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Path-Complete Methods and Analysis of Constrained Switching Systems

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Thesis submitted in partial fulfillment of the requirements for the Ph.D. Degree in Engineering Sciences

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Louvain-la-Neuve, 2017.

Acknowledgments

We are trained, as engineers, to look for solutions to problems. You may understand my excitement when, at the beginning of this thesis, my advisor presented me with a range of questions so much beyond my grasp at that time. This thesis has been a wonderful professional and personal experience where I continuously learned to learn, learned to try and experiment, and learned to explain and teach.

Prof. Raphaël Jungers is an amazing advisor. He communicates his seemingly unlimited energy, passion and curiosity to his students. On the one hand, he grants us a total freedom to experiment in our work, and on the other hand, he keeps challenging us and pushing us ever further in developing our skills. Perfect examples of this are the several times he supported me in the pursuit of grants and awards. Furthermore, he manages to do all of this with a particular attention to the well-being of his students. I am grateful for all he taught me, on both the professional and personal level.
Along with Raphaël, I want to thank the members of my support committee, Prof. Olivier Bonaventure and Prof. Yurii Nesterov, for our interesting discussions and for their feedback and support during the thesis.

I cannot think of a better jury for evaluating my manuscript than the one I had. Many thanks to Profs. Philippe Lefèvre, Nikolaos Athanasopoulos, Geir Dullerud, Maurice Heemels, Yurii Nesterov, and of course my advisor, for their dedication in the feedbacks on the preliminary version of this manuscript and the discussions and ideas we shared during the private defense.

Many thanks to the brilliant researchers I had the pleasure to work with. In this regard, I would like to thank Profs. David Angeli, Nikolaos Athanasopoulos, Geir Dullerud and Gilles Millérioux as well as Ray Essick for their warm welcome in their research labs all around the world.
In particular, Nikos, thank you for the numerous (unexpectedly) long discussions we've had while you were a post-doc of Raphaël. I wish you the best of luck with your new position!

The ICTEAM institute and the Applied Mathematics Department definitely provides a thoroughly enjoyable work environment. It has been a pleasure to exchange with the brilliant academic staff, and I would like to thank in particular Profs. Jean-Charles Delvenne and Julien Hendrickx for sharing ideas and pointers to further my own work.
Thanks to François W., Etienne, Isabelle, Nathalie and Marie-Christine who have helped me so many times regarding technical and administrative issues.
My dear Corentin Vande Kerckhove, sharing an office with you during this thesis has been a pleasure and a priviledge. You are a scholar, a gentleman, and I will miss your magic tricks. I wish you the best of luck for the future,

and that you will continue to create and enjoy new opportunities!
Many thanks to my Eulerian colleages: Adeline, Adrien, Romain, Allan, Nicolas, Benjamin, Corentin, David, Emilie, Estelle, François G., Luc, Pierre-Antoine B., Pierre-Yves C. plus all the others who made this experience so enjoyable. I'm counting on you to keep up with the traditions that make this place so special: bring pies to celebrate any good news, organize challenging yet hilarious team-building events, and keep playing foosball!
Thanks to my fellow officers from the UCLouvain IEEE student branch for trusting me and joining me in the organization of our many events (and for your thorough appreciation of my clever naming schemes)!

Je souhaite aussi remercier, du fond du coeur, mes amis et ma famille pour leur soutien, amour et amitié qui m'ont donné des ailes durant les moments difficiles. En particulier Jag, Jerem, Laurent, Pygou et Steph, Bapt et Sissi, P.A. et Emilie, Géo et Pauline, EPLOL, la promotion MAP2013, les bros, Marianne et Paul, merci pour tout!

Catherine, je te remercie pour cette belle aventure qui a recouvert nos études et doctorats!

Abstract

Switching systems are dynamical systems having several operating modes. For example, consider a controlled system where the controller may take several configurations, e.g. failure or active. Other examples include multi-agent systems where varying communication patterns dictate the dynamics of agents. They also appear as models of cyber-physical systems, which result from the interaction of algorithms and physical devices. These applications are key in motivating researchers to tackle the deep mathematical challenges arising from the study of switching systems.

In this thesis, we study a general class of switching systems where the sequences of modes in time are generated by an automaton. They are known as *constrained switching systems*. We provide theory and algorithms for their stability analysis, discussing the notions of uniform, exponential and dead-beat stability, and including I/O stability and feedback stabilization. In particular, we provide tools to approximate the exponential decay rate of constrained switching systems arbitrarily accurately in finite time.

Our techniques revolve around and expand Path-Complete Methods. These are mathematical tools merging automata theory with multiple Lyapunov functions within a general stability analysis framework. These techniques show potential for tackling further challenges, which pushes us to further analyze them. First, in the context of arbitrary switching systems, we show how these multiple Lyapunov functions can be represented as common Lyapunov functions. Second, we provide algorithms to compare their performances as stability certificates.

This work contributes to a global effort in the study of complex dynamical systems where computer science and systems theory meet.

Contents

Notations

Vectors and Matrices

$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	Real, natural and integer numbers
$\mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{>0}$	Non-negative, positive real vectors in dimension n
$(x)_i$	Entry i in the vector x
$(A)_{i,j}$	Entry at row i and column j in A
$Q \succ 0$, $Q \succeq 0$	Q is positive definite, positive semi-definite
$\mathbf{0}, \mathbf{1}$	Vector (or matrix) full of zeroes, and full of ones
I	Identity matrix

Graphs and Languages

$\langle M \rangle$	Set $\{1, \ldots, M\}$ for the integer M (a.k.a. alphabet)
$\langle M \rangle^k$	Words of length k on the alphabet $\langle M \rangle$
$\mathbf{G} = (S, E)$	Directed labeled graph, with nodes S and edges E
$\mathscr{P}(\mathbf{G})$, $\mathscr{P}^k(\mathbf{G})$	Paths in a graph, paths of length k
$\mathscr{P}_s(\mathbf{G})$, $\mathscr{P}_{s,d}(\mathbf{G})$	Paths from node s, from node s to node d
$\mathscr{L}(\mathbf{G})$	Language of a graph
ϵ	The empty word
$\lvert Z \rvert$	Cardinality of a discrete set Z
$\lvert w \rvert$	The length of the word w
$\lvert p \rvert$	The length of the path p
$w(p), p \in \mathscr{P}(\mathbf{G})$	The word on the path p in the graph $\mathbf{G}$

Acronyms

CJSR	Constrained joint spectral radius
CSS	Constrained switching system
GLF	Graph-Lyapunov function
PCLF	Path-complete Lyapunov function
VLFC	Vector Lyapunov function candidate

Introduction

Our technology is getting smarter. The rise in available computing power paired with recent advances in the fields of computer sciences, communication technologies, and control systems allowed us to create amazing devices. Naturally, as we imagine more and more complex technologies, such as autonomous cars, smart grids, smart cities, etc..., we need new tools for the design and analysis that encompass the complex interactions between the algorithmic and physical worlds. Generally speaking, such systems at the intersection between the software and the physical worlds are referred to as *cyber-physical systems*. Their promising applications trigger a wide and multidisciplinary research effort [CL09, Tab09, GST12] for their analysis and control.

The first step in conducting the analysis of such a complex system is to construct a *mathematical model*. There, hybrid systems are modeling tools that find tremendous applications for the study of cyber-physical systems and more (see [GST12, CL09] for a general exposition). They allow to model and capture the interactions between discrete dynamics (such as the evolution of the steps of an algorithm [CL09]) and continuous dynamics (such as the motion of a vehicle or the dynamics of an electrical circuit). However, the richness of hybrid systems makes them hard to study, and often beyond the reach of classical analysis tools. Our aim in this thesis is to provide innovative tools and general insights for the analysis and understanding of hybrid systems. In order to achieve that, we focus on a rich subclass of hybrid systems known as *switching systems*.

Switching systems [LM99, Jun09, SG11] are hybrid systems where the discrete components boil down to a *switch*. This switch selects, at every time, an active *mode* for the system that drives the evolution of its continuous component. Switching systems arise naturally in applications where the equations describing the evolution of a system cannot be captured adequately by a single model, e.g. for viral dynamics under switching treatments [HVMC11], congestion control in computer networks [SWL06], networked control systems [DHvdWH11, JHK16], consensus and multi-agent systems [BHOT05, OSM04].

The analysis of switching systems presents us with several mathematical challenges and theoretical limitations, in particular for the matter of stability analysis where we face Undecidability, NP-Hardness, etc... [BT00a, BT00b, TB97]. Despite these barriers, several usefull tools and algorithms have been developed for the stability analysis of switching systems, see e.g. [AJPR14, DRI02, LD06b, Jun09, The05, PJ08].

In this thesis, we study first the stability of switching systems *under constrained switching*. This refers to a situation where we know that the sequences of modes in time have to follow some rules (e.g. a mode can be active at max-

imum N time-steps in a row before switching to another mode). These rules are encoded, using language theoretical concepts, by an *automaton*. Related to hybrid systems above, the automaton itself can be used to represent an algorithm that directs the switching system. This setup allows us to describe a vast range of rules appearing in practice in application.
A particular case of constrained switching systems is an *arbitrary switching system*, where any switching sequence may occur, without restrictions. Most of the stability analysis tools mentioned above were first developed for the arbitrary switching case and benefit from a deep analysis and guarantees there. However, prior to our work, while some tools can easily be translated to constrained switching systems, there existed a gap in the theory regarding their performances between the arbitrary and the constrained switching cases. Our work fills this gap, providing the required theoretical foundations through which we may analyze these tools, and allowing to develop them further.

Our tools take their inspirations and generalize *Path-Complete Methods*, showing their applicability outside of the scope of their original presentation in [AJPR14]. There, the authors develop techniques, merging automata theory and systems theory, within a unified stability analysis framework.
In our work, we begin by generalizing them to study the uniform stability, exponential stability and dead-beat stability of constrained switching systems. We then consider switching systems with disturbance inputs, where we provide new characterizations of input-output stability, and systems with control inputs, where we study a class of switching feedback controllers developed in the literature for which our analysis techniques are found to be well adapted.
Motivated by the potential of Path-Complete Methods, we then proceed to study the tools themselves. We first exhibit a fundamental link between them and common Lyapunov functions in the context of arbitrary switching systems, and then provide the first systematic tools for analyzing the conservativeness of the methods.
A more precise outline is drawn below.

Outline

The outline of this thesis is illustrated in Figure 1.

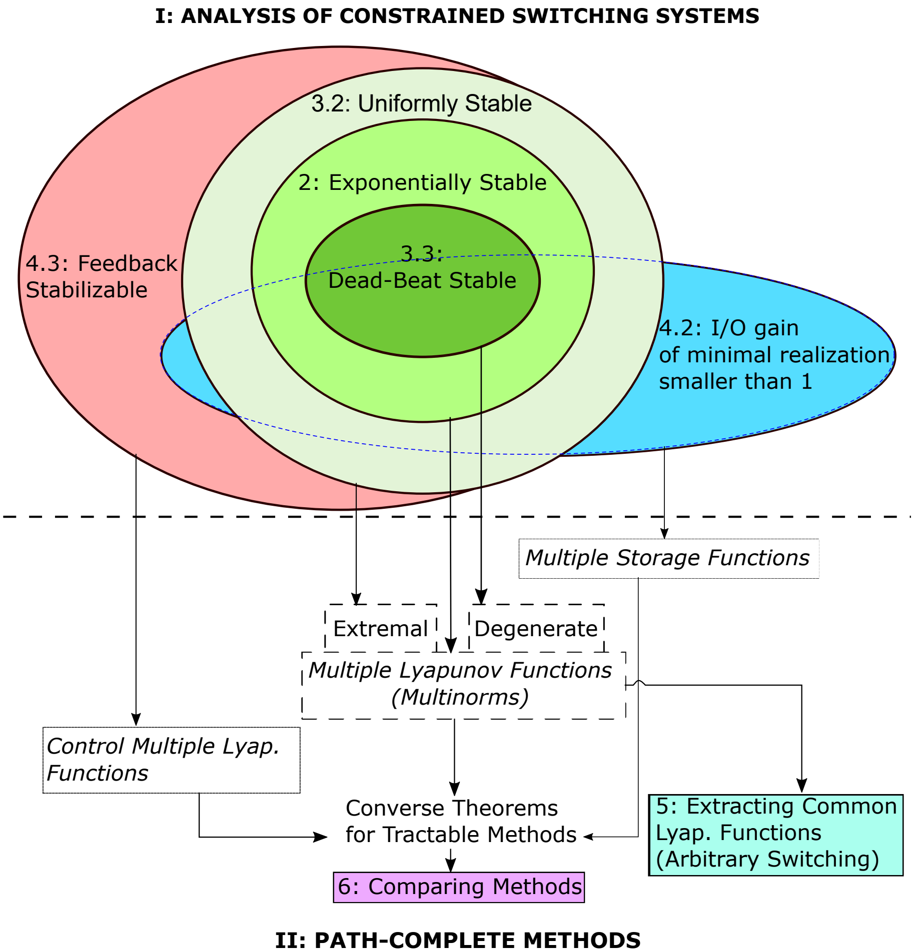


Figure 1: Thesis outline. The upper part represents sets of switching systems studied in Part I. *Exponentially* stable systems are studied first in Chapter 2. We study the respectively weaker and stronger notions of *uniform* and *dead-beat* stability in Chapter 3. In Chapter 4, we study systems with input and outputs. We analyze first their *I/O stability* through the characterization of their I/O gains, then their *stabilization by feedback*. In Part II (lower half of the figure), we put the tools developed in Part I within the general Path-Complete Framework. There, Chapter 5 provides connections to classical Lyapunov theory and common Lyapunov functions techniques, and Chapter 6 provides techniques for comparing the conservativeness of the stability analysis tools.

Chapter 1: Preliminaries

In this first chapter, we introduce concepts that are central to the whole thesis. In particular, we introduce the reader to *constrained switching systems* and to key notions related to their stability, such as norms and Lyapunov function candidates. Our constrained switching systems rely on *directed and labeled graphs* to set rules on the switching sequences.

Part I: Analysis of Constrained Switching Systems

In Part I, we provide tools for the stability analysis of constrained switching systems.

Chapter 2: Approximating the Exponential Growth Rate

We published the research for this chapter in [PEDJ16b, PJ15a].

In this chapter, we first present several key concepts related to the exponential stability of constrained switching systems, and in particular, their exponential growth rate known as the *constrained joint spectral radius (CJSR)* [Dai12]. We then provide an exact characterization of exponential stability using the concept of *multinorm*: multiple Lyapunov functions with exactly one function per node of the graph defining the switching sequences. Building on top of this concept, we generalize several key results existing for arbitrary switching systems and provide a family of algorithms that allow for an arbitrarily accurate estimate of the constrained joint spectral radius. In doing so, we effectively solve the issue of extending the Path-Complete Methods of [AJPR14] to constrained switching systems. This allows us to shed new lights, under the form of converse theorems involving the constrained joint spectral radius, on the performances of existing tools [DB01,LD06b,ELD14,PJ08] for the stability analysis of constrained switching systems.

Chapter 3: Stability and Dead-Beat Stability

We published the research for this chapter in [PMJ17, PJ15b], with the exception of Section 3.2.2.

We are interested in both the uniform and dead-beat stability of constrained switching systems. The results are related with the concept of *non-defectivity*, which with respect to the results of Chapter 2, guarantees the existence of *extremal multinorms* achieving a decay rate *equal to the CJSR.* For the topic of uniform stability, we focus on the case CJSR $= 1$, since CJSR < 1 implies exponential stability, and CJSR > 1 implies instability. In this case, for arbitrary switching system, a sufficient condition for stability is the *irreducibility* of the set of matrices describing the dynamics (i.e. where there are no non-trivial subspaces invariant under the dynamics) [Jun09, Theorem 2.1]. We show that the result does not carry directly to constrained switching systems, and provide two generalizations, leading to decidable sufficient conditions for stability. Dead-beat stability is the property that the state is mapped to the origin in finite time. We show that it amounts to the case CJSR $= 0$ and we provide a polynomial time algorithm for its decision. In this case, the system needs to be defective, so no extremal multinorms exist (except for trivial systems). We illustrate the usage of our algorithm on an application dealing with flatness-based control of an electrical vehicle.

Chapter 4: Systems with Inputs and Outputs
We published the research for this chapter in [EPDJ15] and [PEDJ16a].

We enrich our previous setting and consider discrete-time linear constrained switching systems with inputs and outputs. We are interested in two problems. First, we study the *input-output stability* properties of constrained switching systems. We characterize the $\mathcal{L}_p$-gains of such systems by providing a theoretical framework involving multiple storage functions, similar to the multinorms of Chapter 2. We then focus on the $\mathcal{L}_2$-gain and provide a converse theorem for the existence of quadratic storage functions.
Second, we assume that the input is a control input. We study a class of output-feedback switching controllers whose basis for synthesis relies on LMIs [LD06b, DB01]. We apply the multinorm framework of Chapter 2 and obtain guarantees on the performances of such controllers when obtained through LMIs relative to the ideal decay rate achievable by such controllers.

Part II: Path-Complete Methods

In Part II, we make take a step back and analyze the *Lyapunov methods* developed in Part I through the prism of the *Path-Complete Framework*. This framework has been introduced in [AJPR14] for analyzing Lyapunov methods in the context of arbitrary switching systems.

Chapter 5: The Flexibility of Path-Complete Methods
The results have been published in part in [AAJP17a].

This chapter introduces the Path-Complete Framework in the contexts of Chapters 2, 3 and 4. In these three chapters, we introduced stability/invariance certificates in a general way, by defining a *multiple Lyapunov functions* having to satisfy Lyapunov inequalities encoded in a directed and labeled graph. After a presentation of the framework, we focus on the case of arbitrary switching systems. There, using automata theoretical tools, we give a positive answer to the following question: *can we always represent a Path-Complete Lyapunov function as a common Lyapunov function?* This brigdes an existing gap between the concept of multiple Lyapunov functions and the classical Lyapunov theory.

Chapter 6: Comparison of Path-Complete Methods
The results have been published in [AAJP17a] and [AAJP17b].

The Path-Complete Framework provides a multitude of ways for one to design stability certificates for switching systems. However, in general, theoretical barriers dictate that whenever checking for the existence of a certificate can be done in finite time, that certificate is conservative. This leads us to a very natural question: *which are the certificates that are provably less conservative than others?*
In this chapter, we provide tools to decide and characterize when, given two types of multiple Lyapunov functions within the Path-Complete framework, one leads to less conservative stability certificates than the other. Two approaches are presented. The first is purely combinatorial revealing joint structures in the graphs of Lyapunov inequalities that, when present, allow us to conclude that one tool is more conservative than the other.
The second develops a *linear programming formulation* for solving this problem, extracted from a study of the graphs of Lyapunov inequalities. In particular, this provides a necessary and sufficient condition to decide when, given two classes of Path-Complete Lyapunov functions, if we know that the first provides a stability certificate, we can construct a multiple Lyapunov function of the second class where the pieces are conic combinations of the pieces of the first. This encompasses our first combinatorial criterion. It is a first step towards a systematic comparison of the conservativeness of Path-Complete methods.

1 | Preliminaries

The goal of these preliminaries is to introduce *constrained switching system* and other key concepts involved in their stability analysis.

- In Section 1.1, we recall some basic notions of linear algebra and define the concept of discrete-time switching systems, in general.
- In Section 1.2, we introduce notions relative to the stability of switching systems and Lyapunov methods.
- In Section 1.3 we present the general tools used to define constraints on the switching sequences driving switching systems
- Last, in Section 1.4 we put the concepts above into good use and properly define the concept of constrained switching systems.

The presentation of this chapter covers basic notions and definitions. More advanced concepts, such as the *joint spectral radius* and *Path-Complete Methods*, are introduced in later chapters.

1.1 Vectors, Matrices, and Systems

Vectors and Matrices

We begin with some conventions relative to matrices and vectors. Consider a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times m}$, for some integers n and m.

- For $1 \leq i \leq n$, $(x)_i$ denotes the ith component of the vector.
- For $1 \leq i \leq n$, $1 \leq j \leq m$, $(A)_{i,j}$ denotes the component on the ith row and jth column of A. By convention, a vector of size n is represented as a $n \times 1$ matrix. The transpose of a matrix A is $A^\top$.

Two particular vectors are the *origin* $\mathbf{0} \in \mathbb{R}^n$, which is the vector where all components are 0, and the *unit vector* $\mathbf{1} \in \mathbb{R}^n$, where all components are 1. We use similar notations for the matrices where all components are 0 and 1 respectively. The identity matrix is denoted by $I \in \mathbb{R}^{n \times n}$.
Given two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$, the Kroenecker product of A and B is a matrix of dimension $(mp) \times (nq)$ defined as follows:

$$A \otimes B = \begin{bmatrix} (A)_{1,1}B & \cdots & (A)_{1,n}B \\ \vdots & \ddots & \vdots \\ (A)_{m,1}B & \cdots & (A)_{m,n}B \end{bmatrix}.$$

Subspaces

A *linear subspace* $\mathcal{X} \subseteq \mathbb{R}^n$ is a set of vectors such that if $x \in \mathcal{X}$, then $\alpha x \in \mathcal{X}$ for all $\alpha \in \mathbb{R}$, and for all pair x, y in $\mathcal{X}$, $x + y \in \mathcal{X}$. Given a set of k vectors $\{x_1, x_2, \ldots, x_k\}$,

$$\operatorname{span}\{x_1, x_2, \ldots, x_k\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists \alpha_1, \ldots, \alpha_k \in \mathbb{R} : x = \sum_i \alpha_i x_i \right\}.$$

A set of vectors $x_1, x_2, \ldots, x_k$ *spans* $\mathcal{X}$ if any $y \in \mathcal{X}$ can be obtained as a linear combination of the vectors in the set: $y = \sum_{i=1}^k \alpha_i x_i$, for some $\alpha_i \in \mathbb{R}$, $1 \leq i \leq k$. A *basis* for $\mathcal{X}$ is a set of linearly independent vectors, i.e. where no vector can be expressed as a linear combination of the others, that span the subspace. The *dimension* of $\mathcal{X}$ is the number of vectors in any of its basis.
Given two subspace $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbb{R}^n$, their Minkoswki sum is

$$\mathcal{X} + \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

The *image* of a matrix $A \in \mathbb{R}^{m \times n}$ is the subspace

$$\operatorname{im}(A) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n : y = Ax\}.$$

The *kernel* of a matrix $A \in \mathbb{R}^{m \times n}$ is the subspace

$$\ker(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}.$$

Switching Systems

In this thesis, we study *discrete-time switching dynamical systems.* In general they take the form

$$x(t+1) = f_{\sigma(t)}(x(t)), \tag{1.1}$$

where

- $t \in \mathbb{N}$ represents the time,
- $x(t) \in \mathbb{R}^n$ is the state of the system at time t, and $x(0)$ is called the initial condition of the system, and the sequence of states $x(0), x(1), \ldots$ is a trajectory,
- $\sigma(t)$ is called the *mode* of the system at time t. It takes values in the set $\{1, \cdots, M\}$ for some integer M. In time, the sequence $\sigma(0)\sigma(1)\ldots$ is referred to as a *switching sequence.*
- $f_{\sigma(t)}$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, called the dynamics of the system for mode $\sigma(t)$.

Definition 1.1. *A switching system (1.1) is* linear *if for each mode* $\sigma \in \{1, \ldots, M\}$*, the dynamics are* linear*, i.e*

$$\forall x \in \mathbb{R}^n : f_\sigma(x) = A_\sigma x,$$

where $A_\sigma \in \mathbb{R}^{n\times n}$.

Our aim is to study the stability properties of switching systems *knowing* that the switching sequences follow logical rules. An example for such rules is the following: for all time t, $\sigma(t) = 2$, then $\sigma(t+1) \in \{4, 5\}$. More generally, these rules define sets $\mathscr{L}$ of *switching sequences*, in which we know the switching sequence driving belongs. These sets are defined later, using notions from language theory.

1.2 Stability, Norms, and Lyapunov Functions

Stability

We study several stability notions for *switching systems under constrained switching sequences*, where the switching sequences belong to a given set $\mathscr{L}$. We define notions of stability in a classical way, by making use of functions of class $\mathcal{K}$ and $\mathcal{K}_\infty$ (see e.g. [Son08], [Lib12, Appendix A]; see [Kel14] for a survey on the topic).

Definition 1.2 (Class $\mathcal{K}$ and $\mathcal{K}_\infty$)**.** *The scalar function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is of class* $\mathcal{K}$ *if it is* continuous, positive definite *and* strictly increasing.
The scalar function $\alpha : \mathbb{R}_{\geq 0} \to\to \mathbb{R}_{\geq 0}$ *is of class* $\mathcal{K}_\infty$ *if it is of class* $\mathcal{K}$ *and* unbounded.

Functions of class $\mathcal{K}_\infty$ have several interesting properties. Notice that

- If α is of class $\mathcal{K}_\infty$, the inverse α^{-1} exists and is of class $\mathcal{K}_\infty$.
- If α and β are of class $\mathcal{K}_\infty$, so is their composition $\alpha \circ \beta$.

In the below, $\|x\|$ denotes the *euclidean norm* of the vector $x \in \mathbb{R}^n$: $\|x\| = \sum_{i=1}^n ((x)_i^2)^{1/2}$.

Definition 1.3. *Consider a switching system (1.1) on M modes. Let $\mathscr{L}$ be a* set of switching sequences.
We say that

- *the system is* stable *if there exists a function α of class $\mathcal{K}_\infty$ such that*
$$\forall \sigma(0)\sigma(1)\ldots \in \mathscr{L}, \forall x(0) \in \mathbb{R}^n, \forall t \in \mathbb{N} : \|x(t)\| \leq \alpha(\|x(0)\|); \tag{1.2}$$
- *the system is* asymptotically stable *if it is* stable *and*
$$\forall \sigma(0)\sigma(1)\ldots \in \mathscr{L}, \forall x(0) \in \mathbb{R}^n : \lim_{t\to\infty} x(t) = \mathbf{0}; \tag{1.3}$$
- *the system is* exponentially stable *if there exist constants $\rho < 1$ and $K \geq 1$ such that*
$$\forall \sigma(0)\sigma(1)\ldots \in \mathscr{L}, \forall x(0) \in \mathbb{R}^n : \|x(t)\| \leq K\rho^t\|x(0)\|. \tag{1.4}$$

Remark 1.4. *If the dynamics are* homogeneous*, such as it is the case of* linear switching systems*, stability (1.2) is equivalent*[1] *to*

$$\exists K \geq 1, \forall \sigma(0)\sigma(1)\ldots \in \mathscr{L}, \forall x(0) \in \mathbb{R}^n, \forall t \in \mathbb{N} : \|x(t)\| \leq K\|x(0)\|. \tag{1.5}$$

The stability of a system is also known as *stability in the sense of Lyapunov.* Informally, the stability of a dynamical system can be understood as follows:

If the initial condition is close to *the origin,*
the state remains close to *the origin at all time.*

Lyapunov methods are popular tools to analyze the stability of dynamical systems. They rely on exhibiting the existence of a *Lyapunov function*, a distance-like functions with appropriate properties, that allows to quantify what *close to* means in the above. Lyapunov methods are central in our work, and we introduce important concepts below.
For linear systems, we will make use of *norms* as Lyapunov functions, and then for more general systems, we rely on more general constructions.

[1] For any switching sequence and $x(0) \neq \mathbf{0}$, if the system is stable, we divide the initial condition by $\|x(0)\|$ and we get by homogeneity $\|x(t)\|/\|x(0)\| \leq \alpha(1)$. Hence (1.5) holds with $K = \alpha(1)$.

Norms

Norms are central for the stability analysis of linear switching systems.

Definition 1.5. *A scalar function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* positive semi-definite *if*

$$\forall x \in \mathbb{R}^n : f(x) \geq 0 \text{ and } f(\mathbf{0}) = 0.$$

It is positive definite *if it is positive semi-definite and* $f(x) = 0 \Leftrightarrow x = \mathbf{0}$*.*

Definition 1.6. *A* (semi-)norm *on* $\mathbb{R}^n$ *is a scalar function* $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *that is*

1. *positively homogeneous of order 1, i.e.,*

$$\forall x \in \mathbb{R}^n, \forall \alpha \in \mathbb{R} : \|\alpha x\| = |\alpha|\|x\|,$$

where $|\alpha|$ *is the absolute value of* α*,*

2. *subbadditive, i.e.,*

$$\forall x, y \in \mathbb{R}^n : \|x + y\| \leq \|x\| + \|y\|,$$

3. *positive (semi-)definite.*

The homogeneity and subbadditivity of a norm imply its *convexity*. Subbadditivity is also known as the *triangle inequality*, and norms properly induce a notion of *distance* on $\mathbb{R}^n$.

Lemma 1.7. *Given two norms* $\|\cdot\|$ *and* $\|\cdot\|'$ *on* $\mathbb{R}^n$*, there exists real numbers* $0 < \alpha < \beta$ *such that*

$$\forall x \in \mathbb{R}^n : \alpha\|x\| \leq \|x\|' \leq \beta\|x\|.$$

Definition 1.8 (Matrix Norm)**.** *Given a norm* $\|\cdot\|$ *in* $\mathbb{R}^n$ *and a matrix* $A \in \mathbb{R}^{n\times n}$*, we define the induced matrix norm of* A *as*

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

Induced matrix norms are submultiplicative*: for* $A, B \in \mathbb{R}^{n\times n}$*,*

$$\|AB\| \leq \|A\|\|B\|.$$

In the text, unless specified otherwise, we use $\|A\|$ to denote the induced euclidean norm of the matrix $A \in \mathbb{R}^{n\times m}$.

Definition 1.9. *Consider a square and symmetric matrix* $Q \in \mathbb{R}^{n\times n}$*. The matrix is positive (semi-)definite if the function*

$$\mathbb{R}^n \to \mathbb{R} : x \to x^\top Q x$$

is positive (semi-)definite. We write $Q \succ 0$ *if* Q *is positive definite, and* $Q \succeq 0$ *if it is positive (semi-)definite.*

Definition 1.10. *A* quadratic (semi-)norm *is a (semi-)norm of the form* $x \mapsto (x^\top Q x)^{1/2}$, *where* Q *is a positive (semi-)definite matrix.*

When dealing with non-linear systems, norms may not be the appropriate tool for stability analysis. In that case, we can rely on more general types of functions. In a general setting, a minimal set of properties still need to be satisfied. A suitable class of functions over which to search for proofs of stability are *Lyapunov function candidates*:

Definition 1.11 (Lyapunov function candidates)**.** *A function* $V : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ *is a* Lyapunov function candidate *(LFC) if it is* positive definite *and* radially unbounded*, i.e.* $V(x) \to \infty$ *if* $\|x\| \to \infty$.

Norms are LFCs by definition. More generally, given any $\mathcal{K}_\infty$ function α, the function $x \mapsto \alpha(\|x\|)$ is a LFC.
Lyapunov function candidates are not norms, they do not need to be subbadditive, or homogeneous of degree 1, or convex.

Lemma 1.12 (e.g., [Lib12, Annex A.3])**.** *A function* V *is a Lyapunov function candidate if and only if there are* $\mathcal{K}_\infty$ *functions* α *and* β *such that*

$$\forall x \in \mathbb{R}^n : \alpha(\|x\|) \leq V(x) \leq \beta(\|x\|).$$

1.3 Graphs and Languages

The class of switching systems we study admits a general form of switching sequences, specified by logical rules. In order to define rules on these sequences, we borrow some standard tools from *automata* and *language* theory, see e.g. [HMU06, CL09, Lot02]. More precisely, we discuss switching sequences in terms of *words*, that are sequences of symbols. The formalism is introduced below.

Alphabets, Words and Languages

An *alphabet* is a finite set of *symbols*. These symbols can be of different nature. Here we take them as integers. Given an integer $M \geq 1$, we define the *alphabet* $\langle M \rangle$ as the set of integers

$$\langle M \rangle := \{1, 2, \dots, M\}.$$

A sequence of symbols from an alphabet $\langle M \rangle$ is called a *word*. A word of length k is a finite sequence

$$w = \sigma_1 \sigma_2 \dots \sigma_k, \text{ with } \sigma_i \in \langle M \rangle, 1 \leq i \leq k.$$

Given a word w, its length $|w|$ is defined as the (minimal) number of symbols it contains. Symbols are words of length 1.

Example 1.13. *For an integer $M = 5$, we let $\langle M \rangle = \{1, 2, 3, 4, 5\}$. A word is a sequence of these 5 integers, e.g., $w = 221$ is a word of length 3 on $\langle M \rangle$. Alphabets do not need to be restricted to sets of integers, they can be more general. For example, the set $\{a, b, c\}$ can be considered an alphabet. The sequence $w = aabbc$ is a word, and on that alphabet, it is of length 5. For the alphabet $\{a, aa, bb, c\}$, w is of length 3.*

The set of all words of length k on $\langle M \rangle$ is written $\langle M \rangle^k$.
Given two words w_1 and w_2, with $|w_1| = k$, the *concatenation* of the two words is the word $w = w_1 w_2$, whose first k symbols are those of w_1, which are then followed by those of w_2.
We define the *empty word* ϵ as a word of length $k = 0$. By definition, for any word w, $w = \epsilon w = w\epsilon = w$. Any word w can be written as $w = w^- w' w^+$, where

- the word w^- is a *prefix* (perhaps ϵ),
- the word w' is a *factor* (perhaps ϵ),
- the word w^+ is a *suffix* (perhaps ϵ).

A *language* $\mathscr{L}$ on an alphabet $\langle M \rangle$ is a *set of words* on this alphabet (an alphabet is itself a language). The concatenation (or product) of two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ is

$$\mathscr{L}_1 \mathscr{L}_2 := \{w_1 w_2 \,|\, w_1 \in \mathscr{L}_1, w_2 \in \mathscr{L}_2\}.$$

The *Kleene closure* (see e.g. [CL09, p. 56]) of a language $\mathscr{L}$ is the operation defined as

$$\mathscr{L}^* := \{\epsilon\} \cup \mathscr{L} \cup \mathscr{L}\mathscr{L} \cup \mathscr{L}\mathscr{L}\mathscr{L} \cdots,$$

and the *Kleene plus* of $\mathscr{L}$ is given by

$$\mathscr{L}^+ := \mathscr{L}^* \backslash \{\epsilon\}.$$

Any language $\mathscr{L}$ on the alphabet $\langle M \rangle$ satisfies $\mathscr{L} \subseteq \langle M \rangle^*$, which is the set of all *finite words* over the alphabet $\langle M \rangle$, ϵ included.

