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The Procedural Value

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Abstract

We propose a single-valued solution that extends both the consistent Shapley value of Maschler and Owen (1989) and Raiffa's discrete bargaining solution to a large class of NTU games. Though not axiomatized, the solution is motivated *via* the Nash program. In this respect, we follow an approach that is similar to the one initiated by Hart and Mas-Collel (1996).

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Introduction A game with non-transferable utility (NTU) specifies, for each coalition, the set of utility profiles that its members can guarantee themselves if they cooperate. Various values have been defined in this context, the main ones being defined by Harsanyi (1963), Shapley (1969) and Maschler and Owen (1992). Two subclasses of NTU games were especially studied: the two-player case and the games with transferable utility (TU). Up to now, the litterature was mainly concerned in defining solutions that extend both the Nash (1950) bargaining solution and the Shapley (1953) value. Axiomatic systems were designed in order to better understand and to discriminate among the various solutions (cf. Aumann (1985), Hart (1985) and de Clippel (2002)). Recently, the Nash program was also successfully applied by Hart and Mas-Collel (1996) in order to support the consistent Shapley value of Maschler and Owen (1992).

In what follows, we define a single-valued solution (named the procedural value) that extends both the Shapley TU-value (even the consistent Shapley value of Maschler and Owen (1989)) and Raiffa's discrete bargaining solution (cf. Luce and Raiffa (1957), pages 136-137) to a large class of NTU games. Though not axiomatized, the solution is motivated *via* the Nash program. The bargaining procedures we consider are closely related to those defined in Hart and Mas-Collel (1996).

\$1. Notations Let n be a positive integer and let $N := \{1, \dots, n\}$ be the set of players. $P(N)$ denotes the set of coalitions, that is the set of nonempty subsets of N. $P^*(N)$ denotes the set of coalitions of cardinality greater or equal to two. Elements of $\mathbb{R}^N$ that represent a distribution of payoffs are called *allocations*. Let S be a coalition. For each $i \in N$, $S + i$ (resp. $S - i$) stands for $S \cup \{i\}$ (resp. $S \setminus \{i\}$). $\mathbb{R}^S$ is seen as a subset of $\mathbb{R}^N$: $\mathbb{R}^S := \{x \in \mathbb{R}^N | (\forall i \in N \setminus S) : x_i = 0\}$. Inequalities between vectors in $\mathbb{R}^S$ are of three kinds: $\geq_S$, $>_S$, $>>_S$ (usual definitions). $\mathbb{R}^S_+$ (resp. $\mathbb{R}^S_{++}$) denotes the set of vectors in $\mathbb{R}^S$ with nonnegative (resp. positive) components in S. When X is a subset of $\mathbb{R}^N$, $proj_S(X)$ denotes the projection of X on $\mathbb{R}^S$. When $x \in \mathbb{R}^N$, x_S denotes the projection of x on $\mathbb{R}^S$. When X is a subset of $\mathbb{R}^S$, $int_S(X)$ (resp. $\partial_S X$) denotes the topological interior (resp. frontier) of X in $\mathbb{R}^S$.

\$2. Games A *(cooperative) game* V is a function that associates, to each coalition S, a nonempty closed comprehensive convex strict subset $V(S)$ of $\mathbb{R}^S$ such that:

1. $(\forall S \in P(N))(\forall x \in \partial_S V(S)) : \{\lambda \in \mathbb{R}^S_+ \setminus \{0\} | \lambda.x = \max_{y \in V(S)} \lambda.y\} \subseteq$

$\mathbb{R}^S_{++}$;

2. $(\forall S \in P(N))(\forall i \in N \setminus S) : V(S) \subseteq proj_S(V(S+i))$.

For each $S \in P(N)$, the boundary of $V(S)$ has to be nonlevel according to condition 1. Contrarily to Hart and Mas-Collel (1996), we don't impose any smoothness assumption. The second condition is a weak monotonicity property that is equivalent to the one that appears in Otten *et al.* (1998). For each coalition $S \in P(N)$, each player $i \in N \setminus S$, and each allocation x that is feasible for S (i.e. in $V(S)$), there must exist an allocation x' that is feasible for $S+i$ (i.e. in $V(S+i)$) such that the players in S are treated in the same way with x than with x' (i.e. $x_S = x'_S$). Nevertheless, player i may be treated very badly in x', meaning that x'_i may be very low. Hart and Mas-Collel (1996) uses a stronger monotonicity condition that essentially requires x'_i to be independent of x, for each $i \in S$. Unfortunately, hyperplane games that are not equivalent to some TU-game (cf. definitions hereafter), violate it. A still stronger condition would be to impose some superadditivity. A game V is *(weakly) superadditive* if

$$(\forall S \in P(N))(\forall i \in S) : \{x_S + d_{\{i\}} | x_S \in V(S)\} \subseteq V(S+i),$$

where $d_{\{i\}} := \arg\max_{x \in V(\{i\})} x_i \in V(\{i\})$, for each $i \in N$.

