf algebras is a morphism of bialgebras preserving the antipodes. Note that in the case of associative Hopf algebras, the antipode is unique ($S_L = S = S_R$) and then S is automatically an antihomomorphism of coalgebras and algebras. Moreover, a bialgebra morphism between associative Hopf algebras necessarily preserves the antipode.

Example 2.1 (1) In the symmetric monoidal category $(\text{Set}, \times, \{\star\})$ of sets where σ is the twist morphism (where $\sigma(x, y) = (y, x)$ for any element x of a set X and any element y of a set Y), every object has a coalgebra structure with Δ being the diagonal and ϵ_X the morphism sending every element of X to $\star$. Hence, a non-associative bialgebra (or algebra) is an unital magma, an associative bialgebra (or algebra) is a monoid, an associative Hopf algebra is a group. The case of non-associative Hopf algebras in the category of sets will be treated in detail in Example 5.15.

- (2) In the symmetric monoidal category $(\text{Vect}_K, \otimes, K)$ of vector spaces over a field K where σ is the twist morphism (defined by $\sigma(x \otimes y) = y \otimes x$ for any $x \otimes y \in X \otimes Y$), we recover the notion of K -algebra, K -coalgebra, K -bialgebra and Hopf K -algebra.
- (3) In [8], a symmetric monoidal category was introduced such that Hom-algebras, Hom-coalgebras and Hom-Hopf algebras (see [21]) coincide with the algebras, coalgebras and Hopf algebras in this symmetric monoidal category.
- (4) In [9], the authors showed that Turaev's Hopf group-coalgebras (see [24]) are Hopf algebras in a symmetric monoidal category which they called Turaev category.
- (5) Associative and non-coassociative bialgebras and Hopf algebras in any symmetric monoidal category $\mathcal{C}$ can be seen as non-associative bialgebras and Hopf algebras in $\mathcal{C}^{op}$, the opposite category, which is still a symmetric monoidal category.
- (6) The coquasi-bialgebras and quasi-bialgebras are respectively examples of non-associative bialgebras in Vect_K and in Vect_K^{op} , see [19] for an introduction about these structures. The coquasi-Hopf algebras have different antipode conditions, but under some specific assumptions, it is possible to see them as non-associative Hopf algebras. An example which is both a non-associative Hopf algebra, as we defined, and a coquasi-Hopf algebra is the structure of octonions see [1].

These examples give us a glimpse of some frameworks and cases in which the results of this paper can be applied.

We also recall an important observation. The identity object I in a symmetric monoidal category $\mathcal{C}$, is a Hopf algebra in $\mathcal{C}$. The comultiplication on I is given by the natural isomorphism $\Delta: I \cong I \otimes I$, the counit, the unit and the antipode are the identity maps. Moreover, for any Hopf algebra A , we have unique Hopf algebra morphisms $u_A: I \rightarrow A$ and $\epsilon_A: A \rightarrow I$. Hence, I is the zero object in the category of Hopf algebras in $\mathcal{C}$. Obviously, I is also the zero object in the category of bialgebras in $\mathcal{C}$.

Convention 2.2 For the monoidal product of n copies $A \otimes \cdots \otimes A$, the notation A^n will be used. The same convention will be used for the morphisms, for example, we denote $\alpha \otimes \alpha: A \otimes A \rightarrow B \otimes B$ by $\alpha^2: A^2 \rightarrow B^2$. For the sake of simplicity, “bialgebras” will mean “non-associative bialgebras” (unless the associativity is explicitly mentioned). If it is not explicitly mentioned, the bialgebras and Hopf algebras are considered to be constructed in a general symmetric monoidal category $\mathcal{C}$.

3 Split Extensions of Non-associative Bialgebras

In this section, we introduce a notion of split extension of bialgebras in a symmetric monoidal category $\mathcal{C}$. These extensions have several properties. Among them, the fact that they are equivalent to actions of bialgebras, a restricted version of the Split Short Five Lemma holds for these extensions. Moreover, a split extension of bialgebras is an exact sequence and the definition of split extension of bialgebras generalizes the one of split extension of magmas introduced in [12].

Definition 3.1 Let X and B be bialgebras in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$. An action of B on X is a morphism $\triangleright: B \otimes X \rightarrow X$ in $\mathcal{C}$, such that the diagrams

$$\triangleright \cdot (u_B \otimes 1_X) = 1_X, \quad (3.1)$$

$$\begin{array}{ccc} X & \xrightarrow{u_B \otimes 1_X} & B \otimes X \\ & \searrow & \downarrow \triangleright \\ & & X \end{array}$$

$$\triangleright \cdot (1_B \otimes u_X) = u_X \cdot \epsilon_B, \quad (3.2)$$

$$\begin{array}{ccc} B & \xrightarrow{1_B \otimes u_X} & B \otimes X \\ & \searrow u_X \cdot \epsilon_B & \downarrow \triangleright \\ & & X \end{array}$$

$$(1_B \otimes \triangleright) \cdot (\Delta \otimes 1_X) = (1_B \otimes \triangleright) \cdot (\sigma_{B,B} \otimes 1_X) \cdot (\Delta \otimes 1_X), \quad (3.3)$$

$$\begin{array}{ccccc} B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B \otimes B \otimes X & & \\ \downarrow \Delta \otimes 1_X & & \downarrow 1_B \otimes \triangleright & & \\ B \otimes B \otimes X & \xrightarrow{\sigma_{B,B} \otimes 1_X} & B \otimes B \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X \end{array}$$

$$\epsilon_X \cdot \triangleright = \epsilon_B \otimes \epsilon_X, \quad (3.4)$$

$$\begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ & \searrow \epsilon_B \otimes \epsilon_X & \downarrow \epsilon_X \\ & & I \end{array}$$

$$\Delta \cdot \triangleright = (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma_{B,X} \otimes 1_X) \cdot (\Delta \otimes \Delta). \quad (3.5)$$

$$\begin{array}{ccccc} B \otimes X & \xrightarrow{\quad \triangleright \quad} & & & X \\ \downarrow \Delta \otimes \Delta & & & & \downarrow \Delta \\ B \otimes B \otimes X \otimes X & \xrightarrow{1_B \otimes \sigma_{B,X} \otimes 1_X} & B \otimes X \otimes B \otimes X & \xrightarrow{\triangleright \otimes \triangleright} & X \otimes X \end{array}$$

in $\mathcal{C}$ commute.

Let us note that the last two axioms mean that $\triangleright$ is a morphism of coalgebras. The axiom (3.3) is inspired by the condition (1) that Majid used in [18] to define a Hopf algebra crossed module. It is also what we need to define a bialgebra via the semi-direct product construction (see Theorem 3.3 in [20] and [22] for the construction of the semi-direct product also called smash-product). In particular, we are interested in diagrams of the form

$$X \xleftarrow[\pi_1]{\pi_1} X \rtimes B \xleftarrow[\pi_2]{i_2} B \quad (3.6)$$

in $\mathcal{C}$, where $i_1 = 1_X \otimes u_B$, $i_2 = u_X \otimes 1_B$, $\pi_1 = 1_X \otimes \epsilon_B$, $\pi_2 = \epsilon_X \otimes 1_B$ and $X \rtimes B$ is the object $X \otimes B$, where the bialgebra structure is given by the following morphisms in $\mathcal{C}$,

$$m_{X \rtimes B} = (m \otimes m) \cdot (1_X \otimes \triangleright \otimes 1_B \otimes 1_B) \cdot (1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B) \cdot (1_X \otimes \Delta \otimes 1_X \otimes 1_B)$$

$$u_{X \rtimes B} = u_X \otimes u_B,$$

$$\Delta_{X \rtimes B} = (1_X \otimes \sigma_{X,B} \otimes 1_B) \cdot (\Delta \otimes \Delta),$$

$$\epsilon_{X \rtimes B} = \epsilon_X \otimes \epsilon_B.$$

By combining Figs. 14 and 15 in the appendix, we check that this definition provides a bialgebra structure on $X \otimes B$ making the morphisms i_1 , i_2 and π_2 bialgebra morphisms, and π_1 a coalgebra morphism. Note that we can check that $u_X \otimes u_B$ is the neutral element for the multiplication thanks to the first two axioms (3.1) and (3.2) of the definition of action. The comultiplication of this structure is given as the usual comultiplication of the product of two coalgebras. In particular, the coassociativity of the comultiplication of this semi-direct product is obvious. It is also interesting to remark that the associativity of this structure is not automatic, even if the bialgebras X and B are associative, see Example 4.2.

Let us make some observations about the graph (3.6), which are analogous to the ones made in [12]:

Lemma 3.2 *The graph*

$$X \xleftarrow[\pi_1]{\pi_1} X \rtimes B \xleftarrow[\pi_2]{i_2} B$$

as defined in (3.6), where i_1, i_2, π_2 are morphisms of bialgebras, satisfies the following properties

- (1) $\pi_1 \cdot i_1 = 1_X, \pi_2 \cdot i_2 = 1_B$
- (2) $\pi_1 \cdot i_2 = u_X \cdot \epsilon_B, \pi_2 \cdot i_1 = u_B \cdot \epsilon_X$
- (3) $m_{X \rtimes B} \cdot (i_1 \cdot \pi_1 \otimes i_2 \cdot \pi_2)(1_X \otimes \sigma_{X,B} \otimes 1_B) \cdot (\Delta \otimes \Delta) = 1_X \otimes 1_B$
- (4) $\pi_1 \cdot m_{X \rtimes B} \cdot (i_1 \otimes i_2) = 1_X \otimes \epsilon_B$
- (5) $(1_B \otimes \pi_1) \cdot (1_B \otimes m_{X \rtimes B}) \cdot (1_B \otimes i_2 \otimes i_1) \cdot (\Delta \otimes 1_X) = (1_B \otimes \pi_1) \cdot (1_B \otimes m_{X \rtimes B}) \cdot (1_B \otimes i_2 \otimes i_1) \cdot (\sigma_{B,B} \otimes 1_X) \cdot (\Delta \otimes 1_X)$
- (6) $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (i_1 \otimes i_2 \otimes 1_X \otimes 1_B) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (i_1 \otimes i_2 \otimes 1_X \otimes 1_B)$
- (7) $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (i_1 \otimes 1_X \otimes 1_B \otimes i_2) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (i_1 \otimes 1_X \otimes 1_B \otimes i_2)$
- (8) $m_{X \rtimes B} \cdot (1_X \otimes 1_B \otimes m_{X \rtimes B}) \cdot (1_X \otimes 1_B \otimes i_1 \otimes i_2) = m_{X \rtimes B} \cdot (m_{X \rtimes B} \otimes 1_X \otimes 1_B) \cdot (1_X \otimes 1_B \otimes i_1 \otimes i_2)$
- (9) π_1 is a morphism of coalgebras and π_1 preserves the unit.

Proof The properties (1) and (2) are trivial. The condition (3) is proven via the commutativity of the following diagram.

$$\begin{array}{ccc}
 X \otimes B & \xrightarrow{\Delta \otimes \Delta} & X^2 \otimes B^2 \xrightarrow{1_X \otimes \sigma_{X,B} \otimes 1_B} (X \otimes B)^2 \\
 \parallel & & \searrow \scriptstyle (2.3) \quad 1_X \otimes \epsilon_X \otimes \epsilon_B \otimes 1_B \quad \downarrow \scriptstyle 1_X \otimes \epsilon_B \otimes \epsilon_X \otimes 1_B \\
 X \otimes B & \xlongequal{\quad} & X \otimes B \quad \downarrow \scriptstyle 1_X \otimes u_B \otimes u_X \otimes 1_B \\
 \parallel & & (X \otimes B)^2 \\
 \parallel & & \downarrow \scriptstyle 1_X \otimes \Delta \otimes 1_X \otimes 1_B \\
 X \otimes B & \xrightarrow{1_X \otimes u_B \otimes u_B \otimes u_X \otimes 1_B} & X \otimes B^2 \otimes X \otimes B \\
 \parallel & & \downarrow \scriptstyle 1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B \\
 X \otimes B & \xrightarrow{1_X \otimes u_B \otimes u_X \otimes u_B \otimes 1_B} & X \otimes B \otimes X \otimes B^2 \\
 \parallel & & \downarrow \scriptstyle 1_X \otimes \triangleright \otimes 1_B \otimes 1_B \\
 X \otimes B & \xrightarrow{1_X \otimes u_X \otimes u_B \otimes 1_B} & X^2 \otimes B^2 \\
 \parallel & & \downarrow \scriptstyle m \otimes m \\
 X \otimes B & \xlongequal{\quad} & X \otimes B
 \end{array}$$

$(i_1 \cdot \pi_1) \otimes (i_2 \cdot \pi_2)$
 $m_{X \rtimes B}$

The equality (4) holds since this diagram commutes

$$\begin{array}{ccc}
 X \otimes B & \xrightarrow{1_X \otimes u_B \otimes u_X \otimes 1_B} & (X \otimes B)^2 \\
 \parallel & \searrow (2.5) & \downarrow 1_X \otimes \Delta \otimes 1_X \otimes 1_B \\
 & & X \otimes B^2 \otimes X \otimes B \\
 & \searrow (3.1) & \downarrow 1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B \\
 & & X \otimes B \otimes X \otimes B^2 \\
 & & \downarrow 1_X \otimes \triangleright \otimes (1_B)^2 \\
 & & X^2 \otimes B^2 \\
 & \searrow (2.1) & \downarrow m \otimes m \\
 X \otimes B & \xrightarrow{1_X \otimes u_X \otimes u_B \otimes 1_B} & X \otimes B \\
 \downarrow 1_X \otimes \epsilon_B & & \uparrow 1_X \otimes \epsilon_B \\
 X & &
 \end{array}$$

The property (5) is due to (3.3) and the commutativity of the squares denoted (A) as we can see in the following commutative diagram

$$\begin{array}{ccccc}
 B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B \otimes B \otimes X & \xrightarrow{1_B \otimes i_2 \otimes i_1} & B \otimes (X \otimes B)^2 \\
 \downarrow \Delta \otimes 1_X & & \downarrow 1_B \otimes \triangleright & (A) & \downarrow 1_B \otimes m_{X \times B} \\
 B \otimes B \otimes X & \xrightarrow{\sigma_{B,B} \otimes 1_X} & B \otimes B \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X \otimes B \\
 \downarrow \sigma_{B,B} \otimes 1_X & & \downarrow 1_B \otimes i_2 \otimes i_1 & (A) & \downarrow 1_B \otimes m_{X \times B} \\
 B \otimes B \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X & \xrightarrow{1_B \otimes 1_X \otimes \epsilon_B} & B \otimes X \otimes B \\
 \downarrow 1_B \otimes i_2 \otimes i_1 & & \downarrow 1_B \otimes 1_X \otimes \epsilon_B & & \\
 B \otimes (X \otimes B)^2 & \xrightarrow{1_B \otimes m_{X \times B}} & B \otimes X \otimes B & &
 \end{array}$$

We observe that the squares (A) commute thanks to the unitality of the multiplication and the counitality of the comultiplication, as expressed with commutativity of this diagram

$$\begin{array}{ccc}
 B^2 \otimes X & \xrightarrow{1_B \otimes u_X \otimes 1_B \otimes 1_X \otimes u_B} & B \otimes (X \otimes B)^2 \xrightarrow{1_B \otimes 1_X \otimes \Delta \otimes 1_X \otimes 1_B} B \otimes X \otimes B^2 \otimes X \otimes B \\
 \parallel & & \downarrow 1_B \otimes 1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B \\
 B^2 \otimes X & \xrightarrow{1_B \otimes \Delta \otimes 1_X} & B^3 \otimes X \\
 \downarrow 1_B \otimes \triangleright & & \downarrow 1_B \otimes 1_B \otimes \sigma_{B,X} \\
 & & B^2 \otimes X \otimes B \xrightarrow{1_B \otimes \triangleright \otimes 1_B} B \otimes X \otimes B \xrightarrow{1_B \otimes u_X \otimes 1_X \otimes 1_B \otimes u_B} B \otimes X^2 \otimes B^2 \\
 & & \downarrow 1_B \otimes m \otimes m \\
 & & B \otimes X \otimes B
 \end{array}$$

