Journal of Algebra and Its Applications
(2023) 2350252 (27 pages)

DOI: 10.1142/S0219498823502523

S-protomodularity of the category of cocommutative bialgebras

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Received 18 January 2022
Revised 18 July 2022
Accepted 19 July 2022
Published

Communicated by A. Facchini

We prove that the category of cocommutative bialgebras in any locally presentable symmetric monoidal category is an S-protomodular category with respect to a particular class of split extensions of cocommutative bialgebras. We also obtain the "partial" well-known *Smith is Huq* condition, meaning that two effective S-equivalence relations centralize each other as soon as the normal subobjects associated with them commute in the sense of Huq.

Keywords: Cocommutative bialgebras, split extensions, protomodularity, commutators, symmetric monoidal categories.

Mathematics Subject Classification 2020: 18E13, 16T10, 18M05

0. Introduction

The notion of protomodular category was introduced by Bourn in [4]. The categories of groups, Lie algebras, crossed modules, rings, the dual of the category of sets are examples of protomodular categories. In the pointed case, a category $\mathcal{C}$ is protomodular if and only if the Split Short Five Lemma holds. This means that for any diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "\kappa_1"] \arrow[d, "v"'] & A_1 \arrow[r, shift right, "\alpha_1"'] \arrow[d, "p"'] & B_1 \arrow[l, shift right, "e_1"'] \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "\kappa_2"] & A_2 \arrow[r, shift right, "\alpha_2"'] & B_2 \arrow[l, shift right, "e_2"'] \arrow[r] & 0
\end{tikzcd} \tag{0.1}$$

where κ_i is the kernel of α_i and $\alpha_i \cdot e_i = 1_{B_i}$ for any $i \in \{1, 2\}$, then p is an isomorphism whenever v and g are.

It is well known that the category of internal groups in a finitely complete category is protomodular [4]. In particular, this result implies that the category of cocommutative Hopf algebras in a symmetric monoidal category, that has equalizers, is protomodular, seen as internal groups in the category of cocommutative coalgebras. The category of cocommutative Hopf algebras over a field is even semi-abelian [15].

In this paper, we are interested in cocommutative bialgebras in any locally presentable symmetric monoidal category. As a matter of fact, it was proven in [13] that the category of cocommutative K-bialgebras is not a protomodular category. Similarly, the category of monoids is not a protomodular category. However, a class of split epimorphisms of monoids, called *Schreier split epimorphisms*, turned out to have some very interesting properties, similar to the ones of split epimorphisms of groups. For example, *Schreier split epimorphisms* are equivalent to the actions of monoids, the Split Short Five Lemma holds, etc. [7, 8]. The authors in [7] introduced the notion of S-protomodularity, that is the protomodularity with respect to a class S of split epimorphisms. It implies that we have the Split Short Five Lemma as in (0.1) whenever (α_1, e_1) and (α_2, e_2) are in the class S. The main example is the category of monoids with the class of *Schreier split epimorphisms*.

In [22], we defined a notion of split extensions for (non-associative) bialgebras (and non-associative Hopf algebras) such that they have "group-like" properties. More precisely, we showed that this definition of split extension of (non-associative) bialgebras is equivalent to the notion of action of (non-associative) bialgebras. Moreover, we proved the validity of the Split Short Five Lemma for these kinds of split extensions. In particular, these results restrict to the case of cocommutative bialgebras.

Hence, it is natural to hope that the category of cocommutative bialgebras in any symmetric monoidal category is S-protomodular with respect to the class introduced in [22].

In a finitely complete and pointed category, we can define two types of centrality: the centrality of equivalence relations in the sense of Smith and the centrality of normal monomorphisms in the sense of Huq. Note that these notions of centrality are not independent. If two effective equivalence relations centralize each other (in the sense of Smith) then the corresponding normal monomorphisms necessarily centralize in the sense of Huq [6].

The converse is not true in general. If $\mathcal{C}$ is a category such that any two effective equivalence relations always centralize each other as soon as their normalizations centralize in the sense of Huq, one says that $\mathcal{C}$ satisfies the so-called *Smith is Huq* property [6, 20].

In [18], the authors introduced the notion of *Smith is Huq* for pointed S-protomodular categories. In that context, the *Smith is Huq* property can be expressed as follows: two effective S-equivalence relations centralize each other if and only if their associated normal subobjects commute (in the sense of Huq).

In this paper, we prove that the category of cocommutative bialgebras in any locally presentable symmetric monoidal category is S-protomodular with respect to the class of split extensions introduced in [22], we give a description of the centrality in the sense of Huq for two subbialgebras of a cocommutative bialgebra and we examine the *Smith is Huq* condition for cocommutative bialgebras. The results obtained in this paper generalize results in [8, 18] on the category of monoids.

The layout of this paper is as follows: Sec. 1 contains some preliminaries on cocommutative bialgebras in a symmetric monoidal category.

In Sec. 2, we prove that the category of cocommutative bialgebras in any locally presentable symmetric monoidal category is S_{coc}-protomodular with respect to the class S_{coc} of split extensions of cocommutative bialgebras defined in [22].

In Sec. 3, we describe the Huq commutator of two subbialgebras of a cocommutative bialgebra.

In the last section, we investigate the *Smith is Huq* condition for cocommutative bialgebras by using the results of [18].

1. Preliminaries

1.1. *Bialgebras in a symmetric monoidal category*

We recall that a *monoidal category* is given by a triple $(\mathcal{C}, \otimes, I)$ where $\mathcal{C}$ is a category, $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a bifunctor and I is the identity element (we omit to explicit the three natural isomorphisms, the associator, the right unit and the left unit).

A *braided monoidal category* is a 4-tuple $(\mathcal{C}, \otimes, I, \sigma)$ where $(\mathcal{C}, \otimes, I)$ is a monoidal category and σ is a *braiding.* A braiding consists of a family of natural isomorphisms $\sigma_{X,Y}\colon X \otimes Y \to Y \otimes X$ satisfying

$$\sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes 1_Y) \cdot (1_X \otimes \sigma_{Y,Z}),$$

$$\sigma_{X,Y\otimes Z} = (1_Y \otimes \sigma_{X,Z}) \cdot (\sigma_{X,Y} \otimes 1_Z).$$

A braided monoidal category is called *symmetric* when

$$\sigma^{-1}_{Y,X} = \sigma_{X,Y}. \tag{1.1}$$

In this paper, we omit the indexes of the braiding when this does not bring any confusion.

An *algebra* in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$ is given by an object $A \in \mathcal{C}$ endowed with a morphism $m_A \colon A \otimes A \to A$, called the multiplication. An algebra is associative and unital when there is a morphism $u_A \colon I \to A$ called the unit, such that the following equalities are satisfied

$$m_A \cdot (u_A \otimes 1_A) = 1_A = m_A \cdot (1_A \otimes u_A), \tag{1.2}$$

$$m_A \cdot (m_A \otimes 1_A) = m_A \cdot (1_A \otimes m_A) \tag{1.3}$$

$$\begin{tikzcd} A \arrow[r, "u_A \otimes 1_A"] \arrow[dr, equal] & A \otimes A \arrow[d, "m_A"] & A \arrow[l, "1_A \otimes u_A"'] \arrow[dl, equal] \\ & A. & \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A \arrow[r, "m_A \otimes 1_A"] \arrow[d, "1_A \otimes m_A"] & A \otimes A \arrow[d, "m_A"] \\ A \otimes A \arrow[r, "m_A"] & A. \end{tikzcd}$$

All the algebras that we will consider in this paper are associative and unital. A *morphism of algebras* $f \colon A \to B$ is a morphism in $\mathcal{C}$ such that the following diagrams commute

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "u_A"] \arrow[dr, "u_B"] & A \arrow[d, "f"] \\ & B. \end{tikzcd}$$

A coalgebra is the dual notion of the notion of an algebra. In other words, a coalgebra over $(\mathcal{C}, \otimes, I, \sigma)$ is an object $C \in \mathcal{C}$ with a comultiplication $\Delta_C \colon C \to C \otimes C$. From now on, the coalgebras will always be coassociative, i.e. the following equality holds

$$(\Delta_C \otimes 1_C) \cdot \Delta_C = (1_C \otimes \Delta_C) \cdot \Delta_C \tag{1.4}$$

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"] & C \otimes C \arrow[d, "1_C \otimes \Delta_C"] \\ C \otimes C \arrow[r, "\Delta_C \otimes 1_C"] & C \otimes C \otimes C. \end{tikzcd}$$

We will also assume that the coalgebras are counital, meaning that there exists a morphism $\epsilon_C \colon C \to I$, called counit, satisfying the condition:

$$(\epsilon_C \otimes 1_C) \cdot \Delta_C = 1_C = (1_C \otimes \epsilon_C) \cdot \Delta_C, \tag{1.5}$$

as expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
C & C \otimes C \arrow[l, "\epsilon_C \otimes 1_C"'] \arrow[r, "1_C \otimes \epsilon_C"] & C \\
& C \arrow[u, "\Delta_C"] \arrow[ul, equal] \arrow[ur, equal] &
\end{tikzcd}$$

Similarly, a *morphism of coalgebras* $g\colon C \to D$ is a morphism in $\mathcal{C}$ such that the following two diagrams commute:

$$\begin{tikzcd}
D \otimes D & C \otimes C \arrow[l, "g \otimes g"'] \\
D \arrow[u, "\Delta_D"] & C \arrow[l, "g"] \arrow[u, "\Delta_C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I & D \arrow[l, "\epsilon_D"'] \\
& C. \arrow[ul, "\epsilon_C"] \arrow[u, "g"']
\end{tikzcd}$$

Let notice that, if A and B are algebras, then the monoidal product $A \otimes B$ is also an algebra. The unit of $A \otimes B$ is given by the application $u_A \otimes u_B$ and the multiplication is given by

$$(m_A \otimes m_B) \cdot (1_A \otimes \sigma \otimes 1_B)\colon (A \otimes B) \otimes (A \otimes B) \to A \otimes B.$$

Dually, if C and D are coalgebras, then the object $C \otimes D$ has a coalgebra structure: the counit is the morphism $\epsilon_C \otimes \epsilon_D$ and the comultiplication is given by $(1_C \otimes \sigma \otimes 1_D) \cdot (\Delta_C \otimes \Delta_D)$.

