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Alliances and Technological Partnerships in Contests

Jérôme Dollinger†

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Abstract

This paper analyses the formation of alliances and technological partnerships in contests. Alliances enhance the probability of winning at the cost of sharing the prize if won, while technological partnerships reduce the marginal cost of the effort invested in the contest by the members. When agents cannot form technological partnerships, I find that no alliance can be stable. When agents exhibit extreme free ridding behaviour at equilibrium, the stabilisation of the grand alliance by technological cooperation requires restrictive assumptions on the set of blocking agents. Nevertheless, When the agents manifest less free ridding intentions, the threat of being excluded from a global technological partnership is sufficient to ensure the stability of the grand alliance in the long run. This indicates that when the free ridding behaviours are not extreme, the ability to exclude is a sufficient condition for the global technological cooperation to annihilate competition in contests. In that context, the existence of technological partnerships facilitates the formation of alliances.

Key words: Contests, Alliances, Technological partnerships, Stability.
JEL classification: C70, D72, D74.

1 Introduction

In contests, where individuals vie for a prize, cooperation might improve the chances of winning at the cost of sharing resources. Cooperation between agents can be of different natures. Contestants could gather their efforts by forming an alliance in the contest to improve their chances of winning. If won however, the prize will need to be divided among allies. Technological collaboration might also occur in order to reduce the cost devoted by the agents to the contest. Dominance over technology shall shrink the resources invested in the contest.

Agents cooperate in a contest by forming both alliances and technological partnerships in many different situations. On the one hand, countries or regions fighting over a territory can conclude alliances in order to raise their military power relative to the other belligerents. On the other hand, any country or region involved in the conflict could also share their military resources to diminish their cost of war (exchange of technology, common military R&D projects, weapons delivery, *etc*...).[1] Political parties engaged in an election may not only conclude alliances in order to improve their chances of winning the election, but they might decide to share their networks to mitigate the cost of the political campaign as well.[2] In the process of

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[1]During World War II, while the Allies fought against the Axis for the control of Europe, several technological partnerships were concluded. Great Britain, the US, and the USSR exchanged a lot of weapons, among which military vehicles, artillery, or aircraft.

[2]In the 2022 legislative election in France, most of the parties on the left side of the political spectrum merged into one party : NUPES. These parties pool together their resources during the campaign. They share their networks of financial supports (firms, associations, public figures ...)

voting a bill, lobbies fight to obtain the votes of the policy makers. Some of them may join forces to become a stronger influence in the decision. They could also share their networks and knowledge about an industry to become more credible in the eyes of the policy makers. Having access to an existing network spares the necessary cost to create relationships. In public procurements, whereby firms compete to obtain a public market, some of them might merge their projects to present a more attractive version to the public authority and hence improve their probability of being chosen.[3] At the same time, the candidates can form R&D agreements among them in order to decrease the cost of their project.[4]

This paper aims at understanding the role of different kinds of cooperation in contests. While alliances raise the strength of their members in the contest by pooling their resources together, technological cooperation optimises the usage of resources. In such a context, it is legitimate to wonder how different types of co-operation would alter the course of a contest ? To answer this question, I develop a game theoretic model where n agents can cooperate by forming both alliances and technological partnerships before competing in a rent-seeking contest game. Alliances enable agents to pool their resources together to improve their chances of winning the prize. If won however, the prize will be divided among the allies. So when the number of members in the alliances rises, the expected value of winning the contest decreases while the probability of winning increases. In parallel, agents can form technological partnerships to reduce the marginal cost of the efforts invested in the contest : agents become more efficient. The reduction of the marginal cost is proportional to the number of agents in the technological partnership. However, forming a technological partnership also enhances the efficiency of some adversaries. When agents are able to cooperate by forming both alliances and technological partnerships, deviations from an alliance might be unprofitable if losses in efficiency are expected. If by leaving an alliance an agent would be excluded from a technological partnership, the optimal choice might be to stay in the alliance. Nonetheless, the existing literature on the formation of exclusive groups in contests only considers cooperation through the formation of alliances between agents.

If agents are unable to form technological partnerships among them, no alliance structure is stable in the long run. Can alliance structures be stable when agents cooperate in technology ? The main objective of this work is to understand the role of technological cooperation in the formation of alliances in contests. More precisely, this paper investigates whether technological cooperation mitigates cooperation in contests. To that end, I study the conditions for the technological cooperation to stabilise the grand alliance, whereby all the agents are allies so that there is no competition in the contest anymore.

Stability is first studied in a setting with linear costs, where agents exhibit ex-

as well as their logistic networks (e.g., the agents facilitating the organisation of political meetings and maximising the visibility of such events : mayors, local supporters, contacts in the media...).

[3] Public procurement is an example of a bidding process supervised by a public authority. Coordination among bidders to undermine (rig) the bidding procedure is often referred as Bid Rigging, which is generally an illegal practice. Such collusive actions can take different forms : Losing bidders can be subcontractors in the winning contract, firms can merge into a joint venture to submit a single bid at a suboptimal price...

[4] R&D agreements are here non-drastic innovation, i.e. R&D innovations improve the production process of the goods and services proposed to the public authority and do not develop a new product.

treme free ridding behaviour at equilibrium : Only the most efficient agents invest non-null effort in the contest. In this framework, the grand alliance is never formed in the long run with the existing stability notions. I then propose a generalisation of the concept of contractual stability under the unanimity decision rule, whereby partners of the deviating agents in one structure are able to block the deviation initiated by these agents in the other structure. Among the technological structures composed of equally efficient agents, only the global technological coalition is able to stabilise the grand alliance with this concept. In order to weaken the restrictions for a deviation to occur, I consider the concept of Δ-stable pairs, where the deviating agents might be excluded from their own technological partnership or alliance. For any deviation in the alliance (resp. technological) structure, all the non-deviating technological partners (resp. allies) of the deviating agents are able to exclude the deviating agents from their technological partnership (resp. alliance) if the deviation occurs. When extreme free ridding behaviours emerge, the threat of exclusion is irrelevant for most of the deviating agents. Consequently, the grand alliance is not stabilised by any technological partnerships of identical size when deviating agents can be excluded from their technological partnership or alliance. With extreme free ridding behaviours, the stabilisation of the grand alliance by technological cooperation requires the set of blocking agents to be large in the contract defining the coalitions. The difficulty to stabilise the grand alliance stems from the extreme free ridding behaviours of agents at equilibrium. No punishment is harsh enough to prevent agents investing zero effort in the contest from deviating. In order to study the stability of alliances in contests when there are no such extreme free ridding behaviours at equilibrium, I consider a version of the model with non-linear costs. As pointed out by Epstein and Mealem (2009), contest games with non-linear costs are subject to reduced free ridding behaviours at equilibrium. When agents manifest less free ridding intentions, the threat of being excluded from a global technological partnership is sufficient to ensure the stability of the grand alliance in the long run. This indicates that when the free ridding behaviours are not extreme, the ability to exclude is a sufficient condition for the global technological cooperation to annihilate competition in contests.

The rent-seeking contest games considered in this study originate from the seminal works of Tullock (1967, 1980) on contests between individuals, later axiomatised by Skaperdas (1996). These models have naturally been generalised to contest games between groups of agents. A first line of research develops contest games with *ex-ante* heterogeneous agents belonging to exogenously assembled groups. In a contest game with n agents without rivalry within groups, Esteban and Ray (1999) investigate the relationship between the population distribution of exogenously formed groups and the equilibrium conditions. The absence of rivalry between the members of the winning group in the acquisition of the prize is equivalent to define the prize as a public good. Still considering models where exogenously formed groups engage in a contest for a public good prize, Baik (1993, 2008) and Epstein and Mealem (2009) develop more restrictive settings, enabling them to work on more tractable equilibrium conditions. Nevertheless, the incentives behind the composition of groups are not addressed by these works. Understanding the formation of groups would, however, sharpen the theoretical predictions by identifying the group structures that are more likely to emerge.

This gap has prompted a second strand of literature on contest games studying the formation of alliances. A first approach has been to look at the formation of alliances through sequential processes. Skaperdas (1998) studies the formation of alliances in rent-seeking contests with three agents by comparing the alterations of the winning probability when alliances are formed. In this paper, imbedded contests are considered in order to allow for rivalries within the winning group. This idea of imbedded contests has been implemented in a n players game by Tan and Wang (2010). None of these papers endogenise the formation process of alliances. The development of contest models where agents both strategically choose their effort and form groups has been undertook using coalition theory. Bloch *et al.* (2006), and Sánchez-Pagés (2007) generalise the formalism of Esteban and Ray (1999) to endogenise the formation of exclusive alliances in contests. In these formalisms, agents can only form alliances in the contest, while this paper allows agents to form both alliance and technological partnerships. Extending the formalism à la Baik (1993), Lee and Kim (2021) endogenise the formation of groups with n *ex-ante* heterogenous agents with both an exclusive and an open membership game.[5] The present work differs from Lee and Kim (2021) in several aspects. First, they consider a game where *ex-ante* heterogeneous agents form alliances while remaining silent on the source of this heterogeneity. I propose a model where the agents are *ex-ante* homogeneous and differentiate each other by their respective alliance and technological partnership. Such a formalism enables the agents not only to form exclusive alliances in the contest but also to cooperate through technological partnerships.[6] Second, Lee and Kim (2021) analyse the subgame perfect Nash equilibria of a formation game, whereas I am focusing on the analysis of stable coalition structures likely to emerge in the long run. Finally, they worked on a γ-game (Hart and Kurz, 1983) which is related to γ-stability, while I am working with δ-stability.[7] Furthermore, this paper enriches the literature studying the formation of several types of coalitions. Dollinger, Mauleon and Vannetelbosch (2024) proposed the notion of stable pairs of coalition structures, whereby each coalition structure is stable relative to the other. This concept imposes to study the stability of each structure by fixing the other. With the new concepts considered in this paper, the deviations taking place in one coalition structure might affect the architecture of the other structure.

The remainder of the paper is organised as follows. Section 2 presents the framework where agents can form both alliances and technological partnerships in a contest with linear or quadratic costs. Section 3 presents the different stability concepts used to study the technological structures able to stabilise the grand alliance. Section 4 discusses the robustness of the results to other stability notions and the implications of these results before concluding.

[5]In open membership games, agents announce simultaneously their decision to form coalitions. If several agents emit the same signal, a coalition is formed. Nonmembers can join a coalition without the consent of the existing members. Yi and Shin (2000) and Yi (1997) examine an open membership game with *ex-ante* symmetric agents and found that the grand alliance is a Nash equilibrium. Once agents become *ex-ante* asymmetric, the latter result is not necessarily true, as shown by Belleflamme (2000).

[6]There exists a broad literature studying R&D cooperation through the prism of coalition and network theory. To mention but a few, Bloch (1996), Goyal and Moraga-González (2001), Goyal and Joshi (2003), Mauleon, Sempere-Monerris and Vannetelbosch (2023), Dollinger, Mauleon and Vannetelbosch (2024).

[7]The equilibrium of the δ and γ games presented by Hart and Kurz (1983) has been adapted to any noncooperative model on coalitions formation by Herings, Mauleon and Vannetelbosch (2010) with the notions of δ and γ stability.

2 Model

We consider a contest game where agents can rise her chances of winning a public good prize by forming an alliance. Moreover, agents can decrease the cost of their effort invested in the contest by forming a technological partnership. Bloch *et al.* (2006) develop a simple non-cooperative contest model to analyse the formation of groups where the benefits from cooperation increase the probability of winning at the cost of sharing the prize with allies in case of victory. We extend Bloch *et al.* (2006) framework by allowing the simultaneous formation of both alliances and technological partnerships among agents.

Let $N = \{1, 2, \cdots, n\}$ be the set of $n \geqslant 2$ *ex-ante* identical agents. An alliance structure $A = \{A_1, \cdots, A_m\}$, is a partition of the set of individuals, i.e. $A_i \cap A_j \neq \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{m} A_i = N$. Let $m := \mathrm{card}(A) \in [\![1, n]\!]$ be the number of alliances in A. The cardinality of the j^{th} alliance in A is $a_j \in \mathbb{N}^*$. Let $A(i) \in A$ denotes the alliance including agent i that is of cardinality $a(i) := \mathrm{card}(A(i)) \in \mathbb{N}^*$. The grand alliance and the singleton alliance structures are respectively denoted $A^g = \{N\}$ and $A^s = \{\{1\}, \{2\}, \cdots, \{n\}\}$. The alliance structure composed of $e \leqslant n$ coalitions of the same size is denoted $A^{(e)} = \{A_1, \cdots, A_e\}$, where $a_1 = a_2 = \cdots = a_e = \frac{n}{e} \in \mathbb{N}^*$. We denote $\mathcal{A}$ the set of all the alliance structures of n agents.

In addition to cooperate through alliances, agents can also create technological partnerships mitigating their cost of effort. A technological structure is a collection of partnerships $T = \{T_1, \cdots, T_z\}$, which is a partition of the set of individuals, i.e. $T_i \cap T_j \neq \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{z} T_i = N$. Let $z := \mathrm{card}(T) \in [\![1, n]\!]$ be the number of partnerships in T. The cardinality of the j^{th} coalition in T is $t_j := \mathrm{card}(T_j) \in \mathbb{N}^*$. Let $T(i) \in T$ denotes the technological partnership including agent i that is of cardinality $t(i) \in \mathbb{N}^*$. The agents in an alliance A_j that belong to the largest technological coalitions among all the agents within this alliance are called the most efficient agents in A_j. The global technological coalition and the singleton technological structure are respectively denoted $T^g = \{N\}$ and $T^s = \{\{1\}, \{2\}, \cdots, \{n\}\}$. As for the alliances, let $T^{(e)} = \{T_1, \cdots, T_e\}$ denotes the technological structure composed of $e \leqslant n$ coalitions of the same size, where $t_1 = t_2 = \cdots = t_e = \frac{n}{e} \in \mathbb{N}^*$.[8] We denote $\mathcal{T}$ the set of all the technological coalition structures of n agents.

The pair of coalition structures (A, T) is a collection including an alliance structure A and a technological structure T.[9] A *pair* (A, T) is said to be *symmetric* if the alliance structure is identical to the technological structure, i.e. $A = T$. Similarly, an alliance A_i is *symmetric* with respect to a technological coalition T_j when all the members of A_i are in T_j and *vice versa* : $A_i = T_j$. The interactions between agents are formalised with a two stages model. In the first stage, agents form simultaneously their contest alliances and their technological partnerships. Once the coalitions have been formed,[10] the agents compete in a contest game by choosing their respective level of effort. This second stage is called the *effort game* or the *contest game*.

[8]Note that $T^{(1)} = T^g$ and $T^{(n)} = T^s$. Moreover, for $T^{(e)}$ to be feasible, one must have that $n \in e\mathbb{N}^*$.

[9]The pairs of structures are ordered pairs or $2-$tuples here.

[10]Here, we are going to look at the stability concepts in order to identify the pairs of structures most likely to emerge in the long run. We do not analyse the strong Nash equilibria in a game of coalition formation as it is done in the literature on exclusive membership games. See Lee and Kim (2021) for instance.

