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MULTI-ITEM LOT-SIZING with a JOINT SET-UP COST

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Abstract

We consider a multi-item lot-sizing problem in which there are demands, and unit production and storage costs. In addition production of any mix of items is measured in batches of fixed size, and there is a fixed set-up cost per batch in each period. Suppose that the unit production costs are constant over time, the storage costs are nonnegative, and for any two items the one that has a higher storage cost in one period has a higher storage cost in every period. Then we show that there is a linear program with $O(mT^2)$ constraints and variables that solves the multi-item lot-sizing problem, thereby establishing that it is polynomially solvable, where m is the number of items and T the number of time periods. This generalizes an earlier result of Anily and Tzur who presented a $O(mT^{m+5})$ dynamic programming algorithm for essentially the same problem. Under additional conditions, a similar linear programming result is shown to hold in the presence of backlogging when the batch size is arbitrarily large.

Brief computational results on two instances with varying batch sizes are presented and discussed.

Keywords: Multi-Item Lot-Sizing, Joint Set-Up Cost, Convex Hull, Extended Formulation, Mixed Integer Programming.

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1 Introduction

Often in supply chain optimization multi-items have to be considered together, due to the dependency of the cost structure or the operational constraints on the total quantities replenished and/or transported. In other cases, the exact composition of the individual items is important. Other complicating characteristics of multi-item replenishment problems include capacity limitations, time dependency of demand and cost parameters and the existence of fixed costs.

In this paper we consider a multi-item lot-sizing problem in which production takes place in mixed batches of constant capacity. Apart from the usual unit storage and production costs, there is a fixed cost per batch representing the use of limited capacity resource. An alternative interpretation involves a warehouse/retailer and decisions on the use of trucks of a given capacity to ship from the warehouse so as to (possibly) stock items and satisfy demand forecasts at the retailer. The objective is to find replenishment decisions for all items that satisfy the demand for the items over a finite planning horizon, and minimize the sum of fixed and variable production costs and storage costs.

Dynamic lot-sizing problems with capacities are some of the most frequently used deterministic inventory planning problems. In addition to being directly representative of various production and transportation settings, they often arise as sub-problems of more complex multi-echelon dynamic lot-sizing problems when those are decomposed as part of the solution method.

However, dynamic lot-sizing problems with capacity restrictions are known to be hard problems. When only one batch is allowed in each period and the capacity limitation is time-dependent, even the single item case is known to be NP-Hard, see Florian et al. [5]. When the capacity limitations are time independent, the single item problem is solved in polynomial time, see Florian and Klein [4] and Van Hoesel and Wagelmans [11]. Pochet and Wolsey [8] considered the single item problem with multiple batches, time-varying set-up, inventory holding and variable production costs. They designed an $O(T^3)$ algorithm, where T is the number of periods, which is based on finding a shortest path in an appropriately defined network. Lee [6] addressed the single item multiple batch problem in which there exists a setup cost for ordering in a particular period, in addition to a different setup cost incurred for each batch, and presented an $O(T^4)$ procedure to solve it.

In this paper we consider the problem with multiple items and multiple batches, that was studied recently by Anily and Tzur ([2],[1]), under the following restrictions:
i) the unit production costs of each item are stationary over time. It follows that the total production costs are constant, and can thus be ignored.
ii) the storage costs for each item are stationary over time and non-negative.

Under these assumptions, Anily and Tzur [2] develop a dynamic programming algorithm whose running time is polynomial for a fixed number of items. More precisely, the running time of their algorithm is O(mT^{m+5}), where m is the number of items, and T is the number of periods. Anily and Tzur [1] propose an alternative optimal search algorithm and heuristics for solving the problem. See the references therein for additional background on the problem and a literature review on single and multiple item capacitated dynamic lot-sizing problems, with and without batching considerations.

Here we consider essentially the same problem, except that ii) is relaxed to ii') and iii'):
ii') the storage costs of each item in each period are required to be non-negative. In view of the fact that the production costs are time independent, the non-negativity of the storage costs results in the *Wagner-Whitin* cost property or *non-speculative costs* for each item.
iii') there exists a permutation Π of the set of items such that the $k-th$ item in the permutation is the $k-th$ most expensive item to store in all periods. In other words, there exists an indexing of the items according to the non-increasing ordering of their storage costs that is preserved in all time periods. We call the resulting problem $WW^* - CC - FAM$.

Under these generalized conditions we derive a tight and compact extended formulation for the problem. Here a formulation is *extended* when it involves additional variables, but its projection back into the original space along with the integrality constraints gives precisely the MIP feasible region. It is *compact* if its size is polynomial in the size of the input, and it is *tight* when its projection is precisely the convex hull of the MIP feasible region.