We also recall that a *bialgebra* is a 5-tuple $(B, m_B, u_B, \Delta_B, \epsilon_B)$ where (B, m_B, u_B) is an algebra, $(B, \Delta_B, \epsilon_B)$ is a coalgebra and Δ_B, ϵ_B are algebra morphisms (which is equivalent of asking that m_B, u_B are coalgebra morphisms) i.e.

$$\Delta_B \cdot m_B = (m_B \otimes m_B) \cdot (1_B \otimes \sigma \otimes 1_B) \cdot (\Delta_B \otimes \Delta_B), \tag{1.6}$$

$$\Delta_B \cdot u_B = u_B \otimes u_B, \tag{1.7}$$

$$\epsilon_B \cdot m_B = \epsilon_B \otimes \epsilon_B, \tag{1.8}$$

$$\epsilon_B \cdot u_B = 1_I. \tag{1.9}$$

Moreover, a morphism in $\mathcal{C}$ is a *morphism of bialgebras* if it is a morphism of algebras and coalgebras.

A bialgebra is called cocommutative when its underlying coalgebra structure is cocommutative, it means that

$$\sigma \cdot \Delta_B = \Delta_B. \tag{1.10}$$

The category of cocommutative bialgebras in a symmetric monoidal category $\mathcal{C}$ is denoted by $\mathsf{Bial}_{\mathcal{C},\mathrm{coc}}$.

Example 1.1. (1) In the symmetric monoidal category $(\mathsf{Set}, \times, \{\star\})$ of sets where σ is the twist morphism (where $\sigma(x, y) = (y, x)$ for any element x of a set X and any element y of a set Y), every object X has a unique coalgebra structure with Δ_X being the diagonal and ϵ_X the morphism sending every element to the singleton. Hence, a (cocommutative) bialgebra (or algebra) is a monoid.

(2) In the symmetric monoidal category $(\mathsf{Vect_K}, \otimes, K)$ of vector spaces over a field K where σ is the twist morphism (defined by $\sigma(x \otimes y) = y \otimes x$ for any $x \otimes y \in X \otimes Y$), we recover the notion of K-algebra, K-coalgebra and K-bialgebra.

(3) In [10], a symmetric monoidal category was introduced such that Hom-algebras, Hom-coalgebras and Hom-bialgebras (see [17]) coincide with the algebras, coalgebras and bialgebras in this symmetric monoidal category.

We recall the definition of a locally presentable category introduced in [1].

Definition 1.1. A category is called *locally λ-presentable* (λ a regular cardinal) provided that it is cocomplete, and has a set A of presentable objects such that every object is a λ-direct colimit of objects from A.

A category is *locally presentable* if it is locally λ-presentable for some regular cardinal λ.

The categories of sets Set, vector spaces Vect, K-algebras $\mathsf{Alg_K}$ and K-coalgebras $\mathsf{CoAlg_K}$ are examples of locally presentable categories. A locally presentable category is complete [1]. We refer to [1] for a more complete explanation of the basis of locally presentable categories.

1.2. *Adjunction between cocommutative bialgebras and cocommutative coalgebras*

It is well known that we have an adjunction between K-coalgebras and K-bialgebras [12]. We recall a result of [21, Sec. 4.2]. Let $\mathcal{C}$ be a locally presentable symmetric monoidal category, we have the following adjunction between (cocommutative) bialgebras and (cocommutative) coalgebras;

$$\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{CoAlg}_{\mathcal{C},\mathrm{coc}} \tag{1.11}$$

where U is the forgetful functor and F is the free algebra functor.

In particular, this adjunction implies that U preserves the limits and that a monomorphism of bialgebras is in particular also a monomorphism of coalgebras. This observation will be useful several times in this paper.

1.3. *Limits in the category of cocommutative bialgebras*

We recall the constructions of products, equalizers and pullbacks in the category of cocommutative bialgebras in a symmetric monoidal category that has equalizers.

It is interesting to recall that the categorical product of cocommutative bialgebras is given by the monoidal product.

Proposition 1.1. *In the category $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$, the categorical product of two cocommutative bialgebras A and B is given by the monoidal product $(A\otimes B, \pi_A, \pi_B)$ where the projections $\pi_A\colon A\otimes B \to A$ and $\pi_B\colon A\otimes B\to B$ are defined by $\pi_A := 1_A\otimes \epsilon_B$ and $\pi_B := \epsilon_A \otimes 1_B$.*

$$\begin{tikzcd} A & A\otimes B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\ & C. \arrow[ul, "f"] \arrow[u, dashed, "(f\otimes g)\cdot\Delta_C"] \arrow[ur, "g"'] & \end{tikzcd} \tag{1.12}$$

The equalizers are defined as in [2]. Let $f, g\colon A \to B$ be two morphisms of bialgebras, then the following construction in $\mathcal{C}$ is the equalizer of f and g.

$$\begin{tikzcd} E \arrow[r, "\varepsilon"] & A \arrow[r, shift left, "(g\otimes 1_A)\cdot\Delta_A"] \arrow[r, shift right, "(f\otimes 1_A)\cdot\Delta_A"'] & B\otimes A\ . \end{tikzcd} \tag{1.13}$$

where ε is the equalizer of $(f\otimes 1_A)\cdot\Delta_A$ and $(g\otimes 1_A)\cdot\Delta_A$ in $\mathcal{C}$. We can easily check that this construction is the equalizer of f and g in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$.

Via the definitions of products and equalizers in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$, we define the pullback in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ of two morphisms of bialgebras $f\colon A\to B$ and $g\colon C\to B$ as follows:

$$\begin{tikzcd} A\otimes_B C \arrow[r, "p_C"] \arrow[d, "p_A"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $A\otimes_B C$ is the object in the following equalizer:

$$\begin{tikzcd} A\otimes_B C \arrow[r, "\varepsilon"] & A\otimes C \arrow[r, shift left, "(1_A\otimes g\otimes 1_C)\cdot(1_A\otimes\Delta_C)"] \arrow[r, shift right, "(1_A\otimes f\otimes 1_C)\cdot(\Delta_A\otimes 1_C)"'] & A\otimes B\otimes C \end{tikzcd}$$

and the two projections are defined as $p_A := (1_A\otimes\epsilon_C)\cdot\varepsilon$ and $p_C := (\epsilon_A\otimes 1_C)\cdot\varepsilon$.

1.4. *Special split extensions of cocommutative bialgebras*

In [22], we introduced a notion of split extensions of (non-associative) bialgebras and we showed several properties. Here, we recall this notion and some results in the case of cocommutative bialgebras that will be useful later on. In the context of cocommutative bialgebras, we denote by S_{coc} the class of the "good" split extensions of cocommutative bialgebras.

Definition 1.2. A *split extension of cocommutative bialgebras in* S_{coc} is given by a diagram

$$\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd}, \tag{1.14}$$

where X, A, B are cocommutative bialgebras, κ, α, e are morphisms of bialgebras, such that

(0) λ is a morphism of coalgebras preserving the unit.
(1) $\lambda \cdot \kappa = 1_X$, $\alpha \cdot e = 1_B$,
(2) $\lambda \cdot e = u_X \cdot \epsilon_B$, $\alpha \cdot \kappa = u_B \cdot \epsilon_X$,
(3) $m_A \cdot ((\kappa \cdot \lambda) \otimes (e \cdot \alpha)) \cdot \Delta_A = 1_A$,
(4) $\lambda \cdot m_A \cdot (\kappa \otimes e) = 1_X \otimes \epsilon_B$.

We recall that the conditions $\lambda \cdot \kappa = 1_X$, $\lambda \cdot e = u_X \cdot \epsilon_B$ and the preservation of the unit by λ are consequences of the axiom (4).

Notice that the diagram (1.14) defined above is an exact sequence (see [22, Proposition 3.15]). The terminology of *split extensions* follows the recent paper of [14].

