

Quantisation and Gauge Invariance

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Abstract

Recent developments concerning canonical quantisation and gauge invariant quantum mechanical systems and quantum field theories are briefly discussed. On the one hand, it is shown how diffeomorphic covariant representations of the Heisenberg algebra over curved manifolds of non trivial topology involve topology classes of flat U(1) bundles. On the other hand, through some examples, the recently proposed physical projector approach to the quantisation of general gauge invariant systems is shown to avoid the necessity of any gauge fixing—hence also avoiding the possibility of Gribov problems which usually ensue any gauge fixing procedure—, and is also capable to provide the adequate description of the physical content of gauge invariant systems.

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1 Introduction

Two of the most fundamental tenets of modern physics at the turn of this century are, on the one hand, the general concepts of quantum physics, and on the other hand, the general idea of local or gauge symmetries as the governing principle for all interactions and their properties as well as for their ultimate unification.

Nevertheless, the combination of these two general conceptual frameworks is not straightforward, and is in fact beset by a series of difficult issues. For example, a manifest realisation of a local symmetry requires the introduction of gauge variant degrees of freedom which cannot be associated to physical, *i.e.* gauge invariant observables. Maintaining such degrees of freedom explicit through the quantisation procedure requires tools which in effect cancel the contributions of these non physical configurations to physical observables. On the other hand, reducing these gauge variant degrees of freedom before quantisation often runs counter to a manifest realisation of other symmetries, such as global spacetime symmetries as for example Poincaré invariance, while more importantly still, such a reduction implies a gauge invariant configuration space description whose topology and geometry is typically very intricate and difficult to circumvent in general terms, thereby leading to the possibility of Gribov problems[1]. At the quantum level, the difficulty is then exacerbated still further by the fact that quantisation techniques for manifolds of non trivial topology and geometry are much more involved to develop than for the ordinary and most familiar case of a flat euclidean space.

In this contribution, we wish to discuss two recent developments concerned with such issues. On the one hand, the general classification of representations of the Heisenberg algebra over configuration space manifolds of arbitrary topology and geometry has recently been considered[2], leading to the possibility of inequivalent representations associated to all topologically distinct flat $U(1)$ bundles which may be defined for a base manifold whose first homotopy group is non trivial.

On the other hand, a new approach to the quantisation of constrained systems, which include gauge invariant ones, has recently been suggested[3]. This approach is formulated solely within Dirac's quantisation of such systems[1], does not require any extension (such as in BRST quantisation[1]) nor any reduction (such as in Faddeev's reduced phase space quantisation[1]) of the initial formulation of the system, and uses in an essential way the physical projection operator onto gauge invariant quantum states only. Since the latter operator is defined through integration over the manifold of the group of gauge transformations, it lends itself to general group theoretical methods for the evaluation of its matrix elements, which are also the generating functions for the wave functions of gauge invariant states. Some of the advantages of this new approach to quantisation have already been illustrated in some simple examples, showing[4] among other important points, that this method being free of the necessity of any gauge fixing procedure is also necessarily free of any Gribov of any type[1], a situation which is not achieved with any of the other quantisation methods. Until now, the physical projector approach to gauge invariant systems has been applied to some simple quantum mechanical systems[5, 6] as well as to pure $U(1)$ Chern-Simons theory in 2+1 dimensions[7], one of the simplest examples of a topological quantum field theory[8]. It would indeed be most worthwhile to explore the potential of this new approach to the quantisation of gauge invariant systems in situations much closer to actual theories for the fundamental interactions, either field theories for gravity[9] or the strong and electroweak interactions, as well as any of the systems motivated by studies within string and M-theory[10].

This contribution is organised as follows. In the next section, the discussion of configuration space representations of the Heisenberg algebra over a manifold of arbitrary topology and geometry is recalled. In Sect.3, Dirac's general Hamiltonian formulation of gauge invariant systems is briefly described, at least in the simplest cases of such systems, concluding with the construction of the physical projector. Sect.4 then considers the application of the physical projector to some simple

quantum mechanical gauge invariant systems, showing how the physical spectrum as well as the wave functions of physical states follow straightforwardly, while also circumventing any possibility of a Gribov problem. In Sect.5, the physical projector is brought to bear on the quantisation of the pure U(1) Chern-Simons theory in 2+1 dimensions, leading to a physical spectrum totally independent of the geometry of the 2+1 dimensional space but depending solely on its topology, and in complete agreement with results achieved in the literature through standard quantisation methods[11]. Finally, some further remarks are presented in the Conclusions. By its nature, the presentation of this contribution can only be sketchy; further details are to be found in the original references[2, 5, 6, 7].