Two subclasses of games have been much studied in the literature. A *TU-game* is a function $v : P(N) \to \mathbb{R}$. A game V is *equivalent to the TU-game* v if $V(S) = \{x \in \mathbb{R}^S | \sum_{i \in S} x_i \leq v(S)\}$, for each coalition S. An *hyperplane game* (cf. Maschler and Owen (1989)) is a game for which there exists a pair $(\lambda, v) \in (\times_{S \in P(N)} \mathbb{R}^S_{++}) \times (\mathbb{R}^{P(N)})$ such that $V(S) = \{x \in \mathbb{R}^S | \lambda_S . x \leq v(S)\}$, for each coalition S. Any game that is equivalent to a TU-game, is an hyperplane game.

\$3. The Procedural Value Let V be a game. The *procedural value* $\sigma(V) \in \times_{S \in P(N)} \partial_S V(S)$ of V is defined by induction on the cardinality of the coalitions. For each $i \in N$, $\sigma_{\{i\}}(V) := d_{\{i\}}$. Let then S be a coalition with at least two members and let us assume that $\sigma_T(V)$ has already been defined for each coalition T that is strictly included in S. For each $i \in S$ and each $z \in proj_{S-i}(V(S))$, let

$$x^i(V(S), z) := \arg \max_{x \in V(S) \text{ s.t. } x_{S-i} = z} x_i \qquad \in \partial_S V(S).$$

This vector represents the[1] choice of player $i \in S$, when he has the dictatorial power to choose for S, having though the constraint to give at least

[1] $\arg\max_{x \in V(S) \text{ s.t. } x_{S-i} \geq z} x_i = \arg\max_{x \in V(S) \text{ s.t. } x_{S-i} = z} x_i$, because $V(S)$ is comprehensive and nonlevel.

the utilities as specified in z to the other members of S.
Let us then define the sequence $(w_S(V,k))_{k\in\mathbb{N}\setminus\{0\}}$ in $V(S)$ as follows:

$$\begin{cases} w_S(V,1) := \frac{1}{s}\sum_{i\in S} x^i(V(S),\sigma_{S-i}(V)) \\ (\forall k\in\mathbb{N}\setminus\{0\}) : w_S(V,k+1) := \frac{1}{s}\sum_{i\in S} x^i(V(S), proj_{S-i}(w_S(V,k))). \end{cases}$$

The vector $w_S(V,1)$ is well-defined thanks to condition 2 that appears in the definition of a game.
Finally, we define $\sigma_S(V)$ as follows:

$$\sigma_S(V) := \lim_{k\to\infty} w_S(V,k).$$

The sequence $(w_S(V,k))_{k\in\mathbb{N}\setminus\{0\}}$ being weakly increasing in each component, it is easy to check that the limit is well-defined and that $\sigma_S(V)$ is Pareto-optimal in $V(S)$.

The procedural value can be interpreted and motivated as follows. Let $S\in P^*(N)$. The vector $\sigma_S(V)\in V(S)$ describes the payoff allocation that would be agreed upon, should the players in S cooperate. This agreement is assumed to satisfy some fairness criteria.
Considering $\sigma_S(V)$ as being a fair solution to the bargaining problem[2] $(V(S),d_S)$ would be misleading as the possibility of partial cooperation would not be taken properly into account. Alternatively, we assume that the fairness of any specific allocation in $V(S)$ is influenced by what would happen to the players in $S-i$ if they were all cooperating without some member i of S, that is by the profile of vectors $(\sigma_{S-i}(V))_{i\in S}$ (well-defined by recursion). The importance of this profile of vectors for the determination of $\sigma_S(V)$ was first recognized by Myerson (1980) in the case of TU-games when he introduced the concept of "balanced contributions", and was confirmed afterwards by Hart and Mas-Collel (1996) (cf. their proposition 4) in the case of NTU-games as far as the consistent Shapley value is concerned.
In this context, the allocation $w_S(V,1)$ may look as an *equitable* compromise between the conflicting interests of the members of S. Indeed, it is the rational outcome of an equitable procedure that consists in choosing at random, according to a uniform distribution, a player in S that receives the power to choose for S, having though the constraint to ascertain the participation of the other members of S who have the outside option $\sigma_{S-i}(V)$ at their disposal.
The allocation $w_S(V,1)$ cannot be considered as the *fair* agreement for S

