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Drift estimation by timescale transformation

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Abstract: In this study, we consider a model free control technique for a class of minimum phase nonlinear systems with unknown dynamics. We propose a timescale transformation approach that can be used to estimate the drift and the high frequency gain of the output dynamics of nonlinear systems with a well defined relative degree. The proposed controller provides an effective output regulation mechanism that avoids the need for direct disturbance model estimation and high gain output feedback controllers. A simulation study is conducted to demonstrate the effectiveness of the proposed approach.

Keywords: Extremum-seeking control, model-free control, data-driven control

1. INTRODUCTION

Model-free controller design techniques have recently emerged to address the control of highly uncertain systems (Fliess and Join (2013), Hou and Jin (2013)). In situations where the uncertain systems can be taken off-line for experimentation, it is possible to consider the application of learning techniques such as reinforcement learning (Fu (1970), Sutton et al. (1992), Meyn (2022)). When off-line operation is not possible, the required control system must generate a suitable excitation of the control system to guarantee that key elements of the system dynamics are adequately estimated. Recent developments in the area of data-driven control (De Persis and Tesi (2019), Tanaskovic et al. (2017)) has formalized the excitation requirements for the design of suitable data-driven control systems.

Adaptive control techniques (Krstić et al. (1992), Jiang and Praly (1998), Astolfi et al. (2008)) are suitable when the system is known but subject to unknown parametric uncertainties. In the absence of exact model forms, unknown process nonlinearities can be estimated using a variety of basis functions. Learning strategies such as Gaussian regression (Sanner and Slotine (1991), Liu et al. (2018)) and deep learning (Lillicrap et al. (2015), Liu and Theodorou (2019)) have been exploited in a number of applications to estimate unknown nonlinearities. Neural network based techniques (e.g.,Ge et al. (2013), Farrell (1998)) have been used extensively in the literature to overcome the lack of knowledge of unknown process nonlinearity. The main drawback of neural-network techniques is the presence of an estimation bias that limits the achievable performance of the resulting closed-loop systems. While it is generally argued that this bias can be minimized by increasing the number of basis functions, this requires to increase the effective number of parameters in the closed-loop system which imposes an unresolved issue associated with the persistency of excitation conditions needed to provide an accurate estimation of the unknown nonlinearities.

Extremum-seeking control has emerged as a suitable model-free adaptive estimation that overcome the lack of knowledge of system nonlinearities (Tan et al. (2006), Krstic and Wang (2000)). While its main focus has been the solution of real-time steady-state optimization problems, it has been proposed as an alternative to model-free control techniques. Dual mode techniques have been proposed for the design of control systems that can be used to achieve stabilization of control systems to the unknown optimum of a measured function of interest. Many alternative formulations of the dual mode approach have been proposed in the literature including demodulation techniques (Guay, 2016), Lie bracket averaging techniques (Guay and Atta, 2018) and adaptive estimation techniques (Guay and Burns, 2017). While these techniques can be used to solve general classes of real-time optimization problems for unknown dynamical systems, they are most focussed on the estimation of directional derivative of a measured output along the control vector field. The estimation of the directional derivative along the drift vector fields is never addressed explicitly. This can limit the ability to regulate the output in the presence of exogenous variations that affect the drift dynamics.

In this study, we propose a modulation based technique for the estimation of the directional derivative of a measured output along the drift vector field as well as its derivative along the control vector field. The approach proposed is to introduce a modulation in the form of a timescale transformation. It is shown that the use of sinusoidal variations in the timescale allows one to estimate drift component of the unknown nonlinear dynamics. In this manuscript, we show how this approach can be used to implement feedback linearizing controller for a class of minimum phase nonlinear systems as described in (Freidovich and Khalil, 2008).

The paper is structured as follows. The problem formulation is given in Section 2. The proposed timescale transformation approach is proposed in Section 3. A stability analysis is presented in Section 4. Section 5 presents a brief simulation study. Conclusions are given in Section 6.

2. PROBLEM FORMULATION

We consider a class of unknown nonlinear systems described by the following dynamical system:

$$\dot{x} = f(x) + g(x)u \tag{1a}$$
$$y = h(x) \tag{1b}$$

where $x \in \mathbb{R}^n$ are the state variables, $u \in \mathbb{R}$ is the input variable, and $y \in \mathbb{R}$ is the measured output variable. It is assumed that the function $h : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable with respect to x. In addition, the vector valued functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be sufficiently smooth. The function h is assumed to be unknown. The objective is to bring the time-varying output function to zero using only the measurement y. In addition, we require the following assumptions.