2 Representations of the Heisenberg Algebra

The representation theory of the Heisenberg algebra associated to the real line, and by extension to any flat euclidean space parametrised by cartesian coordinates, is most familiar to anyone having studied quantum theory. Denoting q^α ($\alpha, \beta = 1, 2, \dots, n$) the cartesian coordinates and p_α their canonical conjugate momenta, this algebra is defined by the following commutation relations at the operator level,

$$[\hat{q}^\alpha, \hat{p}_\beta] = i\hbar\delta_\beta^\alpha \quad , \quad (1)$$

as well as the self-adjoint properties $(\hat{q}^\alpha)^\dagger = \hat{q}^\alpha$ and $\hat{p}_\alpha^\dagger = \hat{p}_\alpha$. In particular, up to unitary transformations, there exists essentially a single such representation, corresponding to the usual plane wave representation of that algebra (von Neumann's theorem). However, when the configuration space coordinate system q^α is curvilinear, or when the configuration space possesses a non trivial geometry or even topology, usual canonical quantisation procedures for the associated symplectic Poisson bracket structure through the correspondence principle—leading to the above commutation relations defined locally over phase space—seem to have remained an open issue. However, in the same way as may be developed for an euclidean space[1], the representation theory of the above Heisenberg algebra in the general case may straightforwardly be considered[2].

Indeed, only two basic assumptions are required. On the one hand, that the representation space possesses as a basis all eigenstates $|q\rangle$ of the position operators $\hat{q}^\alpha$ associated to a local coordinate system set-up on the configuration space M , namely $\hat{q}^\alpha|q\rangle = q^\alpha|q\rangle$. On the other hand, that this representation space is equipped with an hermitian positive definite inner product for which the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ are indeed self-adjoint.

From only these two assumptions, it then follows[2] that the general configuration space representation of the Heisenberg algebra is such that the momentum operator matrix elements are given by

$$\langle q|\hat{p}_\alpha|q'\rangle = \frac{-i\hbar}{g^{1/4}(q)} \frac{\partial}{\partial q^\alpha} \left(\frac{1}{g^{1/4}(q)} \delta^{(n)}(q - q') \right) + \frac{1}{\sqrt{g(q)}} A_\alpha(q) \delta^{(n)}(q - q') \quad , \quad (2)$$

while the vanishing commutation relations $[\hat{p}_\alpha, \hat{p}_\beta] = 0$ imply the further restriction

$$A_{\alpha\beta}(q) \equiv \frac{\partial A_\beta(q)}{\partial q^\alpha} - \frac{\partial A_\alpha(q)}{\partial q^\beta} = 0 \quad , \quad \alpha, \beta = 1, 2, \dots, n \quad . \quad (3)$$

In these relations, the positive definite function $g(q)$ specifies the normalisation of the position eigenstates $|q\rangle$ through

$$\langle q|q'\rangle = \frac{1}{\sqrt{g(q)}} \delta^{(n)}(q - q') \quad , \quad (4)$$

while the variables $A_\alpha(q)$ corresponds to a local vector field of vanishing field strength $A_{\alpha\beta}(q)$ defined over the configuration space.

In particular for any quantum states $|\psi\rangle$ and $|\varphi\rangle$, their configuration space wave function representations $\psi(q) \equiv \langle q|\psi\rangle$ and $\varphi(q) \equiv \langle q|\varphi\rangle$ are such that their inner product is given by

$$\langle \psi|\varphi \rangle = \int_M d^n q \sqrt{g(q)} \psi^*(q) \varphi(q) \quad , \quad (5)$$

while the position and momentum operators are realised through the differential operators,

$$\langle q|\hat{q}^\alpha|\psi\rangle = q^\alpha \psi(q) \quad , \quad \langle q|\hat{p}_\alpha|\psi\rangle = \frac{-i\hbar}{g^{1/4}(q)} \left[\frac{\partial}{\partial q^\alpha} + \frac{i}{\hbar} A_\alpha(q) \right] g^{1/4}(q) \psi(q) \quad . \quad (6)$$

The two quantities $g(q)$ and $A_\alpha(q)$ thus parametrise all possible representations of the Heisenberg algebra over M . However, not all these representations are necessarily unitarily inequivalent. Indeed, even though $g(q)$ parametrises the normalisation of the basis states $|q\rangle$ and $A_\alpha(q)$ the configuration space matrix elements of the momentum operators $\hat{p}_\alpha$, we have not yet accounted for the freedom in a possible q -dependent phase redefinition of the basis states, namely $|q\rangle_{(2)} = e^{i\chi(q)/\hbar} |q\rangle$. In fact, such a local phase redefinition is tantamount to a local U(1) gauge transformation of the vector field $A_\alpha(q)$ which is defined through the above parametrisation of the $\hat{p}_\alpha$ matrix elements, from which one finds $A_\alpha^{(2)}(q) = A_\alpha(q) + \partial\chi(q)/\partial q^\alpha$.