We prove the “partial associativity” condition (6) thanks to Fig. 1. With similar computations we can show the “partial associativities” (7) and (8), moreover (9) is clear. $\square$

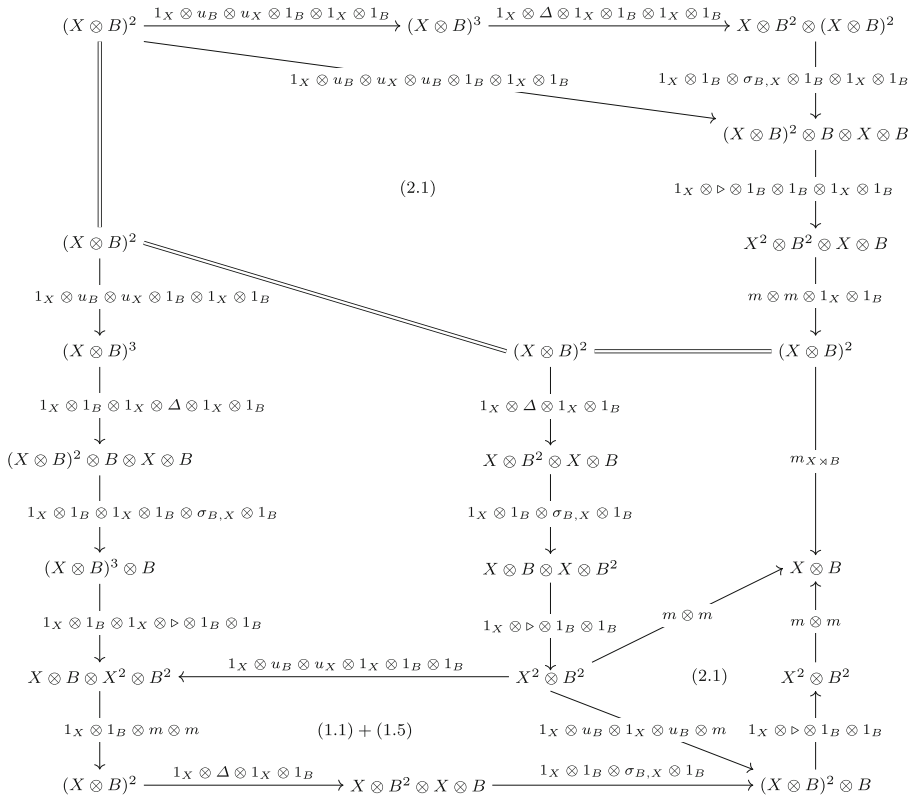


Fig. 1 Condition (6)

We define a split extension of bialgebras by taking inspiration of the above Lemma.

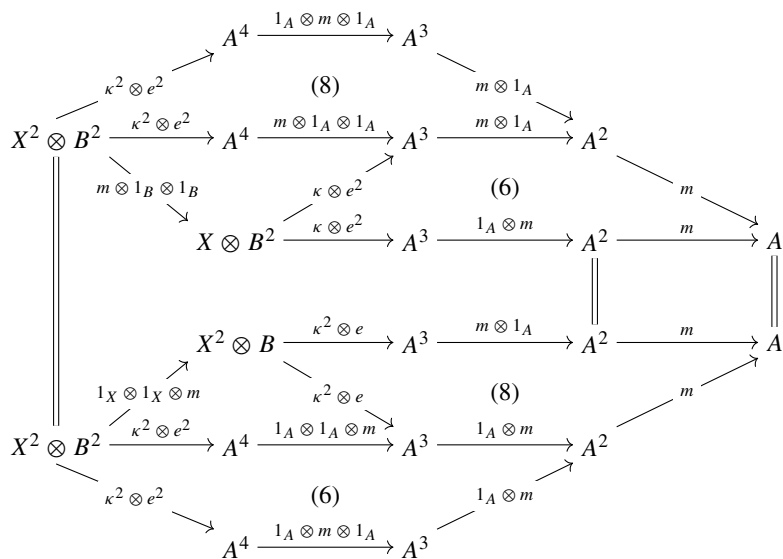
Definition 3.3 A split extension of bialgebras in $\mathcal{C}$ is given by a diagram in $\mathcal{C}$

$$X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B, \quad (3.7)$$

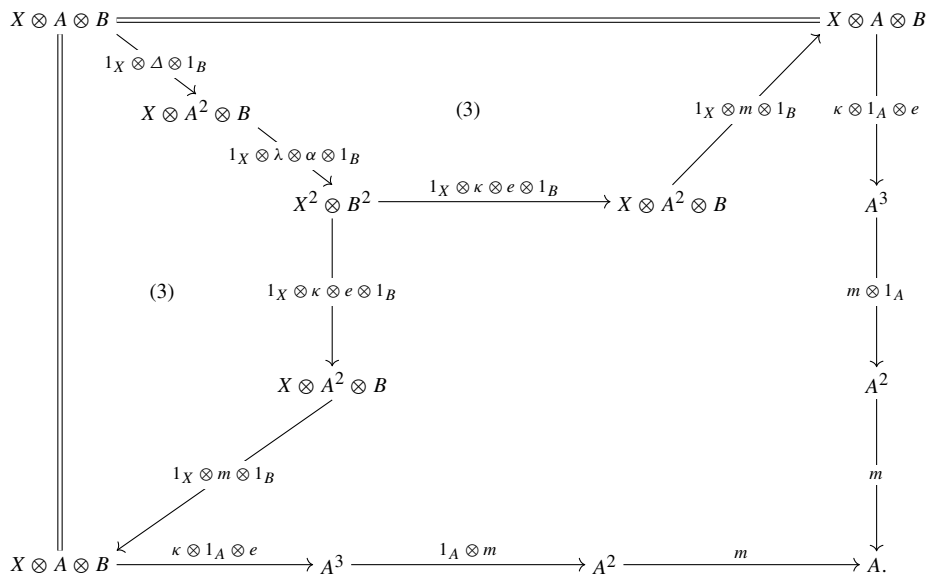
where X, A, B are bialgebras, κ, α, e are morphisms of bialgebras, such that

- (1) $\lambda \cdot \kappa = 1_X, \alpha \cdot e = 1_B$
- (2) $\lambda \cdot e = u_X \cdot \epsilon_B, \alpha \cdot \kappa = u_B \cdot \epsilon_X$
- (3) $m \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta = 1_A$
- (4) $\lambda \cdot m \cdot (\kappa \otimes e) = 1_X \otimes \epsilon_B$
- (5) $(1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\Delta \otimes 1_X) = (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\sigma_{B,B} \otimes 1_X) \cdot (\Delta \otimes 1_X)$
- (6) $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes e \otimes 1_A) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes e \otimes 1_A)$
- (7) $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes 1_A \otimes e) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes 1_A \otimes e)$
- (8) $m \cdot (m \otimes 1_A) \cdot (1_A \otimes \kappa \otimes e) = m \cdot (1_A \otimes m) \cdot (1_A \otimes \kappa \otimes e)$
- (9) λ is a morphism of coalgebras preserving the unit.

Remark 3.4 We notice that the conditions $\lambda \cdot \kappa = 1_X, \lambda \cdot e = u_X \cdot \epsilon_B$ and λ preserving the unit are consequences of the axiom (4). The condition (7) follows from (3), (6) and (8), as we show in the following diagrams, where we use that e and κ are (bi)algebra morphisms.



Hence, by pre-composing with $(1_X \otimes \lambda \otimes \alpha \otimes 1_B) \cdot (1_X \otimes \Delta \otimes 1_B)$ and using the condition (3), we obtain the following diagram



Then, we conclude that condition (7) holds.

In the associative case, the conditions (6), (7) and (8) in Definition 3.3 trivially become redundant. Note that these “partial associativity conditions” are versions of the conditions considered in the article [12]. More precisely, this definition generalizes the notion of split extension of unitary magmas in [12]. Indeed, by taking $\mathcal{C} = \text{Set}$, we obtain exactly the definition introduced in [12].

X' and $p: A \rightarrow A'$ such that the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\
 \downarrow v & & \downarrow p & & \downarrow g \\
 X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B'
 \end{array} \quad (3.8)$$

commutes in $\mathcal{C}$.

We do not need to ask the commutativity of all the squares thanks to this corollary:

Corollary 3.8 *Let (g, v, p) be a morphism of split extensions of bialgebras*

$$\begin{array}{ccccc}
 X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\
 \downarrow v & & \downarrow p & & \downarrow g \\
 X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B'
 \end{array}$$

If $p \cdot \kappa = \kappa' \cdot v$ and $p \cdot e = e' \cdot g$ hold, then the identities $\lambda' \cdot p = v \cdot \lambda$ and $\alpha' \cdot p = g \cdot \alpha$ follow, and conversely.

Proof Let us suppose that $p \cdot \kappa = \kappa' \cdot v$ and $p \cdot e = e' \cdot g$, then by Proposition 3.6, we can prove that $\lambda' \cdot p = v \cdot \lambda$ by checking that $(\lambda' \cdot p) \cdot \kappa = (v \cdot \lambda) \cdot \kappa$ and $(\lambda' \cdot p) \cdot e = (v \cdot \lambda) \cdot e$, which is done in the following identities:

$$\begin{aligned}
 (\lambda' \cdot p) \cdot \kappa &= \lambda' \cdot \kappa' \cdot v = v \cdot (v \cdot \lambda) \cdot \kappa, \\
 (\lambda' \cdot p) \cdot e &= \lambda' \cdot e' \cdot g = \epsilon_B = (v \cdot \lambda) \cdot e.
 \end{aligned}$$

Similarly, one can check that $\alpha' \cdot p = g \cdot \alpha$. $\square$

Proposition 3.9 *Let $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ be a split extension of bialgebras, then the following diagram*

$$\begin{array}{ccccccc}
 A^2 & \xrightarrow{\Delta \otimes 1_A} & A^3 & \xrightarrow{1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)} & A^3 & \xrightarrow{1_A \otimes m} & A^2 \\
 \downarrow m & & & & & & \downarrow \lambda \otimes \lambda \\
 & & & & & & X^2 \\
 & & & & & & \downarrow m \\
 A & \xrightarrow{\lambda} & & & & & X
 \end{array} \quad (3.9)$$

commutes in $\mathcal{C}$.

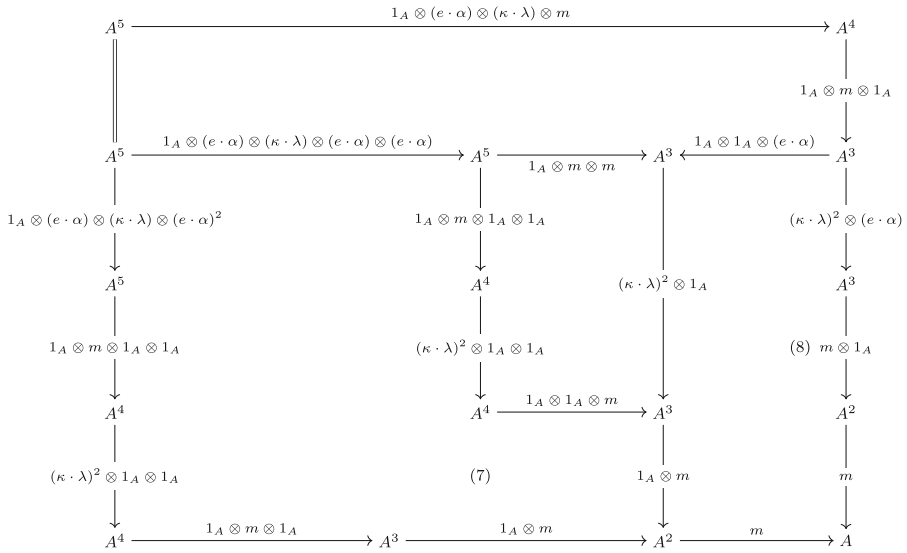


Fig. 2 Diagram (A)

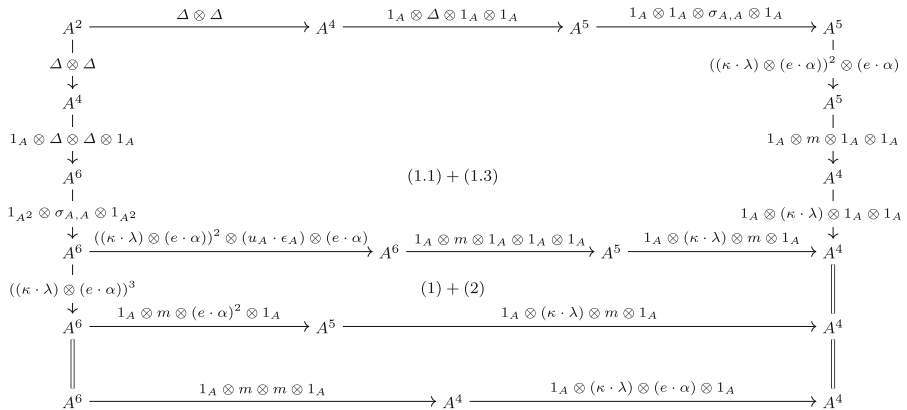


Fig. 3 Diagram (B)

Proof In order to prove the proposition, we make the three Figs. 2, 3 and 4 commute. The Fig. 2 is commutative since e and α are morphisms of (bi)algebras. The Fig. 3 commutes thanks to the counitality of the comultiplication, the unitality of the multiplication and the fact that $(e \cdot \alpha)$ is a morphism of (bi)algebras. In Fig. 4 we use the fact that we work with coalgebra morphisms.

Finally, we combine the Figs. 2, 3 and 4 in Fig. 17 (in the appendix) and we obtain the following equality

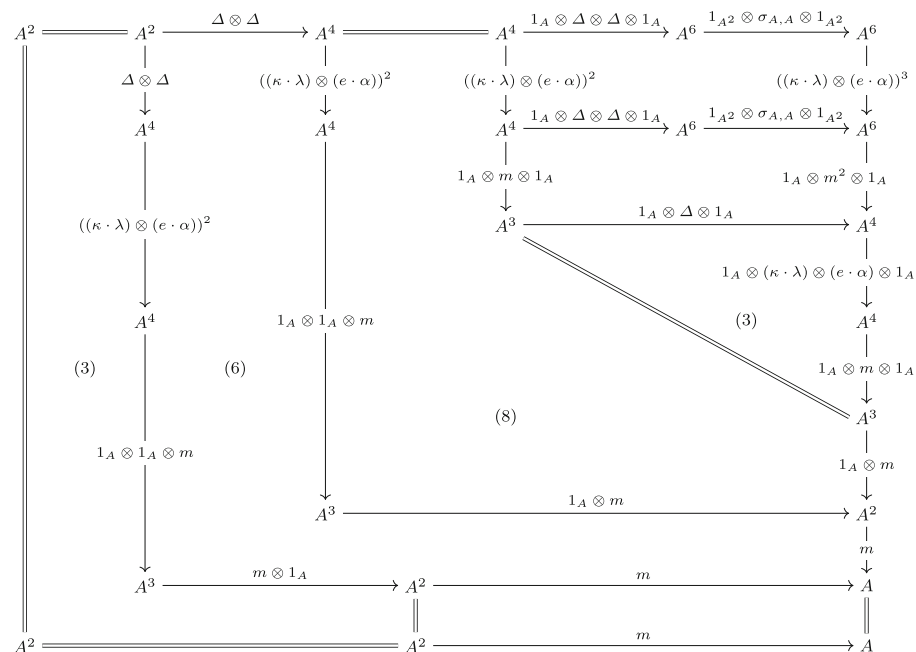


Fig. 4 Diagram (C)

$$\begin{aligned}
 m &= m \cdot (\kappa \otimes e) \cdot (m \otimes 1_B) \cdot (\lambda \otimes \lambda \otimes \alpha) \cdot (1_A \otimes m \otimes 1_A) \cdot (1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda) \otimes m) \\
 &\quad \cdot (1_A \otimes 1_A \otimes \sigma_{A,A} \otimes 1_A) \cdot (1_A \otimes \Delta \otimes 1_A \otimes 1_A) \cdot (\Delta \otimes \Delta).
 \end{aligned}$$

By composing with λ and using the condition (4), we can conclude that

$$\lambda \cdot m = m \cdot (\lambda \otimes \lambda) \cdot (1_A \otimes m) \cdot (1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)) \cdot (\Delta \otimes 1_A).$$

□

Proposition 3.10 Given a split extension $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ of bialgebras, we can construct an action of bialgebras, $\triangleright: B \otimes X \rightarrow X$ defined by

$$\triangleright = \lambda \cdot m \cdot (e \otimes \kappa).$$

Proof We check all the axioms of the definition of actions of bialgebras (Definition 3.1).