In the following, we use the standard notation $L_f h$ and $L_g h$ to denote the Lie (or directional) derivative of the function $h(x)$ in the direction of the vector fields $f(x)$ and $g(x)$, respectively.

Assumption 1. The system (1a) with output (1b) is such that:

$$\|L_g h\| \geq \alpha \tag{2}$$

for some positive constant α.

The assumption imposes a strong relative degree of one for the nonlinear system. This allows one to rewrite the nonlinear system in the Byrnes-Isidori normal form:

$$\begin{aligned} \dot{\eta} =& \phi(\eta, \xi) \\ \dot{\xi} =& L_f h + L_g h u \\ y =& \xi. \end{aligned} \tag{3}$$

where $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^{n-1}$.

It is assumed that there is a steady-state trajectory η_{ss} given as the solution of the dynamical system:

$$\frac{d\eta_{ss}}{d\tau} = \beta\phi(\eta_{ss}, 0).$$

Furthermore, the steady-state solution, η_{ss}, evolves on a compact set $\Omega_{ss} \in \mathbb{R}^{n-1}$.

We define the error variable $\tilde{\eta} = \eta - \eta_{ss}$ with dynamics:

$$\frac{d\tilde{\eta}}{d\tau} = \beta\phi(\tilde{\eta} + \eta_{ss}, \xi) - \beta\phi(\eta_{ss}, 0). \tag{4}$$

Assumption 2. There exists a radially unbounded positive definite function $V_\eta(\tilde{\eta})$ such that for all $\eta \in \mathbb{R}^{n-1}$

$$\frac{\partial V_\eta}{\partial \eta}(\beta\phi(\tilde{\eta} + \eta_{ss}, \xi) - \beta\phi(\eta_{ss}, 0)) \leq -\beta\alpha_1\|\tilde{\eta}\|^2 + \beta\gamma_1\|\tilde{\eta}\|\|\xi\|$$

for positive constants α_1 and γ_1.

3. PROPOSED APPROACH

3.1 Target Control System

The target control system consists of system (3) in closed-loop with the state-feedback controller:

$$w_r = -\frac{1}{\beta L_g h}(\beta L_f h + k\xi) \triangleq \psi(\xi, \tilde{\eta}).$$

This yields the following closed-loop system:

$$\begin{aligned} \frac{d\tilde{\eta}}{d\tau} =& \beta\phi(\tilde{\eta} + \eta_{ss}, \xi) - \beta\phi(\eta_{ss}, 0) \\ \frac{d\xi}{d\tau} =& -k\xi. \end{aligned}$$

Let $V(\xi) = \frac{1}{2}\xi^2$ and define the set $\Omega_\xi = \{\xi \in \mathbb{R} \, | V(\xi) \leq c\}$. By Assumption 2, there exists a constant c_0 and a constant α_0 such that the set $\Omega_\eta = \left\{\eta \in \mathbb{R}^{n-1} \mid V_\eta(\tilde{\eta}) \leq c_0 + \alpha_0 c\right\}$ is positively invariant for the dynamical system (4). Let $\Omega_c = \Omega_\xi \times \Omega_\eta$.

Furthermore, it is assumed that the ideal state feedback controller is bounded on the set Ω_c. That is, there exists a constant M such that:

$$\sup_{\xi \in \Omega_\xi, \eta \in \Omega_\eta} \|w_r(\xi, \eta)\| \leq M.$$

Under this ideal feedback, it can be shown that the closed-loop system achieves converngences, *i.e.,* $\lim_{t\to\infty} \xi(t) = 0$, for any initial condition in Ω_d for $d < c$. The positive constant d is chosen such that the saturated controller

$$w_r = M\text{Sat}\left(\frac{1}{M}\psi(\xi, \eta)\right) \tag{5}$$

operates in the linear range of the saturation function for all $(x, \eta) \in \Omega_d$.

3.2 Timescale transformation

The proposed ideal state-feedback controller cannot be implemented since the quantities $L_f h$ and $L_g h$ are unknown. We propose a timescale transformation approach to provide independent estimates of $L_f h$ and $L_g h$.

The proposed control approach can be described as follows. The rate of change of the output is given:

$$\frac{d\xi}{dt} = L_f h + L_g h u.$$

Using standard extremum seeking control techniques, many alternative methods can be employed to approximate the unknown nonlinearity $L_g h$ by adding some perturbation to the input. However, the drift component $L_f h$ cannot be estimated directly and independently. In the following, we provide a strategy that can be used to estimate the drift component $L_f h$ using perturbation based techniques and a timescale transformation.

