Chapter 6

Performance loss in non-ideal conditions

6.1 Introduction

The performance analysis proposed in Chapter 5 has been carried out with the assumption that the joint detector is operated in ideal conditions. By this, we mean that two important hypotheses are satisfied :

- The clocks of the K transmitters and that of the unique receiver are perfectly synchronized. This implies that the received sequence of samples can be effectively modeled by the filtering of the transmitted symbol sequences by the nominal equivalent channels, plus the additive noise. In other words, *timing synchronization* is ideal.

- The equivalent channels are perfectly known by the receiver, which allows the computation of JD coefficients that are perfectly matched to these equivalent channels. In other words, *channel knowledge* is ideal.

In a practical system, these assumptions are not satisfied. Actually both the clock alignment and the channel coefficients have to be estimated and tracked. Maximum likelihood (ML) estimation techniques can be used to derive efficient estimates of these parameters on the basis of the observed noisy signal. ML timing estimation techniques are analyzed in chapter 7, while the issue of ML channel estimation is beyond the scope of this work. The parameter estimates obtained with these techniques are not ideal. It follows a timing error ε_p for each subchannel (which is actually identical for all subchannels allocated to a given user), and an estimated channel matrix $\underline{\underline{\hat{\mathrm{H}}}}(z) = \underline{\underline{\mathrm{H}}}(z) + \underline{\underline{\tilde{\mathrm{H}}}}(z)$ that differs from the true value $\underline{\underline{\mathrm{H}}}(z)$. It is important to evaluate the robustness of the proposed multiuser FB-based transmission system with respect to these timing and channel estimation errors. The performance degradation associated with these non-ideal conditions can be modeled by means of an excess mean square error (MSE) at the output of the joint detector. In the following, we assume that the timing and channel estimation errors are very small, which allows the use of first order approximation for the computation of the excess MSE. These conditions are satisfied if the system is operated at high SNR.

6.2 Excess MSE in non-ideal conditions

The symbol estimates at the output of an IIR linear MMSE JD operated in non-ideal conditions are given by

$$\underline{\hat{\mathrm{I}}}(z) = \underline{\underline{\hat{\mathrm{W}}}}(z)\,\underline{\mathrm{R}}_{\underline{\varepsilon}}(z) \tag{6.1}$$

where :

- The non-ideal JD coefficients are based on the estimated channel matrix $\underline{\underline{\hat{\mathrm{H}}}}(z)$:

$$\begin{aligned}\underline{\underline{\hat{\mathrm{W}}}}(z) &= \sigma_I^2\,\underline{\underline{\hat{\mathrm{H}}}}^H(1/z^*)\left[\sigma_n^2\,\underline{\underline{\mathrm{E}}}_M + \sigma_I^2\,\underline{\underline{\hat{\mathrm{H}}}}(z)\,\underline{\underline{\hat{\mathrm{H}}}}^H(1/z^*)\right]^{-1} && (6.2)\\ &= \underline{\underline{\mathrm{W}}}(z) + \underline{\underline{\tilde{\mathrm{W}}}}(z) && (6.3)\end{aligned}$$

  where $\underline{\underline{\mathrm{W}}}(z)$ is the optimal linear MMSE JD associated with the true channel matrix $\underline{\underline{\mathrm{H}}}(z)$.

- The asynchronous received sequences are related to the transmitted symbols by

$$\begin{aligned}\underline{\mathrm{R}}_{\underline{\varepsilon}}(z) &= \underline{\underline{\mathrm{H}}}_{\underline{\varepsilon}}(z)\,\underline{\mathrm{I}}(z) + \underline{\mathrm{N}}(z) && (6.4)\\ &= \underline{\mathrm{R}}_{\underline{0}}(z) + \underline{\tilde{\mathrm{R}}}_{\underline{\varepsilon}}(z) && (6.5)\end{aligned}$$

  where the asynchronous channel matrix $\underline{\underline{\mathrm{H}}}_{\underline{\varepsilon}}(z)$ contains the z-transformed sequences $\overline{h}_{pk}(mT + \rho T_s + \varepsilon_k)$ (see (4.38)).

The symbol estimation errors $\underline{\hat{\epsilon}}(z)$ obtained in non-ideal conditions are given by

$$\begin{aligned}\underline{\hat{\epsilon}}(z) &= \underline{\mathrm{I}}(z) - \underline{\hat{\mathrm{I}}}(z) && (6.6)\\ &\approx \underline{\epsilon}(z) - \underline{\underline{\tilde{\mathrm{W}}}}(z)\,\underline{\mathrm{R}}_{\underline{0}}(z) - \underline{\underline{\mathrm{W}}}(z)\,\underline{\tilde{\mathrm{R}}}_{\underline{\varepsilon}}(z) && (6.7)\end{aligned}$$

where $\underline{\epsilon}(z)$ is the estimation error obtained with perfect channel knowledge and timing synchronization. The mean square symbol estimation errors are

$$\Gamma_{\hat{\underline{\epsilon}}\hat{\underline{\epsilon}}}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underline{\underline{\gamma}}_{\hat{\underline{\epsilon}}\hat{\underline{\epsilon}}}(e^{j\Omega})\, \mathrm{d}\Omega. \tag{6.8}$$

From (6.7), the error power spectrum contains the following contributions :

$$\begin{aligned}\underline{\underline{\gamma}}_{\hat{\underline{\epsilon}}\hat{\underline{\epsilon}}}(z) \approx\ & \underline{\underline{\gamma}}_{\underline{\epsilon}\underline{\epsilon}}(z) + \left[\tilde{\underline{\underline{W}}}(z)\, \underline{\underline{\gamma}}_{\underline{R}_0\underline{R}_0}(z)\, \tilde{\underline{\underline{W}}}^H(\frac{1}{z^*})\right] \\ & + \left[\underline{\underline{W}}(z)\, \underline{\underline{\gamma}}_{\tilde{\underline{R}}_\varepsilon\tilde{\underline{R}}_\varepsilon}(z)\, \underline{\underline{W}}^H(\frac{1}{z^*}) + \underline{\underline{\gamma}}_{\underline{\epsilon}\tilde{\underline{R}}_\varepsilon}(z)\, \underline{\underline{W}}^H(\frac{1}{z^*}) + \underline{\underline{W}}(z)\, \underline{\underline{\gamma}}_{\tilde{\underline{R}}_\varepsilon\underline{\epsilon}}(z)\right].\end{aligned} \tag{6.9}$$

The first term is the MMSE obtained in ideal conditions. The second term is the excess MSE due to channel estimation errors. The third term is the excess MSE due to timing synchronization errors. Some cross-terms have been neglected to obtain (6.9), for the following reasons :

- The cross-spectrum between $\underline{\epsilon}(z)$ and $\tilde{\underline{\underline{W}}}(z)\, \underline{R}_0(z)$ is exactly equal to zero thanks to the orthogonality principle.
- The cross-spectrum between $\tilde{\underline{\underline{W}}}(z)\, \underline{R}_0(z)$ and $\underline{\underline{W}}(z)\, \tilde{\underline{R}}_\varepsilon(z)$ represents the coupling between the channel estimation and timing synchronization effects. As both estimation errors are assumed to be uncorrelated, the expectation of these cross-terms is equal to zero.

6.2.1 Excess MSE due to channel estimation errors

Let us now compute how the channel estimation error $\tilde{\underline{\underline{H}}}(z)$ is transformed into an error on the linear JD matrix $\tilde{\underline{\underline{W}}}(z)$. Developing (6.2), we get

$$\begin{aligned}\hat{\underline{\underline{W}}}(z) \approx\ & \sigma_I^2 \left[\underline{\underline{H}}^H(\frac{1}{z^*}) + \tilde{\underline{\underline{H}}}^H(\frac{1}{z^*})\right] \qquad (6.10)\\ & \cdot \left\{\underline{\underline{\gamma}}_{\underline{R}_0\underline{R}_0}(z) + \sigma_I^2 \left[\underline{\underline{H}}(z)\, \tilde{\underline{\underline{H}}}^H(\frac{1}{z^*}) + \tilde{\underline{\underline{H}}}(z)\, \underline{\underline{H}}^H(\frac{1}{z^*})\right]\right\}^{-1} \\ \approx\ & \underline{\underline{W}}(z) + \sigma_I^2\, \tilde{\underline{\underline{H}}}^H(\frac{1}{z^*})\, \underline{\underline{\gamma}}^{-1}_{\underline{R}_0\underline{R}_0}(z) \qquad (6.11)\\ & - \sigma_I^2\, \underline{\underline{H}}^H(\frac{1}{z^*})\, \underline{\underline{\gamma}}^{-1}_{\underline{R}_0\underline{R}_0}(z)\, \sigma_I^2 \left[\underline{\underline{H}}(z)\, \tilde{\underline{\underline{H}}}^H(\frac{1}{z^*}) + \tilde{\underline{\underline{H}}}(z)\, \underline{\underline{H}}^H(\frac{1}{z^*})\right] \underline{\underline{\gamma}}^{-1}_{\underline{R}_0\underline{R}_0}(z)\end{aligned}$$