Hence in conclusion, the variables $A_\alpha(q)$ of vanishing field strength determine a flat U(1) bundle over the configuration space manifold M , while unitarily inequivalent representations of the Heisenberg algebra over M are thus classified in terms of the gauge equivalence classes of flat U(1) bundles over M . The latter classes are characterized through the U(1) holonomies around non contractible cycles in M , namely a mapping of the first homotopy group (or fundamental group) of M into the set of gauge equivalence classes of flat U(1) connections over M , characterized through their holonomies. In particular for a manifold of trivial homotopy group, namely a simply connected one as is the case for the n dimensional euclidean space, there is thus a single representation of the Heisenberg algebra, up to local U(1) unitary phase transformations in configuration space. This conclusion of course corresponds to von Neumann's theorem for the real line, while choosing then the trivial gauge configuration $A_\alpha(q) = 0$ in the associated trivial homology class provides the usual plane wave representation of that algebra.

This conclusion still leaves open the interpretation of the normalisation factor $g(q)$. From (5), it is clear that through the combination $d^n q \sqrt{g(q)}$ this function determines an integration measure over M . In particular, when this integration measure is chosen to be diffeomorphic invariant under coordinate reparametrizations in M , the associated configuration space wave function representation $\psi(q) = \langle q|\psi\rangle$ is itself diffeomorphic covariant, with the position and momentum operators represented as in (6). When M is equipped with a metric structure $g_{\alpha\beta}(q)$, the canonical choice for $g(q) = \det g_{\alpha\beta}(q)$ indeed determines such a diffeomorphic invariant integration measure, thereby leading finally to diffeomorphic covariant representations of the Heisenberg algebra over manifolds of arbitrary topology and geometry. An immediate example is that of curvilinear coordinates in a flat euclidean space, but more involved cases may of course be considered in a likewise manner.

Finally, the condition of self-adjoint momentum operators $\hat{p}_\alpha$ requires that the associated wave function representation of states be such that

$$\int_M d^n q \frac{\partial}{\partial q^\alpha} \left[\sqrt{g(q)} |\psi(q)|^2 \right] = 0 \quad , \quad \alpha = 1, 2, \dots, n \quad , \quad (7)$$

thereby implying restrictions on states when M has boundaries.

Given such a configuration space representation of the Heisenberg algebra, it is also possible to determine the wave functions for momentum eigenstates. However, because of the possibility of non trivial U(1) holonomies in the general case, a network of paths $P(q_0 \rightarrow q)$ connecting any point

q on M to a given point q_0 through a given path, has to be set-up on M in a continuous fashion as a function of q . Having done so, one introduces the path-ordered $U(1)$ holonomies

$$\Omega[P(q_0 \rightarrow q)] = P e^{-\frac{i}{\hbar} \int_{P(q_0 \rightarrow q)} dq^\alpha A_\alpha(q)} \quad , \quad (8)$$

so that momentum eigenstate configuration space wave functions are given by[2]

$$\langle q|p \rangle = \frac{e^{i\varphi(q_0,p)}}{(2\pi\hbar)^{n/2}} \frac{\Omega[P(q_0 \rightarrow q)]}{g^{1/4}(q)h^{1/4}(p)} e^{\frac{i}{\hbar}(q-q_0)\cdot p} \quad , \quad (9)$$

where $\varphi(q_0,p)$ is an arbitrary phase factor, while the function $h(p)$ determines the normalisation of the momentum eigenstates according to

$$\langle p|p' \rangle = \frac{1}{\sqrt{h(p)}} \delta^{(n)}(p - p') \quad . \quad (10)$$

Clearly, in the case of the trivial representation with $A_\alpha(q) = 0$, one recovers the usual plane wave representation of the Heisenberg algebra, extended to include the normalisation factors $g(q)$ and $h(p)$ possibly different from unity in order to ensure proper diffeomorphic covariant properties in the case of non cartesian coordinates over M .

Given these different expressions, it then also becomes possible to set-up a phase space path integral representation of matrix elements of quantum operators, say in configuration space. The ensuing expressions are identical to the usual ones in discretized form, the sole difference appearing through the normalisation and the path dependency of the external states (see the original reference[2] for further details).

Once such a general discussion of representations of the Heisenberg algebra over configuration spaces of arbitrary topology and geometry has been displayed, it is a simple matter to apply it to any given system whose dynamics is determined through some action principle. Based on the latter, the canonical Hamiltonian formulation of the system may be developed, which through canonical quantisation then leads to a certain quantum representation of the quantised system. In the case of a configuration space of non trivial first homotopy group, the choice of representation parametrised in terms of non trivial $U(1)$ holonomies of the flat bundle around the non contractible cycles in M is a matter of physics, in the same way that for example the spin representation of a quantised system invariant under space rotations is a matter of physics. It is also at the level of the Hamiltonian that a possible metric structure over M may appear and thus determine the adequate reparametrisation covariant representation of the Heisenberg algebra to be used. Finally, in correspondence with the given classical Hamiltonian, a two parameter class of hermitian (and possibly self-adjoint) quantum Hamiltonian operators may be introduced[2], which all reduce to the classical one in the limit $\hbar \rightarrow 0$. Again, further developments are left for the interested reader to find in the original reference. The main point this contribution wished to emphasize is the general classification of configuration space representations of the Heisenberg algebra over a manifold of arbitrary topology and geometry in terms of topology and gauge equivalence classes of flat $U(1)$ bundles over that manifold.