It follows, because it is compact, that solving a linear program over the extended formulation can be carried out in polynomial time. What is more, because it is tight, an optimal extreme point solution can be found that is feasible and optimal for the MIP. Thus the problem is in $\mathcal{P}$. The effectiveness of this approach has been demonstrated on a wide variety of lot-sizing problems starting with Eppen and Martin [3]. The forthcoming book of Pochet and Wolsey [10] classifies and presents the state-of-the-art on the formulation and solution of a wide variety of production planning problems by mixed integer programming, and demonstrates the effectiveness of tight extended formulations on several industrial cases.

The contributions of this paper are as follows: we present a linear program with $\mathrm{O}(mT^2)$ constraints and variables that solves problem $WW^*-CC-FAM$, thus showing that it is polynomially solvable for any number of items and periods. Alternatively, the problem can be solved as a linear program with $\mathrm{O}(mT)$ constraints and variables combined with an $\mathrm{O}(mT^2\log T)$ separation algorithm. If backlogging is allowed, a similar result is shown to hold for the problem in which, for each individual item, production is uncapacitated, the storage cost is constant over time, and the backlogging cost is a fixed multiple of the constant storage cost. Computationally we demonstrate numerically the effectiveness of the extended formulations relative to the original formulations.

In Section 2 we formally present the problem, and then give the results and proofs. In Section 3 we consider the backlogging case. In section 4 we discuss briefly some computational results indicating both the practical value and difficulties of using a tight reformulation for problem $WW^*-CC-FAM$.

2 MIP and LP Formulations of $WW^*-CC-FAM$

We begin this Section by presenting the notation. Let T be the number of periods and m be the number of items. We also use:
d_t^i - the demand of item i in period t for $1\le i\le m$, $1\le t\le T$.
C - the batch capacity.
Q_t - fixed cost for producing a full or partial batch in period t for $1\le t\le T$.
h_t^i - the unit storage cost of item i in period t for $1\le i\le m$, $1\le t\le T$.
For convenience, we define $h_t^{m+1}=0$ for all t, and we assume that in each period t, $h_t^1\ge h_t^2\ge \ldots\ge h_t^m\ge h_t^{m+1}=0$.

We use the following decision variables:
Y_t - the number of batches produced in period t.
x_t^i - the production quantity of item i in period t.
s_t^i - the storage quantity of item i at the end of period t, where $s_0^i=0$ for all i.

The problem $WW^*-CC-FAM$ can be formulated as the mixed integer program:

$$\min \textstyle\sum_i\sum_t h_t^i s_t^i+\sum_t Q_tY_t \tag{1}$$
$$s_{t-1}^i+x_t^i=d_t^i+s_t^i \text{ for all } i,t \tag{2}$$
$$\textstyle\sum_i x_t^i\le CY_t \text{ for all } t \tag{3}$$
$$x,s\in\mathbb{R}_+^{mT},\ Y\in\mathbb{Z}_+^T. \tag{4}$$

We introduce the following surrogate product i consisting of the first i items together. Let $S_t^i=\sum_{j=1}^i s_t^j$, $X_t^i=\sum_{j=1}^i x_t^j$, $D_t^i=\sum_{j=1}^i d_t^j$ and $H_t^i=h_t^i-h_t^{i+1}\ge 0$. Note that $\sum_i\sum_t h_t^i s_t^i=\sum_i\sum_t h_t^i(S_t^i-S_t^{i-1})=\sum_i\sum_t(h_t^i-h_t^{i+1})S_t^i=\sum_i\sum_t H_t^iS_t^i$, and $X_t^i\le CY_t$ for all i,t, so we obtain a reformulation:

$$\min \sum_i \sum_t H_t^i S_t^i + \sum_t Q_t Y_t \tag{5}$$
$$S_{t-1}^i + X_t^i = D_t^i + S_t^i \text{ for all } i,t \tag{6}$$
$$X_t^i \leq C Y_t \text{ for all } i,t \tag{7}$$
$$X, S \in \mathbb{R}_+^{mT},\ Y \in \mathbb{Z}_+^T \tag{8}$$
$$S_t^i \geq S_t^{i-1}, X_t^i \geq X_t^{i-1} \text{ for all } i,t, \tag{9}$$

where we use $S_0^i = X_0^i = 0$ for all i, and the last constraints (9) come from the non-negativity of s_t^i and x_t^i respectively.