Moreover, the condition (3.3) for the particular action $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$ is exactly the condition (5) in the definition of split extension of bialgebras (Definition 3.3). $\square$

Consider the diagram

where the two rows are split extensions, the top row by definition and the bottom one by Lemma 3.2, where the action of bialgebras is given by Proposition 3.10. The morphisms φ and ψ in $\mathcal{C}$ are defined by

$$\psi = (\lambda \otimes \alpha) \cdot \Delta,$$

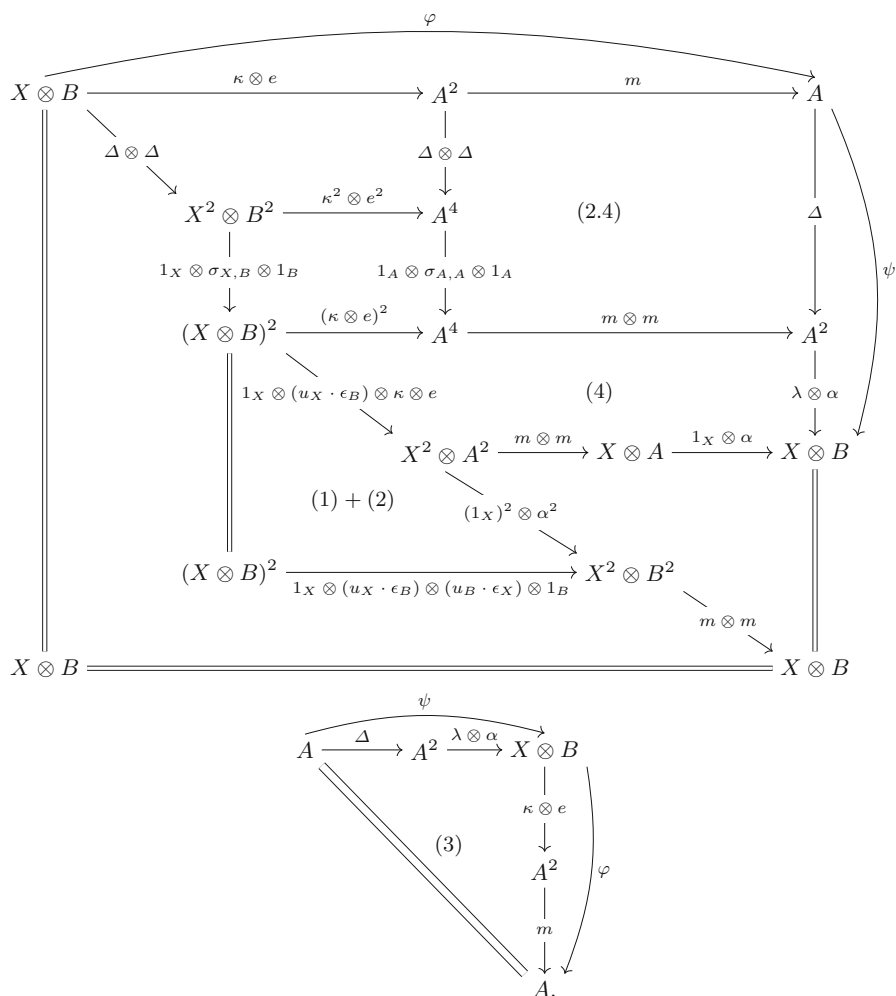
and

$$\varphi = m \cdot (\kappa \otimes e).$$

In the following lemmas, we prove step by step that ψ and φ are isomorphisms of split extensions. First, we prove that they are inverse to each other in $\mathcal{C}$.

Lemma 3.11 *The morphisms φ and ψ in $\mathcal{C}$ are inverse to each other.*

Proof We prove by means of the two following diagrams that $\psi \cdot \varphi = 1_X \otimes 1_B$ and $\varphi \cdot \psi = 1_A$ by using the properties of the split extensions.



□

In order to prove that ψ and φ of (3.10) are morphisms of bialgebras, we need the following technical Lemma. Notice that this Lemma will also be convenient to express the action $\triangleright := \lambda \cdot m \cdot (e \otimes \kappa)$ in the particular case of Hopf algebras (see Remark 5.6).

Lemma 3.12 *Given a split extension $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ of bialgebras, we have*

$$m \cdot (e \otimes \kappa) = m \cdot (\kappa \otimes e) \cdot (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X), \quad (3.11)$$

where $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$.

Proof The equality of the lemma is proven thanks to the commutativity of the diagram below, where we use that α , κ and e are morphisms of bialgebras.

$$\begin{array}{ccccc}
 B \otimes X & \xlongequal{\quad} & B \otimes X & \xrightarrow{e \otimes \kappa} & A^2 & \xrightarrow{m} & A \\
 \Delta \otimes 1_X \downarrow & & (1_B \otimes \sigma_{B,X} \otimes 1_X) \cdot (\Delta \otimes \Delta) \downarrow & & (1_A \otimes \sigma_{A,A} \otimes 1_A) \cdot (\Delta \otimes \Delta) \downarrow & & \Delta \downarrow \\
 B^2 \otimes X & & B \otimes X \otimes B \otimes X & \xrightarrow{(e \otimes \kappa)^2} & A^4 & \xrightarrow{m \otimes m} & A^2 \\
 1_B \otimes \sigma_{B,X} \downarrow & & e \otimes \kappa \otimes 1_B \otimes 1_X \downarrow & & m \otimes \alpha \otimes \alpha \downarrow & & \lambda \otimes \alpha \downarrow \\
 B \otimes X \otimes B & & A^2 \otimes B \otimes X & \xrightarrow{m \otimes 1_B \otimes (u_B \cdot \epsilon_X)} & A \otimes B^2 & \xrightarrow{\lambda \otimes m} & X \otimes B \\
 e \otimes \kappa \otimes 1_B \downarrow & & & & & & \kappa \otimes e \downarrow \\
 A^2 \otimes B & & & & & & A^2 \\
 m \otimes 1_B \downarrow & & & & & & m \downarrow \\
 A \otimes B & \xrightarrow{\lambda \otimes 1_B} & X \otimes B & \xrightarrow{\kappa \otimes e} & A^2 & \xrightarrow{m} & A
 \end{array}$$

(3)

(2.1) + (2.3)

(2.4)

□

Lemma 3.13 $\varphi := m \cdot (\kappa \otimes e): X \otimes B \rightarrow A$ is a morphism of bialgebras.

Proof We use the partial associativity of the split extensions to check that Fig. 5 commutes. Moreover, Fig. 6 commutes since e and κ are (bi)algebra morphisms. Thanks to Figs. 5 and 6 and Lemma 3.12 we have Fig. 7 showing that $\varphi := m \cdot (\kappa \otimes e)$ is a morphism of algebras.

Finally, φ is a morphism of bialgebras since it is also a morphism of coalgebras:

$$\begin{array}{ccccc}
 X \otimes B & \xrightarrow{\kappa \otimes e} & A^2 & \xrightarrow{m} & A \\
 \Delta \otimes \Delta \downarrow & & \Delta \otimes \Delta \downarrow & & \Delta \downarrow \\
 X \otimes X \otimes B \otimes B & \xrightarrow{\kappa^2 \otimes e^2} & A^4 & & A^2 \\
 1_X \otimes \sigma_{X,B} \otimes 1_B \downarrow & & 1_A \otimes \sigma_{A,A} \otimes 1_A \downarrow & & \\
 X \otimes B \otimes X \otimes B & \xrightarrow{(\kappa \otimes e)^2} & A^4 & \xrightarrow{m \otimes m} & A^2
 \end{array}$$

(3.12)

(2.4)

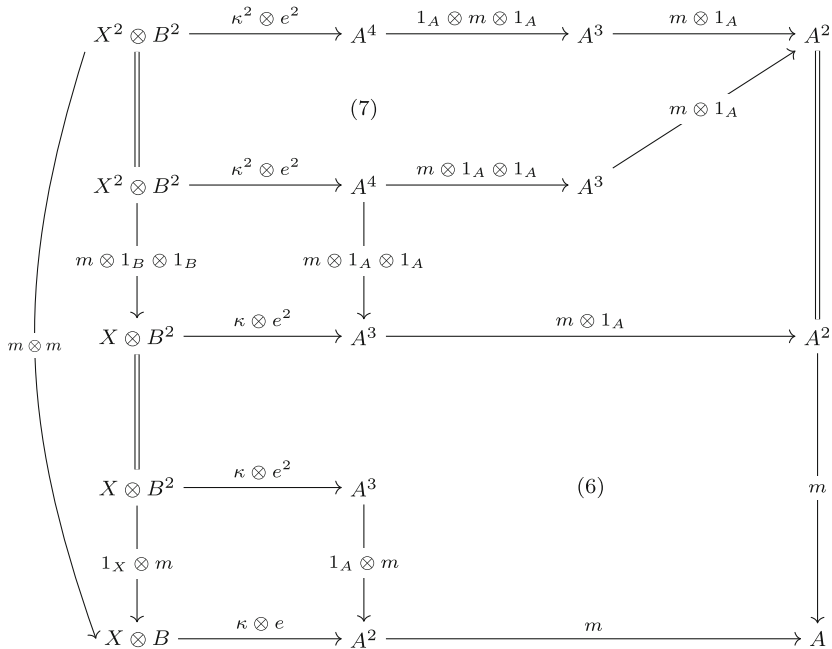


Fig. 6 Second sub-diagram in the proof that φ is a morphism of algebras

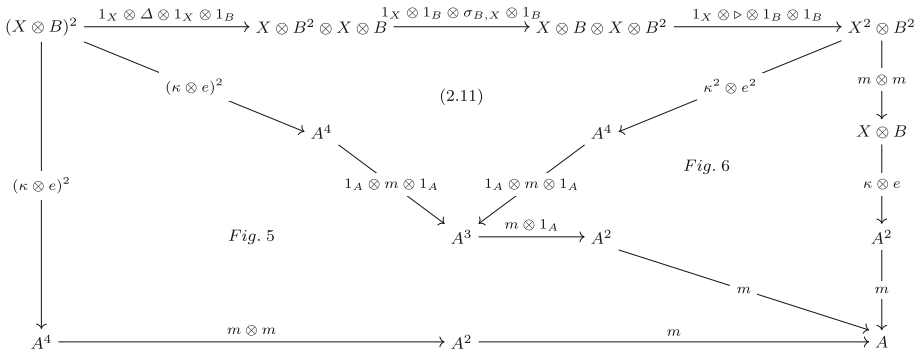


Fig. 7 φ is a morphism of algebras

Proof Thanks to Proposition 3.8, it is enough to check that the four following diagrams of (3.10) commute to conclude that φ and ψ are morphisms of split extensions of bialgebras. We recall that $\psi = (\lambda \otimes \alpha) \cdot \Delta$ and $\varphi = m \cdot (\kappa \otimes e)$. $\square$

Proposition 3.15 Given a split extension $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ of bialgebras, the following properties hold:

- (a) κ is the kernel of α in the category of bialgebras in $\mathcal{C}$;
- (b) α is the cokernel of κ in the category of bialgebras in $\mathcal{C}$;
- (c) e is the kernel of λ in the category of pointed coalgebras in $\mathcal{C}$.

Hence, a split extension of bialgebras is a short exact sequence.

Proof (a) Let $\omega : D \rightarrow A$ be a morphism of bialgebras such that $\alpha \cdot \omega = u_B \cdot \epsilon_D$. We build the morphism $\hat{\omega} : D \rightarrow X$ by setting

$$\hat{\omega} = \lambda \cdot \omega.$$

First, we verify that $\kappa \cdot \hat{\omega} = w$ thanks to the commutativity of the following diagram

$$\begin{array}{ccccc}
 D & \xrightarrow{\quad \Delta \quad} & D^2 & & D \\
 \omega \downarrow & & \downarrow \omega \otimes \omega & \searrow (\kappa \cdot \lambda \cdot \omega) \otimes (u_A \cdot \epsilon_D) & \downarrow \kappa \cdot \lambda \cdot \omega \\
 & & A^2 & \xrightarrow{(\kappa \cdot \lambda) \otimes (e \cdot \alpha)} & A^2 \\
 & \nearrow \Delta & & & \downarrow m \\
 A & & & & A
 \end{array}
 \quad (3.13)$$

(2.1) + (2.3) (3)

Moreover, $\hat{\omega}$ is a coalgebra morphism by construction, and an algebra morphism since the following diagram commutes

$$\begin{array}{ccccccc}
 D^2 & \xrightarrow{\quad m \quad} & & & D & & \\
 \omega^2 \searrow & & & & \downarrow \omega & & \\
 & & A^2 & \xrightarrow{\quad m \quad} & A & & \\
 \Delta \otimes 1_D \nearrow & & \downarrow \Delta \otimes 1_A & & & & \\
 D^2 & \xrightarrow{\quad \Delta \otimes 1_D \quad} & D^3 & \xrightarrow{\quad \omega^3 \quad} & A^3 & & \\
 \omega \otimes (u_A \cdot \epsilon_D) \otimes (\kappa \cdot \lambda \cdot \omega) \searrow & & \downarrow 1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda) & & & & \\
 & & A^3 & & & & \\
 \omega \otimes (\kappa \cdot \lambda \cdot \omega) \nearrow & & \downarrow 1_A \otimes m & & & & \\
 D^2 & \xrightarrow{\quad \omega \otimes (\kappa \cdot \lambda \cdot \omega) \quad} & A^2 & & & & \\
 \omega^2 \searrow & & \downarrow \lambda^2 & & & & \\
 D^2 & \xrightarrow{\quad \omega^2 \quad} & A^2 & \xrightarrow{\quad \lambda^2 \quad} & X^2 & \xrightarrow{\quad m \quad} & X
 \end{array}$$

(2.1) + (2.3) (3.9) (1)

Finally, if there exists another morphism ω' such that $\kappa \cdot \omega' = w$, then by (1)

$$\omega' = \lambda \cdot \kappa \cdot \omega' = \lambda \cdot \omega = \lambda \cdot \kappa \cdot \hat{\omega} = \hat{\omega}.$$

Hence, κ is the kernel of α .

(b) Let $\beta : A \rightarrow C$ be a bialgebra morphism such that $\beta \cdot \kappa = u_C \cdot \epsilon_X$. We define $\tilde{\beta} : B \rightarrow C$ by

$$\tilde{\beta} = \beta \cdot e.$$

This morphism is a bialgebra morphism, and thanks to Proposition 3.6, it is enough to remark that $\tilde{\beta} \cdot \alpha \cdot \kappa = u_C \cdot \epsilon_X = \beta \cdot \kappa$ and $\tilde{\beta} \cdot \alpha \cdot e = \beta \cdot e$, to conclude that $\tilde{\beta} \cdot \alpha = \beta$. Moreover,

if there exists another β' such that $\beta' \cdot \alpha = \beta$ then, thanks to (1) we have

$$\beta' = \beta' \cdot \alpha \cdot e = \beta \cdot e = \tilde{\beta} \cdot \alpha \cdot e = \tilde{\beta}$$

and α is the cokernel of κ

(c) Let $\gamma: Y \rightarrow A$ be a morphism of coalgebras such that $\lambda \cdot \gamma = u_X \cdot \epsilon_Y$. We can verify that the equality $e \cdot \hat{\gamma} = \gamma$ holds for $\hat{\gamma} := \alpha \cdot \gamma: Y \rightarrow B$, using the condition (3) in Definition 3.3 as in diagram (3.13). Moreover, it is clear that $\hat{\gamma}$ is a coalgebra morphism and it is the unique morphism such that $e \cdot \hat{\gamma} = \gamma$. $\square$

Lemma 3.16 *Let (g, v, p) be a morphism of split extensions as in Definition 3.7, then*

$$v \cdot \triangleright = \triangleright \cdot (g \otimes v),$$

where the actions are induced by the split extensions.