where the second order terms were neglected and the following matrix perturbation property was used :

$$\left(\underline{\underline{A}} + \underline{\underline{a}}\right)^{-1} \approx \underline{\underline{A}}^{-1} - \underline{\underline{A}}^{-1}\, \underline{\underline{a}}\, \underline{\underline{A}}^{-1} \tag{6.12}$$

for an arbitrary small perturbation $\underline{\underline{\mathrm{a}}}$ of a given matrix $\underline{\underline{\mathrm{A}}}$. The JD deviation matrix $\underline{\underline{\tilde{\mathrm{W}}}}(z)$ becomes

$$
\begin{aligned}
\underline{\underline{\tilde{\mathrm{W}}}}(z) &\approx \left\{\left[\sigma_I^2\,\underline{\underline{\mathrm{E}}}_N - \underline{\underline{\mathrm{W}}}(z)\,\underline{\underline{\gamma}}_{\underline{\mathrm{R}}_0\underline{\mathrm{R}}_0}(z)\,\underline{\underline{\mathrm{W}}}^H(\frac{1}{z^*})\right]\underline{\underline{\tilde{\mathrm{H}}}}^H(\frac{1}{z^*})\right.\\
&- \left.\underline{\underline{\mathrm{W}}}(z)\,\underline{\underline{\tilde{\mathrm{H}}}}(z)\left[\sigma_I^2\,\underline{\underline{\mathrm{H}}}^H(\frac{1}{z^*})\right]\right\}\underline{\underline{\gamma}}^{-1}_{\underline{\mathrm{R}}_0\underline{\mathrm{R}}_0}(z) \qquad (6.13)\\
&\approx \underline{\underline{\gamma}}_{\underline{\epsilon}\underline{\epsilon}}(z)\,\underline{\underline{\tilde{\mathrm{H}}}}^H(\frac{1}{z^*})\,\underline{\underline{\gamma}}^{-1}_{\underline{\mathrm{R}}_0\underline{\mathrm{R}}_0}(z) - \underline{\underline{\mathrm{W}}}(z)\,\underline{\underline{\tilde{\mathrm{H}}}}(z)\,\underline{\underline{\mathrm{W}}}(z) \qquad (6.14)
\end{aligned}
$$

in which (6.13) comes from the expression of the optimal linear JD in (5.53) and (6.14) comes from the expression of the optimal MMSE in (5.58).

The excess MSE due to channel estimation errors is obtained by introducing (6.14) into the second term of (6.9). This result is valid under the condition that the CE error $\underline{\underline{\tilde{\mathrm{H}}}}(z)$ is small with respect to the true channel $\underline{\underline{\mathrm{H}}}(z)$. If we make the additional assumption that the symbol estimation error $\underline{\epsilon}(z)$, obtained in the absence of a CE error, has a small variance, then the first term in (6.14) can be neglected, which provides a very tractable expression for the CE-related excess MSE :

$$
\begin{aligned}
\Gamma_{\underline{\hat{\epsilon}}\underline{\hat{\epsilon}}}(0) &\approx \Gamma_{\underline{\epsilon}\underline{\epsilon}}(0) + \int_{-\pi}^{\pi}\left\{\underline{\underline{\mathrm{W}}}(e^{j\Omega})\,\underline{\underline{\tilde{\mathrm{H}}}}(e^{j\Omega})\left[\underline{\underline{\mathrm{W}}}^H(e^{j\Omega})\,\underline{\underline{\gamma}}_{\underline{\mathrm{R}}_0\underline{\mathrm{R}}_0}(e^{j\Omega})\,\underline{\underline{\mathrm{W}}}(e^{j\Omega})\right]\right.\\
&\quad .\ \left.\underline{\underline{\tilde{\mathrm{H}}}}^H(e^{j\Omega})\,\underline{\underline{\mathrm{W}}}^H(e^{j\Omega})\right\}\frac{\mathrm{d}\Omega}{2\pi} \qquad (6.15)\\
&\approx \Gamma_{\underline{\epsilon}\underline{\epsilon}}(0) + \int_{-\pi}^{\pi}\left\{\underline{\underline{\mathrm{W}}}(e^{j\Omega})\,\underline{\underline{\tilde{\mathrm{H}}}}(e^{j\Omega})\left[\sigma_I^2\,\underline{\underline{\mathrm{E}}}_N - \underline{\underline{\gamma}}_{\underline{\epsilon}\underline{\epsilon}}(e^{j\Omega})\right]\right.\\
&\quad .\ \left.\underline{\underline{\tilde{\mathrm{H}}}}^H(e^{j\Omega})\,\underline{\underline{\mathrm{W}}}^H(e^{j\Omega})\right\}\frac{\mathrm{d}\Omega}{2\pi} \qquad (6.16)\\
&\approx \Gamma_{\underline{\epsilon}\underline{\epsilon}}(0) + \sigma_I^2\int_{-\pi}^{\pi}\left\{\underline{\underline{\mathrm{W}}}(e^{j\Omega})\left[\underline{\underline{\tilde{\mathrm{H}}}}(e^{j\Omega})\,\underline{\underline{\tilde{\mathrm{H}}}}^H(e^{j\Omega})\right]\underline{\underline{\mathrm{W}}}^H(e^{j\Omega})\right\}\frac{\mathrm{d}\Omega}{2\pi}.
\end{aligned}
$$

Let us model the CE error matrix $\underline{\underline{\tilde{\mathrm{H}}}}(z)$ as $M \times N$ mutually independent white Gaussian sequences of length $L+1$ and with equal variance σ_h^2. The average CE error cross-spectrum is then

$$
\mathrm{E}\left\{\underline{\underline{\tilde{\mathrm{H}}}}(z)\,\underline{\underline{\tilde{\mathrm{H}}}}^H(z)\right\} = (L+1)\,N\,\sigma_h^2\,\underline{\underline{\mathrm{E}}}_M. \qquad (6.17)
$$

Combining (6.15) and (6.17), the average MSE associated with the p-th subchannel is finally

$$
\sigma^2_{\hat{\epsilon}_p} = \sigma^2_{\epsilon_p} + \sigma_I^2\,.\,(L+1)N\,.\,\sigma_h^2\sum_{n=-\infty}^{+\infty} w_p^2(n) \qquad (6.18)
$$

where $w_p(n)$ is the p-th sequence of JD coefficients. The sum $\sum_n w_p^2(n)$ in (6.18) can be interpreted as the sensitivity of the linear MMSE JD with respect to CE errors. It is inversely proportional to the energy associated with subchannel p. If the coefficients $w_p(n)$ are large, then the channel estimation errors may be significantly magnified, resulting in large excess MSE. If, on the other hand, these coefficients are small, then we do not have significant error magnification. The CE error variance σ_h^2, that depends on the selected ML channel estimator, is proportional to the additive noise power. The product $\sigma_h^2 \sum_n w_p^2(n)$ is thus a representation of the inverse SNR available in the p-th subchannel.

These results can be extended to the decision-feedback JD. In that case, both the feedforward coefficients $\underline{\underline{\hat{\mathrm{W}}}}_{df}(z) = \underline{\underline{\mathrm{W}}}_{df}(z) + \underline{\underline{\tilde{\mathrm{W}}}}_{df}(z)$ and the feedback coefficients $\underline{\underline{\hat{\mathrm{B}}}}(z) = \underline{\underline{\mathrm{B}}}(z) + \underline{\underline{\tilde{\mathrm{B}}}}(z)$ can deviate from their optimal values as their computation is based on the non-ideal channel estimate $\underline{\underline{\hat{\mathrm{H}}}}(z) = \underline{\underline{\mathrm{H}}}(z) + \underline{\underline{\tilde{\mathrm{H}}}}(z)$. Using the properties of the optimal DF-JD, it can be shown that the symbol error spectrum with imperfect channel knowledge is

$$\underline{\underline{\gamma}}_{\hat{\epsilon}\hat{\epsilon}}(z) = \underline{\underline{\gamma}}_{\epsilon\epsilon}(z) + \underline{\underline{\mathrm{B}}}(z) \left[\underline{\underline{\tilde{\mathrm{W}}}}(z)\, \underline{\underline{\gamma}}_{\underline{\mathrm{R}}_0\underline{\mathrm{R}}_0}(z)\, \underline{\underline{\tilde{\mathrm{W}}}}^H(\frac{1}{z^*}) \right] \underline{\underline{\mathrm{B}}}^H(\frac{1}{z^*})$$
$$+ \underline{\underline{\tilde{\mathrm{B}}}}(z)\, \underline{\underline{\mathrm{S}}}_c^{-1}(z)\, \underline{\underline{\tilde{\mathrm{B}}}}^H(\frac{1}{z^*}). \quad (6.19)$$

where $\underline{\underline{\mathrm{S}}}_c^{-1}(z)$ is defined in (5.56). There are two contributions to the excess MSE. The first one depends on the error on the linear MMSE JD coefficients $\underline{\underline{\tilde{\mathrm{W}}}}(z)$, whose first-order approximation is given in (6.14). The second term involves the error on the feedback matrix coefficients $\underline{\underline{\tilde{\mathrm{B}}}}(z)$. This error can be approximated as follows. Let $\underline{\underline{\hat{\mathrm{S}}}}_c(z) = \underline{\underline{\mathrm{S}}}_c(z) + \underline{\underline{\tilde{\mathrm{S}}}}_c(z)$ be the key power spectrum (see (5.56)) based on the estimated channel $\underline{\underline{\hat{\mathrm{H}}}}(z)$. The error on the estimated key power spectrum is

$$\underline{\underline{\tilde{\mathrm{S}}}}_c(z) \approx \sigma_n^{-2} \left[\underline{\underline{\tilde{\mathrm{H}}}}^H(\frac{1}{z^*})\, \underline{\underline{\mathrm{H}}}(z) + \underline{\underline{\mathrm{H}}}^H(\frac{1}{z^*})\, \underline{\underline{\tilde{\mathrm{H}}}}(z) \right]. \quad (6.20)$$