Directed Labeled Graphs and their Languages

We now introduce the tools through witch we impose rules on switching sequences. These are *directed, labeled graphs*, such as those represented in Figures 1.1, 1.3a, 1.3b and 1.5. The *generality* of the rules we can model with these graphs is discussed in Remarks 1.20 and 1.24.

A *directed and labeled graph* on an alphabet $\langle M \rangle$ is defined a tuple $\mathbf{G} = (S, E)$ where

1. S is the finite set of *nodes* of the graph,

2. $E \subseteq S \times S \times \langle M \rangle^+$ is a set of directed and labeled *edges*. Given an edge $e = (s, d, w) \in E$, we call the node s the source of the edge, the node d the destination of the edge, and the word $w \in \langle M \rangle^+$ the label of the edge.

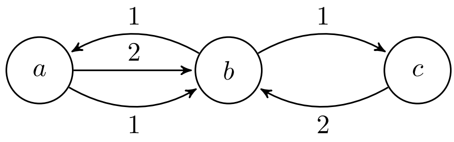


Figure 1.1: A graph on three nodes and five edges. Each edge carries a single symbol. We can not generate the word $22\ldots$ with this graph.

A *path* in a graph $\mathbf{G}$ is a sequence of edges

$$p = \{(s_i, s_{i+1}, w_i)_{i=1,2,\ldots}\}.$$

The *length of a path* p, written $|p|$ is the number of *edges* it contains.
The *source of the path* p is the source node of its first edge, and if $|p| < \infty$, the *destination* of p is the destination node of the last edge. Note that a path of length $|p|$ encounters $|p| + 1$ nodes, possibly with repetition.

Similar to edges, paths carry labels. The *word* $w(p) \in \langle M \rangle^*$ *generated by the path* $p = \{(s_i, s_{i+1}, w_i)_{i=1,2,\ldots,}\}$ is the concatenation of the words on its edges, $w(p) = w_1 w_2 \ldots$. Note that, in general,

$$|w(p)| \geq |p|,$$

since by definition, each edge carries a word of length at least one.
Given two paths p_1 and p_2 where the destination of p_1 is the source of p_2, we write $p_1 p_2$ as the paths starting with the edges of p_1 followed by those of p_2 (concatenation). We have $|p_1 p_2| = |p_1| + |p_2|$ and $w(p_1 p_2) = w(p_1) w(p_2)$.

A *cycle* is a closed path, i.e., where the source and destination nodes are the same. A cycle is *simple* if it can not be expressed as the concatenation of two or more cycles.

Given an integer k, the *set of all paths of length* k in $\mathbf{G}$ is $\mathscr{P}^k(\mathbf{G})$, and the set of all paths is $\mathscr{P}(\mathbf{G}) = \cup_{k=1}^{\infty} \mathscr{P}^k(\mathbf{G})$. Given a pair of nodes s and d of the graph, $\mathscr{P}_s^k(\mathbf{G})$ and $\mathscr{P}_{s,d}^k(\mathbf{G})$ are respectively the set of paths in $\mathbf{G}$ of length k whose source node is s, and the set of paths whose source node is s and destination node is d.

Definition 1.14. *A graph* $\mathbf{G} = (S, E)$ *on the alphabet* $\langle M \rangle$ *is in* expanded form *if*

$$\forall (s, d, w) \in E : w \in \langle M \rangle,$$

i.e., the labels on every edges are of length 1.

Remark 1.15. *We can always cast a graph* $\mathbf{G} = (S, E)$ *into a graph* $\mathbf{G}'$ *in expanded form. To do so, we take any edge* $e = (s, d, w) \in E$*, with* $|w| = k > 1$*,* $w = \sigma_1 \dots \sigma_k$*, and we add* $k-1$ *nodes* $s_{e,1}, \dots, s_{e,k-1}$*. These nodes play the role of intermediate steps along the edge. Then, we replace the edge* e *by removing it and adding* k *edges instead,* $(s, s_{e,1}, \sigma_1)$*,* $(s_{e,i}, s_{e,i+1}, \sigma_{i+1})$ *for* $1 \leq i \leq k-2$*, and* $(s_{e,k-1}, d, \sigma_k)$*.*
The path formed by these new edges has the same source, destination, and label as e*. See Figure 1.2.*

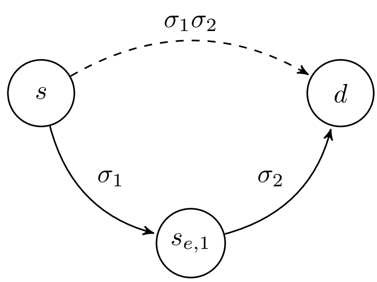


Figure 1.2: The expanded form. In a graph $\mathbf{G}$ with an edge $(s, d, \sigma_1\sigma_2)$ (dashed) having a label of length 2, the expanded form replaces this edge with two edges of length 1 and one intermediate node (plain).

Before defining the central concept of the *language of a graph*, we introduce is the *factor-closure* (see [CL09, p. 56] for the similar notion of prefix-closure) of a language. The *factor-closure* of a language $\mathscr{L}$ on the alphabet $\langle M \rangle$, denoted $\overline{\mathscr{L}}$, is defined as the set of *all the factors of words in* $\mathscr{L}$,

$$\overline{\mathscr{L}} = \{w' \in \langle M \rangle^* \mid \exists w^-, w^+ \in \langle M \rangle^* : w = w^- w' w^+ \in \mathscr{L}\}.$$

Example 1.16. *Take* $M = 2$*. The alphabet* $\langle M \rangle$ *is the set* $\{1, 2\}$*. The set* $\mathscr{L} = \{11, 2\}$ *is a language on this alphabet.*
The Kleene closure $\mathscr{L}^*$ *of* $\mathscr{L}$ *contains any words of the form*

$$w = w_1 w_2 \dots w_k,$$

for some k *finite, and where* $w_i \in \mathscr{L} \cup \{\epsilon\}$*.*
The factor-closure of $\mathscr{L}$ *is* $\overline{\mathscr{L}} = \{\epsilon, 1, 2, 11\}$*. The factor-closure of* $\mathscr{L}^*$*,* $\overline{\mathscr{L}^*}$*, is the set of all words that do not contain the factor* $w = \{212\}$*.*

Definition 1.17. *The* language *of a graph* $\mathbf{G}$ *is the* factor-closure *of* the set of all words generated by some path *in* $\mathbf{G}$*, i.e.,*

$$\mathscr{L}(\mathbf{G}) = \overline{\{w(p) \mid p \in \mathscr{P}(\mathbf{G})\}}.$$

Example 1.18. *Consider the graph* $\mathbf{G} = (S, E)$ *of Figure 1.1. The alphabet for this graph is the set* $\{1, 2\}$*. We have* $S = \{a, b, c\}$ *and*

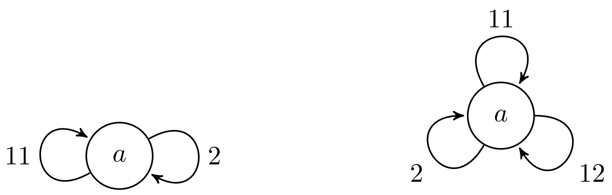
$$E = \{(a, b, 1), (a, b, 2), (b, a, 1), (b, c, 1), (c, b, 2)\}.$$

The language of this graph is

$$\mathscr{L}(\mathbf{G}) = \overline{\{11, 12\}^*} = \{M\}^* \backslash \{w = w^- w' w^+ \in \{M\}^*, w' = 22\}.$$

To see this is observe that for any path p in this graph, we can always add at most one edge e_1 at the beginning and one edge e_2 at the end such that the path $e_1 p e_2$ begins and ends at node b. Hence, if we look at the set factors of words generated by paths that cycle on b, we recover $\mathscr{L}(\mathbf{G})$. Finally, observe that any cycle on b carries a word in $\{11, 12\}^*$.

Example 1.19. *Consider the two graphs $\mathbf{G}_1$ at Figure 1.3a and $\mathbf{G}_2$ at Figure 1.3b. Paths in $\mathbf{G}_1$ generate words in $\{11, 2\}^+$. From there, we can show that*



(a) Graph $\mathbf{G}_1$.

(b) Graph $\mathbf{G}_2$

Figure 1.3: Graphs for Example 1.19.

$$\mathscr{L}(\mathbf{G}_1) = \langle M \rangle^* \backslash \{w^- w' w^+ : w^-, w^+ \in \langle M \rangle^*, w' = 212\}.$$

Paths in $\mathbf{G}_2$ generate words in $\{11, 12, 2\}^+$. One can verify that $\mathscr{L}(\mathbf{G}_2) = \langle M \rangle^*$.

Remark 1.20 (Graphs and Regular Languages)**.** *The graphs we consider differ from classical* automata*, which are commonly defined as a graph $\mathbf{G} = (S, E)$, on top of which starting and accepting nodes are specified (see e.g. [HMU06, Chapter 2]). However, our graphs can be cast into* non-deterministic automata *with $\epsilon-$*transitions *(see e.g. [CL09, Section 2.2.4]). Since automata encode* regular languages*, the set of languages we obtain is a subset of the regular languages.*

A regular language is a languages $\mathscr{L} \in \langle M \rangle^*$ that can be obtained by a finite number of applications of three operations on the language $\{\epsilon\} \cup \langle M \rangle$. These operations are the union*, the* product *(or concatenation of languages), and finally the* Kleene closure *[HMU06, Section 3.1.1].*

Infinite Words Generated by a Graph

The language $\mathscr{L}(\mathbf{G})$ of a graph $\mathbf{G}$ is by definition an *infinite set* of *finite words*. Switching sequences, by definition, are infinite sequences of symbols. Hence, they should be considered as *infinite words*, and special care should be taken when using graphs to define sets of switching sequences.

We refer to the set of all infinite words on $\langle M \rangle$ as $\langle M \rangle^\infty$. We say that a word $w \in \langle M \rangle^\infty$ is *accepted* by a graph $\mathbf{G}$, denoted with a slight abuse of notation as $w \in \mathscr{L}(\mathbf{G})$, if all prefixes of finite length of w are in $\mathscr{L}(\mathbf{G})$, i.e.,

$$w \in \mathscr{L}(\mathbf{G}) \Leftrightarrow \{w^- \in \langle M \rangle^* \,|\, \exists w^+ \in \langle M \rangle^\infty : w = w^- w^+\} \subseteq \mathscr{L}(\mathbf{G}).$$

There is one *necessary and sufficient* condition for a graph $\mathbf{G}$ to accept infinite words: it needs to have at least one strongly connected component.

Definition 1.21. *Given a graph $\mathbf{G} = (S, E)$, a strongly connected component is a subset of nodes $S' \subseteq S$ such that, for any pair of nodes $s, d \in S'$, there is a path in $\mathbf{G}$ from s to d. The graph $\mathbf{G}$ is strongly connected if S is a strongly connected component.*

Throughout this thesis, unless stated otherwise, we make the following assumption:

Assumption 1.22. *The graphs considered herein are strongly connected.*

Remark 1.23. *This assumption can be made without loss of generality. This is due to the fact that the notions of stability considered here are asymptotic, which allows us to restrict ourselves to the* strongly connected components *of any given graph.*

The languages defined through graphs are exactly those that *avoid* a regular language (see Remark 1.24). We say that a word w avoids a language $\mathscr{L}$ if the language does not contain any factor of w.
This is illustrated in e.g. Example 1.18, where the language of a graph is defined as the set of all words that don't have 212 has a factor.
This interpretation of graphs come from computer science, more precisely, symbolic dynamical systems.

Remark 1.24 (Graphs, symbolic dynamical systems, and sofic shifts)**.** *A symbolic dynamical system (see [Lot02, Section 1.5]) is a set W of* two-sided *infinite words, subset of*

$$\langle M \rangle^{\mathbb{Z}} := \{\ldots \sigma_{-2}\sigma_{-1}\sigma_0\sigma_1\sigma_2 \ldots, \sigma_i \in \langle M \rangle, i \in \mathbb{Z}\}.$$

The notions of prefixes, factors and suffixes extend naturally to these words. In a symbolic dynamic system, the set of words avoids *a language $\mathscr{L} \in \langle M \rangle^*$. The graphs considered herein are involved in the concept of* sofic shifts*. These are symbolic dynamical systems that avoid a* regular language *$\mathscr{L}$. One can show that a symbolic dynamical system is a* sofic shift *if and only if it can be encoded in a graph $\mathbf{G} = (S, E)$ with $E \in S \times S \times \{M\}$ (see [Lot02, Proposition 1.5.6]). Here, encoded means that* for any two-sided word $w \in S$*, there is a* two-sided path *generating the word.*
Equivalently, we can that any finite factor of a two-sided word w is in $\mathscr{L}(\mathbf{G})$ - this is where these sofic shift meet with our definitions (see Definition 1.17).

Markovian graphs

As mentioned above and explained further in Section 1.4, we use directed and labeled graphs as a means to encode words and sets of switching sequences for switching systems. In e.g. [LD06b, ELD14, BFT03, KC15, Koz14] the authors consider another setup where the rules on switching sequences are encoded into graphs with exactly *one node per mode of the system* (see Figure 1.4). We call these *Markovian graphs.* We read rules on sequences in such a graph

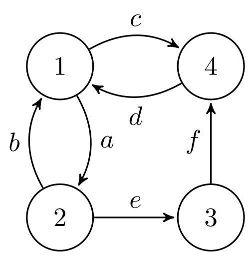


Figure 1.4: Graph taken from [ELD14, Figure 1]. The numbers in the nodes correspond to modes of a switching system, and an edge between two nodes means we can transit between the two corresponding nodes. The letters on the edges serve no other purposes than to facilitate the description of the graph.

has follows. Consider a sequence $\sigma(0)\sigma(1)\ldots$. The sequence is accepted if for any time $t \in \mathbb{N}$, there is an edge between the node with label $\sigma(t)$ and the node with label $\sigma(t+1)$. For example, in the graph of Figure 1.4, we accept a periodic sequence repeating the subsequence “1234”, but no accepted sequence will contain the subsequence “24”.

The graphs we use allow for the representation of richer languages. To be more precise, as discussed in Remark 1.24, graphs $\mathbf{G} = (S, E)$ with labels on the edges can be used to define *sofic-shifts*, i.e. sets of two sided words that avoid a *regular language.* In comparison, Markovian graphs can only encode particular cases of sofic shifts that are *subshifts of finite type* (see [Lot02, p. 24]). Subshifts of finite type are sets of two sided words that avoid a *finite set of words* (in comparison, sofic shifts avoid regular languages that are infinite sets of words). More particularly, the finite sets of words avoided by shifts defined on Markovian graphs are all pairs $\sigma\sigma'$ where no edges exists between the node σ and the node σ'.
In order to observe how this can limit the expression of rules, consider the following set of words on the alphabet $\langle 2 \rangle$:

$$\mathscr{L} = \{w = \sigma_1\sigma_2 \ldots \mid \sigma_i = \sigma_{i+1} = 2 \Rightarrow \sigma_{i+2} = 1.\}$$

This language is generated by the graph $\mathbf{G}$ at Figure 2.1b in Chapter 2. It cannot be generated by a Markovian graph however.

In practice, any Markovian graph can easily be cast as a labeled directed graph such as the ones used herein: it suffice to assign to each edge the label that correspond to its destination node. For example, the language generated by the graph at Figure 1.4 is equivalent to the graph **G** at Figure 1.5.

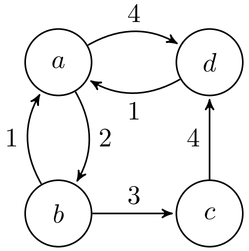


Figure 1.5: A graph generating the same language as that of the Markovian graph of Figure 1.5.

Remark 1.25. *These Markovian graphs arise naturally in the study of* Markov Jump Linear Systems *(MJLS). A Markov Jump Linear System is essentially a system of the form (1.1) with M modes, where a* transition probability matrix $P \in [0,1]^{M\times M}$ *is defined and, given a mode $\sigma(t)$ at time t, the* probability *that the mode at time $t+1$ is σ' is given by $(P)_{\sigma(t),\sigma'}$. Given such a matrix, a graph such as the of Figure 1.4 is obtained by putting an edge between mode i and mode j if $(P)_{i,j} > 0$. MJLS can be found in many applications and their stability analysis has received a lot of attention in the past, see e.g. [LD06b, CFM06] and references therein. In relation with MJLS, the stability analysis tools we develop herein relate to* robust stability notions. *We will be discussing criteria that aim at deciding whether there exists one unstable trajectory, regardless of the probability of occurrence of such a trajectory.*

1.4 Constrained Switching Systems

In this section, we define the class of *constrained switching systems* studied in this thesis.

We assume to have full knowledge of the description of the dynamics f_σ in (1.1). We do not know in advance, nor choose, the switching sequence that is applied to the system. Switching sequences are therefore seen as perturbations, chosen by some adversary that wishes to destabilize the system.

That adversary may have limited options however. We assume that, as in Definition 1.3, he can pick his sequence in a set $\mathscr{L}$ that is known to us.

To describe these sets, we make use of the notions of languages and graphs defined in Section 1.3.

We refer to a discrete-time linear switching system, for which we know the sequences belong to a language generated by a graph **G**, as a (linear) *constrained switching* system.

Definition 1.26. *A linear* constrained switching *system (CSS) on M modes, with a set matrices $\mathbf{A} = \{A_1, \dots, A_M\}$, $A_i \in \mathbb{R}^{n\times n}$, and a graph $\mathbf{G}$ on the alphabet $\langle M\rangle$, is a switching system of the form*

$$x(t+1) = A_{\sigma(t)}x(t),$$

where

$$\forall t \geq 0 : \ \sigma(0)\sigma(1)\dots\sigma(t) \in \mathscr{L}(\mathbf{G}).$$

We refer to the CSS on the set $\mathbf{A}$ and graph $\mathbf{G}$ as a tuple $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$.

These systems are the object of interest for Chapters 2 and 3. In Chapter 4, we consider a more general class with input and outputs and whose state dimension may also vary to fit the dynamics. In Chapters 5 and 6, we move our focus to non-linear switching systems. Nevertheless, the descriptions of these generalization shares all the main elements of the linear constrained switching systems defined above.

Switching sequences and words

Switching sequences are infinite sequences of integers. The set of all such sequences is isomorphic to the set of *infinite words* $\langle M\rangle^\infty$. A switching sequence $\sigma(\cdot) : \mathbb{N} \to \langle M\rangle$ is captured by the infinite word $w = \sigma(0)\sigma(1)\ldots \in \langle M\rangle^\infty$.

For simplicity we allow ourselves to use the terms *switching sequences* to designate both finite and infinite word, with the signification made clear with the context.

Definition 1.27 (Matrix Products)**.** *Given a set of M matrices $\mathbf{A}$ in $\mathbb{R}^{n\times n}$, and a word $w = w_1w_2\dots w_k \in \langle M\rangle^*$, $|w| = k \geq 1$, we define*

$$A_w = A_{w_k} \cdots A_{w_2}A_{w_1}$$

as the product of matrices from $\mathbf{A}$ corresponding to w. For the empty word ϵ, we let $A_\epsilon = I$ be the identity matrix in $\mathbb{R}^{n\times n}$.

When both a switching sequence and an initial condition are given, we can describe a trajectory for the system.

Definition 1.28 (From words to trajectories)**.** *Given a set of M matrices $\mathbf{A}$ in $\mathbb{R}^{n\times n}$, a point $x_0 \in \mathbb{R}^n$ and a finite word $w \in \langle M\rangle^*$,*

$$x(x_0, w) = A_w x_0.$$

For an infinite word $w = \sigma_1\sigma_2\ldots \in \langle M\rangle^\infty$, and $t \in \mathbb{N}$,

$$x(t, x_0, w) = A_{\sigma_t} \cdots A_{\sigma_1} x_0.$$

The next example presents a switching system with simple dynamics, but whose trajectories exhibit interesting behaviors depending on the switching sequences used.

Example 1.29. *We consider a simple system: we take a* scalar *switching system on $M = 3$ modes:*

$$x(t+1) = A_{\sigma(t)}x(t), \mathbf{A} = \{A_1, A_2, A_3\} = \{2,\ 1/4,\ 0\}.$$

We are interested in the asymptotic stability (see Definition 1.3) of the switching system when the switching sequences belong to a given a set $\mathscr{L} \subseteq \langle M\rangle^\infty$ of infinite words*.*

We consider three cases, each one with a different family of switching sequences.

1. *In the first case, we take the language of infinite words $\mathscr{L}_1 = \langle M\rangle^\infty$. This case is referred to as the* arbitrary switching case*.*

   This language can be encoded in the graph $\mathbf{G}$ at Figure 1.6. Since this language contains the sequence (word) $w = \sigma_1\sigma_2\ldots$, where $\sigma_i = 1$ for all i, the system is unstable*. Indeed, for any $x_0 \neq \mathbf{0}$, the trajectories diverge to infinity when applying that sequence.*

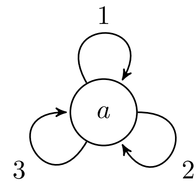


Figure 1.6: A graph $\mathbf{G}$ whose language is $\mathscr{L}(\mathbf{G}) = \langle 3\rangle^*$.

2. *In the second case, we take the language of all words where no two consecutive symbols are the same,*

   $$\mathscr{L}_2 = \{w = \sigma_1\sigma_2\ldots \mid \forall i \geq 1, \sigma_i \neq \sigma_{i+1}\}.$$

   This language can be expressed as the set of all words that do not contain 11, 22 or 33 *as a factor. It can be encoded in the graph $\mathbf{G}$ at Figure 1.7. In this case, the system is stable. To show this, notice first that if mode 3 is chosen at a time t, then for any x_0,*

   $$\forall t' > t, x(t', x_0, w) = \mathbf{0}.$$

   If mode 3 is never chosen, then the system will keep switching between mode 1 and mode 2. In this case, we can rewrite the dynamics as

   $$x_{t+2} = A_2A_1x_t = A_1A_2x_t = \frac{1}{2}x_t,$$

   and the system remains stable.

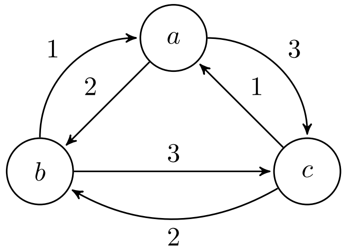


Figure 1.7: Words accepted by this graph are those that do not repeat the same symbol twice in a row.

3. *The last case is inspired from [Dai12, Example 1.3]. The language is constituted from all infinite words that start with a finite number of* 1*, and then switch and remain to mode* 3 *forever. That is, we pick*

$$\mathscr{L}_3 = \{\sigma_1\sigma_2\sigma_3 \ldots \mid \exists k \in \mathbb{N},\, \sigma_i = 1 \text{ if } i \leq k,\, \sigma_i = 3 \text{ if } i > k\},$$

which is represented in the graph **G** *at Figure 1.8 (observe that this graph is not strongly connected).*

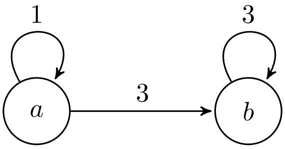


Figure 1.8: A graph **G** for the language $\mathscr{L}_3$ in Example 1.29.

In this case, the system presents an exotic behavior. Trajectories are mapped to the origin in a finite time. Regardless of the initial condition and sequence chosen, after some finite *(but unknown) time, the state is mapped onto the origin by mode 3. However, since that time can be arbitrarily big, we cannot bound the norm of the trajectories, so the system is unstable. More precisely, for any non-zero initial condition* x_0 *and constant* $K \geq 1$*, we can pick a sequence* $w \in \mathscr{L}_3$ *and a time* t *such that*

$$x(t, x_0, w) \geq K x_0.$$

That language can not be encoded in a graph.

Remark 1.30. *What is the cause of the exotic behavior in Example 1.29, in the third case? The problem, as pointed in [Dai12, Example 1.3], is that* $\mathscr{L}_3$ *is not* compact*. Consider a sequence of words* $w_1, w_2, \ldots$ *in* $\mathscr{L}_3$*, where the word* w_k *is the word where the* k *first symbols are* 1*s and all the others are* 3*s. That sequence converges to the word* w^* *where all symbols are 1. The highlighted* *behavior comes from the fact that for each element of the sequence, the system is asymptotically stable, but not for the limit.*

As it turns out, we can not represent this language using graphs. Those that can be represented by graphs are compact, and this prevents the behavior discussed above.

Part I

Analysis of Constrained Switching Systems

2 Approximating the Exponential Growth Rate

In this first chapter we tackle the question of whether or not a constrained switching system (CSS) is exponentially stable. We provide an exact characterization of stability in terms of *mutiple Lyapunov functions*, leading to a family of algorithms for the *arbitrarily accurate approximation of the exponential growth rate of a system.*

The chapter is based on [PEDJ16b] and [PJ15a] and is organized as follows.

- In Section 2.1, we motivate our work regarding the literature, present the notions of stability studied here, and introduce the concept of the *constrained joint spectral radius* (CJSR) which is the exponential growth rate of a switching system.

- In Section 2.2, we present an exact characterization of exponential stability using *multinorms*, that are multiple Lyapunov functions with exactly one *norm* per state of the graph **G**.

- In Sections 2.3, we use *quadratic multinorms* and *convex optimization* to provide approximations of the CJSR with a *fixed accuracy.* In Subsection 2.3.1, we provide a sufficient condition for when the approximation is exact. In Subsection 2.3.2, we compare the approach with an alternative, that first transforms a CSS into a high-dimensional arbitrary switching system and computes a common Lyapunov function there.

- In Section 2.4, we provide algorithms to provide arbitrarily accurate approximations of the CJSR, with bounded time complexity. They rely on computing quadratic multinorms after making use of lifting procedures on the system's description.

2.1 Asymptotic and Exponential Stability

In this chapter, we provide tools and algorithms to decide the exponential stability of constrained switching systems (see Chapter 1, Definition 1.26).

Switching systems find application in many theoretical and engineering related domains. For example, [SWL06] provides a switching system modeling of TCP congestion control, an algorithm regulating data transfer across computer networks. In [HVMC11], viral mitigation schemes are approached through switching systems techniques. Consensus systems (see e.g. [OSM04]), where a set of agents seek to compute a common value, are adequately modelled as switching systems when taking into account variations in the network of interactions of the agents. More applications where switching systems arise can be found in e.g. [Jun09, The05, SG11, LM99, JDDB12] and references therein. In the following example, we introduce switching systems with constraints their switching sequences through an application in *networked control systems.*

Example 2.1. *A typical setup where switching systems arise is in the presence of uncertainty on the parameters of a time-invariant system. The following is inspired from* networked control systems.
A networked control system is a spatially distributed control system where communication between sensors, actuators and computational units is done across a (wireless) communication network, see e.g. [AdJ⁺11, LZA03, CvdWHN06, AK04, JDDB12, JHK16, DHvdWH11]. This has several benefits in terms of maintainability and design flexibility and such systems find their way in many applications, see e.g. the references above.
The downside of this technology is that, in coordinating a control loop across a network, one needs to take into account the unexpected failures from the non-ideal communication network. Consider the following situation. We have a linear dynamical system with input of the form $x(t+1) = Ax(t) + Bu(t)$, with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^d$. We assume that A is unstable.
Given this open-loop system, we wish to implement a linear state feedback controller $u(t) = Kx(t)$, such that overall, the system $x(t+1) = (A + BK)x(t)$ is stable.

In its implementation, the controller is physically separated from the plant. It receives measurements of the state $x(t)$, and sends its input $u(t)$ through a non-ideal *communication channel.*
For this example, we consider that one of two scenarios may occur at any time t.

- *If the communication channel performs as expected, the dynamics are*

$$x(t+1) = (A + BK)x(t) = A_1 x(t).$$

- *In case of a failure, we assume the plant perceives $u(t) = 0$[1], so the*

[1] In Chapters 3 and 4, we consider a similar scenario with an update-and-hold policy at the plant.

dynamics are

$$x(t+1) = Ax(t) = A_2 x(t).$$

Overall, the resulting dynamics can be modeled as a switching system

$$x(t+1) = A_{\sigma(t)} x(t), \sigma(t) \in \{1, 2\},$$

where $\sigma(0)\sigma(1)\ldots$ *is called a* switching sequence*, and is unknown to us.*

Intuitively, the (un)stability of this system depends on the occurences of the unstable mode 2. It is therefore related to the switching sequences of the switching system. We consider two cases.

1. *The first is the* arbitrary switching case*. If we are given no information at all, we may consider that any switching sequences may occur. In particular, the system may undergo a permanent failure, represented with the sequence* $\sigma(t) = 2$ *at all time* $t \geq 0$*. In this case, the system would be unstable.*
   Such analysis is questionable, as there could be some fail-safes preventing a total crash of the communication channel.

2. *The second is a* constrained switching case*. If we assume, for example, that a mechanism prevents a failure from occurring* more than two times in a row*, would the system be unstable? Intuitively, performances may suffer compared to the ideal case without failures, but discussing stability is more complex.*

Both scenarios can be modeled as two different constrained switching systems *(see Definition 1.26), on a same set of matrices* $\mathbf{A} = \{A_1, A_2\}$*.*
The first system, denoted $\boldsymbol{\mathcal{S}}_1 = (\mathbf{G}_1, \mathbf{A})$*, accepts arbitrary switching sequences, which can be encoded in the graph* $\mathbf{G}_1$ *at Figure 2.1a.*
The second system, $\boldsymbol{\mathcal{S}}_2 = (\mathbf{G}_2, \mathbf{A})$*, cannot have switching sequences repeating mode* 2 *more than two times in a row*[2]*. To represent this, we can use the graph* $\mathbf{G}_2$ *at Figure 2.1b.*

In this chapter, we are interested in the *asymptotic and exponential stability* of linear constrained switching systems, defined[3] as follows:

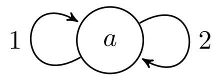
Definition 2.2. *Consider the linear constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{A}, \mathbf{G})$*, with the set of* M *matrices* $\mathbf{A}$ *and the graph* $\mathbf{G}$ *on the alphabet* $\langle M \rangle$*.*

- *The system is* asymptotically stable *if*

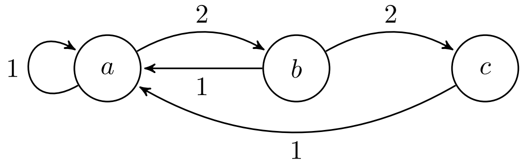
$$\forall x_0, \forall w \in \mathscr{L}(\mathbf{G}), |w| = \infty : \lim_{t \to \infty} x(t, x_0, w) = \mathbf{0}. \qquad (2.1)$$

[2] Such scenarios are usually referred to as *maximum dwell time* constraints.

[3] These are equivalent to Definition 1.3, but recalled for being self-contained. For constrained switching systems, asymptotic stability implies stability in the sense of Definition 1.3.



(a) Graph $\mathbf{G}_1$. Paths in this graph generates a sequence of labels in $\{1,2\}$. We call these sequences *words*. The language of the graph, i.e. the set of all words generated by paths, is $\mathscr{L}(\mathbf{G}_1) = \{1,2\}^*$.



(b) Graph $\mathbf{G}_2$. For this graph, the language is $\mathscr{L}(\mathbf{G}_2) = \{\sigma_1\sigma_2\ldots \in \{1,2\}^* \mid \sigma_i = \sigma_{i+1} = 2 \Rightarrow \sigma_{i+2} = 1\}$

Figure 2.1: Graphs for Example 2.1.

- *The system is* exponentially stable *if there are $\rho < 1$ and $K \geq 1$ such that*
$$\forall x_0, \forall w \in \mathscr{L}(\mathbf{G}) : \|x(x_0, w)\| \leq K\rho^{|w|}\|x_0\|. \tag{2.2}$$
The smallest number ρ such that there is $K \geq 1$ and (2.2) *holds is called the (worst case)* exponential growth rate *of the system.*

Before discussing constrained switching systems, let us discuss the exponential stability of *arbitrary switching systems*. These are switching systems for which any arbitrary sequence of modes is a possible switching sequence. An *arbitrary* switching system on M modes is a special case of constrained switching systems, with a graph $\mathbf{G}$ such that $\mathscr{L}(\mathbf{G}) = \langle M \rangle^*$ such as the one of Figure 2.1a for $M = 2$.

Arbitrary switching systems and the problem of deciding stability under arbitrary switching have received a lot of attention in the past (see e.g., [AL01, AJPR14, AS98, Jun09, LM99, Bra98, SG11, SWM$^+$07]). The stability analysis in the arbitrary switching case is closely related to the question of the approximation and computation of the *joint spectral radius* (JSR) $\hat{\rho}(\mathbf{A})$ (introduced in [RS60]).

Definition 2.3. *The* joint spectral radius *(JSR) of a set of M matrices $\mathbf{A}$ is defined as follows:*
$$\hat{\rho}(\mathbf{A}) = \lim_{k \mapsto \infty} \sup_{w \in \langle M \rangle^k} \|A_w\|^{1/k}.$$

Remark 2.4. *When $\mathbf{A}$ contains a single matrix A, the joint spectral radius of $\mathbf{A}$ coincides with the spectral radius of A. Additionally, the JSR is homogeneous in the set of matrices $\mathbf{A}$: given $\alpha \in \mathbb{R}$, letting $\alpha\mathbf{A} = \{\alpha A, A \in \mathbf{A}\}$, we have*

$\hat{\rho}(\alpha \mathbf{A}) = |\alpha|\hat{\rho}(\mathbf{A})$.
The JSR is also independent of the norm used in its definition due to the equivalence of norms in $\mathbb{R}^{n\times n}$.

The JSR is the worst case growth rate of products of matrices, corresponding to the worst case exponential growth rate (see Definition 2.2) of the trajectories in an arbitrary switching system. Both exponential and asymptotic stability are equivalent to $\hat{\rho}(\mathbf{A}) < 1$, see e.g. [Jun09, Theorem 1].

The joint spectral radius has intrigued a lot of researchers in the past, and will certainly continue to do so in the future, see [Jun09, The05] for surveys on the topic. It appears in several applications additional to the stability of switching systems, e.g. in coding theory [MOS01], for the continuity and regularity of wavelets [DL92], and more...
The JSR also presents major theoretical challenges. In particular, it is surprisingly complex to compute and approximate. We mention two important theoretical barriers.