Our first relaxation is precisely to drop the latter constraints (9) giving:

$$\begin{gathered}\min \sum_i \sum_t H_t^i S_t^i + \sum_t Q_t Y_t\\ S_{t-1}^i + X_t^i = D_t^i + S_t^i \text{ for all } i,t\\ X_t^i \leq C Y_t \text{ for all } i,t\\ X, S \in \mathbb{R}_+^{mT},\ Y \in \mathbb{Z}_+^T.\end{gathered}$$

Letting $D_{tl}^i \equiv \sum_{u=t}^l D_u^i$, and noting that $S_{t-1}^i + C\sum_{u=t}^l Y_u \geq S_{t-1}^i + \sum_{u=t}^l X_u^i = D_{tl}^i + S_l^i \geq D_{tl}^i$, we obtain a second relaxation:

$$\min \sum_i \sum_t H_t^i S_t^i + \sum_t Q_t Y_t \tag{10}$$
$$S_{t-1}^i + C\sum_{u=t}^l Y_u \geq D_{tl}^i \text{ for } 1 \leq t \leq l \leq T,\ i = 1, \ldots, m, \tag{11}$$
$$S \in \mathbb{R}_+^{mT},\ Y \in \mathbb{Z}_+^T. \tag{12}$$

Observation 1 *For the single item case, the set* $X^{WW-CC}(s, y, d, C) =$

$$\begin{gathered}s_{t-1} + C\sum_{u=t}^l y_u \geq d_{tl} \text{ for } 1 \leq t \leq l \leq T,\\ s \in \mathbb{R}_+^T, y \in \mathbb{Z}_+^T\end{gathered}$$

has been studied in [9]. $\mathrm{conv}(X^{WW-CC}(s, y, d, C))$ *is known explicitly.*

This motivates taking as a third relaxation the following linear program:

$$\min\{\sum_i \sum_t H_t^i S_t^i + \sum_t Q_t Y_t : \bigcap_{i=1}^m \mathrm{conv}(X^{WW-CC}(S^i, Y, D^i, C))\}.$$

The next result is largely based on results that can be found in the literature, in particular a compact extended formulation for $\mathrm{conv}(X^{WW-CC}(S^i, Y, D^i, C))$.

Proposition 1 *The linear program:*

$$\min \sum_i \sum_t H_t^i S_t^i + \sum_t Q_t Y_t \tag{13}$$
$$S_{t-1}^i = C\mu_t^i + C\sum_{u=t}^T f_{tu}^i \delta_{tu}^i \text{ for all } i,t \tag{14}$$
$$\sum_{u=t}^l Y_u + \mu_t^i + \sum_{\{u: f_{tu}^i \geq f_{tl}^i\}} \delta_{tu}^i \geq \lfloor D_{tl}^i / C \rfloor + 1 \text{ for all } i,t,l \text{ with } 1 \leq t \leq l \leq T \tag{15}$$
$$\sum_{u=t}^{T+1} \delta_{tu}^i = 1 \text{ for all } i,t \tag{16}$$
$$Y \in \mathbb{R}_+^T, \mu \in \mathbb{R}_+^{mT}, (\delta_{tt}^i, \ldots, \delta_{tT}^i, \delta_{t,T+1}^i) \in \mathbb{R}_+^{T-t+2} \text{ for all } i,t, \tag{17}$$

where $f_{tl}^i = D_{tl}^i / C - \lfloor D_{tl}^i / C \rfloor$ *for all* i,t,l *with* $1 \leq t \leq l \leq T$ *and* $f_{t,T+1}^i = 0$ *for all* i,t, *solves the relaxation (10)-(12).*

Proof: The proof consists of two steps. The first is to show that the integer solutions of the linear program (13)-(17) provide a valid reformulation of (10)-(12). The second is to show that the linear program has a solution with Y integer.

To understand this reformulation, note that in an extreme point solution of (10)-(12), S^i_{t-1} mod C must take one of the values $Cf^i_{tt}, \ldots, Cf^i_{tT}, Cf^i_{t,T+1}$. Introducing corresponding 0-1 variables: $\delta^i_{tu} = 1$ if $S^i_{t-1} \mod C = Cf^i_{tu}$, we have that in every extreme point

$$S^i_{t-1} = C\sum_{u=t}^{T} f^i_{tu}\delta^i_{tu} + C\mu^i_t \tag{18}$$

$$\sum_{u=t}^{T+1} \delta^i_{tu} = 1 \tag{19}$$

$$\delta^i_{t.} \in \{0,1\}^{T-t+2}, \mu^i_t \in \mathbb{Z}^1_+. \tag{20}$$

Now the constraint (11) can be rewritten as:

$$\sum_{u=t}^{l} Y_u + \sum_{u=t}^{T} f^i_{tu}\delta^i_{tu} + \mu^i_t \geq D^i_{tl}/C. \tag{21}$$

Finally it suffices to observe that in the set defined by (19),(20) and (21) the constraint:

$$\sum_{u=t}^{l} Y_u + \sum_{\{u: f^i_{tu} \geq f^i_{tl}\}} \delta^i_{tu} + \mu^i_t \geq \lfloor D^i_{tl}/C \rfloor + 1$$

can be used in place of (21) without changing the set of integer solutions. Now as all the extreme points of (10)-(12) are feasible, and the extreme rays of (10)-(12) and (13)-(17) are the same, we have a valid reformulation.