Let us introduce the timescale $v(\tau)d\tau = dt$ for some strictly positive variable $v(\tau) > 0$. The dynamics can then be written in the new timescale as:

$$\frac{d\xi}{d\tau} = v(\tau)L_f h + v(\tau)L_g h u.$$

If one defines $w(\tau) = v(\tau)u$, we get the system:

$$\frac{d\xi}{d\tau} = v(\tau)L_f h + L_g h w(\tau).$$

The introduction of the new timescale introduces an additional variable $v(\tau)$ that can be used to excite the drift component of the dynamics. The input variables $v(\tau)$ and $w(\tau)$ can be used to allow for the independent estimation of $L_f h$ and $L_g h$.

First, we pose the following timescale transformation:

$$\frac{dt}{d\tau} = \beta + A_1 \sin(\omega_1 \tau) \tag{6}$$

where β, A_1 and ω_1 are positive constants. The timescale transformation $\tau \to t$ is taken such $t(0) = 0$. The transformation provides a diffeomorphism on the real positive line if and only if $\beta > A_1$.

We can then write the output dynamics as:

$$\frac{dy}{d\tau} = (\beta + A_1 \sin(\omega_1 \tau))L_f h + (\beta + A_1 \sin(\omega_1 \tau))L_g hu.$$

Defining the feedback transformation:

$$u = \frac{w}{1 + \frac{A_1}{\beta}\sin(\omega_1 \tau)} \tag{7}$$

leads to output dynamics of the form:

$$\frac{dy}{d\tau} = (\beta + A_1 \sin(\omega_1 \tau))L_f h + \beta L_g h w.$$

The estimation of $L_g h$ requires the use of a dither signal of the input w. The input is given by:

$$w = w_r + A_2 \sin(\omega_2 \tau)$$

where w_r is the input to be assigned, A_2 and ω_2 are positive constants.

The estimation of $L_f h$ and $L_g h$ are obtained using an extremum seeking control approach using two filters. The ideal filter for the estimation of $L_g h$ can be written as follows:

$$\frac{d\xi_2}{d\tau} = -\omega_l \left(\xi_2 - \frac{2}{A_2}\sin(\omega_2 \tau)\frac{dy}{d\tau} \right)$$

where ω_l is a positive constant.

The use of the proposed timescale transformation allows one to realize an estimate of $L_f h$ using the following ideal filter:

$$\frac{d\xi_1}{d\tau} = -\omega_l \left(\xi_1 - \frac{2\beta}{A_1}\sin(\omega_1 \tau)\frac{dy}{d\tau} \right).$$

These two filters are qualified as "ideal" since they are driven by the rate of change of y rather than the output measurement y. In general, the term $\frac{dy}{d\tau}$ is replaced by an estimate. One reliable approach is to use the following high pass filter:

$$\frac{d\eta}{d\tau} = -\omega_h \eta + y(\tau)$$
$$v(\tau) = -\omega_h^2 \eta + \omega_h y(\tau).$$

This differentiator is such that the dynamics of the derivative estimate $v(\tau)$ can be written as follows:

$$\frac{dv}{d\tau} = -\omega_h \left(v(\tau) - \frac{dy}{d\tau} \right).$$

We can define the error signal $\tilde{v} = v(\tau) - \frac{dy}{d\tau}$ with dynamics:

$$\frac{d\tilde{v}}{d\tau} = -\omega_h \tilde{v}(\tau) + \frac{d^2 y}{d\tau^2}.$$

This yields the following expressions for the estimates of $L_f h$ and $L_g h$:

$$\frac{d\xi_1}{d\tau} = -\omega_l \left(\xi_1 - \frac{2\beta}{A_1}\sin(\omega_1 \tau)\tilde{v}(\tau) - \frac{2\beta}{A_1}\sin(\omega_1 \tau)\frac{dy}{d\tau} \right)$$
$$\frac{d\xi_2}{d\tau} = -\omega_l \left(\xi_2 - \frac{2}{A_2}\sin(\omega_2 \tau)\tilde{v}(\tau) - \frac{2}{A_2}\sin(\omega_2 \tau)\frac{dy}{d\tau} \right). \tag{8}$$

As stated above, the ideal controller of the target system requires the inverse of $\beta L_g h$. In this study, we propose a dynamic inversion of its estimate ξ_2 given by:

$$\frac{d\xi_r}{d\tau} = -\omega_\ell \xi_r (\xi_2 \xi_r - 1). \tag{9}$$

The proposed certainty equivalence controller takes the general form:

$$w_r = \hat{\psi}(\xi_r, \xi_1, \xi) = -\xi_r(\xi_1 + k\xi). \tag{10}$$

Remark 1. The dynamic inverse system (13) is susceptible to instability. In general, convergence of ξ_r to ξ_2^{-1} requires that $0 < \xi_r(0)\xi_2(0) < 1$. That is, the initial condition $\xi_r(0)$ must be close to the inverse of the initial condition $\xi_2(0)$. This can always be ensured by choosing the initial conditions accordingly. In the developments below, it will be assumed that the trajectories of (13) are such that the inverse $\xi_r^{-1}(t)$ exists for $t \geq 0$.