The estimated feedback coefficients $\underline{\underline{\hat{\mathrm{B}}}}(z)$ are obtained from the factorization of the estimated power spectrum $\underline{\underline{\hat{\mathrm{S}}}}_c(z)$:

$$\underline{\underline{\hat{\mathrm{S}}}}_c(z) = \underline{\underline{\hat{\mathrm{D}}}}^H(\frac{1}{z^*})\, \underline{\underline{\hat{\Lambda}}}\, \underline{\underline{\hat{\mathrm{D}}}}(z) \rightarrow \underline{\underline{\hat{\mathrm{B}}}}(z) = \underline{\underline{\hat{\mathrm{D}}}}(z). \quad (6.21)$$

The error on the key power spectrum $\underline{\underline{\tilde{S}}}_c(z)$ is related to the error on its spectral factor $\underline{\underline{\tilde{D}}}(z)$ as follows :

$$\underline{\underline{\tilde{S}}}_c(z) \approx \underline{\underline{D}}^H(\frac{1}{z^*})\,\underline{\underline{\Lambda}}\,\underline{\underline{\tilde{D}}}(z) + \underline{\underline{D}}^H(\frac{1}{z^*})\,\underline{\underline{\tilde{\Lambda}}}\,\underline{\underline{D}}(z) + \underline{\underline{\tilde{D}}}^H(\frac{1}{z^*})\,\underline{\underline{\Lambda}}\,\underline{\underline{D}}(z). \qquad (6.22)$$

After left division of Equation (6.22) by $\underline{\underline{D}}^H(\frac{1}{z^*})$ and right division by $\underline{\underline{D}}(z)$, we get

$$\begin{aligned}\underline{\underline{D}}^{-H}(\frac{1}{z^*})\,\underline{\underline{\tilde{S}}}_c(z)\,\underline{\underline{D}}^{-1}(z) \approx & \left[\underline{\underline{\Lambda}}\,\underline{\underline{\tilde{D}}}(z)\,\underline{\underline{D}}^{-1}(z)\right] \\ & + \left[\underline{\underline{\tilde{\Lambda}}}\right] + \left[\underline{\underline{D}}^H(\frac{1}{z^*})\,\underline{\underline{\tilde{D}}}^H(\frac{1}{z^*})\,\underline{\underline{\Lambda}}\right] \qquad (6.23)\end{aligned}$$

The first term in (6.23) is a strictly causal, monic and stable matrix transfer function. The zero-order term in its z-expansion is a lower triangular matrix with '0's on the main diagonal. The second term in (6.23) is a constant diagonal matrix. The third term is strictly anticausal. Actually, any power spectrum can be decomposed into a sum of three such terms. It follows the next expression for the error on the feedback JD coefficients :

$$\underline{\underline{\tilde{B}}}(z) = \underline{\underline{\tilde{D}}}(z) = \underline{\underline{\Lambda}}^{-1}\left[\underline{\underline{D}}^{-H}(\frac{1}{z^*})\,\underline{\underline{\tilde{S}}}_c(z)\,\underline{\underline{D}}^{-1}(z)\right]_+ \underline{\underline{D}}(z) \qquad (6.24)$$

where $[x(z)]_+$ denotes the strictly causal part of the spectrum $x(z)$. Using (6.24) and (5.66), the second contribution to the excess MSE (last term in (6.19)) can finally be written as a function of the channel estimation error $\underline{\underline{\tilde{H}}}(z)$ as follows :

$$\begin{aligned}\underline{\underline{\tilde{B}}}(z)\,\underline{\underline{S}}_c^{-1}(z)\,\underline{\underline{\tilde{B}}}^H(\frac{1}{z^*}) \approx & \;\underline{\underline{\Lambda}}^{-1}\left[\underline{\underline{D}}^{-H}(\frac{1}{z^*})\,\underline{\underline{\tilde{S}}}_c(z)\,\underline{\underline{D}}^{-1}(z)\right]_+ \underline{\underline{\Lambda}}^{-1} \\ & \left[\underline{\underline{D}}^{-H}(\frac{1}{z^*})\,\underline{\underline{\tilde{S}}}_c(z)\,\underline{\underline{D}}^{-1}(z)\right]_+^H \underline{\underline{\Lambda}}^{-1} \qquad (6.25)\end{aligned}$$

where $\underline{\underline{\tilde{S}}}_c(z)$ is given in (6.20).

6.2.2 Excess MSE due to timing synchronization errors

In first order approximation, one can write

$$\underline{\underline{H}}_{\underline{\varepsilon}}(z) \approx \underline{\underline{H}}_{\underline{0}}(z) + \dot{\underline{\underline{H}}}_{\underline{0}}(z)\,\mathrm{Diag}\left(\underline{\varepsilon}\right) + \frac{1}{2}\ddot{\underline{\underline{H}}}_{\underline{0}}(z)\,\mathrm{Diag}\left(\underline{\varepsilon}^2\right) \qquad (6.26)$$

where Diag $(\underline{\varepsilon})$ is a $N \times N$ diagonal matrix containing the timing errors associated with the N subchannels, and $\dot{\underline{\underline{\mathrm{H}}}}_0(z)$ (resp $\ddot{\underline{\underline{\mathrm{H}}}}_0(z)$) contains the z-transformed sequences of equivalent channels first (resp. second) derivatives with respect to the sampling instant. Let us model the timing errors ε_k as zero-mean Gaussian variables with variance $\sigma^2_{\varepsilon_k}$. As the average timing errors are assumed to be equal to zero, the first-order terms are discarded in the next derivations, and only the second-order terms are considered.

The excess MSE due to timing synchronization errors is obtained by introducing (6.26) into the third term of (6.9), together with

$$\tilde{\underline{\mathrm{R}}}_{\underline{\varepsilon}}(z) = \left[\underline{\underline{\mathrm{H}}}_{\underline{\varepsilon}}(z) - \underline{\underline{\mathrm{H}}}_0(z)\right] \underline{\mathrm{I}}(z) \tag{6.27}$$

$$\underline{\epsilon}(z) = \left[\underline{\underline{\mathrm{E}}}_N - \underline{\underline{\mathrm{W}}}(z)\,\underline{\underline{\mathrm{H}}}_0(z)\right] \underline{\mathrm{I}}(z) + \text{additive noise.} \tag{6.28}$$

Keeping only the second order terms, it comes

$$\begin{aligned} \gamma_{\underline{\hat{\epsilon}}\underline{\hat{\epsilon}}}(z) \approx\ & \gamma_{\underline{\epsilon}\underline{\epsilon}}(z) + \sigma_I^2 \left[\underline{\underline{\mathrm{W}}}(z)\,\dot{\underline{\underline{\mathrm{H}}}}_0(z)\,\mathrm{Diag}\left(\underline{\varepsilon}^2\right)\,\dot{\underline{\underline{\mathrm{H}}}}_0^H(\frac{1}{z^*})\,\underline{\underline{\mathrm{W}}}^H(\frac{1}{z^*})\right] \\ +\ & \frac{1}{2}\sigma_I^2 \left\{ \left[\underline{\underline{\mathrm{E}}}_N - \underline{\underline{\mathrm{W}}}(z)\,\underline{\underline{\mathrm{H}}}_0(z)\right] \mathrm{Diag}\left(\underline{\varepsilon}^2\right)\,\ddot{\underline{\underline{\mathrm{H}}}}_0^H(\frac{1}{z^*})\,\underline{\underline{\mathrm{W}}}^H(\frac{1}{z^*}) \right. \\ +\ & \left. \underline{\underline{\mathrm{W}}}(z)\,\ddot{\underline{\underline{\mathrm{H}}}}_0(z)\,\mathrm{Diag}\left(\underline{\varepsilon}^2\right) \left[\underline{\underline{\mathrm{E}}}_N - \underline{\underline{\mathrm{H}}}_0^H(\frac{1}{z^*})\,\underline{\underline{\mathrm{W}}}^H(\frac{1}{z^*})\right] \right\} \end{aligned} \tag{6.29}$$

The third and fourth terms in (6.29) can be neglected if the nominal symbol errors $\underline{\epsilon}(z)$ are small. The timing-related excess MSE is then obtained by integrating the second term in (6.29) around the unit circle. The average symbol mean square errors are finally

$$\sigma^2_{\hat{\epsilon}_p} = \sigma^2_{\epsilon_p} + \sigma_I^2 \sum_{k'=1}^{K} \left[\sum_{n=-\infty}^{+\infty} \sum_{p' \in \mathcal{C}_{k'}} \dot{x}^2_{pp'}(n) \right] \sigma^2_{\varepsilon_{k'}} \tag{6.30}$$

where $\dot{x}_{pp'}(n)$ are the first derivatives with respect to $\varepsilon_{k'}$ of the sequence of interference coefficients $x_{pp'}(n)$. The factor between brackets in (6.30) can be interpreted as the sensitivity of the p-th symbol error variance $\sigma^2_{\epsilon_p}$ with respect to the k'-th timing error $\varepsilon_{k'}$.