In the effort game, each agent i yields a certain level of effort, $x_i \in \mathbb{R}_+$, to win the prize in the contest. The effort level of each agent determines the performance of the agent in the contest. Efforts are costly, so that the cost endured by a agent i in the contest hinges upon the effort level as well as the size of the coalitions in the technological structure : $c\colon [\![1,n]\!] \times \mathbb{R}_+ \times (\mathbb{N}^*)^z \to \mathbb{R}_+$ defined by $c\,(i, x_i, T) = c\left(i, x_i, (t_j)_{j\in[\![1,z]\!]}\right)$.[11] The members of a technological partnership pay only a fraction of the effort they would have invested in the contest without the partnership: $\frac{x_i}{t(i)}$. The technological partnership decreases the marginal effort invested in the contest by each one of its member. The larger the technological partnership of i, the smaller the fraction of the cost of effort paid by i. We assume that the agents have a cost function increasing in their level of effort and decreasing in the size of their own technological partnership. In this paper, the cost function can be linear, $c(i, x_i, T(i)) = \frac{x_i}{t(i)}$, or quadratic, $c(i, x_i, T(i)) = \left(\frac{x_i}{t(i)}\right)^2$, with respect to the net effort invested in the contest by the agent i.[12]

As in Bloch *et al.* (2006), agents in the same alliance pull together their efforts to inflate their probability of winning the prize but only to receive some portion of this prize. Agent i receives a portion of the prize $\pi_i(A_j)$ if her alliance A_j wins the contest.[13] If no agent invest any effort in the contest, it is assumed that every agent has the same probability, $\frac{1}{n}$, of winning the contest. The probability for alliance A_j to win the prize is

$$p_j = \begin{cases} \dfrac{X_j}{X} & \text{if } (x_k)_{k\in N} \neq (0, \cdots, 0) \\ \\ \dfrac{1}{n} & \text{if } (x_k)_{k\in N} = (0, \cdots, 0) \end{cases}$$

where $X := \sum_{k\in N} x_k$ and $X_j = \sum_{k\in A_j} x_k$.[14] In this paper, we consider the *rent seeking contests*, whereby agents compete over a fixed prize $V > 0$. In addition, the prize is assumed to be a public good that is equally shared among the members of the winning alliance (no rivalry within coalitions) :

$$\pi_i(A_j) = \begin{cases} \dfrac{V}{a_j} & \text{if } i \in A_j \\ \\ 0 & \text{if } i \notin A_j \end{cases}$$

It follows that the utility of agent i in the alliance A_j and in the technological structure T is given by : [15]

[11] The cost function $c(\cdot)$ is assumed to be the same for any agent.

[12] It is assumed that the cost of effort of a agent i only depends upon the size of her own technological coalition, $T(i)$, and not on the size of the other technological coalitions in the structure.

[13] The portion of the prize received by i only depends upon her own alliance A_j and not upon the whole alliance structure A.

[14] We consider here the classical Tullock (1980) success function axiomatised by Skaperdas (1996).

[15] In general, there are contests where an agent i benefits from the prize even if her coalition does not win. As an example, in a contest where agents fight over a policy, the adopted policy can increase the utility of a agent i if it is close to the bliss point of i, even though the policy she defended has not been adopted. See the model on policy conflicts in Bloch *et al.* (2006).

$$u_i\Big((x_k)_{k\in N}, A, T\Big) = \sum_{\ell=1}^{m} p_\ell \pi_i(A_\ell) - c(i, x_i, T(i)) = p_j \cdot \frac{V}{a_j} - c(i, x_i, T(i)) \qquad (1)$$
$$\text{s.t. } x_k \geqslant 0 \ \forall k \in N$$

We first analyse the Nash equilibrium of the effort game (second stage). In this stage, agents choose their optimum level of effort to invest in the contest given the pair of structures (A, T). Next proposition shows the existence of a unique Nash equilibrium for the effort game when the costs are linear : $c(i, x_i, T(i)) = \frac{x_i}{t(i)}$.[16]

Proposition 1. *A Nash equilibrium of the effort game with linear costs always exists and is unique.*

The Appendix gathers all the proofs.

The equilibrium conditions in the proof exhibit an extreme case of free riding where only the most efficient agents yield strictly positive effort at equilibrium. Epstein and Mealem (2009) attributed this free riding behaviour to the constant return of the effort invested in the contest in terms of gain in the winning probability. Formally, they discover that this full free rider problem emerge in contest game with linear costs and disappear when the costs are no longer linear (when the cost function is a convex parabola). For these types of costs in general and for the quadratic costs in particular, the existence of Nash equilibrium has been established. See Proposition 2 in Bloch *et al.* (2006).

We now characterise the unique equilibrium of the effort game with linear costs. The following proposition provides the analytical form of this equilibrium.

Proposition 2. *If* $(m-1) \leqslant \frac{\bar{t}_j}{a_j} \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}$ *for any* $j \in L = [\![1, m]\!]$*, the equilibrium winning probability and effort level in each group* $j \in L$*, and the subsequent equilibrium utility of the contest game with linear costs are :*

$$p_j^* = 1 - \frac{a_j(m-1)}{\bar{t}_j \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}}$$

$$x_i^* = \begin{cases} \dfrac{V(m-1)}{b_j \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}} \left[1 - \dfrac{a_j(m-1)}{\bar{t}_j \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}}\right] & \textit{for } i \in B_j \\ 0 & \textit{for } i \in A_j \backslash B_j \end{cases}$$

$$u_i^*(A, T) = \begin{cases} \left[1 - \dfrac{a_j(m-1)}{\bar{t}_j \displaystyle\sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}}\right] \left[\dfrac{V}{a_j} - \dfrac{V(m-1)}{b_j \bar{t}_j \displaystyle\sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}}\right] & \textit{if } i \in B_j \\ \left[1 - \dfrac{a_j(m-1)}{\bar{t}_j \displaystyle\sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}}\right] \dfrac{V}{a_j} & \textit{if } i \in A_j \backslash B_j \end{cases}$$

[16] The uniqueness of the Nash equilibrium comes from the assumption that the costs are linear with respect to x_i. Proposition 1 remains true in the general model with linear costs in x_i as long as $\pi_i(A_j) > 0$ for $i \in A_j$.

where $B_j \subset A_j$ denotes the set of the most efficient agents in A_j and $card(B_j) = b_j$. For any $k \in B_j$, $card(T(k)) := \bar{t}_j$ denotes the size of the largest technological coalition in T to which belong agents from $B_j \subset A_j$.

The condition $(m-1) \leqslant \frac{\bar{t}_j}{a_j} \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}$ ensures that the probability of winning for any alliance is in the interval $[0,1]$. Without this requirement, the optimality conditions found in Proposition 1 are not fulfilled.

Notice that when the largest technological coalition of agents in alliance A_j grows, then necessary the utility of $i \in A_j$ rises at equilibrium. The equilibrium effort x_i^* invested in the contest by the most efficient agents in A_j increases with the size of the technological coalition of these agents. Then, the probability of winning the contest ($p_j = \frac{X_j}{X}$) increases and so does the utility of i. As the less efficient agents produce no effort at equilibrium, the size of their technological coalition does not affect the utility of i.[17]

Remark 1. When the costs are linear in a contest game with heterogeneous agents, some agents and, by extension, entire contest alliances might remain inactive at equilibrium (produce zero effort). Echoing Remark 1 in Lee and Kim (2021), one proves in the present model that all the alliances $(A_1, \cdots, A_m)$ are active at equilibrium in the effort game when $(m-1) < \frac{\bar{t}_j}{a_j} \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}$.

In order to better understand the equilibrium utility function of a agent i when the costs are linear, one can start by looking at the spillover effects of each coalition structure. In the alliance structure, when agents cannot form any technological coalition, the merger of alliances will only decrease the number of alliances in the structure (m). In this case there is a strict positive spillover effect in the alliance structure : the merger of alliances increases the equilibrium utility of the agents outside these alliances. However, once the agents are able to create technological coalitions, the merger of two alliances has no clear (monotonous) effect on the payoffs of the agents outside these two alliances any more. Indeed, the merger between two alliances could increase the largest technological coalition ($\bar{t}_\ell$) of some members of these alliances. This could inverse the effect of the merger on the equilibrium utility of these agents.
To comprehend this point, let us consider the pair of structures $(A = \{A_1, A_2, A_3\}, T = \{T_1, T_2^s\})$, where $a_1 = 1$, $a_2 = a_3 = \frac{n-1}{2}$, $t_1 = \frac{n+1}{2}$, $t_2 = \frac{n-1}{2}$, and where all the agents in $A_1 \cup A_2$ are in T_1 whereas all the agents in A_3 are in T_2^s. In that case, $m = 3$ and $\sum_{\ell \in L} \frac{a_\ell}{\bar{t}_\ell} = \frac{1}{\left(\frac{n+1}{2}\right)} + \frac{\frac{n-1}{2}}{\left(\frac{n+1}{2}\right)} + \frac{n-1}{2}$. If A_2 and A_3 were merging, the resulting pair would be $(A = \{A_1, A_2 \cup A_3\}, T = \{T_1, T_2^s\})$, where $a_2 + a_3 = n-1$, $m = 2$, and $\sum_{\ell \in L} \frac{a_\ell}{\bar{t}_\ell} = \frac{1}{\left(\frac{n+1}{2}\right)} + \frac{n-1}{\left(\frac{n+1}{2}\right)}$. So after the merger, m decreases but $\sum_{\ell \in L} \frac{a_\ell}{\bar{t}_\ell}$ decreases as well. Taking $n = 11$, the utility of the singleton agent in A_1 would be $\left(1 - \frac{2}{6 \cdot (\frac{1}{6} + \frac{5}{6} + 5)}\right)^2 \cdot V \approx 0.892V$ before the merger and $\left(1 - \frac{1}{6 \cdot (\frac{1}{6} + \frac{19}{6})}\right)^2 \cdot V \approx 0.879V$ after. So the utility of this agent, who is outside the merged alliances, is decreasing with the merger in that situation.

However, there are situations where the utility of the agents outside the merged

[17] When $\overline{T}_j$ increases one or some other technological coalitions shrink. The shrinks of other technological coalitions either increases the utility of $i \in A_j$ or leave it unaffected. More precisely, when the size of $\overline{T}_\ell$ decreases for some $\ell \in L$, the utility of $i \in A_j$ increases ; when there is no $\ell \in L$ such that the size of $\overline{T}_\ell$ decreases, the utility of $i \in A_j$ remains unchanged.

alliances always increases with the merger : The mergers between alliances whose most efficient agents are equally efficient. In that case, there is a strict positive spillover effect in alliances as stated by the following property.

Property 1. *Conditional positive spillover effect in alliances*
In the model with linear costs, given any technological structure T, a sufficient condition such that $u_i^*(A,T) < u_i^*\big((A\backslash\{A_1,A_2\}) \cup (A_1 \cup A_2), T\big)$ for any agent $i \notin A_1, A_2$, is that $\bar{t}_1 = \bar{t}_2$.

The reduction in the number m of alliances in a structure without changing the number of most efficient agents in any alliances, reduces the equilibrium aggregate effort invested in the contest. The intuition behind this effect is that after the merger, the most efficient agents in the new alliance (b_j) are more numerous than before the merger. Therefore, they will decrease their equilibrium marginal quantity of effort invested in the contest (x_i^*). Since the quantity of effort yields at equilibrium by the agents outside the merged alliances is unchanged, there is a drop in the overall quantity of effort invested in the contest at equilibrium. Following the decline of the equilibrium aggregate effort (X^*), the equilibrium probability of agents outside the merged alliances will rise $\left(p_j = \dfrac{X_j^*}{X^*}\right)$. Hence the growth of the equilibrium utility for these agents.

Regarding the technological structure, there is no clear (monotonous) spillover effect as well. The formation of a larger technological coalition by some agents can decrease or increase the utility of the agent outside the coalition depending on the pair of structures (A,T). For instance, for an agent outside two technological coalitions T_1 and T_2, if she shares the same alliance than some members of $T_1 \cup T_2$, her utility might increase if these two coalitions were to merge. To illustrate this remark, let us consider a contest between seven agents in $N = \{i,j,k,p,q,r,s\}$, where $A = \Big\{A(i) = \{i,j,k,r,s\}, \{p,q\}\Big\}$ and $T = \Big\{T_1 = \{j\}, T_2 = \{k,r,s\}, \{i\}, \{p,q\}\Big\}$. In that case $u_i^*(A,T) \approx 0.066V < 0.079V = u_i^*\big(A, (T\backslash\{T_1,T_2\}) \cup (T_1 \cup T_2)\big)$. Now, if we consider the pair of structures composed of $A = \Big\{A(i) = \{i,j,k,p,r,s\}, \{q\}\Big\}$ and $T = \Big\{T_1 = \{p\}, T_2 = \{q\}, \{i,j,k,r,s\}\Big\}$ we have $u_i^*(A,T) \approx 0.067V > u_i^*\big(A, (T\backslash\{T_1,T_2\}) \cup (T_1 \cup T_2)\big) \approx 0.042V$.

Even though there is no clear (monotonous) spillovers in the technological partnerships, one can note that the utility of a agent i is always decreasing when agents outside the technological coalition of i *and* outside the alliance of i form a larger technological partnership. The next property establishes formally this conditional negative spillover effect in technological partnerships.

Property 2. *Conditional negative spillover effect in technological partnerships*
In the model with linear costs, given any alliance structure A, for any agent $i \notin T_1, T_2$, if $(T_1 \cup T_2) \cap A(i) = \varnothing$, then it holds that $u_i^*(A,T) \geqslant u_i^*(A, T\backslash\{T_1,T_2\} \cup \{T_1 \cup T_2\})$.

How the gain in the winning probability subsequent to the formation of an alliance would be affected by the gain in efficiency in the cost of effort ? The answer to this question will determine the incentives for agents to create both alliances and technological partnerships in contests. For this purpose, the following section analyses the incentives behind the formation of pairs of structure composed of the grand alliance.

3 Stabilisation of the grand alliance

We start the analysis of stability by considering the linear cost setting, whereby agents experience extreme free riding behaviours at equilibrium : only the most efficient agents of each alliance yield non-null effort.

3.1 Stability with linear costs

One approach to identify a coalition structure likely to emerge in the long run is to seek a set of groups from which no element has incentive to alter the structure of the set. To define precisely a stability concept, one must formally answer several crucial questions : What are the possible deviations from an arbitrary initial structure the agents can make ? Are non-deviating agents affected by the deviations ? In an attempt to answer these questions, Hart and Kurz (1983) associated the stable coalition structures to the strong Nash equilibria of a game where each agent announce a set of partners with whom she wants to be associated. In this game, a set of agents form a coalition together if all of them announce the same largest set of agents. This notion of δ-stable coalition structure assumes that the non-deviating agents that used to be partners with some of the deviating players remain together after the deviation.[18] In this work agents are able to form both alliances and technological partnerships, so the stability notion developed by Hart and Kurz (1983) needs to be extended to pairs of structures.

Before defining any stability concept, one must identify the coalition structures toward which a set of one or several agents is able to deviate. With a single coalition structure, Herings, Mauleon and Vanntelbosch (2010) proposed the notion of δ-obtainable structures.

Definition 1 (Herings, Mauleon and Vanntelbosch, 2010)**.** A coalition structure S' is ***δ*-obtainable** from the coalition structure S via a deviating coalition $D \subset N$, if (i) $\{S'_i \in S' \mid S'_i \subset N \backslash D\} = \{S_i \backslash D \mid S_i \in S,\, S_i \backslash D \neq \varnothing\}$ and (ii) $\exists \{S'_1, \cdots, S'_k\} \subset S'$ such that $\bigcup_{j=1}^{k} S'_i = D$.

Condition (i) states that the coalitions formed by the non-deviating agents are not affected by the deviations of the agents in D. Meaning that if the agents in D leave their respective coalition, the non-deviating agents do not move. Condition (ii) allows the deviating agents in D to form one or several coalitions in the new structure S'. Non-deviating agents do not belong to those new coalitions. A δ-stable structure S can then be defined as a structure such that all the deviations from S to any δ-obtainable structure are blocked by at least one deviating agents.

