On weak action representability, action accessibility and varieties of non-associative algebras

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Workshop in Algebra and Category Theory
Torino, 18/06/2025

Joint work with Xabier García Martínez and Federica Piazza

The semi-abelian context

- Semi-abelian categories were introduced by G. Janelidze, L. Márki and W. Tholen in 2002 in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- A category $\mathcal{C}$ is said to be *semi-abelian* if it is pointed, Barr-exact, Bourn-protomodular with finite coproducts.
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathcal{C}$ such that $\alpha \circ \beta = \mathrm{id}_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

  where, for every object B in $\mathcal{C}$, $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- There is a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X).$$

Varieties of non-associative algebras

- A *non-associative algebra* $(A, \cdot)$ is a vector space A over a field $\mathbb{F}$ equipped with a bilinear operation $A \times A \to A \colon (x, y) \mapsto xy$ called the *multiplication*.
- We denote by **Alg** the category of non-associative algebras, where morphisms are the linear maps which preserve the multiplication.
- An *identity* of an algebra A is a non-associative polynomial $\varphi(x_1, \dots, x_n)$ such that $\varphi(a_1, \dots, a_n) = 0$, for every $a_1, \dots, a_n \in A$.
- Let I be a set of identities. The *variety of non-associative algebras* determined by I is the class $\mathcal{V}$ of all algebras which satisfy all the identities in I.
- Any variety of non-associative algebras $\mathcal{V}$ can be seen as a full subcategory of **Alg**. Any such variety is a semi-abelian category.

Examples

- **AbAlg** is the variety of *abelian* algebras: $xy = 0$.
- **Assoc** is the variety of *associative* algebras: $x(yz) - (xy)z = 0$.
- **CAlg** is the variety of *commutative* algebras: $xy - yx = 0$.
- **ACAlg** is the variety of *anti-commutative* algebras: $xy + yx = 0$.
- **CAssoc** is the variety of *commutative associative algebras*.
- **Lie** is the variety of *Lie algebras*, which is determined by anti-commutativity and the *Jacobi identity*: $xy + yx = 0$ and $x(yz) + y(zx) + z(xy) = 0$.
- **JJord** is the variety of *Jacobi–Jordan algebras*, which is determined by commutativity and the Jacobi identity.
- **Leib** is the variety of *(right) Leibniz algebras*, which is determined by the *(right) Leibniz identity*: $(xy)z - (xz)y - x(yz) = 0$.
- **Alt** is the variety of *alternative algebras*, which is determined by $(yx)x - y(xx) = 0$ and $x(xy) - (xx)y = 0$.

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp**, **Lie** and any abelian category.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
- In any abelian category, the actor of X is 0.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden, C. Vienne, 2021)

Let $\mathcal{V}$ be a variety of non-associative algebras over an infinite field $\mathbb{F}$ with $\mathrm{chat}(\mathbb{F}) \neq 2$. *If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathbf{AbAlg}$ or $\mathcal{V} = \mathbf{Lie}$.*

- More recently, G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ admits a weak representation. This means there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau\colon\ \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

An object T as above is called *weak representing object* of X, the pair (T, τ) is called *weak representation* of $\mathrm{SplExt}(-, X)$ and $(\varphi\colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

Associative algebras and Leibniz algebras

- **G. Janelidze, 2022.** The category **Assoc** is weakly action representable. A weak representing object of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f*-, -*f) \in \mathrm{End}(X)\times\mathrm{End}(X)^{\mathrm{op}} \mid f*(xy) = (f*x)y,$$
$$(xy)*f = x(y*f), x(f*y) = (x*f)y,\ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak representing object of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y),$$
$$D(xy) = D(x)y - D(y)x,\ xd(y) = xD(y),\ \forall x, y \in X\}.$$

A counterexample: Jordan algebras

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible.

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a commutative algebra $(X, \cdot)$ which satisfies the *Jordan identity*:

$$(xy)(xx) = x(y(xx)), \quad \forall x, y \in X.$$

Remark (A. S. Cigoli, S. Mantovani, 2012)

The variety **Jord** of Jordan algebras is not action accessible.

Conclusion: Jord is not weakly action representable.

Actions in varieties of algebras

- Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$.
- Let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$.
- We suppose that $\mathcal{V}$ is *operadic* (all the identities of $\mathcal{V}$ are *multilinear*). This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible*, or equivalently that $\mathcal{V}$ is an *Orzech category of interest*.
- This is equivalent to saying that the λ/μ *rules* hold: there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$\begin{aligned} x(yz) =& \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) + \\ &+ \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx), \end{aligned}$$

and

$$\begin{aligned} (yz)x =& \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) + \\ &+ \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx) \end{aligned}$$

are identities in $\mathcal{V}$.