Definition 1.3. A *morphism of split extensions in* S_{coc} from the split extension $\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd}$ to the split extension $\begin{tikzcd} X' \arrow[r, shift right, "\kappa'"'] & A' \arrow[l, dashed, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] & B' \arrow[l, shift right, "e'"'] \end{tikzcd}$ is given by three morphisms of bialgebras $g\colon B \to B'$, $v\colon X \to X'$ and $p\colon A \to A'$ such that the four squares in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] \arrow[d, "v"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] \arrow[d, "p"'] & B \arrow[l, shift right, "e"'] \arrow[d, "g"] \\ X' \arrow[r, shift right, "\kappa'"'] & A' \arrow[l, dashed, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] & B'. \arrow[l, shift right, "e'"'] \end{tikzcd}$$

Let us remark that the commutativity of all the squares is not required. Notice that if $p \cdot \kappa = \kappa' \cdot v$ and $p \cdot e = e' \cdot g$ hold, then the identities $\lambda' \cdot p = v \cdot \lambda$ and $\alpha' \cdot p = g \cdot \alpha$ follow, and conversely (see [22, Corollary 3.8]).

The two definitions above give rise to the category $\mathsf{SplitExt}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ of split extensions of cocommutative bialgebras in S_{coc}.

We recall a convenient equality.

Lemma 1.1. *Let* $\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd}$ *be a split extension of cocommutative bialgebras in* S_{coc}, *then the following identity holds*:

$$\lambda \cdot m_A = m_X \cdot (\lambda \otimes \lambda) \cdot (1_A \otimes m_A) \cdot (1_A \otimes (e \cdot \alpha) \otimes (\kappa \otimes \lambda)) \cdot (\Delta_A \otimes 1_A). \tag{1.15}$$

Even if the category of cocommutative bialgebras is not protomodular (see [13]), we have an interesting result for the split extensions defined above: a relative form of the Split Short Five Lemma holds in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$.

Theorem 1.1. *Let (g, v, p) be a morphism of split extensions of cocommutative bialgebras S_{coc}*

$$\begin{tikzcd}
X \arrow[r, shift right, "\kappa"'] \arrow[d, "v"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] \arrow[d, "p"'] & B \arrow[l, shift right, "e"'] \arrow[d, "g"] \\
X' \arrow[r, shift right, "\kappa'"'] & A' \arrow[l, dashed, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] & B' \arrow[l, shift right, "e'"']
\end{tikzcd}$$

then p is an isomorphism whenever v and g are.

In [22], it was proved that there is an equivalence between the categories of actions and split extensions of (cocommutative) bialgebras in a symmetric monoidal category. We recall this equivalence and the definition of the objects and morphisms of the category of actions of cocommutative bialgebras.

Definition 1.4. Let X and B be cocommutative bialgebras in a symmetric monoidal category $(\mathcal{C}, \otimes, I, \sigma)$. An *action of bialgebras* is a morphism in $\mathcal{C}$, $\triangleright\colon B \otimes X \to X$, such that

$$\triangleright \cdot (u_B \otimes 1_X) = 1_X,$$

$$\triangleright \cdot (1_B \otimes \triangleright) = \triangleright \cdot (m_B \otimes 1_X),$$

$$\triangleright \cdot (1_B \otimes u_X) = u_X \cdot \epsilon_B,$$

$$\triangleright \cdot (1_B \otimes m_X) = m_X \cdot (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta_B \otimes 1_X \otimes 1_X),$$

$$\epsilon_X \cdot \triangleright = \epsilon_B \otimes \epsilon_X,$$

$$\Delta_X \cdot \triangleright = (\triangleright \otimes \triangleright) \cdot (1_B \otimes \sigma \otimes 1_X) \cdot (\Delta_B \otimes \Delta_X).$$

In other words, an action of bialgebras of B on X is a cocommutative bialgebra in the symmetric monoidal category of B-modules, where B is a cocommutative bialgebra.

Definition 1.5. Let $\triangleright\colon B \otimes X \to X$ and $\triangleright'\colon B' \otimes X' \to X'$ be two actions of cocommutative bialgebras. A *morphism of actions of cocommutative bialgebras* is defined as a pair of morphisms of bialgebras $g\colon B \to B'$ and $v\colon X \to X'$ such that

$$v \cdot \triangleright = \triangleright' \cdot (g \otimes v).$$

There is then the category of actions of cocommutative bialgebras that will be denoted by $\mathsf{Act}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$.

We recall the theorem obtained in [22] and the construction of the functor in the equivalence of categories between $\mathsf{Act}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ and $\mathsf{SplitExt}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$.

Theorem 1.2. *There is an equivalence between the category* $\mathsf{SplitExt}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ *of split extensions of cocommutative bialgebras in* S_{coc} *and the category* $\mathsf{Act}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ *of actions of cocommutative bialgebras.*

Proof. The functor $F\colon \mathsf{SplitExt}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}) \to \mathsf{Act}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ is defined as

$$F\left(\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] \arrow[d, "v"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] \arrow[d, "p"] & B \arrow[l, shift right, "e"'] \arrow[d, "g"] \\ X' \arrow[r, shift right, "\kappa'"'] & A' \arrow[l, dashed, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] & B' \arrow[l, shift right, "e'"'] \end{tikzcd}\right) = \begin{tikzcd} B \otimes X \arrow[r, "\triangleright"] \arrow[d, "g \otimes v"'] & X \arrow[d, "v"] \\ B' \otimes X' \arrow[r, "\triangleright'"] & X', \end{tikzcd}$$

where $\triangleright := \lambda \cdot m_A \cdot (e \otimes \kappa)$. The functor $G\colon \mathsf{Act}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}) \to \mathsf{SplitExt}(\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}})$ is defined as

$$G\left(\begin{tikzcd} B \otimes X \arrow[r, "\triangleright"] \arrow[d, "g \otimes v"'] & X \arrow[d, "v"] \\ B' \otimes X' \arrow[r, "\triangleright'"] & X' \end{tikzcd}\right) = \begin{tikzcd} X \arrow[r, shift right, "i_1"'] \arrow[d, "v"'] & X \rtimes B \arrow[l, dashed, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] \arrow[d, "v \otimes g"] & B \arrow[l, shift right, "i_2"'] \arrow[d, "g"] \\ X' \arrow[r, shift right, "i'_1"'] & X' \rtimes B' \arrow[l, dashed, shift right, "\pi'_1"'] \arrow[r, shift right, "\pi'_2"'] & B', \arrow[l, shift right, "i'_2"'] \end{tikzcd}$$

where $i_1 = 1_X \otimes u_B$, $i_2 = u_X \otimes 1_B$, $\pi_1 = 1_X \otimes \epsilon_B$, $\pi_2 = \epsilon_X \otimes 1_B$ and $X \rtimes B$ is the object $X \otimes B$ where the bialgebra structure is given by the following morphisms of $\mathcal{C}$

$$\begin{aligned} m_{X \rtimes B} = {} & (m_X \otimes m_B) \cdot (1_X \otimes \triangleright \otimes 1_B \otimes 1_B) \cdot (1_X \otimes 1_B \otimes \sigma \otimes 1_B) \\ & \cdot (1_X \otimes \Delta_B \otimes 1_X \otimes 1_B) \end{aligned}$$

$$u_{X \rtimes B} = u_X \otimes u_B,$$
$$\Delta_{X \rtimes B} = (1_X \otimes \sigma \otimes 1_B) \cdot (\Delta_X \otimes \Delta_B),$$
$$\epsilon_{X \rtimes B} = \epsilon_X \otimes \epsilon_B. \qquad \square$$

Let B and X be two cocommutative bialgebras; since the trivial action $\epsilon_B \otimes 1_X : B \otimes X \to X$ is an action of bialgebras as defined in Definition 1.4, we obtain the following corollary.

Corollary 1.1. *Let B and X be two cocommutative bialgebras, the pair (π_2, i_2) in the following product*

$$\begin{tikzcd} X \arrow[r, shift left, "i_1"'] & X \otimes B \arrow[l, shift left, "\pi_1"'] \arrow[r, shift left, "\pi_2"'] & B \arrow[l, shift left, "i_2"'] \end{tikzcd}$$

where $i_1 = 1_X \otimes u_B$, $i_2 = u_X \otimes 1_B$, $\pi_1 = 1_X \otimes \epsilon_B$ and $\pi_2 = \epsilon_X \otimes 1_B$, belongs to S_{coc}.

2. S_{coc}-Protomodularity

It is known that the category of cocommutative bialgebras is not protomodular. However, we have interesting results with respect to the class S_{coc} of split extensions of cocommutative bialgebras, as the Split Short Five Lemma (Theorem 1.1) and the equivalence with the actions (Theorem 1.2).

These results are the motivation to prove that the category is S_{coc}-protomodular, which means protomodular with respect to S_{coc}.

Let $\mathcal{C}$ be a pointed category, and S be a class of split epimorphisms with a chosen section, to recall the definition of an S-protomodular category we give the definition of a strongly split epimorphism.

Definition 2.1. Two morphisms $v: X \to A$ and $w: B \to A$ are *jointly strongly epimorphic* if, whenever they factor through a common monomorphism $\mu: M \to A$, μ is an isomorphism.

$$\begin{tikzcd} X \arrow[rr, "v"] \arrow[dr, "\delta"'] & & A & & B \arrow[ll, "w"'] \arrow[dll, "\gamma"] \\ & & M \arrow[u, tail, "\mu"] & & \end{tikzcd}$$

Definition 2.2. A split epimorphism with a chosen section $A \overset{e}{\underset{\alpha}{\leftrightarrows}} B$ is called *strongly split epimorphism* whenever the kernel $\kappa\colon X \to A$ of α and the section e are *jointly strongly epimorphic*.