Definition 2 (Mauleon, Sempere-Monerris and Vannetelbosch, 2016)**.** A **coalition structure** S is ***δ*-stable** if for any $D \subset N$, for any δ-obtainable S' from S via D, for $i \in D$ such that $u_i^*(S') > u_i^*(S)$, there exists $j \in D$ such that $u_j^*(S') \leqslant u_j^*(S)$.

Without any technological cooperation, the following proposition shows that there exists no δ-stable alliance structure. Specifically, we are looking at the δ-stable alliances when the technological structure is assumed to be the singleton structure. In that case, all the agents are equally efficient.[19]

[18] See section 3.3 for a discussion on γ stability.

[19] In a contest where all the agents are *ex-ante* homogeneous, Bloch *et al.* (2006) showed that the grand alliance is not δ-stable. I extend this result here.

Proposition 3. *When the costs are linear, given that the technological structure is the singleton structure, $T = T^s$, no alliance structure $A \in \mathcal{A}$ is δ-stable for any $n \in \mathbb{N}^*$*

This proposition is proven by showing that in any alliance structure every agent has incentives to unilaterally leave her coalition to become a singleton. Moreover, all the agents have incentives to deviate from the singleton alliance toward the grand alliance.

To know whether the ability for agents to form technological partnerships could stabilise some alliance structures, one need to adopt stability concepts that generalise δ-stability to the formation of pairs of structures. A natural approach to investigate the stability of a pair of structures is to study the stability of each structure fixing the other one. This is the idea behind the concept of δ-stable pairs introduced by Dollinger, Mauleon and Vannetelbosch (2024).

3.1.1 δ-stable pairs

In their study on R&D and MS agreements, Dollinger, Mauleon and Vannetelbosch (2024) propose the concept of δ-stable pair of structures. A pair of coalition structures (S_1, S_2) is δ-stable when at least one deviating agents is not better off after the deviation to any δ-obtainable structure S_1 given the structure S_2, and to any δ-obtainable structure S_2 given S_1.

Definition 3 (Dollinger, Mauleon and Vannetelbosch, 2024)**.** A **pair of coalition structures** (S_1, S_2) is **δ-stable** if:

(i) for any $D \subset N$, for any δ-obtainable S_1' from S_1 via D, for $i \in D$ such that $u_i^*(S_1', S_2) > u_i^*(S_1, S_2)$, then there exists $j \in D$ such that $u_j^*(S_1', S_2) \leqslant u_j^*(S_1, S_2)$,

(ii) for any $D \subset N$, for any δ-obtainable S_2' from S_2 via D, for $i \in D$ such that $u_i^*(S_1, S_2') > u_i^*(S_1, S_2)$, then there exists $j \in D$ such that $u_j^*(S_1, S_2') \leqslant u_j^*(S_1, S_2)$.

The following proposition studies the pairs of structure composed of the grand alliance. It is shown that no pair of structures composed of the grand alliance is δ-stable. The most efficient agents have always incentives to unilaterally leave the grand alliance to become singletons.

Proposition 4. *When the costs are linear, the pairs of structures (A^g, T) are not δ-stable for all $T \in \mathcal{T}$, for all $n \geqslant 3$.*

Agents from the largest technological coalition are at least as efficient as any other agent in the contest (technological advantage). When one of these agent leave the grand alliance to become singleton in the contest, her technological advantage over the other agents is preserved. For this agent, even though her probability of winning the contest declines, she will not have to share the prize with anyone if she won. Proposition 4 shows that the second effect prevails over the first for agents in the largest technological coalition.[20]

Without further restrictions on the possible deviations of the agents, no technological structures can stabilise the grand alliance in the long run. Imposing that

[20] Overall, the expected payoff of a agent in the largest technological coalition rises if she unilaterally leaves the grand coalition.

technological partnerships can only be formed among agents from the same alliance will reduces the number of possible deviations from the technological structures. Therefore, stabilising the alliance structure should be easier under such an assumption. In the following section, the stability of the grand alliance is examined under the assumption that technological partnerships are only formed among allies.

3.1.2 Technological partnerships among allies

Along this section we make the following assumption : agents cannot form technological partnerships with their adversaries in the coalitions formation stage.

Assumption 1. $T(i) \subseteq A(i)\ \forall i \in N$

This assumptions means that technological partners forbid to share knowledge outside their alliances. Under such an *ad-hoc* assumption, some deviations toward δ-obtainable alliances and technological structures are not possible. We need then to define the set of deviations from the alliance structure A as well as from the technological structure T that are feasible with Assumption 1.

The deviations in the technological structure T respecting Assumption 1 are those where the deviating agents form new technological coalitions with their respective allies in A. Furthermore, Assumption 1 also affects the set of possible deviations from the alliance structure A. An evolution in the alliance structure may alter the structure of the technological partnerships and therefore violates Assumption 1. For instance in the pairs of grand coalitions (A^g, T^g), if a agent i leaves the grand alliance A^g to become a singleton, the alliance structure becomes $\{N \backslash \{i\}, \{i\}\}$. In that case, all the members of the alliance $N \backslash \{i\}$ will be part of the same technological partnership (T^g) as their new opponent i. This is not possible under Assumption 1. To take this last issue into consideration we assume that every deviation from the alliance structure A is going to trigger an alteration of the technological structure T so that former allies in the initial alliance will never be in the same technological structure. This leads to the definition of the *feasible technological structure* for a deviation from A to A' via D.

Definition 4 (Feasible technological structure)**.** The feasible technological structure of T to a deviation from A to the δ-obtainable structure A' via D is $T_{A \to A'}$ such that $A'(i) \cap T(i) = T_{A \to A'}(i)$ for all $i \in N$.

The feasible technological structure $T_{A \to A'}$ to a deviation from the alliance structure A to A' via D is the technological structure T such that all the coalitions in T have been split into coalitions only composed of allies in A'.[21] If all the agents of a technological coalition remains allies in A', the alteration of T leaves this coalition unchanged.

We are going to use the concept of δ-stable pairs of structures where the deviations toward any alliance structure A entail the alteration of the technological structure into a feasible technological structure with respect to A. The following proposition shows that the global technological coalition does not stabilise the grand alliance under Assumption 1.

[21] $T_{A \to A'}(i)$ is simply a subset (not necessary a proper subset) of $T(i)$ since $T_{A \to A'}(i) = A'(i) \cap T(i) \subseteq T(i)$.

Proposition 5. *The pair of structures (A^g, T) is not δ-stable under Assumption 1 for any $T \in \mathcal{T}$ and any $n \geqslant 5$.*

This results highlights that restricting the set of obtainable technological structures does not alter the incentives for agents to deviate in the alliance structure. However, instead of imposing *ad-hoc* assumptions, one could impose assumptions about the reactions of agents to the possible deviations. This should enable us to identify the situations where the grand alliance is stabilised.

In the following section, the set of blocking agents is enlarged in a generalisation of the concept of contractual stability under unanimity decision rule.

3.1.3 Contractually stable pairs under unanimity

The concept of δ-stable pairs used in Proposition 4 only allows the deviating agents to block a deviation. We are going to widen the set of agents that are able to block a deviation.[22] The technological partners of a agent grant her access to their technology. In a conflict, a agent is able to use the military technology of her partners in the technological coalition. In public procurement, a firm will have access to the resources of her R&D partners. One could therefore consider that the technological partners of a agent have a say in the modifications of the alliance including this agent, even though they are not necessary members of this alliance. We assume in this section that any alteration in the alliance structure can be blocked by the technological partners of the agents involved in such modifications.

When agents can only form one coalition structure, the notion of contractual stability extend the set of blocking agents to include the former allies of each deviating agent (Mauleon, Sempere-Monerris, Vannetelbosch, 2016).[23] This stability concept is suitable when coalitions can be visualised as contracts binding some or all the members such that modifying a coalition requires the consent of some or all the members of the coalition. Under the unanimity decision rule, a new agent will enter a coalition only if she wishes to come in, her new partners are willing to accept her, and her former partners authorise her to withdraw. Therefore, all the deviating agents as well as all of their former and future partners are able to block a deviation.

In the following, we adjust the notion of contractual stability to situations where agents belong to a pair of coalition structures. When a deviation takes place in the alliance structure, we enable all the technological partners of the deviating agents to block the deviation. That is, for a new alliance to be formed, all the members of this new alliance as well as all the technological partners of these members need to accept the formation of this new coalition.[24] Therefore, if a change in the alliance structure does affect some non-deviating technological partners of the deviating agents, these non-deviating agents are assumed to have the right to cancel the deviation taking place in the alliance structure. Alliances can be seen as contracts binding all the members of the alliances as well as all of their technological partners.

[22] These agents are refers as blocking agents.

[23] The notion of contractual stability originates from Caulier, Mauleon, Sempere-Monerris, Vannetelbosch (2013) and Caulier, Mauleon, Vannetelbosch (2013) in their works on the coalitional networks. The consent of coalition partners is needed for adding or deleting links.

[24] When only singleton technological partnerships have been concluded, $T = T^s$, only the deviating agents are able to block any deviation in the alliance structure. In that case, contractual stability under the unanimity decision coïncide with δ-stability. Therefore, for any $A \in \mathcal{A}$, no pair (A, T^s) is contractually stable under the unanimity decision by Proposition 3.

Definition 5. A pair of coalition structures (A, T) is **contractually stable under the unanimity decision rule** if:

(i) for any $D \subset N$, for any δ-obtainable A' from A via D, for $i \in D$ such that $u_i^*(A', T) > u_i^*(A', T)$, then there exists $j \in \bigcup_{k \in D} T(k)$ such that $u_j^*(A', T) \leqslant u_j^*(A', T)$ and

(i) for any $D \subset N$, for any δ-obtainable T' from T via D, for $i \in D$ such that $u_i^*(A, T') > u_i^*(A, T')$, then there exists $j \in \bigcup_{k \in D} A(k)$ such that $u_j^*(A, T') \leqslant u_j^*(A, T')$.

Condition (i) states that any deviation in the alliance structure can be blocked by either the deviating agents or the technological partners of these deviating agents. Condition (ii) is identical to the second condition in the definition of δ-stable pairs (Definition 3). [25] When bidding firms are competing for the acquisition of a public market, they can conclude R&D agreements as well as merge their projects. Contractual stability under unanimity is assuming that the underlying contract associated to the formation of a joint project by several firms (alliance) enable the technological partners of these firms to veto the entry or the exit of agents in the alliance. The technological partners of a firm are granting access to their R&D technologies to this firm. In return, they are assumed to receive a veto power over the decisions concerning any change in the alliance formed by the firm.

Next Proposition states that the pairs of grand coalitions (A^g, T^g) is contractually stable under the unanimity decision rule.

Proposition 6. *When the costs are linear, the pair of structures (A^g, T^g) is contractually stable under the unanimity decision rule.*

All the agents are indifferent from deviating from the global technological coalition. Moreover, the deviations from the grand alliance are always blocked by the agents ending up in the largest alliances. The largest alliances always includes agents that are able to block a deviation in the alliance structure since all the agents are technological partners ($T = T^g$).

When the technological partners of the deviating agents are able to block, the global technological partnership does stabilise the grand alliance. In Proposition 3, we found that no alliance was δ-stable in the absence of technological partnerships. Proposition 6 highlights that the existence of technological partnerships can stabilise alliances in contests.

The following proposition shows that the global technological coalition $T = T^g$ is the only technological structures composed of equally efficient members that is able to stabilise the grand alliance when contractual stability under unanimity decision rule is considered.

Proposition 7. *When the costs are linear, the pairs of structures $\left(A^g, T^{(e)}\right)$ are not contractually stable under the unanimity decision rule, $n \geqslant 5$, and $e \in [\![2, n]\!]$.*

[25] If we extend the set of blocking agents to include all the allies of the deviating agents for a deviations taking place in the technological structure, the only change in Definition 5 would be an extension of the set of blocking agents considered in Condition (ii). Under this additional restriction, the results proven in this section remain true. This stems from the fact that we only investigate contractual stability for pairs of coalitions composed of the grand alliance, where the members of the grand alliance are indifferent between any technological structure.

When the technological structure is composed of two or more coalitions of equal size, agents have incentives to leave the grand alliance to become singletons in order to increase their expected gain from the contest. By leaving the grand alliance to become singletons, the smaller probability of winning the contest is counterbalanced by the fact that if won, the prize won't be shared. When the technological structure was the grand coalition, the agents from the largest alliances in the obtainable alliance structure were always able to block, which is no longer the case with other technological structures.

The extension of the set of blocking agents to agents that are not deviating agents is a strong assumption. Indeed, some agents that are not directly involved in the formation of new coalitions are able to veto their creation. One might argue that too much power is given to the non-deviating agents. To circumvent this limitation, we are going to consider a stability notion where the non-deviating agents are not able to veto a deviation in one structure anymore. Instead, these non-deviating agents are going to be able to exclude the deviating agents in the other structure.

3.1.4 Stability under threats of exclusion

If some countries involved in a conflict decide to switch sides, they might be excluded from the technological partnerships they used to form with their former allies in retaliation.[26] Similarly, the firms leaving a joint project in a public procurement might not have access to the R&D resources of their former partners anymore.[27] So if the deviation of some agents in a structure is detrimental for their partners in the other structure, one might assume that the deviating agents could be excluded from their coalition in the other structure.

In order to incorporate the ability of deviating agents to force a deviation in our setting, we are developing the notion of Δ-*stable pairs of coalition structures*.[28] If the deviation of some agents in a structure is detrimental for their partners in the other structure, these latter agents are able to issue a veto to the deviation. Since the decision to deviate is made by the deviating agents only, the vetoes could be ignored. In that case, the deviating agents would force the deviation even though their partners in the other structure are opposed to it. Consequently, the deviating agents will be exiled from their coalition in the other structure.

Before defining the solution concept, one must formally define what is meant by excluding the deviating agents from a coalition.

Definition 6 (Adjusted coalitions)**.** In response to a deviation from the alliance structure A [resp. the technological structure T] to the δ-obtainable structure A' [resp. T'] via D, the deviating agents in the coalition T_j [resp. A_j] are excluded from T_j [resp. A_j] when T_j [resp. A_j] is divided into $T_j \backslash D$ and $(T_j \cap D)^s$ [resp. $A_j \backslash D$ and $(A_j \cap D)^s$].

The deviating agents are excluded from their technological coalition T_j after a deviation in the alliance structure A when all the deviating agents from T_j become

[26] In World War II, Romania had no longer access to military technology from the UK after joining the Axis.

[27] Another example could be the alliances of parties during an election. When some parties leave the alliance, the remaining parties might decide to stop sharing their networks of financial and logistic supports with the deviating parties.

[28] This stability concept is a refinement of the concept of δ-stable pairs of structures proposed by Dollinger, Mauleon and Vannetelbosch (2024).

singletons, while the non-deviating agents remains together.

One could have considered other schemes of exclusion. For instance, the exclusion from a technological coalition could totally dissolve the coalition into singletons. In that case, both deviating and non-deviating agents from the technological coalition would become singletons. Another possible scheme could be that the adjustment of a coalition would split the coalition in two new coalitions : the subcoalition with the deviating agents and the subcoalition with the non-deviating agents. The stability results of this section are robust to these alternative schemes of exclusion.

The notion of Δ-stable pairs of coalition structures can now be formally defined. A pair composed of a technological and an alliance structures is Δ-stable if any deviating agent is not better off from the deviation to any δ-obtainable technological structure with the possibility of being excluded in the alliance structure, and to any δ-obtainable alliance structure with the possibility of being excluded in the technological structure.