As in Freidovich and Khalil (2008), the controller (10) must be implemented using a saturation function to avoid any peaking arising from the estimates ξ_1, ξ_2 and the dynamic inverse ξ_r. The controller used in this study is given by

$$w_r = M\text{Sat}\left(\frac{1}{M}\hat{\psi}(\xi_r, \xi_1, \xi) \right). \tag{11}$$

4. STABILITY ANALYSIS

In this section, we present an abridged analysis of the resulting closed-loop system.

4.1 Averaged dynamics

We first compute the averaged system for the two filters (8). This gives:

$$\frac{d\xi_1}{d\tau} = -\omega_l \Bigg(\xi_1 - \frac{2\beta}{A_1}\sin(\omega_1 \tau)\Big((\beta + A_1 \sin(\omega_1 \tau))L_f h + \beta L_g h w_r + \beta L_g h A_2 \sin(\omega_2 \tau) \Big) - \frac{2\beta}{A_1}\sin(\omega_1 \tau)\tilde{v} \Bigg)$$

and

$$\frac{d\xi_2}{d\tau} = -\omega_l \Bigg(\xi_2 - \frac{2}{A_2}\sin(\omega_2 \tau)\Big((\beta + A_1 \sin(\omega_1 \tau))L_f h + \beta L_g h w_r + \beta L_g h A_2 \sin(\omega_2 \tau) \Big) - \frac{2}{A_2}\sin(\omega_2 \tau)\tilde{v} \Bigg).$$

Let $\overline{\omega}$ be the lowest common multiplier of ω_1 and ω_2. Let ω_1 and ω_2 be chosen such that:

$$\frac{\overline{\omega}}{2\pi}\int_0^{\frac{2\pi}{\overline{\omega}}} \sin(\omega_1 \sigma)\sin(\omega_2 \sigma)d\sigma \equiv 0.$$

One can then define the averaged dynamics as:

$$\frac{d\xi_1^a}{d\tau} = -\frac{\omega_l \overline{\omega}}{2\pi}\int_0^{\frac{2\pi}{\overline{\omega}}} \Bigg(\xi_1^a - \frac{2\beta}{A_1}\sin(\omega_1 \sigma)$$

$$\times\Bigg((\beta + A_1\sin(\omega_1\sigma))L_f h^a + \beta L_g h^a w_r + \beta L_g h^a A_2 \sin(\omega_2\sigma)\Bigg) - \frac{2\beta}{A_1}\sin(\omega_1\tau)\tilde{v}\Bigg)d\sigma$$

and

$$\frac{d\xi_2^a}{d\tau} = -\frac{\omega_l\overline{\omega}}{2\pi}\int_0^{\frac{2\pi}{\overline{\omega}}}\Bigg(\xi_2^a - \frac{2}{A_2}\sin(\omega_2\sigma) \times\Bigg((\beta + A_1\sin(\omega_1\sigma))L_f h^a + \beta L_g h^a w_r + \beta L_g h^a A_2 \sin(\omega_2\sigma)\Bigg) - \frac{2}{A_2}\sin(\omega_2\tau)\tilde{v}\Bigg)d\sigma.$$

This yields the following averaged dynamics:

$$\frac{d\xi_1^a}{d\tau} = -\omega_l(\xi_1^a - \beta L_f h^a)$$

and

$$\frac{d\xi_2^a}{d\tau} = -\omega_l(\xi_2^a - \beta L_g h^a).$$

As a result, we obtain an averaged dynamical system that provides filtered estimates of $L_f h$ and $L_g h$. It is important to note that similar results could be achieved using Lie bracket averaging techniques.