- Deciding whether or not a set of matrices $\mathbf{A}$ has $\hat{\rho}(\mathbf{A}) \leq 1$ is *undecidable* (see [BT00a], [Jun09, Theorem 2.6]). This means that for general sets of matrices, there are no algorithms that can output in finite time whether or not $\hat{\rho}(\mathbf{A}) \leq 1$. Keep in mind that, the JSR being *homogeneous* in the matrix set $\mathbf{A}$, this implies that the question $\hat{\rho}(\mathbf{A}) \leq \alpha$ is undecidable as well for any constant $\alpha > 0$.

- *Approximating* the JSR is NP-Hard (see [TB97], [BN05], [Jun09, Theorem 2.4]). Unless $P = NP$, there are no algorithms that, given a set of matrices $\mathbf{A}$ and a relative accuracy $r > 0$, compute an estimate λ satisfying
$$|\lambda - \hat{\rho}(\mathbf{A})| < r\hat{\rho}(\mathbf{A}),$$
in a time polynomial in the bit-sizes of $\mathbf{A}$ and ϵ. In [BN05], the authors show that for a fixed accuracy r and a fixed number of matrices, there are algorithms polynomial in the dimension of the matrices for providing r accurate estimations.

Nevertheless, the research effort towards the computation of the JSR continues (see e.g. [AJPR14, Jun09, BNT05, VHJ14, GP13, VHJ14, BN05] and references therein), and in particular, methods have been developed to approximate the JSR with arbitrary accuracy in finite time (see [AJPR14] for a general survey).

Let us now turn ourselves towards *constrained switching* systems, as we defined them in Definition 1.26. While the main theoretical works has been on arbitrary switching systems, constrained switching systems have not been ignored [SWM+07,LA09], and stability criteria in this more general setting have been studied (see e.g. [LD06b, ELD14, LJP16, AL14b, ASJ17] and references therein).

The stability of a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is characterized by its *constrained joint spectral radius*, introduced by Dai [Dai12].

Definition 2.5 (The CJSR)**.** *The constrained joint spectral radius (CJSR) of a constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ *is defined as*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) = \lim_{k\to\infty} \max_{w\in\mathscr{L}(\mathbf{G}),|w|=k} \|A_w\|^{1/k} \tag{2.3}$$

Remark 2.6. *For two systems* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ *and* $\boldsymbol{\mathcal{S}}' = (\mathbf{G}', \mathbf{A})$ *where* $\mathscr{L}(\mathbf{G}) = \mathscr{L}(\mathbf{G}')$*, we have* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = \hat{\rho}(\boldsymbol{\mathcal{S}}')$*.*

Remark 2.7. *The constrained joint spectral radius benefits from many properties of the joint spectral radius, in particular its homogeneity with respect to the set of matrices* $\mathbf{A}$ *and its continuity [Jun09, 1.2.2.8]. It is also independent of the norm used in (2.3).*

A proof of the following result is given in Annex 2.A.

Theorem 2.8 (Reformulated from [Dai12, Corollary 2.8])**.** *Consider a constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$*. The following are equivalent:*

- *The system is asymptotically stable.*
- *The system is exponentially stable.*
- *Its CJSR satisfies* $\hat{\rho}(\boldsymbol{\mathcal{S}}) < 1$*.*

Additionally, for all $\epsilon > 0$*, there exists* $K \geq 1$ *such that*

$$\forall w \in \mathscr{L}(\mathbf{G}), |w| = t, \forall x_0 \in \mathbb{R}^n : \|A_w x_0\| \leq K(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \epsilon)^t \|x_0\|.$$

To the best of our knowledge, previous works on the stability of constrained switching systems have focused on establishing algorithmically checkable stability conditions, without studying their conservativeness (except when allowing the algorithms to run for an a priori unbounded amount of time.). There is a particular interest in using multiple Lyapunov functions with *quadratic pieces* as stability certificates. For example, in [DB01] and [DRI02], the authors use such a method where a different quadratic function is assigned to each *mode* of the system. The authors focus on the arbitrary switching case there, but their work easily extends. An advantage of the approach is that it leads to a set of LMIs, checkable with convex programming. Refinements have been proposed [BFT03,LD06b,LK09,ELD14], providing hierachies of LMIs, the feasibility of each one being sufficient for stability, and the feasibility of all LMIs starting from some level necessary as well. The performances of these hierarchies is not well understood, and nothing quantitative could be learnt if, e.g., the 10th level of the hierarchy failed to produce a certificate.

Our main contribution in this chapter is a theoretical and algorithmic framework that sheds new light on the works mentioned above, and allows to provide performance guarantees on these methods/hierarchies. Conceptually, it extends the general *Path-Complete Framework* introduced by Ahmadi et al. [AJPR14] , which encompasses and allows for a meta-analysis of the tools above in the arbitrary switching case, to constrained switching systems.

In Section 2.2, we start by establishing necessary and sufficient stability conditions in terms of multiple Lyapunov functions, and then in Sections 2.3 and 2.4, we put them to good use and develop tractable CJSR approximation algorithms.

2.2 Multinorms: CSS Require Multiple Lyapunov Functions

Our first step is to capture the essential stability properties of a constrained switching system by an appropriate form of stability certificate. Since exponential stability is closely related to the CJSR, we seek to characterize this quantity through Lyapunov-like methods.

In the *arbitrary* switching case, stability is equivalent to the existence of a contractive *norm* (see Section 1.1).

Proposition 2.9 (e.g. [Jun09, Proposition 1.4]). *The joint spectral radius of a set of matrices* $\mathbf{A}$ *satisfies*

$$\hat{\rho}(\mathbf{A}) = \inf_{\|\cdot\|} \min_{\gamma} \left\{ \gamma : \forall x \in \mathbb{R}^n,\ \forall A \in \mathbf{A} : \|Ax\| \leq \gamma \|x\| \right\}. \tag{2.4}$$

where the infimum is taken over all vector norms *in* $\mathbb{R}^n$.

A stable arbitrary switching system has $\hat{\rho}(\mathbf{A}) < 1$ (see [Jun09]) and from Proposition 2.9, there exists a norm $\|\cdot\|$ such that $\|Ax\| < \|x\|$, $\forall x \in \mathbb{R}^n$, $\forall A \in \mathbf{A}$. This norm effectively plays the role of a *common Lyapunov function.*

That construction cannot be used for constrained switching systems, as it is straightforward to build stable CSS for which contractive norms *do not* exist as shown in the next example.

Example 2.10. *Consider the set of two* scalar *matrices* $\mathbf{A} = \{A_1, A_2\} = \{2, 1/8\}$. *An arbitrary switching system on this set is unstable, with a* joint spectral radius $\hat{\rho}(\mathbf{A}) = 2$. *Hence, there are no* contractive norms *here, see Proposition 2.9.*

Consider now the graph $\mathbf{G}$ *of Figure 2.2. The periodic system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ *is stable and, applying (2.3),* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = \lim_{t\to\infty} \|(A_1A_2)^t\|^{1/(2t)} = 1/2$.

We fill this gap between arbitrary switching and constrained switching systems by introducing the algebraic concept of *multinorm.*

Definition 2.11 (Multinorm). *A* multinorm $\mathcal{H}$ *for a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ *in dimension* n *is a set of* $|S|$ *norms on* $\mathbb{R}^n$, $\mathcal{H} = \{\|\cdot\|_s,\ s \in S\}$.

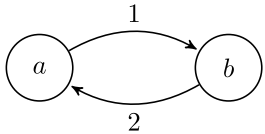


Figure 2.2: A graph $\mathbf{G}$ generated periodic switching sequences

The value $\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}})$ *of a multinorm is defined as*

$$\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}}) := \min_{\gamma}\{\gamma : \forall x \in \mathbb{R}^n, \forall (s,d,w) \in E : \|A_w x\|_d \leq \gamma^{|w|}\|x\|_s\}. \qquad (2.5)$$

Similar objects have been presented in the past, [LD06b, AJPR14, Bra98, Ahm11, DRI02], where multiple Lyapunov functions are considered for characterizing stability where single Lyapunov functions fail to do so. Their role was either to provide a sufficient stability condition under the form of a set of LMIs, or in [AJPR14], to characterize the stability of *arbitrary* switching systems using multiple Lyapunov functions. In comparison, we provide general *necessary and sufficient* conditions for the stability of constrained switching systems, using multiple Lyapunov functions with exactly one *norm* per *node* of $\mathbf{G}$. This sets the ground for further studying the performances of multiple Lyapunov functions in the context of constrained switching systems.

Proposition 2.12. *The* constrained joint spectral radius *(2.3) of a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ *satisfies*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) = \inf_{\mathcal{H}}\{\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}}) : \mathcal{H} \text{ is a multinorm for } \boldsymbol{\mathcal{S}}\}. \qquad (2.6)$$

Proof. We first show that the value of any multinorm for a system is an upper bound of its CJSR.

Consider a multinorm $\mathcal{H} = \{\|\cdot\|_s,\ s \in S\}$ for the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ with value γ. For any path $p \in \mathscr{P}(\mathbf{G})$ between two nodes s and d in S, we have from Definition 2.11,

$$\forall x \in \mathbb{R}^n : \|A_{w(p)}x\|_d \leq \gamma^{|w(p)|}\|x\|_s.$$

Now, take any norm $\|\cdot\| \in \mathbb{R}^n$. By equivalence of norms in $\mathbb{R}^n$, there exists two constants $0 < \alpha < \beta$ such that the inequalities $\alpha\|x\| \leq \|x\|_s \leq \beta\|x\|$ hold for all $x \in \mathbb{R}^n$ and all nodes $s \in S$.
Hence, we can bound the induced matrix norm of the matrix product associated to the path p:

$$\|A_{w(p)}\| = \max_{\|x\|=1} \frac{\|A_{w(p)}x\|}{\|x\|} \leq \frac{\beta}{\alpha} \max_{\|x\|=1} \frac{\|A_{w(p)}x\|_d}{\|x\|_s} \leq \frac{\beta}{\alpha}\gamma^{|w(p)|}. \qquad (2.7)$$

Taking paths of lengths $k \to \infty$ and the $|w(p)|$th root of the above inequality, we obtain $\hat{\rho}(S) \leq \gamma$ from (2.3).

We now show that for any $\delta > 0$ there exists a multinorm of value at most $(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)$. Consider the scaled set of matrices

$$\mathbf{A}' = \{A'_i = A_i/(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta),\ i = 1, \ldots, N\}.$$

The CJSR of $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is a positively homogeneous function of order one in the set $\mathbf{A}$ (see Remark 2.7). The system $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \mathbf{A}')$ is thus stable with $\hat{\rho}(\boldsymbol{\mathcal{S}}') = \hat{\rho}(\boldsymbol{\mathcal{S}})/(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta) < 1$, by Theorem 2.8.
We define, at each node $s \in S$, the following functions which we then prove are norms:

$$\|x\|_s := \sup_w \{\|A'_w x\| : w \in \{w(p),\ p \text{ is a path with source } s\} \cup \{\epsilon\}\}, \quad (2.8)$$

where $\|\cdot\|$ is the euclidean norm. These functions are well-defined (by exponential stability of $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \mathbf{A}')$), sub-additive, homogeneous, and positive definite (since $A'_\epsilon := I$, the identity matrix, and thus $\|x\|_s \geq \|x\|$). Therefore, they are norms. Moreover, for any edge $(s, d, \tilde{w}) \in E$, and all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|x\|_s &:= \sup_w \{\|A'_w x\| : w \in \{w(p),\ p \text{ is a path with source } s\} \cup \{\epsilon\}\}, \\ &\geq \sup_{w'} \{\|A'_{w'} A'_{\tilde{w}} x\| : w' \in \{w(p),\ p \text{ is a path with source } d\} \cup \{\epsilon\}\}, \qquad (2.9) \\ &= \|A'_{\tilde{w}} x\|_d, \end{aligned}$$

where (2.9) is obtained by taking a path p starting with the edge $(s, d, \tilde{w})$. Since $A'_\sigma = A_\sigma/(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)$, we have $\|A_{\tilde{w}} x\|_d \leq (\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)^{|\tilde{w}|} \|x\|_s$ for any $(s, d, \tilde{w}) \in E$. Thus, the value of the multinorm with the norms defined in (2.8) is upper bounded by $\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta$. ∎

The proof of the following is direct from Proposition 2.12.

Theorem 2.13. *A constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is exponentially stable if and only if it admits a multinorm $\mathcal{H}$ with value $\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}}) < 1$.*

Remark 2.14. *From Theorem 2.13, we know that if a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is exponentially stable, it must have a multinorm with value smaller than one. In view of this and Remark 2.6, this means that such a multinorm exists* for all systems $\boldsymbol{\mathcal{S}}' = (\mathbf{G}', \mathbf{A})$ *with $\mathscr{L}(\mathbf{G}') = \mathscr{L}(\mathbf{G})$! Later, the results presented in Section 2.4 show that among these equivalent systems, there must be some for which we can construct multinorms using tractable tools, in particular LMIs, and we provide ways to search for these systems.*

Example 2.15. *Consider the switching system $\boldsymbol{\mathcal{S}}$ of Example 2.10 with switching constrained by the graph of Figure 2.2, with nodes a and b. Take the multinorm*

$$\mathcal{H} = \{\|x\|_a, \|x\|_b\} = \{4\|x\|, \|x\|\},$$

where $\|\cdot\|$ is the euclidean norm (here, the absolute value). That multinorm has a value *of* $\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}}) = 1/2$. *Indeed, for the edge* $(a, b, 1) \in E$, *we have from (2.5)*

$$\forall x \in \mathbb{R} : \|2x\| = \|2x\|_b \leq \frac{1}{2}\|x\|_a = \frac{1}{2}4\|x\|,$$

and for the edge $(b, a, 2) \in E$, *we have*

$$\forall x \in \mathbb{R} : 4\|1/8x\| = \|1/8x\|_a \leq \frac{1}{2}\|x\|_b = \frac{1}{2}\|x\|.$$

Thus, the system is stable from Theorem 2.13. In this case, the value of the multinorm equals the CJSR of the system.

Example 2.16. *Consider the switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ *with* $\mathbf{G}$ *as depicted in Figure 2.3 and the set* $\mathbf{A} = \{A_1, A_2\}$ *with*

$$A_1 = \begin{pmatrix} 0 & 2 \\ -1/8 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1/8 \\ -2 & 0 \end{pmatrix}.$$

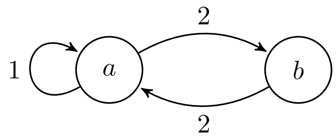


Figure 2.3: Graph for Example 2.16.

The system with arbitrary switching on that set would be unstable, with a JSR $\hat{\rho}(\mathbf{A}) = 2$, *achieved with the periodic word* 12121212 . . .. *That word is not accepted by the graph* $\mathbf{G}$. *A mulitnorm for the constrained switching system proving its stability is given by*

$$\|x\|_a = \left(x^\top \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} x\right)^{1/2}, \|x\|_b = \left(x^\top \begin{pmatrix} 40 & 0 \\ 0 & 1/8 \end{pmatrix} x\right)^{1/2}.$$

See Figure 2.4 for a visualization of the inequalities (2.5) for this example.

Remark 2.17. *Figure 2.4 provides a visual representation of the inequalities (2.5) for Example 2.16. It is based from a geometric interpretation of the inequality, for* $(s, d, w) \in E$,

$$\forall x \in \mathbb{R}^n : \|A_w x\|_d \leq \|x\|_a, \tag{2.10}$$

which goes as follows.
Given a norm $\|\cdot\|$, *let* $X = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ *be its one-level set. The norm is entirely characterized by this level. In fact, for any* convex, symmetric *and* compact *set* $C \subset \mathbb{R}^n$, *the function*

$$\phi_C(x) = inf\{\lambda \geq 0 : x \in \lambda C\},$$

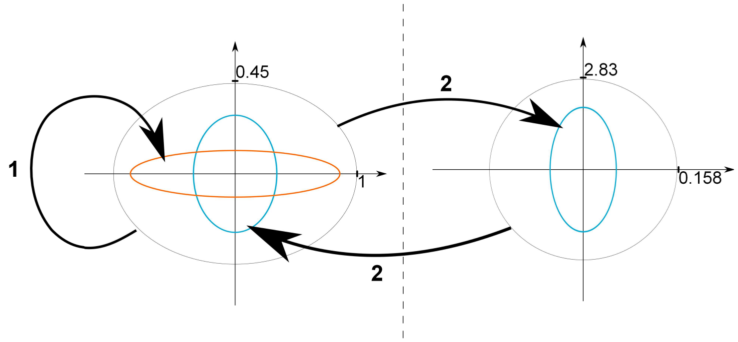


Figure 2.4: Visualization of the inequalities (2.5) for Example 2.16, with a level-set interpretation (see Remark 2.17). In gray on the left and on the right are respectively the one level sets of the norms $\|\cdot\|_a$ and $\|\cdot\|_b$. In orange and blue are images of these sets through the matrices A_1 and A_2 respectively. An arrow from one set to another and labeled with a number indicates that the first set is mapped into the second by that the corresponding mode.

which is the Minkowski function of the set C *(see [BM08, Section 3.1.2]) is a norm.*
The Lyapunov inequalities can be interpreted in terms of inclusions of convex sets. Letting X_s and X_d be respectively the level sets of the norms $\|\cdot\|_s$ and $\|\cdot\|_d$, (2.10) *is equivalent to*

$$A_\sigma X_s \subseteq X_d,$$

where $A_\sigma X_s = \{A_\sigma x \mid x \in X_s\}$.

2.2.1 Extremal Multinorms and Non-Defectivity

Given a system $\boldsymbol{\mathcal{S}}$, Proposition 2.12 guarantees that the value of any multinorm is an upper bound on the CJSR of $\boldsymbol{\mathcal{S}}$. We now investigate the existence of multinorms with values *equal* to the CJSR. The result proposed hereunder generalizes the characterization of *extremal* norms for arbitrary switching systems (see e.g [Jun09, Section 2.1.2]).

Extremal multinorms are linked to the *non-defectivity* of a system. Non-defectivity is a property where the growth of any trajectory cannot be faster than their exponential growth rate, $\hat{\rho}(\boldsymbol{\mathcal{S}})$. In contrast, *defective systems* may present $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, and yet be unstable with a sub-exponential growth rate (see [PJ15]).

Definition 2.18 (Non-Defectivity)**.** *A system $\boldsymbol{\mathcal{S}}$ is non-defective if for any vector norm $\|\cdot\|$, there exists a constant $K \geq 1$ such that for any path p in* $\mathbf{G}$,

$$\|A_{w(p)}x\| \leq K\hat{\rho}(\boldsymbol{\mathcal{S}})^{|w(p)|}\|x\|. \tag{2.11}$$

Theorem 2.19. *A system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ admits an* extremal *multinorm, i.e. a multinorm $\mathcal{H}^* = \{\|\cdot\|_s, s \in S\}$ with $\gamma^*(\mathcal{H}^*, \boldsymbol{\mathcal{S}}) = \hat{\rho}(\boldsymbol{\mathcal{S}})$, if and only if it is non-defective.*

Proof. Consider first the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$. In this case, we observe that (2.11) holds if and only if $A_{w(p)} = \mathbf{0}$ for all paths p in $\mathbf{G}$. If this is the case, then the value of *any* multinorm is 0 as well, and (2.11) holds.

Assume now that $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$. By homogeneity of the CJSR with respect to the set $\mathbf{A}$, we can further assume $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$.

We start with the only-if part. Let $\mathcal{H}^* = \{\|\cdot\|_s, s \in S\}$ be an *extremal* multinorm. Take any norm $\|\cdot\|$. By equivalence of norms in $\mathbb{R}^n$, there are two scalars $\alpha, \beta > 0$ such that for any node $s \in S$, $\alpha\|\cdot\| \leq \|\cdot\|_s \leq \beta\|\cdot\|$. Since $\mathcal{H}^*$ is extremal, for any path p between two nodes s and d, we have $\|A_{w(p)}x\|_d \leq \|x\|_s$ from (2.5), at which point we conclude that (2.11) holds for $K = \beta/\alpha$.
For the if part, consider a system $\boldsymbol{\mathcal{S}}$ with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$. Similarly to the proof of Proposition 2.12, we define at each node $s \in S$ the following *norm*:

$$\|x\|_s^* := \sup_w \{\|A_w x\| : w \in \{w(p),\ p \text{ is a path with source } s\} \cup \{\epsilon\}\},$$

where $\|\cdot\|$ is e.g. the Euclidean norm. It is direct to check that for any edge $(s, d, w) \in E$, $\|A_w x\|_d^* \leq \|x\|_s^*$. Therefore, the multinorm $\mathcal{H} = \{\|\cdot\|_s^*,\ s \in S\}$ is such that $\gamma^*(\mathcal{H}, \boldsymbol{\mathcal{S}}) \leq 1$. From Proposition 2.12, since $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, we conclude that $\mathcal{H}$ is extremal.

∎

Notice that non-defectivity is a property independent of the CJSR of the system. Moreover, the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, non-defectivity is a necessary and sufficient condition of *stability* (see Definition 1.3, and Definition 2.18). This matter is further investigated in Chapter 3, where we provide tools and insights to decide whether or not (2.11) holds true.

2.3 Approximation using Quadratic Norms

Proposition 2.12 provides us with an optimization program to compute the CJSR. However optimizing over general norms is intractable.
In this section, we investigate what happens when we restrict ourselves to using *quadratic norms*, of the form

$$x \mapsto \left(x^\top Q x\right)^{1/2}$$

for a positive definite matrix Q, as in Example 2.16.
These are appealing since we are able to use convex optimization tools to obtain the best CJSR approximation using quadratic norms.

Proposition 2.20. *Consider a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$*, and a constant* $\gamma > 0$*. If there exists quadratic forms* $\{Q_s, s \in S\}$*,* $Q_s \succ 0$*, satisfying the set of linear matrix inequalities (LMIs)*

$$\forall\, (s, d, w) \in E : -A_w^\top Q_d A_w + \gamma^{2|w|} Q_s \succeq 0, \tag{2.12}$$

then $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma$*. Moreover, the LMIs (2.12) can be solved in a number of operations bounded by*

$$\mathcal{O}\left(n^{13/2} \cdot |S|^{7/2} \cdot |E|^{3/2}\right). \tag{2.13}$$

Proof. The upper bound is a direct consequence of Proposition 2.12. The complexity computations are obtained from the reference book [BTN01, p. 424]. The number of variables in the problem (2.12) is $(n(n+1)/2)\,|S|$. The LMIs constraints can be represented by a $n\,(|S| + |E|)$ bloc diagonal symmetric matrix with diagonal blocs of size $n \times n$. ■

These LMIs are a direct extension of those of [DB01, DRI02], appearing naturally here when we consider quadratic multinorms.
In the following, we show that distance between the CJSR and the the best CJSR upper bound which can be obtained using quadratic multinorms is bounded.

Theorem 2.21. *Consider a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$*. The value* $\gamma_Q(\boldsymbol{\mathcal{S}})$ *defined as*

$$\gamma_Q(\boldsymbol{\mathcal{S}}) := \inf_{\gamma}\{\gamma : \exists\{Q_s, s \in S\}, Q_s \succ 0, \text{ satisfying (2.12)}\},$$

satisfies the following inequalities:

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma_Q(\boldsymbol{\mathcal{S}}) \leq \sqrt{n}\,(\hat{\rho}(\boldsymbol{\mathcal{S}})). \tag{2.14}$$

Remark 2.22. *For the purpose of implementation, the optimization program of Theorem 2.21 can be written as*

$$\begin{aligned} \gamma_Q(\boldsymbol{\mathcal{S}}) = & \inf_{\{Q_s, s \in S\}, \gamma} \gamma \\ & \forall (s, d, w) \in E : A_w^\top Q_d A_w \preceq \gamma^{2|w|} Q_s, \\ & \forall s \in S : Q_s \succ 0. \end{aligned} \tag{2.15}$$

This is a quasi-convex *optimization program. It may be solved using for example a bisection procedure or more advanced techniques such as the golden-section procedure in [NP13]. An upper bound on* $\gamma_Q(\boldsymbol{\mathcal{S}})$ *is given*

$$\bar{\gamma} = \max\{\|A\|, A \in \mathbf{A}\},$$

where $\|\cdot\|$ *is the induced euclidean norm. For this choice, computing* $\gamma_Q(\boldsymbol{\mathcal{S}})$ *requires* $\mathcal{O}(\log(\bar{\gamma}))$ *resolutions of the LMIs (2.12).*

Theorem 2.21 is a generalization of the result of Ando and Shih [AS98] existing for arbitrary switching systems.

Proof. The result is obtained from *John's Ellipsoid Theorem* [Joh14] (see [BNT05, PJ08, AS98] for similar approaches for arbitrary switching systems). John's ellipsoid theorem states that for any norm $\|\cdot\|$ of $\mathbb{R}^n$, there exists a *quadratic* norm $\|\cdot\|_Q : x \mapsto \left(x^\top Q x\right)^{1/2}$, with $Q \succ 0$, such that $\forall x \in \mathbb{R}^n : \|x\|_Q \leq \|x\| \leq \sqrt{n}\|x\|_Q$.

Let us take any $\delta > 0$ and a multinorm $\mathcal{H}_\delta = \{\|\cdot\|_s, s \in S\}$ with $\gamma^*(\mathcal{H}_\delta) \leq \hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta$. Such a multinorm exists (Proposition 2.12).
By John's ellipsoid theorem, for each norm in the multinorm $\|\cdot\|_s$, there exist a quadratic norm $\|\cdot\|_{Q,s}$, such that for any edge $(s, d, w) \in E$, $\forall x \in \mathbb{R}^n$, we can write

$$\|A_w x\|_{Q,d} \leq \|A_w x\|_d \leq (\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)\|x\|_s \leq \sqrt{n}(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)\|x\|_{Q,s}.$$

Since the above holds for any edge we can state that, for any $\delta > 0$, the quadratic multinorm $\mathcal{H}_{Q,\delta} = \{\|\cdot\|_{Q,s}, s \in S\}$ satisfies $\gamma^*(\mathcal{H}_{Q,\delta}) \leq \sqrt{n}(\hat{\rho}(\boldsymbol{\mathcal{S}})+\delta)$. Taking $\delta \to 0$ we obtain (2.14).

■

Example 2.23. *For complex valued matrices, the bound $\sqrt{n}$ is known to be tight for all n (see [AS98, Section 3]) in the arbitrary switching case.*

For $n = 2$, the system on the set of real matrices $\mathbf{A} = \{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix},$$

has $\hat{\rho}(\mathbf{A}) = 1$, but $\gamma_Q = \sqrt{2}$. In fact, an extremal norm *for this system is given by $\|x\| = |(x)_1| + |(x)_2|$, and the best quadratic approximation is attained by the euclidean norm. To our knowledge, for $n \geq 3$ the bound has not been proven to be tight.*

In Section 2.4, we develop several algorithms aiming at providing *arbitrarily accurate* estimations of the CJSR, with *finite time complexity*, all having at their common core the optimization program of Theorem 2.21. Before moving further in that direction, we provide in Subsection 2.3.1 a handy sufficient condition that allows us to decide, after having solved the program of Theorem 2.21, whether or not the bound obtained is tight. Additionally, in Subsection 2.3.2, we compare the approach with another one, that transform a CSS into a *high dimensional arbitrary switching system*, on which we can apply classical estimation methods for estimating the joint-spectral radius.

2.3.1 A Sufficient Condition for Exact Estimation

We present a sufficient condition under which the CJSR estimate obtained is *exact*. It relies on the study of *periodic systems*. These can be seen as CSS on graphs that are *cyclic*[4], such as the one in Figure 2.2.

We start with the following observation.

Lemma 2.24. *For any system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and any cycle c of length T in $\mathbf{G}$,*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \geq \rho(A_{w(c)})^{1/|w(c)|},$$

where $\rho(A_{w(c)})$ is the spectral radius of the product $A_{w(c)}$.

Proof. For any induced matrix norm $\|\cdot\|$, the following holds

$$\begin{aligned}
\rho(A_{w(c)})^{1/T} &= \lim_{k\to\infty} \|A^k_{w(c)}\|^{1/k|w(c)|} \\
&\leq \lim_{k\to\infty} \max\{\|A_{w(c)}\|^{1/k|w(c)|} : p \text{ is a path of length } k|w(c)|\} \\
&\leq \hat{\rho}(\boldsymbol{\mathcal{S}}).
\end{aligned}$$

■

Lemma 2.25. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ where $\mathbf{G}$ is cyclic. Then $\gamma_Q(\boldsymbol{\mathcal{S}}) = \hat{\rho}(\boldsymbol{\mathcal{S}})$.*

Proof. Without loss of generality, assume that the matrices are invertible. If $\mathbf{G}$ is cyclic, then it is easy to see that

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) = \rho\left(A_{w(c)}\right)^{1/(|w(c)|)},$$

where c is a simple cycle, visiting all nodes in $\mathbf{G}$ (and $|w(c)| = |S| + 1$).

Now, take any node $s \in S$, and let c_s be the cycle in $\mathbf{G}$ starting and ending at that node. By the definition of the spectral radius of $A_{w(c_s)}$, for any $\delta > 0$, there is a quadratic form Q_s^δ such that

$$A^\top_{w(c_s)} Q_s^\delta A_{w(c_s)} \preceq (\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)^{2(|S|+1)} Q_s^\delta.$$

From that quadratic form, we can construct a multinorm for the all system. Take any node $q \in S$, and let $w(q, s)$ be the word on the *unique* acyclic path from q to s, and take

$$Q_q = \frac{1}{(\hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta)^{2|w(q,s)|}} A^\top_{w(q,s)} Q_p A_{w(q,s)}.$$

We can verify that (2.12) holds for all edges in the graph for $\gamma = \hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta$, with equality when the source of the edge is not s, and inequality when the source is s.

Therefore, we have that for all δ, $\gamma_Q(\boldsymbol{\mathcal{S}}) < \hat{\rho}(\boldsymbol{\mathcal{S}}) + \delta$, and we can conclude. ■

[4] These are graphs where any simple cycle visits all the nodes.

Theorem 2.26 (Sufficient extremality condition)**.** *Let* $\{Q_s,\, s \in S\}$ *be the optimal quadratic forms obtained by applying Theorem 2.21 to the system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$*, corresponding to a multinorm* $\mathcal{H}$ *with value* γ_**. If the set of edges*

$$E' = \left\{(s, d, w) \in E : \lambda_{\min}\left(-A_w^\top Q_d A_w + \gamma_*^{2|w|} Q_s\right) = 0\right\},$$

where $\lambda_{\min}(X)$ *denotes the smallest eigenvalue of the positive definite matrix* X*, forms a* simple *cycle* c *in* $\mathbf{G}$*, then* $\mathcal{H}$ *is* extremal*, and* $\hat{\rho}(S) = \rho(A_{w(c)})^{1/|w(c)|}$.

Proof. The edges in the set E' are those for which the LMI constraints in (2.12) are tight for the multinorm $\mathcal{H}$. Let $\mathbf{G}' = (S, E')$ be the sub-graph of $\mathbf{G}$ where we keep only the edges in E'. By construction, $\mathbf{G}'$ is cyclic.
It is a known fact of convex optimization that removing constraints that *are not* tight at the optimal solution of a given program *does not* affect its optimal solutions. Therefore, it must be the case that the optimal values $\gamma_Q(\boldsymbol{\mathcal{S}})$ and $\gamma_Q(\boldsymbol{\mathcal{S}}')$ obtained by applying Theorem 2.21 respectively to $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ and to the system $\boldsymbol{\mathcal{S}}' = (\mathbf{G}' = (S, E'), \mathbf{A})$ are *equal.* Both are thus upper bounds on $\hat{\rho}(\boldsymbol{\mathcal{S}})$.

However, since $\mathbf{G}' = (S, E')$ is cyclic, then $\gamma_Q(\boldsymbol{\mathcal{S}}') = \hat{\rho}(\boldsymbol{\mathcal{S}}')$, which is the spectral radius of a product of matrices associated with the simple cycle in $\mathbf{G}'$ (see Lemma 2.25). Then Lemma 2.24, implies $\gamma_Q(\boldsymbol{\mathcal{S}}') \leq \hat{\rho}(\boldsymbol{\mathcal{S}})$, and we conclude than $\gamma_Q(\boldsymbol{\mathcal{S}}') = \gamma_Q(\boldsymbol{\mathcal{S}}) = \hat{\rho}(\boldsymbol{\mathcal{S}})$.

■

2.3.2 Lifting to an Arbitrary Switching System

This subsection presents a *lifting procedure* that allows to study the stability properties of a constrained switching systems through an equivalent, but higher dimensional, arbitrary switching system. In [Koz14], Kozyakin introduced this lift to study constrained switching systems where the switching sequence is a realization of a Markov Chain. The results of [Koz14] show the equality between the CJSR of these Markovian systems and the JSR of the arbitrary switching system on the lifted set of matrices. A similar lifting procedure was introduced in parallel by Wang et al. [WRDV14] for the general class of systems described by an automaton $\mathbf{G}$. Wang et al. proved that the stability of a constrained switching system is equivalent to that of the lifted set of matrices presented below. From these results and those of Dai's [Dai12], it is easy to conclude the following: the CJSR of a constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ equals the JSR of the lifted matrix set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, and the boundedness of both the constrained and lifted arbitrary switching systems are equivalent.