3 Gauge Invariant Systems and the Physical Projector

The analysis of the Hamiltonian formulation of constrained systems goes back to Dirac[1] and is very general. Here, only the simplest possible situation will be outlined explicitly, corresponding essentially to systems with gauge symmetries of the Yang-Mills type. Moreover, we shall also assume that all degrees of freedom are Grassmann even variables which thus commute at the classical level.

As is well known, when a system possesses continuous (global) symmetries, by Noether's theorem there exist conserved quantities generating these symmetries, whose Poisson brackets among one another also determine the Lie algebra of the symmetry group, while their Poisson brackets with the Hamiltonian of the system close among themselves. More explicitly, if $H(q, p)$ denotes the latter Hamiltonian while $\phi_a(q, p)$ stand for the symmetry generators, one has

$$\{\phi_a(q, p), \phi_b(q, p)\} = C_{ab}^c \phi_c(q, p) \quad , \quad \{H(q, p), \phi_a(q, p)\} = C_a^b \phi_b(q, p) \quad . \quad (11)$$

Here, C_{ab}^c and C_a^b are specific coefficients assumed to be constants in our discussion (more generally however, these coefficients may be phase space dependent quantities[1]).

In the case of local or gauge symmetries, these properties of the conserved quantities are of course preserved, but in addition, the physical requirement of gauge invariance implies further that these conserved quantities must vanish identically, $\phi_a(q, p) = 0$, for gauge invariant configurations. Moreover, the dynamics of the system in phase space is then governed by the action principle (assuming (q^α, p_α) to define canonically conjugate degrees of freedom[1])

$$S[q, p; \lambda^a] = \int_{t_i}^{t_f} dt [\dot{q}^\alpha p_\alpha - H(q, p) - \lambda^a \phi_a(q, p)] \quad , \quad (12)$$

where $\lambda^a(t)$ are arbitrary Lagrange multipliers for the gauge generator constraints $\phi_a = 0$, which also parametrise the gauge freedom generated by ϕ_a under time evolution. In particular, infinitesimal gauge transformations then correspond to the variations

$$\delta_\epsilon q^\alpha = \{q^\alpha, \epsilon^a \phi_a\} \quad , \quad \delta_\epsilon p_\alpha = \{p_\alpha, \epsilon^a \phi_a\} \quad , \quad \delta_\epsilon \lambda^a = \dot{\epsilon}^a + \lambda^c \epsilon^b C_{bc}^a - \epsilon^b C_b^a \quad , \quad (13)$$

where $\epsilon^a(t)$ are arbitrary infinitesimal functions.

Within Dirac's quantisation of such systems[1], the usual rules of canonical quantisation are simply applied to all the phase space degrees of freedom (q^α, p_α) , leading *a priori* to representation algebras of the Heisenberg type as discussed in the previous section. The restrictions of the gauge invariance are then imposed through the operator constraints

$$\hat{\phi}_a |\psi\rangle = 0 \quad , \quad (14)$$

while time evolution is generated by the time ordered operator

$$S(t_f, t_i) = T e^{-\frac{i}{\hbar} \int_{t_i}^{t_f} dt [\hat{H} + \lambda^a(t) \hat{\phi}_a]} \quad . \quad (15)$$

Clearly, such an operator propagates both gauge invariant as well as gauge variant states, and thus cannot correspond to the physical propagator of the system to which only physical, *i.e.* gauge invariant states would contribute. Usually, in order to achieve that aim, one considers some gauge fixing procedure through which only gauge invariant configurations are maintained in the actual time dependent dynamics. This gauge fixing may be effected before quantisation, namely through the so-called Faddeev reduced phase space approach[1]. Or else, the gauge fixing is effected through the so-called BRST quantisation of the gauge invariant system[1], in which case the original phase space is extended to include degrees of freedom—namely ghosts—of Grassmann parity opposite to that of the original degrees of freedom, in order to compensate for the contributions of the gauge variant configurations and of the degrees of freedom conjugate to the Lagrange multipliers λ^a which are also introduced in order to render the λ^a 's dynamical as well. However, whatever the approach to gauge fixing being implemented, even though the ensuing description is by construction indeed always gauge invariant, it may generally suffer Gribov problems[1], namely it may include some gauge invariant configurations more than once, or not at all as the case may be, clearly

an unacceptable situation if one is to properly include once and only once all physically distinct configurations possibly accessible to the system.

