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Parenting and Eldercare: Positive and Normative Analyses*

Simon Fan Yu Pang Pierre Pestieau

Abstract

Global trends in delayed childbearing and population aging have intertwined parenting and eldercare, necessitating concurrent attention to young children and elderly parents. This paper develops an overlapping-generations model where young adults, exhibiting two-sided altruism, educate their children to promote human capital accumulation and provide caregiving for their aging parents. Education can be attained through financial investments and the implementation of harsh discipline, which demands minimal parental resources but can strain parent-child relations. Eldercare is labor intensive, with its quality decreasing with the frequency of childhood discipline. Our positive analysis suggests that extended longevity may reduce the prevalence of harsh parenting, and enhanced altruism towards the elderly benefits them but can have adverse effects on private savings and children's human capital. We then examine the steady-state first-best solution and the second-best public policies. When intergenerational altruism is limited, we advocate for the idea of taxing labor and subsidizing education from a novel perspective of adjusting parenting styles and promoting eldercare.

JEL Classifications: I21, J13, J24

Keywords: Parenting style; Human capital; Longevity; Long-term care

*Fan: Department of Economics, Lingnan University, Hong Kong (fansimon@ln.edu.hk); Pang: School of Business, Macau University of Science and Technology, Macau (ypang@must.edu.mo). Pestieau: Department of Economics, University of Liege and CORE, Belgium and Toulouse School of Economics, France (P.Pestieau@ulg.ac.be).
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1 Introduction

One of the most critical forms of intergenerational transfers is education. It is commonly argued that young parents nurture and educate their children altruistically, dedicating time and resources to cultivate children's human capital (Becker and Tomes 1986; Heckman, Humphries, and Veramendi 2018). An alternative parenting style, especially prevalent in East Asian communities, is known as "tiger parenting", which involves demanding and stern practices that parents employ to secure their children's educational attainment.[1] The authoritative nature of tiger parenting is under increasing scrutiny as it hinders communication within families and strains the intergenerational relationship. In aging societies worldwide, where the need for filial care intensifies, persistent tensions can erode emotional bonds, thereby compromising the quality of caregiving support received by the elderly (Silverstein et al. 2002; Vangen and Herlofson 2023).

Family dynamics encompasses how parents and children support each other, balance work and personal life, navigate conflicts, and express affection throughout their shared journeys. Consider a three-generation family unit comprising a young child, a working parent, and a retired grandparent. The parent, who harbors altruism towards both the child and the grandparent, finds herself caught between meeting their distinct needs. The imperative of nurturing the child's potential drives her to allocate a portion of her earnings to the child's education. Beyond financial commitments, she may also implement structured disciplinary techniques (e.g., enforcing strict rules or even resorting to corporal punishment), which require minimal time and monetary investments. Concurrently, it is time demanding for the parent to assist the grandparent (Pestieau and Sato 2008; De Donder and Leroux 2017), yet she may find some joy of giving (Cremer, Pestieau, and Ponthière 2012). While the elderly depends on caregiving support provided by his adult child, the perceived quality of this eldercare can be influenced by the way that she was raised. For instance, the elderly who was stern in earlier life may struggle to trust his child to take on care responsibilities with compassion, which diminishes the emotional benefits he obtains; the elderly may also experience remorse for his past harshness, leading him to feel unworthy of such support.

Child rearing and eldercare are often treated as distinct issues in economic literature, with fam-

[1] The concept of a "tiger mother" gained prominence through Yale Law professor Amy Chua's 2001 memoir *Battle Hymn of the Tiger Mother*. Chua believes that practicing strict child-rearing techniques helps to shape young children into disciplined, high-achieving individuals prepared for academic and professional accomplishments.

ilies presumed to face caregiving pressure related to either young children or old parents. However, evolving demographic trends, such as late childbearing and population aging, have led to a potential overlap between these two concerns within families (Zhong and Peng 2024). Dual altruism, in turn, prompts the sandwiched generation to simultaneously attend to both ends of the age spectrum. Elderly caregiving responsibilities can limit parental funds for education, thereby impeding young children's human capital formation and their future earning prospects (Bogan 2015). Conversely, prioritizing children's education may inadvertently overlook the demands of the elderly. The dual obligations impose financial constraints on young parents, compelling them to make sacrifices on employment and private consumption. To better understand the allocation of household resources across three generations, it is essential to investigate how government actions, particularly through educational policies and welfare systems, affect family planning decisions.

This paper aims to present a positive and normative analysis that connects parenting styles to eldercare. We establish an overlapping-generations model, wherein individuals progress through childhood, young adulthood, and old adulthood. Children do not make decisions but accumulate human capital, treating parental inputs as given. Young adults exhibit two-sided altruism: they are concerned with enhancing their children's human capital (known as "impure parental altruism") and find joy in assisting their elderly parents. They are endowed with one unit of time, which they choose to allocate between work and taking care of their parents. Meanwhile, young adults invest in their children's education through financial means and the implementation of harsh disciplinary measures, both facilitating the development of their children's human capital. The elderly allocate their savings towards consumption and desire their children's support and attention; however, engaging in harsh parenting early on can undermine the perceived quality of future eldercare offered by their children.

In Section 2, our analysis starts with a *laissez-faire* economy, where young adults choose work hours, educational spending, parenting style, and savings without policy interventions. We suggest that as life expectancy rises, parents tend to employ harsh tactics less frequently with their children. Driven by longer lifespans, parents become increasingly aware of the role their children will play in providing caregiving support in the future. This foresight prompts them to adjust their upbringing tactics by mitigating aggression and conflicts. A deeper sense of fulfillment derived from caring for old parents strengthens dedication to labor-intensive eldercare, but it can result in a decrease in both financial investments in education and harsh disciplinary measures. In contrast, a heightened focus on child development diverts attention and effort away from eldercare obligations. Both altruistic drives tend to generate undesirable consequences on individual savings and consumption.

In Section 3, we examine the social optimum where a social planner maximizes the individual utility in the steady state. By defining the steady state as the constant level of human capital in each generation, our examination reveals a difference between the *laissez-faire* choices and the first-best optimality conditions. This disparity results largely from impure parental altruism, where parents prioritize their children's human capital but do not directly take into account their children's utility (Becker 1993). In particular, if parental concern about children's human capital is low, educational investments fall short of the Golden-Rule level; likewise, when children's altruism towards parents is weak, eldercare provision dips below the Golden-Rule level. Instruments to restore the first-best outcome are lacking, which prompts us to explore the second-best policies.

Section 4 presents an analysis regarding the optimal policies that a benevolent government can use to improve social welfare. We postulate that the government can craft public policies to assist families in balancing work commitments, consumption, child development, and eldercare. Policy instruments at the government's disposal include taxation and/or subsidies. Our analysis indicates that, under certain circumstances (e.g., when intergenerational altruism is low), the optimal intervention entails taxing labor and subsidizing formal education. Moreover, we conduct a simulation wherein we derive numerical solutions to the optimal income taxation and educational subsidy as well as the individual choices in equilibrium. We then offer numerical comparative static analyses under various parameter configurations.

This study contributes to the burgeoning research on parenting styles. Weinberg (2001) shows that parenting styles tend to be influenced by parental income. Burton, Phipps, and Curtis (2002) present a model of parent-child interactions in which parenting is determined by both the parent's stress and the child's (mis)behavior. Doepke and Zilibotti (2017) explore the effects of socioeconomic environment (e.g., income inequality, division of labor, and technological advancement) on child-rearing practices, attributing the long-term decline of authoritarian parenting to rising returns to independence. Fan, Pang, and Pestieau (2024) construct an OLG model analyzing the relationship between parenting and eldercare, departing from the current model in two main aspects. First, they formalize family eldercare as motivated by societal norms and strategic exchange, contrasting with our focus on children's pure altruism towards their parents (i.e., joy of giving). Second, they incorporate parental time investment in children but abstract from educational expenditure.