[2] $d := \sum_{i\in N} d_{\{i\}}$.

when it is not Pareto optimal. Nevertheless, the equity of $w_S(V,1)$ leads us to consider it as a *partial agreement* that will be renegotiated in the cases where there remains some surplus to be shared.
The way partial agreements are renegotiated, is determined *via* the fair arbitration scheme proposed by Professor Raiffa (cf. Luce and Raiffa (1957), pages 136-137). This scheme uses a similar idea to the one developed previously in order to show that $w_S(V,1)$ could be considered as a reasonable partial agreement for the members of S, namely that the procedure "pick up a dictator at random, with equal probabilities" should determine a new equitable partial agreement starting from each temporary equitable *status quo*. The only difference with the first part of the argument that motivated $w_S(V,1)$, is that each dictator (let's say $i \in S$) is now facing a threat that is determined by the temporary *status quo*, instead of $\sigma_{S-i}(V)$.
New light will be cast on the solution in the next section, as we will see that it is the only rational outcome of some natural bargaining game described through an extensive form.

The procedural value coincides with the consistent Shapley value on the class of hyperplane games (cf. Maschler and Owen (1989); especially lemma 1). *A fortiori*, it coincides with the Shapley TU-value on the class of games that are equivalent to some TU-game. On the other hand, the procedural value coincides, by construction, with Raiffa's discrete solution (cf. Luce and Raiffa (1957), pages 136-137) on the class of two-players games.

The procedural value is scale covariant. If the game V is (weakly) superadditive, then $\sigma_S(V)$ is individually rational for the players in S (i.e. $\sigma_S(V) \geq_S d_S$), for each $S \in P(N)$.

\$4. Non-Cooperative Foundation Let V be a game, let K be a positive integer, and let a be an allocation. The extensive form game $\Gamma(V,K,a)$ is made up of stage-games that are indexed by pairs (S,k), where $S \in P(N)$ denotes the active coalition and $k \in \{0,\ldots,K-1\}$ indicates the number of times S has already been active in the history of the game. The number K thus represents the maximal number of times a coalition can be active. In stage (S,k), a player (let's say i) is chosen at random in S, according to a uniform distribution. He proposes an allocation (let's say x) that is feasible for S (i.e. in $V(S)$). Then, every member of S sequentially (in some prespecified order) chooses wether to accept player i's proposal. If everybody accepts, then the game stops with the allocation x being enforced. If somebody rejects and if $k \in \{0,\ldots,K-2\}$ (resp. $k = K-1$ and $S \in P^*(N)$; resp. $k = K-1$ and $\#S = 1$), then the stage-game

$(S, k+1)$ is played (resp. player i receives the payoff a_i and the stage-game $(S-i, 0)$ is played; resp. player i receives the payoff a_i and the game stops).

Theorem Let V be a game and let a be an allocation. If a is sufficiently small, then

$$\lim_{K\to\infty} (SPE_S(V, K, a))_{S\in P(N)} = \sigma(V),$$

where, for each $S \in P(N)$, $SPE_S(V, K, a)$ denotes the projection on $\mathbb{R}^S$ of the unique subgame perfect equilibrium outcome associated to each subgame of $\Gamma(V, K, a)$ that starts with the stage-game $(S, 0)$.

The extensive form games $\Gamma(V, K, a)$ we consider as relevant bargaining procedures, are closely related to the ones proposed by Hart and Mas-Collel (1996) in order to support the consistent Shapley value of Maschler and Owen (1992). In their framework, the parameter K is replaced by a parameter $\rho \in]0, 1]$ that represents[3] the probability for each player to be excluded of the active coalition and to receive the payoff a_i, when his proposition is rejected by some member of the active coalition. Their extensive form games can potentially last infinitely many period, although this happens with probability zero. They thus have to focus on stationary subgame perfect equilibria (SSPE), as folk-like theorems hold in their model for the larger set of subgame perfect equilibria (SPE). Moreover, they obtain the weaker result that every sequence of SSPE outcomes indexed by ρ converges to some[4] consistent Shapley value, as ρ tends to zero. They don't establish the converse. Our approach also shows the limits of the robustness of their result: a slight variation (out of the class of variations defined in their sixth section) of their bargaining procedure may yield the Shapley value in the TU-case and something different from the consistent Shapley value in the NTU-case.