For the second part, we observe that the 0-1 matrix A associated with the constraints (15)-(16) is totally unimodular. We use the characterization that for all column subsets J, there exists a partition J^+, J^- of J such that

$$|\sum_{u \in J^+} a_{ru} - \sum_{u \in J^-} a_{ru}| \leq 1$$

for every row r of the matrix.

Let J_Y be the columns associated to Y_t variables in J and let J^i_δ be the columns associated to μ^i, δ^i variables in J.
We assign elements of J_Y alternately to J^+ and J^- working from Y_1 up to Y_T.
Now fix t, and let $\tau_t = \min\{u : u \geq t, u \in J_Y\}$.
For each i, treat the variables μ^t_i, δ^i_{tu} in the order $(\mu^i_t, \delta^i_{tu_1}, \delta^i_{tu_2}, \ldots, \delta^i_{tu_{T-t+1}}, \delta^i_{t,T+1})$ where $f^i_{tu_1} \geq f^i_{tu_2} \geq \ldots \geq f^i_{tu_{T-t+1}} \geq f^i_{tu_{T+1}} = 0$. Observe that the corresponding constraint submatrix is a consecutive 1's matrix with 1's in the first column.
Now consider row $r = (i, t, \ell)$ of (15). If $\tau_t \in J^+$, assign the columns of J^i_δ alternately starting with J^-.
We have by construction that as $\tau_t \in J^+$,

$$0 \leq \sum_{u \in J^+ \cap J_Y} a_{ru} - \sum_{u \in J^- \cap J_Y} a_{ru} \leq 1.$$

Also from the assignment to J^i_δ,

$$-1 \leq \sum_{u \in J^+ \cap J^i_\delta} a_{ru} - \sum_{u \in J^- \cap J^i_\delta} a_{ru} \leq 0.$$

By addition we obtain that

$$-1 \leq \sum_{u \in J^+} a_{ru} - \sum_{u \in J^-} a_{ru} \leq 1.$$

If $\tau_t \in J^-$, the argument is similar. Thus the matrix is TU, and the proposed formulation is an integral polyhedron. $\square$

Theorem 2 *The linear program (13)-(17) solves $WW^* - CC - FAM$.*
Explicitly given an optimal extreme point solution of this linear program, it suffices to set $s^i_t = S^i_t - S^{i-1}_t$ and $x^i_t = d^i_t + s^i_t - s^i_{t-1}$ for all i, t to obtain a solution to $WW^* - CC - FAM$.

Proof: Now we need to show that if (S, X, Y) is an optimal extreme point solution of this linear program, then (s, x, Y) solves the original problem. Given $Y \in \mathbb{Z}^T_+$, each Wagner-Whitin solution (X^i, S^i, Y) is uniquely determined, with $S^i_{t-1} = \max_{l \geq t}(D^i_{tl} - C\sum_{u=t}^{l} Y_u)^+$, $X^i_t = \min[D^i_t + S^i_t, CY_t]$ and $X^i_t = D^i_t + S^i_t - S^i_{t-1}$. As $D^i_t \leq D^{i+1}_t$, we see immediately that $S^i_{t-1} \leq S^{i+1}_{t-1}$ which in turn implies that $X^i_t \leq X^{i+1}_t$. It follows that $s^i = S^i - S^{i-1} \geq 0, x^i = X^i - X^{i-1} \geq 0$ and so the vector (s^i, x^i) satisfies the flow conservation constraints, and hence is feasible for the original problem. □

The $WW^* - CC - FAM$ problem with integer upper bounds U_t on the number of batches produced in period t, $1 \leq t \leq T$, can be solved in a similar way by adding to the linear program (13)-(17) the constraints $Y_t \leq U_t$, as the resulting polyhedron remains integral.

Note also that, as there is an $O(T^2 \log T)$ separation algorithm for conv(X^{WW-CC}), another way to solve the linear program (13)-(17) is to take the linear programming relaxation of (5)-(9), and then call the separation algorithm for conv$(X^{WW-CC}(S^i, Y, D^i, C))$ for each surrogate item i.

When $C \geq \sum_{i=1}^{m} \sum_{u=1}^{T} d^i_u = D^m_{1T}$, the problem is uncapacitated. Using the explicit description of the convex hull of the single item uncapacitated problem, we get:

Corollary 3 *For the uncapacitated problem $WW^* - U - FAM$, the linear program*

$$\min \textstyle\sum_i \sum_t H^i_t S^i_t + \sum_t Q_t Y_t \tag{22}$$
$$S^i_{t-1} + \textstyle\sum_{u=t}^{l} D^i_{ul} Y_u \geq D^i_{tl} \text{ for } 1 \leq t \leq l \leq T,\ i = 1, \ldots, m \tag{23}$$
$$S \in \mathbb{R}^{mT}_+, Y \in \mathbb{R}^T_+ \tag{24}$$
$$s^i_t = S^i_t - S^{i-1}_t \text{ for all } i, t \tag{25}$$
$$x^i_t = s^i_t - s^i_{t-1} + d^i_t \text{ for all } i, t \tag{26}$$

solves the problem.