Proof This follows from the fact that (g, v, p) is a morphism of split extensions and p is a morphism of bialgebras, as we can see in the following diagram

$$\begin{array}{ccccccc}
 & & & \triangleright & & & \\
 & & & \curvearrowright & & & \\
 B \otimes X & \xrightarrow{e \otimes \kappa} & A^2 & \xrightarrow{m} & A & \xrightarrow{\lambda} & X \\
 \downarrow g \otimes v & & \downarrow p \otimes p & & \downarrow p & & \downarrow v \\
 B' \otimes X' & \xrightarrow{e' \otimes \kappa'} & A'^2 & \xrightarrow{m'} & A' & \xrightarrow{\lambda'} & X' \\
 & & & \curvearrowleft & & & \\
 & & & \triangleright & & &
 \end{array}$$

$\square$

Lemma 3.17 *Let $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ and*

$X' \xleftarrow[\kappa']{\lambda'} A' \xleftarrow[\alpha']{e'} B'$ be two split extensions of bialgebras, and $g: B \rightarrow B'$ and $v: X \rightarrow X'$ two morphisms of bialgebras. Then the following conditions are equivalent:

- (1) *there exists $p: A \rightarrow A'$ such that (g, v, p) is a morphism of split extensions;*
- (2) *there exists a unique $p: A \rightarrow A'$ such that (g, v, p) is a morphism of split extensions;*
- (3) *$v \cdot \triangleright = \triangleright \cdot (g \otimes v)$.*

Proof Thanks to Proposition 3.6 and Lemma 3.16, we just need to check that 3) $\Rightarrow$ 1). Let us define $\tilde{p}: X \rtimes B \rightarrow X' \rtimes B'$ as

$$\tilde{p} = v \otimes g.$$

It is clear that this morphism is a morphism of coalgebras. Moreover, $\tilde{p}$ is a morphism of algebras as we can see in the following diagram

$$\begin{array}{c}
 \xrightarrow{m_{X \rtimes B}} \\
 (X \otimes B)^2 \xrightarrow{1_X \otimes \Delta \otimes 1_X \otimes 1_B} X \otimes B^2 \otimes X \otimes B \xrightarrow{1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B} (X \otimes B)^2 \otimes B \xrightarrow{1_X \otimes \triangleright \otimes (1_B)^2} X^2 \otimes B^2 \xrightarrow{m^2} X \otimes B \\
 \downarrow (v \otimes g)^2 \quad \downarrow v \otimes g^2 \otimes v \otimes g \quad \downarrow (v \otimes g)^2 \otimes g \quad \downarrow 3) \quad \downarrow v^2 \otimes g^2 \quad \downarrow v \otimes g \\
 (X' \otimes B')^2 \xrightarrow{1_{X'} \otimes \Delta \otimes 1_{X'} \otimes 1_{B'}} X' \otimes B'^2 \otimes X' \otimes B' \xrightarrow{1_{X'} \otimes 1_{B'} \otimes \sigma_{B',X'} \otimes 1_{B'}} (X' \otimes B')^2 \otimes B' \xrightarrow{1_{X'} \otimes \triangleright \otimes (1_{B'})^2} X'^2 \otimes B'^2 \xrightarrow{m'^2} X' \otimes B' \\
 \xrightarrow{m_{X' \rtimes B'}}
 \end{array}$$

Then we can define a morphism $p: A \rightarrow A'$ as $\varphi_{A'} \cdot \tilde{p} \cdot \psi_A$ where φ and ψ are the isomorphisms in Proposition 3.14. In particular, that gives us

$$p = m \cdot (\kappa' \otimes e') \cdot (v \otimes g) \cdot (\lambda \otimes \alpha) \cdot \Delta.$$

Finally, (g, v, p) is a morphism of split extensions of bialgebras. Indeed, thanks to Corollary 3.8 the commutativity of these two diagrams

$$\begin{array}{ccc}
 X & \xrightarrow{\kappa} & A \\
 \Delta \searrow & & \downarrow \Delta \\
 X^2 & \xrightarrow{\kappa \otimes \kappa} & A^2 \\
 1_X \otimes (u_B \cdot \epsilon_X) \searrow & & \downarrow \lambda \otimes \alpha \\
 & & X \otimes B \\
 v \downarrow & & \downarrow v \otimes g \\
 & & X' \otimes B' \\
 & & \downarrow \kappa' \otimes e' \\
 & & A'^2 \\
 & & \downarrow m \\
 X' & \xrightarrow{\kappa'} & A'
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{e} & A \\
 \Delta \searrow & & \downarrow \Delta \\
 B^2 & \xrightarrow{e \otimes e} & A^2 \\
 (u_X \cdot \epsilon_B) \otimes 1_B \searrow & & \downarrow \lambda \otimes \alpha \\
 & & X \otimes B \\
 g \downarrow & & \downarrow v \otimes g \\
 & & X' \otimes B' \\
 & & \downarrow \kappa' \otimes e' \\
 & & A'^2 \\
 & & \downarrow m \\
 B' & \xrightarrow{e'} & A'
 \end{array}$$

suffices to conclude that diagram (3.8) commutes. $\square$

Definition 3.18 Let $\triangleright: B \otimes X \rightarrow X$ and $\triangleright': B' \otimes X' \rightarrow X'$ be two actions of bialgebras. A morphism between them is defined as a pair of morphisms of bialgebras $g: B \rightarrow B'$ and $v: X \rightarrow X'$ such that

$$v \cdot \triangleright = \triangleright' \cdot (g \otimes v).$$

The split extensions of bialgebras (Definition 3.3) endowed with the morphisms of split extensions of bialgebras (Definition 3.7) form the category of split extensions of bialgebras denoted by $\text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$. The actions of bialgebras (Definition 3.1) with the morphisms of actions (Definition 3.18) form the category of actions of bialgebras, denoted by $\text{Act}(\text{BiAlg}_{\mathcal{C}})$.

Theorem 3.19 *Let $\mathcal{C}$ be a symmetric monoidal category. The category $\text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$ of split extensions of bialgebras in $\mathcal{C}$ and the category $\text{Act}(\text{BiAlg}_{\mathcal{C}})$ of actions of bialgebras in $\mathcal{C}$ are equivalent.*

Proof The functor $F: \text{SplitExt}(\text{BiAlg}_{\mathcal{C}}) \rightarrow \text{Act}(\text{BiAlg}_{\mathcal{C}})$ is defined as

$$F \left(\begin{array}{ccccc} X & \xleftarrow{\lambda} & A & \xleftarrow{e} & B \\ \downarrow v & \searrow \kappa & \downarrow p & \searrow \alpha & \downarrow g \\ X' & \xleftarrow{\lambda'} & A' & \xleftarrow{e'} & B' \end{array} \right) = \begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ \downarrow g \otimes v & & \downarrow v \\ B' \otimes X' & \xrightarrow{\triangleright'} & X' \end{array}$$

where $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$ as in Proposition 3.10, and we have a morphism of actions thanks to Lemma 3.16. The functor $G: \text{Act}(\text{BiAlg}_{\mathcal{C}}) \rightarrow \text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$ is defined as

$$G \left(\begin{array}{ccc} B \otimes X & \xrightarrow{\triangleright} & X \\ \downarrow g \otimes v & & \downarrow v \\ B' \otimes X' & \xrightarrow{\triangleright'} & X' \end{array} \right) = \begin{array}{ccccc} X & \xleftarrow{\pi_1} & X \rtimes B & \xleftarrow{i_2} & B \\ \downarrow v & \searrow i_1 & \downarrow p & \searrow \pi_2 & \downarrow g \\ X' & \xleftarrow{\pi'_1} & X' \rtimes B' & \xleftarrow{i'_2} & B' \end{array}$$

where $p := v \otimes g$ is given by Lemma 3.17 and the bialgebra structures of the semi-direct products $X \rtimes B$ and $X' \rtimes B'$ are defined as in (3.6).

We observe that

$$(F \cdot G)(\triangleright) = \pi_1 \cdot m_{X \rtimes B} \cdot (i_2 \otimes i_1) = \triangleright, \quad (3.14)$$

where the last equality holds thanks to the commutativity of the following diagram

$$\begin{array}{ccccccc} B \otimes X & \xrightarrow{u_X \otimes 1_B \otimes 1_X \otimes u_B} & (X \otimes B)^{\otimes 2} & \xrightarrow{1_X \otimes \Delta \otimes 1_X \otimes 1_B} & X \otimes B^2 \otimes X \otimes B & \xrightarrow{1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B} & X \otimes B \otimes X \otimes B^2 \\ \parallel & & & & & & \downarrow 1_X \otimes \triangleright \otimes 1_B \otimes 1_B \\ B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B^2 \otimes X & \xrightarrow{1_B \otimes \sigma_{B,X}} & B \otimes X \otimes B & \xrightarrow{\triangleright \otimes 1_B} & X \otimes B \xrightarrow{u_X \otimes 1_X \otimes 1_B \otimes u_B} X^2 \otimes B^2 \\ & & & & & & \downarrow m \otimes m \\ & & & & & & X \otimes B \\ & & & & & & \downarrow 1_X \otimes \epsilon_B \\ B \otimes X & \xrightarrow{\quad \triangleright \quad} & & & & & X. \end{array} \quad (2.1)$$

(2.3)

Thanks to this observation and the isomorphisms φ and ψ of (3.10), the functors F and G give rise to an equivalence of categories. $\square$

To end this section we prove that a variation of the Split Short Five Lemma holds in $\text{BiAlg}_{\mathcal{C}}$.

Theorem 3.20 *Let (g, v, p) be a morphism of split extensions of bialgebras in a symmetric monoidal category $\mathcal{C}$*

$$\begin{array}{ccccc}
 X & \xleftarrow{\lambda} & A & \xleftarrow{e} & B \\
 \downarrow v & \xrightarrow{\kappa} & \downarrow p & \xrightarrow{\alpha} & \downarrow g \\
 X' & \xleftarrow{\lambda'} & A' & \xleftarrow{e'} & B' \\
 & \xrightarrow{\kappa'} & & \xrightarrow{\alpha'} &
 \end{array} \quad (3.15)$$

then p is an isomorphism whenever v and g are.

Proof Thanks to Theorem 3.19, the diagram (3.15) is canonically isomorphic to

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_1} & X \rtimes B & \xleftarrow{i_2} & B \\
 \downarrow v & \xrightarrow{i_1} & \downarrow v \otimes g & \xrightarrow{\pi_2} & \downarrow g \\
 X' & \xleftarrow{\pi'_1} & X' \rtimes B' & \xleftarrow{i'_2} & B' \\
 & \xrightarrow{i'_1} & & \xrightarrow{\pi'_2} &
 \end{array}$$

It follows that $v \otimes g$ is an isomorphism whenever v and g are. $\square$

4 The Cocommutative and Associative Cases

If we consider cocommutative bialgebras, then the category $BiAlg_{\mathcal{C},coc}$ of cocommutative bialgebras can be seen as the category of internal magmas in the category of cocommutative coalgebras. Indeed, the categorical product of cocommutative coalgebras is given by the tensor product.

In this particular case, the condition (3.3) in Definition 3.1 becomes trivial and the definition can be reformulated explicitly as:

Definition 4.1 Let X and B be cocommutative bialgebras. An action of B on X is a morphism of coalgebras $\triangleright: B \otimes X \rightarrow X$ such that

$$\begin{aligned}
 \triangleright \cdot (u_B \otimes 1_X) &= 1_X, \\
 \triangleright \cdot (1_B \otimes u_X) &= u_X \cdot \epsilon_B.
 \end{aligned}$$

Similarly, thanks to the cocommutativity we can drop the condition (5) in the definition of split extensions of bialgebras (Definition 3.3), and this turns out to be exactly the internal version, in the category of coalgebras, of Definition 1.4 in [12]. Then, in the case of cocommutative bialgebras, the above theorem reduces to the results in Section 4.6 in [12]. Indeed, if $\mathcal{C}$ is a category with finite limits, it is in particular a cartesian monoidal category. Hence, any magma in such a category can be seen as a non-associative cocommutative bialgebra in this category (where the unique comultiplication is the diagonal). Accordingly, the results in [12] become a particular case of our theorem.

It is interesting to note that the multiplication $m_{X \rtimes B}: (X \otimes B) \otimes (X \otimes B) \rightarrow X \otimes B$, defined in (3.6), is not associative in general. Even if X and B are associative bialgebras this structure need not be associative. Let us illustrate this observation with a simple example.

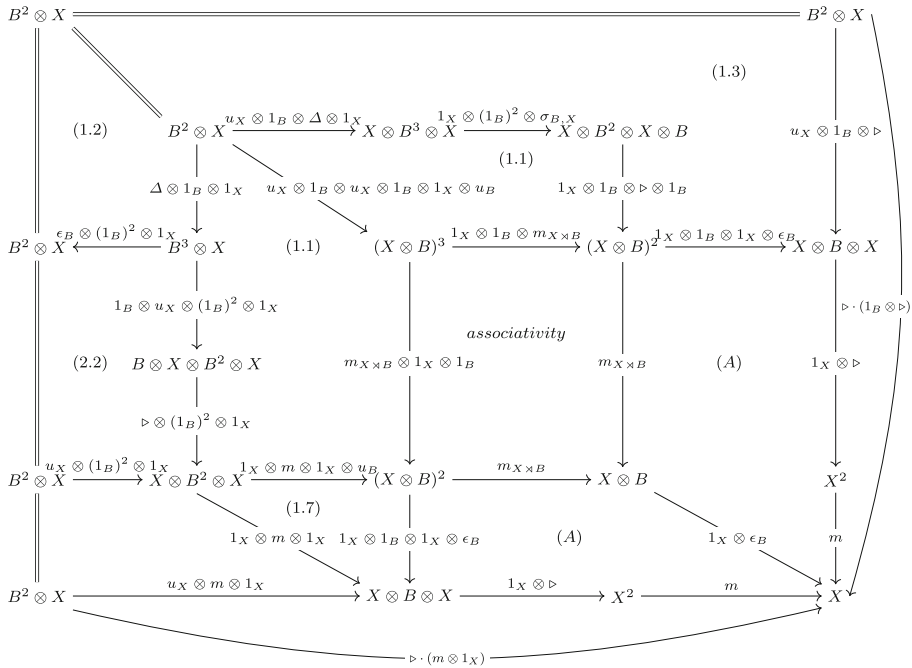


Fig. 8 Associativity implies (4.1)

Example 4.2 Let $\mathcal{C}$ be the category of sets, and we consider the monoids $(\mathbb{N}, +, 0)$ and $(\mathbb{N}_0, \cdot, 1)$. In particular we can construct an action of $(\mathbb{N}_0, \cdot, 1)$ on $(\mathbb{N}, +, 0)$ via the function $\triangleright: \mathbb{N}_0 \times \mathbb{N} \rightarrow \mathbb{N}$ sending (b, x) to x to the power of b , x^b . We can observe that this function satisfies the conditions (3.1) and (3.2) of Definition 3.1 (the other ones being trivial in $\mathcal{C} = \text{Set}$): $x^1 = x$, $0^b = 0$ for any $x \in \mathbb{N}$, $b \in \mathbb{N}_0$. By defining, via this action, the bialgebra structure on $\mathbb{N} \rtimes \mathbb{N}_0$ as $m((x, b), (y, c)) = (x + y^b, b \cdot c)$, this multiplication is non-associative:

$$m(m((0, 2), (1, 1)), (1, 1)) = m((1, 2), (1, 1)) = (1 + 1^2, 2 \cdot 1) = (2, 2)$$

$$m((0, 2), m((1, 1), (1, 1))) = m((0, 2), (2, 1)) = (0 + 2^2, 2 \cdot 1) = (4, 2).$$

We need additional conditions on the actions of two associative bialgebras B and X , to obtain an associative bialgebra structure on $X \rtimes B$. In particular, we have the following Lemma.