For now on, split epimorphisms will mean split epimorphisms with a chosen section.

Definition 2.3 ([7]). Let $\mathcal{C}$ be a pointed finitely complete category and S a class of split epimorphisms stable under pullbacks. $\mathcal{C}$ is said to be S-*protomodular* when the class S is closed under finite limits in the category of split epimorphisms in $\mathcal{C}$, $\mathsf{Pt}(\mathcal{C})$, and any split epimorphism in S is a strongly split epimorphism.

2.1. *Pullback stability*

We prove that the class S_{coc} of split extensions of cocommutative bialgebras in any locally presentable symmetric monoidal category is pullback stable.

We consider the pullback of $X \overset{\lambda}{\underset{\kappa}{\leftrightarrows}} A \overset{e}{\underset{\alpha}{\leftrightarrows}} B$ along the morphism of bialgebras $g\colon C \to B$.

$$\begin{tikzcd} X \arrow[r, shift right, "{(\kappa, u_C\cdot\epsilon_X)}"'] \arrow[d, "1_X"'] & A\otimes_B C \arrow[l, dashed, shift right, "\lambda\cdot p_A"'] \arrow[r, shift right, "p_C"'] \arrow[d, "p_A"] & C \arrow[l, shift right, "{(e\cdot g, 1_C)}"'] \arrow[d, "g"] \\ X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd} \tag{2.1}$$

where $(\kappa, u_C\cdot\epsilon_X)$ and $(e\cdot g, 1_C)$ are the morphisms induced by the universal property of the pullback.

Proposition 2.1. *Let* (2.1) *be the pullback of* $X \overset{\lambda}{\underset{\kappa}{\leftrightarrows}} A \overset{e}{\underset{\alpha}{\leftrightarrows}} B$ *along a morphism of bialgebras* $g\colon C \to B$, *then the upper row of* (2.1):

$$\begin{tikzcd} X \arrow[r, shift right, "{(\kappa, u_C\cdot\epsilon_X)}"'] & A\otimes_B C \arrow[l, dashed, shift right, "\lambda\cdot p_A"'] \arrow[r, shift right, "p_C"'] & C \arrow[l, shift right, "{(e\cdot g, 1_C)}"'] \end{tikzcd}$$

belongs to S_{coc}.

Proof. The conditions (0), (1), (2) and (4) of Definition 1.2 are easily checked. We give the details of the condition (3). To show that

$$m_{A\otimes_B C}\cdot((\kappa, u_C\cdot\epsilon_X)\cdot\lambda\cdot p_A \otimes (e\cdot g, 1_C)\cdot p_C)\cdot\Delta_{A\otimes_B C} = 1_{A\otimes_B C},$$

we compose the two sides of the equality with p_A and p_C.

$$\begin{tikzcd}
(A \otimes_B C) \arrow[r, "\Delta_{A\otimes_B C}"] \arrow[dd, "p_A"'] & (A \otimes_B C)^2 \arrow[rr, "{(\kappa, u_C\cdot\epsilon_X)\cdot\lambda\cdot p_A \otimes (e\cdot g, 1_C)\cdot p_C}"] \arrow[d, "p_A^2"] \arrow[drr, "\kappa\cdot\lambda\cdot p_A\otimes e\cdot g\cdot p_C" description] & & (A \otimes_B C)^2 \arrow[r, "m_{A\otimes_B C}"] \arrow[d, "p_A^2"] & (A \otimes_B C) \arrow[dd, "p_A"] \\
& A^2 \arrow[rr, "\kappa\cdot\lambda\otimes e\cdot\alpha"'] & \arrow[r, phantom, "(3)"] & A^2 \arrow[dr, "m_A"] & \\
A \arrow[ur, "\Delta_A"] \arrow[rrrr, "1_A"] & & & & A
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_B C) \arrow[r, "\Delta_{A\otimes_B C}"] \arrow[dd, "p_C"'] & (A \otimes_B C)^2 \arrow[rr, "{(\kappa, u_C\cdot\epsilon_X)\cdot\lambda\cdot p_A \otimes (e\cdot g, 1_C)\cdot p_C}"] \arrow[drr, "u_C\cdot\epsilon_X\cdot\lambda\cdot p_A\otimes 1_C\cdot p_C" description] & & (A \otimes_B C)^2 \arrow[r, "m_{A\otimes_B C}"] \arrow[d, "p_C^2"] & (A \otimes_B C) \arrow[dd, "p_C"] \\
& (1.2) + (1.5) & & C^2 \arrow[dr, "m_C"] & \\
C \arrow[rrrr, "1_C"] & & & & C
\end{tikzcd}$$

Since p_A and p_C are jointly monic in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ (and then also jointly monic in $\mathsf{CoAlg}_{\mathcal{C},\mathrm{coc}}$ thanks to the adjunction (1.11)) and the cocommutativity implies that $m_{A\otimes_B C}\cdot((\kappa, u_C\cdot\epsilon_X)\cdot\lambda\cdot p_A\otimes(e\cdot g, 1_C)\cdot p_C)\cdot\Delta_{A\otimes_B C}$ is a morphism of coalgebras, we can conclude that the condition (3) holds. Hence, the class S_{coc} of split extensions of cocommutative bialgebras is stable under pullbacks. □

2.2. *Closure under finite limits*

To prove that the split extensions in S_{coc} are closed under finite limits in the category of split epimorphisms of cocommutative bialgebras $\mathsf{Pt}(\mathsf{Bialg}_{\mathcal{C},\mathrm{coc}})$, we prove that they are closed under products and equalizers.

Proposition 2.2. *The class S_{coc} of split extensions of bialgebras in any locally presentable symmetric monoidal category is closed under finite limits.*

Proof. Let $X \underset{\kappa}{\overset{\lambda}{\leftrightarrows}} A \underset{\alpha}{\overset{e}{\leftrightarrows}} B$ and $X' \underset{\kappa'}{\overset{\lambda'}{\leftrightarrows}} A' \underset{\alpha'}{\overset{e'}{\leftrightarrows}} B'$ be two split extensions of bialgebras in S_{coc}. We prove that

$$X\otimes X' \underset{\kappa\otimes\kappa'}{\overset{\lambda\otimes\lambda'}{\leftrightarrows}} A\otimes A' \underset{\alpha\otimes\alpha'}{\overset{e\otimes e'}{\leftrightarrows}} B\otimes B' \tag{2.2}$$

belongs to S_{coc}. Since the construction is made component-wise it is clear that (2.2) belongs to S_{coc} : we can explicitly see the proof of condition (3) via the commutativity of the following diagram

$$\begin{tikzcd}
A \otimes A' \arrow[r, "\Delta_A \otimes \Delta_{A'}"] \arrow[dd, "1_A \otimes 1_{A'}"'] & A^2 \otimes A'^2 \arrow[r, "1_A \otimes \sigma \otimes 1_{A'}"] \arrow[drr, "\kappa \cdot \lambda \otimes e \cdot \alpha \otimes \kappa' \cdot \lambda' \otimes e' \cdot \alpha'"'] \arrow[d, phantom, "(3)"] & (A \otimes A')^2 \arrow[r, "\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes e \cdot \alpha \otimes e' \cdot \alpha'"] \arrow[dr, phantom, "(1.1)"] & (A \otimes A')^2 \arrow[d, "1_A \otimes \sigma \otimes 1_{A'}"] \\
 & & & A^2 \otimes A'^2 \arrow[d, "m_A \otimes m_{A'}"] \\
A \otimes A' \arrow[rrr, "1_A \otimes 1_{A'}"] & & & A \otimes A'
\end{tikzcd}$$

The other conditions hold via similar computations.

The class S_{coc} is also closed under equalizers. We construct the equalizer of two morphisms of split extensions of bialgebras in S_{coc} (g, v, p) and (g', v', p'):

$$\begin{tikzcd}
\hat{E} \arrow[r, shift right, "\tilde{\kappa}"'] \arrow[d, "\hat{\varepsilon}"'] & E \arrow[l, dashed, shift right, "\tilde{\lambda}"'] \arrow[r, shift right, "\tilde{\alpha}"'] \arrow[d, "\varepsilon"'] & E' \arrow[l, shift right, "\tilde{e}"'] \arrow[d, "\varepsilon'"'] \\
X' \arrow[r, shift right, "\kappa'"'] \arrow[d, shift right, "v"'] \arrow[d, shift left, "v'"] & A' \arrow[l, dashed, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] \arrow[d, shift right, "g"'] \arrow[d, shift left, "g'"] & B' \arrow[l, shift right, "e'"'] \arrow[d, shift right, "p"'] \arrow[d, shift left, "p'"] \\
X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"']
\end{tikzcd}$$

where ε, ε' and $\hat{\varepsilon}$ are the equalizers in $\mathsf{Bialg}_{\mathcal{C},\text{coc}}$ of the pairs of morphisms (g, g'), (p, p') and (v, v'), respectively. The morphisms $\tilde{\alpha}$, $\tilde{e}$, $\tilde{\kappa}$ and $\tilde{\lambda}$ are induced by the universal properties of the equalizers. Note that $\tilde{\lambda}$ is induced by the universal property of $\hat{\varepsilon}$ seen as a coequalizer in $\mathsf{CoAlg}_{\mathcal{C},\text{coc}}$ via (1.11). By using the fact that $\varepsilon, \varepsilon'$ and $\hat{\varepsilon}$ are monomorphisms of bialgebras (and hence also of coalgebras) we can conclude that