Definition 2.27 (Kroenecker lift)**.** *Given a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$*, with* $|S|$ *nodes* $\{s_i \in S,\, 1 \leq i \leq |S|\}$ *and* $\mathbf{A} \subset \mathbb{R}^{n\times n}$*, the* Kroenecker lift *of* S *is a matrix set defined as* $\mathbf{A}_{\boldsymbol{\mathcal{S}}} = \{A_{(s_i, s_j, w)} \in \mathbb{R}^{n|S|\times n|S|},\, (s_i, s_j, w) \in E\}$*, with for the edge* (s_i, s_j, w)*,*

$$A_{(s_i, s_j, w)} = \left(\mathbf{e}_j \mathbf{e}_i^\top\right) \otimes A_w,$$

where $\mathbf{e}_k \in \mathbb{R}^{|S|}$ *is the* k*th vector of the canonical basis of* $\mathbb{R}^{|S|}$*, and* $\otimes$ *is the Kroenecker product.*

Remark 2.28. *By construction, if we take two edges* (s,d,σ) *and* (s',d',σ')*, the product* $A_{(s',d',\sigma')}A_{(s,d,\sigma)} \neq 0$ *only if* $s' = d$*. All products of matrices in* $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ *that are nonzero correspond to paths in* $\mathbf{G}$*.*

In Subsections 2.4.1, 2.4.2 and 2.4.3, the methods we present for approximating the CJSR are based on the ones of [BNT05, LD06b, PJ08], that have been developed for arbitrary switching systems. It is natural to compare our methods with a direct application of the ones of [BNT05, LD06b, PJ08] for approximating $\hat{\rho}(\boldsymbol{\mathcal{S}}) = \hat{\rho}(\mathbf{A}_{\boldsymbol{\mathcal{S}}})$. We will focus on [BNT05], which approximates the JSR of a set of matrices using Theorem 2.21 with a single quadratic form. The conclusions naturally carry on to the other methods of [PJ08, LD06b].

Proposition 2.29. *Consider a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S,E), \mathbf{A})$ *and its associated set* $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$*. For any* $\gamma > 0$*, the following two are equivalent.*

- *There exists* $Q_{\boldsymbol{\mathcal{S}}} \succ 0$ *of dimension* $n|S| \times n|S|$ *such that* $\forall A_{(s,d,w)} \in \mathbf{A}_{\boldsymbol{\mathcal{S}}}$*,*

$$A_{(s,d,w)}^\top Q_{\boldsymbol{\mathcal{S}}} A_{(s,d,w)} - \gamma^{2|w|} Q_{\boldsymbol{\mathcal{S}}} \preceq 0,$$

- *The system* $\boldsymbol{\mathcal{S}}$ *has a quadratic multinorm with value at most* γ*.*

Proof. Without loss of generality, we consider that the graph $\mathbf{G}$ is in expanded form, that $\gamma = 1$, and finally, that the dimension of the system is $n = 1$ (the proof focuses on the structure of the LMI, which is independent of the dimension). Also, in order to ease the notations, we define the set of nodes S as a set of integers, $S = \{1, \ldots, |S|\}$.

For any edge $e = (s,d,\sigma) \in E$, we have $A_e \in \mathbb{R}^{|S|\times|S|}$, and we let $B_e = A_e^\top Q_{\boldsymbol{\mathcal{S}}} A_e - Q_{\boldsymbol{\mathcal{S}}}$.
If we inspect the entries of $A_{(s,d,\sigma)}$, we have that $(A_{(s,d,\sigma)})_{k,\ell} = \delta_{d,k}\delta_{s,\ell}A_\sigma$, where where $\delta_{i,j} = 1$ if and only if $i = j$ (Kroenecker delta).
If we inspect the entries of B_e, we have for any $1 \leq k, \ell \leq |S|$

$$\begin{aligned}(B_{(s,d,\sigma)})_{k,\ell} &= \sum_{i',j'} (A_{(s,d,\sigma)}^\top)_{k,i'} (Q_{\boldsymbol{\mathcal{S}}})_{i',j'} (A_{(s,d,\sigma)})_{j',\ell} - (Q_{\boldsymbol{\mathcal{S}}})_{k,\ell}, \\ &= \delta_{k,s}\delta_{\ell,s}\left((A_\sigma^\top (Q_{\boldsymbol{\mathcal{S}}})_{d,d} A_\sigma\right) - (Q_{\boldsymbol{\mathcal{S}}})_{k,\ell}.\end{aligned}$$

For the only if part, $Q_{\boldsymbol{\mathcal{S}}} \succ 0$ and $B_e \preceq 0$. Therefore, for all $1 \leq k \leq |S|$, $(Q_{\boldsymbol{\mathcal{S}}})_{k,k} > 0$, and for any $e = (s,d,\sigma) \in E$, $(B_e)_{s,s} = A_\sigma^\top (Q_{\boldsymbol{\mathcal{S}}})_{d,d} A_\sigma - (Q_{\boldsymbol{\mathcal{S}}})_{i,i} \leq 0$. These inequalities prove that we extract a quadratic multinorm for $\boldsymbol{\mathcal{S}}$ using the quadratic forms $(Q_{\boldsymbol{\mathcal{S}}})_{s,s}$, $1 \leq s \leq |S|$.

For the if part of the proof, we take a quadratic multinorm with value lower than 1. Let $Q_i > 0, 1 \leq s \leq |S|$, be the quadratic forms associated to each node. We reconstruct $Q_{\boldsymbol{\mathcal{S}}}$ by setting $(Q_{\boldsymbol{\mathcal{S}}})_{k,\ell} = Q_k$ if $k = \ell$, and $(Q_{\boldsymbol{\mathcal{S}}})_{k,\ell} = 0$ else. ∎

Applying Theorem 3.1 to $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ therefore produces the same estimate as if it was applied on S. However, when using $\mathbf{A}_{\boldsymbol{\mathcal{S}}} \subset \mathbb{R}^{n|S| \times n|S|}$, the complexity (2.13) reads $\mathcal{O}\left((n|S|)^{(13/2)}|S|^{(3/2)}\right)$, a significant increase compared to the previous $\mathcal{O}\left(n^{(13/2)}|S|^{(7/2)}|S|^{(3/2)}\right)$. Moreover, the accuracy bounds (2.14) become worse, due to the larger dimension of $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$.

2.4 Lifting Procedures to Increase Accuracy

Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and a relative error $r > 0$, we construct algorithms that compute an estimate γ of $\hat{\rho}(\boldsymbol{\mathcal{S}})$ satisfying

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma \leq (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}}).$$

Remark that for any $r \geq \sqrt{n} - 1$, the optimization program of Theorem 2.21 provides us with a valid estimate.

All techniques developed here share the same broad structure.

1. We are given $r > 0$ and a system $\boldsymbol{\mathcal{S}}$.
2. Using this information, we compute a *lifted system* $\boldsymbol{\mathcal{S}}'$, using one of the techniques of this section.
3. We compute $\gamma_Q(\boldsymbol{\mathcal{S}}')$ using Theorem 2.21.
4. We have $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma_Q(\boldsymbol{\mathcal{S}}') \leq (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}})$.

Note that, since Theorem 2.21 plays a central role for all techniques above, the results of Subsection 2.3.1 can be applied to all techniques below.

In the below, we consider graph in *expanded form* (see Definition 1.14), for the sake of exposition.

2.4.1 The T-Product Lift

This first method allows to provide arbitrarily accurate estimates of the CJSR, with the cost of constructing a system on a graph with a large amount of edges. A T-product lift is a construction that takes a graph $\mathbf{G}$, and outputs a graph $\mathbf{G}'$, with the same language, same set of nodes but where each edge corresponds to a path of lenght T in the original graph.

Definition 2.30 (T-product lift)**.** *Given a graph* $\mathbf{G} = (S, E)$ *and an integer* T*, the* T-product lift *of* $\mathbf{G}$ *is a graph* $\mathbf{G}^T = (S', E')$ *defined as follows:*

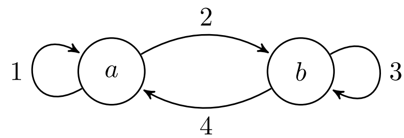
1. *The sets of nodes in* $\mathbf{G}^T$ *is that of* $\mathbf{G}$*,* $S' := S$*;*
2. *The set of edges* E' *is defined as follows:*

$$(s, d, w) \in E' \Leftrightarrow \exists p \in \mathscr{P}^T_{s,d}(\mathbf{G}),\ w = w(p).$$

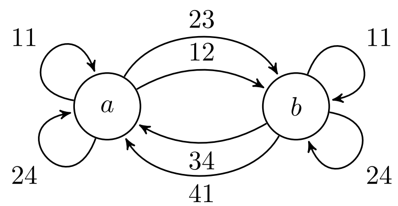
Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, the T-product lift of $\boldsymbol{\mathcal{S}}$ is given by $\boldsymbol{\mathcal{S}}^T = (\mathbf{G}^T, \mathbf{A})$.

Remark 2.31. *Remark that $\mathscr{L}(\mathbf{G}) = \mathscr{L}(\mathbf{G}^T)$. Hence, a T-product lift of a system still represents the* same *dynamics, with the same stability properties - except in aspects that concern* multinorms *(see Remark 2.14). It is true that if the system is stable, so is any of its T-product lift, but every one of these might (and many will!) have a quadratic multinorm as a stability certificate (see Theorem 2.33).*

Example 2.32. *The 2-product lift of the automaton of Figure 2.5a is presented on Figure 2.5b.*



(a) A graph $\mathbf{G}$ for Example 2.32.



(b) The T-lift (T = 2) of the graph $\mathbf{G}$ at Figure 2.5a. It has the same set of nodes, and each edge corresponds to a path of length 2 in $\mathbf{G}$.

Figure 2.5

Theorem 2.33. *The optimal value $\gamma_Q(\boldsymbol{\mathcal{S}}^T)$ obtained by applying Theorem 2.21 to the system $\boldsymbol{\mathcal{S}}^T = (\mathbf{G}^T, \mathbf{A})$ satisfies $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma_Q(\boldsymbol{\mathcal{S}}^T) \leq n^{1/(2T)}\hat{\rho}(\boldsymbol{\mathcal{S}})$.*

Proof. The system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ describes dynamics of the form

$$x(t+1) = A_{\sigma(t)}x(t),\ \sigma(t) \in \langle M \rangle,\ \sigma(0)\sigma(1)\ldots \in \mathscr{L}(\mathbf{G}).$$

Given an integer T, let us study the constrained switching system $\boldsymbol{\mathcal{S}}'$ for the dynamics

$$y(k+1) = A_{w(k)}y(k),\ w(k) \in \langle M \rangle^T,\ w(0)w(1)\ldots \in \mathscr{L}(\mathbf{G}).$$

With a slight abuse of notations, we can write $\boldsymbol{\mathcal{S}}'$ as the tuple $((\mathbf{G}^T)', \mathbf{A}^T)$, where $\mathbf{A}^T = \{A_w,\ w \in \langle M \rangle^T\}$, and where $(\mathbf{G}^T)'$ is the same as $\mathbf{G}^T$, *except that each word on each edge is now considered as single symbol of length one.* More precisely, we define a new alphabet with M^T symbols, one per word in

$\langle M \rangle^T$, and each corresponding to a matrix in $\mathbf{A}^T$. The graph $(\mathbf{G}^T)'$ uses these symbols. Observe that

$$\hat{\rho}(\boldsymbol{\mathcal{S}}') = \max_{w \in \mathscr{L}(\mathbf{G}), |w| = kT} \|A_w\|^{1/k},$$

since words of length k for $\boldsymbol{\mathcal{S}}'$ correspond to words of length kT for $\boldsymbol{\mathcal{S}}$.

We now prove that $\hat{\rho}(\boldsymbol{\mathcal{S}}') = \hat{\rho}(\boldsymbol{\mathcal{S}})^T$. Define for all $k \geq 1$,

$$\hat{\rho}_k(\boldsymbol{\mathcal{S}}) = \max_{w \in \mathscr{L}(\mathbf{G}), |w| = k} \|A_w\|^{1/k}.$$

By construction, we have $\hat{\rho}_k(\boldsymbol{\mathcal{S}}') = \hat{\rho}_{kT}(\boldsymbol{\mathcal{S}})^T$, and by continuity of the exponentiation, we obtain $\hat{\rho}(\boldsymbol{\mathcal{S}})^T = \lim_{k \to \infty} \hat{\rho}_{kT}(\boldsymbol{\mathcal{S}})^T = \lim_{k \to \infty} \hat{\rho}_k(\boldsymbol{\mathcal{S}}') = \hat{\rho}(\boldsymbol{\mathcal{S}}')$.

From Theorem 2.21, we know that $\gamma_*(\boldsymbol{\mathcal{S}}') \leq \sqrt{n}\hat{\rho}(\boldsymbol{\mathcal{S}}') = \sqrt{n}\hat{\rho}(\boldsymbol{\mathcal{S}})^T$. To conclude, it suffice to show that

$$\gamma_Q(\boldsymbol{\mathcal{S}}') = \gamma_Q(\boldsymbol{\mathcal{S}}^T)^T.$$

To do so, we write (2.12) for both systems $\boldsymbol{\mathcal{S}}^T$ and $\boldsymbol{\mathcal{S}}'$. Since the labels on both graphs coincide edge by edge - but are considered of length 1 in $\boldsymbol{\mathcal{S}}'$ and length T in $\boldsymbol{\mathcal{S}}^T$ - all inequalities are the same, except that γ is raised to the power $2T$ for $\boldsymbol{\mathcal{S}}^T$, but only to the power 2 for $\boldsymbol{\mathcal{S}}'$ (since words of $\boldsymbol{\mathcal{S}}^T$ are symbols of $\boldsymbol{\mathcal{S}}'$). This concludes the proof. ■

Remark 2.34. *The bound $n^{1/(2T)}$ remains tight for the arbitrary switching system on the set of matrices defined in Example 2.23.*

The proof of Theorem 2.33 involves concepts similar to *finite steps Lyapunov functions* see e.g. [AL14a, GGLW14] or [BFT03, Theorem 2]. In fact, a natural interpretation of using the T-product lift is to search for Lyapunov functions but require them to decrease only after every T steps.

With the following, we are able to provide the *complexity* of computing a CJSR approximation with relative error $r > 0$.

Corollary 2.35. *Consider a system $\boldsymbol{\mathcal{S}}$ and a relative error $r > 0$. For any*

$$T(r) \geq \frac{\log(n)}{2\log(1+r)},$$

the estimate $\gamma_Q(\boldsymbol{\mathcal{S}}^T)$ obtained by applying Theorem 2.21 to the T-product lift of $\boldsymbol{\mathcal{S}}$ is within the range

$$[\hat{\rho}(\boldsymbol{\mathcal{S}}),\, (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}})].$$

Proof. The parameter $T(r)$ is obtained by solving for T in $n^{1/(2T)} \leq 1 + r$. ■

Putting together Proposition 2.20 and Remark 2.22, and taking into account that the number of edges in the T-product lift of $\mathbf{G} = (S, E)$ is $\mathcal{O}(|E|^T)$, we have the following.

Corollary 2.36. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ and a relative error $\epsilon > 0$, one can obtain an estimate of the CJSR within $[\hat{\rho}(\boldsymbol{\mathcal{S}}), (1 + r)\hat{\rho}(\boldsymbol{\mathcal{S}})]$ with a number of operations bounded by*

$$\mathcal{O}\left(\log(\bar{\gamma}) n^{13/2} |S|^{7/2} |E|^{(3T(r)/2)}\right),$$

with $\bar{\gamma} = \max\{\|A\|, A \in \mathbf{A}\}$, and $T(r)$ as in Corollary 2.35.

Remark 2.37. *It is interesting to notice that overall, given a graph $\mathbf{G}$ and $r > 0$, the complexity of obtaining a CJSR approximation with relative error at most $1 + r$ is not polynomial using our algorithms. For obtaining algorithms polynomial in n, with all other parameters fixed (accuracy, number of nodes and edges in the graphs), one may apply the method of [BN05] to approximate the JSR set of matrices $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ presented in Subsection 2.3.2.*

2.4.2 Path-Dependent Methods

Path-Dependent Methods have been introduced by Lee and Dullerud [LD06b] for the stability analysis of switching systems with constrained switching sequences. Their idea, which follows a similar idea to that of Bliman and Ferrari-Trecate [BFT03], is to build a multiple Lyapunov function that associates a different quadratic form to each *switching sequence of a length* $K \geq 0$, and K is called the memory of the technique. In [ELD14], an extension is provided, where on top of the memory, a look-ahead (assume we have knowledge of the current and K future modes) can be included. Here, we are going to focus on the original approach in [LD06b] to remain concise.

An interesting feature of this approach is a converse theorem it provides: if a system is exponentially stable, then there must be some value for the parameter K such that the method provides a stability certificate. Similar statements can be made about the T-product lift of Subsection 2.4.1, and about the $[d]$-lift presented next in Subsection 2.4.3. If $\hat{\rho}(\boldsymbol{\mathcal{S}}) < 1$ and since the CJSR approximations provided are asymptotically tight, there is a finite value T (or d) for which the approximation algorithm will return an upper bound on the CJSR lower then 1. Given this parallel, it is natural to ask whether the methods of [LD06b, BFT03] present similar approximation bounds to that of Theorems 2.33 and 2.48.

Remark 2.38. *In the works [LD06b, ELD14, BFT03, KC15, FJ14], the constraints on the switching sequences are are defined in formalism which is different than ours, which may induce some confusions regarding the current presentation (see the discussion in Section 1.3).*
For a system on M modes they consider a matrix $W \in \{0, 1\}^M$. If $(W)_{i,j} = 1$,

this means we are allowed the transition from $\sigma(t) = i$ *to* $\sigma(t+1) = j$ *in the switching sequences. In our case, using such a matrix translates into a graph* $\mathbf{G} = (S, E)$ *on* M nodes*, with* $S = \{s_i, i \in \{1, \ldots, M\}\}$*, defined as follows*

$$(W)_{i,j} = 1 \Leftrightarrow (s_i, s_j, i) \in E,$$

meaning that all edges with a same source node have a same label.
With this structure, note that for any sequence of modes w *of* length 2 or more*, there is a* unique *path* p *in the graph with* $w(p) = w$*. In our setting, depending on the graph, several paths may carry the same sequences of symbols.*

Remark 2.39. *As mentioned before, for a given memory* K*, the authors of [LD06b] attach* one *Lyapunov function* per switching sequence of length K*. Hence, for* $K = 0$ *and since switching sequences correspond to words in* $\langle M \rangle^*$*, the authors look for a* common Lyapunov function *that is attached to the sequence/word* ϵ *(empty word). In doing so, their approach is overly conservative for* $K = 0$ *(this is reported in [ELD14, Remark 13]), as they don't take into account the constraints on the switching sequences.*
The constraints start being considered as soon as $K = 1$*. There, within their formalism, the authors attach a Lyapunov function per node of the graph (since they correspond to modes, see Remark 2.38).*
In order to avoid these issues, we attach Lyapunov functions to paths *in the graph instead of* words *in* $\mathscr{L}(\mathbf{G})$*, and define paths of length* 0 *as the nodes of the graph. Hence, a path-dependent Lyapunov function attached to paths of length* $K = 0$ *correspond to words of length* $K = 1$ *in the original setting of [LD06b].*

We first define a lifting procedure, augmenting the graph $\mathbf{G}$ of a system such that when we apply Theorem 2.21 on that new graph, we compute a multiple Lyapunov function with one function per path of length $K \geq 0$. By convention, we associate paths of length 0 with nodes in the graph.

Definition 2.40 (K-path-dependent Lifting)**.** *Given a graph* $\mathbf{G} = (S, E)$ *on the alphabet* $\langle M \rangle$*, and an integer* $K \geq 0$*, the* K*-path-dependent lift of* $\mathbf{G}$ *is the graph* $\mathbf{G}_K = (S_K, E_K)$ *constructed as follows: for* $K = 0$*,* $\mathbf{G}_K = \mathbf{G}$*, else*

1. *Build a* node $s_p \in S_K$ *per* path p *of length* K *in* $\mathbf{G}$*.*
2. *For any two nodes* s_p *and* s_q *in* S_K*, corresponding to paths* p *and* q *in* $\mathscr{P}^K(\mathbf{G})$*, we have*

$$\begin{aligned}(s_p, s_q, \sigma) \in E_K \Leftrightarrow \\ & p = \{e_1,\, e_2,\, \ldots, e_{K-1},\, e_K\}, \\ & q = \{e_2,\, e_3,\, \ldots, e_K,\, (s, d, \sigma)\},\end{aligned}$$

where $(s, d, \sigma) \in E$*.*

Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$*, the* K*-path-dependent lift of* $\boldsymbol{\mathcal{S}}$ *is* $\boldsymbol{\mathcal{S}}_K = (\mathbf{G}_K, \mathbf{A})$*.*

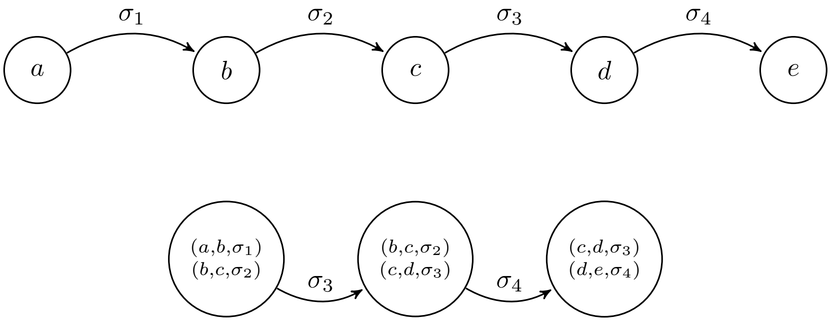


Figure 2.6: Constructing the path-dependent lift for $K = 2$. Above, a path of length 4 in the graph $\mathbf{G}$. This path leads to 3 nodes in the graph $\mathbf{G}_K$, one per path of length 2 in the above. We construct an edge between two nodes in $\mathbf{G}_3$ if and only if the last edge for the source node is the first edge of the destination node. The label on this edge is that of the last edge of the destination node.

Remark 2.41. *The construction of Definition 2.40 is illustrated in Figure 2.6.*

Example 2.42. *Figure 2.7 presents the 1-path-dependent lift,* $\mathbf{G}_K$ *for* $K = 1$*, of the graph* $\mathbf{G}$ *of Figure 2.5a. There are 4 nodes in* $\mathbf{G}_K$*, each one as been marked with the corresponding edge of* $\mathbf{G}$*. In the case* $K = 1$*, there is an edge between two nodes in the lifted graph the two corresponding edges forms a path (the destination of the first edge is the source of the second).*

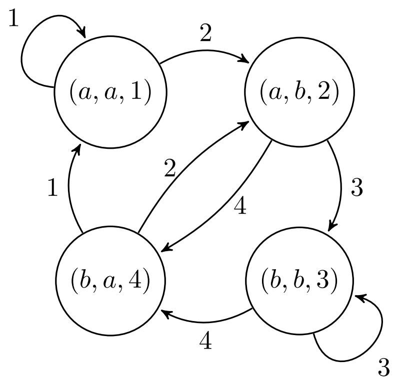


Figure 2.7: The 1-path-dependent lift of the graph at Figure 2.5a.

Remark 2.43. *The* K*-path-dependent lift* $\mathbf{G}_K$ *has the following property. Take a node* $s \in S_K$*, which corresponds to a path* p *in* $\mathbf{G}$*. Then all paths in* $\mathbf{G}_K$

ending at s carry the same word $w(p)$. If we take $\mathbf{G}$ as in Figure 2.1a, Path-Dependent lifts generate the well known Debruijn Graphs.

Theorem 2.44. *Consider a system $\boldsymbol{\mathcal{S}}$ and an integer $T \geq 1$. Let $\gamma_Q(\boldsymbol{\mathcal{S}}^T)$ be the CJSR estimate obtained by applying Theorem 2.21 to the T-product lift $\boldsymbol{\mathcal{S}}^T$, and let $\gamma_Q(\boldsymbol{\mathcal{S}}_{T-1})$ be obtained by applying Theorem 2.21 to the $(T-1)$-path-dependent lift $\boldsymbol{\mathcal{S}}_{T-1}$. We have*

$$\gamma_Q(\boldsymbol{\mathcal{S}}_{T-1}) \leq \gamma_Q(\boldsymbol{\mathcal{S}}^T).$$

Proof. For $T = 1$, $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_{T-1} = \boldsymbol{\mathcal{S}}^T$, so the claim holds. Assume now $T \geq 2$.

We will show that given a quadratic multinorm for $\boldsymbol{\mathcal{S}}^T$ with value γ, we can always construct a quadratic multinorm for $\boldsymbol{\mathcal{S}}_{T-1}$ with value $\gamma' \leq \gamma$. A path of length T is a sequence of edges

$$p = \{e_i = (s_{i-1}, s_i, \sigma_i) \in E, 1 \leq i \leq T\},$$

and for such paths, we let s_0 be the source of p, s_T its destination, and the ith symbol in $w(p)$ is σ_i.

A quadratic multinorm with value γ for the T-product lift $\boldsymbol{\mathcal{S}}^T$ associates a quadratic form $Q_s \succ 0$ per node in $\mathbf{G}$ such that, for all paths p, with a source node s_0 and a destination node s_T,

$$A_{w(p)}^\top Q_{s_T} A_{w(p)} - \gamma^{2T} Q_{s_0} \preceq 0. \tag{2.16}$$

To the same path p corresponds an edge in the $(T-1)$-path-dependent lift, between a node associated to the path $p_1 = \{e_1, \ldots, e_{T-1}\}$ and a node associated to the path $p_2 = \{e_2, \ldots, e_T\}$, both being paths of length $T-1$. For a quadratic multinorm of the $\boldsymbol{\mathcal{S}}_{T-1}$ with value γ to exist, there must be a quadratic form $Q_p \succ 0$ per path p of length $T-1$, such that the following holds:

$$\begin{aligned} A_{\sigma_T}^\top Q_{\{e_2,\ldots,e_T\}} A_{\sigma_T} & \\ - \gamma^2 Q_{\{e_1,\ldots,e_{T-1}\}} &\preceq 0. \end{aligned} \tag{2.17}$$

Letting $R = Q^{-1}$ denote the inverses of the quadratic forms in the corresponding lift, by a Schurr complement, the LMIs (2.16) are equivalent to

$$A_{w(p)} R_{s_0} A_{w(p)}^\top - \gamma^{2T} R_{s_T}, \preceq 0, \tag{2.18}$$

and the LMIs (2.17) to

$$A_{\sigma_T} R_{\{e_1,\ldots,e_{T-1}\}} A_{\sigma_T}^\top - \gamma^2 R_{\{e_2,\ldots,e_T\}} \preceq 0. \tag{2.19}$$

Assume that we have a solution $\{R_s, v \in S\}$ to (2.18). Given any path $p = (e_1, \ldots, e_{T-1})$, with labels $\sigma_1, \ldots, \sigma_{T-1}$ that visits the nodes $s_0, \ldots, s_{T-1}$ in $\mathbf{G}$, define

$$\begin{aligned}R_{\{e_1,\ldots,e_{T-1}\}} = R_{s_{T-1}}\\ &+ \gamma^{-2} A_{\sigma_{T-1}} R_{s_{T-2}} A^\top_{\sigma_{T-1}}\\ &+ \gamma^{-4} A_{\sigma_{T-2}\sigma_{T-1}} R_{s_{T-3}} A^\top_{\sigma_{T-2}\sigma_{T-1}}\\ &+ \cdots\\ &+ \gamma^{-2(T-1)} A_{\sigma_1\ldots\sigma_{T-1}} R_{s_0} A^\top_{\sigma_1\ldots\sigma_{T-1}}.\end{aligned}$$

Injecting these quadratic forms in (2.19), we obtain (2.18), and thus we have a quadratic multinorm for S_{T-1}, with value at most γ. ∎

Corollary 2.45. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and an integer $M \geq 1$, the value γ_* obtained by applying Theorem 2.21 to the M-Path-Dependent lift of S satisfies*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma_* \leq n^{1/(2(M+1))} \hat{\rho}(\boldsymbol{\mathcal{S}}).$$

Remark 2.46. *The CJSR estimates obtained by K-path-dependent methods decrease monotonically with K. This is not necessarily true for the T-product lift.*

To our knowledge, in general, there exist no methods/construction to produce switching systems that attain the accuracy bound $n^{1/(2(K+1))}$ for the K-path-dependent methods, for all K. In practice, path-dependent methods provide very efficient bounds, often better than the T-product lift (although it takes more time to compute, as discussion in Subsection 2.5). For future work, it would be interesting either to provide (and analyze) such examples, or to further refine the accuracy guarantees of path-dependent methods.

2.4.3 Sum-of-Squares Methods

In this subsection we generalize the technique presented in [PJ08] to constrained switching systems and devise a CJSR approximation scheme relying on an algebraic lifting of the set of matrices $\mathbf{A}$. The procedure generating the lifted space is called the *[d]-lift*.

Given an integer d, the $[d]$-lift of the vector $x \in \mathbb{R}^n$ is the vector $x^{[d]} \in \mathbb{R}^{C(n+d-1,d)}$ of monomials of degree d, where

$$C(k,\ell) = \frac{k!}{\ell!(k-\ell)!}$$

is the number of combinations of ℓ elements in a set of k elements, with one entry per monomial of degree d on n coefficient, and is defined as follows: for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \{0, \ldots, d\}$ and $\sum \alpha_i = d$,

$$(x^{[d]})_\alpha = \sqrt{\frac{d!}{\alpha_1!\alpha_2!\cdots\alpha_n!}} \prod_{i=1}^n (x_i)^{\alpha_i}.$$

The scaling of the monomial are chosen such that $\left\|x^{[d]}\right\| = \|x\|^d$ for the euclidean norm in appropriate dimension. Furthermore, the *[d]-lift* is well defined for linear maps, i.e. given a matrix A, we can define $A^{[d]}$ such that $A^{[d]}x^{[d]} = (Ax)^{[d]}$. This definition extends to sets of matrices $\mathbf{A}^{[d]}$.

Definition 2.47 ([d]-lift)**.** *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and an integer $d \geq 1$, the $[d]$-lift of $\boldsymbol{\mathcal{S}}$ is the system*

$$\boldsymbol{\mathcal{S}}^{[d]} = (\mathbf{G}, \mathbf{A}^{[d]}),$$

with

$$\mathbf{A}^{[d]} = \{A^{[d]} \mid A \in \mathbf{A}\}.$$

Theorem 2.48. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and an integer $d \geq 1$. The value $\gamma_Q(\boldsymbol{\mathcal{S}}^{[d]})$ obtained by applying Theorem 2.21 to the system $\boldsymbol{\mathcal{S}}^{[d]}$ satisfies*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma_Q^{1/d}(\boldsymbol{\mathcal{S}}^{[d]}) \leq C(n+d-1, d)^{1/(2d)}\hat{\rho}(\boldsymbol{\mathcal{S}}).$$

Proof. Since $\|x^{[d]}\| = \|x\|^d$ holds for the euclidean norm, we have $\hat{\rho}(\boldsymbol{\mathcal{S}}^{[d]}) = \hat{\rho}(\boldsymbol{\mathcal{S}})^d$ from (2.3). Given the dimension of the set $\mathbf{A}^{[d]}$, Theorem 2.21 produces an estimate such that $\hat{\rho}(\boldsymbol{\mathcal{S}}^{[d]}) \leq \gamma_Q(\boldsymbol{\mathcal{S}}^{[d]}) \leq C(n+d-1, d)^{1/2}\hat{\rho}(\boldsymbol{\mathcal{S}}^{[d]})$. ∎

In constrast with the method of Subsection 2.4.1, we now conserve the graph of the constrained system, but approximate its CJSR by using potentially non-convex approximations of multinorms that are obtained from homogeneous sum-of-squares polynomials with degree $2d$, $d \geq 1$ being an integer to be chosen. These polynomials have the form $x \mapsto (x^{[d]})^\top Q x^{[d]}$, where $Q \succ 0$ is a quadratic form in the lifted space.

Remark 2.49. *There again, to our knowledge, the approximation guarantees are only conservative. Apart from the trivial case where $d = 1$, we do not know of a way to construct systems for which the CJSR approximation reaches the upper bound $C(n+d-1, d)^{1/(2d)}$.*

2.5 Numerical Examples

Example 2.50. *We put ourselves on a scenario alike that of Example 2.1. We consider the dynamics of a plant that may experience controller failures, and take the switching system*

$$x(t+1) = \big(A + BK_{\sigma(t)}\big) = A_{\sigma(t)}x(t),$$

with

$$A = \begin{pmatrix} 0.94 & 0.56 \\ 0.14 & 0.46 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.20}$$

and $K_{\sigma(t)} = (k_{1,\sigma(t)}, k_{2,\sigma(t)})$. *The control gains switch to represent 4 failures modes. When everything works as expected,* $\sigma(t) = 1$, *and*

$$K_1 = \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} -0.49 & 0.27 \end{pmatrix}. \tag{2.21}$$

The second and third modes correspond respectively to a failure of the first and second part of the controller, with $K_2 = (0, k_2)$ *and* $K_3 = (k_1, 0)$. *The last mode is the total failure case with* $K_4 = (0, 0)$. *As a constraint, we consider that a same part of the controller never fails twice in a row. We obtain a constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, *on the set of* $M = 4$ *matrices* $\mathbf{A} = \{A_\sigma, \sigma \in \langle M \rangle\}$, *with the graph* $\mathbf{G} = (S, E)$ *depicted in Figure 2.8. We wish to provide*

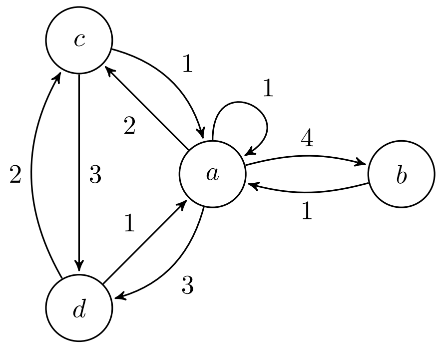


Figure 2.8: Graph for Example 2.50.

accurate estimations of the CJSR of the system using the results of Section 2.4. We begin with the T-product lifts and the (T-1)-path-dependent lifts for $T = 1, \ldots, 7$. *Both methods share the same accuracy guarantees for each values of* T, *landing estimates withing* $[\hat{\rho}(\boldsymbol{\mathcal{S}}), (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}})]$, *with* $1 + r = \sqrt{2}$ *for* $T = 1$ *to* $1 + r \simeq 1.051$ *for* $T = 7$. *Table 2.1 shows the sizes of the lifted systems used for producing the CJSR estimates. From there, we can see that* T*-product lift is faster for computing its estimates. However, the estimates of the (T-1)-path-dependent lift are of better quality (from Theorem 2.44).*

For the T*-product and* $T - 1$ *path-dependent lifts, the sizes of the graphs are*

	Nodes	Edges	Dimension	Bound
$\boldsymbol{\mathcal{S}}$	$\lvert S \rvert$	$\lvert E \rvert$	n	$n^{1/2}$
$\boldsymbol{\mathcal{S}}^T$	$\lvert S \rvert$	$\lvert \mathscr{P}^T(\mathbf{G}) \rvert$	n	$n^{1/(2T)}$
$\boldsymbol{\mathcal{S}}_T$	$\lvert \mathscr{P}^T(\mathbf{G}) \rvert$	$\lvert \mathscr{P}^{T+1}(\mathbf{G}) \rvert$	n	$n^{1/(2(T+1))}$
$[d]$-lift	$\lvert S \rvert$	$\lvert E \rvert$	$C(n+d-1, d)$	$C(n+d-1, d)^{1/(2d)}$

Table 2.1: Example 2.50. Size of the (lifted) systems and the corresponding bound on the CJSR estimation.

further represented in Figure 2.9.