3 The Backlogging Case: Uncapacitated

In this Section we consider the backlogging version of the joint set-up problem. In order to be able to obtain similar results to those in Section 2 we have to remove the capacity restrictions on each item, and also limit ourselves to the case of stationary nonnegative storage costs, and backlogging costs that are proportional to the storage costs. Under such conditions, it can be shown that a) in a period t with a joint set-up, at least d^i_t of each item is produced, and b) given two consecutive joint set-up periods t and ℓ, the stock of all items is depleted simultaneously in a certain period τ where $t \leq \tau \leq \ell - 1$.

This problem, which we denote by $WW^* - U - B, FAM$, can be formulated as the following mixed integer program:

$$z = \min \textstyle\sum_i \sum_t h^i s^i_t + \sum_i \sum_t b^i r^i_t + \sum_t Q_t Y_t$$
$$s^i_{t-1} - r^i_{t-1} + x^i_t = d^i_t + s^i_t - r^i_t \text{ for all } i, t$$
$$\textstyle\sum_i x^i_t \leq M Y_t \text{ for all } t$$
$$x, s, r \in \mathbb{R}^{mT}_+,\ Y \in \mathbb{Z}^T_+,$$

where $h^i \geq h^{i+1} \geq 0$ and $b^i = \tau h^i$ for all i, and $M = C \geq D^m_{1T} = \sum_{i,t} d^i_t$.

Proceeding as before, with $R_t^i = \sum_{j=1}^i r_t^j$ and $B^i = b^i - b^{i+1} \geq 0$, we obtain a first relaxation:

$$\begin{gathered}\min \sum_i \sum_t H^i S_t^i + \sum_i \sum_t B^i R_t^i + \sum_t Q_t Y_t \\ S_{t-1}^i - R_{t-1}^i + X_t^i = D_t^i + S_t^i - R_t^i \text{ for all } i,t \\ X_t^i \leq M Y_t \text{ for all } i,t \\ X,S,R \in \mathbb{R}_+^{mT},\ Y \in \mathbb{Z}_+^T,\end{gathered}$$

and a second relaxation:

$$\min \sum_i \sum_t H^i S_t^i + \sum_i \sum_t B^i R_t^i + \sum_t Q_t Y_t \tag{27}$$

$$S_{t-1}^i + M \sum_{u=t}^l Y_u + R_l^i \geq D_{tl}^i \text{ for } 1 \leq t \leq l \leq T,\ i = 1,\dots,m, \tag{28}$$

$$S,R \in \mathbb{R}_+^{mT},\ Y \in \mathbb{Z}_+^T. \tag{29}$$

Observation 2 *For the single item case, the set* $X^{WW-U-B}(s,r,y,d) =$

$$\begin{gathered}s_{t-1} + M \sum_{u=t}^l y_u + r_l \geq d_{tl} \text{ for } 1 \leq t \leq l \leq T, \\ s,r \in \mathbb{R}_+^T, y \in \mathbb{Z}_+^T\end{gathered}$$

has been studied in [9]. $\mathrm{conv}(X^{WW^*-U-B}(s,r,y,d))$ *is known explicitly.*

Thus a third and final relaxation is the linear program:

$$\min\{\sum_i \sum_t (H^i S_t^i + B^i R_t^i) + \sum_t Q_t Y_t : \bigcap_{i=1}^m \mathrm{conv}(X^{WW-U-B}(S^i, R^i, Y, D^i))\}.$$

for which an explicit description is

$$\min \sum_i \sum_t H^i S_t^i + \sum_i \sum_t B^i R_t^i + \sum_t Q_t Y_t \tag{30}$$

$$\alpha_t^i + Y_t + \beta_t^i = 1 \text{ for all } \ i,t \text{ with } D_t^i > 0 \tag{31}$$

$$S_{t-1}^i \geq \sum_{u=t}^l D_u^i (\alpha_u^i - \sum_{j=t}^{u-1} Y_j) \text{ for } 1 \leq t \leq l \leq T,\ i = 1,\dots,m \tag{32}$$

$$R_l^i \geq \sum_{u=t}^l D_u^i (\beta_u^i - \sum_{j=u+1}^l Y_j) \text{ for } 1 \leq t \leq l \leq T,\ i = 1,\dots,m \tag{33}$$

$$S,R,\alpha,\beta \in \mathbb{R}_+^{mT},\ Y \in [0,1]^T. \tag{34}$$

Theorem 4 *The linear program (30)-(34) solves* $WW^* - U - B, FAM$.
Explicitly given an optimal extreme point solution of this linear program, it suffices to set $s_t^i = S_t^i - S_t^{i-1}, r_t^i = R_t^i - R_t^{i-1}$ *and* $x_t^i = d_t^i + s_t^i - r_t^i - s_{t-1}^i + r_{t-1}^i$ *for all* i,t *to obtain a solution to* $WW^* - U - B, FAM$.