$$\begin{tikzcd}
\hat{E} \arrow[r, shift right, "\tilde{\kappa}"'] & E \arrow[l, dashed, shift right, "\tilde{\lambda}"'] \arrow[r, shift right, "\tilde{\alpha}"'] & E' \arrow[l, shift right, "\tilde{e}"']
\end{tikzcd} \tag{2.3}$$

belongs to S_{coc}. We prove that the condition (3) in Definition 1.2 is satisfied for (2.3). Let us first observe that the following diagram commutes:

$$
\begin{tikzcd}
E \arrow[r, "\Delta_E"] \arrow[dd, "\varepsilon"] & E^2 \arrow[r, "\tilde{\lambda}\otimes\tilde{\alpha}"] \arrow[d, "\varepsilon^2"] & \hat{E}\otimes E' \arrow[r, "\tilde{\kappa}\otimes\tilde{e}"] \arrow[d, "\hat{\varepsilon}\otimes\varepsilon'"] & E^2 \arrow[r, "m_E"] \arrow[d, "\varepsilon^2"] & E \arrow[dd, "\varepsilon"] \\
 & A'^2 \arrow[r, "\lambda'\otimes\alpha'"] & X'\otimes B' \arrow[r, "\kappa'\otimes e'"] \arrow[d, phantom, "(3)"] & A'^2 \arrow[dr, "m_{A'}"] & \\
A' \arrow[ur, "\Delta_{A'}"] \arrow[rrrr, "1_{A'}"'] & & {} & & A'
\end{tikzcd}
$$

Since ε is a monomorphism and $m_E \cdot (\tilde{\kappa} \cdot \tilde{\lambda} \otimes \tilde{e} \cdot \tilde{\alpha}) \cdot \Delta_E$ is a morphism of coalgebras, we can conclude that condition (3) holds thanks to the adjunction (1.11). We give also an explicit proof of the condition (4) via the commutativity of this diagram:

$$
\begin{tikzcd}
\hat{E}\otimes E' \arrow[rr, "\tilde{\kappa}\otimes\tilde{e}"] \arrow[dr, "\hat{\varepsilon}\otimes\varepsilon'"] \arrow[dd, "1_{\hat{E}}\otimes\epsilon"'] & & E^2 \arrow[r, "m_E"] \arrow[d, "\varepsilon^2"] & E \arrow[r, "\tilde{\lambda}"] \arrow[d, "\varepsilon"] & \hat{E} \arrow[dd, "\hat{\varepsilon}"] \\
 & X'\otimes B' \arrow[r, "\kappa'\otimes e'"] \arrow[drrr, "1_{X'}\otimes\epsilon_{B'}"] & A'^2 \arrow[r, "m_A"] & A' \arrow[dr, "\lambda'"] \arrow[d, phantom, "(4)"] & \\
\hat{E} \arrow[rrrr, "\hat{\varepsilon}"] & & & {} & X'
\end{tikzcd}
$$

Since $\hat{\varepsilon}$ is a monomorphism of coalgebras and $\tilde{\lambda} \cdot m_E \cdot (\tilde{\kappa} \otimes \tilde{e})$ is a coalgebra morphism, we can conclude that (2.3) satisfies condition (4). The other conditions are straightforward to check. $\square$

2.3. *Strongly split epimorphisms*

It was proven in [22] that for a split extension of (cocommutative) bialgebras in S_{coc}

$$
\begin{tikzcd}
X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"']
\end{tikzcd}
$$

κ and e are jointly epimorphic. Now we prove that for cocommutative biagebras in a locally presentable symmetric monoidal category, κ and e are jointly strongly epimorphic and hence any split extension of cocommutative bialgebras in S_{coc} is a strongly split epimorphism.

Proposition 2.3. *Any split extension of cocommutative bialgebras in S_{coc}, in a locally presentable symmetric monoidal category, is a strongly split epimorphism.*

Proof. Let $\begin{tikzcd} X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd}$ be a split extension of cocommutative bialgebras in S_{coc} and $\mu\colon M \to A$ be a monomorphism of bialgebras. Let κ and e factor through μ:

$$\begin{tikzcd} X \arrow[rr, "\kappa"] \arrow[dr, "\delta"'] & & A & & B \arrow[ll, "e"'] \arrow[dl, "\gamma"] \\ & & M \arrow[u, tail, "\mu"] & & \end{tikzcd}$$

We can form the following commutative diagram

$$\begin{tikzcd} X \arrow[r, shift right, "\delta"'] \arrow[d, "1_X"'] & M \arrow[l, dashed, shift right, "\lambda \cdot \mu"'] \arrow[r, shift right, "\alpha \cdot \mu"'] \arrow[d, "\mu"'] & B \arrow[l, shift right, "\gamma"'] \arrow[d, "1_B"] \\ X \arrow[r, shift right, "\kappa"'] & A \arrow[l, dashed, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] & B \arrow[l, shift right, "e"'] \end{tikzcd}$$

We prove that the upper row is in the class S_{coc} by checking all the conditions of Definition 1.2. In particular, via the commutativity of the following diagram

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[dd, "\mu"'] & M^2 \arrow[rr, "\delta \cdot \lambda \cdot \mu \otimes \gamma \cdot \alpha \cdot \mu"] \arrow[d, "\mu^2"'] \arrow[drr, "\kappa \cdot \lambda \cdot \mu \otimes e \cdot \alpha \cdot \mu" description] & & M^2 \arrow[r, "m_M"] \arrow[d, "\mu^2"'] & M \arrow[dd, "\mu"] \\ & A^2 \arrow[rr, "\kappa \cdot \lambda \otimes e \cdot \alpha"'] & & A^2 \arrow[dr, "m_A"] & \\ A \arrow[ur, "\Delta_A"] \arrow[rrrr, "1_A"] & & (3) & & A \end{tikzcd}$$

we conclude that $\begin{tikzcd} X \arrow[r, shift right, "\delta"'] & M \arrow[l, dashed, shift right, "\lambda \cdot \mu"'] \arrow[r, shift right, "\alpha \cdot \mu"'] & B \arrow[l, shift right, "\gamma"'] \end{tikzcd}$ satisfies condition (3) since μ is a monomorphism of bialgebras (and then also a monomorphism of coalgebras (1.11)) and $m_M \cdot (\delta \cdot \lambda \cdot \mu \otimes \gamma \cdot \alpha \cdot \mu) \cdot \Delta_M$ is a coalgebra morphism thanks to the cocommutativity. The other conditions of Definition 1.2 can be proven by direct computation: for example, condition (4) holds via the commutativity of the following diagram

$$
\begin{tikzcd}
X \otimes B \arrow[r, "\delta \otimes \gamma"] \arrow[dr, "\kappa \otimes e"] \arrow[dd, "1_X \otimes 1_B"'] \arrow[ddr, phantom, "(4)"] & M^2 \arrow[r, "m_M"] \arrow[d, "\mu^2"] & M \arrow[d, "\mu"] \\
 & A^2 \arrow[r, "m_A"] & A \arrow[d, "\lambda"] \\
X \otimes B \arrow[rr, "1_X \otimes \epsilon_B"] & & X.
\end{tikzcd}
$$

To conclude, $X \underset{\delta}{\overset{\lambda \cdot \mu}{\leftrightarrows}} M \underset{\alpha \cdot \mu}{\overset{\gamma}{\leftrightarrows}} B$ belongs to S_{coc} and then the thesis follows from the Split Short Five Lemma (Theorem 1.1). □

Thanks to the previous results we obtain the following theorem:

Theorem 2.1. *The category of cocommutative bialgebras in any locally presentable symmetric monoidal category is an S_{coc}-protomodular category with respect to the class S_{coc} of split extensions of cocommutative bialgebras defined in Definition* 1.2.

Proof. This theorem holds thanks to Propositions 2.1, 2.2 and 2.3. □

Remark 2.1. This theorem generalizes the result of [7] saying that the category of monoids is S-protomodular with respect to the class of Schreier split epimorphisms. Indeed, if $\mathcal{C}$ is the locally presentable symmetric monoidal category $(\mathsf{Set}, \times, \{\star\})$, then S_{coc} becomes the class of the Schreier split epimorphisms and $\mathsf{BiAlg}_{\mathsf{Set},\mathrm{coc}}$ the category of monoids.

3. Huq Commutator

In this section, we recall the notion of centrality in the sense of Huq in a pointed category with binary products. Moreover, we give an explicit description of this centrality for two subbialgebras of a cocommutative bialgebra.

Let $\mathcal{C}$ be a pointed category with binary products; we say that two subobjects $x\colon X \to A$ and $y\colon Y \to A$ commute (or centralize) in the sense of Huq [16] if there exists a morphism $p\colon X \times Y \to A$ such that the following diagram commutes

$$
\begin{tikzcd}
X \arrow[r, "{(1,0)}"] \arrow[dr, "x"'] & X \times Y \arrow[d, dashed, "p"] & Y \arrow[l, "{(0,1)}"'] \arrow[dl, "y"] \\
 & A &
\end{tikzcd}
$$

Then we denote by $[X, Y] = 0$ the fact that the subobjects X and Y commute.

Note that in $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$, if it exists, p is unique thanks to Corollary 1.1 and Proposition 2.3.