Lemma 4.3 *Let X and B be two associative bialgebras, then $m_{X \rtimes B}: (X \otimes B) \otimes (X \otimes B) \rightarrow X \otimes B$ is associative as well if and only if the following conditions are satisfied*

$$\triangleright \cdot (m \otimes 1_X) = \triangleright \cdot (1_B \otimes \triangleright), \quad (4.1)$$

$$\triangleright \cdot (1_B \otimes m) = m \cdot (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma_{B, X} \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X). \quad (4.2)$$

Proof Via Fig. 8, we show that if $m_{X \rtimes B}$ is associative then (4.1) is immediately satisfied. With similar computations, we can show that the associativity also gives the condition (4.2). The other implication is given by the commutativity of the diagram in Fig. 16 in the appendix (it is a direct computation using the conditions (4.1) and (4.2)).

The trapezoids (A) commute thanks to (2.3) and (2.6), as it is shown in the following diagram

$$\begin{array}{c}
 (X \otimes B)^2 \xrightarrow{m_{X \rtimes B}} X \otimes B \\
 \downarrow 1_X \otimes \Delta \otimes 1_X \otimes 1_B \quad \downarrow 1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B \quad \downarrow 1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B \\
 X \otimes B^2 \otimes X \otimes B \xrightarrow{1_X \otimes 1_B \otimes \sigma_{B,X} \otimes 1_B} X \otimes B \otimes X \otimes B^2 \xrightarrow{1_X \otimes \triangleright \otimes (1_B)^2} X^2 \otimes B^2 \xrightarrow{(2.6)} X \otimes B \\
 \downarrow 1_X \otimes 1_B \otimes 1_X \otimes \epsilon_B \quad \downarrow 1_X \otimes 1_B \otimes \epsilon_B \otimes 1_X \otimes \epsilon_B \quad \downarrow 1_X \otimes 1_B \otimes \epsilon_B \otimes 1_X \otimes \epsilon_B \quad \downarrow 1_X \otimes \epsilon_B \\
 X \otimes B \otimes X \xrightarrow{(2.3)} X \otimes B \otimes X \xrightarrow{1_X \otimes \triangleright} X^2 \xrightarrow{m} X
 \end{array}$$

□

Remark 4.4 In the symmetric monoidal category of sets, the conditions (4.1) and (4.2) correspond to the conditions (1.2) and (1.4) (considered in the more restricted setting of groups).

Let us consider associative bialgebras. We define the categories $\text{Act}(\text{AssBiAlg}_{\mathcal{C}})$ and $\text{SplitExt}(\text{AssBiAlg}_{\mathcal{C}})$. An object in $\text{Act}(\text{AssBiAlg}_{\mathcal{C}})$, is an action of associative bialgebras (Definition 3.1) satisfying (4.1) and (4.2), the morphisms are the morphisms of $\text{Act}(\text{BiAlg}_{\mathcal{C}})$. The category $\text{SplitExt}(\text{AssBiAlg}_{\mathcal{C}})$ is a full subcategory of $\text{SplitExt}(\text{BiAlg}_{\mathcal{C}})$ since the conditions (6), (7) and (8) become redundant. In particular, since Definition 3.3 is a generalization of the split extensions of magmas introduced in [12] (which are a generalization of the “Schreier split epimorphisms” of monoids), it is clear that the split extensions of associative bialgebras generalize the notion of “Schreier split epimorphisms” of monoids introduced in [7].

Corollary 4.5 *Let $\mathcal{C}$ be a symmetric monoidal category. There is an equivalence between $\text{SplitExt}(\text{AssBiAlg}_{\mathcal{C}})$ the category of split extensions of associative bialgebras in $\mathcal{C}$ and $\text{Act}(\text{AssBiAlg}_{\mathcal{C}})$ the category of actions of associative bialgebras in $\mathcal{C}$.*

Proof It is clear by applying Lemma 4.3 and Theorem 3.19. □

5 Split Extensions of Non-associative Hopf Algebras

In this section, we consider a similar result for non-associative Hopf algebras. We prove an equivalence between the category of split extensions of non-associative Hopf algebras and the category of actions of non-associative Hopf algebras.

Convention 5.1 *For the sake of simplicity, in this section “Hopf algebra” will mean “non-associative Hopf algebra” (unless the associativity is explicitly mentioned).*

Definition 5.2 A split extension of Hopf algebras is a split extension of bialgebras

$$X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B \quad (5.1)$$

such that X, A, B are Hopf algebras and κ, α, e are morphisms of Hopf algebras, with an additional condition of associativity (condition (9')) and an additional condition about the left and right antipodes (conditions (10') and (11')). More precisely, the split extension (5.1) satisfies

- (1') $\lambda \cdot \kappa = 1_X, \alpha \cdot e = 1_B$
- (2') $\lambda \cdot e = u_X \cdot \epsilon_B, \alpha \cdot \kappa = u_B \cdot \epsilon_X$
- (3') $m \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta = 1_A$

- (4') $\lambda \cdot m \cdot (\kappa \otimes e) = 1_X \otimes \epsilon_B$
 (5') $(1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\Delta \otimes 1_X) = (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\sigma_{B,B} \otimes 1_X) \cdot (\Delta \otimes 1_X)$
 (6') $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes e \otimes 1_A) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes e \otimes 1_A)$
 (7') $m \cdot (m \otimes 1_A) \cdot (\kappa \otimes 1_A \otimes e) = m \cdot (1_A \otimes m) \cdot (\kappa \otimes 1_A \otimes e)$
 (8') $m \cdot (m \otimes 1_A) \cdot (1_A \otimes \kappa \otimes e) = m \cdot (1_A \otimes m) \cdot (1_A \otimes \kappa \otimes e)$
 (9') $m \cdot (m \otimes 1_A) \cdot (e \otimes 1_A \otimes \kappa) = m \cdot (1_A \otimes m) \cdot (e \otimes 1_A \otimes \kappa)$
 (10') $S_L \cdot \lambda \cdot m \cdot (e \otimes \kappa) = \lambda \cdot m \cdot (e \otimes \kappa) \cdot (1_B \otimes S_L),$
 (11') $\epsilon_B \otimes S_R = \lambda \cdot m \cdot (e \otimes \kappa) \cdot (S_R \otimes S_R) \cdot (1_B \otimes \lambda) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa) \cdot (\Delta \otimes 1_X),$
 (12') λ is a morphism of coalgebras preserving the unit.

The following definition is inspired by the definition given in [18] in the case of associative Hopf algebras.

Definition 5.3 Let X and B be Hopf algebras, $\triangleright: B \otimes X \rightarrow X$ is an action of Hopf algebras if it is an action of bialgebras such that the following additional conditions are satisfied

$$\triangleright \cdot (1_B \otimes \triangleright) = \triangleright \cdot (m \otimes 1_X), \quad (5.2)$$

$$\triangleright \cdot (1_B \otimes m) = m \cdot (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma_{B,X} \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X), \quad (5.3)$$

$$\triangleright \cdot (1_B \otimes S_L) = S_L \cdot \triangleright, \quad (5.4)$$

$$\triangleright \cdot (S_R \otimes S_R) \cdot (1_B \otimes \triangleright) \cdot (\Delta \otimes 1_X) = \epsilon_B \otimes S_R. \quad (5.5)$$

These conditions can be expressed by the commutativity of the four diagrams below.

$$\begin{array}{c}
 \begin{array}{ccc}
 B \otimes B \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X \\
 \downarrow m \otimes 1_X & & \downarrow \triangleright \\
 B \otimes X & \xrightarrow{\triangleright} & X
 \end{array} \quad (5.2) \\
 \\
 \begin{array}{ccccc}
 B \otimes X \otimes X & \xrightarrow{\Delta \otimes 1_X \otimes 1_X} & B^2 \otimes X^2 & \xrightarrow{1_B \otimes \sigma_{B,X} \otimes 1_X} & (B \otimes X)^2 \\
 \downarrow 1_B \otimes m & & & & \downarrow \triangleright \otimes \triangleright \\
 & & & & X^2 \\
 & & & & \downarrow m \\
 B \otimes X & \xrightarrow{\triangleright} & & & X
 \end{array} \quad (5.3) \\
 \\
 \begin{array}{ccc}
 B \otimes X & \xrightarrow{\triangleright} & X \\
 \downarrow 1_B \otimes S_L & & \downarrow S_L \\
 B \otimes X & \xrightarrow{\triangleright} & X
 \end{array} \quad (5.4) \qquad
 \begin{array}{ccccc}
 B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B^2 \otimes X & \xrightarrow{1_B \otimes \triangleright} & B \otimes X \\
 \downarrow \epsilon_B \otimes S_R & & & & \downarrow S_R \otimes S_R \\
 X & \xleftarrow{\triangleright} & & & B \otimes X
 \end{array} \quad (5.5)
 \end{array}$$

Note that whenever $S_L = S_R$, the condition (5.5) follows from (5.2) and (5.4). We notice that, when we consider associative Hopf algebras, the conditions (5.4) and (5.5) are trivially satisfied thanks to the uniqueness of the antipode.

We define the map $\Theta: B \otimes X \rightarrow X \otimes B$ as the composition

$$\Theta := (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X).$$

We will use this map to obtain shorter computations. We re-formulate the conditions (5.3), (3.1), (5.2) and (3.2) in terms of Θ . These new conditions will help us to prove that the semi-direct product is a Hopf algebra when we construct it with an action as defined above (Definition 5.3).

Lemma 5.4 *Let $\triangleright: B \otimes X \rightarrow X$ be an action of Hopf algebras, the morphism $\Theta := (\triangleright \otimes 1_B) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X)$ satisfies the following conditions*

$$(m \otimes 1_B) \cdot (1_X \otimes \Theta) \cdot (\Theta \otimes 1_X) = \Theta \cdot (1_B \otimes m), \quad (5.6)$$

$$\Theta \cdot (u_B \otimes 1_X) = 1_X \otimes u_B, \quad (5.7)$$

$$(1_X \otimes m) \cdot (\Theta \otimes 1_B) \cdot (1_B \otimes \Theta) = \Theta \cdot (m \otimes 1_X), \quad (5.8)$$

$$\Theta \cdot (1_B \otimes u_X) = u_X \otimes 1_B, \quad (5.9)$$

The left diagram consists of two parts. The top part is a commutative diagram with nodes $B \otimes X \otimes X$, $X \otimes B \otimes X$, $X \otimes X \otimes B$, $B \otimes X$, $X \otimes B$, and $X \otimes B \otimes B$. The bottom part is a commutative diagram with nodes $B \otimes B \otimes X$, $B \otimes X \otimes B$, $X \otimes B \otimes B$, $B \otimes X$, and $X \otimes B$. The right diagram consists of two parts. The top part is a commutative diagram with nodes X , $B \otimes X$, $X \otimes B$, and X . The bottom part is a commutative diagram with nodes B , $B \otimes X$, $X \otimes B$, and B .

Proof We only show the two first equalities since the computations are similar. First, we prove (5.6) via the following diagram, where the key part is given by (5.3).

$$\begin{array}{c}
\Theta \cdot (1_B \otimes m) \\
\begin{array}{ccccccc}
B \otimes X^2 & \xrightarrow{\Delta \otimes 1_X \otimes 1_X} & B^2 \otimes X^2 & \xrightarrow{1_B \otimes 1_B \otimes m} & B^2 \otimes X & \xrightarrow{1_B \otimes \sigma_{B,X}} & B \otimes X \otimes B \xrightarrow{\triangleright \otimes 1_B} X \otimes B \\
\downarrow \Delta \otimes 1_X \otimes 1_X & & \downarrow \Delta \otimes 1_B \otimes 1_X \otimes 1_X & & \downarrow 1_B \otimes \sigma_{B,XX} & & \downarrow 1_B \otimes m \otimes 1_B \\
B^2 \otimes X^2 & \xrightarrow{1_B \otimes \Delta \otimes 1_X \otimes 1_X} & B^3 \otimes X^2 & \xrightarrow{1_B \otimes 1_B \otimes \sigma_{B,XX}} & B^2 \otimes X^2 \otimes B & & \\
\downarrow 1_B \otimes \sigma_{B,X} \otimes 1_X & & \downarrow 1_B \otimes \sigma_{BB,X} \otimes 1_X & & \downarrow 1_B \otimes \sigma_{B,X} \otimes 1_X \otimes 1_B & & \\
(B \otimes X)^2 & \xrightarrow{1_B \otimes 1_X \otimes \Delta \otimes 1_X} & B \otimes X \otimes B^2 \otimes X & \xrightarrow{1_B \otimes 1_X \otimes 1_B \otimes \sigma_{B,X}} & (B \otimes X)^2 \otimes B & \xrightarrow{\triangleright^2 \otimes 1_B} & X^2 \otimes B \\
\downarrow \triangleright \otimes 1_B \otimes 1_X & & \downarrow 1_X \otimes \Delta \otimes 1_X & & \downarrow 1_X \otimes 1_B \otimes \sigma_{B,X} & & \downarrow 1_X \otimes \triangleright \otimes 1_B \\
X \otimes B \otimes X & \xrightarrow{1_X \otimes \Delta \otimes 1_X} & X \otimes B^2 \otimes X & \xrightarrow{1_X \otimes 1_B \otimes \sigma_{B,X}} & (X \otimes B)^2 & \xrightarrow{1_X \otimes \triangleright \otimes 1_B} & X^2 \otimes B \\
& & & & \downarrow 1_X \otimes \Theta & & \\
& & & & X^2 \otimes B & &
\end{array}
\end{array}$$

(2.2) (5.3)

The condition (3.1) provides directly the condition (5.7) as we can see in the following diagram

$$\begin{array}{ccc}
X & \xrightarrow{u_B \otimes 1_X} & B \otimes X \\
& \searrow u_B \otimes 1_X \otimes u_B & \downarrow \Delta \otimes 1_X \\
& & B^2 \otimes X \\
& & \downarrow 1_B \otimes \sigma_{B,X} \\
& & B \otimes X \otimes B \\
& & \downarrow \triangleright \otimes 1_B \\
& & X \otimes B \\
X & \xrightarrow{1_X \otimes u_B} & X \otimes B.
\end{array}$$

(2.5) (3.1)

The other equalities follow from (5.2) and (3.2), the proofs being similar to the ones given above. $\square$