Our paper also complements the theoretical literature on aging and informal care for the elderly. Previous theories focus on the impact of increased longevity on human capital accumulation and economic development (see, for example, Cervellati and Sunde 2005, Hazan and Zoabi 2006, and de la Croix and Ponthière 2010), whilst our model investigates this effect from the perspective of parenting styles and delves into the consequence on the eldercare provision. Cremer, Pestieau, and Ponthière (2012) introduce and review three alternative motives of long-term care within families, namely exchange contract, societal norm, and pure altruism (see Klimaviciute and Pestieau (2023) for a recent literature review). In particular, Canta and Cremer (2019) hold that parents can commit to a rule specifying a bequest conditional on the level of eldercare. Barigozzi, Cremer, and Roeder (2020) suggest that social expectation plays a crucial role in determining the dynamics of family eldercare, which contributes to the emergence of the gender gap in care provision. We address that the level of caregiving support depends positively on the extent of filial altruism, as discussed by Cremer, Pestieau, and Roeder (2016). Our model adopts a life-course perspective that the quality of eldercare provision is grounded in the history of the parent-child relationship.

2 The Model

Consider an overlapping-generations small open economy where a unit mass of identical individuals live for childhood, young adulthood, and old adulthood. Children do not make any decision and accumulate human capital by treating parental inputs as given. Young adults are endowed with one unit of time, which they allocate between work and care of their parents. Alongside this time-allocation decision, they each raise a child, providing education through financial investments in learning and the enforcement of harsh discipline. Assume that the practice of harsh discipline does not consume parental time or material resources. In addition, young adults make financial planning regarding private consumption, savings, and educational expenditures for their children subject to the constraint of their incomes. Old adults retire and have a lifespan shorter than one unit of time, denoted by the parameter $\pi \in (0, 1)$, where an increase in π means aging. They derive utility from old-age consumption as well as from their children's caregiving support.

Denote individuals who spend their young adulthood in period t as generation t. In period t, a representative member of generation t has a human capital stock of H_t, which is governed by her parental human capital stock (H_{t-1}), the frequency of harsh discipline implemented by her parent (d_{t-1}), and the amount of educational expenditure of her parent (e_{t-1}), as expressed by

$$H_t = \Phi\left(H_{t-1}, d_{t-1}, e_{t-1}\right). \tag{1}$$

Each input exhibits diminishing returns, namely

$$\frac{\partial \Phi}{\partial H_{t-1}} > 0 \qquad \frac{\partial \Phi}{\partial d_{t-1}} > 0 \qquad \frac{\partial \Phi}{\partial e_{t-1}} > 0 \qquad \frac{\partial^2 \Phi}{\partial H_{t-1}^2} < 0 \qquad \frac{\partial^2 \Phi}{\partial d_{t-1}^2} < 0 \qquad \frac{\partial^2 \Phi}{\partial e_{t-1}^2} < 0. \tag{2}$$

Note that for generation $t-1$, H_{t-1} is a state variable, while d_{t-1} and e_{t-1} are choice variables. In particular, a high value of d_{t-1} means harsh parenting, and a low value of d_{t-1} means loving parenting. We assume the relationship between the two educational methods satisfies a non-negative second-order partial derivative, namely

$$\frac{\partial^2 \Phi}{\partial d_{t-1} \partial e_{t-1}} \geqslant 0. \tag{3}$$

In period t, the young adult allocates a fraction $a_t \in (0, 1)$ of time to caring for her parent when her parent is alive over a period of length π. She experiences a pure joy of giving, as depicted by the function $g(\pi a_t)$, where $g' > 0$ and $g'' < 0$. The aggregate production function of the economy is given by $F(K_t, L_t)$, where F represents a constant-returns-to-scale technology, K_t is the capital stock, and $L_t = (1 - \pi a_t)H_t$ denotes the labor supply in efficiency units. We assume that in this small open economy, the prices of production factors are exogenously given. Specifically, the wage rate per efficiency unit of labor is fixed at w, while the intertemporal interest rate is fixed at r. A young adult, who earns a total income of $w(1 - \pi a_t)H_t$, faces a budget constraint as expressed by

$$c_t^y + s_t + e_t = w(1 - \pi a_t)H_t, \tag{4}$$

where c_t^y and s_t denote her private consumption and saving, respectively. The young adult obtains utility of $u(c_t^y)$ from consumption.

In period $t + 1$, each member of generation t becomes old and spends her income from young-age savings on consumption without leaving bequests. With c_t^o denoting the old-age consumption level per unit of time, the old-age budget constraint implies that

$$\pi c_t^o = (1+r)s_t. \tag{5}$$

For every unit of time, the old adult derives utility from consumption, denoted as $u(c_t^o)$, and from her child's aid, denoted as $v(a_{t+1}, d_t)$. We assume that the parental enjoyment from receiving aid from the child increases with the duration of caregiving (a_{t+1}) and decreases with the frequency of harsh parenting (d_t). Formally, the following partial derivatives are satisfied:

$$\frac{\partial v}{\partial a_{t+1}} > 0 \qquad \frac{\partial^2 v}{\partial a_{t+1}^2} \leqslant 0 \qquad \frac{\partial v}{\partial d_t} < 0 \qquad \frac{\partial^2 v}{\partial d_t^2} \leqslant 0. \tag{6}$$

In essence, eldercare provision exhibits weakly diminishing returns, while harsh parenting causes severe disutility consequences.

In sum, the lifetime utility of a member of generation t is given by

$$U_t = u(c_t^y) + \gamma g\left(\pi a_t\right) + \alpha H_{t+1} + \beta\pi\left[u(c_t^o) + v(a_{t+1}, d_t)\right], \tag{7}$$

where $\gamma > 0$ reflects the intensity of altruism towards her parent, and $\alpha > 0$ measures the degree of parental altruism linked to the child's human capital formation. The final term captures the old-age utility, where $\beta > 0$ is the time preference factor. Inserting (1) – (5) into (7) rewrites the individual lifetime utility as

$$U_t = u\left[\left(1-\pi a_t\right) w H_t - s_t - e_t\right] + \gamma g\left(\pi a_t\right) + \alpha\Phi(H_t, d_t, e_t) + \beta\pi\left[u\left(\frac{(1+r)s_t}{\pi}\right) + v(a_{t+1}, d_t)\right].$$

We proceed to analyze the *laissez-faire* equilibrium. The individual maximizes U_t by choosing (d_t, e_t, s_t, a_t) in young adulthood. The optimal choices are in accordance with the following four first order conditions:

$$\left(\frac{\partial v}{\partial d_t}\right)^{-1}\frac{\partial\Phi}{\partial d_t} = -\frac{\beta\pi}{\alpha}, \tag{8}$$

$$u'\left(c_t^y\right) = \beta(1+r)u'\left(c_t^o\right) = \alpha\frac{\partial\Phi}{\partial e_t} = \frac{\gamma}{wH_t}g'\left(\pi a_t\right). \tag{9}$$

For analytical tractability, we further make the following assumption, which holds throughout this paper:

$$\frac{\partial^2 v}{\partial d_t \partial a_{t+1}} = 0. \tag{10}$$

The next proposition examines the relationship between longevity and parental behaviors:

Proposition 1 *d_t and e_t decrease with π if*

$$\frac{\partial^2 \Phi}{\partial d_t^2}\frac{\partial^2 \Phi}{\partial e_t^2} \geqslant \left(\frac{\partial^2 \Phi}{\partial d_t \partial e_t}\right)^2. \tag{11}$$