Proposition 3.1. *In $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$, the following are equivalent for $x\colon X \to A$ and $y\colon Y \to A$ two subobjects of a cocommutative bialgebra A:*

(i) $m_A \cdot \sigma \cdot (x \otimes y) = m_A \cdot (x \otimes y)$

(ii) $[X, Y] = 0$, *i.e. there exists a (unique) morphism of bialgebras $p\colon X \otimes Y \to A$ such that the following diagram commutes*

$$
\begin{tikzcd}
X \arrow[r, "1_X \otimes u_Y"] \arrow[dr, "x"'] & X \otimes Y \arrow[d, dashed, "p"] & Y \arrow[l, "u_X \otimes 1_Y"'] \arrow[dl, "y"] \\
& A &
\end{tikzcd}
\tag{3.1}
$$

Proof. (i) $\Rightarrow$ (ii), we define $p : X \otimes Y \to A$ by $p := m \cdot (x \otimes y)$. It is easy to see that this map makes the diagram (3.1) commute. Moreover, thanks to (i) we can prove that p is a morphism of algebras (and hence a morphism of bialgebras) as proved in the following diagram;

$$
\begin{tikzcd}
(X \otimes Y)^2 \arrow[rr, "1_X \otimes \sigma \otimes 1_Y"'] \arrow[rrr, bend left, "m_{X\otimes Y}"] \arrow[dr, "(x\otimes y)^2"] \arrow[dd, "(x\otimes y)^2"'] \arrow[ddd, bend right, "p^2"'] & & X^2 \otimes Y^2 \arrow[r, "m_X \otimes m_Y"'] \arrow[dr, "x^2 \otimes y^2"] & X \otimes Y \arrow[r, "x \otimes y"'] \arrow[rr, bend left, "p"] & A^2 \arrow[r, "m_A"'] & A \arrow[ddd, "1_A"] \\
& A^4 \arrow[rr, "1_A \otimes \sigma \otimes 1_A"] \arrow[d, phantom, "\text{(i)}"] & & A^4 \arrow[ur, "m_A^2"] \arrow[d, "1_A \otimes m_A \otimes 1_A"] \arrow[rr, phantom, "(1.3)"] & & \\
A^4 \arrow[rrr, "1_A \otimes m_A \otimes 1_A"] \arrow[d, "m_A^2"] \arrow[drrr, phantom, "(1.3)"] & & & A^3 \arrow[drr, "m_A \cdot (m_A \otimes 1_A)"] & & \\
A^2 \arrow[rrrrr, "m_A"] & & & & & A
\end{tikzcd}
$$

On the other way around, since p is a morphism of bialgebras which makes (3.1) commute, we can make the following diagram commute:

$$
\begin{tikzcd}
X \otimes Y \arrow[rrrr, "x \otimes y"] \arrow[ddr, "1_X \otimes u_Y \otimes u_X \otimes 1_Y"'] \arrow[dddd, "1_X \otimes 1_Y"'] & & & & A^2 \arrow[dddd, "m_A"] \\
 & \arrow[r, phantom, "(1.2)"] & A^4 \arrow[urr, "m_A^2"] & & \\
 & (X \otimes Y)^2 \arrow[ur, "(x \otimes y)^2"] \arrow[uurrr, bend right, "p^2"'] \arrow[drr, "1_X \otimes \sigma \otimes 1_Y"] & & & \\
 & & & X^2 \otimes Y^2 \arrow[d, "m_X \otimes m_Y"] & \\
X \otimes Y \arrow[urrr, "1_X \otimes u_X \otimes u_Y \otimes 1_Y"] \arrow[rrr, "1_X \otimes 1_Y", "(1.2)"'] \arrow[drrr, "u_X \otimes 1_Y \otimes 1_X \otimes u_Y"'] \arrow[dddd, "\sigma"'] & & & X \otimes Y \arrow[r, "p"] & A \\
 & & & X^2 \otimes Y^2 \arrow[u, "m_X \otimes m_Y"'] & \\
 & (X \otimes Y)^2 \arrow[urr, "1_X \otimes \sigma \otimes 1_Y"] \arrow[dr, "(x \otimes y)^2"] \arrow[ddrrr, bend left, "p^2"] & & & \\
 & \arrow[r, phantom, "(1.2)"] & A^4 \arrow[drr, "m_A^2"] & & \\
Y \otimes X \arrow[rrrr, "y \otimes x"'] \arrow[uur, "u_X \otimes 1_Y \otimes 1_X \otimes u_Y"] & & & & A^2 \arrow[uuuu, "m_A"']
\end{tikzcd}
$$

and conclude that (i) holds. □

Note that we obtained a similar result in the case of cocommutative Hopf algebras in the paper [15].

4. Smith is Huq

Now we would like to compare the notions of centrality in the sense of Huq and in the sense of Smith.

We first recall the notion of centrality of two equivalence relations in the sense of Smith in any category with pullbacks [6].

Definition 4.1. Let (R, r_0, r_1) and (S, s_0, s_1) be two equivalence relations over the same object X. We denote the pullback of s_0 along r_1 by

$$\begin{tikzcd} R \times_X S \arrow[r, "p_2"] \arrow[d, "p_1"] & S \arrow[d, "s_0"] \\ R \arrow[r, "r_1"] & X. \end{tikzcd}$$

A connector between R and S is an arrow $\hat{p}\colon R \times_X S \to X$ such that

- $xS\hat{p}(x, y, z)$ and $zR\hat{p}(x, y, z)$,
- $\hat{p}(x, x, y) = y$ and $\hat{p}(x, y, y) = x$,
- $\hat{p}(x, y, \hat{p}(y, u, v)) = \hat{p}(x, u, v)$ and $\hat{p}(\hat{p}(x, y, u), u, v) = \hat{p}(x, y, v)$,

where the above expressions are defined. When such an arrow exists, we say that the relations R and S centralize (in the sense of Smith).

By the same powerful technique as in [3] (Metatheorem 0.1.3), it is enough to give this definition in set-theoretical terms.

Note that this notion of centrality of equivalence relations is not independent of the centrality in the sense of Huq. If two effective equivalence relations centralize each other (in the sense of Smith) then it is true that the normal subobjects associated with them, called normalizations [4], centralize in the sense of Huq [6].

The converse is not true in general. If $\mathcal{C}$ is a category such that any two effective equivalence relations always centralize each other as soon as their normalizations centralize in the sense of Huq, we will say that $\mathcal{C}$ satisfies the *Smith is Huq* property [6].

For example, the categories of groups and cocommutative Hopf algebras over a field satisfy this condition [15].

In the paper [18], the authors studied this condition in the context of S-protomodular categories.

They considered pointed S-protomodular categories with respect to a class of split epimorphisms S that are stable under composition and such that any product projection, i.e. any such diagram

$$\begin{tikzcd} X \arrow[r, shift right, "i_1"'] & X \times B \arrow[l, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] & B, \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

where $i_1 := (1_X, 0)$ and $i_2 := (0, 1_B)$, belongs to the class S.

In that context, the condition *Smith is Huq* means that two effective S-equivalence relations centralize each other (in the sense of Smith) if and only if their normalizations commute in the sense of Huq. The category of monoids (but also the categories of monoids with operations [19]) satisfies this property, with respect to the class of Schreier split epimorphisms.

First, we prove that the class S_{coc} is stable under composition.

Proposition 4.1. *The class S_{coc} of split extensions of cocommutative bialgebras in any locally presentable symmetric monoidal category is closed under composition.*

Proof. Let $X \underset{\kappa}{\overset{\lambda}{\leftrightarrows}} A \underset{\alpha}{\overset{e}{\leftrightarrows}} B$ and $Y \underset{\kappa'}{\overset{\lambda'}{\leftrightarrows}} B \underset{\alpha'}{\overset{e'}{\leftrightarrows}} C$ be two composable split extensions in S_{coc}. We can build the following diagram

$$Z \underset{\hat{\kappa}}{\overset{\hat{\lambda}}{\leftrightarrows}} A \underset{\alpha' \cdot \alpha}{\overset{e \cdot e'}{\leftrightarrows}} C$$

where $\hat{\kappa}\colon Z \to A$ is the kernel of $\alpha' \cdot \alpha$ and $\hat{\lambda}$ is the factorization through $\hat{\kappa}$ of the morphism of coalgebras $m_A \cdot (\kappa \cdot \lambda \otimes e \cdot \kappa' \cdot \lambda' \cdot \alpha) \cdot \Delta_A$. In particular, we have the following equality:

$$\hat{\kappa} \cdot \hat{\lambda} = m_A \cdot (\kappa \cdot \lambda \otimes e \cdot \kappa' \cdot \lambda' \cdot \alpha) \cdot \Delta_A. \tag{4.1}$$