Proof: The proof again consists of two parts. The first is to show that this linear program with Y integer is a valid reformulation of the relaxation (27)-(29) and that the linear program has a solution with Y integer. The second is to show that the s,r,x vector is actually a feasible solution of the original problem.

For the first part we use a different formulation to that proposed above, specifically the extended formulation:

$$\min \sum_i \sum_t H^i S_t^i + \sum_i \sum_t B^i R_t^i + \sum_t Q_t Y_t \tag{35}$$
$$\alpha_t^i + Y_t + \beta_t^i = 1 \text{ for all } \ i,t \text{ with } D_t^i > 0 \tag{36}$$
$$S_{t-1}^i = \sum_{u=t}^n D_u^i \delta_{tu}^i \text{ for all } \ i,t \tag{37}$$
$$R_l^i = \sum_{u=1}^l D_u^i \gamma_{ul}^i \text{ for all } \ i,l \tag{38}$$
$$\delta_{tl}^i + \sum_{u=t}^{l-1} Y_u - \alpha_l^i \geq 0 \text{ for } 1 \leq t \leq l \leq T, \ i = 1,\ldots,m \tag{39}$$
$$\gamma_{tl}^i + \sum_{u=t+1}^{l} Y_u - \beta_t^i \geq 0 \text{ for } 1 \leq t \leq l \leq T, \ i = 1,\ldots,m \tag{40}$$
$$\gamma, \delta \geq 0, S^i, R^i, \alpha^i, \beta^i \in \mathbb{R}_+^T \text{ for } i = 1,\ldots,m, \ Y \in [0,1]^T. \tag{41}$$

The proof that this provides a valid formulation for $X^{WW-U-B}(S,Y,R,D)$ and the corresponding matrix is TU when $m=1$, see [9], carries over immediately to the case of arbitrary m. Thus the linear program (35)-(41) and its projection (30)-(34) have a solution with Y integer.

For the second step, we observe that with $Y \in \{0,1\}^T$ fixed, the hypothesis on the storage and backlogging costs ensures that for each composite item i in the interval between two production periods t,l with $t < l$, there exists a period τ such that there is an optimal solution in which the demands in the interval $[t+1,\tau]$ are satisfied from production in t and thus from stock, while the demands in the interval $[\tau+1, l-1]$ are satisfied from production in l and thus by backlogging. Thus the composite stocks S_t^i, backlogs R_t^i and production levels X_t^i are nondecreasing in i, and the corresponding s,r,x solutions are feasible. □

It is easy to construct examples which show that this result no longer holds for the constant capacity case with backlogging WW^*-CC-B, FAM even under the very restrictive conditions imposed above on the storage and backlog costs.

4 Computation

The goal here is not at all a thorough computational study, but rather to look at the effectiveness of the formulations and to point out one or two surprises. We have taken a single instance of $WW^*-CC-FAM$ with 30 items and 50 periods, and three different capacity values. The instance without backlogging is generated as follows: $d_t^i = \lceil 5 * rand \rfloor$ ($\lceil x \rfloor$ denotes the closest integer to x) for all i,t, $h(i,t) = h(i-1,t) + 0.05 * rand$ for all i,t with $i > 1$, $h(i,1) = 0.05 + 0.1 * rand$ for all i and $Q_t = 75 + 50 * rand$ for all t, where $rand$ is generated each time uniformly in $[0,1]$.

The options we consider are addition of a complete reformulation of conv($X^{WW^*-CC-FAM}$) which has $\mathrm{O}(mT^2)$ constraints and variables, an approximate version depending on a parameter TK with $0 \leq TK \leq T$ with $\mathrm{O}(mT \times TK)$ constraints and variables, the complete reformulation of conv($X^{WW^*-U-FAM}$) which has $\mathrm{O}(mT^2)$ constraints but only $\mathrm{O}(mT)$ variables, or an approximate version with $\mathrm{O}(mT \times TK)$ constraints and $\mathrm{O}(mT)$ variables. More precisely the mixed integer program fed to the Xpress-MP optimizer is the original formulation (1)-(4) plus a TK approximate version of (13)-(17), and/or a TK approximate version of (22)-(24) with S_t^i replaced by $\sum_{j=1}^i s_t^j$.

In general TK is a parameter indicating a level of approximation in the reformulation, specifically the approximate formulation includes a complete linear reformulation $WW-CC/U$ for each composite item over each set of TK consecutive time periods. When $TK=0$, it gives the original MIP formulation and when $TK=T$ it gives the complete convex hull. See Van Vyve and Wolsey [12] for a discussion of the approximate reformulations and Pochet, Van Vyve and Wolsey [7] for a description of $LS-LIB$, a library of reformulations, and its use.