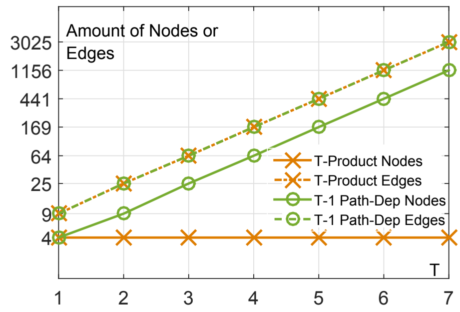


Figure 2.9: Example 2.50. Size of the graphs $\mathbf{G}^T$ and $\mathbf{G}_{T-1}$, respectively the T-product and $(T-1)$-path-dependent lifts of the graph $\mathbf{G}$ at Figure 2.8.

The tools are implemented in Matlab (see Annex 2.B). The execution times to obtain the estimates for these two techniques are represented in Figure 2.10.

Finally, Figure 2.11 presents the estimates obtained using both the T-product

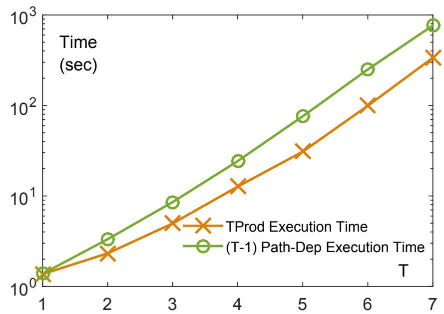


Figure 2.10: Example 2.50. Execution times for producing the estimates.

and $T-1$-path-dependent lifts, for $1 \leq T \leq 7$. Using these estimates, note that we can also compute lower-bounds *on the CJSR. Indeed, if $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq \gamma \leq (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}})$, it is true that $\gamma/(1+r) \leq \hat{\rho}(\boldsymbol{\mathcal{S}})$. The lower bounds corresponding to our techniques are also represented in Figure 2.11. The best CJSR estimate so far is obtained using the 6-Path-Dependent lift, and proves that $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq 0.9748\ldots$. This proves the stability of the system.*

We can go one step further and compute the exact value of the CJSR *using Theorem 2.26. The results of Figure 2.11 indicate that the sufficient condition could only be met by the $T-1$-path-dependent lift, for $T = 5, 6$ or 7. In this case, for $T = 7$, the conditions of Theorem 2.26 are met. The obtained simple cycle corresponds to the switching sequence repeating the labels* $\{2, 3, 1, 1, 1, 1, 2, 1\}$

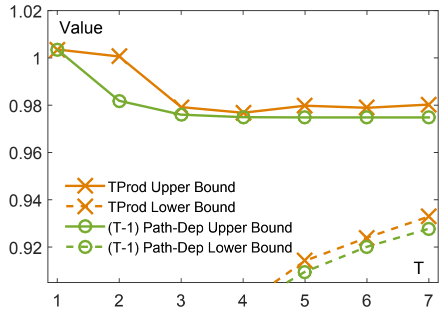


Figure 2.11: Example 2.50. Using the $T-1$-path-dependent lifts, we produce better estimates. The estimates produced by the T-product lifts need not decrease monotonically as T increase. Lower bounds computed from the relative accuracy of the CJSR estimates.

and reaches the CJSR. Indeed, we have (Lemma 2.24)

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \geq \rho(A_1 A_2 A_1^4 A_3 A_2)^{1/8} \simeq 0.9478\ldots,$$

which matches our best CJSR estimate.

Let us now turn ourselves to SOS techniques (Subsection 2.4.3). There, we estimate the CJSR using the $[d]$-lift, for $1 \leq d \leq 10$. For these values, we get estimates within the range $[\hat{\rho}(\boldsymbol{\mathcal{S}}), (1+r)\hat{\rho}(\boldsymbol{\mathcal{S}})]$, with $1+r$ ranging from $\sqrt{2}$ for $d=1$ to approximately $1+r = 1.1275$ for $d = 10$. The time taken to produce estimates is represented at Figure 2.12. The values of the estimates themselves are represented at Figure 2.13, with lower bounds on Figure 2.13a. The lower bounds are computed using the accuracy guarantees of Theorem 2.48.
On Figure 2.13b, we represent the error made by the technique, namely the quantity

$$\frac{\rho' - \hat{\rho}(\boldsymbol{\mathcal{S}})}{\hat{\rho}(\boldsymbol{\mathcal{S}})},$$

where ρ' is the value of the estimate.

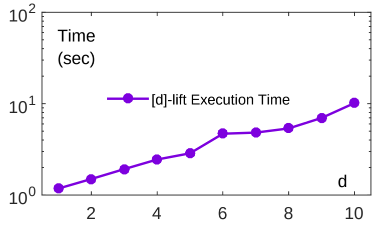
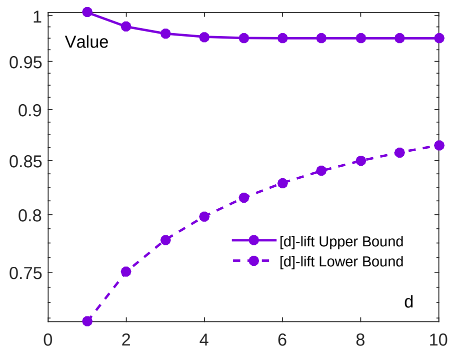
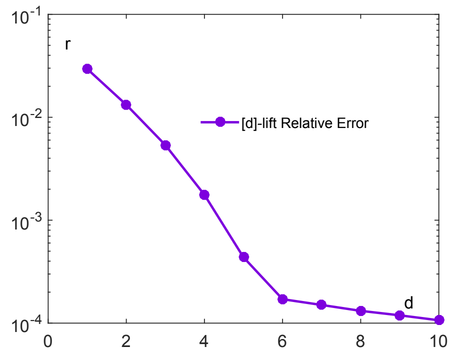


Figure 2.12: Example 2.50. $[d]$-lift execution time for producing the estimate.



(a) $[d]$-lift upper and lower bounds.



(b) Relative error of the estimates obtained using the $[d]$-lift.

Figure 2.13: Example 2.50. Producing CJSR estimates using the $[d]$-lift.

It seems that the $[d]$-lift is more efficient *than the other techniques, as it produce very accurate estimates* a-posteriori *in a relatively short time. While this is true, notice that this is not necessarily true* a-priori*: the accuracy guarantee $1 + r = 1.1275$ for $d = 10$ is achieved using the T-product lift for $T = 3$. Moreover, Figure 2.13b really shows that, while the estimates gets really close to the true CJSR estimates, an error is still present, which is not the case for the 6-Path-Dependent lift.*

Example 2.51. *We consider the matrices A and B as in* (2.20)*, and K as in* (2.21)*. Following the setup of Example 2.1, we construct the set of matrices $\mathbf{A} = \{A_1 = A + BK,_2 = A\}$ and take the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ with the graph $\mathbf{G}$ at Figure 2.1b.*

We can show numerically that this system does not have a quadratic multinorm (see Appendix 2.B for an exposition of a Matlab toolbox we developed in that endevaour). However, we can show that it is stable using Sum-of-Square

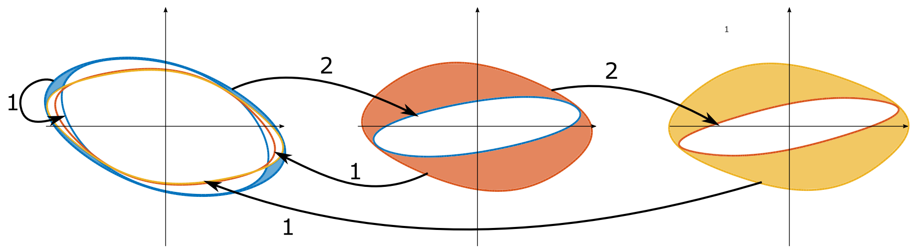


Figure 2.14: Example 2.51. The level sets of the SOS functions for the three nodes of the graph at Figure 2.1b. An arrow shows how the level set of its source is mapped by the mode of the label into the level set of its destination.

functions of degree $2d$ *with* $d = 2$.

Figure 2.14 illustrates the level set of these sum-of-square functions[5]. *It represents a set representation of the certificate in a manner similar to what is done in Example 2.16 using Remark 2.16.*

Example 2.52. *We now consider a consensus system where we seek to compute the rate of convergence of agents to consensus. The study of consensus systems is a typical scenario where switching systems arise [OSM04]. To put it simply, we consider* n *of agents in a network, each agent initially holding a value* $(x(0))_i$. *The agents then adapt their own values through interactions with the others, and we aim at understanding under which conditions they will reach* consensus*: a situation where the values of all agents are equal.*

A consensus system takes the form

$$x(t+1) = A(t)x(t),$$

where $A(t)$ *depends on interactions at time* t. *This matrix is nonnegative at each time and each one of its rows* $A(t)$ *sums to one (i.e. it is stochastic). Naturally, in the case where there are only a finite number of possible matrices* $A(t)$, *the dynamics considered form a switching system. The interactions between agents can be represented in a directed and weighted graph* $H = (\langle n \rangle, E)$ *as follows. If at time* t, *the graph* $H(t)$ *with edges* $E(t)$ *is chosen, then* $(A(t))_{i,j} = r > 0 \leftrightarrow (i,j,r) \in E$. *As an example, a graph* H_1 *is given at Figure 2.15, corresponding to the matrix*

$$A_1 = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \\ 0.2 & 0 & 0 & 0 & 0.8 \end{pmatrix} \tag{2.22}$$

[5] Or more precisely, we plot the level sets of the functions $x \mapsto \left((x^{\lceil 2 \rceil})^\top Q_s (x^{\lceil 2 \rceil})\right)^{1/4}$, where $x^{\lceil 2 \rceil}$ is the lifting of x defined in Subsection 2.4.3, and Q_s is the quadratic form in $\mathbb{R}^{3\times 3}$ parameterizing the SOS function at node s in the graph.

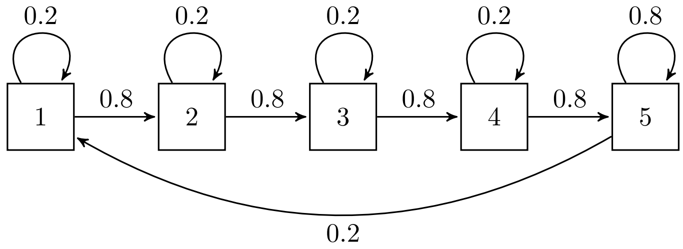


Figure 2.15: Example 2.52. Graph H_1 of interactions between 5 agents, corresponding to the matrix A_1 at (2.22).

We say that a system converges to consensus *if for any initial condition $x(0)$, the state $x(t)$ converges to a vector parallel to $\mathbf{1}$. Equivalently, if we let $P \in \mathbb{R}^{n \times n-1}$ be an orthogonal projection matrix on the subspace orthogonal to $\mathbf{1}$, convergence to consensus is the fact that $y(t) = Px(t) \to \mathbf{0}$ as time goes to infinity. This allows us to see consensus as the exponential stability of a switching system.*

There are classical sufficient conditions for consensus to arise [BHOT05, OT09,OT08]. In our current case, we are particularly interested in the bounded inter-connectivity times *assumption [OT09]*[6]*:* there is a bound T such that at any time t, *letting $E(t)$ is the set of edges of the graph $H(t)$, the graph*

$$\tilde{H}(t) = (\langle n \rangle, E(t) \cup E(t+1) \cup \cdots \cup E(t+T))$$

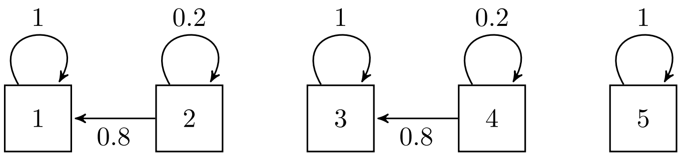
is strongly connected.

In the present example, we consider a consensus system with 3 interaction graphs: H_1 in Figure 2.15, and H_2 and H_3 at Figure 2.16a and 2.16b. In order to enforce *the* bounded interconnectivity times *assumption, we impose that for some integer $N \geq 1$, the interactions H_1 arise at least once every N time steps. This can be represented by a graph $\mathbf{G}_N = (S, E)$ as depicted in Figure 2.17. With this in mind, we are certain that consensus will arise, and wish to compute the* rate of consensus *for several choices of N. This is done by evaluating the rate of convergence to the origin of the part of the state x which is* orthogonal to $\mathbf{1}$. *In order to do so, we first construct a set of matrices $\mathbf{A} = \{A_1, A_2, A_3\}$, with A_i corresponding to the graph H_i above, and then construct a second set*

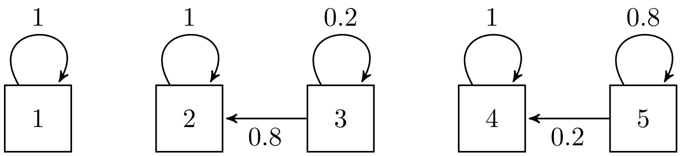
$$\mathbf{A}' = \{B_i = P^\top A_i P \mid A_i \in \mathbf{A}\},$$

where $P \in \mathbb{R}^{5 \times 4}$ is an orthogonal projection on the subspace span$(\mathbf{1})^\perp$, i.e. $P^\top \mathbf{1} = 0$ and $P^\top P = I \in \mathbb{R}^{4 \times 4}$.

[6] In general, the bounded inter-connectivity times assumption alone is not sufficient for consensus. Since we are dealing with a finite number of possible interaction graph and that for each one of these, every agent as a self loop with positive weight, we also satisfy Assumption 1 in [OT09]. In this case, consensus is guaranteed see [OT09, Theorem 2.4].



(a) Interaction graph H_2.



(b) Interaction graph H_3.

Figure 2.16: Example 2.52. Graphs of interaction.

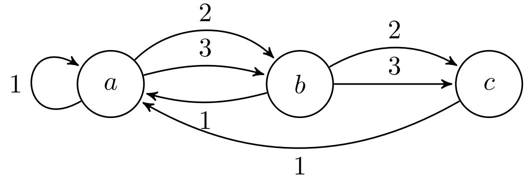


Figure 2.17: Example 2.52. Graph $\mathbf{G}_N$, for $N = 3$, that is used to enforce that the agents interact according to H_1 at least once every $N = 3$ steps.

Finally, given an integer N, we construct the constrained switching system $\boldsymbol{\mathcal{S}}_N = (\mathbf{G}_N, \mathbf{A}')$, and seek to approximate its constrained joint spectral radius. The results are presented in Table 2.18.

N	Quadratic	5-Path-Dependent Lift	6-Product Lift	6d-SOS
1	0.7273	0.7273	0.7273	0.7273
2	0.9872	0.9293	0.9487	0.9065
3	1.0738	0.9882	1.0088	0.9875
4	1.0969	0.9991	1.0138	1.0076

Figure 2.18: Example 2.52. Approximating the rate of convergence to consensus when the interactions of graph H_1 are guaranteed to trigger at least once every N steps. Results are rounded up at the 4th decimal.

2.6 Summary

We provide a characterization of the exponential stability of constrained switching systems (switching systems whose switching sequences are generated by a graph). Our main results are as follows.

Section 2.1: Asymptotic and Exponential Stability.

For constrained switching systems, asymptotic and exponential stability are characterized by the *constrained joint spectral radius* (CJSR) of the system (Theorem 2.8).

Section 2.2: Multinorms: CSS Require Multiple Lyapunov Functions.

We characterize the CJSR in terms of *multinorms* (Proposition 2.12). A multinorm is a set of one norm per node of the graph. Stability is equivalent to the existence of a multinorm satisfying decrease conditions along the edges of $\mathbf{G}$ (Theorem 2.13).

Section 2.3: Approximation using Quadratic Norms.

We approximate multinorms as a proxy to approximate the CJSR. For quadratic multinorms, this provides us a convex optimization program, and we establish approximation guarantees on the CJSR (Theorem 2.21).

Section 2.4: Lifting Procedures to Increase Accuracy.

We provide three lifting procedures: the T-product lift (Subsection 2.4.1), the T-path-dependent lift (Subsection 2.4.2) and the $[d]$-lift (Subsection 2.4.3), that transform a system into another one with equivalent stability properties. Applying quadratic multinorm approximations on the liftings, we obtain arbitrarily accurate estimations of the CJSR (Theorem 2.33 and Corollary 2.36; Corollary 2.45; Theorem 2.48).

We propose the following two questions as bases for future work.

Question 2.1. *What are the worst-case performances for the quadratic multinorm estimation method?*

The problem is as follows. We are given a graph $\mathbf{G} = (S, E)$ on M modes, an integer $n \geq 1$, and a number $r > 1$. Is there a set of matrices $\mathbf{A} = \{A_1, \ldots, A_M\}$, $A_i \in \mathbb{R}^{n \times n}$, such that for $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$,

$$\frac{\gamma_Q(\boldsymbol{\mathcal{S}})}{\hat{\rho}(\boldsymbol{\mathcal{S}})} > r, \tag{2.23}$$

with γ_Q as defined in Theorem 2.33? As variants, we may focus on graphs that are T-product lifts or T-path-dependent lifts of a given graph (in particular,

the graph $\mathbf{G}$ of Figure 2.1a, with $M = 2$), or on estimates obtained using the $[d]$-lift rather than quadratic multinorms.
Of course, we already know upper bounds on r such that no system can be built to satisfy (2.23), see Sections 2.3 and 2.4. Moreover, while these bounds relate only to the dimension n of the system, Legat et al. recently introduced bounds involving the graph itself rather than the matrices (see [LJP16]). Nevertheless, the general question remains open.

Question 2.2. *Are their more efficient lifting techniques?*

This question is related to the previous one. As pointed out in Table 2.1, the lifting schemes involved here tend to produce very large graphs for disminishing returns in terms of the accuracy they provide.
In practice, however, it is often the case that the edges that correspond to tight constraints for the estimation of γ_Q are located only in a small part of the graph. Can we construct greedy and efficient lifting techniques, that would transform only this part of the graph and preserve the accuracy guarantees of the T-product and/or T-path-dependent lift?

Appendix

2.A On the CJSR

In this appendix, we provide a short proof for Theorem 2.8 since it is central for our work.

From Definition 2 in [Dai12], we know that the limit (2.3) converges. Since the limit converges, $\forall \epsilon > 0$, $\exists T_\epsilon \geq 0$, $\forall t \geq T_\epsilon$,

$$\hat{\rho}(\boldsymbol{\mathcal{S}})^t \leq \max_{w \in \mathscr{L}(\mathbf{G}), |w|=t} \|A_w\| \leq (\hat{\rho}(\boldsymbol{\mathcal{S}}) + \epsilon)^t.$$

From there, it is easy to see that if $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 1$, there are trajectories that grow unbounded.

To show that $\hat{\rho}(\boldsymbol{\mathcal{S}}) < 1$ implies (exponential) stability, it suffice to take $\epsilon < 1 - \hat{\rho}(\boldsymbol{\mathcal{S}})$ in the above. Indeed, we then get that for all accepted words,

$$\lim_{t \to \infty} \max_{w \in \mathscr{L}(\mathbf{G}), |w|=t} \|A_w\| = 0.$$

Exponential stability is then acquired since there can only be a finite amount of products of length $t \leq T_\epsilon$ with $\|A_w\| > (\hat{\rho}(\boldsymbol{\mathcal{S}}) + \epsilon)^t$.

Consider now the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) \geq 1$. For all $t \geq 1$, we take x_*^t as

$$x_*^t \in \arg \max_{\|x\|=1} \max_{w \in \mathscr{L}(\mathbf{G}), |w|=t} \|A_w x\|.$$

We extract from the sequence $\{x_*^t\}_{t \geq 1}$ a subsequence converging to a point x_*, $\|x_*\| = 1$. From this point, there is a switching sequence satisfying

$$\lim_{t \to \infty} \max_{w \in \mathscr{L}(\mathbf{G}), |w|=t} \|A_w x_*\| \geq 1/2.$$

Thus, the system is not asymptotically stable when $\hat{\rho}(\boldsymbol{\mathcal{S}}) \geq 1$. This concludes the proof of Theorem 2.8.

2.B Software and The CSSystem Toolbox

We developed a Matlab toolbox for tackling CJSR approximation problems. The toolbox is available on Matlab central, and relies in part on the Yalmip [Lof05] tool for encoding the LMIs. Search for "*the CSSystem Toolbox*", or follow the link below:

`https://www.mathworks.com/matlabcentral/fileexchange/52723-the-cssystem-toolbox?s_tid=srchtitle`

It was developed[7] at the UCLouvain for that purpose, and provides all the tools needed for encoding CSS, using the tools of Theorem 2.21 and computing the liftings of Section 2.4.

The toolbox itself is very well documented and illustrated with several demo-introductions. In this appendix, we wish to show how to use it for solving the problem of Example 2.1. We take the numerical values (matrices) of Subsection 2.5.

First, we define the data. Here, we will study the system of Example 2.51. The graph of Figure 2.1b has 3 nodes and 6 edges, it is represented as a $|E| \times 3$ array, where each row correspond to an edge[8]. In the toolbox, nodes are denoted by integer. Here node a in the figure is node 1 in the code, b is 2, c is 3. The matrix set $\mathbf{A} = \{A_1, A_2\}$ is defined as $A_1 = A + BK$ and $A_2 = A$, with A, B and K_1 as in Example 2.50. The following code generates a CSSystem object for us to use, that corresponds to $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$.

```
% First , we build the graph on 6 edges.
graph = [1 2 1; 1 1 2; 2 3 1; 2 1 2; 3 1 2];
% Now we need our matrices , 2 modes.
A = [0.94 0.56; 0.14, 0.46];
B = [0 ; 1];
K = [-0.49, 0.27];
A1 = A+B*K;
A2 = A;
% We use a cell to encode the matrix set.
matrix_set = {A1,A2};
% For constructing the system, we simply write
S = CSSystem(graph, matrix_set);
```

For debugging purposes, we provide a tool to visualize the graph of the system. It interfaces with the *dot* program, available at `www.graphviz.org`. Once a CSSystem object is constructed, the following code outputs the drawing at Figure 2.19.

```
S.view()
```

We can now proceed to check wether or not the system is stable. To do so, the algorithm of Theorem 2.21 is implemented under the name “cjsrQuadMultinorm”.

[7]My own codes, prototypes and experience helped to bootstrap the project for the constrained switching systems part. For the optimization, improvements, and general development of the toolbox, credits are due to Damien Scieur, Léopold Cambier, and Benoit Legat, who I would like to thank again for providing us with such an elegant tool.

[8]The toolbox allows for a more general representation for the edges, taking into account labels of size more than 1. This is, however, out of our scope here.

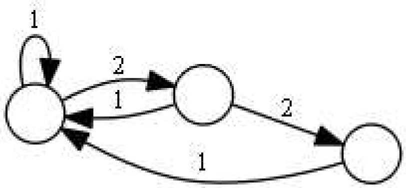


Figure 2.19: The output of “S.view()”.

```
[gammaQ, Quads] = S.cjsrQuadMultinorm()
```

Under the default options, the steps of the bisection procedure of Remark 2.22 appear in the console. The output “gammaQ” is the bound $\gamma_Q(\boldsymbol{\mathcal{S}})$, and here we have

$$\gamma_Q(\boldsymbol{\mathcal{S}}) = 1.0027.$$

The output “Quads” is a cell with one element per node of $\mathbf{G}$, and each element contains the positive definite matrix for the corresponding quadratic function.

Out of curiosity, we may as well visualize the level sets of these norms (Figure 2.20a).

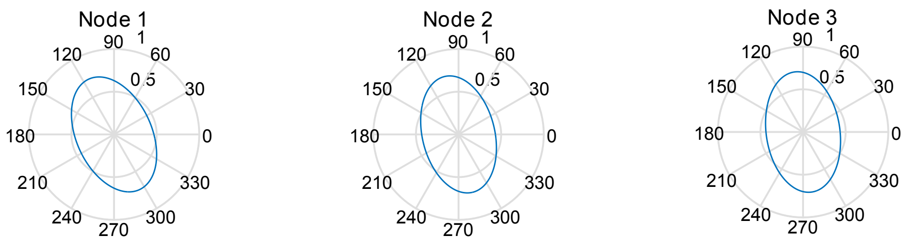
```
S.viewMultinorm('quad',Quads)
```

The upper bound on the CJSR is greater than one - so we can not deduce stability. It is also smaller than $\sqrt{2}$ - so we can not deduce instability either. We therefore need to refine our estimate, using a lifting procedure.

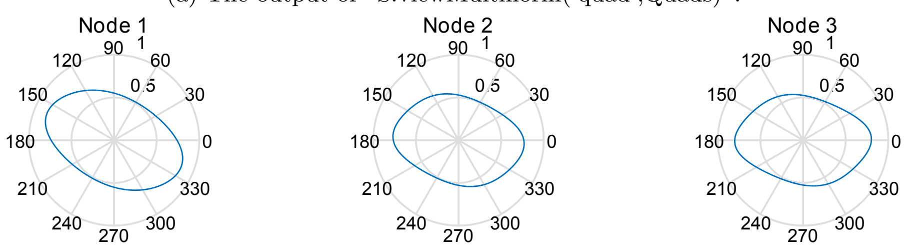
As an illustration, we compute and then estimate the CJSR using the 2-product lift, the 1-path-dependent lift, and the [2]-lift (SOS functions of degree 4). To compute the lifts we use the following.

```
S_2ProdLift = S.TProductLift(2);
S_1PathDep = S.MDependantLift(1);
S_2SosLift = S.sosLift(2);
```

These produce other CSSystem objects. The philosophy here is that the quadratic CJSR approximation tool is defined for any system. In order to refine the approximation of the CJSR of one system, the lifting procedures generates other systems, on which to run the approximation tools.



(a) The output of “S.viewMultinorm(’quad’,Quads)”.



(b) The output of “S_2SosLift.viewMultinorm(’sos’,Q2Sos,2)”.

Figure 2.20

```
% Computing the lifts
S_2ProdLift = S.TProductLift(2);
S_1PathDep = S.MDependantLift(1);
S_2SosLift = S.sosLift(2);
% Estimating the CJSR
[gamma2Prod,Q2prod] = S_2ProdLift.cjsrQuadMultinorm()
[gamma1PDep,Q1PDep] = S_1PathDep.cjsrQuadMultinorm()
[gamma2Sos,Q2Sos] = S_2SosLift.cjsrQuadMultinorm()
```

This time, the estimations show that the system is stable. The estimate for the 2-product lift is 0.9842, for the 1-path dependent it is 0.9839, and it is $\sqrt{0.9805} = 0.9902$ using sum-of-square functions. Note that, when using sum-of-squares for degree d, we need to take the dth root of the output estimate, since we don't actually keep track of whether or not the system was lifted when we compute the estimates.

We end this example by having a look at the level sets of the SOS functions corresponding to our third estimate in Figure 2.20b.

The above illustrates only a few of the features of the toolbox. We invite the curious reader to download it and run the demos for a full exposition.

3 Stability and Dead-Beat Stability

In Chapter 2, we provided tools to compute the exponential growth rate, a.k.a. the *constrained joint spectral radius* (CJSR), of constrained-switching systems. We now put ourselves at the frontiers of exponential stability: when the CJSR is 1 and when it is 0. In the first case, we devise sufficient conditions that guarantee the boundedness of all trajectories of a system, and provide tools to decide them. The second case, we show it corresponds to dead-beat stability, and provide a polynomial time algorithm to decide this.

The chapter is based on [PMJ17] and [PJ15b] and is organized as follows.

- Section 3.1 motivates our work with respect to the literature on the arbitrary switching case, and presents the properties we wish to capture.
- In Section 3.2, we set ourselves in the case where the exponential growth rate of a system equals 1, and generalize the concept of *irreducibility*. We provide two sufficient conditions for the boundeness of a system. The first, intuitive, is obtained from the lifting procedure of Section 2.3.2. The second has weaker requirements, and involves only a subset of the nodes of the graph in a constrained switching system. An algorithm to decide irreducibility is provided, taking its roots in the algorithmic framework of Chapter 2.
- In Section 3.3, we move to the other extreme, when the CJSR is 0. We provide a polynomial-time algorithm for deciding this case, which corresponds to the dead-beat stability of a constrained-switching system (all trajectories vanish to the origin in finite time).

3.1 Stability, Non-defectivity and Dead-Beat Stability

We assume the reader is familiar with both the concept of constrained-switching system, introduced in Definition 1.26, and the *constrained joint spectral radius* (CJSR) (Definition 2.5).

This second chapter deals with sufficient conditions for the stability of constrained switching systems. However, here, our focus is no longer on *exponential* and *asymptotic* stability (cfr. Chapter 2), but on both the weaker notion of *stability* (boundedness of trajectories), and the stronger notion of *dead-beat stability.*

Definition 3.1. *Consider the linear constrained-switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, with the set of M matrices $\mathbf{A}$ and the graph $\mathbf{G}$ on the alphabet $\langle M \rangle$.*

- *The system is* stable *if*

$$\exists K \geq 1, \forall x_0 \in \mathbb{R}^n, \forall w \in \mathscr{L}(\mathbf{G}) : \|x(x_0, w)\| \leq K\|x_0\|. \tag{3.1}$$

- *The system is* dead-beat stable *if there is a time $T \geq 1$ such that*

$$\forall x_0 \in \mathbb{R}^n, \forall w \in \mathscr{L}(\mathbf{G}), |w| \geq T : \|x(x_0, w)\| = 0. \tag{3.2}$$

The dead-beat stability of a system implies its exponential stability (Definition 2.2), which implies its stability. These implications can be related as well to the constrained joint spectral (Definition 2.5) radius of a system.

The constrained joint spectral radius of a system is its maximal exponential growth rate (Theorem 2.8). In the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$, we show later that the system is dead-beat stable. If $\hat{\rho}(\boldsymbol{\mathcal{S}}) < 1$, the system is both asymptotically and exponentially stable (see Theorem 2.8). If $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$ is more complicated. This is a necessary condition for stability, but not a sufficient one (see [PJ15] and references therein for more about unstable arbitrary switching systems with JSR = 1). Finally, if $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 1$, then we can show that the system possesses an unbounded trajectory whose growth rate is exponential. This is illustrated at Figure 3.1.

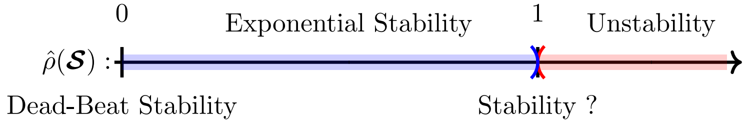


Figure 3.1: Stability and the CJSR of a constrained switching system.

In this chapter, we study the cases $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$ and $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$. For the first case, we highlight sufficient conditions for stability. For the second, we show it corresponds to dead-beat stability and provide a polynomial time algorithm for deciding it. As it was the case in Chapter 2, most of the theoretical works has been developed in the case of arbitrary switching systems and allow for a good understanding there. For constrained switching systems, these matters have been left unexplored so far.

Remark 3.2. *Stability and* non-defectivity *(see Subsection 2.2.1) are linked. Indeed, a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ *with* $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$ *is* non-defective *if and only if (Definition 2.18) the system* $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \mathbf{A}')$ *with* $\mathbf{A}' = \{A/\hat{\rho}(\boldsymbol{\mathcal{S}}), A \in \mathbf{A}\}$ *is stable.*

In Section 3.2, we put ourselves in the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$. If $\boldsymbol{\mathcal{S}}$ is an arbitrary switching system with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, then there is a well-known condition for its stability: the *irreducibility* of $\mathbf{A}$.

Definition 3.3. *A set* $\mathbf{A} \subset \mathbb{R}^{n\times n}$ *of matrices is* irreducible *if for any non-trivial linear subspace* $\mathcal{X} \subset \mathbb{R}^n$*, i.e. with* $0 < \dim(\mathcal{X}) < n$*, there is a matrix* $A \in \mathbf{A}$ *such that* $A\mathcal{X} \not\subset \mathcal{X}$*. That is, the matrices in* $\mathbf{A}$ *do not share a non-trivial common invariant subspace of* $\mathbb{R}^n$*.*

Remark 3.4. *For the case* $n = 2$*, irreducibility is equivalent to the fact that the matrices in* $\mathbf{A}$ *do not share a common eigenvector.*

Proposition 3.5 (e.g., [Jun09, Theorem 2.1])**.** *If a set of matrices* $\mathbf{A}$ *with* $\hat{\rho}(\mathbf{A}) = 1$ *is irreducible, then there is a uniform bound on the norm of all products of matrices taken from* $\mathbf{A}$*.*

Irreducibility is known to be decidable (see [AP04] and references therein). Moreover, irreducibility implies the existence of *extremal* and *Barabanov* norms for arbitrary switching systems. The existence of such norms is central for the stability analysis of arbitrary switching systems [Wir05, Jun09, GP13].

Section 3.2 is focused around providing a proper generalization of both irreducibility and Proposition 3.5 for constrained switching systems. As shown in Example 3.6, Proposition 3.5 does not generalize directly to constrained switching systems.

Example 3.6. *We construct an* unstable *constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$*, with* $M = 2$ *modes, which has both* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$ *and an irreducible set of matrices. The existence of such systems proves that irreducibility is not sufficient for the stability of constrained switching systems with* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$*. The set* $\mathbf{A} = \{A_1, A_2\}$ *is defined as*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.3}$$

The graph $\mathbf{G}$ *is given in Figure 3.2. The language of this graph* $\mathscr{L}(\mathbf{G})$ *is the set of all words that do not have* 121 *has a factor. Since* A_1 *and* A_2 *do not share common eigenvectors, the set* $\mathbf{A}$ *is irreducible.*

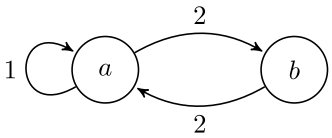


Figure 3.2: Graph for Example 3.6.