6.3 Sensitivity to timing offsets in a single-user system

The effect of a timing offset on the performance of a single-user FB-based transmission system has been analyzed in [63]. The present chapter extends these results by considering paraunitary filter banks and assuming an ideal half Nyquist pulse shaping filter in the transmitter, a static frequency selective channel, and various IIR receivers. The independence of the average timing sensitivity to the selected FB is demonstrated. This study is restricted to a single-user scenario. Actually, the results can be generalized easily to a multiuser system, but in that case, the global timing sensitivity is strongly dependent on the selected signature allocation.

6.3.1 Baseband FB modulation

6.3.1.1 Timing sensitivity

In all generality, the symbol estimate at the output of the detector can be written as :

$$\hat{I}_{p,\varepsilon}(n) = J_{p,\varepsilon}(n) + w_{p,\varepsilon}(n). \tag{6.31}$$

where $J_{p,\varepsilon}(n) = \sum_{p'=0}^{N-1} \sum_{m=-\infty}^{+\infty} x_{pp',\varepsilon}(m)\, I'_p(n-m)$ and $w_{p,\varepsilon}(n)$ represents the contribution of the additive noise.

The interference coefficients $x_{pp',\varepsilon}(m)$ may be obtained by sampling a given set of continuous-time interference waveforms $x_{pp',a}(t)$, that depend on the FB, the channel and the chosen detector, at the symbol rate, with a timing error ε:

$$x_{pp',\varepsilon}(m) = x_{pp',a}(mT + \varepsilon) \tag{6.32}$$

In the z domain, the estimation errors are:

$$\underline{\epsilon}_{\varepsilon}(z) = \underline{\hat{\mathrm{I}}}_{\varepsilon}(z) - \underline{\mathrm{I}}(z) = \left(\underline{\underline{\mathrm{X}}}_{\varepsilon}(z) - \underline{\underline{\mathrm{E}}}_N\right) \underline{\mathrm{I}}(z) + \underline{\mathrm{W}}_{\varepsilon}(z) \tag{6.33}$$

where matrix $\underline{\underline{\mathrm{X}}}_{\varepsilon}(z)$ has element (p, p') given by the z-transform of the sequence $x_{pp',\varepsilon}(m)$. The error covariance matrix is given by

$$\underline{\underline{\mathrm{R}}}_{\epsilon\epsilon}(\varepsilon) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\gamma}}_{\epsilon_\varepsilon\epsilon_\varepsilon}(e^{j\Omega})\, d\Omega = \left[\underline{\underline{\mathrm{R}}}_{JJ}(\varepsilon) + \underline{\underline{\mathrm{R}}}_{ww}\right] + \underline{\underline{\mathrm{R}}}_{II} - \left[\underline{\underline{\mathrm{R}}}_{I\hat{I}}(\varepsilon) + \underline{\underline{\mathrm{R}}}_{\hat{I}I}(\varepsilon)\right]. \tag{6.34}$$

The first two terms give the total variance of the symbol estimates built at the receiver, including the additive noise. The error variance on each output is now the sum of two contributions:

$$\sigma^2_{\epsilon_p} = x_p(\varepsilon)\,\sigma_I^2 + y_p\,\sigma_n^2 \tag{6.35}$$

The first term represents the residual interference between symbol streams, the ISI inside each symbol stream and the self-interference $(x_{pp,\varepsilon}(0) - 1)$.

The second term gives the contribution of the additive noise: its variance is independent of the timing error. The proportionality factors $x_p(\varepsilon)$ and y_p may be computed from equation (6.34). The interference factor may be decomposed into:

$$x_p(\varepsilon) = x_{p_1}(\varepsilon) + x_{p_2}(\varepsilon) + 1 \tag{6.36}$$

with the following definitions:

$$\begin{aligned} x_{p_1}(\varepsilon) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{X}}}_{\varepsilon}(e^{j\Omega})\underline{\underline{\mathrm{X}}}_{\varepsilon}^{H}(e^{j\Omega})\, d\Omega\right)_{pp} \\ &= \sum_{m=-\infty}^{\infty}\sum_{p'=0}^{N-1} |x_{pp',\varepsilon}(m)|^2 \end{aligned} \tag{6.37}$$

$$x_{p_2}(\varepsilon) = -\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{X}}}_{\varepsilon}(e^{j\Omega}) + \underline{\underline{\mathrm{X}}}_{\varepsilon}^{H}(e^{j\Omega})\, d\Omega\right)_{pp} = -2\, x_{pp,\varepsilon}(0) \tag{6.38}$$

For the receivers investigated in this paper, it can be shown that the $x_p(\varepsilon)$ functions are even, with a global minimum at 0. For small timing errors, the following approximation is valid:

$$x_p(\varepsilon) \approx x_p(0) + \frac{\varepsilon^2}{2}\ddot{x}_p(0), \qquad \ddot{x}_p(\varepsilon) = \frac{d^2 x_p(\varepsilon)}{d\varepsilon^2} \tag{6.39}$$

The signal to noise plus interference ratio (SINR) at the output of the p^{th} symbol stream is given by:

$$\rho_p(\varepsilon) = \frac{\sigma_I^2}{\sigma_{e_p}^2(\varepsilon)} \approx \frac{\sigma_I^2}{\sigma_I^2\left[x_p(0) + \frac{\varepsilon^2}{2}\ddot{x}_p(0)\right] + y_p\,\sigma_n^2} \tag{6.40}$$

With the last approximation, the 3 dB loss offset is inversely proportional to the initial signal to noise ratio and to the second derivative:

$$\varepsilon_{p,3dB} \approx \sqrt{\frac{2}{\rho_p(0)\,\ddot{x}_p(0)}} \tag{6.41}$$

In the sequel, we propose to use this value $\ddot{x}_p(0)$ as a measurement of the timing sensitivity. We just need to remember that the real sensitivity also depends on the initial signal to noise ratio at perfect synchronization $\rho_p(0)$.

An interesting interpretation of this second derivative is obtained by observing that:

$$\ddot{x}_p(0) \approx 2\sum_{p'=0}^{N-1}\sum_{n=-\infty}^{\infty} |\dot{x}_{pp',a}(nT)|^2 \tag{6.42}$$

In other words, the timing sensitivity is equal to the sum of the interference waveform squared slopes sampled at the symbol rate. The last approximation is valid if $x_{pp',a}(nT) \approx \delta(p-p')\delta(n)$, that is to say if the interference level is low.

Let us show that the coefficients $x_{p_1}(\varepsilon)$ can be expressed as the sum of a few sinusoïds. Using (6.37) and $\Omega = 2\pi f T$, we write:

$$x_{p_1}(\varepsilon) = \left(T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \underline{\underline{\mathrm{X}}}_{\varepsilon}(f) \underline{\underline{\mathrm{X}}}_{\varepsilon}^{H}(f)\, df \right)_{pp} \tag{6.43}$$

From (6.32), the discrete time spectra $\underline{\underline{\mathrm{X}}}_{\varepsilon}(f)$ are linked to the continuous time spectra $\underline{\underline{\mathrm{X}}}_{a}(f)$ by:

$$\underline{\underline{\mathrm{X}}}_{\varepsilon}(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \underline{\underline{\mathrm{X}}}_{a}(f - \frac{n}{T})\, e^{2\pi j \varepsilon (f - \frac{n}{T})} \tag{6.44}$$

By substitution of (6.44) into (6.43), we get:

$$x_{p_1}(\varepsilon) = \left\{ \underline{\underline{\mathrm{R}}}_{xx}(0) + \sum_{\Delta n=1}^{M} 2\Re \left[e^{2\pi j \varepsilon \frac{\Delta n}{T}} \underline{\underline{\mathrm{R}}}_{xx}(\Delta n) \right] \right\}_{pp} \tag{6.45}$$

where the sequence of matrices $\underline{\underline{\mathrm{R}}}_{xx}(\Delta n)$ is defined as:

$$\underline{\underline{\mathrm{R}}}_{xx}(\Delta n) = \frac{1}{T} \int_{-\infty}^{\infty} \underline{\underline{\mathrm{X}}}_{a}(f)\, \underline{\underline{\mathrm{X}}}_{a}^{H}(f - \frac{\Delta n}{T}) df \tag{6.46}$$

and $M = \lfloor N(1+\alpha) \rfloor$. It has been explicitly assumed that the spectra $X_{pp'}(f)$, including the transmit filter, are zero for $|f| > \frac{1+\alpha}{2T_c}$.