Proof. See Appendix. ■

Proposition 1 states that with increasing longevity, parents tend to engage in less harsh tactics towards their children. Driven by the prospect of longer lifespans, parents grow increasingly mindful of their future caregiving needs (Pestieau 2022). This foresight prompts them to consider the pivotal role their adult children will assume in offering care and assistance as they age. Recognizing that fostering strong familial bonds increases caregiving support provided by their children in their later years, parents are inclined to adjust their behaviors in early life by reducing hostility and discord to prevent familial rifts. This implication is broadly consistent with the observations from the United States. For example, in 2020, people living in Hawaii, Washington, and Minnesota have the country's longest life expectancy at birth, yet children in the three states are least affected by maltreatment; in contrast, life expectancies in Mississippi, West Virginia, and Kentucky stand out as the lowest in the nation, while their child victimization rates are alarmingly high.[2]

Proposition 1 also posits a potential negative relationship between life expectancy and educational spending, supported by three rationales. First, a longer lifespan leads to increased life-cycle savings, prompting young adults to limit their dedication towards their children's education. Second, with longer-lived parents requiring more eldercare, young adults will allocate additional time to caregiving, which decreases their total earnings. This income effect leads them to curtail educational funds for their children. Third, in light of weak complementarity between e_t and d_t (refer to condition (3)), the lower value of d_t may hinder the marginal contribution of e_t to H_{t+1}, resulting in a decrease in educational investments received by children.

[2]See the US life expectancy data from National Center for Health Statistics (2023) and the child maltreatment data from https://www.acf.hhs.gov/cb/data-research/child-maltreatment.

In short, Proposition 1 predicts that an increase in life expectancy can result in a decrease in discipline and educational investments. We can further infer from equation (1) that, all else being equal, this has a detrimental effect on the human capital stock of the younger generation. Empirical literature has presented evidence supporting this inference. For example, Acemoglu and Johnson (2007) empirically establish that rising longevity may exert a negative influence on human capital and income per capita. Fraumeni, He, Li, and Liu (2019) show that population aging significantly reduces the growth rates of human capital in China between 1985 and 2014.

We develop the following proposition to analyze the effect of intergenerational altruism on the distribution of care and attention within the family structure.

Proposition 2 *Suppose that condition (11) holds.*
(i) $(e_t,\, d_t)$ *increase with* α *and* $(a_t,\, s_t)$ *fall with* α.
(ii) a_t *increases with* γ *and* $(e_t,\, d_t,\, s_t)$ *decrease with* γ.

Proof. See Appendix. ■

Proposition 2 explores how shifting priorities, shaped by concerns over future potential in children versus eldercare needs in parents, affect parenting styles and household resources allocation. When young adults are increasingly concerned about their children (larger α), prioritizing human capital development becomes paramount, which manifests through financial investments in education and the implementation of structured discipline (larger e_t and d_t). Yet this heightened concern for children necessitates young parents to increase their labor supply to boost earnings. As a result, they direct less attention towards elderly parents who also need care and assistance (smaller a_t).

Conversely, as the concerns of young adults turn to the well-being of their aging parents, family eldercare takes precedence. The escalating concern for the elderly (larger γ) leads to a reallocation of resources to ensure caregiving support of aging parents under some conditions (larger a_t), which in turn generates adverse labor market consequences. Van Houtven, Coe, and Skira (2013) find that female caregivers (i.e., daughters) decrease work hours and face a lower wage in the United States. Furthermore, when the imperative to prioritize eldercare amplifies, another potential consequence emerges – a lack of attention to children's developmental needs: the diversion of time from income-generating endeavors reduces efforts given to children's upbringing and education (smaller d_t and e_t). This implication also aligns with earlier empirical evidence: Bogan (2015) shows that having an elderly dependent in the household significantly mitigates the likelihood of investing funds in offspring education in the United States.

By examining the intergenerational distribution of care and resources, Proposition 2 also sheds light on its consequence and implication for the working-age population. Young adults find themselves sandwiched between the dual responsibilities of supporting aging parents and raising their own children. Greater altruism towards parents diverts time from work to eldercare, thereby reducing their incomes and hence savings. Conversely, heightened altruism towards children results in a greater commitment to educational spending and hence crowds out savings. In short, an increase in intergenerational altruism – in terms of either α or γ – can impose a significant financial strain on the young, leaving them less funds for future retirement (smaller s_t) and taking a toll on their own well-being. Empirically, Bogan (2015) presents evidence that the American households with both children and elderly dependents have much less stockholding in order to smooth consumption.

Finally, we conduct the comparative static analyses of (e_t, d_t) with respect to H_t (a state variable in period t) in the next proposition:

Proposition 3 *$(e_t,\, d_t)$ increase with H_t if condition (11) and the following condition hold:*

$$\frac{\partial^2 \Phi}{\partial e_t \partial H_t} > 0, \qquad \frac{\partial^2 \Phi}{\partial d_t \partial H_t} > 0. \tag{12}$$

Proof. See Appendix. ■

Proposition 3 states that under reasonable conditions, parents with a higher human capital stock are inclined to invest more in their children's education through both educational expenditure and disciplinary measures. Condition (12) indicates that parental human capital plays a complementary role to educational expenditure and disciplinary measures in developing children's human capital. Such a complementarity contributes to the positive relationship between H_t and (e_t, d_t). An extensive body of theoretical and empirical literature has shown that parental human capital is positively related to parental educational funds (Becker 1993; Brown 2006; Kornrich and Furstenberg 2013). There are also some empirical studies demonstrating the positive link between parental socioeconomic status and the use of harsh discipline in the household. For example, Bronfenbrenner (1958) finds that higher socioeconomic status parents exert greater pressure on their children and are more likely to use psychological techniques of discipline in the US between 1930 and 1955. In a Chinese sample, Wang and Liu (2014) show that more mothers and fathers in the high socioeconomic status group implement corporal punishment on their children.

3 First-best Optimality

In this section, we aim to examine social optimum. We define social welfare in period t as the lifetime utility of generation t. As usual in this kind of problem (Hammond 1987; Harsanyi 1995), we eliminate the influences of the two-sided altruism by laundering out both altruistic terms (i.e., $\alpha = \gamma = 0$). The social welfare in period t can thus be rewritten as $u(c_t^y) + \beta\pi[u(c_t^o) + v(a_{t+1}, d_t)]$. Consider that the utilitarian social planner aims to maximize the steady-state social welfare, where the steady state is defined as below:

Definition 1 *The economy achieves the steady state when the human capital stock is constant (i.e., $H_t = H$ for all t).*

Equation (1) captures the evolution of human capital stock over time. By Definition 1, the steady state condition can be written as $H = \Phi(H, d, e)$.

In the first-best world, the social planner makes the same choices for each generation, namely (d, e, a, s), in accordance with the following Lagrangian:

$$\mathcal{L} = u[(1 - \pi a)wH - e - s] + \beta\pi u\left(\frac{(1+r)s}{\pi}\right) + \beta\pi v(a, d) - \lambda[H - \Phi(H, d, e)],$$

where λ denotes the Lagrange multiplier associated with human capital formation. The first order conditions with respect to (d, e, a, s) can be rearranged as

$$\left(\frac{\partial v}{\partial d_t}\right)^{-1}\frac{\partial \Phi}{\partial d_t} = -\frac{\beta\pi}{\alpha}, \tag{13}$$

$$u'\left(c_t^y\right) = \beta(1+r)u'\left(c_t^o\right) = \alpha\frac{\partial \Phi}{\partial e_t} = \frac{\gamma}{wH_t}g'\left(\pi a_t\right). \tag{14}$$

By comparing (13) and (14) with (8) and (9), we establish the following proposition to characterize the conditions under which the decentralized equilibrium outcome aligns with the social optimum:

Proposition 4 *The social optimum coincides with the laissez-faire equilibrium if and only if*

$$\alpha = \lambda, \qquad \gamma = \frac{\beta}{g'(\pi a)} \frac{\partial v}{\partial a}. \tag{15}$$

Proposition 4 demonstrates the sufficient and necessary condition for a decentralized economy to achieve the social optimum. If parental altruism is limited ($\alpha < \lambda$), then the steady-state parental choices of d and e are smaller than desired (Proposition 2(i)). To increase the educational investments received by young children, it may be necessary to introduce a subsidy on formal education. Also, discipline should be more frequently employed to ensure that its marginal benefit of human capital accumulation is equal to its marginal cost of eldercare. For the social planner, a small value of α will lead to overly high levels of a and s, which implicates that eldercare provision and private savings should be discouraged.