Nevertheless, Gribov problems are a consequence of gauge fixing, so that if the latter may be circumvented, Gribov problems would simply not be an issue anymore. The recently proposed physical projector approach to the quantisation of constrained systems[3] indeed provides such a framework which avoids the apparent necessity of gauge fixing, while at the same ensuring a proper inclusion of all physically distinct configurations and thus avoiding Gribov problems altogether[4]. In fact, the physical projector approach is simply set within Dirac's formulation briefly described above, and uses the projector onto the gauge invariant components of any quantum state of the quantised system. The projector itself is obtained by integrating over the group of gauge transformations the transformations of states generated by the gauge generators $\hat{\phi}_a$,

$$\mathcal{E} = \int dU(\theta^a) e^{-\frac{i}{\hbar} \theta^a \hat{\phi}_a} . \quad (16)$$

Here, θ^a are coordinates over the group manifold, while $dU(\theta^a)$ is the group invariant integration measure normalised such that $\mathcal{E}$ indeed obeys the properties of a projection operator,

$$\mathcal{E}^2 = \mathcal{E} \quad , \quad \mathcal{E}^\dagger = \mathcal{E} . \quad (17)$$

In the case of a compact gauge group, this definition of $\mathcal{E}$ is sufficient. When the gauge group is non compact, thereby leading to a spectrum of the gauge generators $\hat{\phi}_a$ which is not purely discrete, further considerations need to be applied[3].

Given the physical projector $\mathcal{E}$, the definition of the physical evolution operator propagating gauge invariant states only is clearly,

$$S_{\text{phys}}(t_f, t_i) = \mathcal{E} S(t_f, t_i) \mathcal{E} = e^{-\frac{i}{\hbar} (t_f - t_i) \hat{H}} \mathcal{E} = \mathcal{E} e^{-\frac{i}{\hbar} \mathcal{E} \hat{H} \mathcal{E}} \mathcal{E} , \quad (18)$$

making it explicit that indeed only gauge invariant states contribute to this operator both as external states as well as intermediate ones. In particular, in the case of a compact Lie group of gauge transformations, as implicitly assumed in this discussion having taken the structure coefficients C_{ab}^c to be constants, it is clear that the evaluation of matrix elements of such a physical operator, and of the physical projector itself, immediately implies group theory techniques for the calculation of various group invariants.

In the remainder of this contribution, different examples of the application of the physical projector quantisation of gauge invariant systems are briefly described.

4 Some Quantum Mechanical Gauge Invariant Systems

One of the simplest examples of quantum mechanical systems possessing a local gauge invariance is that of a $\text{SO}(2)=\text{U}(1)$ local gauge invariance in a plane defining two degrees of freedom. However, this situation being far too simple by itself, the invariance under time dependent $\text{SO}(2)$ rotations may be coupled to translations in a direction perpendicular to the plane of rotations, thereby leading finally to a system with three degrees of freedom of cartesian coordinates $(x(t), y(t), z(t))$, whose dynamics is governed by the Lagrange function[12],

$$L = \frac{1}{2} \left[(\dot{x} + g\xi y)^2 + (\dot{y} - g\xi x)^2 + (\dot{z} - \xi)^2 \right] - V(\sqrt{x^2 + y^2}) . \quad (19)$$

Here, $\xi(t)$ is a gauge degree of freedom, g is a gauge coupling constant, while the potential $V(\sqrt{x^2 + y^2})$ is a rotation invariant quantity which for later purposes is taken to be a spherically symmetric harmonic potential $V = \omega^2(x^2 + y^2)/2$.

Since gauge transformations in this system correspond to time dependent SO(2) rotations in $(x(t), y(t))$ coupled to time dependent translations in $z(t)$, it should be clear that the generator of this local abelian gauge symmetry is given by

$$\phi = p_z + gp_\theta \quad , \quad (20)$$

where p_z is the momentum conjugate to the coordinate z while p_θ is that conjugate to the angular variable θ associated to polar coordinates (r, θ) in the (x, y) plane. Similarly, the gauge invariant Hamiltonian of the system simply reads,

$$H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + \frac{1}{2}p_z^2 + \frac{1}{2}\omega^2 r^2 \quad , \quad (21)$$

which clearly has a vanishing Poisson bracket with the gauge generator ϕ , since the variables (r, p_r) , (θ, p_θ) and (z, p_z) each form a pair of canonically conjugate phase space degrees of freedom.

Canonical quantisation of the system is straightforward enough[12, 6], and is best developed in the helicity basis[5] related to the SO(2) symmetry in the (x, y) plane. Given the creation and annihilation operators $a_{x,y}^{(\dagger)}$ associated to the cartesian coordinates (x, y) in the usual way, the helicity creation and annihilation operators are defined by

$$a_\pm = \frac{1}{\sqrt{2}}[a_x \mp ia_y] \quad , \quad a_\pm^\dagger = \frac{1}{\sqrt{2}}[a_x^\dagger \pm ia_y^\dagger] \quad . \quad (22)$$

One then finds

$$\hat{H} = \frac{1}{2}\hat{p}_z^2 + \hbar\omega[a_+^\dagger a_+ + a_-^\dagger a_-] \quad , \quad \hat{\phi} = \hat{p}_z + \hbar g[a_+^\dagger a_+ - a_-^\dagger a_-] \quad . \quad (23)$$