We prove that this construction belongs to S_{coc}. The condition (3) is proven via the commutativity of the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[dr, "\Delta_A"'] \arrow[ddd, "1_A"'] & & A^2 \arrow[rr, "\hat{\kappa} \cdot \hat{\lambda} \otimes 1_A"] \arrow[drr, "\Delta_A \otimes 1_A"] & & A^2 \arrow[rr, "1_A \otimes e \cdot e' \cdot \alpha' \cdot \alpha"] & & A^2 \arrow[r, "m_A"] & A \\
 & A^2 \arrow[u, phantom, "(1.4)"] \arrow[rrr, "1_A \otimes \Delta_A"] \arrow[d, "1_A \otimes \alpha"] & & & A^3 \arrow[r, "\kappa \cdot \lambda \otimes e \cdot \kappa' \cdot \lambda' \cdot \alpha \otimes 1_A"] \arrow[u, phantom, "(4.1)"] & A^3 \arrow[u, "m_A \otimes 1_A"] \arrow[r, "1_A \otimes e \cdot e' \cdot \alpha' \cdot \alpha"'] & A^3 \arrow[u, "m_A \otimes 1_A"] \arrow[d, "1_A \otimes m_A"] \arrow[r, phantom, "(1.3)"] & \\
 & A \otimes B \arrow[rr, "1_A \otimes \Delta_B"] \arrow[rrrr, bend right, "\kappa \cdot \lambda \otimes 1_A"'] & & A \otimes B^2 \arrow[r, "\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes e' \cdot \alpha'"'] & A \otimes B^2 \arrow[r, "1_A \otimes m_B"] \arrow[d, phantom, "(3)"] & A \otimes B \arrow[r, "1_A \otimes e"'] \arrow[d, phantom, "(3)"] & A^2 \arrow[uur, "m_A"'] & \\
A \arrow[rrrrrrr, "1_A"] & & & & & & & A \arrow[uuu, "1_A"']
\end{tikzcd}$$

The condition (4) holds thanks to the commutativity of the two diagrams in Fig. 1, where $\overline{\alpha \cdot \hat{\kappa}}$ is the factorization of $\alpha \cdot \hat{\kappa}$ through the kernel κ' of α':

$$\kappa' \cdot \overline{\alpha \cdot \hat{\kappa}} = \alpha \cdot \hat{\kappa}, \tag{4.2}$$

and the fact that $\hat{\lambda} \cdot m_A \cdot (\hat{\kappa} \otimes e \cdot e')$ and $(1_Z \otimes \epsilon_C)$ are coalgebras morphisms and $\hat{\kappa}$ is a monomorphism of coalgebras. $\square$

Thanks to this proposition and Corollary 1.1, $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ is a category in which we can apply the results obtained in [18]. To apply the results of [18], we recall the notions of S-equivalence relation and reflexive-multiplicative graph.

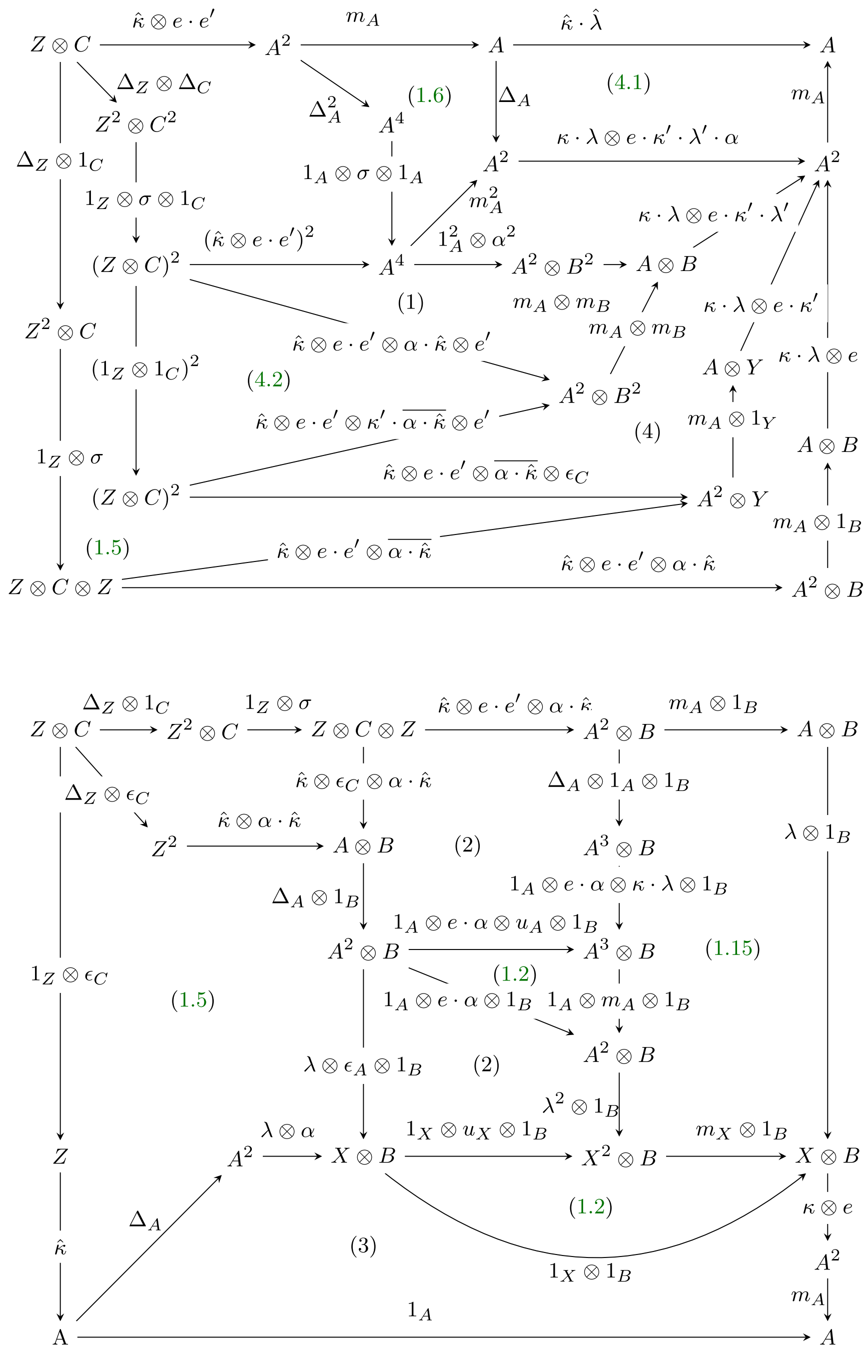

Fig. 1. Condition (4) of the composition of two split extensions in S_{coc}.

Definition 4.2. An equivalence relation

$$\begin{tikzcd} A \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\beta"'] & B \arrow[l, "e" description] \end{tikzcd}$$

is an S-equivalence relation if the split epimorphism (α, e) is in S.

Definition 4.3 ([11]). In a category $\mathcal{C}$ with pullbacks, a reflexive graph, denoted by $\begin{tikzcd} A_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "\iota" description] \end{tikzcd}$, together with a morphism $c\colon A_1 \times_{A_0} A_1 \to A_1$ is called a *reflexive-multiplicative graph*

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "c"] & A_1 \arrow[r, shift left=2, "\delta"] \arrow[r, shift right=2, "\gamma"'] & A_0 \arrow[l, "\iota" description] \end{tikzcd}, \tag{4.3}$$

where $A_1 \times_{A_0} A_1$ is the (object part of the) following pullback

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & A_1 \arrow[d, "\gamma"] \\ A_1 \arrow[r, "\delta"'] & A_0 \end{tikzcd}$$

and c is a multiplication that is required to satisfy the identities

$$c \cdot (1_{A_1}, \iota \cdot \delta) = 1_{A_1} = c \cdot (\iota \cdot \gamma, 1_{A_1}), \tag{4.4}$$

where $(1_{A_1}, \iota \cdot \delta)\colon A_1 \to A_1 \times_{A_0} A_1$ and $(\iota \cdot \gamma, 1_{A_1})\colon A_1 \to A_1 \times_{A_0} A_1$ are induced by the universal property of the pullback $A_1 \times_{A_0} A_1$.

To apply [18, Theorem 4.2], we need the following result.

Proposition 4.2. *Every reflexive graph*

$$\begin{tikzcd} & X' \arrow[d, shift right=1, "\kappa'"'] & \\ X \arrow[r, "\kappa"'] & A \arrow[l, dashed, bend right=0, shift right=2, "\lambda"'] \arrow[u, dashed, shift right=1, "\lambda'"'] \arrow[r, shift left=2, "\alpha"] \arrow[r, shift right=2, "\beta"'] & B \arrow[l, "e" description] \end{tikzcd}$$

such that $\begin{tikzcd} X \arrow[r, "\kappa"'] & A \arrow[l, dashed, shift right=2, "\lambda"'] & B \arrow[l, shift right=1, "e"'] \end{tikzcd}$ ($A \xrightarrow{\alpha} B$) *and* $\begin{tikzcd} X' \arrow[r, "\kappa'"'] & A \arrow[l, dashed, shift right=2, "\lambda'"'] & B \arrow[l, shift right=1, "e"'] \end{tikzcd}$ ($A \xrightarrow{\beta} B$) *belong to* S_{coc} *and* $[X, X'] = 0$, *is a reflexive-multiplicative graph.*

Proof. Let consider the following pullback

$$\begin{tikzcd} A \otimes_B A \arrow[r, "p_2"'] \arrow[d, "p_1"'] & A \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & B \end{tikzcd}$$

where $A \otimes_B A$ is the object in the following equalizer:

$$A \otimes_B A \xrightarrow{\ \varepsilon\ } A \otimes A \underset{(1_A \otimes \alpha \otimes 1_A)\cdot(\Delta_A \otimes 1_A)}{\overset{(1_A \otimes \beta \otimes 1_C)\cdot(1_A \otimes \Delta_A)}{\rightrightarrows}} A \otimes B \otimes A\,. \tag{4.5}$$