When adult children exhibit a low level of altruism towards their aging parents ($\gamma < \frac{\beta}{g'(\pi a)} \frac{\partial v}{\partial a}$), the *laissez-faire* economy will see an eldercare deficit in the steady state (i.e., an overly small a). To ensure adequate support and assistance for the elderly, the social planner chooses to reallocate young adults' time from work to family eldercare. In this case, each young adult should have less private savings and less funds for her child's education.

It is worth noting that variables such as a and d, which are not readily observable for obvious reasons, can only be controlled indirectly. For instance, subsidizing educational investments affects the practice of harsh discipline through the function Φ. Taxing earnings can be an effective way to make eldercare provision more attractive. In the next section, we resort to a second-best approach, showing that government intervention can improve the decentralized outcome.

4 Second-best Optimum

In this section, we consider that a benevolent government designs public policies to help families balance eldercare, child development, and work commitment. Policy tools at disposal include an income tax (at a rate of τ), an educational subsidy (θ), and a demogrant (A). In period t, the interactions between the government and generation t proceed in two stages. The government moves first to maximize the steady-state welfare by optimally choosing (τ, θ, A) subject to the balanced budget, as follows:

$$\tau\left(1-\pi a_t\right)wH_t+\theta e_t=A, \tag{16}$$

where τ and θ are a tax if positive and a subsidy if negative, and $A\geqslant 0$. By observing the policies, generation t maximizes the their utility by choosing (d_t, e_t, a_t, s_t) during young adulthood. Note that in this section, e_t is interpreted as the amount of educational investment received by generation $t+1$, where their parents contribute $(1+\theta)e_t$ while the government contributes $-\theta e_t$.

4.1 Optimal Public Policies

In this subsection, we present a theoretical analysis of the optimal policies and the corresponding equilibrium. By backward induction, we first examine the individual optimal choices given the policies. Introducing the three policy instruments into the lifetime utility of generation t obtains

$$\begin{aligned}U_t=&\,u[(1-\tau)(1-\pi a_t)wH_t-s_t-\left(1+\theta\right)e_t+A]+\gamma g\left(\pi a_t\right)+\alpha H_{t+1}\\&+\beta\pi\left[u\left(\frac{(1+r)s_t}{\pi}\right)+v(a_{t+1},d_t)\right].\end{aligned}$$

Given (a_{t+1}, H_t; τ, θ, A), each member of generation t maximizes U_t by choosing (d_t, e_t, a_t, s_t). In the steady state, their optimal choices satisfy

$$\left(\frac{\partial v}{\partial d}\right)^{-1}\frac{\partial H}{\partial d}=-\frac{\beta\pi}{\alpha}, \tag{17}$$

$$u'(c^y)=\beta(1+r)u'(c^o)=\frac{\alpha}{1+\theta}\frac{\partial H}{\partial e}=\frac{\gamma g'(\pi a)}{(1-\tau)wH}. \tag{18}$$

which indicates that the optimal solutions to (d,e,a,s) are a function of (τ,θ,A).

Next, we probe into the optimal design of policies by the benevolent government. The government chooses policy tools (τ, θ, A) to maximize the steady-state social welfare, taking into account the individual optimal responses. Specifically, set up the Lagrangian in the steady state, where the two altruistic components have been removed (i.e., $\alpha=\gamma=0$):

$$\begin{aligned}\mathfrak{L}=&\,u[(1-\pi a)(1-\tau)wH-s-(1+\theta)e+A]+\beta\pi\left[u\left(\frac{(1+r)s}{\pi}\right)+v(a,d)\right]\\&+\mu\left[\tau\left(1-\pi a\right)wH+\theta e-A\right],\end{aligned}$$

where μ is the multiplier associated with the government's budget constraint (16). Taking the first order conditions with respect to (τ, θ, A) and then using (17) and (18) derives the interior solutions as follows:

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial \tau} &= [(1-\pi a)(1-\tau)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \tau}+\frac{\partial H}{\partial e}\frac{\partial e}{\partial \tau}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a}{\partial \tau} \\
&\quad -(1-\pi a)wHu'(c^y)+\mu[(1-\pi a)wH+\Delta_\tau]=0, \qquad (19)\\
\frac{\partial \mathfrak{L}}{\partial \theta} &= [(1-\pi a)(1-\tau)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \theta}+\frac{\partial H}{\partial e}\frac{\partial e}{\partial \theta}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a}{\partial \theta} \\
&\quad -eu'(c^y)+\mu(e+\Delta_\theta)=0, \qquad (20)\\
\frac{\partial \mathfrak{L}}{\partial A} &= [(1-\pi a)(1-\tau)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial A}+\frac{\partial H}{\partial e}\frac{\partial e}{\partial A}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a}{\partial A} \\
&\quad +u'(c^y)-\mu(1-\Delta_A)=0, \qquad (21)
\end{aligned}$$

where $\Delta_z = \tau(1-\pi a)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial z}+\frac{\partial H}{\partial e}\frac{\partial e}{\partial z}\right)-\tau\pi wH\frac{\partial a}{\partial z}+\theta\frac{\partial e}{\partial z}$ for $z\in\{\tau,\theta,A\}$.

We can now use these conditions to rewrite them in compensated terms with the notation $\frac{\partial x^C}{\partial z} = \frac{\partial x}{\partial z}+\frac{dA}{dz}\frac{\partial x}{\partial A}$, where $x=\{\mathfrak{L},d,e,a\}$, $z\in\{\tau,\theta\}$ and the superscript C denotes compensation. By compensation, we look at the impact of the income tax rate but also at the impact of the educational spending implied by the government's budget constraint (16).

$$\begin{aligned}
\frac{\partial \mathfrak{L}^C}{\partial \tau} &= [(1-\tau)(1-\pi a)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial e}\frac{\partial e^C}{\partial \tau}+\frac{\partial H}{\partial d}\frac{\partial d^C}{\partial \tau}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a^C}{\partial \tau}+\mu\Delta_\tau^C,\\
\frac{\partial \mathfrak{L}^C}{\partial \theta} &= [(1-\tau)(1-\pi a)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial e}\frac{\partial e^C}{\partial \theta}+\frac{\partial H}{\partial d}\frac{\partial d^C}{\partial \theta}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a^C}{\partial \theta}+\mu\Delta_\theta^C,
\end{aligned}$$

where $\Delta_z^C=\Delta_\tau+\frac{dA}{d\tau}\Delta_A$. At the point $\tau=\theta=A=0$, we have

$$\left.\frac{\partial \mathfrak{L}^C}{\partial \tau}\right|_{\tau=\theta=0} = [(1-\pi a)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial e}\frac{\partial e^C}{\partial \tau}+\frac{\partial H}{\partial d}\frac{\partial d^C}{\partial \tau}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a^C}{\partial \tau}, \quad (22)$$

$$\left.\frac{\partial \mathfrak{L}^C}{\partial \theta}\right|_{\tau=\theta=0} = [(1-\pi a)wu'(c^y)-\alpha]\left(\frac{\partial H}{\partial e}\frac{\partial e^C}{\partial \theta}+\frac{\partial H}{\partial d}\frac{\partial d^C}{\partial \theta}\right)+\pi\left[\beta\frac{\partial v}{\partial a}-\gamma g'(\pi a)\right]\frac{\partial a^C}{\partial \theta}. \quad (23)$$