Starting from an action of Hopf algebras, we can define a split extension of Hopf algebras

$$X \xleftarrow[\pi_1]{\pi_1} X \rtimes B \xleftarrow[\pi_2]{\pi_2} B \quad (5.10)$$

where the structure of $X \rtimes B$ is given by

$$\begin{aligned}
m_{X \rtimes B} &= (m \otimes m) \cdot (1_X \otimes \Theta \otimes 1_B) \\
u_{X \rtimes B} &= u_X \otimes u_B, \\
\Delta_{X \rtimes B} &= (1_X \otimes \sigma_{X,B} \otimes 1_B) \cdot (\Delta \otimes \Delta), \\
\epsilon_{X \rtimes B} &= \epsilon_X \otimes \epsilon_B,
\end{aligned}$$

$$S_{X \rtimes B_L} = \Theta \cdot (S_L \otimes S_L) \cdot \sigma_{X,B},$$

$$S_{X \rtimes B_R} = \Theta \cdot (S_R \otimes S_R) \cdot \sigma_{X,B}.$$

Thanks to Lemma 3.2, we already know that (5.10) is a split extension of bialgebras. It is also easy to check that i_1 , i_2 and π_2 are morphisms of Hopf algebras as it is shown in the following diagrams for the left antipode, similar computations work for the right antipode,

$$\begin{array}{ccc}
 B \xrightarrow{u_X \otimes 1_B} X \otimes B \xrightarrow{\sigma_{X,B}} B \otimes X & & X \xrightarrow{1_X \otimes u_B} X \otimes B \xrightarrow{\sigma_{X,B}} B \otimes X \\
 \downarrow S_L & & \downarrow S_L \\
 B \xrightarrow{1_B \otimes u_X} B \otimes X & & X \xrightarrow{u_B \otimes 1_X} B \otimes X \\
 \parallel & (5.9) & \parallel \\
 B \xrightarrow{u_X \otimes 1_B} X \otimes B & & X \xrightarrow{1_X \otimes u_B} X \otimes B \\
 & & \downarrow \Theta \\
 & & X \otimes B
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes B \xrightarrow{\sigma_{X,B}} B \otimes X \xrightarrow{S_L \otimes S_L} B \otimes X & & B \otimes X \xrightarrow{S_L \otimes S_L} B \otimes X \\
 \downarrow \epsilon_X \otimes 1_B & & \downarrow \Delta \otimes 1_X \\
 B & \xrightarrow{1_B \otimes \epsilon_X} & B^2 \otimes X \\
 \downarrow S_L & & \downarrow 1_B \otimes \sigma_{B,X} \\
 B & \xleftarrow{\epsilon_B \otimes \epsilon_X \otimes 1_B} & B \otimes X \otimes B \\
 \parallel & (3.4) & \downarrow \triangleright \otimes 1_B \\
 B & \xleftarrow{\epsilon_X \otimes 1_B} & X \otimes B
 \end{array}$$

Furthermore, thanks to (3.3) one can show that $S_{X \rtimes B_L}$ and $S_{X \rtimes B_R}$ are antihomomorphisms of coalgebras and thanks to (3.3), (5.4), (5.5), (5.6) and (5.8) one can show that they are antihomomorphisms of algebras. Moreover, we check that the above construction satisfies the antipode conditions (2.8) thanks to the following two diagrams

$$\begin{array}{ccc}
 X \otimes B \xrightarrow{\Delta \otimes 1_B} X^2 \otimes B \xrightarrow{1_X \otimes S_R \otimes 1_B} X^2 \otimes B^2 \xrightarrow{1_X \otimes \sigma_{X,B} \otimes 1_B} (X \otimes B)^2 & & X \otimes B \xrightarrow{1_X \otimes 1_B \otimes S_{X \rtimes B_R}} X \otimes B^2 \xrightarrow{1_X \otimes 1_B \otimes S_R^2} X^2 \otimes B^2 \xrightarrow{1_X \otimes 1_B \otimes \Theta} (X \otimes B)^2 \\
 \parallel & & \parallel \\
 X^2 \otimes B^2 \xrightarrow{1_X \otimes S_R \otimes 1_B} X^2 \otimes B^2 \xrightarrow{1_X \otimes 1_X \otimes m} X^2 \otimes B \xrightarrow{1_X \otimes \sigma_{X,B}} X \otimes B \otimes X & & X^2 \otimes B^2 \xrightarrow{1_X \otimes \Theta} X^2 \otimes B \xrightarrow{m \otimes m} X \otimes B \\
 (2.8) & & (5.8) \\
 X \otimes B \xrightarrow{\Delta \otimes 1_B} X^2 \otimes B \xrightarrow{1_X \otimes S_R \otimes \epsilon_B} X^2 \otimes B \xrightarrow{1_X \otimes 1_X \otimes u_B} X^2 \otimes B \xrightarrow{m \otimes 1_B} X \otimes B & & \\
 & & (u_X \otimes u_B) \cdot (\epsilon_X \otimes \epsilon_B)
 \end{array}$$

$$\begin{array}{c}
\begin{array}{ccccccc}
X \otimes B & \xrightarrow{\Delta^2} & X^2 \otimes B^2 & \xrightarrow{\Delta^2} & (X \otimes B)^2 & \xrightarrow{\sigma_{X,B} \otimes 1_B} & B \otimes X^2 \otimes B \\
& \searrow \Delta^2 & \searrow \sigma_{X,B} \otimes 1_B & \searrow \sigma_{X,B} \otimes 1_B & \searrow \sigma_{X,B} \otimes 1_B & \searrow \sigma_{X,B} \otimes 1_B & \searrow \sigma_{X,B} \otimes 1_B \\
& & X^2 \otimes B^2 & \xrightarrow{S_L \otimes 1_X \otimes S_L} & X^2 \otimes B^2 & \xrightarrow{1_X \otimes 1_B} & X \otimes B^2 \\
& & & & & & \searrow \sigma_{X,B} \otimes 1_B \\
& & & & & & B \otimes X \otimes B \\
& & & & & & \searrow \Theta \otimes 1_B \\
& & & & & & X^2 \otimes B^2 \\
& & & & & & \searrow m \otimes m \\
& & & & & & X \otimes B
\end{array} \\
(2.8) \qquad \qquad \qquad (5.9) \qquad \qquad \qquad (5.6)
\end{array}$$

$(u_X \otimes u_B) \cdot (\epsilon_X \otimes \epsilon_B)$

Moreover, the conditions (5.8) and (5.6) imply that $X \xleftarrow[\pi_1]{\pi_2} X \rtimes B \xleftarrow[\pi_2]{\pi_1} B$ satisfies the condition (9') of the Definition 5.2 as it is shown in the diagram below.

$$\begin{array}{c}
\begin{array}{ccccccc}
(B \otimes X)^2 & \xrightarrow{1_X \otimes (1_B \otimes 1_X)^2 \otimes u_B} & (X \otimes B)^3 & \xrightarrow{1_X \otimes 1_B \otimes 1_X \otimes \Theta \otimes 1_B} & X \otimes B \otimes X^2 \otimes B^2 & \xrightarrow{1_X \otimes 1_B \otimes m^2} & (X \otimes B)^2 \\
& & & & & & \searrow 1_X \otimes \Theta \otimes 1_B \\
& & & & & & X^2 \otimes B^2 \\
& & & & & & \searrow m \otimes m \\
& & & & & & X \otimes B
\end{array} \\
(2.1) \qquad \qquad \qquad (5.6) \qquad \qquad \qquad (5.8) \qquad \qquad \qquad (2.1)
\end{array}$$

Finally, the conditions (10') and (11') hold thanks to (5.4), (5.5) and (3.14), and we can conclude that (5.10) is a split extension of Hopf algebras as defined in Definition 5.2.

On the other hand, if we have a split extension of Hopf algebras, we can define an action of Hopf algebras. Thanks to Proposition 3.9 and condition (9'), we can prove two identities which are crucial properties, for our purposes, of a split extension of Hopf algebras.

Lemma 5.5 *Let $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ be a split extension of Hopf algebras, we have*

$$\begin{aligned}
\lambda \cdot m \cdot (e \otimes \kappa) \cdot (m \otimes 1_X) &= \lambda \cdot m \cdot (e \otimes (\kappa \cdot \lambda)) \cdot (1_B \otimes m) \cdot (1_B \otimes e \otimes \kappa), \quad (5.11) \\
\lambda \cdot m \cdot (e \otimes \kappa) \cdot (1_B \otimes m) &= m \cdot (\lambda \otimes \lambda) \cdot (m \otimes m) \cdot (e \otimes \kappa \otimes e \otimes \kappa)
\end{aligned}$$

$$\cdot (1_B \otimes \sigma_{B,X} \otimes 1_X) \cdot (\Delta \otimes 1_X \otimes 1_X), \quad (5.12)$$

$$\begin{array}{ccccc}
B^2 \otimes X & \xrightarrow{1_B \otimes (\lambda \cdot m \cdot (e \otimes \kappa))} & B \otimes X & & \\
\downarrow m \otimes 1_X & & \downarrow \lambda \cdot m \cdot (e \otimes \kappa) & & \\
B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} & X & & \\
\downarrow \Delta \otimes 1_X \otimes 1_X & & \downarrow \lambda \cdot m \cdot (e \otimes \kappa) & & \\
B \otimes X \otimes X & \xrightarrow{\Delta \otimes 1_X \otimes 1_X} B^2 \otimes X^2 \xrightarrow{1_B \otimes \sigma_{B,X} \otimes 1_X} (B \otimes X)^2 & & & \\
\downarrow 1_B \otimes m & & \downarrow (\lambda \cdot m \cdot (e \otimes \kappa))^2 & & \\
B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} X & & & \\
\downarrow & & \downarrow m & & \\
B \otimes X & \xrightarrow{\lambda \cdot m \cdot (e \otimes \kappa)} X & & &
\end{array}$$

Proof Thanks to Proposition 3.9, the result follows as we can check by means of the Figs. 9 and 10, where we use that e, α, κ are morphisms of bialgebras.

This lemma implies that the action of bialgebras defined by (3.10) ($\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$) satisfies the conditions (5.2) and (5.3). Hence, this action becomes an action of Hopf algebras since the conditions (5.4) and (5.5) are given by the conditions (10') and (11').

$$\begin{array}{c}
\begin{array}{ccccccc}
B^2 \otimes X & \xrightarrow{1_B \otimes e \otimes \kappa} & B \otimes A^2 & \xrightarrow{1_B \otimes m} & B \otimes A & \xrightarrow{1_B \otimes \lambda} & B \otimes X \xrightarrow{e \otimes \kappa} A^2 \\
& \searrow 1_B \otimes e \otimes \kappa & & & & & \nearrow (e \cdot \alpha) \otimes (\kappa \cdot \lambda) \\
& & B \otimes A^2 & \xrightarrow{1_B \otimes m} & B \otimes A & \xrightarrow{e \otimes 1_A} & A^2
\end{array} \\
(1.1) + (1.2) \\
\begin{array}{ccccccccccc}
B^2 \otimes X & \xrightarrow{\Delta \otimes e \otimes \kappa} & B^2 \otimes A^2 & \xrightarrow{(1_B)^2 \otimes m} & B^2 \otimes A & \xrightarrow{(u_X \cdot \epsilon_B) \otimes e \otimes 1_A} & X \otimes A^2 & \xrightarrow{1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)} & X \otimes A^2 & \xrightarrow{1_X \otimes m} & X \otimes A & \xrightarrow{1_X \otimes \lambda} & X^2 \xrightarrow{m} X \\
& & \downarrow e \otimes e \otimes 1_A & & & & & & & & & & \downarrow m \\
& & A^3 & \xrightarrow{1_A \otimes (e \cdot \alpha) \otimes (\kappa \cdot \lambda)} & A^3 & \xrightarrow{1_A \otimes m} & A^2 & \xrightarrow{\lambda \otimes \lambda} & X^2 & \xrightarrow{m} & X \\
& & \uparrow \Delta \otimes 1_A & & & & & & & & & \uparrow \lambda \\
& & A^2 & & & & & & & & & A
\end{array} \\
(2.9) \\
\begin{array}{ccccccc}
B^2 \otimes X & \xrightarrow{e \otimes e \otimes \kappa} & A^3 & \xrightarrow{1_A \otimes m} & A^2 & \xrightarrow{m} & A \\
& \searrow e \otimes e \otimes \kappa & & & & & \\
& & A^3 & \xrightarrow{m \otimes 1_A} & A^2 & &
\end{array} \\
(9') \\
\begin{array}{ccccccc}
B^2 \otimes X & \xrightarrow{m \otimes 1_X} & B \otimes X & \xrightarrow{e \otimes \kappa} & A^2 & \xrightarrow{m} & A
\end{array}
\end{array}$$

Fig. 9 Proof of (5.11)

Fig. 10 Proof of (5.12)

Remark 5.6 The construction of the action of Hopf algebras given by $\triangleright = \lambda \cdot m \cdot (e \otimes \kappa)$ can be reformulated without λ , when we compose it by κ . Indeed, by pre-composing by $(1_B \otimes 1_X \otimes e) \cdot (1_B \otimes 1_X \otimes S_R) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X)$ and post-composing by m , the two components of the equality (3.11), we obtain the following equality

$$\kappa \cdot \triangleright = m \cdot (m \otimes 1_A) \cdot (e \otimes \kappa \otimes e) \cdot (1_B \otimes 1_X \otimes S_R) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X). \quad (5.13)$$

We notice that thanks to the condition (8'), this is equivalent to

$$\kappa \cdot \triangleright = m \cdot (1_A \otimes m) \cdot (e \otimes \kappa \otimes e) \cdot (1_B \otimes 1_X \otimes S_R) \cdot (1_B \otimes \sigma_{B,X}) \cdot (\Delta \otimes 1_X).$$

When the symmetric monoidal category is Vect_K , κ can be viewed as an inclusion and (5.13) gives us a way to construct the action without λ .

For the sake of clarity, we give an explicit description of the morphisms of split extensions and actions of Hopf algebras in $\mathcal{C}$.

Definition 5.7 A morphism of split extensions of Hopf algebras from

$X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ to $X' \xleftarrow[\kappa']{\lambda'} A' \xleftarrow[\alpha']{e'} B'$ is given by three morphisms of Hopf algebras $g: B \rightarrow B'$, $v: X \rightarrow X'$ and $p: A \rightarrow A'$ such that the following diagram commutes in $\mathcal{C}$

$$\begin{array}{ccccc} X & \xleftarrow[\kappa]{\lambda} & A & \xleftarrow[\alpha]{e} & B \\ v \downarrow & & p \downarrow & & g \downarrow \\ X' & \xleftarrow[\kappa']{\lambda'} & A' & \xleftarrow[\alpha']{e'} & B' \end{array} \quad (5.14)$$

Definition 5.8 Let $\triangleright: B \otimes X \rightarrow X$ and $\triangleright': B' \otimes X' \rightarrow X'$ be two actions of Hopf algebras. A morphism between them is defined as a pair of morphisms of Hopf algebras $g: B \rightarrow B'$ and $v: X \rightarrow X'$ such that

$$v \cdot \triangleright = \triangleright' \cdot (g \otimes v).$$

The actions of Hopf algebras (Definition 5.3) endowed with the morphisms of actions of Hopf algebras (Definition 5.8) form the category $\text{Act}(\text{Hopf}_{\mathcal{C}})$ of actions of Hopf algebras in $\mathcal{C}$. The split extensions of Hopf algebras with the morphisms given by Definition 5.7 form the category $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ of split extensions of Hopf algebras in $\mathcal{C}$.

Theorem 5.9 Let $\mathcal{C}$ be a symmetric monoidal category. There is an equivalence between $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ the category of split extensions of Hopf algebras in $\mathcal{C}$ and $\text{Act}(\text{Hopf}_{\mathcal{C}})$ the category of actions of Hopf algebras in $\mathcal{C}$.

Proof Let (g, v, p) be a morphism in $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$, then it is clear that (v, g) is a morphism in $\text{Act}(\text{Hopf}_{\mathcal{C}})$. On the other hand, if (v, g) is a morphism of actions of Hopf algebras, the triple $(g, v, v \otimes g)$ is a morphism of split extensions of Hopf algebras since $v \otimes g$ preserves the antipode. Moreover, the isomorphisms $\varphi := m \cdot (\kappa \otimes e)$ and $\psi := (\lambda \otimes \alpha) \cdot \Delta$ (in (3.14)) form an isomorphism in $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ since they are morphisms of Hopf algebras, as we can see in the following diagram

$$\begin{array}{ccccccc}
 X \otimes B & \xrightarrow{\sigma_{X,B}} & B \otimes X & \xrightarrow{S_L \otimes S_L} & B \otimes X & \xrightarrow{\Delta \otimes 1_X} & B^2 \otimes X \xrightarrow{1_B \otimes \sigma_{B,X}} B \otimes X \otimes B \xrightarrow{\triangleright \otimes 1_B} X \otimes B \\
 \downarrow \kappa \otimes e & & & & \downarrow e \otimes \kappa & & \downarrow \kappa \otimes e \\
 & & & & A^2 & & A^2 \\
 & & & & \downarrow m & & \downarrow m \\
 A^2 & \xrightarrow{m} & A & \xrightarrow{S_L} & A & \xlongequal{\quad\quad\quad} & A
 \end{array} \quad (3.11)$$

where the left square commutes since S_L is an antihomomorphism of algebras (a similar computation holds for S_R). In conclusion, we obtain our statement thanks to the observations about the split extension (5.10) and Lemma 5.5. $\square$

Remark 5.10 Whenever we consider Hopf algebras such that $S_L = S_R$, the semi-direct product in (5.10) also satisfies this property ($S_{X \rtimes B_L} = S_{X \rtimes B_R}$) and Theorem 5.9 can be restricted to such Hopf algebras. Moreover, the condition (11') in Definition 5.2 is trivially satisfied thanks to (10') and (5.11), and as we have already noticed the condition (5.5) in Definition 5.3 always holds.