These expressions make obvious what the physical spectrum of the system is, and how to construct the associated quantum excitations of the Fock vacuum. In particular, working in the configuration space representation of the Heisenberg algebra associated to the polar coordinates (r, θ, z) , it is then straightforward to determine the wave functions of all gauge invariant states[6]. For example, the physical energy spectrum is thus given by

$$E_{n_\pm} = \frac{1}{2}p^2 + \hbar\omega(n_+ + n_- + 1) \quad , \quad \text{with} \quad p = -\hbar g(n_+ - n_-) \quad , \quad (24)$$

$n_\pm$ being of course the integer excitation numbers of the modes of left- or right-handed SO(2) helicity.

For the present system, the spectrum of the gauge generator $\hat{\phi}$ being continuous (because of the $\hat{p}_z$ contribution), the definition of the physical projector requires first to consider the projector $\mathcal{E}_\delta$ onto those states whose $\hat{\phi}$ eigenvalue lies within the interval $[-\delta, +\delta]$, $\delta > 0$ being a positive number taken to be as small as may be required. The operator $\mathcal{E}_0$ projecting onto those states such that $\hat{\phi} = 0$ is then obtained as[3, 6, 7]

$$\mathcal{E}_0 = \lim_{\delta \rightarrow 0} \frac{\pi\hbar}{\delta} \mathcal{E}_\delta = \int_{-\infty}^{+\infty} d\gamma e^{\frac{i}{\hbar}\gamma\hat{\phi}} \quad . \quad (25)$$

Configuration space matrix elements of the physical propagator of the system, $S_{\text{phys}}(t_f, t_i) = e^{-i(t_f - t_i)\hat{H}/\hbar} \mathcal{E}_0$, are then readily determined. An explicit calculation[6] finds,

$$\begin{aligned} \langle r_f, \theta_f, z_f | S_{\text{phys}}(t_f, t_i) | r_i, \theta_i, z_i \rangle &= \frac{\omega}{2i\pi\hbar \sin \omega \Delta t} e^{\frac{i\omega \cos \omega \Delta t}{2\hbar \sin \omega \Delta t}(r_f^2 + r_i^2)} \times \\ &\times \sum_{\ell=-\infty}^{+\infty} e^{-i\frac{\pi}{2}|\ell|} e^{i\ell(\varphi_f - \varphi_i)} e^{-\frac{i}{2}\hbar \Delta t g^2 \ell^2} J_{|\ell|}\left(\frac{\omega r_f r_i}{\hbar \sin \omega \Delta t}\right) \quad , \end{aligned} \quad (26)$$

where $\Delta t = t_f - t_i$ and $\varphi_{f,i} = \theta_{f,i} - g z_{f,i}$. In fact, given this final expression, it is possible[6] to show that each single physical state of the system contributes to these matrix elements once and only once as an intermediate state, thereby establishing that the physical projector approach is indeed necessarily free of any Gribov problem. Moreover, it is also possible from this expression of the physical propagator to extract the wave functions for each of the physical states, the above matrix elements playing somehow the rôle of generating functions for these wave functions.

Other similar examples of gauge invariant quantum mechanical systems may be considered. For instance, rather than introducing the $z(t)$ degree of freedom associated to the coupling of the time dependent local $SO(2)$ gauge transformations with time dependent translations in the z direction, one may consider a whole collection of N similar $(x_i(t), y_i(t))$ $SO(2)$ invariant pairs of degrees of freedom ($i = 1, 2, \dots, N$), which are all rotated by the same time dependent angle irrespective of the value of the index i . $SO(2)$ gauge invariance then requires that the total $SO(2)$ angular momentum of such systems be vanishing, thereby leading to a compact group of gauge transformations. Here again when introducing an identical $SO(2)$ invariant harmonic potential for all these degrees of freedom, the quantisation of such systems is straightforward, while the physical projector approach readily leads to the determination of the physical spectrum and of the wave functions of its physical states, whether in configuration space or any other choice of basis for quantum states, such as coherent states[5].

Such considerations have been worked out in all details for the $SO(2)$ case, and to some extent for the $SO(3)$ case[4]. Generalisations to all Lie algebras are possible[4], and would be worth exploring. In particular, it would be extremely interesting to combine techniques of coherent states with those of combinatorics in group theory, which is possible now through the use of the physical projector. Indeed, as the above construction of the latter operator should have made clear, matrix elements of the physical projector are given by integrals over the group manifold of the exponentiated generators, hence leading to specific combinations of character invariants in the group. Unravelling the connection between these characters and the quantities to be integrated over the group manifold may provide new combinatorial identities for the representation theory of compact Lie algebras. Exceptional Lie algebras may be particularly fascinating cases in that respect, beginning with the case of G_2 .