It is straightforward to see that in equilibrium, $\tau>0$ when the right-hand-side of equation (22) is positive, and $\theta<0$ when the right-hand-side of equation (23) is negative. Suppose that the direct effect of τ on a outweighs its indirect effects on (e,d), and likewise, the direct effect of θ on e dominates its indirect effects on (a,d). Based on (22) and (23), we develop the next proposition:

Proposition 5 *The government taxes earnings* ($\tau > 0$) *and subsidize education* ($\theta < 0$) *when*

$$\alpha < (1 - \pi a) w u'(c^y), \qquad \gamma < \frac{\beta}{g'(\pi a)} \frac{\partial v}{\partial a}. \tag{24}$$

Proposition 5 demonstrates that a benevolent government should implement a policy of taxing labor and subsidizing formal education under certain circumstances. Condition (24) specifies that this policy is particularly pertinent when the parameters of intergenerational altruism remain low. In particular, when parents show limited altruism towards their children (small α), the government is expected to promote and subsidize education to address the deficit in parental attention. When children are not quite altruistic towards their parents (small γ), the optimal public policy involves discouraging labor supply and encouraging eldercare, achievable through imposing income taxes on adult children. Cremer, Pestieau, and Roeder (2016) make a similar point in the optimal design of a social long-term care insurance in the presence of two-sided altruism.

4.2 A Numerical Example

Given the analytical framework in Section 4.1, we undertake a simulation in this subsection to reveal and compare the optimal policies under various configurations. Our objective is to highlight the effects of varying degrees of aging and intergenerational altruism on the structure of optimal policy and the resulting individual choices. To illustrate, let us begin with specific functional forms of Φ, u, v, and g, which are given as follows:

$$u(c) = \ln c, , \tag{25a}$$

$$g(\pi a_t) = \ln(\pi a_t), \tag{25b}$$

$$H_{t+1} = \sqrt{H_t}(d_t^\rho + e_t^\delta) \quad \text{where } \rho, \delta \in (0, 1), \tag{25c}$$

$$v(a_{t+1}, d_t) = a_{t+1} - \frac{\eta}{2} d_t^2 \quad \text{where } \eta > 0. \tag{25d}$$

It is easy to verify that expression (25c) satisfies conditions (2), (3), (11), and (12). Also, expression (25d) ensures that conditions (6) and (10) are met.

With the above explicit functional forms, we rewrite the lifetime utility of generation t under certain government policies as

$$U_t = \ln[(1-\tau)(1-\pi a_t)wH_t - s_t - (1+\theta)e_t + A] + \gamma \ln a_t + \alpha\sqrt{H_t}(d_t^\rho + e_t^\delta)$$
$$+\beta\pi\left(\ln s_t + a_{t+1} - \frac{\eta}{2}d_t^2\right) + \beta\pi\ln(1+r) + (\gamma - \beta\pi)\ln\pi. \tag{26}$$

Given (τ, θ, A), in the steady state, the optimal interior solutions to (H, d, e, a, s) satisfy

$$\sqrt{H} = d^\rho + e^\delta, \tag{27a}$$
$$d^{2-\rho} = \frac{\alpha\rho}{\beta\pi\eta}\sqrt{H}, \tag{27b}$$
$$\frac{1+\gamma+\beta\pi}{\alpha\delta\sqrt{H}}e^{1-\delta} + e = \frac{(1-\tau)wH + A}{1+\theta}, \tag{27c}$$
$$s = \frac{\beta\pi(1+\theta)}{\alpha\delta\sqrt{H}}e^{1-\delta}, \tag{27d}$$
$$a = \frac{\gamma s}{\beta(1-\tau)\pi^2 wH}, \tag{27e}$$

which indicates that (H, d, e, a, s) are all a function of (τ, θ, A). The government chooses (τ, θ, A) to maximize the social welfare U. This maximization problem, which can be expressed by

$$\max_{\tau,\theta,A} \ \alpha H + (1+\beta\pi)\ln s + \beta\pi a + \gamma \ln a - \frac{\beta\pi\eta}{2}d^2,$$

determines the optimal policies.

To demonstrate the solutions to the optimal policies and the associated equilibrium outcomes, we proceed to perform a numerical exercise based on the following set of parameters:

$$\pi = 0.5 \quad \alpha = \gamma = 0.5 \quad \beta = 0.9 \quad \rho = \delta = 0.15 \quad \eta = 5 \quad w = 2 \quad r = 1.$$

The simulation results are summarized in the first column of Table 1. The government sets an optimal income tax at the rate of 31.3% while offering a transfer payment of 0.317 to each individual. Each individual receives the educational funding totaling 1.720, with 22% being covered by her parent and the remaining 78% being subsidized by the government. Moreover, each individual allocates 50.6% of her time to family eldercare during her parent's lifetime. Out of her total earnings of $(1-\pi a)H = 2.650$, she saves an amount of 1.111, representing a saving rate of 41.9%.

Table 1: The Optimal Policies and Equilibrium Results Under Various Configurations

Variables	[1] Benchmark	[2] $\pi = 0.4$	[3] $\alpha = 0.7$	[4] $\gamma = 0.7$
τ	0.313	0.315	0.296	0.289
θ	-0.780	-0.764	-0.741	-0.782
A	0.317	0.353	0.081	0.184
H	3.548	3.606	3.805	3.471
d	0.224	0.254	0.274	0.223
e	1.720	1.720	2.223	1.520
a	0.506	0.665	0.465	0.632
s	1.111	0.946	1.122	1.002

The next three columns record the results of our comparative static analyses. Column [2] differs from column [1] only by a decrease in π from 0.5 to 0.4. With a decline in life expectancy, the government opts for a higher rate of the optimal income tax (e.g., $31.5\% > 31.3\%$), which in turn discourages individuals from working (i.e., $1 - \pi a$ falls from 0.747 to 0.734). While the steady-state human capital stock increases slightly from 3.548 to 3.606, the educational investment per child remains relatively stable. The government covers a smaller proportion (76.4%), with parents contributing the remaining 23.6%. Due to the shortened life expectancy, parents tend to save less (i.e., $0.946 < 1.111$) and impose harsh discipline more frequently (i.e., $0.254 > 0.224$).

Column [3] differs from column [1] only in that α increases from 0.5 to 0.7, which reflects a growing emphasis among parents on promoting their children's human capital accumulation. Both the frequency of harsh discipline and the amount of educational investments experience an uptick, which helps to achieve a higher level of the steady-state human capital (i.e., $3.805 > 3.548$). Also, the government supports the educational initiatives by allocating a budget of $-\theta e = 1.647$ towards education and by reducing the income tax rate (i.e., $29.6\% < 31.3\%$). The shift towards prioritizing children's educational advancement results in a decreased attention towards elderly parents, which is manifested by the reduction of a from 0.506 to 0.465.

In column [4], the sole change from column [1] is that γ rises from 0.5 to 0.7. As adult children develop a heightened concern about the well-being of their aging parents, they choose to dedicate a greater portion of their time in eldercare when their parents are alive (i.e., $0.632 > 0.506$). As their commitments to work decrease, the government responds by lowering the rate of income tax (i.e., $28.8\% < 31.3\%$) to prevent further discouragement of labor supply. However, the increased joy of giving towards elderly parents comes at a cost for young children: parents impose less discipline and cut back on educational investments. While parents' educational expenditure, as measured by $(1+\theta)e$, drops from 0.378 to 0.331, the government reduces its subsidy on formal education ($-\theta e$) from 1.342 to 1.189. With a societal reorientation away from nurturing the younger generation, the human capital stock ends with a lower level (i.e., $3.471 < 3.548$).