In the case of associative Hopf algebras, we can define $\text{SplitExt}(\text{AssHopf}_{\mathcal{C}})$ and $\text{Act}(\text{AssHopf}_{\mathcal{C}})$. The actions of associative Hopf algebras are actions of Hopf algebras where the conditions (5.4) and (5.5) always hold thanks to the uniqueness of the antipode. A split extension of associative Hopf algebras is the same as in $\text{SplitExt}(\text{Hopf}_{\mathcal{C}})$ where the conditions (6'), (7'), (8') and (9') become trivial. Moreover, the conditions (10') and (11') are not required, they become properties that any split extension of associative Hopf algebras has.

Corollary 5.11 Let $\mathcal{C}$ be a symmetric monoidal category. There is an equivalence between $\text{SplitExt}(\text{AssHopf}_{\mathcal{C}})$ the category of split extensions of associative Hopf algebras in $\mathcal{C}$ and $\text{Act}(\text{AssHopf}_{\mathcal{C}})$ the category of actions of associative Hopf algebras in $\mathcal{C}$.

Let us notice that, since any morphism in $\text{SplitExt}(\text{Hopf}_C)$, $\text{SplitExt}(\text{AssHopf}_C)$ and $\text{SplitExt}(\text{AssBiAlg}_C)$ is a morphism in $\text{SplitExt}(\text{BiAlg}_C)$, the Split Short Five Lemma also holds in these categories.

In some sense, this result, in the associative Hopf algebras, is similar to a property obtained for the *exact cleft sequences* of associative Hopf K -algebras (with bijective antipodes) investigated by [3] (see Lemma 3.2.19). We would like to emphasize the differences and shared properties between the definition of an exact cleft sequence of associative Hopf algebras and the definition of split extension of associative Hopf algebras (Definition 5.2), in the symmetric monoidal category Vect_K of vector spaces. First, we recall the definition of an exact cleft sequence of associative Hopf algebras [2].

Definition 5.12 A sequence of morphisms of associative Hopf algebras

$$A' \xrightarrow{\iota} C' \xrightarrow{\pi} B' \quad (5.15)$$

is *exact* if

- (1) ι is injective,
- (2) π is surjective,
- (3) $\ker(\pi) = C'\iota(A')^+ (\ker(\pi))$ is the kernel in Vect_K and $\iota(A')^+ = \{x \in \iota(A') \mid \epsilon_{C'}(x) = 0\}$,
- (4) $\iota(A') = L\ker(\pi) = \{x \in C' \mid (\pi \otimes 1_{C'}) \cdot \Delta(x) = u_{B'} \otimes x\}$.

Definition 5.13 Let (5.15) be an exact sequence of associative Hopf algebras, then the sequence (ι, π)

$$A' \xleftarrow[\iota]{\xi} C' \xleftarrow[\pi]{\chi} B' \quad (5.16)$$

is *cleft* if and only if there exist a morphism of A' -modules $\xi : C' \rightarrow A'$ (i.e. the equality $\xi \cdot m \cdot (\iota \otimes 1_{C'}) = m \cdot (1_{A'} \otimes \xi)$ holds) and a morphism of B' -comodules $\chi : B' \rightarrow C'$ (i.e. the equality $(\pi \otimes 1_{C'}) \cdot \Delta \cdot \chi = (1_{B'} \otimes \chi) \cdot \Delta$ is satisfied) such that the following two equations hold

$$\xi \cdot \chi = u_{A'} \cdot \epsilon_{B'}, \quad (5.17)$$

$$m \cdot ((\iota \cdot \xi) \otimes (\chi \cdot \pi)) \cdot \Delta = 1_{C'}. \quad (5.18)$$

Remark that in [2,3] the above definition is not the definition of an exact cleft sequence, but it is equivalent to it by Lemma 3.1.14 in [2]. It is straightforward to observe that the conditions (5.17) and (5.18) of the sequence (5.16) are the same as the conditions (2')

and (3') in Definition 5.2. Moreover, let $X \xleftarrow[\kappa]{\lambda} A \xleftarrow[\alpha]{e} B$ be a split extension of associative Hopf algebras, then λ is a X -module morphism (thanks to Proposition 3.9) and e is a B -comodule morphism.

However, there are major differences. Indeed, a split extension of associative Hopf algebras (Definition 5.2) is (in general) not exact in the sense of [2]. Conversely, the Hopf algebra morphism π in the exact cleft sequence (5.16) is not a split epimorphism of Hopf algebras, since χ is neither a morphism of algebras nor a morphism of coalgebras (see [3] for such an example). Then, it is clear that one definition does not imply the other and vice versa. Nevertheless, there are sequences of associative Hopf algebras that are exact cleft sequences and split extensions of associative Hopf algebras. For example, any exact sequence of associative

Hopf algebras (5.15) such that π is a split epimorphism is an example of both definitions (see example 2) in $Vect_K$.

To end this paper, we investigate the two main symmetric monoidal categories of interest: Set and $Vect_K$. On the one hand, we specify our results in Set .

Split Extensions of Hopf Algebras in the Category of Sets

Example 5.14 Any split extension of groups (1.1) is a split extension of associative Hopf algebras when the symmetric monoidal category is Set . In particular, Corollary 5.11 becomes the well-known equivalence of categories between split extensions of groups and group actions.

Example 5.15 In Set , non-associative Hopf algebras will be structures given by a set G , with a non-associative multiplication, a neutral element 1, left inverses and right inverses such that

$$g_L^{-1}g = 1 = gg_R^{-1}. \quad (5.19)$$

In particular, the non-zero octonions [10,15] are equipped with a non-associative multiplication satisfying (5.19). This structure is quite general. A special case is given by the structure of loops since any loop satisfies (5.19). We can describe what split extensions of this algebraic structure should be in order to be equivalent to actions of such an algebraic structure. Indeed, a split extension should be a split morphism of these algebraic structures

$X \xrightarrow[\kappa]{} A \xleftarrow[\alpha]{} B$ such that the following conditions are satisfied for any $a \in A, b \in B, x \in X$

- (3') $(a(e \cdot \alpha)(a_R^{-1}))(e \cdot \alpha)(a) = a$
- (6') $(\kappa(x)e(b))a = \kappa(x)(e(b)a)$
- (7') $(\kappa(x)a)e(b) = \kappa(x)(ae(b))$
- (8') $a(\kappa(x)e(b)) = (a\kappa(x))e(b)$
- (9') $(e(b)a)\kappa(x) = e(b)(a\kappa(x))$
- (10') $e(b_R^{-1})_L^{-1} = e(b)$
- (11') $\left(e(b_R^{-1})\left(e(b_R^{-1})_R^{-1}(x_R^{-1}e(b_R^{-1}))\right)\right)e(b_R^{-1})_R^{-1} = x_R^{-1}.$

The other conditions are trivially satisfied with $\lambda(a) = a(e \cdot \alpha)(a_R^{-1})$.

On the other hand, in the symmetric monoidal category $Vect_K$ of vector spaces over a field K , we give some particular cases of associative split extensions of Hopf algebras.

Split Extensions of Hopf Algebras in the Category of Vector Spaces

(1) We consider a split epimorphism α of associative Hopf K -algebras,

$$HKer(\alpha) \xrightarrow{\kappa_\alpha} A \xleftarrow[\alpha]{} B, \quad (5.20)$$

where $HKer(\alpha)$ is the kernel of α in the category Hopf_K of Hopf K -algebras, κ_α stands for the equalizer in $Vect_K$ of $(1_A \otimes u_B \otimes 1_A) \cdot \Delta$ and $(1_A \otimes \alpha \otimes 1_A) \cdot (\Delta \otimes 1_A) \cdot \Delta$,

$$HKer(\alpha) \xrightarrow{\kappa_\alpha} A \xrightarrow[(1_A \otimes \alpha \otimes 1_A) \cdot (\Delta \otimes 1_A) \cdot \Delta]{(1_A \otimes u_B \otimes 1_A) \cdot \Delta} A \otimes B \otimes A. \quad (5.21)$$

We recall that the equalizer in Vect_K of $f, g: A \rightarrow B$ is given by $\{a \in A \mid f(a) = g(a)\}$. We also define the following equalizers

$$\begin{aligned} LKer(\alpha) &\xrightarrow{\kappa_{\alpha,L}} A \xrightarrow[(\alpha \otimes 1_A) \cdot \Delta]{u_B \otimes 1_A} B \otimes A, \\ RKer(\alpha) &\xrightarrow{\kappa_{\alpha,R}} A \xrightarrow[(1_A \otimes \alpha) \cdot \Delta]{1_A \otimes u_B} A \otimes B. \end{aligned}$$

Proposition 5.16 A split epimorphism (5.20) satisfying the condition $HKer(\alpha) = LKer(\alpha)$ is an extension of associative Hopf algebras (Definition 5.2) in the symmetric monoidal category Vect_K .

Proof First, we recall that for any morphism α in AssHopf_K it is well-known that the following conditions are equivalent (see [2])

- $HKer(\alpha) = LKer(\alpha)$,
- $HKer(\alpha) = RKer(\alpha)$,
- $LKer(\alpha) = RKer(\alpha)$,
- $LKer(\alpha)$ is an associative Hopf algebra,
- $RKer(\alpha)$ is an associative Hopf algebra.

Let $A \xleftarrow[\alpha]{e} B$ be a split epimorphism of associative Hopf algebras satisfying the condition $HKer(\alpha) = LKer(\alpha)$. Since A is an associative Hopf algebra, we can define the following section of $\kappa_\alpha: HKer(\alpha) \rightarrow A$

$$\lambda = m \cdot (1_A \otimes (S \cdot e \cdot \alpha)) \otimes \Delta.$$

First, we use the condition $HKer(\alpha) = RKer(\alpha)$ on the kernel to prove that λ factors through $HKer(\alpha)$,

By using that λ factors through $RKer(\alpha) = HKer(\alpha)$, we prove that $\Delta \cdot \lambda = (\lambda \otimes \lambda) \cdot \Delta$ in Fig. 11. Indeed, the central rectangle commutes since $RKer(\alpha) = HKer(\alpha)$, the commutativity of the part (A) is clarified in Fig. 13 (in the appendix).

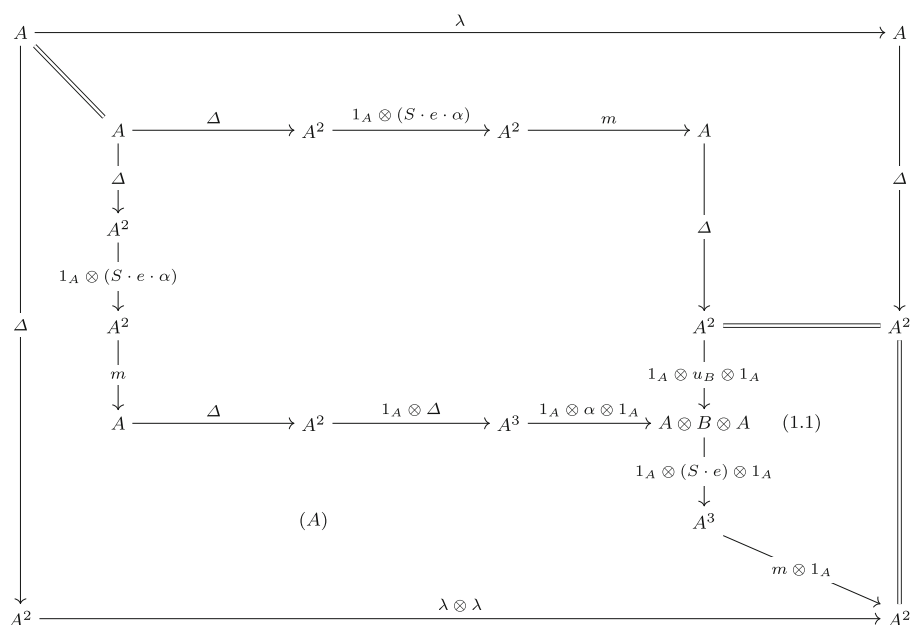
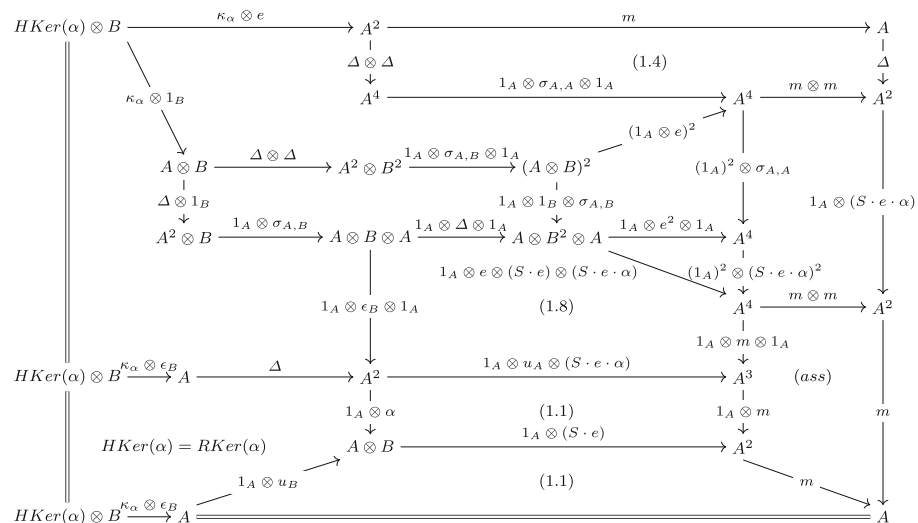

 Fig. 11 λ is a morphism of coalgebras


Fig. 12 Condition (4')

The condition (3') is trivially respected. The condition (4') is also satisfied by this definition of λ thanks to the commutativity of Fig. 12, where we use that $HKer(\alpha) = RKer(\alpha)$,

The last condition (5') is left to the reader, to prove it we use the fact that $\lambda \cdot m \cdot (e \otimes \kappa_\alpha)$ factors through $HKer(\alpha)$.

To conclude, it is a split extension of associative Hopf algebras. $\square$

Notice that this proposition can be extended to any symmetric monoidal category with equalizers that are preserved by all endofunctors on $\mathcal{C}$ of the form $- \otimes X$ and $X \otimes -$.

2) In the symmetric monoidal category $(Vect_K, \otimes, K)$, an exact sequence of associative Hopf algebras (Definition 5.12)

$$A' \xrightarrow{\iota} C' \xrightleftharpoons[\pi]{e} B', \quad (5.22)$$

such that π is a split epimorphism of Hopf algebras, is an exact cleft sequence and a split extension of associative Hopf algebras. Indeed, since the condition 4) in Definition 5.12 is equivalent to the condition $LKer(\pi) = HKer(\pi)$ [2], this example is a particular case of Proposition 5.16. Due to Definition 5.12 the sequence (5.22) has to be isomorphic to the following one,

$$HKer(\pi) \xrightarrow{\kappa\pi} C' \xrightleftharpoons[\pi]{e} \frac{C'}{C'(LKer(\pi))^+},$$

where $HKer(\pi) = LKer(\pi)$.

3) If we consider cocommutative associative Hopf K -algebras, then we can drop the condition $HKer(\alpha) = RKer(\alpha)$ in Proposition 5.16. So any split epimorphism of cocommutative associative Hopf algebras induces a split extension as defined in 5.2 (and an exact cleft sequence). The Corollary 5.11 becomes the well-known equivalence between points over B and B -module Hopf algebras [25].