5 U(1) Chern-Simons Theory and the Physical Projector

Given the space available, let us now turn to an example of a quantum field theory, rather than a quantum mechanical system, whose gauge freedom is so large that gauge invariance projects down a finite number of physical configurations only from the initially infinite number of degrees of freedom. Such theories are dubbed topological quantum field theories, since their sets of physical states only depend on the topology of the underlying manifold on which they are constructed[8]. As a matter of fact, the physical projector approach is also apt to handle the intricacies of such systems[7], as briefly described in this section.

Among the simplest of such topological quantum field theories are the $U(1)$ Chern-Simons theories in 2+1 dimensions, whose action principle reads

$$S = N_k \int_M dx^0 dx^1 dx^2 \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \quad . \quad (27)$$

Here N_k is a normalisation factor taking quantised values at the quantum level[8] (hence the notation), while $A_\mu(x^\mu)$ ($\mu, \nu = 0, 1, 2$) is a $U(1)$ gauge connection defined over the 2+1 dimensional base manifold M . That this system is topological in character is obvious from two facts, The first is that the above action is defined irrespective of a metric structure on M ; only a differentiable

structure is required. The second fact is that the equations of motion of the system reduce to

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \quad . \quad (28)$$

Hence, the dynamics of the system is that of flat U(1) bundles over M , whose characterization is provided entirely through the U(1) holonomies of the gauge field $A_\mu(x^\mu)$ in M , which indeed is purely a topological issue determined by the topology of M .

For the purpose of canonical quantisation, let us restrict to topologies of the form $M = \Sigma \times \mathbb{R}$, where Σ is an arbitrary compact Riemann surface, later on to be taken to be a torus T (note that the 2+1 dimensional split does not refer to a metric signature, but to such a topology split in the diffeomorphic structure of M). In such a case, the holonomies of A_μ reduce to the holonomy classes around the cycles defining a basis of the first homology group of Σ . These holonomies only involve the zero mode contributions of the gauge field, so that the entire physical content of the system is restricted to lie entirely within its zero mode sector. Hence in order to present results, let us only concentrate on that sector, even though the whole canonical quantisation procedure may be applied to all degrees of freedom, while local U(1) gauge transformations homotopic to the trivial one (*i.e.* “small” gauge transformations) may be used[7] to gauge away any non zero mode configuration for the field $A_\mu(x^\mu)$.

Specifically, let us consider the case of the torus T , having only two homology cycles. The coordinate system associated to these cycles enables a mode decomposition[7] of the field $A_\mu(x^\mu)$. Under large gauge transformations characterized by the two integer holonomies k_1 and k_2 associated to these two cycles, the zero modes A_1 and A_2 transform according to

$$A'_1 = A_1 + 2\pi k_1 \quad , \quad A'_2 = A_2 + 2\pi k_2 \quad . \quad (29)$$

On the other hand, these two degrees of freedom define in fact the phase space of the system, with in particular the commutation relations

$$[\hat{A}_1, \hat{A}_2] = i \frac{\hbar}{2N_k} \quad . \quad (30)$$

Hence this system is distinguished by having a phase space which itself is a compact 2-torus of volume $(2\pi)^2$, quite a unique feature not directly amenable to usual quantisation techniques. Moreover, the above commutation relation shows that the number of physical states must also be given by $(2\pi)^2 / (2\pi\hbar / (2N_k)) = 4\pi N_k / \hbar$, a first indication that the factor N_k indeed needs to be quantised at the quantum level ($N_k > 0$ is assumed; when N_k is negative, the rôles of the coordinates x^1 and x^2 are interchanged).

A coherent state quantisation of the system requires some further structure to be introduced beyond the mere topological and diffeomorphic ones, namely a complex structure parametrised by a complex parameter $\tau = \tau_1 + i\tau_2$ such that $\tau_2 > 0$. The associated annihilation and creation operators are then defined by

$$\alpha = \sqrt{\frac{N_k}{\hbar\tau_2}} \left[-i\tau\hat{A}_1 + i\hat{A}_2 \right] \quad , \quad \alpha^\dagger = \sqrt{\frac{N_k}{\hbar\tau_2}} \left[i\bar{\tau}\hat{A}_1 - i\hat{A}_2 \right] \quad . \quad (31)$$

Having restricted our considerations to the zero mode sector alone, the construction of the physical projector relates to large gauge transformations acting on the zero modes only. From the gauge transformations in (29), this projector is found[7] to be given by $\mathcal{E} = \sum_{k_1, k_2=-\infty}^{+\infty} \hat{U}(k_1, k_2)$ with

$$\hat{U}(k_1, k_2) = C(k_1, k_2) \exp \left\{ \frac{2i\pi}{\hbar} \sqrt{\frac{\hbar N_k}{\tau_2}} \left[(\bar{\tau}k_1 - k_2)\alpha + (\tau k_1 - k_2)\alpha^\dagger \right] \right\} \quad . \quad (32)$$

Here, $C(k_1, k_2)$ is a cocycle factor such that the group composition law is obeyed for the above operator representation of large gauge transformations. The latter requirement implies that[7]

$$C(k_1, k_2) = e^{i\pi k k_1 k_2} \quad , \quad \text{with} \quad N_k = \frac{\hbar}{4\pi} k \quad , \quad (33)$$

where k is an arbitrary strictly positive integer, hence also to be equal to the number of physical states.