5 Conclusion

Historically, many parents, particularly in East Asian societies, have favored a strict and authoritarian approach to child-rearing in pursuit of academic excellence. This demanding style, although requiring minimal time and monetary investments, can strain the parent-child relationship, leading to a persistent negative influence on the care provided by children to their parents in later years. As life expectancies continue to rise, along with an increased likelihood of encountering physical and cognitive challenges in old age, there will be a marked trend towards adopting more nurturing and engaging parenting practices.

This paper establishes an OLG model analyzing the relationship between parenting styles, human capital formation, and family eldercare. In our model, parents educate their children in young adulthood and become dependent, demanding caregiving support from children in old adulthood. While educational funding and disciplinary measures are both effective in facilitating children's human capital development, the later exerts a detrimental effect on the perceived quality of eldercare received from children in later life. Young adults have two-sided altruism: they are concerned by their children's educational attainment and at the same time they have joy of giving when aiding their elderly parents.

We first examine the *laissez-faire* decisions without any policy intervention. Population aging, as measured by the lengthening of lifetime, tends to discourage harsh parenting approaches. Balancing the imperative of nurturing the potential of the upcoming generation with the duty of caring for aging parents poses a challenge for young adults. A heightened focus on eldercare fulfillment can diminish their educational expenditures for their children and engagement in disciplinary measures. Conversely, a greater emphasis on child development can detract from the care provided to the elderly. Both altruistic priorities adversely affect personal savings and consumption.

We then delve into the normative aspect of what the optimal public policy should be concerning individual choices. Our evaluation reveals that decentralized equilibrium coincidentally aligns with first-best optimality, rooted in parents' impure altruism towards their children. We outline the conditions under which the government should tax labor and subsidize formal education, proposing that such policies enhance social welfare especially when intergenerational altruism remains low. This result is not novel, as the idea of taxing labor and subsidizing education has been well established in the literature. What distinguishes this paper is its distinctive rationale behind this policy recommendation: aimed at adjusting parenting styles and promoting eldercare. While this study assumes homogeneity among individuals, future research can incorporate individual differences to provide a more nuanced understanding of these dynamics.

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Appendix: Mathematical Proofs

Proof of Proposition 1

Totally differentiating (8) with respect to π, d_t, e_t, and a_{t+1} obtains

$$\left[\left(\frac{\partial v}{\partial d_t}\right)^{-1}\frac{\partial^2\Phi}{\partial d_t^2}-\left(\frac{\partial v}{\partial d_t}\right)^{-2}\frac{\partial^2 v}{\partial d_t^2}\frac{\partial\Phi}{\partial d_t}\right]dd_t-\left(\frac{\partial v}{\partial d_t}\right)^{-2}\frac{\partial^2 v}{\partial d_t\partial a_{t+1}}\frac{\partial\Phi}{\partial d_t}da_{t+1}$$
$$+\left(\frac{\partial v}{\partial d_t}\right)^{-1}\frac{\partial^2\Phi}{\partial d_t\partial e_t}de_t=-\frac{\beta}{\alpha}d\pi$$
$$\Leftrightarrow\quad\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{de_t}{d\pi}+\Omega\frac{dd_t}{d\pi}=-\frac{\beta}{\alpha}\frac{\partial v}{\partial d_t},\tag{A1}$$

where $\Omega:=\dfrac{\partial^2\Phi}{\partial d_t^2}+\left(-\dfrac{\partial v}{\partial d_t}\right)^{-1}\dfrac{\partial^2 v}{\partial d_t^2}\dfrac{\partial\Phi}{\partial d_t}\leqslant\dfrac{\partial^2\Phi}{\partial d_t^2}<0$. Totally differentiating each equality of (9) with respect to (d_t, e_t, a_t, s_t) and π obtains

$$\frac{\beta(1+r)^2}{\pi}\frac{ds_t}{d\pi}-\frac{\gamma g''(\pi a_t)\pi}{wH_tu''(c_t^o)}\frac{da_t}{d\pi}=\frac{\beta(1+r)^2s_t}{\pi^2}+\frac{\gamma g''(\pi a_t)a_t}{wH_tu''(c_t^o)},\tag{A2}$$
$$\frac{\beta(1+r)^2}{\pi}\frac{ds_t}{d\pi}-\frac{\alpha}{u''(c_t^o)}\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{dd_t}{d\pi}-\frac{\alpha}{u''(c_t^o)}\frac{\partial^2\Phi}{\partial e_t^2}\frac{de_t}{d\pi}=\frac{\beta(1+r)^2s_t}{\pi^2},\tag{A3}$$
$$\left[wH_t+\frac{\gamma g''(\pi a_t)}{wH_tu''(c_t^y)}\right]\pi\frac{da_t}{d\pi}+\frac{ds_t}{d\pi}+\frac{de_t}{d\pi}=-\left[wH_t+\frac{\gamma g''(\pi a_t)}{wH_tu''(c_t^y)}\right]a_t.\tag{A4}$$

Combining the above four equations and rearranging yields:

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{de_t}{d\pi}=-\underbrace{\frac{s\Omega_t}{\pi\Psi_t}}_{-}+\underbrace{\frac{\beta}{\alpha}\frac{\partial v}{\partial d_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t},\tag{A5}$$
$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{dd_t}{d\pi}=-\underbrace{\frac{\beta}{\alpha}\frac{\partial v}{\partial d_t}\frac{\partial^2\Phi}{\partial e_t^2}}_{-}-\underbrace{\frac{\beta}{\alpha\Psi_t}\frac{\partial v}{\partial d_t}}_{+}+\underbrace{\frac{s}{\pi\Psi_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t}.\tag{A6}$$

where $\Psi_t:=\alpha\left[\frac{\pi}{\beta(1+r)^2u''(c_t^o)}+\frac{1}{u''(c_t^y)}+\frac{(wH_t)^2}{\gamma g''(\pi a_t)}\right]<0$. If condition (11) holds true, then we have $\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2>0$ is met, and hence, the term in the bracket on the left-hand-side of (A5) as well as (A6) is positive. If condition (3) holds true, then the right-hand-side of (A5) as well as (A6) is positive, and therefore, we have $\frac{dd_t}{d\pi}<0$ and $\frac{de_t}{d\pi}<0$.

Proof of Proposition 2

Totally differentiating (8) with respect to α, d_t, e_t, and a_{t+1} obtains

$$\left(\frac{\partial v}{\partial d_t}\right)^{-1}\left(\frac{\partial^2\Phi}{\partial d_t^2}dd_t+\frac{\partial^2\Phi}{\partial d_t\partial e_t}de_t\right)-\left(\frac{\partial v}{\partial d_t}\right)^{-2}\frac{\partial\Phi}{\partial d_t}\left(\frac{\partial^2 v}{\partial d_t^2}dd_t+\frac{\partial^2 v}{\partial d_t\partial a_{t+1}}da_{t+1}\right)=\frac{\beta\pi}{\alpha^2}d\alpha$$

$$\Leftrightarrow\quad \Omega_t\frac{dd_t}{d\alpha}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{de_t}{d\alpha}=\frac{\beta\pi}{\alpha^2}\frac{\partial v}{\partial d_t}. \tag{A7}$$