The first contribution to the timing sensitivity is finally obtained as a sum:

$$\ddot{x}_{p_1}(0) = -(\frac{2\pi}{T_c})^2 \sum_{\Delta n=1}^{M} 2(\frac{\Delta n}{N})^2 \Re \left[\underline{\underline{\mathrm{R}}}_{xx}(\Delta n) \right]_{pp} \tag{6.47}$$

The second contribution to the timing sensitivity can also be expressed as a function of the continuous time spectra $\underline{\underline{\mathrm{X}}}_{a}(f)$:

$$\ddot{x}_{p_2}(0) = -2 \frac{d^2 x_{pp,a}(t)}{dt^2} = 2 \int_{-\infty}^{\infty} (2\pi f)^2 \left[\underline{\underline{\mathrm{X}}}_{a}(f) \right]_{pp} df \tag{6.48}$$

6.3.1.2 The matched filter receiver

The matched filter receiver has been introduced in Section 5.2. Notations are reminded here in a single-user context, and with an additional timing offset ε.

The continuous-time interference waveforms associated with a matched filter bank are

$$x_{pp'}(t) = h_p(-t) \otimes h_{p'}(t) \tag{6.49}$$

The matched filter outputs are obtained by first filtering the signal with $c(-t)$ in the analog domain and sampling the result at a rate $1/T_c$:

$$q_\varepsilon(n) = [r(t) \otimes c(-t)]\,(nT_c + \varepsilon) \tag{6.50}$$

where ε is the timing offset caused by a non-perfect synchronization. The corresponding polyphase components $q_{\varepsilon,\rho}(n) = q_\varepsilon(nN + \rho)$ are expressed, with z transforms, as:

$$\underline{\mathrm{Q}}_\varepsilon(z) = \underline{\underline{\mathrm{G}}}_\varepsilon(z)\,\underline{\mathrm{V}}(z) + \underline{\mathrm{U}}_\varepsilon(z). \tag{6.51}$$

The polyphase matrix $\underline{\underline{\mathrm{G}}}_\varepsilon(z)$ is Toeplitz with element (i, j) given by the z transform of $g(nT + (i-j)T + \varepsilon)$. The noise vector spectrum $\underline{\underline{\gamma}}_\nu(z) = \frac{N_0}{2}\,\underline{\underline{\mathrm{G}}}_0(z)$ does not depend on the sampling phase thanks to the noise stationarity. The matched filter outputs are finally computed by passing the sequence $q_\varepsilon(n)$ through a bank of analysis filters matched to the synthesis bank:

$$\begin{aligned}
\hat{\underline{\mathrm{I}}}_\varepsilon(z) &= \underline{\underline{\mathrm{S}}}^H(\frac{1}{z^*})\,\underline{\mathrm{Q}}_\varepsilon(z) \\
&= \left(\underline{\underline{\mathrm{S}}}^H(\frac{1}{z^*})\,\underline{\underline{\mathrm{G}}}_\varepsilon(z)\,\underline{\underline{\mathrm{S}}}(z)\right)\,\underline{\mathrm{I}}(z) + \underline{\mathrm{W}}_\varepsilon(z) \qquad (6.52) \\
&= \underline{\mathrm{J}}_\varepsilon(z) + \underline{\mathrm{W}}_\varepsilon(z) \qquad (6.53)
\end{aligned}$$

with $\underline{\mathrm{W}}_\varepsilon(z) = \underline{\underline{\mathrm{S}}}^H(\frac{1}{z^*})\,\underline{\mathrm{U}}_\varepsilon(z)$. The matched filter estimation errors are defined by $\underline{\epsilon}_\varepsilon(z) = \hat{\underline{\mathrm{I}}}_\varepsilon(z) - \underline{\mathrm{I}}(z)$. Assuming a paraunitary FB, the error covariance spectrum writes:

$$\underline{\underline{\gamma}}_{\epsilon_\varepsilon \epsilon_\varepsilon}(z) = \underline{\underline{\mathrm{S}}}^H(\frac{1}{z^*})\left[\sigma_I^2\left(\underline{\underline{\mathrm{G}}}_\varepsilon(z) - \underline{\underline{\mathrm{E}}}_N\right)\left(\underline{\underline{\mathrm{G}}}^H_\varepsilon(\frac{1}{z^*}) - \underline{\underline{\mathrm{E}}}_N\right) + \sigma_n^2\,\underline{\underline{\mathrm{G}}}_0(z)\right]\underline{\underline{\mathrm{S}}}(z) \tag{6.54}$$

Equation (6.54) implies that the arithmetic mean of the error variances does not depend on the FB. As a consequence, we may expect that the average timing sensitivity $\frac{1}{N}\sum_{p=0}^{N-1}\ddot{x}_p(0)$ should also be independent of the FB. To check this, let us derive a more adequate form for the general results shown in (6.47) and (6.48).

The $\underline{\underline{\mathrm{R}}}_{JJ}$ covariance matrix appears to be:

$$\underline{\underline{\mathrm{R}}}_{JJ}(\varepsilon) = \frac{\sigma_I^2}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega}) \left[\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})\underline{\underline{\mathrm{G}}}_\varepsilon^H(e^{j\Omega})\right] \underline{\underline{\mathrm{S}}}(e^{j\Omega})\, d\Omega \tag{6.55}$$

The channel polyphase matrix $\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})$ is derived from the analog channel transmittance $G(f)$ as:

$$\left[\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})\right]_{pq} = \frac{1}{T}\sum_{n=-\infty}^{+\infty} G(f-\frac{n}{T})\, e^{2\pi j(\varepsilon+[p-q]T)(f-\frac{n}{T})} \tag{6.56}$$

It follows:

$$\begin{aligned}
\left[\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})\,\underline{\underline{\mathrm{G}}}_\varepsilon^H(e^{j\Omega})\right]_{pq} &= \sum_{k=1}^{N}\left[\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})\right]_{pk}\left[\underline{\underline{\mathrm{G}}}_\varepsilon^*(e^{j\Omega})\right]_{qk} \\
&= \frac{1}{T^2}\sum_{k=1}^{N}\sum_{n=-\infty}^{\infty}\sum_{\Delta n=-\infty}^{+\infty} G(f-\frac{n}{T})G(f-\frac{n}{T}-\frac{\Delta n}{T}) \\
&\qquad e^{2\pi j\varepsilon\frac{\Delta n}{T}}\, e^{2\pi jT_c(p-k)(f-\frac{n}{T})}\, e^{2\pi jT_c(k-q)(f-\frac{n}{T}-\frac{\Delta n}{T})} \\
&= \frac{1}{T^2}\sum_{\Delta n=-M}^{M} e^{2\pi j\varepsilon\frac{\Delta n}{T}}\left[\sum_{n=-\infty}^{\infty} G(f-\frac{n}{T})G(f-\frac{n}{T}-\frac{\Delta n}{T})\right. \\
&\qquad \left. e^{2\pi jT_c(p-q)(f-\frac{n}{T})}\left(\sum_{k=1}^{N} e^{2\pi j(k-q)(-\frac{\Delta n}{N})}\right)\right]
\end{aligned} \tag{6.57}$$

Using the following property:

$$\sum_{k=1}^{N} e^{2\pi j(k-q)(-\frac{\Delta n}{N})} = \begin{cases} N & \text{if } \Delta n = lN, l\in\mathcal{Z} \\ 0 & \text{otherwise}\end{cases}, \tag{6.58}$$

it appears that the non-zero terms in the last equation are limited to $\Delta n = 0, \pm N$. That is,

$$\underline{\underline{\mathrm{G}}}_\varepsilon(e^{j\Omega})\,\underline{\underline{\mathrm{G}}}_\varepsilon^H(e^{j\Omega}) = \underline{\underline{\mathrm{R}}}_{gg}(e^{j\Omega}) + e^{2\pi j\frac{\varepsilon}{T_c}}\,\underline{\underline{\mathrm{R}}}_{gg+}(e^{j\Omega}) + e^{-2\pi j\frac{\varepsilon}{T_c}}\,\underline{\underline{\mathrm{R}}}_{gg-}(e^{j\Omega}) \tag{6.59}$$

where the following polyphase matrices are defined:

- $\underline{\underline{\mathrm{R}}}_{gg}(e^{j\Omega})$ is the polyphase matrix of $\frac{1}{T_c}(g(t) \otimes g(t))$
- $\underline{\underline{\mathrm{R}}}_{gg+}(e^{j\Omega})$ is the polyphase matrix of $\frac{1}{T_c}(g(t) \otimes g(t)e^{2\pi j \frac{t}{T_c}})$
- $\underline{\underline{\mathrm{R}}}_{gg-}(e^{j\Omega})$ is the polyphase matrix of $\frac{1}{T_c}(g(t) \otimes g(t)e^{-2\pi j \frac{t}{T_c}})$

The last two matrices are non zero only if the excess bandwidth α is non zero. For parity reasons, computation of equation (6.55) simplifies into:

$$\begin{aligned}
\underline{\underline{\mathrm{R}}}_{JJ}(\varepsilon) &= \left[\frac{\sigma_I^2}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\, \underline{\underline{\mathrm{R}}}_{gg}(e^{j\Omega})\, \underline{\underline{\mathrm{S}}}(e^{j\Omega})\, d\Omega\right] \\
&+ 2\cos(\frac{2\pi\varepsilon}{T_c}) \left[\frac{\sigma_I^2}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\, \underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega})\, \underline{\underline{\mathrm{S}}}(e^{j\Omega})\, d\Omega\right] \\
&= \underline{\underline{\mathrm{R}}}_{JJ}(0) + 2(\cos(\frac{2\pi\varepsilon}{T_c}) - 1) \\
&\left[\frac{\sigma_I^2}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\, \underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega})\, \underline{\underline{\mathrm{S}}}(e^{j\Omega})\, d\Omega\right]
\end{aligned}$$

where $\underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega})$ is defined as the polyphase matrix of $\frac{1}{T}(g(t) \otimes g(t)\cos(2\pi\frac{t}{T_c}))$. By comparison with the general expression (6.45), we observe that the use of a paraunitary FB reduces the size of the summation to only one term.