Definition 7 (Δ-stable pairs of coalition structures)**.** A pair of coalition structures (A, T) is Δ-stable if:

(i) for any $D \subset N$, for any δ-obtainable A' from A via D, for $i \in D$ such that $u_i^*(A', T) > u_i^*(A, T)$, $\exists j \in D$ such that $u_j^*(A', T_D^{\#}) \leqslant u_j^*(A, T)$,
where $T_D^- = \{T_\ell \in T \mid T_\ell \cap D \neq \varnothing \text{ and such that } \forall k \in T_\ell \backslash D, u_k(A', T) < u_k(A, T)\}$, and
$T_D^{\#} = \{T_\ell \backslash D, (T_\ell \cap D)^s \ \mid T_\ell \in T_D^-\} \cup \big(T \backslash T_D^-\big)$

(ii) for any $D \subset N$, for any δ-obtainable T' from T via D, for $i \in D$ such that $u_i^*(A, T') > u_i^*(A, T)$, $\exists j \in D$ such that $u_j^*(A_E^{\#}, T') \leqslant u_j^*(A, T)$,
where $A_D^- = \{A_\ell \in A \mid A_\ell \cap D \neq \varnothing \text{ and such that } \forall k \in A_\ell \backslash D, u_k(A, T') < u_k(A, T)\}$, and
$A_D^{\#} = \{A_\ell \backslash D, (A_\ell \cap D)^s \ \mid A_\ell \in A_D^-\} \cup \big(A \backslash A_D^-\big)$

For a deviation taking place in the alliance structure A [resp. technological structure T], T_D^- [resp. A_D^-] denotes the set of technological coalitions [resp. alliances] containing some deviating agents, where *every* non-deviating agents are worse off with the deviation occurring in A [resp. T]. The technological structure [resp. alliance structure] where the deviating agents from the technological partnerships [resp. alliances] in T_D^- [resp. A_D^-] heve been excluded is denoted $T_D^{\#}$ [resp. $A_D^{\#}$].

For a pair (A, T) to be Δ-stable, the deviations toward any δ-obtainable alliance structure A' needs to be blocked by at least one deviating agent that can be excluded from her technological coalition in T : Condition (i). In addition, the deviations toward any δ-obtainable technological structure T' needs to be blocked by at least one deviating agent under the threat of being excluded from her alliance in A : Condition (ii). For any deviation in A, if *all* the non-deviating technological partners of some deviating agents in A are worse off with the deviation, they will exclude their deviating partners from their respective technological coalitions. If there is at least one non-deviating technological partners of every deviating agent in A that is not worse off with the deviation, no exclusion takes place and T remains unchanged. Similarly, when the deviation takes place in the technological structure some deviating agents are excluded from their alliances when *all* of their non-deviating allies are worse off with the deviation taking place in T. It is important to note that the exclusion needs the consent of *all* the partners of the deviating agents to be triggered.[29]

For any deviation in one structure, the non-deviating partners of the deviating agents in the other structure can issue a veto to the deviation. The deviating agents can still deviate in the first structure at the cost of being excluded from their coalition in the other structure.[30] To illustrate the exclusion protocol, let us examine the following example. Among nine agents $\{i_1, \cdots, i_9\}$, four of them want to deviate from A toward A' in the alliance structure : $D = \{i_1, i_4, i_5, i_8\}$. Let $T = \{T_1 = \{i_1, i_2, i_3\}, T_2 = \{i_4, i_5, i_6, i_7\}, T_3 = \{i_8, i_9\}\}$ be the technological structure. Let us assume that the agents $i_3,\ i_6,\ i_7,$ and i_9 are worse off with the deviation taking place in the alliance structure given $T : \forall\ j \in \{i_3, i_6, i_7, i_9\},\ u_j(A', T) < u_j(A, T)$. Then, all the non-deviating agents in T_2 (i.e. i_6 and i_7) as well as all the non-deviating agents in T_3 (only i_9) are against the deviation from A to A'. Therefore, the deviating agents i_4 and i_5 from T_2 as well as the deviating agent i_8 from T_3 are excluded from their respective technological coalition. In T_1 however, the non-deviating agent i_2 is not worse off with the deviation in the alliance structure. So no exclusion can occur in T_1 as at least one non-deviating agent does not suffer from the deviation in A. Eventually, after the deviation in the alliance structure, the adjusted technological structure will be $T_D^{\#} = \{T_1, T_2\backslash\{i_4, i_5\}, \{i_4, i_5\}^s, T_3\backslash\{i_8\}, \{i_8\}\}$.

With the notion of Δ-stability, the deviations taking place in one structure might affect the other structure at the same time. Intuitively, this concept identifies the stable situations when the formation of each pair composed of an alliance and a technological partnership is constrained by threats of exclusion. Each of these pairs can be seen as a contract imposing on the members of each alliance and technological coalition not to harm their partners in each coalition or be subject to be excluded in the other coalition. As an example, if one of the belligerents involved in a conflict decide to switch sides, her former allies might decide to stop sharing their military technologies as well as their arsenals with her. Similarly, if some regions decide to undertake military R&D collaborations with a foe, their allies might decide to exclude them from their common military alliance.

The next proposition shows that (A^g, T^g) is not Δ-stable because any agent has incentives to leave the grand alliances regardless of exclusion from the global technological partnership.

Proposition 8. *When the costs are linear, the pair of structures (A^g, T^g) is not Δ-stable for any $n \geqslant 3$*

When deviating agents in one structure can be excluded from the other structure, one might wonder whether technological structures with equally efficient agents can stabilising A^g. The following proposition shows that the technological structure composed of two or more coalitions of equal size does not stabilise A^g in the sense of Δ-stability as each agent always has incentives to leave the grand alliance regardless of exclusion.

[29]In δ-stable structures, the set of blocking agents are only the deviating agents. The notion of contractual stability proposed by Mauleon *et. al* (2016) extends the set of blocking agents to some of the members of the initial coalition of the deviating agents. With Δ-stability, the non-deviating agents cannot block the deviation. They can only issue a veto that could lead to the exclusion of the deviating agents in the other structure. So the notion of Δ-stable pairs structures is different from the concept of contractually stable structures.

[30]We assume that *all* the non-deviating partners of some deviating agents must issue their veto in order to trigger the exclusion.

Proposition 9. *When the costs are linear, the pair of structures* $(A^g, T^{(e)})$ *is not* Δ*-stable for any* $e \in [\![2, n]\!]$ *and any* $n \geqslant 3$

As with δ-stability for pairs, no technological structures of equally efficient agents can stabilise the grand alliance in the long run.[31]

The following result investigate the stability of symmetric pairs where the agents are equally efficient.

Proposition 10. *When the costs are linear, the pair of structures* $(A^{(e)}, T^{(e)})$ *is not* Δ*-stable for any* $e \geqslant 2$ *and for any* $n \geqslant 8$

Proposition 10 is proven by showing that the agents have always incentives either to merge two technological structures ($e \geqslant 3$), or unilaterally leave their alliance ($e = 2$).

Remark 2. It is of interest to know whether the threats of exclusion considered in the concept of Δ-stability are credible or not. Let us suppose that a agent wishes to deviate in the alliance structure, but her technological partners suffer from this deviation. The deviating agent is therefore threaten to be excluded from her technological coalition. Nevertheless, the non-deviating partners of this agent might also suffer by triggering the exclusion. With that information, the deviating agent would deviate knowing that the exclusion will never occur : The threat is not credible.
I prove that the threats of exclusion are always credible for the deviations from the pairs of structures (A^g, T^g) for any $n \in \mathbb{N}^*$.

The extreme free ridding equilibrium behaviours of agents within alliances completely offset the losses of the agents due to an exclusion. The equilibrium utility of agents when the costs are linear only depends upon the technological structure of their most efficient allies (see Proposition 2). Therefore, if a deviating agent i have formed an alliances with allies that are as efficient as she were before the deviation, being excluded from her own technological partnerships after the deviation will not affect her equilibrium utility. In that case, the exclusion never dissuades the agent i from deviating. This might change however when there is less free ridding behaviours at equilibrium.

3.2 Stability with quadratic costs

Epstein and Mealem (2009) showed that the extreme free ridding behaviours arising in the linear cost formalism are mitigated when the contest success function becomes a generalised logit, which is identical to consider non-linear costs. Such a change in the effort game ensures a non-null relative equilibrium effort for any two agents : There is no extreme free ridding behaviour at equilibrium anymore.

To see whether the grand alliance can be stabilised with less free ridding behaviours at equilibrium, we consider quadratic costs instead of linear costs in the

[31]For all the results studying the pairs composed of the grand alliance A^g, Condition (ii) in the definition of Δ-stability is trivially fulfilled. Indeed, when $A = A^g$, i.e. $m = 1$, all the agents are indifferent between every technological structure T. For these results, substituting Condition (ii) by Condition (ii) of the definition of δ-stability (Definition 3) would not affect the proofs. So in that case, these results would remain true.

model: $c(i, x_i, T(i)) = \left(\frac{x_i}{t(i)}\right)^2$.

With this cost structure, the existence of a Nash equilibrium is obtained exactly as the proof of Proposition 2 in Bloch *et al.* (2006). As in this paper, unlike with linear costs, the uniqueness is not ensure.

Before analysing stability, we start by noticing that Property 1 remains true with quadratic costs.

Property 3. *Conditional positive spillover effect in alliances*
In the model with quadratic costs, given any technological structure T, a sufficient condition such that $u_i^*(A, T) < u_i^*\big((A \backslash \{A_1, A_2\}) \cup (A_1 \cup A_2), T\big)$ for any agent $i \notin A_1, A_2$, is that $\bar{t}_1 = \bar{t}_2$.

As with linear costs, no alliance structure is stable when agents cannot form any technological partnership.

Proposition 11. *When the costs are quadratic, given that the technological structure is the singleton structure, $T = T^s$, no alliance structure $A \in \mathcal{A}$ is δ-stable for $n \geqslant 2$*

Exactly as with Proposition 3, the agents from non singleton alliances always have incentives to leave their alliances to become singletons, while the grand coalition is more profitable than the singleton alliance.

When agents can form technological partnership, Proposition 4 remains valid for $T = T^{(e)}$ for any $e \geqslant 1$: Technological partnerships where all the agents are equally efficient does not stabilise the grand alliances without further restrictions.

Proposition 12. *When the costs are quadratic, the pairs of structures $(A^g, T^{(e)})$ is not δ-stable for any $e \geqslant 1$ and any $n \geqslant 3$.*

However, the model with quadratic costs reduces the free ridding behaviours at equilibrium as pointed out by Epstein and Mealem (2009). In the linear setting, the less efficient agents in each alliance are investing no effort in the contest game. So the cost of effort endured by these agents is null as well. Therefore, the reduction of the cost of effort obtained from the technological partnership does not affect these agents. With quadratic cost, these extreme free ridding behaviours disappear at equilibrium. The cost of being excluded from a technological partnership is consequently never null. The exclusion of a agent from her technological partnership should therefore have a stronger effect compare to the model with linear costs. The next proposition shows that this magnifying effect is enough for the global technological partnership to stabilise the grand alliance when exclusions are possible.

Proposition 13. *When the costs are quadratic, the pairs of structures (A^g, T^g) is Δ-stable for any $n \geqslant 2$.*

This proposition indicates that when exclusions are possible, global cooperation in technology between agents in a situation with limited free ridding behaviours between partners can completely annihilate competition in any contest.

4 Discussion

4.1 δ-stability versus γ stability

The existence of stable pairs of structures heretofore has been established using δ-stability. Hart and Kurz (1983) also proposed the notion of γ-stability. The notion based upon δ-stability are appropriate when working on contests. As an example, when a coalition of countries lost an ally during a conflict, the remaining alliance seldom collapses. For instance, the changes in alliances during World War II (Finland, Romania, Bulgaria, Italy) did not immediately cause the collapse of neither the Allies nor the Axis. Similarly, a technological partnership between two countries might end without ending independent technological partnerships. For instance, the end of the R&D cooperation started in WWII between the West and the USSR at the outbreak of the Cold War did not stop military R&D collaborations between the USSR and its allies (e.g., the Sino-Soviet treaty), nor did it stops the technological cooperation between the US and the UK in the West. However, nothing precludes to adopt notions based on γ-stability.

Contrary to δ-stability, where the non-deviating agents do not move after the deviation, γ-stability imposes that the non-deviating agents that used to be partners with some deviating players becomes singletons after the deviation.

Definition 8 (Herings, Mauleon and Vanntelbosch, 2010)**.** An coalition structure S' is γ-**obtainable** from the coalition structure S via a coalition D, $D \subseteq N$, if (i) $S'(j) = \{j\}$ for any $j \in \{S_i \backslash D \mid S_i \in S,\, S_i \backslash D \neq \varnothing\}$ and (ii) $\exists \{S'_1, \cdots, S'_k\} \subseteq S'$ such that $\bigcup_{j=1}^{k} S'_i = D$.

Condition (i) means that if the firms in coalition D leave their respective coalition in S, the non-deviating agents in S become singletons. Condition (ii) allows the deviating firms in D to form one or several coalitions in the new coalition structure S'. Non-deviating agents do not belong to those new coalitions. We can then adapt Definition 3 to γ-obtainable structures.

Definition 9 (Dollinger, Mauleon and Vannetelbosch, 2024)**.** A **pair of coalition structures** (S_1, S_2) is $\boldsymbol{\gamma}$**-stable** if:

(i) for any $D \subset N$, for any γ-obtainable S'_1 from S_1 via D, for $i \in D$ such that $u^*_i(S'_1, S_2) > u^*_i(S_1, S_2)$, then there exists $j \in D$ such that $u^*_j(S'_1, S_2) \leqslant u^*_j(S_1, S_2)$,

(ii) for any $D \subset N$, for any γ-obtainable S'_2 from S_2 via D, for $i \in D$ such that $u^*_i(S_1, S'_2) > u^*_i(S_1, S_2)$, then there exists $j \in D$ such that $u^*_j(S_1, S'_2) \leqslant u^*_j(S_1, S_2)$.

Generally, the number of coalitions in a γ-obtainable structure from D is larger than in the associated δ-obtainable structure from D. So, for the agents, deviating is much more costly with γ-stability than it is with δ-stability. Therefore, the grand alliance is much easier to stabilised with γ-stability than with δ-stability, regardless of the assumptions on the costs.

Proposition 14. *The pairs of structures $(A^g, T^{(e)})$ is γ-stable for any $e \in \mathbb{N}^*$ and any $n \geqslant 2$.*

When γ-stability is considered, symmetric technological structures always stabilise the grand alliance. In that context, competition in contests where agents are equally efficient are prone to end in the long run.

4.2 Harshness of the contest

Among all the possible outcomes of the formation stage (pairs of structures), one might want to know in which situations the contest is at its higher level of intensity. In our setting, the intensity (or harshness) of a contest is described by the overall amount of effort invested by agents at the equilibrium of the contest game : X^*. It is assumed in this section that the value of X^* indicates the harshness of a contest: The higher the value of X^*, the higher the overall effort invested by all the agents, the more intense the contest. In conflicts, the higher the overall war effort invested by the belligerents, the more destructive the conflict. For public procurements, the lower the global effort invested by firms in their respective projects, the less aggressive the competition for a public market.