The averaged system's dynamics can be expressed in the new timescale as follows:

$$\begin{aligned}\frac{d\tilde{\eta}^a}{d\tau} &= \beta\phi(\tilde{\eta}^a + \eta_{ss}^a, \xi^a) - \beta\phi(\eta_{ss}^a, 0)\\ \frac{d\xi^a}{d\tau} &= \beta L_f h^a + \beta L_g h^a w_r^a.\end{aligned} \quad (12)$$

The averaged controller can be written in terms of the averaged state variables using the dynamical inversion algorithm:

$$\frac{d\xi_r^a}{d\tau} = -\omega_\ell \xi_r^a(\xi_2^a \xi_r^a - 1). \quad (13)$$

and the controller

$$w_r^a = -M\mathrm{Sat}\left(\frac{\xi_r^a}{M}(\xi_1^a + k\xi^a)\right). \quad (14)$$

We define the set

$$S^a = \{\xi^a \in \mathbb{R},\ \xi_1^a \in \mathbb{R},\ \xi_r^a \in \mathbb{R}\, | |w_r| < M\}.$$

For any initial conditions in the set S^a, the controller operates in the linear region of the saturation function.

If one rewrites the output dynamics as follows:

$$\frac{d\xi^a}{d\tau} = (\beta L_f h^a - \xi_1^a) + \xi_1^a + \beta L_g h^a w_r^a$$

one can conclude that, upon substitution of w_r^a for states in the set S^a, one obtains:

$$\frac{d\xi^a}{d\tau} = (\beta L_f h^a - \xi_1^a) + (\beta L_g h^a - {\xi_r^a}^{-1})w_r^a - k\xi^a.$$

4.2 Stability of the Averaged Dynamics

First, we consider the output dynamics subject to the feedback (14) inside the saturation. (Note: in the following, we drop the superscript a. The quantities should be understood as trajectories of the averaged closed-loop system.) This yields:

$$\frac{d\xi}{d\tau} = (\beta L_f h - \xi_1) + (\beta L_g h - {\xi_r}^{-1})w_r - k\xi.$$

We define the functions $L_f h = \alpha_1(\tilde{\eta} + \eta_{ss}, \xi)$ and $L_g h = \alpha_2(\tilde{\eta} + \eta_{ss}, \xi)$. We note that at the equilibirum $(\xi, \tilde{\eta}) = (0, 0)$. The corresponding equilibrium expressions for $L_f h$ and $L_g h$ are defined as:

$$L_f h_{ss} = \alpha_1(\eta_{ss}, 0),\ L_g h_{ss} = \alpha_2(\eta_{ss}, 0).$$

It follows that the equilibirum for the filters and the dynamic inversion dynamics are $\xi_{1,ss} = L_f h_{ss}$, $\xi_{2,ss} = L_g h_{ss}$ and $\xi_{r,ss} = L_g h_{ss}^{-1}$.

Next, we pose the Lyapunov function candidate, $V_1 = \frac{1}{2}\xi^2 + \frac{1}{2}(\xi_1 - \beta L_f h)^2 + \frac{1}{2}(\xi_2 - \beta L_g h)^2$. For all initial conditions in Ω_c, there exist positive constants L_1, L_2, L_3 and L_4 such that: $\left\|\frac{\partial\alpha_1}{\partial\xi}\right\| \le L_1$, $\left\|\frac{\partial\alpha_1}{\partial\eta}\right\| \le L_2$, $\left\|\frac{\partial\alpha_2}{\partial\xi}\right\| \le L_3$ and $\left\|\frac{\partial\alpha_2}{\partial\eta}\right\| \le L_4$. Furthermore, we have that $\|w_r\| \le M$.

This leads to the following inequality:

$$\begin{aligned}\frac{dV_1}{d\tau} &\le -k\xi^2 - \omega_l(\xi_1 - \beta L_f h)^2 - \omega_l(\xi_2 - \beta L_g h)^2\\ &+ \|\xi\|\|\xi_1 - \beta L_f h\| + M\|\xi\|\|\xi_r^{-1} - \beta L_g h\|\\ &+ L_1\beta(\xi_1 - \beta L_f h)^2 + kL_1\beta\|\xi_1 - \beta L_f h\|\|\xi\|\\ &+ L_1 M\beta\|\xi_1 - \beta L_f h\|\|\xi_r^{-1} - \beta L_g h\|\\ &+ \beta^2 L_2\|\xi_1 - \beta L_f h\|\|\phi(\tilde{\eta} + \eta_{ss}, \xi)\|\\ &+ L_3\beta\|\xi_2 - \beta L_g h\|\|\xi_1 - \beta L_f h\| + k\beta L_3\|\xi_2 - \beta L_g h\|\|\xi\|\\ &+ L_3\beta M\|\xi_2 - \beta L_g h)\|\|\xi_r^{-1} - \beta L_g h\|\\ &+ \beta^2 L_4\|\xi_2 - \beta L_g h\|\|\phi(\tilde{\eta} + \eta_{ss}, \xi)\|.\end{aligned}$$

If one considers the dynamic inverse algorithm (13), we pose the Lyapunov function candidate: $V_{inv} = \frac{1}{2}(\xi_r^{-1} - (\beta L_g h))^2$.