The second derivative of the continuous time interference waveforms can be expressed as:

$$\left.\frac{d^2 x_{pp,a}(t)}{dt^2}\right|_{t=0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\underline{\underline{\mathrm{R}}}_{\ddot{g}}(e^{j\Omega})\underline{\underline{\mathrm{S}}}(e^{j\Omega})\right]_{pp} d\Omega \tag{6.60}$$

where $\underline{\underline{\mathrm{R}}}_{\ddot{g}}(e^{j\Omega})$ is the polyphase matrix of $\ddot{g}(t)$. An alternative expression is obtained from (6.49):

$$\left.\frac{d^2 x_{pp,a}(t)}{dt^2}\right|_{t=0} = \int_{-\infty}^{\infty} (2\pi j f)^2 |H_p(f)|^2\, df = -W_p^2 \frac{\mathcal{E}_p}{\sigma_I^2} \tag{6.61}$$

where $\mathcal{E}_p$ and W_p^2 are the received symbol energy and mean square bandwidth defined in (4.32) and (4.33).

Using these results, we can now compute the second derivative of the error covariance matrix with respect to the timing offset, and derive the timing sensitivity for each symbol stream $\ddot{x}_p(0)$:

$$\ddot{\underline{\underline{\mathrm{x}}}}_p(0) = \mathrm{diag}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\left[(\frac{2\pi}{T_c})^2 \underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega}) - \underline{\underline{\mathrm{R}}}_{\ddot{g}}(e^{j\Omega})\right]\underline{\underline{\mathrm{S}}}(e^{j\Omega})d\Omega\right) \tag{6.62}$$

From the last expression, we conclude again that the average timing sensitivity does not depend on the selected FB, as already expected.

6.3.1.3 Baseband PAM and AWGN channel

Let us analyse in more details the general result presented above on a simple example: a single baseband PAM modulation ($N = 1$, $\underline{\underline{S}}(z) = 1$) and a frequency-flat channel channel ($c(t) = \delta(t-\tau)$). In this scenario, the channel correlation corresponds to a full raised cosine filter. The corresponding spectrum $G(f)$ is recalled here for convenience:

$$G(f) = \begin{cases} T_c & |f| < f_1 \\ \frac{T_c}{2}\left(1+\cos\left[\frac{\pi T_c}{\alpha}(|f|-f_1)\right]\right) & f_1 \le |f| \le f_2 \\ 0 & |f| > f_2 \end{cases} \qquad (6.63)$$

with $f_1 = \frac{1-\alpha}{2T_c}$ and $f_2 = \frac{1+\alpha}{2T_c}$, α being the roll-off factor. The received symbol energy is $Es = \sigma_I^2 \int_{-\infty}^{\infty} G(f)\, df = \sigma_I^2$. At perfect synchronism, the interference disappears and we get $\rho(0) = \sigma_I^2/\sigma_n^2$.

Equation (6.45) becomes:

$$\begin{aligned} x_{p_1}(\varepsilon) &= \left[\frac{1}{T_c}\int_{-\infty}^{\infty} |G(f)|^2\, df\right] + 2\cos(\frac{2\pi\varepsilon}{T_c})\left[\frac{1}{T_c}\int_{-\infty}^{\infty} G(f)G^*(f-\frac{1}{T_c})\, df\right] \\ &= (1-\frac{\alpha}{4}) + \frac{\alpha}{4}\cos(\frac{2\pi\varepsilon}{T_c}) \qquad (6.64) \end{aligned}$$

where (6.63) has been used to obtain the last expression. The first contribution to the timing sensitivity is thus:

$$\ddot{x}_{p_1}(0) = -(\frac{2\pi}{T_c})^2\,\frac{\alpha}{4} \qquad (6.65)$$

Equation (6.48) simplifies into:

$$\ddot{x}_{p_2}(0) = 2\int_{-\infty}^{\infty} (2\pi f)^2\, G(f)\, df = (\frac{2\pi}{T_c})^2 \left[(\frac{1}{2} - \frac{4}{\pi^2})\alpha^2 + \frac{1}{6}\right] = 2W^2 \qquad (6.66)$$

using (6.63).

These results are illustrated on figure 6.1. The dotted line gives the evolution of the term (6.66) with α: it appears that the concavity of the Nyquist waveform at the origin increases with the rolloff. The continuous line gives the total timing sensitivity $\ddot{x}_p(0)$, which decreases with the rolloff. With

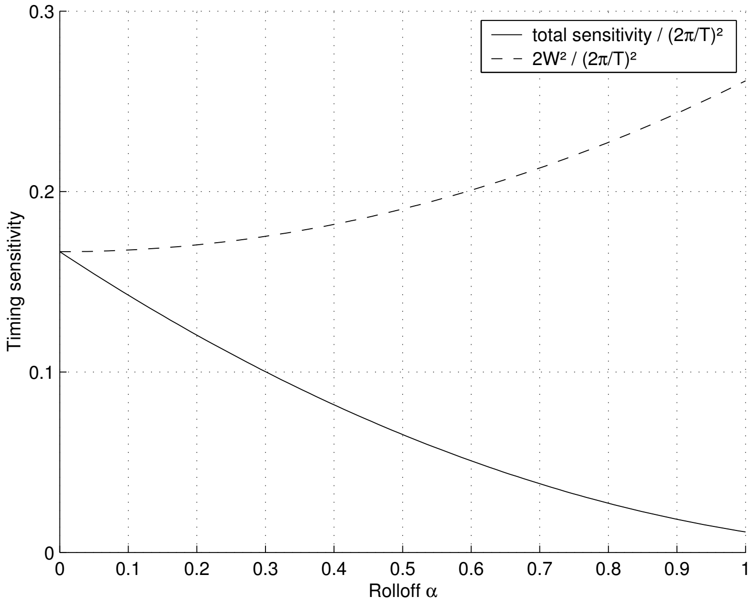

Figure 6.1: *Timing sensitivity vs. roll-off for a single PAM modulation*

the interpretation given in (6.42), this reflects the decrease of the Nyquist waveform slopes around the zero-crossing points at higher α.

Figure 6.2 gives the profile of the SINR $\rho(\varepsilon)$ as a function of the relative timing error ε/T_c. The different sets of curves correspond to various levels of additive noise: $\rho(0) = 10$ to 50 dB with a 5 dB step. Following (6.41), the timing sensitivity is highly dependent on this parameter. The different line styles correspond to distinct roll-off values: $\alpha = 0.1$, 0.3 and 0.5. A larger sensitivity at low α can be observed on the curves.

6.3.1.4 DMT and Hadamard modulations

Let us now investigate the effect of a timing error on DMT and Hadamard transmission schemes, with the simple AWGN channel.

Intuitively, we may expect a higher timing sensitivity for the higher tones and the codes with many transitions, as their spectrum is located at higher frequencies. This is true for the term $\ddot{x}_{p_2}(0) = 2W_p^2$. But concerning the first term $\ddot{x}_{p_1}(0)$, the effect is inverted: filters with higher spectral components will be more influenced by the roll-off region, and the resulting timing sensitivity is decreased. In the light of (6.42), this means that the

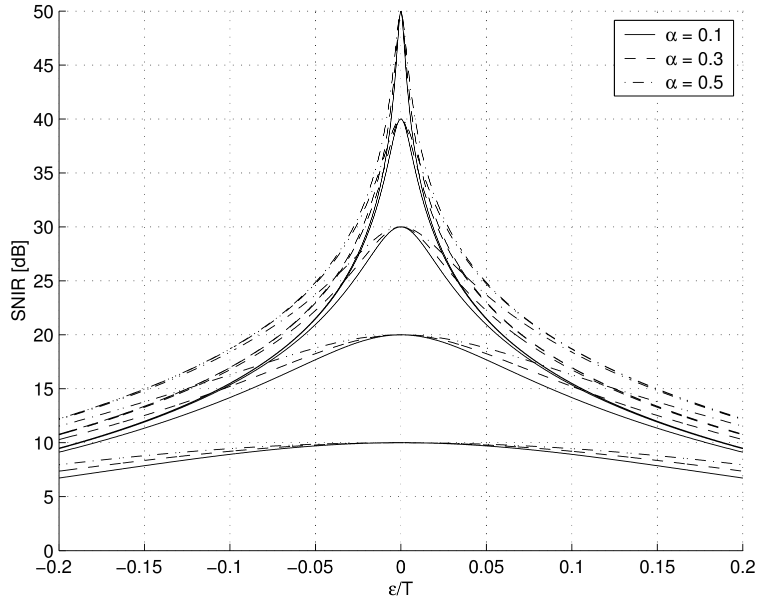

Figure 6.2: *SINR vs. timing error for a single PAM modulation*

average squared slopes of the interference waveforms actually decrease for the higher tones and the codes with more transitions, due to the roll-off effect.