6 Conclusion

To sum up, we defined the category $SplitExt(BiAlg_{\mathcal{C}})$ and proved that this category is equivalent to the category $Act(BiAlg_{\mathcal{C}})$. Moreover, we proved that a suitable version of the Split Short Five Lemma holds when the split extensions occurring in it belong to the category $SplitExt(BiAlg_{\mathcal{C}})$. These results were proved to hold also in the category of Hopf algebras, and we gave some examples of split extensions of Hopf algebras in the categories of sets and of vector spaces. It is worthwhile to observe that any isomorphism $\gamma : A \rightarrow A$ of Hopf algebras determines a split extension in the sense of Definition 5.2 as indicated in the following diagram

$$I \xrightleftharpoons[u]{\epsilon_A} A \xrightleftharpoons[\gamma]{\gamma^{-1}} A.$$

This elementary example motivates the study of internal structures in the context of non-associative bialgebras and Hopf algebras. Indeed, thanks to this example, a discrete reflexive graph

$$I \xrightleftharpoons[u]{\epsilon_A} A \xrightleftharpoons[1_A]{1_A} A, \quad (6.1)$$

is a reflexive graph in $Hopf_{\mathcal{C}}$ such that its “legs” are in $SplitExt(Hopf_{\mathcal{C}})$. Such an internal structure is different from the internal structure called $pre-cat^1\text{-Hopf algebra}$ in [11]. Indeed, the discrete reflexive graph (6.1) is not a $pre-cat^1\text{-Hopf algebra}$ without asking that A is cocommutative. The example (6.1) suggests that the adequate internal notion corresponding to a precrossed module of Hopf Algebras (as defined in [18]) is the one of a reflexive graph

such that one of the two “legs” is a split extension of Hopf algebras. In a forthcoming paper, we will construct an equivalence of categories between these two structures, and we will investigate the equivalence of categories between Hopf crossed modules (as defined in [18]) and internal structures that we will call *cat¹-Hopf algebras*. Similarly to what we did in this paper we will work with non-associative bialgebras, associative bialgebras, non-associative Hopf algebras and associative Hopf algebras in any symmetric monoidal category.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgements The author would like to warmly thank her supervisors Marino Gran and Joost Vercruysse for all the advice in the realization of this paper. Many thanks also to George Janelidze for the suggestion to explore split extensions and actions beyond the vector spaces case, in the context of monoidal categories. This led to a real improvement of this paper. The author also would like to thank Manuela Sobral for the invitation to the University of Coimbra and for the useful discussions. The author thanks the anonymous referee for his/her useful remarks and suggestions. The author’s research is supported by a FRIA (*Fonds pour la formation à la recherche dans l’industrie et dans l’agriculture*) doctoral grant no. 27485 of the *Communauté française de Belgique*.

Declarations

Conflicts of interest/Competing interests NA

Availability of data and material NA

Code availability NA

Appendix

This appendix contains five figures given below. The monoidal product is denoted by juxtaposition. Figure 13 is used in the proof of Proposition 5.16. By combining the diagrams of Figs. 14 and 15, we show that the structure of $X \rtimes B$ as defined in (3.6) gives a bialgebra structure. Thanks to the commutativity of the diagram of Fig. 16, we can conclude that whenever X and B are associative bialgebras and (4.1) and (4.2) are satisfied $m_{X \rtimes B}$ as defined in (3.6) is associative, which is a part of the proof of Lemma 4.3. Finally, the commutativity of Fig. 17 allows one to prove Proposition 3.9.

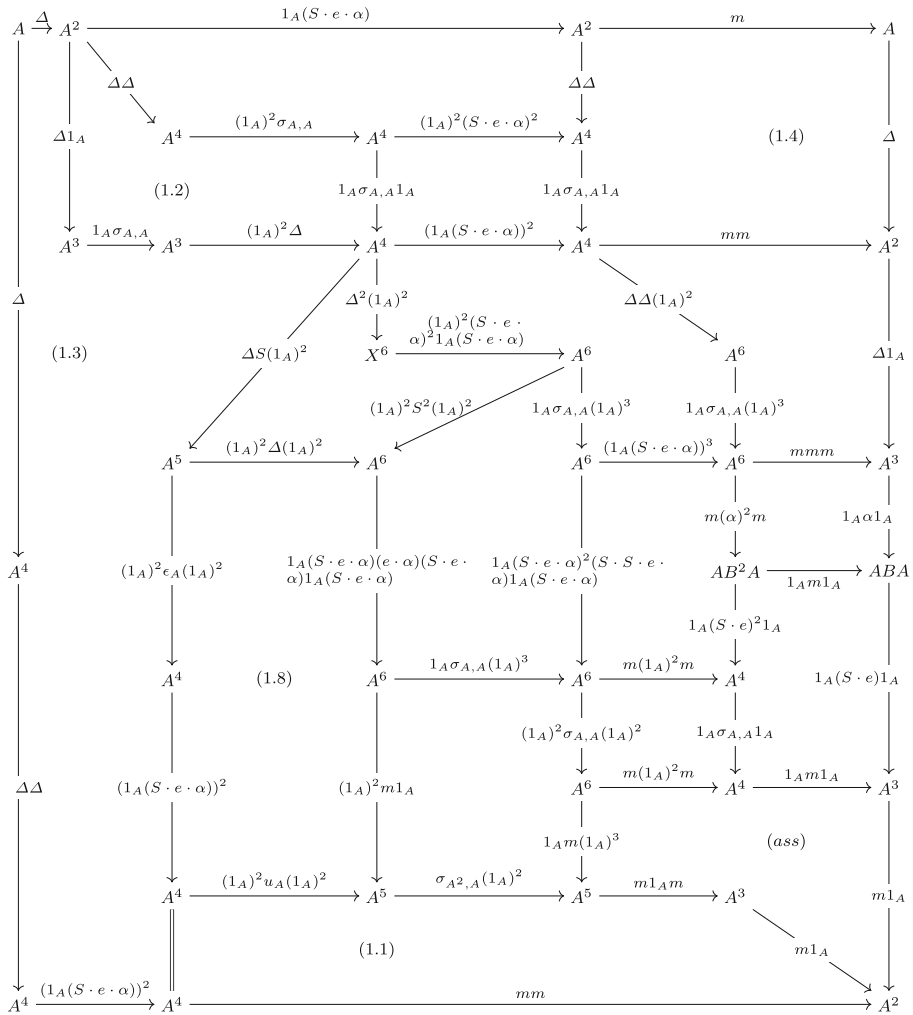


Fig. 13 Commutation of the diagram (A)

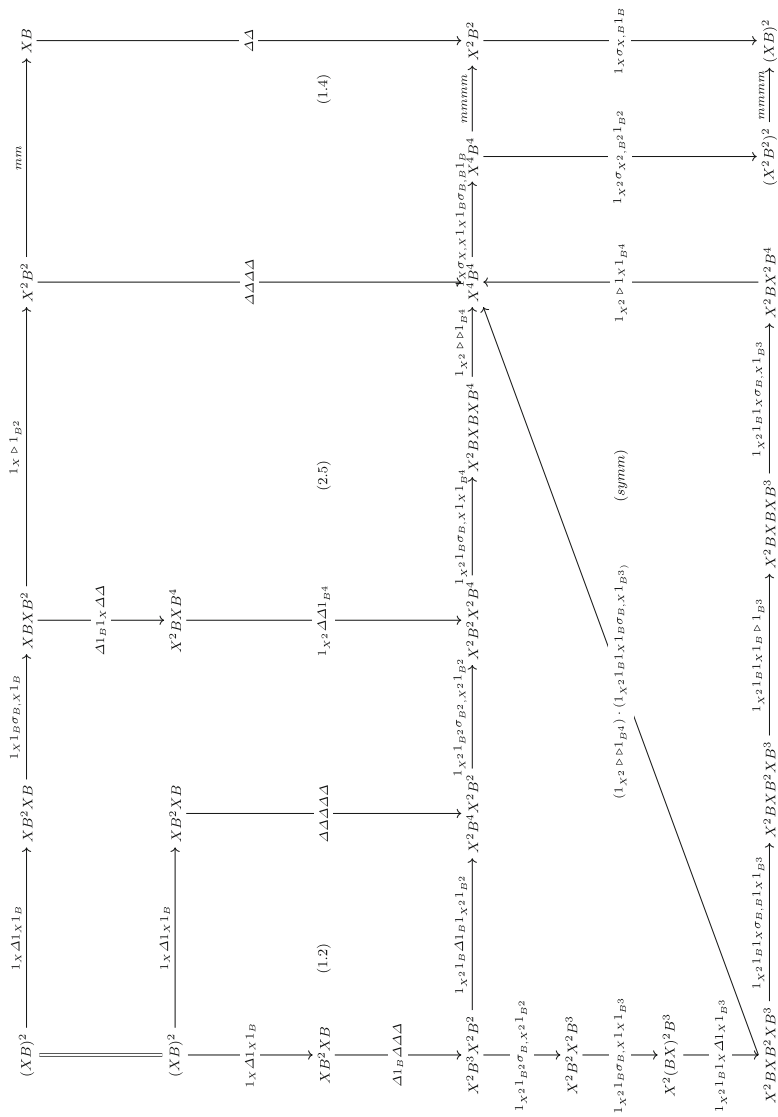


Fig. 14 The semi-direct product is a bialgebra: part 1





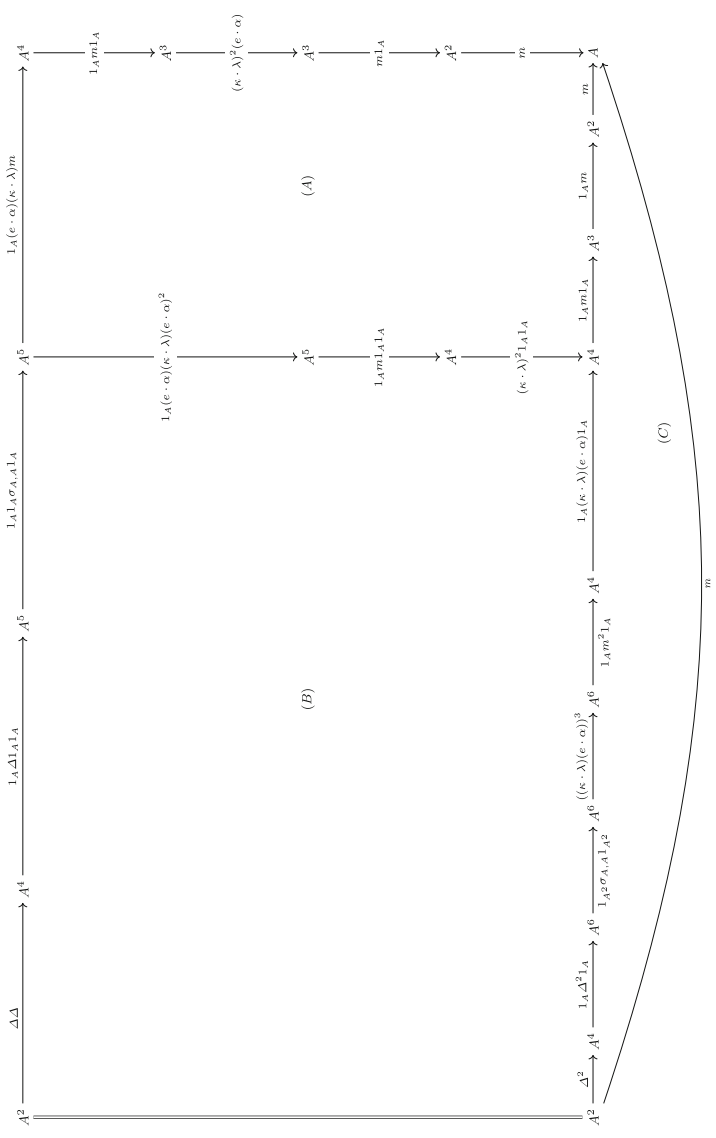


Fig. 17 Combination of the three diagrams (A), (B) and (C)

References

1. Albuquerque, H., Majid, S.: Quasialgebra structure of the octonions. *J. Algebra* **220**(1), 188–224 (1999)
2. Andruskiewitsch, N.: Notes on extensions of Hopf algebras. *Can. J. Math.* **48**(1), 3–42 (1996)
3. Andruskiewitsch, N., Devoto, J.: Extensions of Hopf algebras. *Algebra i Analiz.* **7**(1), 22–61 (1995)
4. Böhm, G.: Crossed modules of monoids II. *Appl. Categ. Struct.* **28**, 601–653 (2020)
5. Bourn, D., Janelidze, G.: Protomodularity, descent, and semidirect products. *Theory Appl. Categ.* **4**(2), 37–46 (1998)
6. Borceux, F., Janelidze, G., Kelly, G.M.: Internal object actions. *Comment. Math. Univ. Carol.* **46**(2), 235–255 (2005)
7. Bourn, D., Martins-Ferreira, N., Montoli, A., Sobral, M.: Schreier split epimorphisms in monoids and semirings. *Textos de Matemática Série B, Departamento de Matemática da Universidade de Coimbra* **45**, (2014)
8. Caenepeel, S., Goyvaerts, I.: Monoidal Hom–Hopf Algebras. *Commun. Algebra* **39**(6), 2216–2240 (2011)
9. Caenepeel, S., De Lombaerde, M.: A categorical approach to Turaev’s Hopf group-coalgebras. *Commun. Algebra* **34**(7), 2631–2657 (2006)
10. Cayley, A.: On Jacobi’s Elliptic functions, in reply to the Rev. Brice Bronwin; and on Quaternions. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **26**(3), 208–211 (1845)
11. Fernandez Vilaboa, J.M., Lopez Lopez, M.P., Villanueva Novoa, E.: Cat^1 -Hopf algebras and crossed-modules. *Commun. Algebra* **35**(1), 181–191 (2006)
12. Gran, M., Janelidze, G., Sobral, M.: Split extensions and semidirect products of unitary magmas. *Comment. Math. Univ. Carolin.* **60**(4), 509–527 (2019)
13. Gran, M., Kadjjo, G., Vercruysse, J.: A torsion theory in the category of cocommutative Hopf algebras. *Appl. Categ. Struct.* **24**, 269–282 (2016)
14. Gran, M., Sterck, F., Vercruysse, J.: A semi-abelian extension of a theorem by Takeuchi. *J. Pure Appl. Algebra* **223**(10), 4171–4190 (2019)
15. Grave, J.: On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **26**(3), 315–320 (1845)
16. Janelidze, G., Márki, L., Tholen, W.: Semi-Abelian categories. *J. Pure Appl. Algebra* **168**, 367–386 (2002)
17. Mac Lane, S.: *Categories for the Working Mathematician*. Springer (1978)
18. Majid, S.: Strict quantum 2-groups, preprint, [arXiv:1208.6265](https://arxiv.org/abs/1208.6265) (2012)
19. Majid, S.: *Foundations of Quantum Group Theory*. Cambridge Univ. Press (1995)
20. Majid, S.: Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicross product construction. *J. Algebra* **130**, 17–64 (1990)
21. Makhlouf, A., Silvestrov, S.D.: Hom–Lie admissible Hom-coalgebras and Hom–Hopf algebras. In: *Generalized Lie Theory in Mathematics, Physics and Beyond*. Springer, Berlin, Chap. 17, pp. 189–206 (2009)
22. Molnar, R.: Semi-direct products of Hopf algebras. *J. Algebra* **47**, 29–51 (1977)
23. Patchkoria, A.: Crossed semimodules and Schreier internal categories in the category of monoids. *Georgian Math. J.* **5**(6), 575–581 (1998)
24. Turaev, V.G.: Homotopy field theory in dimension 3 and crossed group-categories, preprint [arXiv:math.GT/0005291](https://arxiv.org/abs/math.GT/0005291) (2000)
25. Vespa, C., Wambst, M.: On some properties of the category of cocommutative Hopf algebras. *North-Western Eur. J. Math.* 21–37 (2018)