Condition (1) in the proof of the theorem gives an upper bound on the allocation a that is sufficiently low in order to get the result. This upper bound is scale and translation covariant. It is thus independent of the particular utility functions chosen to represent the underlying von Neumann/Morgenstern preferences. The fact that each a_i shouldn't be too large, is clearly a necessary condition for the theorem to be valid: we need to give some incentives to the proposers to promote cooperation. For instance, for each positive integer K, the unique SPE outcome of $\Gamma(V, K, d)$ is d for

[3]In fact, Hart and Mas-Collel (1996) uses the greek letter ρ in order to represent the probability of the complementary event.

[4]The consistent Shapley value is not necessarily single-valued.

the subadditive two-players game V defined by: $V(\{1\}) := \{x \in \mathbb{R}^{\{1\}} | x_1 \leq 1\}$, $V(\{2\}) := \{x \in \mathbb{R}^{\{2\}} | x_2 \leq 1\}$, and $V(\{1,2\}) := \{x \in \mathbb{R}^{\{1,2\}} | x_1 + x_2 \leq 0\}$. Nevertheless, the theorem holds with $a = d$ (which is a very intuitive choice) when we restrict ourselves to (weakly) superadditive games.

\$5. Proofs Before proving the main theorem, we establish a few useful lemmas.

Lemma 1 *Let V be a game, let $S \in P^*(N)$, and let $i \in S$. Then, $proj_{S-i}(V(S))$ is open in $\mathbb{R}^{S-i}$.*

Proof: Let $Y := proj_{S-i}(V(S))$. We note that Y is a convex and comprehensive subset of $\mathbb{R}^{S-i}$. Let us assume that Y is not open in $\mathbb{R}^{S-i}$. We fix $y \in Y \setminus int_{\mathbb{R}^{S-i}}(Y)$. By the supporting hyperplane theorem, let $\lambda \in \mathbb{R}_+^{S-i} \setminus \{0\}$ be a vector that is orthogonal to Y at y. Then, λ is also orthogonal to $V(S)$ at $x^i(V(S), y)$. However, $\lambda_i = 0$. This comes in contradiction with the fact that $V(S)$ is nonlevel.

$\square$

Lemma 2 *Let V be a game, let $S \in P^*(N)$, and let $i \in S$. Then, the function $\psi_i : proj_{S-i}(V(S)) \rightarrow \mathbb{R}$ defined by:*

$$\psi_i(y) := \max_{x \in V(S) \ s.t. \ x_{S-i} = y} x_i,$$

for each $y \in proj_{S-i}(V(S))$, is continuous.

Proof: Indeed, ψ_i is a concave function defined on an open subset of $\mathbb{R}^{S-i}$ (cf., for instance, theorem D on page 93 in Roberts and Varberg (1973)).

$\square$

Lemma 3 *Let V be a game. For each positive integer K, let*

$$(\hat{w}_S(V, K, k))_{k \in \{1,\ldots,K\}, \ S \in P(N)}$$

be the finite sequence defined as follows:

$$\begin{cases} (\forall i \in N)(\forall k \in \{1, \ldots, K\}) : \hat{w}_{\{i\}}(V, K, k) := d_{\{i\}} \\ (\forall S \in P^*(N)) : \hat{w}_S(V, K, 1) := \frac{1}{s} \sum_{i \in S} x^i(V(S), \hat{w}_{S-i}(V, K, K)) \\ (\forall S \in P^*(N))(\forall k \in \{2, \ldots, K\}) : \\ \qquad \hat{w}_S(V, K, k) := \frac{1}{s} \sum_{i \in S} x^i(V(S), proj_{S-i}(\hat{w}_S(V, K, k-1)). \end{cases}$$

Then, $\lim_{K \rightarrow \infty} \hat{w}_S(V, K, K) = \sigma_S(V)$, for each $S \in P(N)$.

Proof: (by induction on the cardinality of S) The result is obvious for each coalition of cardinality 1. Let us thus consider a coalition $S \in P^*(N)$, and let us assume that we already proved the result for each coalition T that is strictly included in S.
For each $\epsilon > 0$, let $z(\epsilon) := \sigma_S(V) - \epsilon 1_S \in V(S)$ and let

$$A_\epsilon := \{x \in \mathbb{R}^S | (\forall i \in S) : x_i \in]z_i(\epsilon), \psi_i(z_{S-i}(\epsilon))[\}.$$