Figure 6.3 gives the timing sensitivity profile vs. the tone index p for a DMT 512 transmission scheme. As expected, we observe an increase for the second term $\ddot{x}_p(0)$ (upper curves). The total sensitivity (lower curves) first increases with the tone index, and finally decreases for the higher tones. This effect appears faster at high roll-off values. The curves corresponding to sine and cosine filters are almost superimposed, with an exact matching for $p = N/2$.

Figure 6.4 gives the same results for a Hadamard 512 FB transmission scheme. The codes are ordered with their spectral content: the first code has no transition, while the last code has the highest number of transitions.

In both schemes, and for any roll-off factor α, it can be verified that the average $\ddot{x}_p$ corresponds to the single PAM value reported on figure 6.1.

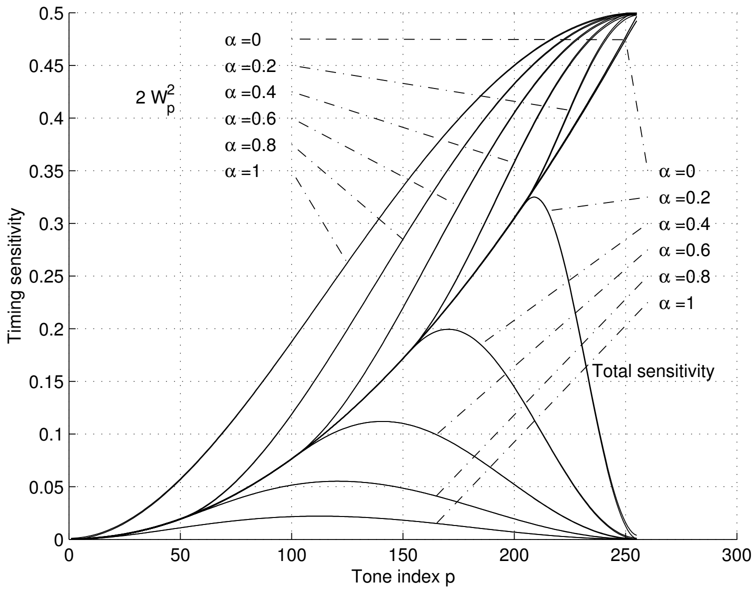

Figure 6.3: *Timing sensibility vs. tone index for a DMT 512 FB scheme*

6.3.1.5 MMSE receivers

Let us now analyze the timing error sensitivity associated with IIR MMSE equalizers. The equalizers are computed at modem start-up by assuming perfect synchronism.

For a paraunitary FB $\underline{\underline{S}}(z)$, it can be shown that the estimation error covariance spectrum writes:

$$\underline{\underline{\gamma}}_{\underline{\epsilon}_\varepsilon \underline{\epsilon}_\varepsilon}(z) = \underline{\underline{B}}(z)\,\underline{\underline{S}}^H(\frac{1}{z^*}) \left[\underline{\underline{\gamma}}^{(0)}_{\underline{\epsilon}_\varepsilon \underline{\epsilon}_\varepsilon}(z)\right] \underline{\underline{S}}(z)\,\underline{\underline{B}}^H(\frac{1}{z^*}) \tag{6.67}$$

with

$$\underline{\underline{\gamma}}^{(0)}_{\underline{\epsilon}_\varepsilon \underline{\epsilon}_\varepsilon}(z) = \Big[\sigma_I^2 \left(\underline{\underline{S}}_{c_0}^{-1}(z)\,\underline{\underline{G}}_\varepsilon(z) - \underline{\underline{E}}_N\right) \left(\underline{\underline{G}}_\varepsilon^H(\frac{1}{z^*})\,\underline{\underline{S}}_{c_0}^{-1}(z) - \underline{\underline{E}}_N\right) + \sigma_n^2\,\underline{\underline{S}}_{c_0}^{-1}(z)\,\underline{\underline{G}}_0(z)\,\underline{\underline{S}}_{c_0}^{-1}(z)\Big] \tag{6.68}$$

and

$$\underline{\underline{S}}_{c_0}(z) = \sigma_I^{-2}\,\underline{\underline{E}}_N + \frac{2}{N_0}\,\underline{\underline{G}}_0(z). \tag{6.69}$$

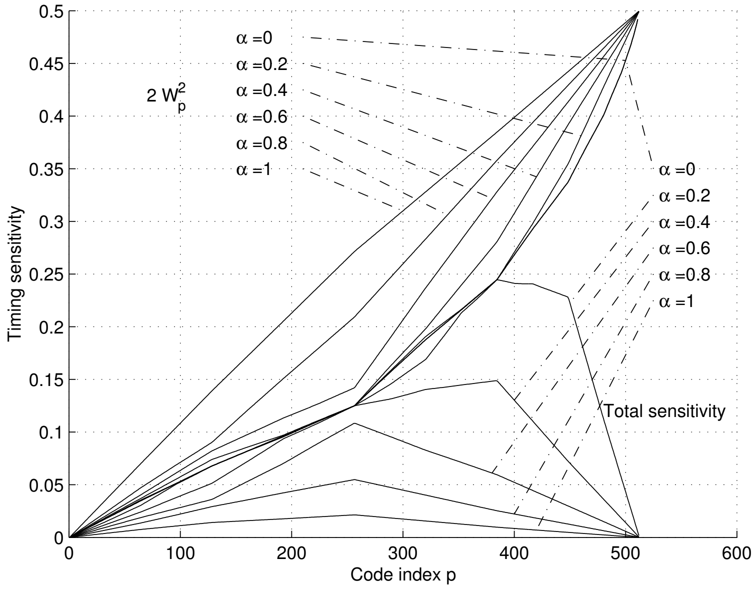

Figure 6.4: *Timing sensibility vs. code index for a Hadamard 512 FB scheme*

As pointed out in Section 1.3.3, it is known that the arithmetic mean of the error variances at the output of a linear JD ($\underline{\underline{\mathrm{B}}}(z) = \underline{\underline{\mathrm{E}}}_N$) is independent of the selected FB. Equation (6.67) shows that this property remains valid even in the presence of an arbitrary timing error ε.

We finally provide a way to compute the timing sensitivity of the presented MMSE receivers:

$$\ddot{\underline{\mathrm{x}}}_p(0) = \mathrm{diag}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{B}}}(e^{j\Omega})\,\underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\,\underline{\underline{\mathrm{S}}}_{\ddot{g}}(e^{j\Omega})\,\underline{\underline{\mathrm{S}}}(e^{j\Omega})\,\underline{\underline{\mathrm{B}}}^H(e^{j\Omega})d\Omega\right) \quad (6.70)$$

with

$$\underline{\underline{\mathrm{S}}}_{\ddot{g}}(e^{j\Omega}) = \left[(\frac{2\pi}{T_c})^2\,\underline{\underline{\mathrm{S}}}_{c_0}^{-1}(e^{j\Omega})\,\underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega})\,\underline{\underline{\mathrm{S}}}_{c_0}^{-1}(e^{j\Omega}) - \underline{\underline{\mathrm{S}}}_{c_0}^{-1}(e^{j\Omega})\,\underline{\underline{\mathrm{R}}}_{\ddot{g}}(e^{j\Omega})\right] \quad (6.71)$$

From (6.70), it appears that the average timing sensitivity is independent of the selected FB in the linear case.