- We fix a set of multilinear identities

$$\Phi_{k,i}(x_1, \dots, x_k) = 0, \quad k = \deg(\Phi_{k,i}),\ i = 1, \dots, n,$$

  which determine $\mathcal{V}$.
- It is possible to find a *partial* algebra $\mathcal{E}(X) \leq \mathrm{End}(X)^2$, that is a vector space $\mathcal{E}(X)$ together with a bilinear *partial* operation

$$\langle -, - \rangle \colon \Omega \subseteq \mathcal{E}(X) \times \mathcal{E}(X) \to \mathcal{E}(X),$$

  which *weakly represents* split extensions with kernel X.

The partial algebra $\mathcal{E}(X)$

We define

$$\mathcal{E}(X) = \{f = (f * -, - * f) \in \mathrm{End}(X)^2 \mid \Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0\}$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j$, $fx := f * x$, $xf := x * f$, together with a bilinear map

$$\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2 \colon (f, g) \mapsto h = (h * -, - * h)$$

where

$$\begin{aligned} x * h =& \lambda_1(x * f) * g + \lambda_2(f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &+ \lambda_5(x * g) * f + \lambda_6(g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x =& \mu_1(x * f) * g + \mu_2(f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &+ \mu_5(x * g) * f + \mu_6(g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden, C. Vienne, 2025)

Let $\mathcal{V}$ be a variety and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)).$$

- *The morphism $U(B) \to \mathcal{E}(X)\colon b \mapsto (b * -, - * b)$ belongs to $\mathrm{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0$.*
- *If $\mathcal{E}(X)$ is an object of $\mathcal{V}$, then it is a weak representing object of X.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Examples

- If $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X.
- If $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X.
- If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$ is a weak representing object of X.
- If $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) \cong \mathrm{Bim}(X)$ is a weak representing object of X.
- If $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X.
- If $\mathcal{V} = \mathbf{JJord}$, then $\mathcal{E}(X) \cong \mathrm{ADer}(X)$ is an external actor of X.

Question: does the converse of the implication

$$\textit{weakly action representable category} \Rightarrow \textit{action accessible category}$$

hold?

Proposition (J. R. A. Gray, 2025)

Let $\mathcal{C}$ be an action representable category and let $\mathcal{X}$ be a Birkhoff subcategory of $\mathcal{C}$. Suppose there exist two monomorphisms $m\colon S \rightarrowtail B$, $m'\colon S \rightarrowtail B'$ in $\mathcal{X}$, two monomorphisms $u\colon B \rightarrowtail D$, $u'\colon B' \rightarrowtail D$ in $\mathcal{C}$, an object X of $\mathcal{X}$ and a monomorphism $v\colon D \rightarrowtail [X]$ such that:

- *m and m' cannot be amalgamated in $\mathcal{X}$.*
- *The pair (u, u') is an amalgam of (m, m'), i.e., $um = u'm'$.*
- *The split extensions with kernel X corresponding to vu and vu' are in $\mathcal{X}$.*

Then, the category $\mathcal{X}$ is not weakly action representable.

Remark (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Any action representable category has the *amalgamation property*.

Example (J. R. A. Gray, 2025)

- Let $\mathcal{C} = \mathbf{Grp}$ and let $\mathcal{X} = \mathbf{Sol}_t(\mathbf{Grp})$ with $t \geq 3$.
- There exist two monomorphisms $m, m' \colon S \rightarrowtail B$. with S abelian and B 2-nilpotent, which cannot be amalgamated in any solvable group D (B. H. Neumann, 1960).
- It is possible to apply J. R. A. Gray's result.

Theorem (J. R. A. Gray, 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Grp})$ *is not weakly action representable for any* $t \geq 3$.

- Consider the following groups

$$S = \langle a, b \mid [a, b] = a^5 = b^5 = 1 \rangle,$$
$$B = \langle x, a, b \mid [x, a] = b^{-1}, [x, b] = x^5 = a^5 = b^5 = 1 \rangle,$$
$$B' = \langle y, a, b \mid [y, b] = a^{-1}, [y, a] = y^5 = a^5 = b^5 = 1 \rangle.$$

- The obvious inclusions of S inside B and B' cannot be amalgamated in any solvable group.
- There exist an abelian group X and two monomorphisms $\psi\colon B \rightarrowtail \mathrm{Aut}(X)$ and $\psi'\colon B' \rightarrowtail \mathrm{Aut}(X)$ agreeing on S, such that the corresponding split extensions are at most 3-solvable.
- It is possible to choose X, ψ, ψ' in such a way the split extensions corresponding to ψ and ψ' are 3-nilpotent ($\Rightarrow$ 2-solvable).

Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_2(\mathbf{Grp})$ *and* $\mathbf{Nil}_k(\mathbf{Grp})$ $(k \geq 3)$ *are not weakly action representable.*

What about varieties of algebras?

- **Question:** does the converse of the implication

  weakly action representable category $\Rightarrow$ *action accessible category*

  hold in the context of varieties of algebras over a field?
- The answer is **NO**.

Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Lie})$ *and* $\mathbf{Nil}_k(\mathbf{Lie})$ *are not weakly action representable for any* $t \geq 2$ *and* $k \geq 3$.

- **Open question:** is the variety $\mathbf{Nil}_2(\mathbf{Grp})$ weakly action representable?

Unitary algebras

Definition

A variety of non-associative algebras $\mathcal{V}$ is *unitary closed* if for any algebra X of $\mathcal{V}$, the algebra $\tilde{X}$ obtained by adjoining to X the external unit 1, together with the identities $x \cdot 1 = 1 \cdot x = x$, is still an object of $\mathcal{V}$.

Examples

- **Assoc**, **CAssoc**, **Alt** are unitary closed.
- **Lie**, **Leib**, **JJord** are not unitary closed.

Remark

The subcategory $\mathcal{V}_1$ of unitary algebras of $\mathcal{V}$ is an *ideally exact category*.

Ideally exact categories

Definition (G. Janelidze, 2024)

Let $\mathcal{C}$ be a category with pullbacks, with initial object 0 and terminal object 1. $\mathcal{C}$ is *ideally exact* if it is Barr exact, protomodular, has finite coproducts and the unique morphism $0 \to 1$ is a regular epimorphism.

- **Examples: Ring**, **CRing**, **MVAlg**, **PrAlg**, **GAlg**, any cotopos...

Theorem (G. Janelidze, 2024)

TFAE:

- *$\mathcal{C}$ is ideally exact.*
- *$\mathcal{C}$ is Barr-exact, has finite coproducts and there exists a monadic functor $\mathcal{C} \to \mathcal{V}$, where $\mathcal{V}$ is a semi-abelian category.*
- *There exists a monadic functor $\mathcal{C} \to \mathcal{V}$, where $\mathcal{V}$ is a semi-abelian category, such that the underlying functor of the corresponding monads preserves regular epimorphisms and kernel pairs.*

Remark

There is a semi-abelian category $\mathcal{V}$ and a monadic adjunction with cartesian unit

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}.$$

Examples

- One may take $\mathcal{V} = (\mathcal{C} \downarrow 0)$ with U and F defined in the obvious way.
- If $\mathcal{V}$ is a unitary closed variety of non-associative algebras, then the monadic adjunction with cartesian unit is given by

$$\mathcal{V}_1 \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}$$

where U forgets the unit and $F(X) = \mathbb{F} \ltimes X$, for any X in $\mathcal{V}$.

Action representability of ideally exact categories

Definition (G. Janelidze, 2024)

An ideally exact category $\mathcal{C}$ is *action representable* if the functor

$$\mathrm{SplExt}_{\mathcal{V}}(-, U(X)) \cong \mathrm{Act}_{\mathcal{V}}(-, U(X)) \colon \mathcal{V}^{\mathrm{op}} \to \mathbf{Set}$$

is representable for any object X of $\mathcal{C}$.

Example (G. Janelidze, 2024)

The categories **Ring** and **CRing** are action representable. The actor of a unitary (commutative) ring X is isomorphic to X itself.

Action representability of categories of unitary algebras

Theorem (M. M., F. Piazza, 2025)

- *The categories* $\mathbf{Assoc}_1$, $\mathbf{CAssoc}_1$, $\mathbf{Alt}_1$, $\mathbf{Pois}_1$, $\mathbf{CPois}_1$ *are action representable.*
- *The actor of a (commutative) associatve algebra* X *is isomorphic to* X *itself.*
- *The actor of an alternative algebra* X *is isomorphic to* X *itself.*
- *The actor of a (commutative) Poisson algebra* X *is isomorphic to the Poisson subalgebra* $\mathrm{Z}(X)$ *of* X*.*

Grazie per l'attenzione!

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