Regardless of the assumption on the costs (linear or quadratic), when the grand coalition is formed, the utility of each agent is $u_i^*(A^g, T) = \frac{V}{n}$ for any $T \in \mathcal{T}$, meaning that $x_i^* = 0$ for any $i \in N$ and so $X^* = 0$. All the agents in the contest are allies when the grand alliance is formed, there is no more contest. In the model with linear costs, $X^* = \frac{V(m-1)}{\sum_{\ell=1}^{m} \frac{a_\ell}{t_\ell}} > 0$ for any $A \neq A^g$. In the model with quadratic costs, when $T = T^{(e)}$ and $t(i) = t$ for all $i \in N$, i.e. all the agents are equally efficient, we have[32] $X^* = t\sqrt{\frac{V(m-1)}{2}} > 0$.

Propositions 3 and 11 show that no alliance is stable when agents are not able to form technological partnerships regardless of the assumptions on the costs. Therefore, in these situations the intensity of the contest will always be positive. This is no longer the case when agents can form technological partnerships under certain conditions (Propositions 13 and 14). So allowing agent to share technological resources can mitigate the intensity of a contest. Furthermore, Proposition 13 highlights that under threats of exclusion the only stable symmetric pair of structures is the pair composed of the grand alliance and the global technological coalition, provided that there is no extreme free ridding behaviours. Furthermore, Proposition 14 proves that with the notion of γ-stability, every technological structures composed of equally efficient agents stabilise the grand alliance. These results indicate that balancing the technological efficiency of each agent involved in a contest reduces the intensity of the contest.

4.3 Implications of the results

We now briefly discuss the implications of our results on several applications of the model.

Without extreme free ridding behaviours, the grand alliance in a conflict (universal peace in Bloch *et al.*, 2006) might be reached in the long run if (1) all the agents involved in the conflict are part of the same technological partnership, and (2) any deviation from the grand alliance can induce an exclusion from the global technological partnership (Proposition 13). So one factor explaining the absence of conflict between members of the European Union is the important and increasing collaboration in military R&D between all the EU members (Karampekios, Oikonomou and Carayannis, 2018). Moreover, the trade of military equipments between *all* the

[32] See the proof of Proposition 11.

NATO members (Callado-Muñoz, Hromcovà and Utrero-Gonzàlez, 2019) homogenises the military technology of each member of the alliance. According to our model, this explains the stability of a military alliance like NATO over time. Nonetheless, in the presence of extreme free ridding behaviours, our formalism indicates that the existence of a fierce competition in terms of military technology (competition between equally efficient groups of agents) does not stabilise global peace even under the threat of exclusion from technological partnerships (Propositions 7 and 9). So, arms races between certain equally efficient countries (e.g., the USA *vs* USSR, or China *vs* the USA) cannot sustain peace in the long run.

When all the bidders involved in a public procurement merge their projects together (grand alliance) we are in the most serious case of bid rigging, where there is no more competition for the acquisition of a public market. This situation leads to the selection of suboptimal (overpriced) projects by the public authority. We showed that no alliance structure is stable when the agents cannot form any technological partnership (Propositions 3 and 11) unlike the situation where agents can create technological partnerships under some conditions (Propositions 13 and 14). These results indicate that the existence of technological coalitions can be a facilitating factor for the formation of alliances in contests. Therefore, our formalism predicts that industries where (i) all the firms cooperate in R&D, and/or (ii) all the firms are equally efficient, are prone to the formation of bid riggings in the occurrence of public procurements. To minimise the risk of bid rigging according to our model, the public authorities should favour industries with heterogeneous firms in terms of efficiency. So, bolstering competition in terms of R&D in an industry should reduce the risk of collusion for the acquisition of public markets.

The situations where all the lobbies in an industry contact the policy makers as one (Industry-wide lobbies) represents the cases where a lobby has the strongest influence over the policy makers decisions. Our model suggests that an industry-wide lobby (grand alliance) can be sustained when all the lobbies exchange their knowledge among them (global technological coalition). Therefore, if one desires to diminish the influence of an industry-wide lobby, making all the knowledge about an industry public and costless would undermine the lobby. When all the lobbies share information on the industry, they have no incentive to leave this grand alliance because in doing so they might no longer have access to this knowledge (Proposition 13). However, making such an information public and costless would make the exclusion from the global technological coalition harmless for any lobby. This would in turn increases the incentives for the lobbies to leave the grand alliance (Propositions 5 and 11.

4.4 Conclusion

This work aims at shedding lights on the impact of different types of cooperations in contests. The existing literature about contests between groups only studies the formation of alliances. I take one step further by developing a model of contests where agents are able to create both alliances and technological partnerships. Alliances enhance the probability of winning at the cost of sharing the prize if won, while technological partnerships reduce the marginal cost of the effort invested in the contest by their members. The gains in strength provided by an alliance can thus be magnified with the efficiency improvements obtained from the technological partnership. These overlapping effects are taken into consideration to identify pairs of alliance and technological structures likely to emerge in the long run.

This paper enriches the literature studying the formation of several types of coalitions. First, the concept of contractually stable alliance developed in Mauleon, Sempere-Monerris, and Vannetelbosch (2016) is extended to pairs of structures by allowing partners of the deviating agents to block the deviation that take place in the other structure. I also propose the concept of Δ-stable pairs, whereby the deviating agents are the only ones who can block a deviation in one structure under the threat of being excluded in the other structure.

In the model with linear costs, where agents exhibit extreme free ridding behaviours, it is found that the stabilisation of the grand alliance by technological cooperation among equally efficient agents requires restrictive assumptions on the set of blocking agents. The pairs composed of the grand alliance and the global technological structure is contractually stable under unanimity decision rule. It is shown that no other technological structure where all agents are equally efficient does stabilise the grand alliance. Nonetheless, when agents exhibit extreme free ridding behaviours at equilibrium, the global technological cooperation is not sufficient to stabilise the grand alliance even under the threat of exclusion: the pair composed of the grand alliance and the global technological coalition is not Δ-stable when the costs are linear. This stems from the fact that at equilibrium, the reduction of the cost for an agent only depends upon the size of the technological coalition(s) of her most efficient ally(ies). Therefore, excluding agents investing zero effort in the contest from their technological coalition without excluding their allies does not affect them. The threat of exclusion becomes relevant, however, in a model with quadratic costs as no extreme free ridding behaviours emerge in such a setting. In that context, the global technological coalition stabilises the grand alliance under the threat of exclusion : the pair composed of the grand alliance and the global technological coalition is Δ-stable when the costs are quadratic. This indicates that when the free ridding behaviours are not extreme, the ability to exclude is a sufficient condition for the global technological cooperation to annihilate competition in contests. Moreover, with the alternative notion of γ-stability, the grand alliance is always stable when all the agents are equally efficient. As no alliance structure is stable when the agents cannot form any technological partnerships, all these findings suggest that the existence of technological coalitions is a facilitating factor for the formation of alliances in contests.

Appendix

Proof of Proposition 1. $u_i\Big((x_k)_{k\in N}, A, T\Big)$ is at least C^2 as it is a linear combination of C^2 functions. Then, one can quickly verify that $u_i\Big((x_k)_{k\in N}, A, T\Big)$ is strictly concave in $x_i \in \mathbb{R}^+$ as long as $V > 0$. Thus, the Nash equilibrium of the effort game $(x_1^*, \cdots, x_n^*) = (x_i^*)_{i\in N}$ is characterised by the Karush, Kuhn, Tucker optimality conditions :

$$\begin{cases} \dfrac{X - X_j}{X^2} V_{i,j} - 1 = 0 & \text{for } x_i^* > 0 \\ \dfrac{X - X_j}{X^2} V_{i,j} - 1 \leqslant 0 & \text{for } x_i^* = 0 \\ j \in [\![1, m]\!] \end{cases} \tag{A.2.1}$$

where $V_{i,j} = \dfrac{V \cdot t(i)}{a_j} \in \mathbb{R}_+^*$ is the share of the prize obtained by individual $i \in A_j$ if alliance A_j wins the contest, which is independent of x_i. We are going to show that the system (A.2.1) has one unique solution.

We know that the terms of the sequence $(V_{i,j})_{i \in A_j}$ are ordered for any $j \in [\![1, m]\!]$ as $\mathbb{R}_+^*$ is a totally ordered field. Thus, $\overline{V_j} := \sup_{i \in A_j} V_{i,j}$ is well defined for any $j \in [\![1, m]\!]$, and for all $i \in A_j$ we have

$$\frac{X - X_j}{X^2} V_{i,j} - 1 \leqslant \frac{X - X_j}{X^2} \overline{V_j} - 1$$

So only the agents with the largest $V_{i,j}$ in A_j will produce positive efforts x_i^* at equilibrium. All the agents $i \in A_j$ such that $V_i < \overline{V_j}$ face a binding positivity constraint and consequently play the corner solution $x_j^* = 0 = \inf \mathbb{R}_+$.[33] Moreover, if agents k and ℓ are both members of A_j and are both among the most efficient agents in A_j, we have $\dfrac{X - X_j}{X^2} V_{k,j} - 1 = \dfrac{X - X_j}{X^2} V_{\ell,j} - 1 = \dfrac{X - X_j}{X^2} \overline{V_j} - 1$. So for any alliance A_j, the agents with the highest normalised prize $\overline{V_j}$ will produce the same level of effort at equilibrium: $x_j^* > 0$. Each alliance is therefore composed of two types of members : the most efficient agents who produce strictly positive effort at equilibrium, and the others who produce no effort. So for each alliance A_j, we have $X_j = b_j x_j^*$, where b_j denotes the number of most efficient agents in A_j producing the level of effort x_j^* at equilibrium.

From these observations, (A.2.1) is equivalent to the following system :

$$\begin{cases} \dfrac{X - b_j x_j^*}{X^2} \overline{V_j} - 1 = 0 \\ j \in [\![1, m]\!] \end{cases}$$

This system is composed of m principal unknowns, $(x_j)_{j \in [\![1,m]\!]}$, and m principal equations, for whom there is one unique solution. ■

Proof of Proposition 2. As in the previous proof, let $V_i = \frac{V \cdot t(i)}{a_j} \in \mathbb{R}_+^*$ denotes the normalised prize for agent $i \in A_j$. For any alliance A_ℓ, let $V_{(\ell)}$ denotes the highest normalised prize among the agents in alliance A_ℓ. First, we compute the

[33] The necessary optimality conditions of Karush-Kuhn-Tucker impose that the cone including all the admissible solutions to maximise the objective function is spaned by the binding constraints only. Here, we showed that the binding constraints are the ones associated to the most efficient agents in an alliance. Therefore, the admissible solutions to maximise the utility function are independent of the value of the effort yielded by the less efficient agents in each coalition. Therefore, these agents will have incentives to choose an effort level as small as possible: $x_i^* = 0$.

equilibrium effort of the agents with the normalised prize $V_{(\ell)}$ in alliance A_ℓ, for any $\ell \in L$. For these agents, (A.2.1) becomes

$$X_\ell = X - \frac{X^2}{V_{(\ell)}} \ \forall \ell \in L \tag{A.2.2}$$

Summing over $\ell \in L$ it yields:

$$X^* = \frac{m-1}{\sum_{\ell \in L} \frac{1}{V_{(\ell)}}} \tag{A.2.3}$$

Substituting (A.2.3) in (A.2.2), it follows for all $\ell \in L$:

$$X_\ell^* = X^*\left[1 - \frac{X^*}{V_{(\ell)}}\right]$$

Thus, we have for the alliance $j \in L$:

$$p_j^* = 1 - \frac{X^*}{V_{(j)}} \tag{A.2.4}$$

Let $B_j \subset A_j$ be the set of the agents belonging to the largest technological coalitions in A_j. So $\text{card}(B_j) = b_j$ denote the number of most efficient agents in A_j. Thus, for all $k \in B_j$, $\text{card}(T(k)) := \bar{t}_j$ denotes the size of the largest technological coalitions to which belong agents from A_j.[34] With these notations in mind, (A.2.4) becomes :

$$p_j^* = 1 - \frac{a_j(m-1)}{\bar{t}_j \sum_{\ell=1}^m \frac{a_\ell}{\bar{t}_\ell}}$$

This probability must be in $[0,1]$. Since, $\frac{a_j(m-1)}{\bar{t}_j \sum_{\ell=1}^m \frac{a_\ell}{\bar{t}_\ell}} \geqslant 0$, one must have that

$$(m-1) < \frac{\bar{t}_j}{a_j} \sum_{\ell=1}^m \frac{a_\ell}{\bar{t}_\ell} \quad \forall j \in L \tag{A.2.5}$$

By construction of the unique Nash equilibrium of the effort game (Proposition 1) : (i) $x_i = 0$ for $i \in A_j \backslash B_j$ and (ii) $X_j = b_j x_{(j)}^* + 0$, where $x_{(j)}^*$ denotes the identical equilibrium effort level of agents in B_j. So the equilibrium level of effort of $i \in A_j$ for any $j \in L$ is :

$$x_i^* = \begin{cases} \frac{X_j^*}{b_j} = \frac{X^*}{b_j}\left[1 - \frac{X^*}{V_{(j)}}\right] = \frac{X^*}{b_j} p_j^* & \text{for } i \in B_j \\ 0 & \text{for } i \in A_j \backslash B_j \end{cases}$$

$$x_i^* = \begin{cases} \frac{V(m-1)}{b_j \sum_{\ell=1}^m \frac{a_\ell}{\bar{t}_\ell}}\left[1 - \frac{a_j(m-1)}{\bar{t}_j \sum_{\ell=1}^m \frac{a_\ell}{\bar{t}_\ell}}\right] & \text{for } i \in B_j \\ 0 & \text{for } i \in A_j \backslash B_j \end{cases}$$

[34] The largest technological coalition(s) to which belong agents from A_j is $\overline{T}_j = \max_{i \in A_j} \text{card}(T(i))$, where $\text{card}(\overline{T}_j) = \bar{t}_j$. The set of the most efficient agents in A_j is $B_j = \text{card}\left(\overline{T}_j \cap A_j\right)$. The most efficient agents in A_j might belong to different technological coalitions of the same size.