It can be shown that the rate of change of V_{inv} satisfies the inequality:

$$\begin{aligned}\frac{dV_{inv}}{d\tau} &\le -\omega_l(\xi_r^{-1} - \beta L_g h)^2 + \omega_l\|\xi_r^{-1} - \beta L_g h\|\|\xi_2 - \beta L_g h\|\\ &+ L_3\beta\|\xi_r^{-1} - \beta L_g h\|\|\xi_1 - \beta L_f h\|\\ &+ \beta L_3 M\|\beta L_g h - \xi_r^{-1}\|^2 + \beta k L_3\|\beta L_g h - \xi_r^{-1}\|\|\xi\|\\ &+ \beta^2 L_4\|\xi_r^{-1} - \beta L_g h\|\|\phi(\tilde{\eta} + \eta_{ss}, \xi)\|\end{aligned}$$

Then, we define $V_2 = V_1 + V_{inv}$. By the uniform Lipschitz property of the error $\tilde{\eta}$ dynamics, there exist positive constants L_ϕ and L_{ss} such that $\|\phi(\eta, \xi) - \phi(\eta, 0)\| \le L_\phi\|\xi\|$ and $\|\phi(\tilde{\eta} + \eta_{ss}, \xi) - \phi(\eta_{ss}, \xi)\| \le L_{ss}\|\tilde{\eta}\|$. In addition, we note, that it is to show for any choice of $k > 0$ and $\omega_l > 0$, there exists a value of $\beta_1 > 0$ such that for all $0 < \beta \le \beta_1$, the rate of change of V_2 satisfies the inequality:

$$\begin{aligned}\frac{dV_2}{d\tau} &\le -k_1\xi^2 - k_2(\xi_1 - \beta L_f h)^2 - k_3(\xi_2 - \beta L_g h)^2\\ &- k_4(\xi_r^{-1} - \beta L_g h)^2 + \beta^2 L_\phi L_2\|\xi_1 - \beta L_f h\|\|\xi\|\\ &+ \beta^2 L_{ss} L_2\|\xi_1 - \beta L_f h\|\|\tilde{\eta}\| + \beta^2 L_4 L_\phi\|\xi_2 - \beta L_g h\|\|\xi\|\\ &+ \beta^2 L_4 L_{ss}\|\xi_2 - \beta L_g h\|\|\tilde{\eta}\| + \beta^2 L_4 L_\phi\|\xi_r^{-1} - \beta L_g h\|\|\xi\|\\ &+ \beta^2 L_4 L_{ss}\|\xi_r^{-1} - \beta L_g h\|\|\tilde{\eta}\|\end{aligned}$$

for positive constants k_1, k_2, k_3 and k_4. Finally, we consider the Lyapunov function candidate $V_T = V_2 + V_\eta$. It can be shown that there exists β_2 such that for every $\beta \leq \min[\beta_1, \beta_2] = \beta^*$:

$$\frac{dV_T}{d\tau} \leq - k_1'\xi^2 - k_2'(\xi_1 - \beta L_f h)^2 - k_3'(\xi_2 - \beta L_g h)^2 - k_4'(\xi_r^{-1} - \beta L_g h)^2 - k_5'\|\tilde{\eta}\|^2$$

for positive constants $k_1\prime$, k_2', k_3', k_4' and k_5'.

The stability holds for every initial condition $(\tilde{\eta}, \xi) \in \Omega_d$ and $(\xi_1, \xi_2, \xi_r) \in S^a$. We identify elements of S^a with level sets of the Lyapunov function V_T as follows. The feedback $\hat{\psi}(\xi_1, \xi_r, \xi)$ is such that:

$$\begin{aligned}|\hat{\psi}| \leq& |\xi_r - (\beta L_g h)^{-1}|(|\xi_1| + k|\xi|) + |(\beta L_g h)^{-1}|(|\xi_1| + k|\xi|)\\ \leq& |\xi_r - (\beta L_g h)^{-1}||\xi_1 - \beta L_f h| + |\xi_r - (\beta L_g h)^{-1}||\beta L_f h|\\ &+ k|\xi_r - (\beta L_g h)^{-1}||\xi| + |(\beta L_g h)^{-1}||\xi_1 - \beta L_f h|\\ &+ |(\beta L_g h)^{-1}|(|\beta L_f h| + k|\xi|) = B_T(\xi, \eta, \xi_1, \xi_2, \xi_r).\end{aligned}$$

Thus, one can use the right hand side of the last inequality to show that there exists a level set of the Lyapunov function $V_T \leq \beta_T$ for which $|\hat{\psi}| < M$ by taking $B_T(\xi, \eta, \xi_1, \xi_2, \xi_r) < M$. It is defined as follows:

$$\Omega_T = \left\{(\tilde{\eta}, \xi) \in \Omega_d,\ (\xi_1, \xi_r) \in \mathbb{R}^2 \,|\, V_T < \beta_T\right\}.$$

Furthermore, one can show that there exists a constant β_0 such that:

$$\frac{dV_T}{d\tau} \leq - \beta_0 V_T.$$

Thus, the equilibrium of the closed-loop averaged system is exponentially stable in the τ timescale for initial conditions in the set Ω_T.