Totally differentiating each equality of (9) with respect to (d_t, e_t, a_t, s_t) and α obtains

$$\frac{\beta(1+r)^2u''(c_t^o)}{\pi}\frac{ds_t}{d\alpha}-\frac{\gamma\pi g''(\pi a_t)}{wH_t}\frac{da_t}{d\alpha}=0, \tag{A8}$$

$$\left[\frac{\beta(1+r)^2u''(c_t^o)}{\pi u''(c_t^y)}+1\right]\frac{ds_t}{d\alpha}+\pi wH_t\frac{da_t}{d\alpha}+\frac{de_t}{d\alpha}=0, \tag{A9}$$

$$\frac{\partial^2\Phi}{\partial e_t^2}\frac{de_t}{d\alpha}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{dd_t}{d\alpha}-\frac{\gamma\pi g''(\pi a_t)}{\alpha wH_t}\frac{da_t}{d\alpha}=-\frac{1}{\alpha}\frac{\partial\Phi}{\partial e_t}. \tag{A10}$$

Combining the above four equations and rearranging yields:

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{da_t}{d\alpha}=\underbrace{\frac{wH_t}{\gamma\pi g''(\pi a_t)\Psi_t}}_{+}\left(\underbrace{\frac{\partial\Phi}{\partial e_t}}_{-}\Omega_t+\underbrace{\frac{\beta\pi}{\alpha}\frac{\partial v}{\partial d_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right), \tag{A11}$$

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{ds_t}{d\alpha}=\underbrace{\frac{\pi}{\beta(1+r)^2u''(c_t^o)\Psi_t}}_{+}\left(\underbrace{\frac{\partial\Phi}{\partial e_t}}_{-}\Omega_t+\underbrace{\frac{\beta\pi}{\alpha}\frac{\partial v}{\partial d_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right), \tag{A12}$$

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{de_t}{d\alpha}=\underbrace{-\frac{1}{\alpha}}_{-}\left(\underbrace{\frac{\partial\Phi}{\partial e_t}}_{-}\Omega_t+\underbrace{\frac{\beta\pi}{\alpha}\frac{\partial v}{\partial d_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right), \tag{A13}$$

$$\frac{dd_t}{d\alpha}=\underbrace{\frac{\beta\pi}{\alpha^2\Omega_t}\frac{\partial v}{\partial d_t}}_{+}-\underbrace{\frac{1}{\Omega_t}}_{-}\frac{de_t}{d\alpha}\frac{\partial^2\Phi}{\partial d_t\partial e_t}, \tag{A14}$$

Equations (A11) – (A13) indicate that when both conditions (11) and (3) hold true, we have $\frac{da_t}{d\alpha}<0$, $\frac{ds_t}{d\alpha}<0$, $\frac{de_t}{d\alpha}>0$, and $\frac{dd_t}{d\alpha}>0$. When condition (3) is satisfied and $\frac{de_t}{d\alpha}>0$, we have $\frac{dd_t}{d\alpha}>0$ in equation (A14).

Totally differentiating (8) with respect to γ, d_t, e_t, and a_{t+1} obtains

$$\left(\frac{\partial v}{\partial d_t}\right)^{-1}\left(\frac{\partial^2\Phi}{\partial d_t^2}dd_t+\frac{\partial^2\Phi}{\partial d_t\partial e_t}de_t\right)-\left(\frac{\partial v}{\partial d_t}\right)^{-2}\frac{\partial\Phi}{\partial d_t}\left(\frac{\partial^2 v}{\partial d_t^2}dd_t+\frac{\partial^2 v}{\partial d_t\partial a_{t+1}}da_{t+1}\right)=0$$

$$\Leftrightarrow\quad \Omega_t\frac{dd_t}{d\gamma}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{de_t}{d\gamma}=0. \tag{A15}$$

Totally differentiating each equality of (9) with respect to (d_t, e_t, a_t, s_t) and γ obtains

$$\frac{\beta(1+r)^2u''(c_t^o)}{\pi}\frac{ds_t}{d\gamma}-\frac{\gamma\pi g''(\pi a_t)}{wH_t}\frac{da_t}{d\gamma}=\frac{g'(\pi a_t)}{wH_t}, \tag{A16}$$

$$\left[\frac{\beta(1+r)^2u''(c_t^o)}{\pi u''(c_t^y)}+1\right]\frac{ds_t}{d\gamma}+\pi wH_t\frac{da_t}{d\gamma}+\frac{de_t}{d\gamma}=0, \tag{A17}$$

$$\frac{\partial^2\Phi}{\partial e_t^2}\frac{de_t}{d\gamma}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{dd_t}{d\gamma}-\frac{\gamma\pi g''(\pi a_t)}{\alpha wH_t}\frac{da_t}{d\gamma}=\frac{g'(\pi a_t)}{\alpha wH_t}. \tag{A18}$$

Combining the above four equations and rearranging yields:

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{de_t}{d\gamma}=\underbrace{\frac{\Omega_t}{\Psi_t}\frac{wH_tg'(\pi a_t)}{\gamma g''(\pi a_t)}}_{+}, \tag{A19}$$

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{da_t}{d\gamma}=\underbrace{\frac{-g'(\pi a_t)\Omega_t}{\gamma\pi g''(\pi a_t)\Psi_t}}_{+}$$

$$-\underbrace{\frac{\alpha g'(\pi a_t)}{\gamma\pi g''(\pi a_t)\Psi_t}}_{+}\left[\underbrace{\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2}_{+}\right]\left[\underbrace{\frac{1}{u''(c_t^y)}+\frac{\pi}{\beta(1+r)^2u''(c_t^o)}}_{-}\right], \tag{A20}$$

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{ds_t}{d\gamma}$$

$$=\underbrace{\frac{\alpha\pi wH_tg'(\pi a_t)}{\beta(1+r)^2\gamma g''(\pi a_t)u''(c_t^o)\Psi_t}}_{-}\left[\underbrace{\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2}_{+}\right], \tag{A21}$$

$$\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{dd_t}{d\gamma}=-\underbrace{\frac{wH_tg'(\pi a_t)}{\gamma g''(\pi a_t)\Psi_t}}_{-}\frac{\partial^2\Phi}{\partial d_t\partial e_t}. \tag{A22}$$

It is straightforward to see from equations (A19) – (A21) that $\frac{de_t}{d\gamma}<0$, $\frac{da_t}{d\gamma}>0$, and $\frac{ds_t}{d\gamma}<0$ under condition (11). Furthermore, we have $\frac{dd_t}{d\gamma}<0$ when both conditions (11) and (3) hold true.

Proof of Proposition 3

Totally differentiating (8) with respect to H_t, d_t, e_t, and a_{t+1} obtains

$$\begin{aligned}
&\left(\frac{\partial v}{\partial d_t}\right)^{-1}\left(\frac{\partial^2\Phi}{\partial d_t^2}dd_t+\frac{\partial^2\Phi}{\partial d_t\partial e_t}de_t+\frac{\partial^2\Phi}{\partial d_t\partial H_t}dH_t\right)\\
&-\left(\frac{\partial v}{\partial d_t}\right)^{-2}\frac{\partial\Phi}{\partial d_t}\left(\frac{\partial^2 v}{\partial d_t^2}dd_t+\frac{\partial^2 v}{\partial d_t\partial a_{t+1}}da_{t+1}\right)=0\\
&\Leftrightarrow\quad \Omega_t\frac{dd_t}{dH_t}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{de_t}{dH_t}=-\frac{\partial^2\Phi}{\partial d_t\partial H_t}. \qquad \text{(A23)}
\end{aligned}$$