Moreover, we can verify that $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$*. Indeed, for all* $k \geq 1$*, the sequence* $\sigma(1), \ldots, \sigma(k)$ *maximizing the norm* $\|A_{\sigma(k)} \cdots A_{\sigma(1)}\|$ *in (2.3) is simply* $\sigma(t) = 1$ *for all* $t = 1, \ldots, k$*. Equation (2.3) thus boils down to the classical Gel'fand formula for computing the spectral radius of* A_1*, which is equal to* 1*.*
However, for the same switching sequence, we have that $\|A_1^t\| \to \infty$ *as* $t \to \infty$*, and so* $\boldsymbol{\mathcal{S}}$ *is unstable.*
We conclude that Proposition 3.5 does not generalize as it is to constrained switching systems.

In Section 3.3, we put ourselves in the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$. From the results of Dai, we known that this implies the asymptotic stability of any constrained switching system. However, in the arbitrary switching case, this is also equivalent to *dead-beat* stability (Theorem 3.7) and we can decide $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ in polynomial time.

Theorem 3.7 (e.g. [Jun09], Section 2.3.1)**.** *The joint spectral radius of a set of* N *matrices* $\mathbf{A}$ *in dimension* n *is zero if and only if all products of size* n *of matrices in* $\mathbf{A}$ *are equal to zero.*

To the best of our knowledge, no extensions of these results to constrained switching systems have been proposed yet. And again, it appears that Theorem 3.7 does not hold for constrained switching systems in general.

Example 3.8. *The* scalar *(*$n = 1$*) system on the two matrices* $A_1 = 1$ *and* $A_2 = 0$ *with the cyclic graph of Figure 3.3 sees all of its trajectories vanish to the origin after 2 steps, and so its CJSR is* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ *from Definition 2.5. We conclude that Theorem 3.7 does not hold when constraints are added on*

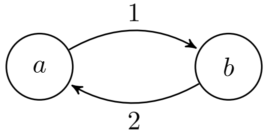


Figure 3.3: A simple cyclic system on two modes.

switching sequences.

3.2 Generalizing Irreducibility

In this section, we provide sufficient conditions for the stability of systems with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, generalizing the concept of irreducibility and Proposition 3.5 to constrained switching systems.

We begin, in Subsection 3.2.1, by an intuitive condition that takes its roots in the lifting procedure of Subsection 2.3.2. We transform the CSS into an arbitrary switching system with the *same* stability properties, and ask ourselves how the irreducibility of the corresponding set of matrices can be translated in terms of properties of the original CSS. From there, we provide an algorithm to decide this property (and irreducibility as well), with a technique reminiscent of the search for quadratic multinorms presented in Section 2.3.

Then, in Subsection 3.2.3, we relax our first condition by introducing the concept of *nodal irreducibility.* We show that the relaxed condition is decidable as well.

3.2.1 Graph Irreducibility

In Subsection 2.3.2, Definition 2.27, we present a lifting procedure, introduced in [Koz14, WRDV14] that takes a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ and produces a set of matrices $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ such that the stability properties of the system $\boldsymbol{\mathcal{S}}$ are equivalent to those of an *arbitrary switching system* on the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$. For completeness, the definition of $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is reminded here:

By a slight abuse of notation, we give a number to each node of S: $S = \{s_i, 1 \leq i \leq |S|\}$.

Then, the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}} \subset \mathbb{R}^{(|S|n)\times(|S|n)}$ is defined as the set

$$\mathbf{A}_{\boldsymbol{\mathcal{S}}} = \{A_{(s_i,s_j,w)},\, (s_i, s_j, w) \in E\},$$

with

$$A_{(s_i,s_j,w)} = \left(\mathbf{e}_j \mathbf{e}_i^\top\right) \otimes A_w,$$

where $\mathbf{e}_i$ is the ith element of the canonical basis of $\mathbb{R}^{|S|}$

The stability of the arbitrary switching system on the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is equivalent to that of the system $\boldsymbol{\mathcal{S}}$ itself, and this leads us to a very natural idea. Instead of devising complex stability conditions for constrained switching systems $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, we may just consider the irreducibility of the matrix set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, which would directly imply the stability of $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$.

Definition 3.9. *A system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ *is* graph-reducible *if there are* $|S|$ *subspaces* $\{\mathcal{X}_s,\, s \in S\}$, *such that*

$$\forall (s, d, w) \in E :\, A_w \mathcal{X}_s \subseteq \mathcal{X}_d,$$

where $0 < \sum_{s\in S} \dim(\mathcal{X}_s) < |S|n$. *If it is not graph-reducible, then it is graph-irreducible.*

Proposition 3.10. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, the following are equivalent.*

1. *The set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is reducible.*
2. *The system $\boldsymbol{\mathcal{S}}$ is graph-reducible.*

Proof. (**2.** $\Rightarrow$ **1.**): An invariant subspace under the matrices in $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is obtained as the cartesian product of the subspaces $\mathcal{X}_s$, $s \in S$. Let this cartesian product be $\mathcal{X}_{\boldsymbol{\mathcal{S}}}$. It is by construction non-trivial in $\mathbb{R}^{|S|n}$, and for any $A_{(s_i,s_j,w)} \in \mathbf{A}_{\boldsymbol{\mathcal{S}}}$, we have

$$A_{(s_i,s_j,w)}\mathcal{X}_{\boldsymbol{\mathcal{S}}} = (\mathbf{e}_j\mathbf{e}_i^\top) \otimes A_w \begin{pmatrix} \mathcal{X}_{s_1} \\ \vdots \\ \mathcal{X}_{s_i} \\ \vdots \\ \mathcal{X}_{s_j} \\ \vdots \\ \mathcal{X}_{s_{|S|}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \vdots \\ A_w\mathcal{X}_{s_i} \\ \vdots \\ \mathbf{0} \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{X}_{s_1} \\ \vdots \\ \mathcal{X}_{s_i} \\ \vdots \\ \mathcal{X}_{s_j} \\ \vdots \\ \mathcal{X}_{s_{|S|}} \end{pmatrix}, \tag{3.4}$$

which concludes this part of the proof.

(**1.** $\Rightarrow$ **2.**): Assume by contradition that $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is reducible, but that that (2.) does not hold. That means that there are non-trivial invariant subspaces $\mathcal{Y}$ for the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, but they can not be written as a cartesian product of one subspace of $\mathbb{R}^n$ per node of the graph as in (3.4).

This is absurd for the following reason: given an invariant subspace $\mathcal{Y}$ of a set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ in dimension n, for all edge $e \in E$, $A_e\mathcal{Y} \subseteq \mathcal{Y}$. Therefore, we have

$$\forall e_1, e_2 \in E,\ A_{e_2}(A_{e_1}\mathcal{Y}) \subseteq A_{e_2}\mathcal{Y} \subseteq \sum_{e \in E} A_e\mathcal{Y},$$

which is the Minkowski sum of the subspaces $A_e\mathcal{Y}$, $e \in E$. Since the above holds for all e_1 we have

$$\forall e_2 \in E,\ A_{e_2}\left(\sum_{e \in E} A_e\mathcal{Y}\right) \subseteq \sum_{e \in E} A_e\mathcal{Y},$$

so the set $\sum_{e \in E} A_e\mathcal{Y}$ is invariant. Because of the structure of the matrices in $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, this set can be expressed as a cartesian product of one subspace per node of the graph. This contradiction concludes the proof. ■

Paired with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, the irreducibility of $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is sufficient for the stability of $\boldsymbol{\mathcal{S}}$. However, it is not difficult to provide systems $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ where

1. the set $\mathbf{A}$ is *irreducible*,

2. the system itself is *stable* with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$ and

3. the system is not graph-irreducible.

This is shown in the next example.

Example 3.11. *Consider a system on two matrices* $\mathbf{A} = \{A_1, A_2\}$ *with*

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and constrained by the graph of Figure 3.4. It is easy to check that the norm

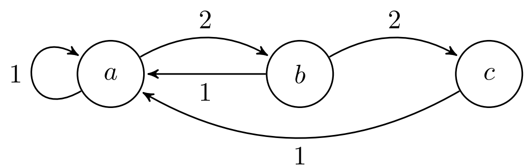


Figure 3.4: We cannot activate mode 2 more than twice in a row.

of any product of these two matrices is 1, hence we both have stability, *and* $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$. *Moreover, observe that the set* $\mathbf{A}$ *is irreducible, due to the presence of the matrix* A_1.

For this system, there are 6 matrices in the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, *one per edge of the graph* $\mathbf{G}$ *of Figure 3.4. For constructing the set* $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, *we let* s_1 *be the node* a, s_2 *be the node* b *and* s_3 *be the node* c *in Figure 3.4.*

We have for example

$$A_{(s_1,s_2,2)} = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\top} \right) \otimes A_2 = \begin{pmatrix} 0 & 0 & 0 \\ A_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After constructing the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, *it is easy to verify that the subspace of* $\mathbb{R}^6$ *defined as*

$$\mathcal{X} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^6,\ x \in \mathbb{R}^2,\ y \in \operatorname{im}(A_2),\ z \in \operatorname{im}(A_2) \right\},$$

is invariant under all matrices in $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$. *Is is non-trivial with dimension 4 (recall that* A_2 *has rank 1 here).*

We conclude that for the system considered here, which has a irreducible set of matrices and is stable, the lifted set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ *is reducible.*

3.2.2 Low-Rank Optimization for Deciding Irreducibility

We make a quick detour to discuss further the question of how to decide irreducibility.

Irreducibility plays an important role in this chapter, as we seek to extend the notion to constrained switching systems. This subsection provides an algorithm for deciding irreducibility, which is tightly related to the tools used in Section 2.3, for the problem of approximating the CJSR.

Knowing whether a finite set of matrices $\mathbf{A}$ is irreducible is decidable (see e.g. [AP04] and references therein). That is, given a *finite* set of matrices $\mathbf{A}$, there exists an algorithm, running in finite time, returning whether the set is irreducible or not. However, to our knowledge it is yet unknown whether it can be decided efficiently. The proposed methods often rely on analyzing the structure of the Lie-Algebras generated by product of matrices computed the set. For example, in [GI99, AGI00, JP15] the authors focus on sets of two matrices, and based their results on the application of the so-called *Shemesh Criterion* [She84], a tool that allows to decide whether or not two matrices share common *eigenvectors*, on a set of matrices whose size is exponential on the size of the original matrices.

The results we present below, while providing us with a promising lead towards efficient decision of irreducibility, came first as an answer to the following question: "*What happens if, in Chapter 2, instead of using* norms *as our basic tools for characterizing stability, we allowed ourselves to use non-trivial* semi-norms *(by non-trivial, we mean that the semi-norm is non-zero for at least one $x \in \mathbb{R}^n$)?*". Similar questions have been studied in [Dai11], where a link between Lyapunov inequalities involving semi-norms and invariant subspaces has been discussed (see [Dai11, Section 3.1]).

As it turns out, the framework of Chapter 2 can be put to good use to provide necessary and sufficient irreducibility conditions for sets of matrices. This leads to an efficient heuristic, which deserves further study.
In the below, we focus on deciding the irreducibility of a set $\mathbf{A}$ of matrices, and the conditions developed herein naturally extend to constrained-switching systems $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ through the lift $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$.

Theorem 3.12. *Consider a set of matrices $\mathbf{A}$ in $\mathbb{R}^{n\times n}$, and a subspace $\mathcal{X}$ of dimension $0 < d < n$. The following are equivalent:*

1. *The subspace is invariant under all matrices in $\mathbf{A}$:*
$$\forall A \in \mathbf{A} : A\mathcal{X} \subseteq \mathcal{X}.$$

2. *Let $\gamma = \max\{\|A\| : A \in \mathbf{A}\}$, where $\|A\|$ is the induced euclidean norm of A. There exists a positive* semi-definite *matrix $Q \succeq 0$ of rank $n - d$ such that*
$$\forall A \in \mathbf{A} : A^\top Q A \preceq \gamma^2 Q, \tag{3.5}$$
where $x^\top Q x = 0 \Leftrightarrow x \in \mathcal{X}$.

Proof. (**1.**) $\Rightarrow$ (**2.**): Consider the subspace $\mathcal{X}^{\perp}$ orthogonal to $\mathcal{X}$, of dimension $0 < n - d < n$. Let $K \in \mathbb{R}^{n \times n-d}$ be an orthogonal basis of $\mathcal{X}^{\perp}$, i.e., $\mathrm{im}(K) = \mathcal{X}^{\perp}$ and $K^{\top}K = I \in \mathbb{R}^{n-d \times n-d}$.

We claim that we satisfy (2.) with

$$Q = KK^{\top},$$

which is a projection operator on the subspace $\mathcal{X}^{\perp}$.
First, observe that $x^{\top}Qx = 0 \Leftrightarrow x \in \mathcal{X}$ by construction. Second, we need to show that

$$\forall x \in \mathbb{R}^n, \forall A \in \mathbf{A} : x^{\top}A^{\top}(KK^{\top})Ax \leq \gamma^2 x^{\top}(KK^{\top})x, \tag{3.6}$$

where $\gamma = \max\{\|A\| : A \in \mathbf{A}\}$ for the euclidean norm.

Decompose any point $x \in \mathbb{R}^n$ as the sum

$$x = (I - KK^{\top})x + KK^{\top}x = x_{\mathcal{X}} + x_{\mathcal{X}^{\perp}},$$

where $x_{\mathcal{X}} \in \mathcal{X}$ is the projection of x on $\mathcal{X}$, and $x_{\mathcal{X}^{\perp}} \in \mathcal{X}^{\perp}$ that of x on $\mathcal{X}^{\perp}$. By definition, for any $x \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^{n-d}$ such that $x_{\mathcal{X}^{\perp}} = Ky$. By construction, we have

$$\forall x \in \mathbb{R}^n : K^{\top}x = K^{\top}x_{\mathcal{X}^{\perp}}, \, K^{\top}Ax = K^{\top}(Ax)_{\mathcal{X}^{\perp}}.$$

Since A is invariant for the subspace $\mathcal{X}$, we also have

$$(Ax)_{\mathcal{X}^{\perp}} = Ax_{\mathcal{X}^{\perp}} = AKy$$

for some $y \in \mathbb{R}^{n-d}$.
Therefore (3.6) is equivalent to the following:

$$\forall y \in \mathbb{R}^{n-d}, \forall A \in \mathbf{A} : y^{\top}(K^{\top}A^{\top}K)^{\top}(K^{\top}AK)y \leq \gamma^2 y^{\top}(K^{\top}K)^{\top}(K^{\top}K)y.$$

We are now able to conclude. The matrix $K^{\top}K$ is the identity in $\mathbb{R}^{n-d}$. Hence, the inequality is equivalent to

$$\|K^{\top}AK\|^2 \leq \gamma^2,$$

which is true by construction since K is orthogonal and satisfies $\|K\| \leq 1$.

(**2.**) $\Rightarrow$ (**1.**): We wish to show that if (3.5) holds when $x^{\top}Qx = 0 \Leftrightarrow x \in \mathcal{X}$, then $\mathcal{X}$ is an invariant subspace for the set $\mathbf{A}$. Assume by contradiction that there is a matrix $A \in \mathbf{A}$ and a point $x \in \mathcal{X}$ with $Ax \notin \mathcal{X}$. Then we have

$$x^{\top}A^{\top}QAx > \gamma^2 x^{\top}Qx = 0.$$

This contradicts (3.5), and thus $\mathcal{X}$ needs to be invariant.

■

The following corollary is the extension of Theorem 3.12 to constrained switching systems.

Corollary 3.13. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, with $\mathbf{A}$ in $\mathbb{R}^{n\times n}$, and subspaces $\{\mathcal{X}_s, s \in S\}$, each one of dimension $0 \leq d_s \leq n$, with $0 < \sum_{s\in S} d_s < n$. The following are equivalent:*

1. *The following invariance properties hold:*
$$\forall (s, d, w) \in E : A_w \mathcal{X}_s \subseteq \mathcal{X}_d.$$

2. *Let $\gamma = \max\{\|A_w\| : (s, d, w) \in E\}$. There exists positive* semi-definite *matrices $\{Q_s \succeq 0, s \in S\}$, with Q_s of rank $n - d_s$ such that*
$$\forall (s, d, w) \in E : A_w^\top Q_d A_w \preceq \gamma^2 Q_s \tag{3.7}$$
where
$$x^\top Q_s x = 0 \Leftrightarrow x \in \mathcal{X}_s.$$

Proof. Obtained by applying Theorem 3.12 and Proposition 3.10 to the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$. ∎

Theorem 3.12 provides us with an algorithm for deciding the irreducibility of a matrix set.

Definition 3.14. *Consider a set of M matrices $\mathbf{A}$ in $\mathbb{R}^{n\times n}$. We say that a subspace $\mathcal{X}$ is a* maximal non-trivial invariant *subspace of $\mathbf{A}$ if it is invariant over the matrices in $\mathbf{A}$, of dimension $0 < d < n$, and for all non-trivial invariant subspace $\mathcal{Y}$ of $\mathbf{A}$,*
$$\dim(\mathcal{Y}) \leq \dim(\mathcal{X}).$$

Corollary 3.15. *Consider a set of matrices $\mathbf{A}$ in $\mathbb{R}^{n\times n}$. The set is reducible if and only if the following low-rank optimization program has a solution $n > d^* > 0$:*
$$\begin{aligned} d^* = \min_Q\ & \mathit{rank}(Q) \\ & \forall A \in \mathbf{A} : A^\top Q A \preceq \gamma^2 Q, \\ & Q \succeq 0, \\ & \mathit{trace}(Q) \geq 1, \end{aligned} \tag{3.8}$$
where $\gamma = \max\{\|A\| : A \in \mathbf{A}\}$, and $\|A\|$ is the induced euclidean norm of A. When $n > d^* > 0$, $n - d^*$ is the dimension of the maximal non-trivial invariant subspace of $\mathbf{A}$.

Proof. By Theorem 3.12, there is a matrix Q of rank d satisfying (3.8) if and only if $\mathbf{A}$ has an invariant subspace of dimension $n - d$. Also, the set $\mathbf{A}$ is reducible if and only if it has a maximal non-trivial invariant subspace of dimension $0 < \tilde{d} < n$. In the program (3.8), the rank of Q is therefore bounded below by $n - \tilde{d}$, since the trace constraint prevents the solution $Q = \mathbf{0}$. From the proof of Theorem 3.12, we know we can construct Q achieving this rank. ∎

In general, solving such low-rank programs is NP-hard [VB96]. The usual approximation technique for these problems, which is minimizing the *nuclear norm* of the matrix instead of its rank, will not work here because of the trace constraint. Different approaches can be taken to solve this issue. Hence, further work is needed to establish whether or not efficient formulations exists. Nevertheless, independent of the algorithm itself, it remains interesting to see that with a slight twist of a tool dedicated for the estimation of the exponential growth rate of a system, we obtain an algorithm for deciding structural properties relative to its trajectories.

3.2.3 Linear Connectivity and Nodal Reducibility

Let us move back to the task at hand: finding sufficient conditions for the stability of a constrained switching system when its CJSR is equal to one. In Definition 3.9, we provide a first generalization of irreducibility for constrained switching systems. However, we observe in Example 3.11, that the generalization is sensitive to the graph used. We now present a novel set of conditions that highlight the fact that, in a graph, not all nodes are *as important* as the others. The sufficient conditions we present now must hold true only for a subset of nodes.

The first part of this Subsection is focused on the presentation of Theorem 3.22, providing us with novel sufficient conditions for stability, under the form of another generalization of irreducibility. In connection to the criterion of Subsection 3.2.1, we show that our new conditions are a relaxation of the previous ones. The proof of Theorem 3.22 is detailed afterwards. Then, in Subsection 3.2.4 we prove the decidability of the condition.

Our second generalization of irreducibility relies on two concepts, involving an interconnection between the graph $\mathbf{G}$ and the set $\mathbf{A}$ of a system.

Definition 3.16 (Nodal irreducibility)**.** *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, a* node $s \in S$ *is* irreducible *if for any* non-trivial *linear subspace $\mathcal{X} \subset \mathbb{R}^n$ there is a cycle $c \in \mathscr{P}_{s,s}(\mathbf{G})$ such that $A_{w(c)}\mathcal{X} \not\subset \mathcal{X}$.*

For a system with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, it must be the case that there exists a sequence of paths in $\mathbf{G}$, $\{p_k \in \mathscr{P}^k_s(\mathbf{G})\}_{k=1,2,\ldots}$, starting at some node $s \in S$ such that, for all $k = 1, 2, \ldots$, $A_{w(p_k)} \neq 0$. For these products of matrices to exist, the graph needs arbitrarily long paths, which is guaranteed by its strong connectivity. On top of this, the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ must possess a connectivity beyond the one brought in a graph theoretical sense by $\mathbf{G}$.

Definition 3.17 (Linear Connectivity)**.** *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ and two nodes $s, d \in S$, s is* linearly connected *to d if there exists $p \in \mathscr{P}_{s,d}(\mathbf{G})$ such that $A_{w(p)} \neq \mathbf{0}$.*
The system $\boldsymbol{\mathcal{S}}$ *is* linearly connected *if all pairs of nodes are linearly connected.*

Proposition 3.18. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$. If the set $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is irreducible, then $\boldsymbol{\mathcal{S}}$ is linearly connected, and all nodes in S are irreducible.*

Proof. Assume that $\boldsymbol{\mathcal{S}}$ is not linearly connected. There exists two different nodes, s and d in S, such that

$$\forall p \in \mathscr{P}_{s,d}, \forall x \in \mathbb{R}^n : A_{w(p)}x = \mathbf{0}.$$

In this case, we can build subspaces satisfying to Proposition 3.10 - (2.) as follows. We let $\mathcal{X}_s = \mathbb{R}^n$, ans $\mathcal{X}_d = \{\mathbf{0}\}$. Then, for all the others nodes $r \in S$,

$$\mathcal{X}_r = \{x \in \mathbb{R}^n \mid \forall p \in \mathscr{P}_{r,d} : A_{w(p)}x = \mathbf{0}\}.$$

Hence, the irreducibility of $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ implies linear connectivity.

Assume now that there is some node $s \in S$ which is not irreducible. Then, there is a non-trivial subspace $\mathcal{X}_s \subset \mathbb{R}^n$, such that

$$\forall c \in \mathscr{P}_{s,s}, \forall x \in \mathbb{R}^n : A_{w(c)}\mathcal{X}_s \subseteq \mathcal{X}_s.$$

Here again, we can build subspaces satisfying to Proposition 3.10 - (2.). The subspace at node s is $\mathcal{X}_s$ as before and for the other nodes $r \in S$

$$\mathcal{X}_r = \{x \in \mathbb{R}^n \mid \forall p \in \mathscr{P}_{r,s} : A_{w(p)}x \in \mathcal{X}_s\}.$$

Hence, the irreducibility of $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ also implies the irreducibility of all nodes. ■

Our main result in this section can be roughly stated as follows. If we can guarantee that all the trajectories of a system with CJSR = 1 encounter irreducible nodes regularly, then the system is stable. In particular, this implies that not all nodes need to be irreducible. To formalize the result, we borrow the concept of unavoidability from automata theory (see e.g. [Lot02, Proposition 1.6.7]).

Definition 3.19. *Given a graph $\mathbf{G} = (S, E)$, a subset S^* of S is said to be unavoidable if any cycle in $\mathbf{G}$ encounters at least one node in S^*.*

Lemma 3.20. *Given a graph $\mathbf{G} = (S, E)$, let S^* be an unavoidable set of nodes. Any path p of length $|S|$ encounters at least 1 node in S^*.*

Proof. Such a path p must visit $|S| + 1$ nodes. At least one node in S appears twice in p, implying that p contains a cycle. Therefore, p visits an unavoidable node in S^*. ■

Remark 3.21. *Unavoidable sets of nodes are classically known in graph-theory as* feedback vertex sets*: sets of nodes whose removal leave a graph acyclic. As it turns out, the problem of finding a feedback vertex set of cardinality smaller than a constant k is, in general, NP-complete. It is the 7th among the famous Karp's 21 NP-complete problems (see [Kar72]).*

Theorem 3.22. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ that is linearly connected and has an unavoidable set of irreducible nodes. If $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, then the system is stable.*

The conditions guaranteeing stability in Theorem 3.22 are split in two sets of conditions.

In Theorem 3.22 we are asking that a property be true not at all node, but only at those that are unavoidable. Intuitively (and literally), unavoidable nodes play a central role for the trajectories of a system, as they are going to be visited frequently. Observe that given a graph $\mathbf{G}$, there may be several sets of unavoidable nodes, and in particular the set S of all nodes is unavoidable.

Example 3.23. *Consider the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ of Example 3.11. Recall that the system is stable, with $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 1$, but $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ is* reducible*, and hence Proposition 3.10 does not prove stability. We show that the conditions of Theorem 3.22 holds for this system.*

The system is linearly connected*. Also, any set of nodes containing node "a" is unavoidable. Moreover, one can check that the node "a" is* nodal irreducible*, since the matrices A_1 and A_1A_2 do not share common eigenvectors. We conclude that Theorem 3.22 holds here for if we take the singleton $\{a\}$ as our set of nodes, and provides us with a sufficient stability condition.*

We now move to the proof of Theorem 3.22, where we are going to elaborate on properties of unavoidable and irreducible nodes to construct trajectories with high growth rate.

Proof of Theorem 3.22

We start with two lemmas. When clear from the context, we use the shorthand notation $\mathscr{P}$ instead of $\mathscr{P}(\mathbf{G})$ to refer to paths of a graph $\mathbf{G}$. The meanings of subscript and superscript remains the same (respectively, source/destinations pairs, and length).

Lemma 3.24. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, linearly connected and with an unavoidable set of irreducible nodes $S^* \subseteq S$. At any irreducible node $s \in S^*$, the subspace defined as*

$$\mathcal{K}_s = \{x \in \mathbb{R}^n \mid \exists K < \infty, \forall p \in \mathscr{P}_s(\mathbf{G}) : \|A_{w(p)}x\| < K\} \tag{3.9}$$

is either $\mathbb{R}^n$ or $\{\mathbf{0}\}$. Moreover, if $\boldsymbol{\mathcal{S}}$ is unstable, then for any irreducible node $s \in S^*$, $\mathcal{K}_s = \{\mathbf{0}\}$.

Proof. It is clear that $\mathcal{K}_s$ is a linear subspace. We prove the first part by contradiction. Take an irreducible node $s \in S^*$, and assume first that $\mathcal{K}_s \neq \mathbb{R}^n$, and second that there is a non-zero vector $x \in \mathcal{K}_s$. Since s is irreducible, there must be $c \in \mathscr{P}_{s,s}$ such that $y = A_{w(c)}x \notin \mathcal{K}_s$. Indeed, if there was no such cycle, the subspace defined as

$$\text{span}\{x, \cup_{c\in\mathscr{P}_{s,s}} A_{w(c)}x\} \subseteq \mathcal{K}_s$$

would be invariant at s, contradicting the irreducibility of the node. This implies the existence of unbounded trajectories from the point y, which in turn implies the contradiction $x \notin \mathcal{K}_s$.

We now move to the second part of the Lemma and assume that $\boldsymbol{\mathcal{S}}$ is unstable. In that case, there exists a node s (not necessarily irreducible) and a sequence of paths $\{p_k \in \mathscr{P}_s^k\}_{k=1,2,...}$ such that $\lim_{k\to\infty} \|A_{w(p_k)}\| = \infty$. For $k \geq |S|$, p_k must have visited an *irreducible* node $d' \in S^*$ (Lemma 3.20). Therefore, we have $\mathcal{K}_{d'} = \{\mathbf{0}\}$ from the first part of the Lemma.
If d' is the only irreducible node, then the proof is done. If not, take another node $s' \in S^*$. From the linear connectivity of $\boldsymbol{\mathcal{S}}$, there must be a path $p \in \mathscr{P}_{s',d'}$ and a vector $x \in \mathbb{R}^n$ such that $A_{w(p)}x \neq \mathbf{0}$, and since $\mathcal{K}_{d'} = \{\mathbf{0}\}$, $A_p x \notin \mathcal{K}_w$. Therefore it must be the case that $x \notin \mathcal{K}_{s'}$ and so $\mathcal{K}_{s'} \neq \mathbb{R}^n$. From the first part of the Lemma, we must conclude $\mathcal{K}_{s'} = \{\mathbf{0}\}$. Repeating the above for all irreducible nodes, we then conclude that $K_s = \{\mathbf{0}\}$ for all $s \in S^*$. ∎

Lemma 3.25. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, linearly connected and with an unavoidable set of irreducible nodes $S^* \subset S$. If $\boldsymbol{\mathcal{S}}$ is unstable then for any scalar $\ell > 1$, there must exist an integer $T_\ell \geq 1$ such that for any irreducible node $s \in S^*$ and any $x \in \mathbb{R}^n$,*

$$\exists p \in \mathscr{P}_s, |p| \leq T_\ell : \|A_{w(p)}x\| \geq \ell \|x\|. \tag{3.10}$$

Proof. Assume by contradiction that the opposite of the Lemma's claim is true. There exists a scalar $\ell > 1$ such that for all $T \geq 1$, there is an irreducible node $s_T \in S^*$ and a vector $x_T \neq \mathbf{0}$ satisfying

$$\forall p \in \mathscr{P}_{s_T}, |p| \leq T : \|A_{w(p)}x_T\| < \ell \|x_T\|.$$

We can further assume, without loss of generality, that for all T, $\|x_T\| = 1$. Since there are a finite amount of nodes in S^*, there must be a sequence $\{T_k\}_{k=1,2,...}$ and a node $s^* \in S^*$, such that $s_{T_k} = s^*$ for all $k = 1, 2, \ldots$. The corresponding vectors $\{x_{T_k}\}_{k=1,2,...}$ form a *sequence* of vectors on the unit ball $\{x \in \mathbb{R}^n : \|x\| = 1\}$. Since the unit ball is a *compact* set, there exists a subsequence of $\{x_{T_k}\}_{k=1,2,...}$ converging, as k goes to infinity, to a limit point x_* on the unit ball. This limit point is such that

$$\forall p \in \mathscr{P}_{s^*} : \|A_{w(p)}x_*\| \leq \ell \|x_*\|,$$

which implies by definition that $x_* \in \mathcal{K}_{s^*}$. By Lemma 3.24, this contradicts the fact that, since s^* is irreducible, it should have $\mathcal{K}_{s^*} = \{\mathbf{0}\}$.

∎

Proof of Theorem 3.22. We consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ which is linearly connected and with an unavoidable set of irreducible nodes $S^* \subset S$. We will show that if $\boldsymbol{\mathcal{S}}$ is *unstable*, then $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 1$. Theorem 3.22 will then be proven by contraposition.

To do so, assume that $\boldsymbol{\mathcal{S}}$ is *unstable*. We begin by proving a result similar to Lemma 3.25, but which only involves irreducible nodes as the source and destination nodes of the paths in (3.10).

Claim: There exists an integer $T^* \geq 1$ such that, for any irreducible node $s \in S^*$, and any non-zero vector $x \in \mathbb{R}^n$,

$$\exists d \in S^*, \exists p \in \mathscr{P}_{s,d}, |p| \leq T^* : \|A_{w(p)}x\| \geq 2\|x\|. \tag{3.11}$$

For proving the Claim, we first define $K = \max_{A \in \mathbf{A}} \|A\|$, and $K > 1$ since we assumed $\boldsymbol{\mathcal{S}}$ to be unstable. Since induced matrix norms are submultiplicative, we have that for all $p \in \mathscr{P}$, $\|A_{w(p)}\| \leq K^{|w(p)|}$.

Observe that Lemma 3.25 holds under our current set of assumptions. Hence, for any choice of the constant ℓ, there is some length T_ℓ and for each irreducible node, there are paths p of length at most T_ℓ satisfying

$$\ell \leq \|A_{w(p)}\| \leq K^{|w(p)|},$$

where the right side inequality comes from the definition of K.
If we define $W := \max_{e \in E} |w(e)| \geq 1$, we always satisfy the property that for any path $p \in \mathscr{P}(\mathbf{G})$, $|w(p)|/W \leq |p|$. Therefore, for the choice $\ell = 2K^{(|S|W)}$, we can guarantee that the length of the paths in (3.10) is at least $|S| + 1$.
We now invoke the concept of unavoidability. Lemma 3.20 guarantees that an irreducible node is encountered within the last $|S|$ edges of the paths considered above. Take any irreducible node s and any $x \in \mathbb{R}^n$. Let p be a path satisfying (3.10) for $\ell = 2K^{(|S|W)}$. Let $s' \in S^*$ be the last irreducible node encountered along the path, and let $d \in S$ be the destination of the path. Assume for the moment that $s' \neq d$. We can decompose our path in $p = p_1p_2$, with $p_1 \in \mathscr{P}_{s,s'}$ and $p_2 \in \mathscr{P}_{s',d}$, and with $|p_2| \leq |S|$. Note that since p_2 has less that $|S|$ edges, and each of these has a word of length less then W, then $|w(p_2)| \leq |S|W$.
By definition of p, from our choice of ℓ, and by submultiplicativity of the norm, we have

$$2K^{(|S|W)}\|x\| \leq \|A_{w(p_2)}A_{w(p_1)}x\| \leq K^{|w(p_2)|}\|A_{w(p_1)}x\| \leq K^{(|S|W)}\|A_{w(p_1)}x\|.$$

If $s' = d$, the discussion is even simpler and we let $p_1 = p$, $\|A_{w(p_1)}x\| \geq 2K^{(|S|W)}\|x\|$. We have shown the existence of a path p_1 whose origin is s and destination is an irreducible node s' such that $\|A_{w(p_1)}x\| \geq 2\|x\|$. This being done from an arbitrary irreducible node s and arbitrary $x \in \mathbb{R}^n$, we conclude that our Claim holds true. We now prove that $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 1$. Take any irreducible node $s_0 \in S^*$ and any non-zero $x \in \mathbb{R}^n$. From (3.11), there exists a sequence $\{s_k\}_{k=1,2,\ldots}$ of nodes in S^* and a corresponding sequence of paths $\{p_k\}_{k=1,2,\ldots}$, each with a length $T_k \leq T^*$ and with $p_k \in \mathscr{P}^{T_k}_{s_{k-1},s_k}$, that satisfy for all $k = 1, 2, \ldots,$

$$\|A_{w(p_k)} \cdots A_{w(p_1)}x\| \geq 2^k\|x\|.$$

We then obtain a lower bound on $\hat{\rho}(\boldsymbol{\mathcal{S}})$ from (2.3):

$$\begin{aligned}\hat{\rho}(\boldsymbol{\mathcal{S}}) &\geq \lim_{k\to\infty}\left(\|A_{w(p_k)}\cdots A_{w(p_1)}\|^{1/\sum T_k}\right)\\ &\geq \lim_{k\to\infty}\left(\|A_{w(p_k)}\cdots A_{w(p_1)}\|^{1/(kT^*)}\right) \geq 2^{1/T^*} > 1.\end{aligned}$$

We conclude that, if a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, linearly connected and with an unavoidable set of nodes is *unstable*, then $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 1$. If $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq 1$, the system must be stable. ■

3.2.4 Deciding Linear Connectivity and Nodal Irreducibility

In this subsection, we show that knowing whether a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is linearly connected and has an unavoidable set of irreducible nodes is decidable. To do so, we show that the irreducibility of a single node in the system is equivalent to the irreducibility of a finite set of matrices, and that a similar result holds for linear connectivity.