Next, we use the Lyapunov function V_T to assess the stability of the averaged system in the original timescale. Using the analysis presented above, it follows that the rate of change of V_T in the original timescale is such that:

$$\frac{dV_T}{dt} \leq - \frac{\beta_0}{\beta + A_1 \sin(\omega_1 t)} V_T.$$

The following upper bound is readily:

$$\frac{dV_T}{dt} \leq - \frac{\beta_0}{\beta + A_1} V_T.$$

As a result, one can conclude that the equilibrium point of the averaged closed-loop system is exponentially stable in the original timescale.

4.3 Stability of the closed-loop system

To establish the stability properties of our nominal system, one can focus on the averaging of the filter equations for $\xi_1(t)$ and $\xi_2(t)$. The remainder of the analysis considers a Lyapunov stability analysis of the closed-loop system.

To do so, we pose the Lyapunov function candidate $V_F = V_T + \frac{1}{2}\tilde{v}^2$ where V_T was defined above. We also rewrite the filter dynamics are follows:

$$\frac{d\xi_1}{d\tau} = -\omega_l(\xi_1 - \beta L_f h + M_1(\tau) + N_1(\tau)\tilde{v})$$

and

$$\frac{d\xi_2}{d\tau} = -\omega_l(\xi_2 - \beta L_g h + M_2(\tau) + N_2(\tau)\tilde{v})$$

where $N_1(\tau) = \frac{2\beta}{A_1}\sin(\omega_1\tau)$, $N_2(\tau) = \frac{2}{A_2}\sin(\omega_2\tau)$,

$$\begin{aligned}M_1(\tau) =& - \frac{2\beta^2}{A_1}\sin(\omega_1\tau)L_f h - \beta(2\sin(\omega_1\tau)^2 - 1)\beta L_f h\\ &- \frac{2\beta}{A_1}\sin(\omega_1\tau)L_g h(w_r + A_2\sin(\omega_2\tau)),\end{aligned}$$

and

$$\begin{aligned}M_2(\tau) =& - \frac{2}{A_2}\sin(\omega_2\tau)(\beta + A_1\sin(\omega_1\tau))L_f h\\ &- \frac{2\beta}{A_2}\sin(\omega_2\tau)L_g h w_r - \beta(2\sin(\omega_2\tau)^2 - 1)L_g h).\end{aligned}$$

We note that $M_1(\tau)$ and $M_2(\tau)$ are both uniformly bounded on Ω_d. Hence, there exist positive constants $\bar{M}_1$ and $\bar{M}_2$ such that $\|M_1(\tau)\| \leq \beta\bar{M}_1$ and $\|M_2(\tau)\| \leq \beta\bar{M}_2$. (Note that the bound on $M_2(\tau)$ utilizes the fact that $A_1 = \rho\beta$ where $\rho \in (0, 1)$). Similarly, we have that $\|N_1(\tau) \leq \frac{2\beta}{A_1}$ and $\|N_2(\tau)\| \leq \frac{2}{A_2}$. Following the analysis above, one can write:

$$\begin{aligned}\frac{dV_F}{d\tau} \leq& -k_1'\xi^2 - k_2'\|\xi_1 - \beta L_f h\|^2 - k_3'\|\xi_2 - \beta L_g h\|^2\\ &- k_4'\|\xi_r^{-1} - \beta L_g h\|^2 - k_5'\|\tilde{\eta}\|^2 + \|M_1(\tau)\|\|\xi_1 - \beta L_f h\|\\ &+ \|M_2(\tau)\|\|\xi_2 - \beta L_g h\| + \|N_1(\tau)\|\|\tilde{v}\|\|\xi_1 - \beta L_f h\|\\ &+ \|N_2(\tau)\|\|\tilde{v}\|\|\xi_2 - \beta L_g h\| - \omega_r\tilde{v}^2 + \tilde{v}\frac{d^2y}{d\tau^2}.\end{aligned}$$