One can note that for $i \in B_j$, we have $x_i^* \geqslant 0$ for any $m \in \mathbb{N}^*$ using (A.2.5) and the fact that $b_j \geqslant 1$ as $\mathbb{R}_+^*$ is a totally ordered field.[35]

It follows,

$$u_i^*(A,T) = \begin{cases} \left[1 - \dfrac{a_j(m-1)}{\bar{t}_j \sum\limits_{\ell=1}^{m} \dfrac{a_\ell}{\bar{t}_\ell}}\right] \left[\dfrac{V}{a_j} - \dfrac{V(m-1)}{b_j \bar{t}_j \sum\limits_{\ell=1}^{m} \dfrac{a_\ell}{\bar{t}_\ell}}\right] & \text{if } i \in B_j \\ \left[1 - \dfrac{a_j(m-1)}{\bar{t}_j \sum\limits_{\ell=1}^{m} \dfrac{a_\ell}{\bar{t}_\ell}}\right] \dfrac{V}{a_j} & \text{if } i \in A_j \backslash B_j \end{cases}$$

■

Proof of Remark 1. In the linear cost model, let suppose that the contest alliance A_j is inactive, i.e. for all $i \in A_j$, $x_i = 0$. In Proposition 1, we show that the members h of A_j with the highest V_h have incentives to produce strictly positive effort when the other agents in A_j produce no effort. More specifically, Proposition 2 highlights that for the members h of A_j with the highest V_h, we have $x_h^* > 0 \iff p_j^* > 0$. In that case, if A_j is inactive, the members h of A_j will have incentives to unilaterally deviate to produce strictly positive effort. Therefore, the situations where at least one contest alliance is inactive cannot be a Nash equilibrium of the second stage effort game as long as $p_j^* > 0 \iff (m-1) < \frac{\bar{t}_j}{a_j} \sum_{\ell=1}^{m} \frac{a_\ell}{\bar{t}_\ell}$. This is always verified when $m \in \{1,2\}$ ($m = 1$ is the situation where the grand alliance is formed). ■

Proof of Property 1. Let us consider the alliances A_1 and A_2 the alliances of respective size a_1 and a_2. Let $\bar{t}_1 = \bar{t}_2$ denotes the size of the largest technological coalition in T to which belong agents from A_1 and A_2 respectively. Let $L := [\![1,m]\!]$. Let $A' = A \backslash \{A_1, A_2\}) \cup (A_1 \cup A_2)$. Let m be the number of alliances in A. So $m' = m - 1$ is the number of alliances in A'. The equilibrium utility of agent $i \notin A_1 \cup A_2$ in A is:

$$u_i^*(A,T) = \begin{cases} \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j \left(\sum\limits_{\ell \in L\backslash\{1,2\}} \dfrac{a_\ell}{\bar{t}_\ell} + \dfrac{a_1+a_2}{\bar{t}_1}\right)}\right] \left[\dfrac{V}{a_j} - \dfrac{V(m-1)}{b_j \bar{t}_j \left(\sum\limits_{\ell \in L\backslash\{1,2\}} \dfrac{a_\ell}{\bar{t}_\ell} + \dfrac{a_1+a_2}{\bar{t}_1}\right)}\right] & \text{if } i \in B_j \\ \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j \left(\sum\limits_{\ell \in L\backslash\{1,2\}} \dfrac{a_\ell}{\bar{t}_\ell} + \dfrac{a_1+a_2}{\bar{t}_1}\right)}\right] \dfrac{V}{a_j} & \text{if } i \in A_j \backslash B_j \end{cases}$$

Similarly, when A_1 and A_2 merge, we have:

[35] $x_i^* = 0$ if and only if $p_j^* = 0$.

$$u_i^*(A',T) = \begin{cases} \left[1 - \dfrac{(m'-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus\{1,2\}} \frac{a_\ell}{\bar{t}_\ell} + \frac{a_1+a_2}{\bar{t}_1}\right)}\right]\left[\dfrac{V}{a_j} - \dfrac{V(m'-1)}{b_j\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus\{1,2\}} \frac{a_\ell}{\bar{t}_\ell} + \frac{a_1+a_2}{\bar{t}_1}\right)}\right] & \text{if } i \in B_j \\ \left[1 - \dfrac{(m'-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus\{1,2\}} \frac{a_\ell}{\bar{t}_\ell} + \frac{a_1+a_2}{\bar{t}_1}\right)}\right]\dfrac{V}{a_j} & \text{if } i \in A_j\setminus B_j \end{cases}$$

One can note that $\sum_{\ell\in L}\frac{a_\ell}{\bar{t}_\ell}$ is the same in A or in A'. Moreover, $u_i^*(A,T)$ is strictly decreasing in m given the other variables and $m' = m-1 < m$. Thus, $u_i^*(A',T) > u_i^*(A,T)$. ■

Proof of Property 2. Let us consider the alliances T_1 and T_2 the alliances of respective size t_1 and t_2. Let $L := [\![1,m]\!]$. Let $I \subset L$ be the set of indexes of the alliances composed of at least one agent from $T_1 \cup T_2$. Let $T' = T\setminus\{T_1,T_2\}) \cup (T_1 \cup T_2)$. The utility of $i \notin T_1 \cup T_2$ in (A,T) is:

$$u_i^*(A,T) = \begin{cases} \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus I} \frac{a_\ell}{\bar{t}_\ell} + \sum_{\ell\in I}\frac{a_\ell}{\bar{t}_\ell}\right)}\right]\left[\dfrac{V}{a_j} - \dfrac{V(m-1)}{b_j\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus I} \frac{a_\ell}{\bar{t}_\ell} + \sum_{\ell\in I}\frac{a_\ell}{\bar{t}_\ell}\right)}\right] & \text{if } i \in B_j \\ \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus I} \frac{a_\ell}{\bar{t}_\ell} + \sum_{\ell\in I}\frac{a_\ell}{\bar{t}_\ell}\right)}\right]\dfrac{V}{a_j} & \text{if } i \in A_j\setminus B_j \end{cases}$$

If $t_1 + t_2 \leqslant \bar{t}_\ell$ for all $\ell \in I$, it is immediate to see that $u_i^*(A,T) = u_i^*(A,T')$.

If there exists a subset $J \subset I$ such that $t_1 + t_2 > \bar{t}_j$ for all $j \in J$, one can rewrite the equilibrium utility of agent $i \notin T_1 \cup T_2$ as follows:

$$u_i^*(A,T') = \begin{cases} \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus J}\frac{a_\ell}{\bar{t}_\ell} + \frac{1}{(t_1+t_2)}\sum_{\ell\in J}a_\ell\right)}\right]\left[\dfrac{V}{a_j} - \dfrac{V(m-1)}{b_j\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus J}\frac{a_\ell}{\bar{t}_\ell} + \frac{1}{(t_1+t_2)}\sum_{\ell\in J}a_\ell\right)}\right] & \text{if } i\in B_j \\ \left[1 - \dfrac{(m-1)a_j}{\bar{t}_j\left(\displaystyle\sum_{\ell\in L\setminus J}\frac{a_\ell}{\bar{t}_\ell} + \frac{1}{(t_1+t_2)}\sum_{\ell\in J}a_\ell\right)}\right]\dfrac{V}{a_j} & \text{if } i\in A_j\setminus B_j \end{cases}$$

Since $t_1 + t_2 > \bar{t}_j$ for all $j \in J$, it is immediate that $u_i^*(A,T) > u_i^*(A,T')$.

■

Proof of Proposition 3. In the pairs of structures (A, T^s), all the agents are equally efficient. Moreover, $\sum_{\ell=1}^{m} a_\ell = n$. So the utility of agent i from the alliance $A_j \in A$ is :

$$u_i^*(A,T^s) = \frac{V}{a_j}\left(1 - \frac{(m-1)a_j}{n}\right)\left(1 - \frac{m-1}{n}\right)$$

Let $A' = (A\setminus A_j) \cup (A_j\setminus\{i\}) \cup \{i\}$ be the structure where i unilaterally leave her alliance A_j in A to become a singleton. The utility of i in A' is :

$$u_i^*(A',T^s) = V\left(1 - \frac{m}{n}\right)^2$$

Thus, for $a_j \geqslant 2$, we have :

$$u_i^*(A,T^s) - u_i^*(A',T^s) = 1 + \frac{1}{n} + a_j\left(\frac{1}{n} + \frac{1}{n^2} - 1\right) + \frac{m}{n}\left(a_j - \frac{2a_j}{n} - 1\right) := f(a_j,m,n)$$

One can show that $f(a_j,m,n)$ is strictly negative for any $n \geqslant 3$ and $m \in [\![1, n-1]\!]$ when $a_j \geqslant 2$

Since the utility of i only depends upon the size of her own alliance and not on the size of the other alliances in the structure when $T = T^s$, the observation that $u_i^*(A,T^s) - u_i^*(A',T^s) < 0$ implies that the agent i will have incentives to leave her alliance $A_j \in A$ to become a singleton for any structure $A \in \mathcal{A}$. Ergo no structure except the singleton structure can be δ-stable for any $n \geqslant 3$ any $m \in [\![1, n-1]\!]$ and when $a_j \geqslant 2$ because every non singleton agents have incentive to unilaterally leave their coalition to become singletons. But all the agents in the singleton structure A^s have incentive to deviate toward the grand alliance A^g when $n \geqslant 2$:

$$u_i^*(A^s,T^s) = \frac{V}{n^2} < \frac{V}{n} = u_i^*(A^g,T^s)\ \forall n \geqslant 2$$

Eventually, no alliance structures is δ-stable for $n \geqslant 3$. ■

Proof of Proposition 4. Let order the technological coalitions in $T : T = (T_{(1)}, T_{(2)}, \cdots, T_{(z)})$ such that $t_{(1)} \leqslant t_{(2)} \leqslant \cdots \leqslant t_{(z)}$. Let consider an agent i from the largest technological coalition in $T : T_{(z)}$. Since there is only one alliance in the pair of structures (A^g, T), $m = 1$. So, the utility of i for any $T \in \mathcal{T}$ is

$$u_i^*(A^g, T) = \frac{V}{n}$$

Given T, we are going to verify whether $i \in T_{(z)}$ have incentives to deviate from the grand alliance $A = A^g$ to become a singleton. In other word, we are seeking whether i has incentives to block the deviation from (A^g, T) toward $(\{N\backslash\{i\}, \{i\}\}, T)$. One can note that in $(\{N\backslash\{i\}, \{i\}\}, T)$, the most efficient agents in $N\backslash\{i\}$ are either in $T_{(z)}$ if $t_{(z)} > 1$, or are singletons like i if $t_{(z)} = 1$. Given T, we have

$$u_i^*(\{N\backslash\{i\}, \{i\}\}, T) = V\left(\frac{n-1}{n}\right)^2 > \frac{V}{n} \; \forall n \geqslant 3$$

Therefore, given T, the agent from the largest technological coalition in T have always incentives to unilaterally leave the grand alliance A^g to become a singleton. Hence, (A^g, T) is not a δ-stable pair for any $T \in \mathcal{T}$ and for $n \geqslant 3$.
N.B. In this proof, i is a member of the largest technological coalition in T. In the proof of Proposition 6, the deviation from (A^g, T^g) toward $(\{N\backslash\{i\}, \{i\}\}, \{N\backslash\{i\}, \{i\}\})$, i belong to the smallest and not the largest technological coalition in $T = \{N\backslash\{i\}, \{i\}\}$. ■

Proof of Proposition 5. First of all, when $T \neq T^g$, Assumption 1 is violated. Therefore, the only possible δ-stable pair of structure composed of A^g under Assumption 1 is (A^g, T^g)

We start by proving the following lemma to reduce the deviations we need to study.

Lemma 1. *Under Assumption 1, any profitable deviation from (A^g, T^g) leads to a symmetric pair of structures.*

Proof of Lemma 1. (1) When $A = A^g$, for any $i \in N$, we have that $u_i^*(A^g, T) = u_i^*(A^g, T') = \frac{V}{n}$ for any $T' \neq T$. Therefore, from $T = T^g$, any δ-obtainable technological structure T' provides the same utility to any agent as T^g.

(2) Let A' be a δ-obtainable structure from some alliance structure $A \in \mathcal{A}$ via D. We say that an alliance $A_i \in A$ is *split* when there exists a k-tuple $(A'_1, \cdots, A'_k) \in (A')^k$ such that (i) $A'_j \subseteq A_i \; \forall j \in [\![1, k]\!]$, and (ii) $\bigcup_{j=1}^k A'_j = A_i$. The coordinates of the the collection $(A'_1, \cdots, A'_k)$ are the *pieces* of A_i, and the collection $(A'_1, \cdots, A'_k)$ is the *partition* of A_i.
The following lemma emphasises that when an alliance A_ℓ is identical to some technological structure T_k, if A_ℓ is split after some deviation in the alliance structure A, the feasible technological structure associated to this deviation includes technological coalitions that are symmetric with respect to each pieces of A_ℓ.

Lemma 2. *If a deviation from A to A' induces a split of an alliance A_ℓ into the partition $(A'_1, \cdots, A'_k)$, and if A_ℓ is symmetric with respect to some technological coalition T_m in T, then $T_{A\to A'}(i) = A'(i)$ for any $i \in A'_j$ and any $j \in [\![1, k]\!]$.*

Proof. Let $i \in A_\ell$. By Definition 4, $T_{A\to A'}(i) = A'(i) \cap T(i) = A'(i) \cap A(i)$ since $T(i)$ and $A(i)$ are symmetric. As $A_\ell \in A$ is split in A', we know that $A'(i) \subseteq A(i)$ for any $i \in A_\ell$. Thence, $T_{A\to A'}(i) = A'(i)$ for all $i \in A_\ell$. ■

Since all the deviations from the grand alliance imply a split of A^g, this lemma implies that, under Assumption 1, all the deviations from A^g to $A' \in \mathcal{A}$ entail an alteration of T into $T_{A^g\to A'}$ that is symmetric with respect to A: $T_{A^g\to A'}(i) = A(i) \cap N = A(i)$ forall $i \in N$.

So under Assumption 1, the only possible deviations from (A^g, T^g) are either the pairs of structures (A^g, T'), where T' is δ-obtainable from T, or the symmetric pairs of structures. But from (1), it has been shown that the deviations in the technological structure are not profitable for any deviating agents as every agent is indifferent to any deviation in T when $A = A^g$. ■

By Lemma 1, when a agent i leaves the grand alliance A^g, the global technological coalition is altered so that the new pair of coalition structure is

$$\left(A = \{N \backslash \{i\}, \{i\}\}, T_{A^g\to A} = \{N \backslash \{i\}, \{i\}\}\right)$$

In that new pair of structures, the utility of i is :

$$u_i^*(A, T_{A^g\to A}) = \frac{V}{4} > \frac{V}{n} \quad \forall n \geqslant 5$$

■

Proof of Proposition 6. In $(A,T) = (A^g, T^g)$, all the agents can block the deviations as they are all technological partners.

Let us start with the deviations from the global technological coalition T^g. For any $i \in A$, we have

$$u_i^*(A^g, T) = \frac{V}{n} \quad \forall T \in \mathcal{T}$$

So every agent is indifferent between deviating in $T \neq T^g$ or staying in the global technological coalition. Therefore, any agent will block the deviation from T^g to every δ-obtainable structure $T \in \mathcal{T} \backslash \{T^g\}$.

Let us now investigate the deviation from the grand alliance A^g. Let $A = (A_{(m)}, \cdots, A_{(1)})$ where $a_{(1)} \leqslant \cdots \leqslant a_{(m)}$. Let $i \in A_{(m)}$. We have: [36]

$$u_i^*(A, T^g) = \frac{V}{a_{(m)}} \left(1 - \frac{(m-1)a_{(m)}}{n}\right)\left(1 - \frac{m-1}{n}\right)$$

So i will block the deviation from A^g to A when:

[36] In that case, since the technological structure is the global technological coalition (T^g), we have $\bar{t}_\ell = n$ for all $\ell \in [\![1, m]\!]$ and $b_j = a_j \ \forall j \in [\![1, m]\!]$.

$$\frac{V}{a_{(m)}}\left(1-\frac{(m-1)a_{(m)}}{n}\right)\left(1-\frac{m-1}{n}\right) \leqslant \frac{V}{n}$$
$$\iff a_{(m)} \geqslant \frac{n(n-m+1)}{n+(m-1)(n-m+1)} \tag{A.2.6}$$

But $A_{(m)}$ is the largest alliances in A composed of m coalitions, so $a_{(m)} \geqslant \frac{n}{m}$.[37] Moreover

$$\frac{n(n-m+1)}{n+(m-1)(n-m+1)} = \frac{n}{m} \cdot \frac{(n-m+1)m}{n+(m-1)(n-m+1)}$$

where $(n-m+1)m-\big(n+(m-1)(n-m+1)\big) = n-m+1-n \leqslant 0$ $\forall m \in [\![1,n]\!]$ $\forall n \geqslant 1$. So

$$\frac{(n-m+1)m}{n+(m-1)(n-m+1)} \leqslant 1$$

Hence

$$\frac{n}{m} \geqslant \frac{(n-m+1)m}{n+(m-1)(n-m+1)}$$

So for any $m \in [\![1,n]\!]$ and any $\forall n \geqslant 1$

$$a_{(m)} \geqslant \frac{n}{m} \geqslant \frac{n(n-m+1)}{n+(m-1)(n-m+1)}$$

So (A.2.6) is always true for $m \in [\![1,n]\!]$ and for $n \geqslant 1$. Thus agents from the largest coalition in any δ-obtainable structure from A^g will block any deviation from $A = A^g$ in (A^g, T^g).[38]

■

Proof of Proposition 7. Let $T^{(e)} = (T_1, \cdots, T_z)$ be the structure such that $\text{card}(T_\ell) = \frac{n}{e}$ for any $\ell \in [\![1,z]\!]$. So $e \in [\![2,n]\!]$.[39]

Let $A = (A_{(m)}, \cdots, A_{(1)})$, where $a_{(1)} \leqslant \cdots \leqslant a_{(m)}$, be the alliance structure where $A_{(m)}$ is a coalition of non-blocking agents. For instance, when only the agents from one technological coalition can block a deviation to A, the coalition of the non-blocking agents is $A_{(m)} = \bigcup_{i=1}^{e-1} T_i$ with $a_{(m)} = \frac{e-1}{e}n$.
Let $i \in A_{(m-1)}$, where $a_{(m-1)} \in [\![1, n - \frac{e-1}{e}n = \frac{n}{e}]\!]$ and $m \in [\![2, \frac{n}{e}+1]\!]$. Then i will block the deviation from A^g when:

$$u_i^*(A,T) \leqslant \frac{V}{n} = u_i^*(A^g,T) \iff a_{(m-1)} \geqslant \frac{n(n-m+1)}{n+(m-1)(n-m+1)} \tag{A.2.7}$$

[37] The smaller value of $a_{(m)}$ is notably reached in the situation where A is such that $a_{(i)} = \frac{n}{m}$ for all $i \in [\![1,m]\!]$.