Within the coherent state representation of the system, it is now possible to determine its physical spectrum content as well as the coherent state wave functions for these gauge invariant states, from the simple application of the physical projector onto the space of quantum states. An explicit analysis[7] finds that the number of physical states is indeed equal to the integer k which quantises the normalisation factor N_k , while the obtained wave functions agree completely with results established previously in the literature[11] (see the original references[7, 11] for explicit expressions). By construction, these physical states $|r\rangle$ ($r = 0, 1, \dots, k-1$) are gauge invariant under large gauge transformations. However, to which extent a dependency on the complex structure τ of the torus T arises also needs to be assessed. The explicit analysis[11, 7] of this issue finds that this further requirement of modular invariance in the parameter τ implies that the integer k must also be even, while the whole set of physical states then provides a single irreducible representation of the modular group of the Riemann surface Σ . This is thus how the topological invariance of the original classical theory is characterized at the quantum level. The dependency on the complex structure through modular classes is a consequence of a conformal anomaly of the quantised system[8].

6 Conclusions

As mentioned in the Introduction, the purpose of this contribution is to highlight some recent results concerned with the issues raised by the canonical quantisation of systems whose configuration space is not euclidean and flat or is parametrized by curvilinear coordinates, and of systems possessing local gauge invariances. A general classification of representations of the Heisenberg algebra in the first case was briefly described, which implies the appearance of topology classes of flat $U(1)$ bundles over configuration space, together with a diffeomorphic covariant wave function representation of quantum states. With respect to gauge invariant systems, a new approach[3] to their quantisation has been discussed, and shown through explicit examples to lead to a perfectly adequate framework avoiding any gauge fixing and thereby also the possibility of Gribov problems. Other advantages of the physical projector approach have not been emphasized however, such as for example the fact that it is also apt to address the issues raised by the quantisation of compact phase spaces with isometries, as illustrated through the $U(1)$ Chern-Simons theory example.

The next stage of explorations based on the physical projector is now to be initiated, aiming towards a deeper insight into the non perturbative dynamics of theories ever closer to those of the actual material universe and its quantum excitations, whether within the realm of quantum field theories or other inception instigated by the recent developments in string and M-theory.

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References

- [1] For a detailed discussion and references to the literature, see for example, J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics* (Leuven University Press, Leuven, 1991).
- [2] J. Govaerts and V.M. Villanueva, *Topology Classes of Flat $U(1)$ Bundles and Diffeomorphic Covariant Representations of the Heisenberg Algebra*, preprint [quant-ph/9908014](#) (August 1999).
- [3] J.R. Klauder, *Ann. Phys.* **254**, 419 (1997);
J.R. Klauder, *Nucl. Phys.* **B547**, 397 (1999).
- [4] J. Govaerts, *J. Phys.* **A30**, 603 (1997).
- [5] J. Govaerts and J.R. Klauder, *Ann. Phys.* **274**, 251 (1999).
- [6] V.M. Villanueva, J. Govaerts and J.-L. Lucio-Martinez, *Quantisation without Gauge Fixing: Avoiding Gribov Ambiguities through the Physical Projector*, preprint [hep-th/9909033](#) (September 1999).
- [7] J. Govaerts and B. Deschepper, *The Physical Projector and Topological Quantum Field Theories: $U(1)$ Chern-Simons Theory in 2+1 Dimensions*, preprint [hep-th/9909221](#) (September 1999).
- [8] E. Witten, *Comm. Math. Phys.* **117**, 353 (1988);
E. Witten, *Comm. Math. Phys.* **121**, 351 (1989).
- [9] J.R. Klauder, *Non Canonical Quantization of Gravity. I. Foundations of Affine Quantum Gravity*, preprint [gr-qc/9906013](#) (June 1999).
- [10] For a recent discussion, see for example, J.H. Schwarz, *TASI Lectures on non BPS D-Brane Systems*, preprint [hep-th/9908144](#) (August 1999).
- [11] M. Bos and V.P. Nair, *Phys. Lett.* **B223**, 61 (1989);
M. Bos and V.P. Nair, *Int. J. Mod. Phys.* **A5**, 959 (1990);
J.M.F. Labastida and A.V. Ramallo, *Phys. Lett.* **B227**, 92 (1989);
S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326**, 108 (1989);
H. Murayama, *Z. Phys.* **C48**, 79 (1990).
- [12] R. Friedberg, T.D. Lee, Y. Pang and H.C. Ren, *Ann. Phys.* **246**, 381 (1996).