We define the set $\Omega = \Omega_T \times \left\{\tilde{v}^2 < M_v\right\}$. First, one can easily conclude that $\left|\frac{d^2y}{d\tau^2}\right| \leq L$ on the set Ω. As a result, one can write:

$$\begin{aligned}\frac{dV_F}{d\tau} \leq& -k_1'\xi^2 - k_2'\|\xi_1 - \beta L_f h\|^2 - k_3'\|\xi_2 - \beta L_g h\|^2\\ &- k_4'\|\xi_r^{-1} - \beta L_g h\|^2 - k_5'\|\tilde{\eta}\|^2 + \beta\bar{M}_1\|\xi_1 - \beta L_f h\|\\ &+ \bar{M}_2\|\xi_2 - \beta L_g h\| + \frac{2\beta}{A_1}\|\tilde{v}\|\|\xi_1 - \beta L_f h\|\\ &+ \frac{2}{A_2}\|\tilde{v}\|\|\xi_2 - \beta L_g h\| - \omega_r\tilde{v}^2 + L\|\tilde{v}\|.\end{aligned}$$

This implies that there exist positive constants k_2'', k_3'', k_4'', k_6' and $\bar{M}$ such that:

$$\begin{aligned}\frac{dV_T}{d\tau} \leq& - k_1'\xi^2 - k_2''\|\xi_1 - \beta L_f h\|^2 - k_3''\|\xi_2 - \beta L_g h\|^2\\ &- k_4''\|\xi_r^{-1} - \beta L_g h\|^2 - k_5'\|\tilde{\eta}\|^2 - k_6'\tilde{v}^2 + \beta\bar{M} + \frac{L^2}{k_6'}.\end{aligned}$$

Repeating as above, we can find a positive constant β_0' such that:

$$\frac{dV_T}{d\tau} \leq - \beta_0' V_T + \beta\bar{M} + \frac{L^2}{k_6'}.$$

Thus, we conclude that the equilibrium of the closed-loop system is a practically stable equilibrium for initial conditions in Ω_T.

5. SIMULATION STUDY

We consider the following second order nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + 5x_2^2,\\ \dot{x}_2 &= x_1 + x_2 - x_2^3 - (2 + x_1)u\\ y &= x_2 - 1 + \sin(15t)/2.\end{aligned}$$

The objective is to stabilize the system and achieve regulation of the measured output to $y = 0$. We implement the proposed control system with the following tuning parameters: $\beta = 0.11$, $A_1 = 0.1$, $A_2 = 1$, $\omega_1 = 550$, $\omega_2 = 250$, $\omega_{l1} = 2$, $\omega_h = 2000$ and $k = 5$.

The closed-loop trajectories of the state variables x_1, x_2 and the output y are shown in Figure 1. We note that the resulting control system achieves regulation of the output. The performance of the drift term, $L_f h$, estimation is shown in Figure 2. The proposed technique provide an adequate unstructured identification of this unknown time-varying nonlinearity.

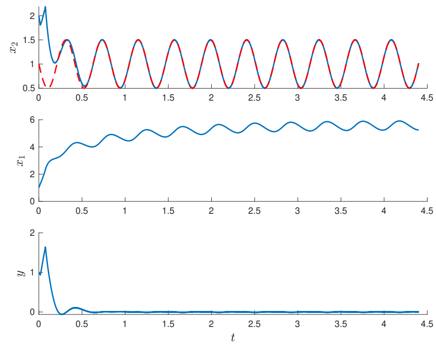

Fig. 1. Closed-loop trajectories of the system. The state x_2 is shown in the upper plot, x_1 in the middle plot and the output y, in the bottom plot.

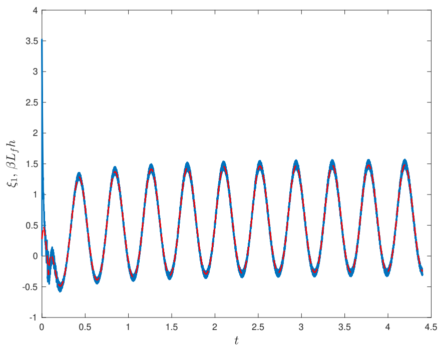

Fig. 2. Estimation of the drift, $\beta L_f h$. The true value is shown as the dashed (- -) red line and the estimate is shown in blue.

6. CONCLUSION

This study proposes a new strategy for the estimation of drift dynamics in unknown nonlinear dynamics. The approach expresses the system in a new time scale that introduces an artificial dither. The resulting timescale dither can be used to provide an estimate of signals in the output dynamics that do not depend on the inputs. This approach is used to implement a state-feedback controller that provides compensation of the drift dynamics.

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