[38] The agents in the largest coalition in a δ-obtainable structure from A^g are not necessary the deviating agents. But in any case, these agents are technological partners of the deviating agents as $T = T^g$.

[39] $e = n$ is the situation where $T = T^s$, which has been considered in Proposition 3. The case $e = 1$ is the situation where $T = T^g$ studied in Proposition 6.

Let us consider the situation where $a_{m-1} = 1$ and so $a_{(k)} = 1$ for any $k \leqslant m-1$. Thus, $m = \frac{n}{e} + 1$ and for any $e \in [\![2, n]\!]$ (A.2.7) becomes:

$$a_{(m-1)} \geqslant \frac{n(n-m+1)}{n+(m-1)(n-m+1)} = \frac{n(n-\frac{n}{e})}{n+\frac{n}{e}(n-\frac{n}{e})} = \frac{n^2\left(1-\frac{1}{e}\right)}{n+\frac{n^2}{e}\left(1-\frac{1}{e}\right)}$$

But

$$n^2\left(1-\frac{1}{e}\right) - n - \frac{n^2}{e}\left(1-\frac{1}{e}\right) > 0 \iff \left(\frac{e-1}{e}\right)^2 n > 1$$

Beside $\underset{e \in [\![2, n-1]\!]}{\operatorname{argmin}} \frac{e-1}{e} = \{e = 2\}$ and for $e = 2$, we have $\left(\frac{e-1}{e}\right)^2 n = \frac{n}{4} > 1\ \forall n \geqslant 5$. So for all $e \in [\![2, n]\!]$ and all $n \geqslant 5$:

$$\frac{n^2\left(1-\frac{1}{e}\right)}{n+\frac{n^2}{e}\left(1-\frac{1}{e}\right)} > 1 = a_{(m-1)}$$

Which is a contradiction. So inequality (A.2.7) cannot be true for a deviation from A^g to A for agent $i \in A_{m-1}$. Therefore, given $T^{(e)}$, there exists a profitable deviation from A^g to A. This shows that $(A^g, T^{(e)})$ is not contractually stable for any $e \in [\![2, n]\!]$ and any $n \geqslant 5$. ■

Proof of Proposition 8. Let us start by investigating the situation when the deviation happened on the technological coalition. The utility of an agent i in the pair of structures (A^g, T), for any $T \in \mathcal{T}$ is :

$$u_i^*(A^g, T) = \frac{V}{n} = u_i^*(A^g, T^g)$$

Thus, all the non-deviating partners are indifferent between the pairs of structures before and after the deviation. So the alliance structure will not be adjusted. Furthermore, as all the deviating agents are indifferent to deviate, every deviation from $(A = A^g, T = T^g)$ toward $(A = A^g, T')$ will be blocked by every deviating agents regardless of the δ-obtainable structure T' from T.

Let us now turn into the analysis of the deviations from the grand alliance $A = A^g$ when the global technological coalition is not adjusted. Given the technological structure $T = T^g$ and the fact that $\sum_{\ell=1}^m a_\ell = n$, the condition for a agent $i \in A(i)$ to block the deviation from (A^g, T^g) toward (A, T^g) for $A \in \mathcal{A}$ is :

$$\begin{aligned} u_i^*(A, T^g) = \left(1 - \frac{(m-1)a(i)}{\sum_{\ell=1}^m a_\ell}\right)\left(\frac{V}{a(i)} - \frac{V(m-1)}{a(i)\sum_{\ell=1}^m a_\ell}\right) &\leqslant \frac{V}{n} \\ \iff a(i) \geqslant \frac{n(n-m+1)}{m(n-m+2)-1} \end{aligned} \tag{A.2.8}$$

But one can notice that when $a(i) = 1$ and $a_{(m)} = n-1$ (the largest coalition in A is the non-deviating coalition), (A.2.8) is false for $n \geqslant 3$ as $1 \leqslant \frac{n-1}{2}$. Therefore, the deviation of i from A^g toward $A = \{\{i\}, N \backslash \{i\}\}$ where $a(i) = 1$ is always profitable for i.

If the non-deviating agents in A decide to exclude i from T^g, the technological structure will be adjusted to become $T_D^{\#} = (T^g)_D^{\#} = (N\backslash\{i\}, \{i\})$, where the deviating agent becomes singleton while the non-deviating agents stay together. In that situation, the utility of i is

$$u_i^*(A, T_D^{\#}) = V\left(1 - \frac{1}{2}\right)^2 = \frac{V}{2}$$

which is larger than $\frac{V}{n}$ for $n \geqslant 3$. Therefore, when $n \geqslant 3$, each agents have incentives to leave the grand alliance in (A^g, T^g) to become singleton, even if they are excluded from the global technological partnership. ■

Proof of Proposition 9. Let A be a δ-obtainable alliance from A^g with m alliances ($\text{card}(L) = m$), where i belong to an alliance of size a_j. In $T^{(e)}$, all the agents are equally efficient. So the utility of agent i in $(A, T^{(e)})$ without any exclusion on the technological structure, is as follows.

$$u_i(A, T^{(e)}) = V\left(1 - \frac{a_k(m-1)}{n}\right)\left(\frac{1}{a_k} - \frac{m-1}{na_k}\right)$$

Let us consider $A^{\#} = \{N\backslash\{i\}, \{i\}\}$, the δ-obtainable alliance from A^g where the only deviating agent i leave the grand coalition. Let suppose without loss of generality that i belong to the technological coalition T_1 in $T^{(e)}$. In that situation, the utility of i without any exclusion becomes :

$$u_i(A^{\#}, T^{(e)}) = V\left(1 - \frac{1}{n}\right)^2 < \frac{V}{n} \quad \forall n \geqslant 3$$

If i is excluded from her technological coalition, the technological structure becomes $T_D^{\#} = \{\{i\}, T_1\backslash\{i\}, T_2, \cdots, T_e\}$, where $t_2 = \cdots = t_e = \frac{n}{e}$. We have

$$\begin{aligned}
u_i(A^{\#}, T_D^{\#}) &= V\left(1 - \frac{1}{1 + \frac{e(e-1)(n-1)}{n} + \frac{e(n-1)}{n-e}}\right)^2 \\
&= V\left(\frac{e(e-1)(n-1)(n-e) + e(n-1)n}{e(e-1)(n-1)(n-e) + e(n-1)n + n(n-e)}\right)^2 \\
&:= V\left(\frac{d_1}{d_2}(e)\right)^2
\end{aligned}$$

As $d_2 > d_1 > 0$ and $\dfrac{d_1}{d_2}$ is strictly decreasing in $e \in [\![2, n]\!]$, we have

$$\frac{d_1}{d_2}(e) \geqslant \frac{d_1}{d_2}(2) = \frac{2(n-1)(n-2) + 2n(n-1)}{2(n-1)(n-2) + 2n(n-1) + n(n-2)} \geqslant \frac{1}{\sqrt{n}} \quad \forall n \geqslant 3$$

So i has incentives to leave A^g even if she is excluded from here technological coalition consequently to the deviation.

■

Proof of Proposition 10. Let us study the deviations taking place from the technological structure $T^{(e)}$. When the alliance structure is $A = A^{(e)}$, then $a_\ell = a = \frac{n}{e}$ for all $\ell \in [\![1, m = e]\!]$. The utility of an agent $i \in A_j$ can therefore be rewritten as follows.

$$u_i^*(A^{(e)}, T^{(e)}) = \begin{cases} \left[1 - \dfrac{a(e-1)}{a\displaystyle\sum_{\ell=1}^{e}\dfrac{\bar{t}_j}{\bar{t}_\ell}}\right]\left[\dfrac{Ve}{n} - \dfrac{V(e-1)}{a^2\displaystyle\sum_{\ell=1}^{e}\dfrac{\bar{t}_j}{\bar{t}_\ell}}\right] & \text{if } i \in B_j \\ \left[1 - \dfrac{a(e-1)}{a\displaystyle\sum_{\ell=1}^{e}\dfrac{\bar{t}_j}{\bar{t}_\ell}}\right]\dfrac{Ve}{n} & \text{if } i \in A_j \backslash B_j \end{cases}$$

So if $A^{(e)}$ remains unaffected by the deviation from $T^{(e)}$, the changes in the utility of every deviating agents only come from the change in the term $\sum_{\ell=1}^{e}\frac{\bar{t}_j}{\bar{t}_\ell}$. Moreover, the utility of any agent is strictly increasing with respect to $\sum_{\ell=1}^{e}\frac{\bar{t}_j}{\bar{t}_\ell}$.

For $e \geqslant 3$. Let us consider the deviation from $T^{(e)} = \{T_1, \cdots, T_e\}$ toward $T' = \{T^{(e)}\backslash\{T_j, T_k\}, T_j \cup T_k\}$, where the agents in T_j and T_k are merging their technological coalitions. Since $A^{(e)}$ is symmetric with respect to $T^{(e)}$, there is no non-deviating agent that are allies with the deviating agents. So no exclusion can be triggered. So $A^{(e)}$ remains unchanged in response to a deviation from $T^{(e)}$ toward T'. On the one hand, in $(A^{(e)}, T^{(e)})$ we have $\sum_{\ell=1}^{e}\frac{\bar{t}_j}{\bar{t}_\ell} = e \cdot 1 = e$ for all $j \in [\![1, e]\!]$. On the other hand, in $(A^{(e)}, T')$ we have for the members of T_j : $\sum_{\ell=1}^{e}\frac{\bar{t}_j}{\bar{t}_\ell} = (e-2)2 + 2$. But $(e-2)2 + 2 - e > 0 \iff e > 2$. Thus, for $i \in T_j$ in T', we have $u_i^*(A^{(e)}, T^{(e)}) < u_i^*(A^{(e)}, T')$. By a symmetric reasoning, we have $u_j^*(A^{(e)}, T^{(e)}) < u_j^*(A^{(e)}, T')$ for any agent $j \in T_k$ in T'. So for $e \geqslant 3$, the agents in $T^{(e)}$ have always incentives to merge two technological coalitions. Therefore, for $e \geqslant 3$, the pair $(A^{(e)}, T^{(e)})$ is not Δ-stable

For $e = 2$. The pair $\left(A^{(2)} = \{A_1, A_2\}, T^{(2)} = \{T_1, T_2\}\right)$ is composed of symmetric structures of two coalitions of equal size. Unlike in the previous case, there is no incentive for any agent to merge the two technological coalitions to form the global technological coalition.[40] As all the agents are members from alliances and technological partnerships of the same size, we can assume without loss of generality that $i \in A_1$ is a member of T_1. Let us consider the deviation from $A^{(2)}$ toward $A = \{\{i\}, A_1\backslash\{i\}, A_2\}$. Let $j \in A_1\backslash\{i\}$. The utility of j in $(A, T^{(2)})$ is

$$\begin{aligned} u_j^*(A, T^{(2)}) &= V\left(1 - \frac{2\left(\frac{n}{2}-1\right)}{n}\right)\left(\frac{1}{\frac{n}{2}-1} - \frac{2}{\frac{n}{2}n}\right) \\ &= V\left(1 - \frac{n-2}{n}\right)\left(\frac{2}{n-2} - \frac{4}{n^2}\right) \leqslant V\left(\frac{n-1}{n^2}\right) = u_j^*(A^{(2)}, T^{(2)}) \quad \forall n \geqslant 6 \end{aligned}$$

[40] $u_i^*(A^{(2)}, T^{(2)}) = \frac{V(n-1)}{n^2} \geqslant u_i^*(A^{(2)}, T^g) = \frac{V}{2n}\ \forall n \geqslant 2$.