Several points are worth noting for comparison:

i) for most multi-item problems with set-ups on the individual items, the mixed integer programming systems such as Xpress-MP or Cplex are now often very effective at generating general cuts, and often obtain bounds close to those obtainable using cuts based on the problem structure.

ii) for most multi-item problems with set-ups on the individual items, the approximation parameter TK needed to obtain the bound generated by adding the convex hull of the single item subproblems is typically small compared to the size T of the time horizon.

4.1 No Backlogging

Results for the instance of $WW^* - CC - FAM$ with three values of the capacity $C \in \{250, 120, 50\}$ are presented in Table 1 below. In the first column we indicate the value of the capacity C, in the next two columns we indicate the choice of TK used to approximate the convex hull of $WW - CC$ and $WW - U$ respectively for each composite item. In the next three columns we specify the number of rows r and columns c of the resulting LP. The next four columns show the LP value of the instance after reformulation, the value obtained after the addition of the Xpress-MP cuts, the best IP value found and the best lower bound on termination. The last three columns give the number of nodes in the tree, the total time in seconds and the final duality gap. The maximum run time was set to be 300 seconds. Note that the instance has just 50 integer variables.

C	TK(CC)	TK(U)	r	c	LP	XLP	IP	LB	nodes	secs	gap
250	0	0	1550	3050	1574.2	2170.5	3175.7	2654.1	23100	300	16.5%
250	0	5	8737	3050	3174.9	3174.9	3174.9	3174.9	1	2	0%
250	5	0	11750	11750	3174.9	3174.9	3174.9	3174.9	1	7	0%
120	0	0	1550	3050	3145.5	3455.3	3899.2	3645.0	24700	300	6.5%
120	0	10	15117	3050	3693.3	3716.4	3885.3	3817.4	5700	300	1.7%
120	5	10	25377	11750	3857.0	3859.6	3885.3	3885.3	438	129	0%
120	10	0	18200	18200	3869.8	3870.3	3885.3	3885.3	101	70	0%
120	15	0	23900	23900	3884.3	3884.3	3885.3	3885.3	5	27	0%
120	20	0	28850	28850	3885.3	3885.3	3885.3	3885.3	1	20	0%
50	0	0	1550	3050	7411.6	7471.6	7579.3	7509.3	51300	300	0.9%
50	10	0	18200	18200	7536.6	7538.8	7585.5	7560.0	1800	300	0.3%
50	15	0	23900	23900	7560.6	7562.7	7578.2	7878.2	248	44	0%
50	20	0	28850	28850	7568.1	7569.6	7578.2	7878.2	36	14	0%
50	25	0	33050	33050	7576.6	7577.9	7578.2	7878.2	3	12	0%
50	30	0	36500	36500	7578.2	7878.2	7578.2	7878.2	1	6	0%

Table 1: An Instance of $WW^* - CC - FAM$ with 30 items and 50 periods

We observe that when $C = 250$, adding the uncapacitated reformulation of $WW^* - U - FAM$ with $TK = 5$, the instance is solved by linear programming without branching, so it appears that this instance is essentially uncapacitated.

For $C = 120$, using either the Xpress-MP default including system cuts or adding the uncapacitated reformulation based on $WW^* - U - FAM$, the instance is unsolved after 300 seconds with a gap of the order of 6.5% and 1.7% respectively. Thus the capacitated reformulation appears to be necessary to solve this instance. The test with $TK = 5$ for $WW^* - CC - FAM$ and $TK = 10$ for $WW^* - U - FAM$ is motivated by the idea that a formulation of reasonable size might be obtained by combining a weaker formulation over a significant number of periods with a stronger formulation over a small number of periods. Here the instance is solved in 438 nodes, but no obvious conclusion can be drawn especially in comparison to the following test. Adding the reformulation of $WW^* - CC - FAM$ with value of $TK = 10$ leads to an LP bound of 3869.8 fairly close to the optimal value of 33885.3, so that the instance is then solved in 101 nodes by branching. If we use the approximation parameter $TK = 15$, optimality is proved with 5 nodes and with $TK = 20$ an optimal solution is attained just by solving the linear program at the top node.

For the instance with $C = 50$, things are different. Because of the tighter capacity, the gap between the LP relaxation of the original MIP is smaller, and the relative performance of the default Xpress-MP MPS system is better. On the other hand to solve the problem to optimality using linear programming, a large value of the parameter TK is essential. In fact, we see that for this instance a value of $TK = T/2 = 25$ is insufficient, but a value of $TK = 30$ suffices. Though we know from Theorem 2 that $TK = T = 50$ always suffices, we have come across very similar instances for which, even with $TK = T - 1 = 49$, the approximate version of conv($X^{WW^*-CC-FAM}$) does not solve the instance without branching.