We begin by a technical lemma. We believe it to be folklore in linear algebra, and present it in a form insightful for the proof of Theorem 3.27.

Lemma 3.26. *Let $\mathbf{A} = \{A_1, A_2, \ldots\}$ be a (possibly infinite) set of matrices in $\mathbb{R}^{n\times n}$. Given a non-trivial subspace $\mathcal{X} \subset \mathbb{R}^n$ of dimension $1 \leq d < n$ and $x \in \mathcal{X}$, $x \neq \mathbf{0}$, if for all products of matrices of length $t \leq d$,*

$$A_{\sigma(t)}\cdots A_{\sigma(1)}x \in \mathcal{X},$$

with $A_{\sigma(i)} \in \mathbf{A}$, $1 \leq i \leq t$, then for any product A^* from matrices in $\mathbf{A}$,

$$A^*x \in \mathcal{X}.$$

Proof. We show by contradiction that the shortest product under which the image of x no longer lies in $\mathcal{X}$ must be of length at most d. Assume that $T > d$ is the length of a shortest product such that

$$A^*A_{\sigma(d)}\cdots A_{\sigma(1)}x \notin \mathcal{X}, \tag{3.12}$$

where A^* is itself a product of length $T - d$ of matrices in $\mathbf{A}$.
Since the vectors $x, A_{\sigma(1)}x, \ldots, A_{\sigma(d)}\cdots A_{\sigma(1)}x$ are $d+1$ vectors that belong to $\mathcal{X}$ which is of dimension d, we can state the following:

$$\exists\, 1 \leq k \leq d: \; A_{\sigma(k)}\cdots A_{\sigma(1)}x \in \operatorname{span}\left\{x, A_{\sigma(1)}x, \ldots, A_{\sigma(k-1)}\cdots A_{\sigma(1)}x\right\}. \tag{3.13}$$

Assume for the moment that $k = d$. From (3.12) and (3.13), we can write

$$A^*A_{\sigma(d)}\cdots A_{\sigma(1)}x \in \operatorname{span}\left\{A^*x, A^*A_{\sigma(1)}x, \ldots, A^*A_{\sigma(d-1)}\cdots A_{\sigma(1)}x\right\} \nsubseteq \mathcal{X}.$$

After observing that all products of matrices involved in the span on the right are of length at most $T-1$, and that at least one vector in this span does not lie in $\mathcal{X}$, we conclude that T is not the shortest length.
In the case $k < d$, we can repeat the above process by first multiplying (3.13) on the left by $\prod_{i=k+1}^{d} A_{\sigma(i)}$ and then by A^*. The conclusion remains unchanged, and we have proved the Lemma by contradiction. ■

Theorem 3.27. *Consider a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S,E), \mathbf{A})$, *with* $\mathbf{A} \subset \mathbb{R}^{n\times n}$. *Given an irreducible node* s *and a non-trivial linear subspace* $\mathcal{X} \subset \mathbb{R}^n$, *the* shortest cycle $c \in \mathscr{P}_{s,s}(\mathbf{G})$ *such that* $A_{w(c)}\mathcal{X} \nsubseteq \mathcal{X}$ *is of length*

$$|c| \leq 1 + n(|S|-1).$$

If a pair of nodes s, d *is linearly connected, then the shortest path* $p \in \mathscr{P}_{s,d}(\mathbf{G})$ *with* $A_{w(p)} \neq 0$ *is of length*

$$|p| \leq 1 + n(|S|-2).$$

Proof. We start with the first part of the theorem. Consider an irreducible node s, a non-trivial subspace $\mathcal{X}$ and a cycle $c_s \in \mathscr{P}_{s,s}(\mathbf{G})$ *with the smallest length* satisfying

$$A_{w(c_s)}\mathcal{X} \nsubseteq \mathcal{X}. \tag{3.14}$$

We are now going to establish two properties of c_s, allowing us to show that $|c_s| \leq 1 + n(|S|-1)$.

Claim 1 : The cycle c_s *is* simple *on the node* s.

The simplicity of c_s means that there are no partitions $c_s = c_s^1 c_s^2$ where c_s^1, $c_s^2 \in \mathscr{P}_{s,s}$. If there was such a partition, then either $A_{w(c_s^1)}\mathcal{X} \not\subset \mathcal{X}$, or $A_{w(c_s^2)}\mathcal{X} \not\subset \mathcal{X}$. Thus, c_s would not be a shortest cycle satisfying (3.14).

Claim 2 : The cycle c_s *never visits a node* $r \neq s$ *more than* n *times.*

This part of the proof relies on Lemma 3.26 and its proof.
Assume by contradiction that c_s visits a node $r \neq s$ for $T > n$ times. We can decompose the cycle as

$$c_s = p_{s,r} c_r p_{r,s},$$

where $p_{s,r} \in \mathscr{P}_{s,r}$ and $p_{r,s} \in \mathscr{P}_{r,s}$ are paths that visit s and r at most once as source or destination; and c_r is a cycle on the node r. In turn, we can write

$$c_r = c_r^1 \dots c_r^{T-1},$$

where the c_r^k, $k = 1, \dots, T-1$ are *simple cycles* on r[1]. Note that the above does imply that r is visited T times overall in c_s. This is illustrated in Figure 3.5. We now define the subspaces $\mathcal{X}_d = A_{w(p_{s,d})}\mathcal{X}$ and $\mathcal{Y} = \{x \in \mathbb{R}^n : A_{w(p_{d,s})}x \in \mathcal{X}\}$.

[1]They cannot be decomposed in smaller cycles on the node r in particular.

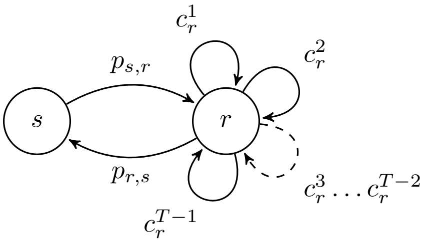


Figure 3.5: Decomposing c_s in a path to r, simple cycles on r, and then a path back to s.

Since c_s satisfies (3.14) and is of shortest length, these subspaces must satisfy to the following two relations:

$$\mathcal{X}_d \subseteq \mathcal{Y},$$
$$A_{w(c_s)}\mathcal{X}_d \not\subseteq \mathcal{Y},$$

and consequently $1 \leq \dim(\mathcal{Y}) \leq n-1$. The second relation above is equivalent to

$$\left(A_{w(c_d^{T-1})} \cdots A_{w(c_d^{n})}\right) A_{w(c_d^{n-1})} \cdots A_{w(c_d^{1})}\mathcal{X}_d \not\subseteq \mathcal{Y},$$

and from the proof of Lemma 3.26, there must exist $1 \leq k \leq n-2$ such that

$$\left(A_{w(c_d^{T-1})} \cdots A_{w(c_d^{n})}\right) A_{w(c_d^{k})} \cdots A_{w(c_d^{1})}\mathcal{X}_d \not\subseteq \mathcal{Y}.$$

This contradicts the fact that c_s is of shortest length. Indeed, defining $c_r' = c_r^1 \dots c_r^k c_r^n \dots c_r^{T-1}$, we have that $c_s' = p_{s,r}c_r'p_{r,s}$ satisfies (3.14) and is such that $|c_s'| < |c_s|$. This concludes the proof of the second claim.

Having established these properties, we can easily compute an upper bound on the length of c_s: there are at most $n(|S|-1)$ visits to nodes other than s, requiring a path of length at most $n(|S|-1)-1$. To form a simple cycle on s from this path, we add two edges from and to s, respectively at the beginning and the end of the path, giving us a cycle of length $1+n(|S|-1)$ at most.

The second part of the theorem relates to linear connectivity. The proof is similar in its ideas, so we will only highlight the main differences. Consider two linearly connected nodes s and d and let the path $p_{s,d} \in \mathscr{P}_{s,d}$ be a shortest path satisfying $A_{w(p_{s,d})} \neq 0$. Similarly to Claim 1, we can show that $p_{s,d}$ cannot visit neither s or d more than once each (as origin and destination, resp.). Claim 2 still holds, any node r visited by $p_{s,d}$ cannot be visited more than n times. The proof proceeds by writing $p_{s,d} = p_{s,r}c_rp_{r,s}$ as before, defining $\mathcal{Y} = \ker(A_{w(p_{r,d})})$, and applying Lemma 3.26 to escape $\mathcal{Y}$ from $\mathrm{im}(A_{w(p_{s,r})})$. Finally, $p_{s,d}$ visits $|S|-2$ nodes n times at maximum, to which we need to add two edges for connecting s and d, giving us a bound of $|p_{s,d}| \leq 1+n(|S|-2)$. ■

Based on the above result and on the fact that irreducibility (Definition 3.3) is decidable, we are able to formulate a decision algorithm for verifying that a system is linearly connected and has an unavoidable set of irreducible nodes. We present this algorithm in three distinct parts. First (Algorithm 1) checks for the linear connectivity of the system, second (Algorithm 2) marks all the irreducible nodes, and third (Algorithm 3) checks the unavoidability of the set of irreducible nodes.

Algorithm 1 Checking linear connectivity of $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$

```
for all pairs of nodes s, d ∈ S do
    if ∀p ∈ 𝒫_{s,d}, |p| ≤ 1 + n(|S| − 2) : A_{w(p)} = 0 then
        Return False                          ▷ the pair s,d is disconnected.
    end if
end for
Return True                                   ▷ all pairs are Linearly Connected.
```

Algorithm 2 Marking the irreducible nodes of $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$

```
Initialize set of irreducible nodes: S* = {}.
for all nodes s ∈ S do
    Initialize set of matrices C_s = {}.
    for all cycles c ∈ 𝒫_{s,s}, with |c| ≤ 1 + n(|S| − 1) do
        Update the matrix set: C_s ← C_s ∪ A_{w(c)}.
    end for
    if the set C_s is irreducible then
        Mark the node s as irreducible: S* ← S* ∪ s.
    end if
end for
Return S*
```

Algorithm 3 Checking unavoidability of a subset $S^* \subset S$ in the graph $\mathbf{G} = (S, E)$

```
Construct the graph G′(S\S*, E′) by removing all nodes in S* from G.
if G′ has no cycle then
    Return True                               ▷ the set S* is unavoidable
end if
Return False
```

Before ending this subsection, we show that the bounds of Theorem 3.27 are *tight*.

Proposition 3.28. *For all integers $n, m \geq 1$, one can build a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$, $\mathbf{A} \subset \mathbb{R}^{n\times n}$, $|S| = m$, for which a cycle of length $1 + n(m-1)$*

is needed to exit a particular non-trivial subspace $\mathcal{X} \subset \mathbb{R}^n$ from a particular node $s \in S$. This holds even when **A** *contains two matrices.*

Proof. We take any graph **G** on m nodes from the *Černỳ automata* family (see Figure 3.6). These automata (directed labeled graph) are related to the well-known Černỳ's conjecture, which is still the subject of active research (a state-of-the-art survey is presented in [Vol08]). Černỳ automata possess $M = 2$

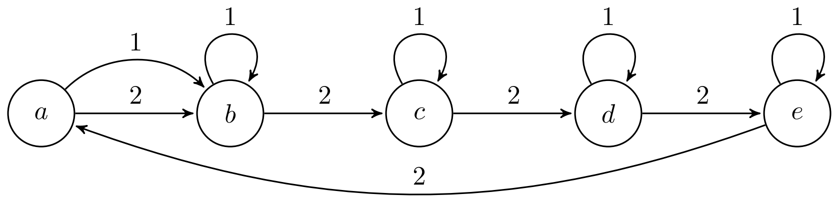


Figure 3.6: A Černỳ automaton on 5 nodes. In this example, the particular node v is taken as the node on the left without self-loop.

labels, and we define following the set of matrices $\mathbf{A} = \{A_1, A_2\} \in \{0,1\}^{n \times n}$:

$$(A_1)_{i,i+1} = 1,\ i = 1 \cdots n-1,$$
$$(A_2)_{n,1} = 1,$$

with the rest of the entries equal 0 for both matrices.

The particular node $s \in S$ is taken to be the only node without self loop (in Figure 3.6, it is node a). The non-trivial subspace $\mathcal{X}$ is defined as the span of the basis vector $e_1 = \begin{pmatrix}1 & 0 & 0 & \ldots\end{pmatrix}^\top \in \mathbb{R}^n$. Notice that $e_1 \in \ker(A_1)$, $A_2 e_1 \in \ker(A_2)$, $A_2 e_1 \not\subset \mathcal{X}$, and finally $A_1^{n-1} A_2 e_1 = e_1$.
For this system, the shortest cycle $c \subset \mathscr{P}_{s,s}$ on s such that $y = A_{w(c)} x \notin \mathcal{X}$ has the associated product

$$A_{w(c)} = A_2\,(A_1^{n-1} A_2)^{m-1},$$

which is indeed a product of length $1 + n(m-1)$.

■

Remark 3.29. *Regarding linear connectivity property, a result similar to Proposition 3.2.4 is obtained following the same ideas. The bound of Theorem 3.27, which is $1 + n(m-2)$ if we follow the notations of the Proposition, is attained through the same family of graphs, with the same matrices, by taking the source node as the node s without self loop, and the destination node as the one pointing at it with the edge labeled 2 ($d \in S$ such that $(d, s, 2) \in E$. This is node e in Figure 3.6).*

3.3 Deciding Dead-Beat Stability in Polynomial Time

We now turn our focus towards our second goal, that is to provide a polynomial time algorithm for deciding, given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, whether $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ or $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$. We prove that, as it is the case for arbitrary switching systems, $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ is equivalent to dead-beat stability, and provide a polynomial algorithm for deciding $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$.

Our main theoretical result in this section is the generalization of Theorem 3.7 from arbitrary to constrained switching systems.

Theorem 3.30. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ if and only if for all paths p of length $n|S|$, $A_{w(p)} = \mathbf{0}$.*

Proof. The proof is direct from the relation between the CJSR of a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and the JSR of the lift $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$ (Definition 2.27). From Theorem 3.7, we know that the JSR of the lifted system equals zero if and only if all products whose length equals the dimension of the matrices $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, which equals $n|S|$, are all equal to zero. This is equivalent to asking that for all paths p of length $n|S|$, $A_{w(p)} = \mathbf{0}$. ■

The result indicates that $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ is decidable. This is known for the case of arbitrary switching systems, and L. Gurvits [Gur96] proposed (without proof) a polynomial time algorithm for checking whether the joint spectral radius of a set of matrices is zero or not. A proof is presented in [Jun09, Section 2.3.1].

Proposition 3.31 (*Gurvits' iteration [Gur96]*)**.** *Let $\mathbf{A} \subset \mathbb{R}^{n\times n}$ be a finite set of matrices, and let $U_0 = I$ be the identity matrix in $\mathbb{R}^{n\times n}$. For $k = 1, 2, \ldots$, let*

$$U_k = \sum_{A\in\mathbf{A}} A^\top U_{k-1} A.$$

The joint spectral radius of $\mathbf{A}$ is zero if and only if $U_n = \mathbf{0}$.

The algorithm below is obtained through an application of Proposition 3.31 to $\mathbf{A}_{\boldsymbol{\mathcal{S}}}$, and allows to bypass the construction of the lifted system itself.

Proposition 3.32 (*Generalized Gurvits' iteration*)**.**
Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$. Define, for all nodes $s \in S$, $U_0^s = I$, where I is the identity matrix in $\mathbb{R}^{n\times n}$. For $k = 1, 2, \ldots$, for all nodes s, let

$$U_k^s = \sum_{(s,d,w)\in E} A_w^\top U_{k-1}^d A_w.$$

We have $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ if and only if $U^s_{n|S|} = \mathbf{0}$ for all $s \in S$.

Proof. By construction, we have

$$U_k^s = \sum_{p \in \mathscr{P}_s^k(\mathbf{G})} A_{w(p)}^\top A_{w(p)}.$$

These matrices are sums of positive semi-definite matrices. They equal zero if and only if all the matrices in the sum are zero. Therefore, we have $U_{n|S|}^s = 0$ if an only if for all paths $p \in \mathscr{P}_s^{n|S|}$, $A_{w(p)} = \mathbf{0}$. The conclusion follows immediately. ■

We now relate the case $\hat{\rho}(\boldsymbol{\mathcal{S}}) = 0$ with our previous results, in particular Theorem 3.22, and through it, to the non-defectivity property defined in Chapter 2, Section 2.2.1. More precisely, the next proposition provides a sufficient condition for non-defectivity. Remark that extremal norms (Theorem 2.19) may exist in the case CJSR = 0 at the condition that all matrices in the matrix set defining the system are $\mathbf{0}$.

Proposition 3.33. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{A})$ linearly connected and with an unavoidable set of irreducible nodes. Then, $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$ and there exists a constant $K > 0$ such that*

$$\forall x \in \mathbb{R}^n, \forall p \in \mathscr{P}(\mathbf{G}) : \|A_{w(p)}x\| \leq K\hat{\rho}(\boldsymbol{\mathcal{S}})^{|w(p)|}\|x\|. \tag{3.15}$$

Proof. The proof that $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$ is extracted from Theorem 3.27 and from [Jun09, Lemma 2.2]. This second Lemma states that *the joint spectral radius* of an *irreducible set of matrices* is greater than 0.
Now, take any irreducible node s of the system. From Theorem 3.27, the set of matrices

$$\mathbf{A}_s = \{A_{w(c)} \mid c \in \mathscr{P}_{s,s}(\mathbf{G}),\, |c| \leq 1 + n(|S| - 1)\},$$

is irreducible. From Lemma 2.2 in [Jun09] the joint spectral radius $\hat{\rho}(\mathbf{A}_s)$ of this set is greater than zero. Therefore, it must be the case that $\hat{\rho}(\boldsymbol{\mathcal{S}}) > 0$ as well.
The second part is due to the homogeneity of the CJSR (2.3). By scaling all matrices of the set $\mathbf{A}$ of the system $\boldsymbol{\mathcal{S}}$ by the positive constant $1/\hat{\rho}(\boldsymbol{\mathcal{S}})$, the scaled system has a CJSR of 1. This scaled system then satisfies to the hypothesis of Theorem 3.22, and by the definition of stability (3.1) and homogeneity of the norm, we obtain

$$\forall x \in \mathbb{R}^n, \forall p \in \mathscr{P}(\mathbf{G}) : \hat{\rho}(\boldsymbol{\mathcal{S}})^{-|w(p)|}\|A_{w(p)}x\| \leq K\|x\|,$$

for some constant K. ■

We now move onto an example to show the applicability and efficiency of the framework. The example proposed below corresponds to a real-world case study.

3.3.1 A Left Inverter For An Electrical Vehicle

Deciding dead-beat stability comes in handy for designing flatness-based controllers for switching systems. The vehicule under consideration here has been deeply studied in [Esp15]. Actually, it is the prototype developed by the Research Center for Automatic Control of Nancy which is annually involved in the European Shell Eco-Marathon race in the Plug-in (battery) category. It is modeled here as a two dimensional non-linear dynamical system, with state variables $(x(t))_1$ and $(x(t))_2$ denoting respectively the position and the velocity at time t. The state space model is

$$x(t+1) \quad = \quad \begin{pmatrix} 1 & a(T_t) \\ 0 & b(T_t, (x(t))_2) \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ c(T_t) \end{pmatrix} u(t) \tag{3.16}$$

with time-varying entries $a(T_t)$, $b(T_t, (x(t))_2)$ and $c(T_t)$ as a function of T_t and/or $(x(t))_2$ and of physical parameters (mass, dimensions, aerodynamics, ...) of the vehicle. The quantity T_t is the *sampling period* that may be either constant or time-varying. We denote with $\ell(t)$ the vector

$$\ell(t) = \big(a(T_t), b(T_t, (x(t))_2), c(T_t)\big).$$

Let us consider the position $(x(t))_1$ as the measured output $y(t)$, that is $y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$. Hence, the system (3.16) can be expressed as

$$\begin{aligned} x(t+1) &= A(\ell(t))x(t) + B(\ell(t))u(t), \\ y(t) &= Cx(t). \end{aligned} \tag{3.17}$$

We aim at designing a *left inverter* for (3.16). A left inverter is a dynamical system able to give, for every time t, the input $u(t)$ from a sequence of measures $y(t), y(t-1), \ldots$ This left inverter allows us to estimate the control $u(t)$ from the measures $y(t), y(t-1), \ldots$ delivered by the sensors. When placed on-board, it is involved in an actuator default detection module that plays a central role for the sake of safety.

Left inversion for discrete-time linear switched (and by extension Parameter Varying) systems has been addressed in several papers for different purposes [SH06, MD07, MD09, PM12]. There, a criterion for the convergence in finite time of the true input $\tilde{u}(t)$, delivered by the left inverter to the system, to the ideal input $u(t)$ is guaranteed by the dead-beat stability of an auxiliary system

$$\epsilon(t+1) = Q(\ell(t), \ldots, \ell(t+r))\epsilon(t), \tag{3.18}$$

where $\epsilon = u - \tilde{u}$ is the reconstruction error, and where the integer r corresponds to the *relative degree* of the system. The relative degree is the minimum number of iterations r such that the output $y(t+r)$ depends explicitely on the input $u(t)$ [PM12]. For the system (3.17), the relative degree is $r = 2$ since we have product $CA(\ell(t+1))B(\ell(t)) \neq 0$.

The matrix $Q(\ell(t),\dots,\ell(t+r))$ describes the dynamics of the reconstruction error is given by

$$Q(\ell(t),\dots,\ell(t+r)) = \\ A(\ell(t)) - B(\ell(t))\big(A(\ell(t+1))B(\ell(t))\big)^{-1}A(\ell(t+1))A(\ell(t)).$$

It turns out that in our case, $Q(\ell(t),\ell(t+1),\ell(t+2))$ depends only on the sampling periods at times t and $t+1$, that is

$$Q(\ell(t),\ell(t+1),\ell(t+2)) = \begin{pmatrix} 1 & a(T_t) \\ -a(T_{t+1})^{-1} & -a(T_{t+1})^{-1}a(T_t) \end{pmatrix} = Q(T_t,T_{t+1}).$$

In this case study, the sampling period T_t takes 2 possible values T^1 or T^2 according to the range of variation of the velocity of the vehicle. The faster the vehicle, the lower the sampling period. Then, we can rewrite the auxiliary system (3.18) as

$$\epsilon(t+1) = Q_{\sigma(t)}\epsilon(t), \tag{3.19}$$

which takes the form of a switched linear system that assigns to each sequence (T_t,T_{t+1}) starting at time t a mode $\sigma(t)\in\{1,2,3,4\}$. This explicitly induces constraints on the switching sequences. Indeed, let us consider that at a given time t, the mode $\sigma(t)$ corresponds to $(T_t,T_{t+1}) = (T_t,T^2)$. At time $t+1$, we need a mode $\sigma(t+1)$ corresponding to a sequence of the form $(T_{t+1},T_{t+2}) = (T^2,T_{t+2})$. Therefore, the auxiliary system can be expressed as a system $\boldsymbol{\mathcal{S}} = (\mathbf{G},\mathbf{A})$ on $N=4$ modes. A valid graph for the system is given in Figure 3.7.

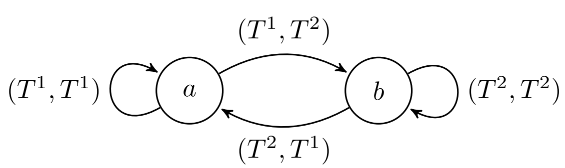


Figure 3.7: Every mode $\sigma(t)$ of the inverse system corresponds to a sequence $(T_t,T_{t+1})\in\{T^1,T^2\}^2$ of the original system. There are 4 of such pairs, and 4 possible transitions between these pairs.

For our system, Theorem 3.30 is verified and the algorithm of Proposition 3.32 confirms that $\hat\rho(\boldsymbol{\mathcal{S}}) = 0$. This can be verified by hand, noticing that for any valid switching sequence (where two following modes are of the form (T_t,T_{t+1}), (T_{t+1},T_{t+2})), we have $Q(T_{t+1},T_{t+2})Q(T_t,T_{t+1}) = 0$, for all $t\geq 0$ when following the switching rules. The auxiliary system is here *dead-beat* stable.

Notice that the *arbitrary switching system* on those modes is not dead-beat stable:

$$Q(T^2,T^2)Q(T^1,T^1) = \begin{pmatrix} 1-a(T^2)a(T^1)^{-1} & a(T^1)-a(T^2) \\ -a(T^2)^{-1}-a(T^1)^{-1} & -a(T^2)^{-1}a(T^1)-1 \end{pmatrix} \neq 0.$$

3.4 Summary

We put ourselves at the limits of the exponential stability of constrained switching systems, devising sufficient conditions for stability when the CJSR $= 1$, and providing a polynomial time algorithm for deciding CJSR $= 0$.

Section 3.1: Stability, Non-defectivity and Dead-Beat Stability.
In the case of arbitrary switching systems, sufficient conditions of stability when JSR $= 1$ (namely *irreducibility*, Proposition 3.5) and algorithms for deciding when JSR $= 0$ (Theorem 3.7) have been established in the literature. These do not extend directly to constrained switching systems (Examples 3.6 and 3.8).

Section 3.2: Generalizing Irreducibility.
The main contributions here are generalizations of irreducibility. We begin with a property called graph-irreducibility (Definition 3.9), which corresponds to the irreducibility of an arbitrary switching system, in high dimension, with the same stability properties as those of the system considered (Proposition 3.10). This leads to a simple and intuitive sufficient stability condition, but it does not fully exploit the structure of the systems (Example 3.11). We then introduce the concepts of *nodal irreducibility* and *linear connectivity*, allowing to obtain a more general sufficient stability condition (Theorem 3.22). The decidability of the condition is established (Theorem 3.27).
Additional to the above, we formulate a low-rank optimization program to decide irreducibility (Corollary 3.15), which seeks to compute *semi-norms* satisfying Lyapunov inequalities similar to that explored in Chapter 2, Proposition 2.20.

Section 3.3: Deciding Dead-Beat Stability in Polynomial Time.
We finally study the case CJSR $= 0$, show that it is equivalent to dead-beat stability (Theorem 3.7), and provide a polynomial time-algorithm for its decision (Proposition 3.32).

We highlight two problems to be addressed for further work.

Question 3.1. *How hard is it to decide irreducibility?*

As mentioned in Subsection 3.2.2, it is unclear, to this date, whether or not irreducibility can be decided efficiently (i.e., in polynomial time in the bit-size of a given set of matrices). Deciding this property is useful for CJSR estimation schemes (see Chapter 2) for several reasons. First, it guarantees the non-defectivity of a system. Second, we know that for irreducible arbitrary switching systems, approximating the JSR can be done by focusing on the invariant subspaces (and their orthogonal) [Jun09, Proposition 1.5]. The generalization of this result follows from the usage of the lifting procedure presented in Subsection 3.2.1.

Question 3.2. *Can we provide a generalization of irreducibility for constrained switching systems in a manner independent of the topology of the graph?*

Nodal irreducibility is interesting in the sense that it shows that not all nodes in a graph are as important in the stability analysis of constrained switching systems. In particular, we can focus on nodes encountered by any cycle in the graph.
However, a problem remains: the definition of unavoidability (Definition 3.19) relates only to the topology of the graph (the presence of cycles). Hence, in any graph, we can artificially add unavoidable nodes, sometimes in such a way that they are reducible, and we cannot use Theorem 3.22 anymore.
We conjecture that the following property could replace unavoidability in Theorem 3.22:
Given a graph $\mathbf{G} = (S, E)$*, a set* $S^* \subseteq S$ *is* $\mathscr{L}(\mathbf{G})$*-unavoidable if for any infinite word* $w \in \mathscr{L}(\mathbf{G})$*, there is* $p \in \mathscr{P}^\infty(\mathbf{G})$ *with* $w = w(p)$ *such that* p *visits nodes in* S^* *infinitely often.*

4 Systems with Inputs and Outputs

In this chapter, we consider systems with an external input u, acting either as a *disturbance* or as a *control input*. We deal with two topics. First, we study the *input-output* stability properties of linear constrained switching systems. Second, we study switching output-feedback controllers for stabilizing the system. The tools we develop in both cases can be seen as extensions of the *multinorm* framework of Chapter 2.
This chapter is based on the conference papers [PEDJ16a] and [EPDJ15], and is organized as follows.

- In Section 4.1, we use an example, based on a networked control system with communication failures, to introduce several questions to be tackled in this chapter.
- In Section 4.2, we study the input-output stability of constrained switching systems. We provide a full characterization of the so-called $\mathcal{L}_p$-gains of constrained switching systems using a generalization of *multinorms* as multiple storage functions. The do so, we rely on the concept of minimal realizations for constrained switching systems (introduced in [PBVS13]). We then investigate the application of this framework to approximate the $\mathcal{L}_2$-gain of switching systems, and discuss the reach of the results.
- In Section 4.3, we study the stabilization of constrained switching systems by state feedback, with a focus on *path-dependent methods* [LD06b, ELD14]. We approach the problem from the point of view of multinorms and the stability analysis results of Chapter 2. This provides us with converse theorems for the existence of linear *edge-dependent* output feedback stabilizing controllers - that can be synthesized through LMIs.

4.1 Systems with Input and Output

The situation presented below is similar to that of Example 2.1 and introduces several stability analysis problems tackled in this chapter, centered around *input-output* stability (Section 4.2) and output feedback stabilization (Section 4.3).
We consider a networked control system subject to data losses in its communication channels. As a reminder, networked control systems are spatially distributed systems where components communicate through a (non-ideal) communication network. Such an architecture is interesting in terms of maintainability and scalability, but presents several issues that need to be dealt with when considering non-ideal networks. Switching systems have shown to be particularly relevant for their study [AdJ+11, LZA03, CvdWHN06, AK04, JDDB12, JHK16, DHvdWH11].

We have a linear dynamical system with input of the form

$$x(t+1) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times d}$ is the *input-to-state* matrix, and $u(t) \in \mathbb{R}^d$ is, here, a control input. We assume that the dynamics are free of additional disturbances. Moreover, we assume A is unstable, and that B is full row-rank ($Bu = \mathbf{0} \Leftrightarrow u = \mathbf{0}$).
Given this open-loop system, we implement a *linear state feedback* controller such that $u(t) = Kx(t)$, such that overall, the system

$$x(t+1) = (A + BK)x(t)$$

is stable.
The controller is implemented as follows.

1. At time t, the state $x(t)$ is measured and sent to a computing device through a communication channel.
2. Then, the computer computes the input $u(t) = Kx(t)$, then sends it to the actuator through a second communication channel.
3. The actuator receives the new input from the computer, updates his input, and applies $u(t)$ *until the next update.*

We are interested in modeling and analyzing the system when the step (2.) above may fail, due to a compromised communication channel. In this case, the actuator does not update its input.
If at time t the communication channel fails, then the dynamics are given by

$$x(t+1) = Ax(t) + Bu(t-1) = Ax(t) + BKx(t-1),$$

where $u(t-1)$ is the input stored at the actuator at time t, that is not updated to the value $u(t)$. Letting $\tilde{u}(t)$ be the input stored at the actuator, we model the situation as a switching system on $M = 2$ modes in dimension $n + d$. The first mode $\sigma = 1$ corresponds to the case where the communication channel operates as expected. We let

$$\tilde{x}(t) = \begin{pmatrix} x(t+1) \\ \tilde{u}(t+1) \end{pmatrix},$$

and have for this mode,

$$\tilde{x}(t+1) = A_1\tilde{x}(t) = \begin{pmatrix} A + BK & \mathbf{0} \\ K & \mathbf{0} \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{u}(t) \end{pmatrix}. \tag{4.1}$$

The second mode $\sigma = 2$ corresponds to the case where the communication channel fails to update the input. In this case, we write

$$\tilde{x}(t+1) = A_2\tilde{x}(t) = \begin{pmatrix} A & B \\ \mathbf{0} & I \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{u}(t) \end{pmatrix}. \tag{4.2}$$

From these two matrices, we form the set $\mathbf{A} = \{A_1, A_2\}$. For our example, we consider a constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, where $\mathbf{G}$ is presented at Figure 4.1. It implies that mode 2 cannot be active more than twice in a row.

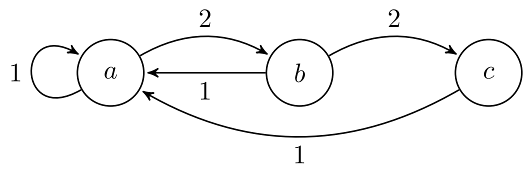


Figure 4.1: Graph $\mathbf{G}$, mode 2 cannot be active more than twice in a row.

We are interested in answering several questions.

1. Is this constrained switching system exponentially stable? To answer this question, we can rely on the tools of Chapter 2.

2. Assume that the system has been operating for some time at mode 1, and then at time $t = 0$, we introduce the possibility for failures as above. The initial conditions are of the form

$$\tilde{x}(0) = B_0 x_0 = \begin{pmatrix} I \\ K \end{pmatrix} x_0,\ x(0) \in \mathbb{R}^n.$$

   Is the system exponentially stable *from such initial conditions*? We discuss this problem in Subsection 4.2.3. To do so, we rely on systems with input and outputs and the concept of *minimal realizations*. We can compute a system $\boldsymbol{\mathcal{S}}'$ from $\boldsymbol{\mathcal{S}}$ where every state is reachable from such an initial condition. We can then check its stability using the tools of Chapter 2.