So the technological structure will be adjusted to become $T_D^{\#} = \{\{i\}, T_1 \backslash \{i\}, T_2\}$ when $n \geqslant 6$. In that case, the utility of agent i is as follows

$$u_i(A, T_D^{\#}) = \frac{V}{9} > V\left(\frac{n-1}{n^2}\right) \quad \forall n \geqslant 8$$

Eventually, this shows that the deviation from $(A^{(2)} = \{A_1, A_2\}, T^{(2)})$ toward $(A_D^{\#} = \{A_1^s, A_2\}, T''')$ is profitable for all the deviating agents for $n \geqslant 8$. So $(A^{(2)}, T^{(2)})$ is not Δ-stable when $n \geqslant 8$. ■

Proof of Remark 2. When $A = A^g$, all the agents are indifferent between every structure T. Therefore, no exclusion occurs in the alliance structure resulting from a deviation from T^g.
Let us study the deviations from A^g toward some δ-obtainable structure A'. Let $i \in A(i) = A_j$ be a non-deviating agent. We have

$$u_i^*(A', T^g) = \frac{V}{a(i)}\left(1 - \frac{a(i)(m-1)}{n}\right)\left(1 - \frac{m-1}{n}\right)$$

If the exclusions are triggered, the only possible adjusted technological structure is $T_D^{\#} = \{N \backslash D, D^s\}$, where D is the set of deviating agents. After the deviation from A^g, all the non-deviating agents are grouped in one alliance. Moreover, all these non-deviating agents are equally efficient regardless of the occurrence of exclusions. Therefore, once the exclusions occur, we have $\bar{t}_j = a(i)$ and $b_j = a(i)$ for a non-deviating agent $i \in A(i) = A_j$. It follows:

$$u_i^*\left(A', T_D^{\#}\right) = \frac{V}{a(i)}\left(1 - \frac{a(i)(m-1)}{a(i)(1+n-a(i))}\right)\left(1 - \frac{m-1}{a(i)(1+n-a(i))}\right)$$

One can notice that $a(i)(1+n-a(i)) \geqslant n$ for any $n \in \mathbb{N}^*$ as $a(i) < n$ for any δ-obtainable structure A' from A^g. So, we have $u_i^*(A', T^g) \leqslant u_i^*\left(A', T_D^{\#}\right)$ $\forall n \in \mathbb{N}^*$. ■

Proof of Property 3. Let $A \in \mathcal{A}$ be an arbitrary alliance structure composed of m alliances. We start by computing the equilibrium value of $u_i^*(A, T^{(e)})$ given $(A, T^{(e)})$ for $e \geqslant 1$. In $T^{(e)} = \{T_1, \cdots, T_e\}$, we have $t_1 = \cdots = t_e := t$. Then,

$$u_i(A, T^{(e)}) = \frac{X_j}{X}\frac{V}{a_j} - \left(\frac{x_i}{t}\right)^2$$

where a_j denotes the size of the alliance of i. The first order condition is

$$x_i = \frac{X - X_j}{X^2}\frac{Vt^2}{2a_j} \quad \forall i \in A_j \tag{A.2.9}$$

Summing over the coalition A_j, we obtain

$$X_j = \frac{Vt^2 X}{2X^2 + Vt^2} \tag{A.2.10}$$

As all the agents are equally efficient, we can sum (A.2.10) over $j \in [\![1, m]\!]$:

$$X = (m-1)\frac{Vt^2}{2X} \iff X^* = t\sqrt{\frac{V(m-1)}{2}}$$

From (A.2.10) it follows $X_j^* = \dfrac{Vt^2 X^*}{2(X^*)^2 + Vt^2} = \dfrac{t}{m}\sqrt{\dfrac{V(m-1)}{2}}$ and from (A.2.9) $x_i^* = \dfrac{X^* - X_j^*}{(X*)^2}\dfrac{Vt^2}{2a_j} = \dfrac{t}{ma_j}\sqrt{\dfrac{V(m-1)}{2}}$. Eventually we have

$$u_i^*(A, T^{(e)}) = \frac{X_j^*}{X^*}\frac{V}{a_j} - \left(\frac{x_i^*}{t}\right)^2 = \frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right)$$

Let us denote by $u_i^*(m)$ the utility of agent i in $(A, T^{(e)})$ for $A \in \mathcal{A}$ with respect to m. Let $m' < m$. We have

$$\frac{u_i^*(m')}{u_i^*(m)} = \frac{m}{m'}\frac{2a_j - 1 + \frac{1}{m'}}{2a_j - 1 + \frac{1}{m}} > 1$$

for any $m' < m$. So u_i^* is strictly decreasing in m.
Let $A' = A \backslash \{A_1, A_2\} \cup (A_1 \cup A_2)$, where A is composed of m alliances. So $m' = m-1$ is the number of alliances in A'. Since u_i^* is strictly decreasing in m, we have $u_i^*(A, T^{(e)}) < u_i^*(A', T^{(e)})$.

■

Proof of Proposition 11. When $e = n$, the equilibrium utility of agent $i \in A_j$ with the pair (A, T^s) can be obtained from the computations in the proof of Property 3. Let $A' = (A \backslash A_j) \cup (A_j \backslash \{i\}) \cup \{i\}$ be the structure where i unilaterally leave her alliance A_j in A to become a singleton. The utility of i in A' is :

$$u_i^*(A', T^s) = \frac{V}{m+1}\left(1 - \frac{m}{2(m+1)}\right)$$

It follows

$$\operatorname{sgn}\left(u_i^*(A, T^s) - u_i^*(A', T^s)\right) = -\operatorname{sgn}\left(a_j^2\frac{m+2}{2(m+1)^2} - \frac{a_j}{m} + \frac{m-1}{2m^2}\right)$$

The roots of the convex polynomial $P[a_j] = a_j^2\frac{m+2}{2(m+1)^2} - \frac{a_j}{m} + \frac{m-1}{2m^2}$ are

$$a_j(m) = \frac{(m+1)^2}{m(m+2)}\left(1 \pm \sqrt{1 - \frac{(m+2)(m-1)}{(m+1)^2}}\right) \leqslant a_j(m=2) \in \{0.04, 0.25\} < 1$$

So for any $m \geqslant 2$, we have $u_i^*(A, T^s) - u_i^*(A', T^s) < 0$. Therefore, any agent i has always incentives to leave her alliance to become singletons.

Moreover,

$$u_i(A^s, T^s) = \frac{V}{n}\left(1 - \frac{n-1}{2n}\right) < \frac{V}{n} \quad \forall n \geqslant 2$$

This shows that no alliance is δ-stable when $T = T^s$ with quadratic costs.

■

Proof of Proposition 12. From the proof of Property 3, the utility of $i \in A_j$ in $(A, T^{(e)})$ for $e \geqslant 1$ is

$$u_i^*(A, T^{(e)}) = \frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right)$$

Let $A' = \{N \backslash \{i\}, \{i\}\}$ be the alliance where i is singleton and all the remaining agent are grouped together. In such an alliance, $a_j = a(i) = 1$ and $m = 2$. So we have,

$$u_i^*(A', T^{(e)}) = V\frac{3}{8} \geqslant \frac{V}{n} = u_i^*(A^g, T^{(e)}) \ \forall \ n \geqslant 3$$

So every agent has incentives to leave the grand alliance in the pair of structures $(A^g, T^{(e)})$.

■

Proof of Proposition 13. When the alliance structure is the grand alliance structure, for any any technological structure $T \in \mathcal{T}$, we have for any agent i belonging to the technological coalition of size $t(i)$:

$$u_i(A^g, T) = \frac{V}{n} - \left(\frac{x_i}{t(i)}\right)^2$$

In that case, $x_i^* = 0$ for every $i \in N$. Therefore, every agent will block every deviation from T^g when the alliance structure is A^g.

Now, the deviations in the alliance structure when $T = T^g$ are going to be investigated. Let us start by considering the case where there is only deviating agents, $D = N$.

No exclusion are to be feared in that situation as there is no non-deviating agent. Let $i \in A_j$ be a agent from the largest coalition. From the proof of Proposition 11, the utility of i in (A, T^g) is (case $e = 1$)

$$u_i^*(A, T^g) = \frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right)$$

The agent i will block a deviation from A^g to $A \in \mathcal{A}$ if and only if

$$\frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right) \leqslant \frac{V}{n} = u_i^*(A^g, T^g) \tag{A.2.11}$$

Since $\frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right)$ is strictly decreasing with respect to both $m \geqslant 2$ and $a_j \geqslant 1$, the largest value for $u_i^*(A, T^g)$ is either when $m = 2$ or when $a_j = 1$.

When $m = 2$, the largest coalition in A is at least of size $\frac{n}{2}$. When $a_j = 1$, this means that $m = n$ as a_j is the size of the largest alliance. So the alliance structures for whom $u_i^*(A, T^g)$ reaches its maximum are either $A' = \{A_1, A_2\}$ where $a_1 = a_2 = \frac{n}{2}$, or $A'' = A^s$.

With A', we have

$$u_i^*(A', T^g) = \frac{V}{n}\left(1 - \frac{1}{2n}\right) < \frac{V}{n} = u_i^*(A^g, T^g) \ \ \forall n \geqslant 1$$

as $\frac{1}{2n} \in]0,1[$ for all $n \in \mathbb{N}^*$.
For A^s, we have

$$u_i^*(A^s, T^g) = \frac{V}{n}\left(1 - \frac{n-1}{2n}\right) < \frac{V}{n} \quad \forall n \geqslant 2$$

Ergo (A.2.11) is true for any $A \neq A^g$ and any $n \geqslant 2$.

Now, let us consider the situation where some agents do not deviate from A^g toward some alliance structure $A \in \mathcal{A}$, $D \subsetneq N$. In that case, if the largest coalition in A is composed of deviating agents, the previous case shows that these agents will block the deviation if the technological structure is not adjusted. Moreover, if the largest alliance in A is the coalition of non-deviating agents[41], they will all suffer from the deviation so that the technological structure will be adjusted. We only need to show that if the technological structure is adjusted, some deviating agents will block the deviation from A^g toward A.

The adjusted technological structure from T^g is $T_D^{\#} = \{N \backslash D, D^s\}$. Let $k \in N\backslash D$ be a member of the alliance A_ℓ and $i \in D^s$ be a member of A_j in $A \in \mathcal{A}$. Let $\text{card}(N\backslash D) = n - d := \alpha$. We have

$$\begin{aligned} u_i(A, T_D^{\#}) &= \frac{X_j}{X}\frac{V}{a_j} - x_i^2 \\ u_k(A, T_D^{\#}) &= \frac{X_j}{X}\frac{V}{a_j} - \left(\frac{x_i}{\alpha}\right)^2 \end{aligned}$$

The equilibrium conditions are

$$\begin{cases} x_i &= \dfrac{X - X_j}{X^2}\dfrac{V}{2a_j} \\ x_k &= \dfrac{X - X_\ell}{X^2}\dfrac{V\alpha^2}{2a_\ell} \end{cases}$$

Summing condition on x_i (resp. x_k) over A_j (resp. A_ℓ) we obtain

$$X_j = \frac{VX}{2X^2 + V} \tag{A.2.12}$$

$$X_\ell = \frac{V\alpha^2 X}{2X^2 + V\alpha^2} \tag{A.2.13}$$

From (A.2.12),

$$\sum_{j \neq \ell} X_j = \frac{VX(m-1)}{2X^2 + V}$$

Together with (A.2.13) it yield

[41] In any deviation from A^g to some obtainable alliance A, the non-deviating agents, if any, are grouped into one single alliance in A.

$$X = \sum_{j \neq \ell} X_j + X_\ell = 4x^2 - 2xV(m-2) - V^2\alpha^2(m-1) = 0$$

where $x := X^2$. The only positive real root of this polynomial is $x_+ = \frac{V(m-2)}{2} + \frac{1}{4}\sqrt{V^2(m-2)^2 + 4V^2\alpha^2(m-1)}$. Hence,

$$X^* = \sqrt{x_+} = \sqrt{\frac{V}{4}\left[(m-2) + \sqrt{(m-2)^2 + 4\alpha^2(m-1)}\right]}$$

So from (A.2.12) and (A.2.13), it follows

$$\begin{cases} x_i^* &= \dfrac{VX^*}{a_j(2(X^*)^2 + V)} \\ x_k^* &= \dfrac{V\alpha^2 X^*}{a_\ell(2(X^*)^2 + V\alpha^2)} \end{cases}$$

Then,

$$u_i(A, T^{\#}) = \frac{V^2}{a_j(2(X^*)^2 + V)}\left[1 - \frac{(X^*)^2}{a_j(2(X^*)^2 + V)}\right]$$

Let A_j be the largest alliances composed of deviating agents. As the technological structure is adjusted, we have $d \in [\![1, n-1]\!]$. Moreover, $m \in [\![2, n]\!]$. We know that $2(X^*)^2 + V = \frac{V}{2}\left(m + \sqrt{(m-2)^2 + 4(n-d)^2(m-1)}\right) := \frac{V}{2}\zeta(m, d)$. Therefore the utility of $i \in A_j$ in A can be written as follows.

$$u_i(A, T^{\#}) = \frac{2}{a_j\zeta}\left(1 - \frac{1}{2a_j} + \frac{1}{a_j\zeta}\right) := g(m, d, a_j)$$

But $\zeta(m, d)$ is strictly increasing with respect to m and strictly decreasing with respect to d as $d < n$. So $g(m, d, a_j)$ is strictly decreasing in m and strictly increasing in d. Thence we have

$$\max_{\substack{m \in [\![2,n]\!] \\ d \in [\![1,n-1]\!]}} g(m, d, a_j) = g(2, n-1, a_j)$$

Furthermore, a_j is the size of the largest coalition of deviating agents. So when $m = 2$ and $d = n-1$, we necessary have $a_j = n-1$: we are in $A = \{N \backslash \{s\}, \{s\}\}$ where the agent s is the only non-deviating agent. In that case, $\zeta(m = 2, d = n-1) = 4$ and

$$g(m, d, a_j) \leqslant g(2, n-1, n-1) = \frac{1}{2(n-1)}\left(1 - \frac{1}{4(n-1)}\right) < \frac{1}{n} \text{ for any } n \geqslant 2$$

So for all $m \in [\![2, n]\!]$, all $d \in [\![1, n-1]\!]$, and any $n \geqslant 2$, we have for every $A \in \mathcal{A}$

$$u_i(A, T^{\#}) \leqslant u_i(A^g, T^g) = \frac{V}{n}$$

When the technological structure is adjusted due to a deviation in the alliance structure, the deviating agents belonging to the largest deviating alliance will always block the deviation from A^g.

Eventually, we show that every deviation from (A^g, T^g) in the alliance and the technological structures are going to be block under the threat of exclusion. This means that (A^g, T^g) is Δ-stable when the costs are quadratic and $n \geqslant 2$.

■

Proof of Proposition 14. First of all, when $A = A^g$, the equilibrium utility of every agent is $\frac{V}{n}$ for any $T \in \mathcal{T}$ regardless whether the costs are linear or quadratic.[42] Therefore, every agent is indifferent between $T^{(e)}$ and any other technological coalition T. So when $A = A^g$, all the deviations from $T^{(e)}$ are blocked.

Let us now investigate the deviation from A^g when $T = T^{(e)}$.

We start with the linear costs model. Let $A \in \mathcal{A}$ be an arbitrary coalition structure. For any deviation from A^g to A, all the non-deviating agents end up singletons in A. Therefore, if there is a non singleton alliance in A it is necessary an alliance composed of deviating agents. Let i be a deviating agent from the largest alliance in A of size a_j. Let $T = T^{(e)}$. The utility of i in $(A, T^{(e)})$ is (cf Proposition 2)

$$u_i^*(A, T^{(e)}) = \frac{V}{a_j}\left(\frac{1}{a_j} - \frac{m-1}{n}\right)\left(\frac{n-m+1}{n}\right)$$

In $T^{(e)}$ all agents are equally efficient. So given a_j, the most profitable alliance structure for i, $A^{\#}$, is the structure where all the agents outside a_j are grouped together by Property 1. However, since $a_j \geqslant a_k$ for any $k \in [\![1, m]\!]$, we have

$$A^{\#} = \{A_1, A(i)\}, \text{ where } a_1 = a_j = \frac{n}{2}$$

So it follows

$$u_i^*(A, T^{(e)}) \leqslant u_i^*(A^{\#}, T^{(e)}) = \frac{V}{n}\frac{n-1}{n} < \frac{V}{n} = u_i^*(A^g, T^{(e)}) \ \ \forall n \geqslant 2$$

So i has always incentives to block the deviations from A^g when $T = T^{(e)}$.

Now, let us consider the model with quadratic costs. The utility of i in $(A, T^{(e)})$ is (cf proof of Property 3)

$$u_i^*(A, T^{(e)}) = \frac{V}{ma_j}\left(1 - \frac{m-1}{2ma_j}\right)$$

All the agents are equally efficient in $T^{(e)}$. So exactly as with the linear costs, given a_j, the most profitable alliance structure for i is $A^{\#}$ by Property 3. Then

[42] See Proposition 2 for linear costs and the proof of Property 3 for quadratic costs with $m = 1$ and $a_j = n$.

$$u_i^*(A, T^{(e)}) \leqslant u_i^*(A^{\#}, T^g) = \frac{V}{n}\left(\frac{2n-1}{2n}\right) < \frac{V}{n} = u_i^*(A^g, T^{(e)}) \quad \forall n \geqslant 2$$

With quadratic costs, the agents from the largest alliance also has incentives to block the deviations from A^g when $T = T^{(e)}$.

Therefore, regardless on the assumptions on the costs, $(A^g, T^{(e)})$ is γ-stable for any $e \in \mathbb{N}^*$ and any $n \geqslant 2$. ■

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