4.2 With Backlogging

Results for an instance of WW^*-CC-B, FAM with four values of the capacity $C \in \{300, 250, 120, 50\}$ are presented in Table 2 below. The demands are generated as for the instance without backlogging, the storage costs for each item are generated using $h(i) = h(i-1) + 0.05 * rand$ for all $i > 1$ with $h(1) = 0.1$, the backlog cost is three times the storage cost and $Q_t = 100$ for all t.

For all but the last two rows of Table 2, the maximum run time is again 300 seconds. For the last two rows it is 3600 seconds. A ** indicates that the Xpress-MP heuristic and cut generation are switched off when using the $WW - CC - B$ reformulation as the system cuts are ineffective, and the heuristic solutions are not of good quality.

C	TK(CC)	TK(U)	r	c	LP	XLP	IP	LB	nodes	secs	gap
300	0	5	17420	7550	3316.0	3316.0	3316.0	3316.0	1	11	0%
250	0	5	17420	7550	3316.1	3316.2	3316.2	3316.2	1	13	0%
120	0	5	17420	7550	3806.8	3837.2	3953.4	3862.5	1300	300	2.3%
**120	3	0	34756	39110	3910.5	-	3952.2	3912.8	58	300	1.0%
**120	5	0	91489	60230	3921.8	-	-	3921.8	-	300	-
50	0	0	1550	4550	7632.0	7725.8	7849.6	7759.4	19700	300	1.1%
**50	3	0	42548	39110	7790.6	-	8007.4	7792.8	178	300	2.7%
**50	5	0	110075	60230	7797.3	-	-	-	-	300	-
**120	3	0	34756	39110	3910.5	-	3928.1	3928.1	503	1744	0%
**50	3	0	42548	39110	7790.6	-	7937.2	7796.0	2000	3600	1.8%

Table 2: An Instance of $WW^* - CC - B, FAM$ with 30 items and 50 periods

As expected from Theorem 4, when the problem is uncapacitated, or C is large, the case with $C \in \{250, 300\}$, the $WW^* - U - B, FAM$ reformulation for each composite item permits the problem to be solved very quickly.

The case with smaller C is more complicated. We now have the choice between using the reformulation of $WW^* - U - B, FAM$, which we know to be tight when the capacity is very large, or else a valid formulation consisting of the convex hull of solutions of $WW - CC - B$ for each surrogate item. The trouble is that the latter formulation is too large $O(T^3) \times O(T^3)$ per item to be useable directly. Even using an approximation, it is only of manageable size for approximation parameter $TK \leq 5$. For $C = 120$, a choice of $TK = 3$ provides an effective compromise between strength of the linear programming bound and size, allowing the problem to be solved in under 30 minutes. However for the instance with $C = 50$, this does not work, and the best result is obtained with the default MPS system providing a solution guaranteed to be within 1.1% of optimal in 300 seconds.

References

[1] S. Anily and M. Tzur. Algorithms for the multi-item capacitated dynamic lot sizing problem. Iibr working paper, Faculty of Management, Tel Aviv University, 2004.

[2] S. Anily and M. Tzur. Shipping multiple-items by capacitated vehicles - an optimal dynamic programming approach. *Transportation Science*, 39:233–248, 2005.

[3] G.D. Eppen and R.K. Martin. Solving multi-item lot-sizing problems using variable definition. *Operations Research*, 35:832–848, 1987.

[4] M. Florian and M. Klein. Deterministic production planning with concave costs and capacity constraints. *Management Science*, 18:12–20, 1971.

[5] M. Florian, J.K. Lenstra, and A.H.G. Rinnooy Kan. Deterministic production planning: Algorithms and complexity. *Management Science*, 26:669–679, 1980.

[6] C.Y. Lee. A solution to the multiple set-up problem with dynamic demand. *IIE Transactions*, 21:266–270, 1989.

[7] Y. Pochet, M. Van Vyve, and L.A. Wolsey. LS-LIB: a library of reformulations, cut separation algorithms and primal heuristics in a high-level modeling language for solving MIP production planning problems. Discussion Paper 2005/47, CORE, Université catholique de Louvain, 2005.

[8] Y. Pochet and L.A. Wolsey. Lot-sizing with constant batches: Formulation and valid inequalities. *Mathematics of Operations Research*, 18:767–785, 1993.

[9] Y. Pochet and L.A. Wolsey. Polyhedra for lot-sizing with Wagner-Whitin costs. *Mathematical Programming*, 67:297–324, 1994.

[10] Y. Pochet and L.A. Wolsey. *Production Planning by Mixed Integer Programming.* Springer, 2006.

[11] C.P.M. van Hoesel and A.P.M. Wagelmans. An $O(T^3)$ algorithm for the economic lot-sizing problem with constant capacities. *Management Science*, 42:142–150, 1996.

[12] M. Van Vyve and L.A. Wolsey. Approximate extended formulations. *Mathematical Programming B*, 00:0–1, 2005.