6.3.2 Extension to CAP FB modulation

6.3.2.1 The matched filter receiver

To implement a FB-CAP matched filter receiver, the outputs of two quadrature matched filters are first sampled at a rate $1/T_c$ with a timing error ε :

$$\begin{pmatrix} q_{a,\varepsilon}(n) \\ q_{b,\varepsilon}(n) \end{pmatrix} = \left[\begin{pmatrix} c(-t) \\ \tilde{c}(-t) \end{pmatrix} \otimes r(t) \right] (nT_c + \varepsilon). \tag{6.72}$$

The polyphase components of the sequences $q_{a,\varepsilon}(n)$ and $q_{b,\varepsilon}(n)$ are given by

$$\begin{bmatrix} \underline{\mathrm{Q}}_{a,\varepsilon}(z) \\ \underline{\mathrm{Q}}_{b,\varepsilon}(z) \end{bmatrix} = \begin{bmatrix} \underline{\underline{\mathrm{G}}}_{\varepsilon}(z) & \underline{\underline{\tilde{\mathrm{G}}}}_{\varepsilon}(z) \\ -\underline{\underline{\tilde{\mathrm{G}}}}_{\varepsilon}(z) & \underline{\underline{\mathrm{G}}}_{\varepsilon}(z) \end{bmatrix} \begin{bmatrix} \underline{\mathrm{V}}_{a}(z) \\ \underline{\mathrm{V}}_{b}(z) \end{bmatrix} + \begin{bmatrix} \underline{\mathrm{U}}_{a,\varepsilon}(z) \\ \underline{\mathrm{U}}_{b,\varepsilon}(z) \end{bmatrix} \tag{6.73}$$

where $\underline{\underline{\mathrm{G}}}_{\varepsilon}(z)$ and $\underline{\underline{\tilde{\mathrm{G}}}}_{\varepsilon}(z)$ are the polyphase matrices associated with the waveforms $g(t)$ and $\tilde{g}(t)$, respectively, with

$$\begin{array}{lllllll} g(t) & = & c(t) \otimes c(-t) & = & \tilde{c}(t) \otimes \tilde{c}(-t) & = & [\Gamma_f(t)\,\cos(\omega_0 t)] \otimes \Gamma_c(t) \\ \tilde{g}(t) & = & \tilde{c}(t) \otimes c(-t) & = & -c(t) \otimes \tilde{c}(-t) & = & [\Gamma_f(t)\,\sin(\omega_0 t)] \otimes \Gamma_c(t) \end{array} \tag{6.74}$$

where $\Gamma_f(t)$ and $\Gamma_c(t)$ are the baseband Nyquist pulse and physical channel autocorrelation functions, respectively.

The matched filter outputs are finally obtained by using a bank of analysis filters:

$$\begin{bmatrix} \underline{\hat{\mathrm{I}}}_{a,\varepsilon}(z) \\ \underline{\hat{\mathrm{I}}}_{b,\varepsilon}(z) \end{bmatrix} = \underline{\underline{\mathrm{S}}}^{H}(\frac{1}{z^*}) \begin{bmatrix} \underline{\mathrm{Q}}_{a,\varepsilon}(z) \\ \underline{\mathrm{Q}}_{b,\varepsilon}(z) \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{J}}_{a,\varepsilon}(z) \\ \underline{\mathrm{J}}_{b,\varepsilon}(z) \end{bmatrix} + \begin{bmatrix} \underline{\mathrm{W}}_{\varepsilon}(z) \\ \underline{\tilde{\mathrm{W}}}_{\varepsilon}(z) \end{bmatrix} \tag{6.75}$$

Using the paraunitary property, the error covariance spectra write:

$$\begin{aligned} \underline{\underline{\gamma}}_{\epsilon_{a,\varepsilon}\epsilon_{a,\varepsilon}}(z) = \underline{\underline{\mathrm{S}}}^{H}(\frac{1}{z^*}) \Big\{ \sigma_I^2 \Big[\Big(\underline{\underline{\mathrm{G}}}_{\varepsilon}(z) - \underline{\underline{\mathrm{E}}}_{N} \Big) \Big(\underline{\underline{\mathrm{G}}}_{\varepsilon}^{H}(\frac{1}{z^*}) - \underline{\underline{\mathrm{E}}}_{N} \Big) \\ + \underline{\underline{\tilde{\mathrm{G}}}}_{\varepsilon}(z)\, \underline{\underline{\tilde{\mathrm{G}}}}_{\varepsilon}^{H}(\frac{1}{z^*}) \Big] + \sigma_n^2\, \underline{\underline{\mathrm{G}}}_{0}(z) \Big\}\, \underline{\underline{\mathrm{S}}}(z) = \underline{\underline{\gamma}}_{\epsilon_{b,\varepsilon}\epsilon_{b,\varepsilon}}(z) \end{aligned} \tag{6.76}$$

It appears again that the average error variance does not depend on the selected FB. Furthermore, for a given basis function p selected in the FB, the same error variance is obtained on each of the quadrature components a and b. An explicit expression of the $\underline{\underline{\mathrm{R}}}_{JJ}(\varepsilon)$ covariance matrix that appears in the error variance equation can be found as in the baseband scenario. The result is:

$$\underline{\underline{\mathrm{R}}}_{JJ}(\varepsilon) = \underline{\underline{\mathrm{R}}}_{JJ}(0) + 2(\cos(\frac{2\pi\varepsilon}{T_c}) - 1)\left[\frac{\sigma_I^2}{2\pi}\int_{-\pi}^{\pi} \underline{\underline{\mathrm{S}}}^H(e^{j\Omega})\left[\underline{\underline{\mathrm{R}}}_{gg_c}(e^{j\Omega}) + \underline{\underline{\mathrm{R}}}_{\tilde{g}\tilde{g}_c}(e^{j\Omega})\right]\underline{\underline{\mathrm{S}}}(e^{j\Omega})\,d\Omega\right] \quad (6.77)$$

The resulting timing sensitivity can be derived consequently.

6.3.2.2 Single CAP and AWGN channel

Considering a single CAP transmission scheme ($N = 1$) and a frequency flat physical channel, the channel correlation corresponds to a cosine modulated full raised cosine filter $g_c(t)$. The corresponding spectrum $G_c(f)$ and its Hilbert transform $\tilde{G}_c(f)$ are:

$$\begin{aligned} G_c(f) &= \frac{1}{2}\left[G(f - f_0) + G(f + f_0)\right] \\ \tilde{G}_c(f) &= \frac{j}{2}\left[-G(f - f_0) + G(f + f_0)\right] \end{aligned} \quad (6.78)$$

with $G(f)$ given in (6.63). This implies the following property, which can be checked easily:

$$\int_{-\infty}^{\infty} G_c(f)\, G_c(f-\Delta_f) + \tilde{G}_c(f)\, \tilde{G}_c(f-\Delta_f)\, df = \int_{-\infty}^{\infty} G(f)\, G(f-\Delta_f) \quad (6.79)$$

We first compute the noise-free estimate variance $\sigma_{J_p}^2(\varepsilon)$ as a function of the timing error:

$$\begin{aligned} x_{p_1}(\varepsilon) = \frac{\sigma_J^2(\varepsilon)}{\sigma_I^2} &= \left[\frac{1}{T_c}\int_{-\infty}^{\infty} |G_c(f)|^2 + |\tilde{G}_c(f)|^2\, df\right] \\ &+ 2\cos(\frac{2\pi\varepsilon}{T_c})\left[\frac{1}{T_c}\int_{-\infty}^{\infty} G_c(f)G_c^*(f - \frac{1}{T_c}) + \tilde{G}_c(f)\tilde{G}_c^*(f - \frac{1}{T_c})\, df\right] \\ &= (1 - \frac{\alpha}{4}) + \frac{\alpha}{4}\cos(\frac{2\pi\varepsilon}{T_c}) \end{aligned} \quad (6.80)$$

using (6.63) and (6.79). This result is exactly the same as in the baseband PAM scenario (6.64): the CAP modulation has no influence on $\sigma_J^2(\varepsilon)$. The corresponding contribution to the timing sensitivity is given by (6.65).

The second contribution to the timing sensitivity is computed as follows:

$$\begin{aligned} \ddot{x}_{p_2}(0) &= 2\int_{-\infty}^{\infty} (2\pi f)^2\, G_c(f)\, df && (6.81)\\ &= (\frac{2\pi}{T_c})^2 \left[(\frac{1}{2}-\frac{4}{\pi^2})\alpha^2+\frac{1}{6}\right]+2\,(2\pi f_0)^2 = 2W^2 && (6.82) \end{aligned}$$

Here the effect of the CAP modulation is sensible : a new term is added, which increases the original timing sensitivity proportionally to the square of the carrier frequency. This is obvious as the modulated interference waveform contains higher spectral components.

6.4 Conclusion

In this Chapter, it has been shown how the performance computation of the MMSE joint detectors introduced in Chapter 5 could be refined to take into account the respective effects of non-ideal timing synchronization and non-ideal channel estimation. These effects translate into an excess mean square error (MSE) at the output of the joint detector. For a steady-state system operating at high SNR, these effects are just second-order effects. The derivation of the excess MSE is interesting, however, as it reflects the robustness of the considered system against timing and channel estimation errors. Equations (6.18) and (6.30) give the respective approximate excess MSE obtained at high SNR (i.e. when the nominal symbol estimation errors are small) : these expressions allow an intuitive interpretation.

The concept of timing sensitivity has been further developed in a single-user context. The variance of the symbol estimation error at the output of a given detector varies with the timing error ε, with a global minimum for $\varepsilon = 0$. The second-order derivative of this function with respect to the timing error can thus be interpreted as a timing sensitivity. The timing sensitivity has been shown to be equivalent to the sum of the interference waveform squared slopes sampled at the symbol rate. The average timing sensitivity of a given paraunitary FB-based system has been shown to be independent of the selected FB (at least for the matched filter and linear MMSE receivers). Finally, the distribution of timing sensitivities as a function of the signature index has been analyzed for the DMT and Hadamard filter banks, for different values of the shaping filter roll-off.