Totally differentiating each equality of (9) with respect to (d_t, e_t, a_t, s_t) and H_t obtains

$$\frac{\beta(1+r)^2}{\pi}\frac{ds_t}{dH_t}-\frac{\gamma\pi g''(\pi a_t)}{wH_t u''(c_t^o)}\frac{da_t}{dH_t}=-\frac{\gamma g'(\pi a_t)}{wH_t^2 u''(c_t^o)}, \qquad \text{(A24)}$$

$$\left[\frac{\beta(1+r)^2u''(c_t^o)}{\pi u''(c_t^y)}+1\right]\frac{ds_t}{dH_t}+\pi wH_t\frac{da_t}{dH_t}+\frac{de_t}{dH_t}=(1-\pi a_t)w, \qquad \text{(A25)}$$

$$\frac{\partial^2\Phi}{\partial e_t^2}\frac{de_t}{dH_t}+\frac{\partial^2\Phi}{\partial d_t\partial e_t}\frac{dd_t}{dH_t}-\frac{\gamma\pi g''(\pi a_t)}{\alpha wH_t}\frac{da_t}{dH_t}=-\left[\frac{\gamma g'(\pi a_t)}{\alpha wH_t^2}+\frac{\partial^2\Phi}{\partial e_t\partial H_t}\right]. \qquad \text{(A26)}$$

Combining the above four equations and rearranging yields

$$\begin{aligned}
&\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{dd_t}{dH_t}=-\underbrace{\left(\frac{1}{\Psi_t}+\frac{\partial^2\Phi}{\partial e_t^2}\right)}_{-}\frac{\partial^2\Phi}{\partial d_t\partial H_t}\\
&+\left\{\frac{\partial^2\Phi}{\partial e_t\partial H_t}-\underbrace{\frac{w}{\Psi_t}\left[(1-\pi a_t)-\frac{g'(\pi a_t)}{g''(\pi a_t)}\right]}_{-}\right\}\frac{\partial^2\Phi}{\partial d_t\partial e_t}, \qquad \text{(A27)}
\end{aligned}$$

$$\begin{aligned}
&\left[\Omega_t\frac{\partial^2\Phi}{\partial e_t^2}-\left(\frac{\partial^2\Phi}{\partial d_t\partial e_t}\right)^2+\frac{\Omega_t}{\Psi_t}\right]\frac{de_t}{dH_t}=\underbrace{-\Omega_t}_{+}\frac{\partial^2\Phi}{\partial e_t\partial H_t}\\
&+\left\{\frac{\partial^2\Phi}{\partial d_t\partial H_t}+\underbrace{w\frac{\Omega_t}{\Psi_t}\left[(1-\pi a_t)-\frac{g'(\pi a_t)}{g''(\pi a_t)}\right]}_{+}\right\}\frac{\partial^2\Phi}{\partial d_t\partial e_t}, \qquad \text{(A28)}
\end{aligned}$$

If conditions (11), (3), and (12) are satisfied, then $\frac{dd_t}{dH_t}>0$ and $\frac{de_t}{dH_t}>0$.

Deriving Equations (19) – (21)

Taking the first order conditions of $\mathfrak{L}$ with respect to (τ, θ, A) obtains

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial \tau} &= u'(c^y)\Bigg[-(1-\pi a)wH - (1-\tau)\pi wH\frac{\partial a}{\partial \tau} + (1-\pi a)(1-\tau)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \tau}\right) \\
&\quad -\frac{\partial s}{\partial \tau} - (1+\theta)\frac{\partial e}{\partial \tau}\Bigg] + \beta\left[u'(c^o)(1+r)\frac{\partial s}{\partial \tau} + \pi\frac{\partial v}{\partial a}\frac{\partial a}{\partial \tau} + \pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial \tau}\right] \\
&\quad +\mu\left[(1-\pi a)\,wH + \tau(1-\pi a)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \tau}\right) - \tau\pi wH\frac{\partial a}{\partial \tau} + \theta\frac{\partial e}{\partial \tau}\right] = 0, \\
\Leftrightarrow \quad \frac{\partial \mathfrak{L}}{\partial \tau} &= (1-\tau)(1-\pi a)wu'(c^y)\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \tau} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \tau}\right) + \beta\pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial \tau} - (1+\theta)u'(c^y)\frac{\partial e}{\partial \tau} \\
&\quad +\pi\left[\beta\frac{\partial v}{\partial a} - (1-\tau)wHu'(c^y)\right]\frac{\partial a}{\partial \tau} + \left[\beta(1+r)u'(c^o) - u'(c^y_t)\right]\frac{\partial s}{\partial \tau} \\
&\quad -(1-\pi a)wHu'(c^y) + \mu\left[(1-\pi a)\,wH + \Delta_\tau\right] = 0, \qquad \text{(A29)} \\
\frac{\partial \mathfrak{L}}{\partial \theta} &= u'(c^y)\Bigg[-\pi wH(1-\tau)\frac{\partial a}{\partial \theta} + (1-\pi a)(1-\tau)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \theta}\right) - \frac{\partial s}{\partial \theta} \\
&\quad -e - (1+\theta)\frac{\partial e}{\partial \theta}\Bigg] + \beta\left[u'(c^o)(1+r)\frac{\partial s}{\partial \theta} + \pi\frac{\partial v}{\partial a}\frac{\partial a}{\partial \theta} + \pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial \theta}\right] \\
&\quad +\mu\left[-\tau\pi wH\frac{\partial a}{\partial \theta} + \tau(1-\pi a)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \theta}\right) + e + \theta\frac{\partial e}{\partial \theta}\right] = 0 \\
\Leftrightarrow \quad \frac{\partial \mathfrak{L}}{\partial \theta} &= (1-\tau)(1-\pi a)wu'(c^y)\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial \theta} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial \theta}\right) + \beta\pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial \theta} - (1+\theta)u'(c^y)\frac{\partial e}{\partial \theta} \\
&\quad +\pi\left[\beta\frac{\partial v}{\partial a} - (1-\tau)wHu'(c^y)\right]\frac{\partial a}{\partial \theta} + \left[\beta(1+r)u'(c^o) - u'(c^y_t)\right]\frac{\partial s}{\partial \theta} \\
&\quad -eu'(c^y) + \mu\,(e + \Delta_\theta) = 0, \qquad \text{(A30)} \\
\frac{\partial \mathfrak{L}}{\partial A} &= u'(c^y)\Bigg[-\pi wH(1-\tau)\frac{\partial a}{\partial A} + (1-\pi a)(1-\tau)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial A} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial A}\right) \\
&\quad -\frac{\partial s}{\partial A} - (1+\theta)\frac{\partial e}{\partial A} + 1\Bigg] + \beta\left[u'(c^o)(1+r)\frac{\partial s}{\partial A} + \pi\frac{\partial v}{\partial a}\frac{\partial a}{\partial A} + \pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial A}\right] \\
&\quad +\mu\left[-\tau\pi wH\frac{\partial a}{\partial A} + \tau(1-\pi a)w\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial A} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial A}\right) + \theta\frac{\partial e}{\partial A} - 1\right] = 0 \\
\Leftrightarrow \quad \frac{\partial \mathfrak{L}}{\partial A} &= (1-\tau)(1-\pi a)wu'(c^y)\left(\frac{\partial H}{\partial d}\frac{\partial d}{\partial A} + \frac{\partial H}{\partial e}\frac{\partial e}{\partial A}\right) + \beta\pi\frac{\partial v}{\partial d}\frac{\partial d}{\partial A} - (1+\theta)u'(c^y)\frac{\partial e}{\partial A} \\
&\quad +\pi\left[\beta\frac{\partial v}{\partial a} - (1-\tau)wHu'(c^y)\right]\frac{\partial a}{\partial A} + \left[\beta(1+r)u'(c^o) - u'(c^y_t)\right]\frac{\partial s}{\partial A} \\
&\quad +u'(c^y) + \mu\,(-1 + \Delta_\theta) = 0. \qquad \text{(A31)}
\end{aligned}$$

Substituting (17) and (18) into (A29) – (A31) obtains (19) – (21).