By defining $c\colon A \otimes_B A \to A$ by

$$c = m_A \cdot (\kappa \cdot \lambda \otimes 1_A) \cdot \varepsilon$$

we can verify that this reflexive graph is multiplicative thanks to Figs. 2 and 3. Indeed, these diagrams imply the following equality:

$$m_A \cdot (\kappa \cdot \lambda \otimes 1_A) \cdot \varepsilon \cdot m_{A \otimes_B A} = m_A \cdot m_A^2 \cdot (\kappa \cdot \lambda \otimes 1_A)^2 \cdot \varepsilon^2$$

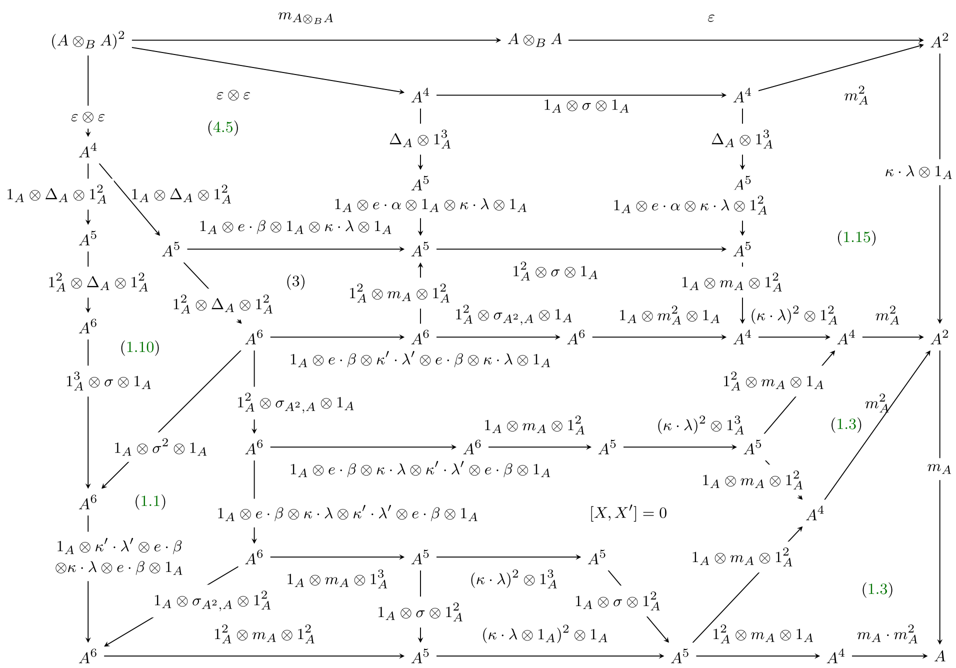

Fig. 2. The morphism c is a morphism of bialgebras (part 1).

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$$
\begin{tikzcd}
(A \otimes_B A)^2 \arrow[r, "\varepsilon \otimes \varepsilon"] \arrow[d, dashed, "\varepsilon \otimes \varepsilon"'] & A^4 \arrow[r, "(\kappa \cdot \lambda \otimes 1_A)^2"] \arrow[d, dashed, "1_A \otimes \Delta_A \otimes 1_A^2"'] \arrow[dr, phantom, "(3)"] & A^4 \arrow[rrr, "m_A^2"] \arrow[drr, phantom, "(1.3)"] & & & A^2 \arrow[dddddd, "n_A"] \\
A^4 \arrow[d, dashed, "1_A \otimes \Delta_A \otimes 1_A^2"'] \arrow[r, phantom, "(1.5)"] & A^5 \arrow[r, "\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes e \cdot \beta \otimes \kappa \cdot \lambda \otimes 1_A"] \arrow[d, dashed, "1_A^2 \otimes \Delta_A^2 \otimes 1_A"'] & A^5 \arrow[u, "1_A \otimes m_A \otimes 1_A^2"'] \arrow[r, "1_A^2 \otimes m_A \otimes 1_A"] \arrow[d, dashed, "1_A^2 \otimes \Delta_A^2 \otimes 1_A"'] \arrow[dr, phantom, "(1.6)"] & A^4 \arrow[r, "1_A^5"] \arrow[dd, "1_A^2 \otimes \Delta_A \otimes 1_A"'] \arrow[ddr, phantom, "(3)"] & A^4 \arrow[ur, "m_A^2"] & \\
A^5 \arrow[d, dashed, "1_A^2 \otimes \Delta_A \otimes 1_A^2"'] & A^7 \arrow[d, dashed, "1_A^3 \otimes \sigma \otimes 1_A^2"'] & A^7 \arrow[d, dashed, "1_A^3 \otimes \sigma \otimes 1_A^2"'] & & & \\
A^6 \arrow[d, "1_A^3 \otimes \sigma \otimes 1_A"'] \arrow[dr, "1_A^3 \otimes \sigma \otimes 1_A"] & A^7 \arrow[r, "\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes (e \cdot \beta \otimes \kappa \cdot \lambda)^2 \otimes 1_A"] \arrow[d, "1_A^5 \otimes \epsilon_A \otimes 1_A"'] \arrow[dr, phantom, "(1) + (2)"] & A^7 \arrow[r, "1_A^2 \otimes m_A^2 \otimes 1_A"] \arrow[d, "1_A^4 \otimes (e \cdot \alpha)^2 \otimes 1_A"'] & A^5 \arrow[r, "1_A^2 \otimes \kappa \cdot \lambda \otimes e \cdot \alpha \otimes 1_A"] & A^5 \arrow[uu, "1_A^2 \otimes m_A \otimes 1_A"'] & \\
A^6 \arrow[dd, "1_A \otimes \kappa' \cdot \lambda' \otimes e \cdot \beta \otimes \kappa \cdot \lambda \otimes e \cdot \beta \otimes 1_A"'] & A^6 \arrow[dr, "\kappa \cdot \lambda \otimes \kappa' \cdot \lambda' \otimes e \cdot \beta \otimes \kappa \cdot \lambda \otimes e \cdot \beta \otimes 1_A"'] & A^7 \arrow[r, "1_A^2 \otimes m_A^2 \otimes 1_A"] & A^5 \arrow[ur, "1_A^2 \otimes \kappa \cdot \lambda \otimes 1_A^2"] \arrow[dr, phantom, "(1.2)"] & & \\
& & A^6 \arrow[u, "1_A^5 \otimes u_A \otimes 1_A"'] \arrow[d, dashed, "(1_A^2 \otimes \kappa \cdot \lambda \otimes 1_A^2) \cdot (1_A^2 \otimes m_A \otimes 1_A^2)"'] & & & \\
A^6 \arrow[r, "1_A^2 \otimes m_A \otimes 1_A^2"'] & A^5 \arrow[r, "(\kappa \cdot \lambda \otimes 1_A)^2 \otimes 1_A"'] & A^5 \arrow[r, "1_A^2 \otimes m_A \otimes 1_A"'] & A^4 \arrow[rr, "m_A \cdot m_A^2"] & & A
\end{tikzcd}
$$

Fig. 3. The morphism c is a morphism of bialgebras (part 2).

which by definition of c means that c is a morphism of algebras (and then also a morphism of bialgebras). □

This result and [18, Theorem 4.2] imply the following result.

Theorem 4.1. *In* $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$, *let* S_{coc} *be the class of split extensions of cocommutative bialgebras in any locally presentable symmetric monoidal category as defined in Definition* 1.2, *two effective* S_{coc}*-equivalence relations centralize each other* (*in the sense of Smith*) *if and only if their normalization commute in the sense of Huq.*

By restricting the above theorem to the locally presentable symmetric monoidal category $(\mathsf{Set}, \times, \{\star\})$, we obtain the result on monoids of [18]. Note that, in [18], they also prove this result for every category of monoids with operations.

5. Conclusion

In [22], we introduced a class of split extensions of (cocommutative) bialgebras, called S_{coc}, such that their category is equivalent to the category of actions of (cocommutative) bialgebras and we proved the Split Short Five Lemma when we restrict it to the split extensions of (cocommutative) bialgebras. In this paper, we use this class S_{coc} to prove that $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ is S_{coc}-protomodular, where $\mathcal{C}$ is a locally presentable symmetric monoidal category. It implies that we can now apply the theory and results of [9, 5] to obtain new results for cocommutative bialgebras. For example, it follows that $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ is S_{coc}-Mal'tsev [9, 5], hence any S_{coc}-reflexive relation is transitive. Another result of this paper is the description of the notion of centrality in the sense of Huq for cocommutative bialgebras. Moreover, we prove that $\mathsf{BiAlg}_{\mathcal{C},\mathrm{coc}}$ satisfies the "partial" *Smith is Huq* condition, meaning that two effective S_{coc}-equivalence relations centralize each other as soon as their normalization commute in the sense of Huq. Note that Theorems 4.1 and 2.1 generalize results in [7, 18] in the category of monoids.

Acknowledgments

The author would like to thank Professor Marino Gran (UCLouvain) for suggesting this question as well as for all the advice in the realization of this paper. The author thanks the anonymous referee for his/her useful remarks. The author's research is supported by a FRIA doctoral grant no. 27485 of the *Communauté française de Belgique*.

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