We note that A_ϵ is an open subset of $\mathbb{R}^S$ that contains $\sigma_S(V)$ (since $V(S)$ is nonlevel). In addition, for each open set O around $\sigma_S(V)$, there exists $\epsilon > 0$ such that $A_\epsilon \subseteq O$.
We end the proof by showing that

$$(\forall \epsilon > 0)(\exists K \in \mathbb{N} \setminus \{0\})(\forall K' \geq K) : \hat{w}_S(V, K', K') \in A_\epsilon.$$

Let $\epsilon > 0$. Let K_1 be such that $w_S(V, K_1) \in A_\epsilon$. Thanks to the inductive hypothesis and lemma 2, there exists a $K_2 \geq K_1$ such that $\hat{w}_S(V, K', K_1) \in A_\epsilon$, for each $K' \geq K_2$. As $\hat{w}_S(V, K', K') \geq \hat{w}_S(V, K', K_1)$ and $\hat{w}_S(V, K', K') \in V(S)$, this implies that $\hat{w}_S(V, K', K') \in A_\epsilon$, for each $K' \geq K_2$.

□

Proof of the theorem: Let a be an allocation such that

$$a_i \leq \min(d_i, \min_{S \in P^*(N) \text{ s.t. } i \in S} \inf_{L \in \mathbb{N} \setminus \{0\}} \psi_i(\hat{w}_{S-i}(V, L, L)), \tag{1}$$

for each $i \in N$. The right-hand side of the inequality appearing in (1) is a real number. Indeed, the function ψ_i is continuous thanks to lemma 2, and the set $\{\hat{w}_{S-i}(V, L, L) | L \in \mathbb{N} \setminus \{0\}\} \cup \{\sigma_{S-i}(V)\}$ is compact thanks to lemma 3.
Let K be a positive integer. We prove by backward induction that

$$(\forall S \in P(N))(\forall k \in \{0, \ldots, K-1\}) : SPE_S(V, K, a, k) = \hat{w}_S(V, K, K-k), \tag{2}$$

where, for each $(S, k) \in P(N) \times \{0, \ldots, K-1\}$, $SPE_S(V, K, a, k)$ denotes the projection on $\mathbb{R}^S$ of the unique SPE outcome associated to each subgame of $\Gamma(V, K, a)$ that starts with the stage-game (S, k).
Property (2) is obviously satisfied whenever S is a singleton. So, let us consider a subgame of $\Gamma(V, K, a)$ that starts with the stage-game (S, k), for some pair $(S, k) \in P^*(N) \times \{0, \ldots, K-1\}$ such that property (2) is satisfied for each subgame of $\Gamma(V, K, a)$ that starts with a stage-game that follows the stage-game (S, k). Let us focus, in addition, on the case where

player $i \in S$ has been selected by nature. Let $x_S \in V(S)$ be the proposition he makes in some SPE of $\Gamma(V, K, a)$, and let y_S be the expected payoff (as specified by property (2) and by a_i, if $k = K - 1$) of the members of S, if some of them rejects player i's proposal. We note that $y_S \in V(S)$. This fact is obvious if $k \neq K - 1$, and is a consequence of condition (1) if $k = K - 1$. The following steps are easy to check.

Step 1 If $x_S >>_S y_S$, then all the members of S accepts the proposition of player i;
Step 2 If every member of S accepts the proposition of player i, then it cannot be that $x_{S-i} >_{S-i} y_{S-i}$. Indeed, suppose that x_S is accepted. If $x_{S-i} >_{S-i} y_{S-i}$, then there would exist $x'_S \in V(S)$ such that $x'_{S-i} >>_{S-i} y_{S-i}$ and $x'_i > x_i$. So, player i would be strictly better off by proposing x'_S instead of x_S (using step 1);
Step 3 If there exists some $i \in S$ such that $x_i < y_i$, then some member of S rejects player i's proposal in any SPE;
Step 4 If $y_S \notin \partial V(S)$, then every member of S accepts player i's proposal. Indeed, if somebody rejects it, then player i receives y_i and has a profitable deviation to some $z \in V(S)$ that strictly Pareto S-dominates y_S (by step 1);
Step 5 If $y_S \notin \partial V(S)$, then $x = x^i(V(S), y_{S-i})$ (by steps 2, 3, and 4);
Step 6 If $y_S \in \partial V(S)$, then the projection on $\mathbb{R}^S$ of the SPE outcome must be y_S (which equals $x^i(V(S), y_{S-i})$), as each player j in S can guarantee himself y_j;

On the other hand, the profile of actions "player i proposes $x^i(V(S), y_{S-i})$ and every member of S accepts player i's proposal" is part of an SPE.
It is then easy to conclude the proof, thanks to lemma 3.

$\square$

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