3. Assume that the system is exponentially stable. Further assume now that, when the communication channel fails (mode 2), disturbances affect the control input. For mode $\sigma = 2$, we have

$$\tilde{x}(t+1) = \begin{pmatrix} A & B \\ \mathbf{0} & I \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} v(t) = A_2\tilde{x}(t) + B_2 v(t), \tag{4.3}$$

where $v(t) \in \mathbb{R}^n$ are disturbances. We are interested in knowing whether or not, when starting from $\tilde{x}(0) = \mathbf{0}$, and using here the intuitive words of Sontag [Son08] discussing *input-output stability*, whether given a *bounded*, *small*, sequence of disturbances $v(0), v(1), \ldots$, the generated sequence $x(0), x(1), \ldots$ with

$$x(t) = \begin{pmatrix} I & \mathbf{0} \end{pmatrix} \tilde{x}(t) = C\tilde{x}(t), \tag{4.4}$$

is itself "bounded"/"small". We study this problem of *input-output* stability in Section 4.2.

4. Finally, assume that we are able to track, for the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, the mode $\sigma(t)$ at any time t. This corresponds to checking whether or not the communication channel is up or down at that time. Then, whenever $\sigma(t) = 1$, we know we can send an input to the actuator. Can we take into account e.g. the fact that $\sigma(t-1) = 2$, and adapt the input and correct the system's behavior? In this case, we consider for mode 1,

$$\tilde{x} = \begin{pmatrix} A + BK & \mathbf{0} \\ K & \mathbf{0} \end{pmatrix} \tilde{x} + \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} u'(t) = A_1\tilde{x}(t) + B_1 u'(t), \tag{4.5}$$

where $u'(t)$ is a corrective input. The mode 2 remains unchanged.
We investigate the case where the corrective input has a form alike $u'(t) = K_{\sigma(t-1)\sigma(t)}\tilde{x}$. We provide converse theorems for the existence of stabilizing controllers of that form in Section 4.3, based on the converse theorems and CJSR approximation framework of Chapter 2.

4.2 Input-Output Stability

In this section, we study switching systems of the form

$$\begin{aligned} x(t+1) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}v(t), \\ z(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}v(t), \end{aligned} \tag{4.6}$$

where $x \in \mathbb{R}^n$, $v \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$ are respectively the state of the system, a *disturbance input*, and a *performance output*. Similarly to the case considered in Chapter 2 and 3, the mode at time t takes value in the alphabet $\langle M \rangle = \{1, \ldots, M\}$ for some integer $M \geq 1$. To complete the definition of the system, we consider a directed and labeled graph $\mathbf{G} = (S, E)$ on the alphabet $\langle M \rangle$, such that the switching sequences satisfy

$$\forall t \geq 0, \sigma(0)\sigma(1)\ldots\sigma(t) \in \mathscr{L}(\mathbf{G}).$$

Given a system (4.6), we let

$$\mathbf{\Sigma} = \{(A_\sigma, B_\sigma, C_\sigma, D_\sigma)_{\sigma=1,\ldots,M}\},$$

and use the notation $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{\Sigma})$ to denote the system (4.6) where the switching sequences are constrained by $\mathbf{G}$.

We use the bold character $\mathbf{v}$ to denote an *input sequence*,

$$\mathbf{v} = v(0), v(1), \ldots,$$

and the set $\mathcal{L}_p$ is defined as follows

$$\mathcal{L}_p := \left\{ \mathbf{v} = v(0), v(1), \ldots \mid \|\mathbf{v}\|_p = \left(\sum_{t=0}^{\infty} \|v(t)\|_p^p \right)^{1/p} < +\infty \right\}.$$

For these systems, we are interested in the analysis and computation of the $\mathcal{L}_p$ *induced* gain ($\mathcal{L}_p$-gain):

$$\gamma_p(\boldsymbol{\mathcal{S}}) := \left(\sup_{\substack{\mathbf{v} \in \mathcal{L}_p,\, \sigma(\cdot) \in \mathscr{L}(\mathbf{G}), \\ x(0) = \mathbf{0}}} \frac{\sum_{t=0}^{\infty} \|z(t)\|_p^p}{\sum_{t=0}^{\infty} \|v(t)\|_p^p} \right)^{1/p}, \tag{4.7}$$

where $z(t)$ is the output of the system at time t for the initial condition $x(0) = \mathbf{0}$, the switching sequence $\sigma(\cdot)$, and the input sequence $\mathbf{v}$.

Remark 4.1. *A common interpretation of the gains (and in particular, of the $\mathcal{L}_2$-gain) is the worst case output over input energy ratio. It is a measure of a system's ability to dissipate the energy of a disturbance - the smaller it is, the lesser the input can perturb the system.*
In particular, a gain smaller than 1 is understood as the fact that an attacker *that seeks to perturb the system always* pays *more input energy than energy he can* generate *out of the system.*

As noted in [NVD16], the $\mathcal{L}_2$-gain is one of the most studied performance measures. Some approaches link the gain of a switching system to the individual performances of its LTI modes (see e.g. [CBG11, Hes04]); and in [ELD14, LD06a], a hierarchy of LMIs is presented to decide whether $\gamma_2 \leq 1$ (using the so-called path-dependent methods mentioned in Section 2.4.2).
In [PZSH11], a generalized version of the gain is studied through generating functions. For continuous time systems, [HH10] gives a tractable approach based on the computation of a storage function. For the general $\mathcal{L}_p$*-gain*, we

refer to [NVD16] that relies on an operator-theoretic approach (see also [DL99]). Our approach is inspired from the work [PEDJ16b] on *stability analysis* (reported in Chapter 2). There, multiple Lyapunov functions with pieces that are *norms* are used for studying (exponential) stability. In Subsection 4.2.1, we provide a characterization of the $\mathcal{L}_p$-gain using *multiple storage functions*, where pieces are now the pth power of norms. We rely on two non-conservative assumptions, namely that the switching system be *internally stable* and in a *minimal realization*, discussed in detail in Subsection 4.2.2. Then, in Subsection 4.2.4, we narrow our focus to the $\mathcal{L}_2$-gain, providing approaches to its computation, and converse theorems for the existence of quadratic storage functions.

For the problems of input-output stability we consider (i.e., computation of gains), the input-output maps of the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{\Sigma})$ are central. We use the following notations to refer to them. Consider a set $\mathbf{\Sigma}$, and a word $w \in \langle M \rangle^*$. If $w = \epsilon$, recall that we let $A_w = A_\epsilon = I$ be the identity matrix in $\mathbb{R}^n$. If w is of length $k \geq 1$, then we let w_i, $1 \leq i \leq k$, be the ith symbol in w, and define the state transition matrix

$$A_w := A_{w_k} \cdots A_{w_1}, \tag{4.8}$$

the input-to-state map

$$B_w := \begin{pmatrix} A_{w_k} \cdots A_{w_2} B_{w_1} & A_{w_k} \cdots A_{w_3} B_{w_2} & \dots & B_{w_k}) \end{pmatrix}, \tag{4.9}$$

the state-to-output map

$$C_w := \begin{pmatrix} C_{w_1} \\ C_{w_2} A_{w_1} \\ \vdots \\ C_{w_2} A_{w_{k-1}} \cdots A_{w_1} \end{pmatrix}, \tag{4.10}$$

and finally, the input-to-output map

$$D_w := \begin{pmatrix} D_{w_1} & \mathbf{0} & \mathbf{0} \\ C_{w_2} B_{w_1} & D_{w_2} & \mathbf{0} \\ \vdots & \vdots \ \ddots & \mathbf{0} \\ C_{w_k} A_{w_{k-1}} \cdots A_{w_2} B_{w_1} \dots & \dots & D_{w_k} \end{pmatrix}. \tag{4.11}$$

For an *infinite word* w, we define the *operators* A_w, B_w, C_w, D_w similarly, with the last three being operators on $\mathcal{L}_p$.

Remark 4.2. *For any $p \geq 1$ and $k \geq 1$,*

$$\max_{w \in \mathscr{L}^k(\mathbf{G})} \|D_w\|_p = \max_{w \in \mathscr{L}^k(\mathbf{G})} \left(\max_{v(0),\dots,v(k-1)} \frac{\sum_{t=0}^{k-1} \|z(t)\|_p^p}{\sum_{t=0}^{k-1} \|v(t)\|_p^p} \right)^{1/p}$$

is a lower-bound *on the $\mathcal{L}_p$-gain. Indeed, we can see that*

$$\max_{w\in\mathscr{L}^k(\mathbf{G})} \|D_w\|_p \leq \max_{w\in\mathscr{L}^k(\mathbf{G})} \left(\max_{\substack{v(0),\ldots,v(k-1)\\ v(k)=v(k+1)=\ldots=\mathbf{0}}} \frac{\sum_{t=0}^{\infty} \|z(t)\|_p^p}{\sum_{t=0}^{\infty} \|v(t)\|_p^p} \right)^{1/p} \leq \gamma_p(\boldsymbol{\mathcal{S}}),$$

since any finitely supported sequence $v(0), v(1), \ldots$ is in $\mathcal{L}_p$. Furthermore, any sequence in $\mathcal{L}_p$ can be approximated arbitrarily closely by finitely supported sequences[1]*. Therefore, we have*

$$\lim_{k\to\infty} \max_{w\in\mathscr{L}^k(\mathbf{G})} \|D_w\|_p = \gamma_p(\boldsymbol{\mathcal{S}}).$$

For the sake of simplicity, unless stated otherwise, we consider here only graphs in expanded form (Definition 1.14) (where, on each edge, the label is in $\langle M \rangle$). The extension to more general graphs is immediate, relying on the matrices A_w, B_w, C_w and D_w above.

4.2.1 Extremal Storage Functions for the $\mathcal{L}_p$-gain

We give an exact characterization of the $\mathcal{L}_p$-gains through a concept similar to the multinorms developped in Chapter 2. In order to do so, we make two assumptions.

Assumption 4.3 (Internal stability)**.** *The system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{\Sigma})$ is* internally stable*, i.e., for*

$$\mathbf{A} = \{A_\sigma \mid (A_\sigma, B_\sigma, C_\sigma, D_\sigma) \in \mathbf{\Sigma}\},$$

the constrained switching system $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \mathbf{A})$ is exponentially stable (Definition 2.2).

It is well known that whenever a (time-varying) system is internally stable its gains are bounded [DL99]. The second assumption is that the system is *minimal*. This is linked with the concept of minimal realization for switching systems (which we discuss in Subsection 4.2.2). The following can be found in [PBVS13, Theorem 1][2].

Definition 4.4 (Minimality)**.** *The system $\boldsymbol{\mathcal{S}}' = (\mathbf{G} = (S, E), \mathbf{\Sigma})$ is* minimal *if, for all $s \in S$,*

$$\mathcal{B}_s = \sum_{\substack{\pi\in\mathscr{P}(\mathbf{G}),\\ \text{dest. of } \pi \text{ is } s}} \operatorname{im}(B_{w(\pi)}), \quad \mathcal{C}_s = \bigcap_{\substack{\pi\in\mathscr{P}(\mathbf{G}),\\ \text{src. of } \pi \text{ is } s}} \ker(C_{w(\pi)}), \tag{4.12}$$

where B_w and C_w are given in (4.9), (4.10), we have $\mathcal{B}_s = \mathbb{R}^n$, $\mathcal{C}_s = \{\mathbf{0}\} \subset \mathbb{R}^n$.

[1] For any sequence $\mathbf{v}$ in $\mathcal{L}_p$, and any $\delta > 0$, there is a sequence $\mathbf{v}'$ such that $\sum \|v(i) - v'(i)\| \leq \delta$, and some integer $k \geq 1$ such that $v'(k+i) = 0$ for all $i = 1, 2 \ldots$.

[2] In that reference, minimality is first defined in terms of dimensions of the systems having equivalent input-output maps. [PBVS13, Theorem 1] show this to be equivalent to Definition 4.4.

The subspace $\mathcal{B}_s$ represents the *span* of the set of all points

$$\{y \mid \exists \pi \in \mathscr{P}(\mathbf{G}),\ \text{dest. of } \pi \text{ is } s\ , \exists v \in \mathbb{R}^{|w(\pi)|d} : y = B_{w(\pi)} v\},$$

which is the span of all points that are reachable from the origin with paths ending at s. Note that the set of reachable points may not be itself a subspace (see [GSL+01] for more on the topic). The subspace $\mathcal{C}_s$ is the set

$$\{x \mid \forall \pi \in \mathscr{P}(\mathbf{G}),\ \text{src. of } \pi \text{ is } s\ : C_{w(\pi)} x = \mathbf{0}\}$$

that is the set of states x that may *only* produce the output $\mathbf{0}$ for all paths starting at s in the absence of disturbances.

Assumption 4.5. *The system $\boldsymbol{\mathcal{S}}$ is minimal.*

We now show that multinorms (see Definition 2.11) provide a characterization of the $\mathcal{L}_p$-gain of constrained switching systems. They play the role of *multiple storage functions.*

Theorem 4.6. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$ satisfying Assumptions 4.3 and 4.5. Its $\mathcal{L}_p$-gain satisfies*

$$\begin{aligned} \gamma_p(\boldsymbol{\mathcal{S}}) = \min \gamma :& \qquad (4.13)\\ &\exists \{V_s : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, s \in S\} \text{ s.t. } \forall x \in \mathbb{R}^n, \forall v \in \mathbb{R}^d, \forall (a, b, \sigma) \in E :\\ &V_b^p(A_\sigma x + B_\sigma v) + \|C_\sigma x + D_\sigma v\|_p^p \leq V_a^p(x) + \gamma^p \|v\|_p^p, \qquad (4.14)\end{aligned}$$

where for all $s \in S$, V_s is a norm *on $\mathbb{R}^n$.*

Moreover, there *always* exists an *extremal multinorm* acting as storage function for internally stable and minimal systems, i.e. that attains the value $\gamma = \gamma_p(\boldsymbol{\mathcal{S}})$ in (4.13).

Definition 4.7. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$. A set of norms $\{V_s : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, s \in S\}$ is* extremal *for the $\mathcal{L}_p$-gain if*

$$\forall (a, b, \sigma) \in E,\ V_b^p(A_\sigma x + B_\sigma v) + \|C_\sigma x + D_\sigma v\|_p^p \leq V_a^p(x) + \gamma_p(\boldsymbol{\mathcal{S}})^p \|v\|_p^p.$$

Theorem 4.8. *Consider a system $(\mathbf{G} = (S, E), \boldsymbol{\Sigma})$, an integer $p \geq 1$, and $\gamma \geq \gamma_p(\boldsymbol{\mathcal{S}})$. At each node $s \in S$, define the function*

$$\begin{aligned} F_{s,\gamma,p}(x) &= \sup_{\substack{\pi \in \mathscr{P}_s^\infty(\mathbf{G}),\\ \mathbf{v} \in \mathcal{L}_p}} \|C_{w(\pi)} x + D_{w(\pi)} \mathbf{v}\|_p^p - \gamma^p \|\mathbf{v}\|_p^p,\\ &= \sup_{\substack{\pi \in \mathscr{P}_s^\infty(\mathbf{G}),\\ \mathbf{v} \in \mathcal{L}_p}} \sum \|z(t)\|_p^p - \gamma^p \sum \|v(t)\|_p^p. \qquad (4.15)\end{aligned}$$

The functions $\{F_{s,\gamma,p}^{1/p}(x),\ s \in S\}$ are norms, and satisfy for all $(a, b, \sigma) \in E$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^d$,

$$F_{b,\gamma,p}(A_\sigma x + B_\sigma w) + \|C_\sigma x + D_\sigma v\|_p^p \leq F_{a,\gamma,p}(x) + \gamma_p^p(\mathbf{G}, \boldsymbol{\Sigma}) \|v\|_p^p. \qquad (4.16)$$

In particular, for $\gamma = \gamma_p(\boldsymbol{\mathcal{S}})$ the set is an extremal multinorm for the system.

Proof of Theorems 4.6 and 4.8. We show that, if a multinorm satisfies 4.14 for some γ, then $\gamma \geq \gamma_p(\boldsymbol{\mathcal{S}})$. Then, we prove 4.8, and finally, and conclude that since any multinorm provides an upper bound on the gain, and that there is one that reaches the gain, Theorem 4.6 holds as well.

Assume (4.14) holds, take $x(0) = \mathbf{0}$, an infinite path $\pi \in \mathscr{P}^\infty(\mathbf{G})$, and a sequence $\mathbf{v} \in \mathcal{L}_p$. Let $s(0), s(1), \ldots$ be the sequence of nodes visited by π, with $s(0)$ being the source node. Unfolding the inequality, we obtain for all $T \geq 1$,

$$\begin{aligned} V(x(0))_{s(0)}^p &+ \gamma^p(\|v(0)\|_p^p + \sum_{t=1}^{\infty} \|v(t)\|_p^p) \\ &\geq V(x(1))_{s(1)}^p + \|z(0)\|_p^p + \gamma^p(\sum_{t=1}^{\infty} \|v(t)\|_p^p) \geq \cdots \\ &\geq V(x(T))_{s(T)}^p + \sum_{t=0}^{T-1} \|z(t)\|_p^p + \sum_{t=T}^{\infty} \|v(t)\|_p^p \geq \sum_{t=0}^{T-1} \|z(t)\|_p^p. \end{aligned}$$

Therefore, for all $T \geq 1$, $(\sum_{t=0}^{T-1} \|z(t)\|_p^p)/(\sum_{t=0}^{\infty} \|v(t)\|_p^p) \leq \gamma^p$, and this being independent of π and $\mathbf{v} \in \mathcal{L}_p$, we get $\gamma_p(\boldsymbol{\mathcal{S}}) \leq \gamma$.

We now move onto the proof of Theorem 4.8. Proving (4.16) is easy, so we focus on proving that for any $\gamma \geq \gamma_p(\boldsymbol{\mathcal{S}})$, the functions $F_{s,\gamma,p}^{1/p}$ are norms. We omit the subscripts γ and p in what follows. The functions have the following properties:

1. $\mathbf{F_s(0) = 0}$. Taking a disturbance $\mathbf{v} = \mathbf{0} \in \mathcal{L}_p$, we observe that $F_s(\mathbf{0}) \geq 0$. Assume by contradiction that there is $K > 0$ such that $F_s(\mathbf{0}) \geq K$. There must be a path π and a sequence $\mathbf{v} \in \mathcal{L}_p$ such that

$$\sum_{t=0}^{\infty} \|z(t)\|_p^p \geq K/2 + \gamma^p \sum_{t=0}^{\infty} \|v(t)\|_p^p > \gamma^p \sum_{t=0}^{\infty} \|v(t)\|_p^p.$$

   Thus, $\gamma < \gamma_p(\boldsymbol{\mathcal{S}})$ (see (4.7)), a contradiction.

2. $\mathbf{F_s(x)}$ **is positive definite**. By Assumption 4.5, for any node $s \in S$, for any $x \in \mathbb{R}^n$, there is a path $\pi \in \mathscr{P}(\mathbf{G})$, with source s, such that $C_{w(\pi)} x \neq \mathbf{0}$. Thus, $F_s(x) > 0$ is guaranteed by taking $\mathbf{v} = \mathbf{0} \in \mathcal{L}_p$.

3. $\mathbf{F_s(x)}$ **is convex and positively homogeneous of order p**. Given a path $\pi \in \mathscr{P}(\mathbf{G})$ of finite length, define the output sequence

$$\mathbf{z}_\pi(x_0, \mathbf{v}) = \begin{pmatrix} z(0) \\ \vdots \\ z(|\pi|-1) \end{pmatrix} = C_{w(\pi)} x_0 + D_{w(\pi)} \begin{pmatrix} v(0) \\ \vdots \\ v(|\pi|-1) \end{pmatrix}.$$

   Clearly, $\mathbf{z}_\pi(\alpha x_0, \alpha \mathbf{v}) = \alpha \mathbf{z}_\pi(x_0, \mathbf{v})$ for any $\alpha \in \mathbb{R}$. Therefore, for any

$\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have

$$F_s(\alpha x) = \sup_{\pi, \mathbf{v} \in \mathcal{L}_p} \sum_t \|z_t(\alpha x, \mathbf{v})\|_p^p - \gamma_p^p |v(t)|_p^p,$$
$$= \sup_{\pi, \alpha \mathbf{v}' \in \mathcal{L}_p} \sum_t |\alpha|^p \|z_t(x, \mathbf{v}')\|_p^p - \gamma_p^p \alpha^p \|v_t'\|_p^p,$$

Since $\|\mathbf{z}_\pi(x_0, \mathbf{v})\|_p^p = \sum_t \|z_t(x_0, \mathbf{v})\|_p^p$ is convex in x_0, the convexity of $F_s(x)$ is guaranteed.

4. $\mathbf{F_s(x)} < +\infty$, $\forall \mathbf{x} \in \mathbb{R}^\mathbf{n}$. Take any finite path π such that the destination of π is s, and $x \in \mathrm{im}(B_w(\pi))$. Let s' be the source of π, and let $\mathbf{v}$ be such that $x = B_{w(\pi)} \left(v_0^\top, \ldots, v_{|\pi|-1}^\top \right)^\top$. We have that

$$F_{s'}(\mathbf{0}) = 0 \geq \sum_{t=0}^{|\pi|-1} (\|z(t)\|_p^p - \gamma_p^p \|v(t)\|_p^p) + F_s(x),$$

and thus $F_s(x)$ is bounded. Now, take any $x \in \mathbb{R}^n$. By Assumption 4.5, we know that x is a linear combination of points in the reachable sets $\mathrm{im}(B_{w(\pi)})$, for some path π ending at s. Thus, by convexity, $F_s(x) < +\infty$.

■

In the context of stability analysis, extremal multinorms exists if and only if the system is *non-defective* (see Theorem 2.19), for which we know only sufficient conditions (see Chapter 3). In the present context, our assumptions can be made without loss of generality.

4.2.2 Minimal Realizations and Internal Stability

Under our assumptions, we prove the existence of extremal storage functions for the $\mathcal{L}_p$-gain, for any p. We discuss the generality these assumptions. We begin with internal stability (Assumption 4.3).

If we omit internal stability we face theoretical barriers: the boundedness of the $\mathcal{L}_p$-gain becomes *undecidable.*

Proposition 4.9. *Given a switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$ and $p \geq 1$, the question of whether or not $\gamma_p(\boldsymbol{\mathcal{S}}) < +\infty$ is* undecidable.

Proof. Consider an arbitrary switching system with two modes $\mathbf{A} = \{A_1, A_2\}$ and $\|A_i\| > 1$ for $i \in \{1, 2\}$. We know that the *boundedness* of the trajectories of that system is undecidable (see [BT00a], and also Chapter 3). Consider now the set

$$\boldsymbol{\Sigma} = \{\{A_1, \mathbf{0}, \mathbf{0}, \mathbf{0}\}, \{A_2, \mathbf{0}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{0}, I, I, \mathbf{0}\}\},$$

which is

$$x(t+1) = A_\sigma x(t), \; z(t) = \mathbf{0}, \; \text{for } \sigma \in \{1, 2\} \; ,$$

and

$$x(t+1) = v(t),\ z(t) = x(t),\ \text{for } \sigma = 3.$$

For that system, the only way to excite the state $x(0) = \mathbf{0}$ with an input is to pick switching sequences that begin with $\sigma(0) = 3$. Also, the only possibility to generate an ouput at any time t is to pick again $\sigma(t) = 3$. At that time, the state resets and we have $x(t+1) = v(t)$.
From there, we have the following expression for the $\mathcal{L}_p$-gain of $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$:

$$\gamma_p(\boldsymbol{\mathcal{S}}) = \sup_{\substack{v(0),T,\\ \sigma(0),\ldots,\sigma(T-1)\in\{1,2\}^T}} \frac{\|A_{\sigma(T-1)}\cdots A_{\sigma(0)}v(0)\|_p}{\|v(0)\|_p}.$$

This is bounded if and only if there is a uniform bound on the norm of all products of matrices from $\mathbf{A}$. Since this is undecidable, the boundedness of the gain is undecidable as well. ■

In practice, if one seeks to study the input-output stability of a system, he need first to know that the system is internally stable. The tools of Chapter 2 provide algorithms to that end.
Internal stability alone is only a *sufficient* condition for the boundedness of the gain. For example, it is easy to construct unstable systems having, at each nodes, non-trivial stable subspaces. For input-output stability, if only the stable subspace is reachable, the gain remains bounded. It is precisely to avoid such considerations that minimality (Assumption 4.5) comes into the picture.

A minimal realization is a classical concept in systems' theory (see e.g. [DP13, Section 2.5]). The notion has been studied for discrete-time linear switching systems in [PBVS13,PvS09]. Given a system $\boldsymbol{\mathcal{S}}$, a minimal realization of $\boldsymbol{\mathcal{S}}$ is a system $\boldsymbol{\mathcal{S}}'$ with the *same input-output maps* (more precisely, for each $w \in \mathscr{L}(\mathbf{G})$, the matrices D_w are the same for both systems), but with a minimal dimension for the state space.
However, we need to redefine the notion of *state space* for discussing minimal realizations. The minimal realization of a constrained-switching system as in (4.6) *may not always be* itself a system of the form $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$. In general, minimal realizations are switching systems such that *a different state space is attached at each node of* $\mathbf{G}$[3].

Definition 4.10. *A* rectangular *constrained switching system is a tuple* $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma}, \mathbf{n})$*. The graph* $\mathbf{G}$ *is strongly connected with nodes* S *and directed labeled edges* E*. Edges are of the form* $(a, b, \sigma) \in E$*, where* $\sigma \in \{1, \ldots, |E|\}$ *is a* unique *label, each edge having a different one. The* dimension vector $\mathbf{n} = (n_s)_{s\in S}$ *assigns a* dimension *to each node* $s \in S$*. The set* $\boldsymbol{\Sigma} = \{\{A_\sigma, B_\sigma, C_\sigma, D_\sigma\}_{\sigma\in 1,\ldots,|E|}\}$ *satisfies, for* $(a, b, \sigma) \in E$,

$$A_\sigma \in \mathbb{R}^{n_b\times n_a},\ B_\sigma \in \mathbb{R}^{n_b\times d},\ C_\sigma \in \mathbb{R}^{m\times n_a},\ D_\sigma \in \mathbb{R}^{m\times d}.$$

[3] Petreczky and van Schuppen refer to such systems as discrete-time linear hybrid systems. A minimal realizations preserves the input output maps, and minimizes the sum of the dimensions of the state spaces at the nodes (see [PvS09, Definition 4 and 5]).

Remark 4.11. *The systems we have considered so far are a particular case of these* rectangular *systems, where the dimension of each node is the same. Also, the proofs, concepts and ideas we develop for our systems $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$ generalize naturally to this class of systems. Notice that by construction for any path $\pi \in \mathscr{P}(\mathbf{G})$, the word $w(\pi)$ is such that the matrices $A_{w(\pi)}$, $B_{w(\pi)}$, $C_{w(\pi)}$ and $D_{w(\pi)}$ are well defined in the context of rectangular systems, and so are the subspaces $\mathcal{B}_s$ and $\mathcal{C}_s$ of (4.12).*

Definition 4.12 (Adapted from [PvS09, Theorem 1])**.** *A rectangular system $(\mathbf{G} = (S, E), \boldsymbol{\Sigma}, \mathbf{n})$, $\mathbf{n} = (n_s)_{s \in S}$, is said to be* minimal *if, for all $s \in S$, $\mathcal{B}_s = \mathbb{R}^{n_s}$, and $\mathcal{C}_s = \{\mathbf{0}\} \subset \mathbb{R}^{n_s}$.*

Example 4.13. *The system (4.3-4.4) is not minimal. First, let us cast it as a rectangular system. We take the graph $\mathbf{G}$ of Figure 4.2. Here, each edge has a unique label, which for this example are represented by $1', \ldots, 5'$. In relation to the system (4.3-4.4), we have that e.g. the mode 1 from the original system corresponds to the modes $1'$, $3'$ and $5'$, which are given by*

$$\left\{ \begin{pmatrix} A + BK & \mathbf{0} \\ K & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} I & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \end{pmatrix} \right\},$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$, $K \in \mathbb{R}^{d \times n}$. Mode 2 corresponds to modes $2'$ and $4'$, which are given by

$$\left\{ \begin{pmatrix} A & B \\ \mathbf{0} & I \end{pmatrix}, \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} I & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \end{pmatrix} \right\}.$$

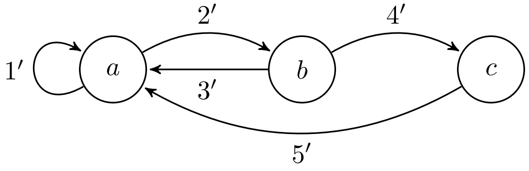


Figure 4.2: Graph $\mathbf{G}'$ for Example 4.13.

To see that the system is not a minimal realization, observe that at node c in the graph,

$$\mathcal{C}_c \supseteq \left\{ \begin{pmatrix} \mathbf{0} \\ v \end{pmatrix}, v \in \mathbb{R}^d \right\},$$

and so the system is not minimal by definition since $\mathcal{C}_c \neq \{\mathbf{0}\}$.

We could, however, make it minimal at the effort of a cleaner modelization. For example, if $A + BK$ is invertible, we can have a system with the same input output maps, but where we can reduce the dimension at node a to be that of the true state of the system $x = C\tilde{x}$. First, we can let mode $1'$ be

$$\left\{ \left(A + BK \right), \mathbf{0}, I, \mathbf{0} \right\},$$

which corresponds to the expected behavior of the system *neglecting network failures.*

For mode $2'$, we have

$$x(t+1) = Ax(t) + Bu(t-1) + Bv(t), \tilde{u}(t+1) = u(t-1)$$

where we must have

$$x(t) = (A+BK)x(t-1), u(t-1) = Kx(t-1).$$

Therefore, we can write, for mode $2'$, $x(t+1) = (A + BK(A+BK)^{-1})x(t)$, with $\tilde{u}(t-1) = K(A+BK)^{-1}x(t)$. In matrix form is given by the parameters

$$\left\{ \begin{pmatrix} A + BK(A+BK)^{-1} \\ K(A+BK)^{-1} \end{pmatrix} \in \mathbb{R}^{n+d \times n}, \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}, I, \mathbf{0} \right\}.$$

In turn, mode $3'$ can be written as

$$\left\{ \begin{pmatrix} A+BK & \mathbf{0} \end{pmatrix}, \mathbf{0}, \begin{pmatrix} I & \mathbf{0} \end{pmatrix}, \mathbf{0} \right\},$$

where the dynamics need to map $\tilde{x}$ onto x.
For mode $4'$, we can take

$$\left\{ \begin{pmatrix} A & B \end{pmatrix}, B, \begin{pmatrix} I & \mathbf{0} \end{pmatrix}, \mathbf{0} \right\}.$$

Finally, we can take mode $5'$ to be the same as mode $1'$.

The rectangular system we obtain has dimensions $n_a = n$, $n_b = n + d$, and $n_c = n$. Note that the only assumption on the dynamics so far is the invertibility of $A + BK$. With a few more assumptions on the dynamics, we can show the rectangular system is a minimal realization.

When modeling a system, obtaining minimal realizations is generally not the most intuitive thing to do. For example, in Section 4.1, we just split the model in the description of the different modes, and connected them through the graph $\mathbf{G}$, which is intuitive and straightforward. Minimal realizations are one step ahead, the result of a processing step that may involve both the algebraic properties of the modes of the system with the combinatorics of the graphs. The good news is that algorithms to compute these realizations exists. The tools we present here can be deduced from those introduced by Petreczky et al. [PBVS13, PvS09], and adapted to constrained switching systems here.

In order to compute a minimal realization for a system $(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n})$ we first compute the subspaces $\mathcal{B}_s$ and $\mathcal{C}_s$ of Definition 4.12.

Proposition 4.14 (Construction of the subspaces $\mathcal{C}_s$)**.** *Given a rectangular system $(\mathbf{G} = (S, E), \mathbf{\Sigma}, \mathbf{n})$, define for all node $s \in S$*

$$X_{s,1} = \sum_{(s,b,\sigma) \in E} C_\sigma^\top C_\sigma; \, \mathcal{C}_{s,1} = \ker(X_{a,1}).$$

For $k = 1, 2, \ldots$, consider the following iteration

$$X_{s,k+1} = \sum_{(s,b,\sigma)\in E} A_\sigma^\top X_{b,k} A_\sigma;\ \mathcal{C}_{s,k} = \ker(\sum_{t=1}^{k} X_{b,k}).$$

For all node $s \in S$, the sequence $\mathcal{C}_{s,k}$ converges and in particular $\mathcal{C}_s = \mathcal{C}_{s,K}$, $K = \sum_{s\in S} n_s$.

Proof. By construction, for any $k \geq 1$, $X_{s,k}$ is the set of all $x \in \mathbb{R}^{n_s}$ such that for any path π of length k starting at node s, $C_{w(\pi)}x = \mathbf{0}$. If at some point $\mathcal{C}_{s,k+1} = \mathcal{C}_{s,k}$ for all $s \in S$, then the iteration has converged. Before that, at any iteration, there must be at least one node $s \in S$ for which

$$\dim\ker(X_{s,\ell+1}) \geq \dim\ker(X_{s,\ell}).$$

Since $\dim\ker(X_{s,\ell}) \leq n_s$ at any step ℓ, an upper bound on the number of iterations before convergence is $K = \sum_{s\in S} n_s$. ■

We can use the procedure above to construct $\mathcal{B}_s$ as well. We have

$$\mathcal{B}_s = \bigcap_{\substack{\pi\in\mathscr{P}(\mathbf{G}),\\ \text{dest. of } \pi \text{ is } s}} \ker(B_{w(\pi)}^\top).$$

For a given system $(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n})$ it suffice to apply Proposition 4.14 to the *dual system* $(\mathbf{G}^\top = (S, E^\top), \mathbf{\Sigma}^\top, \mathbf{n})$, defined with

$$E^\top = \{(b, a, \sigma) \mid (a, b, \sigma) \in E\},$$

and

$$\mathbf{\Sigma}^\top = \{(A_\sigma^\top, C_\sigma^\top, B_\sigma^\top, D_\sigma^\top) \mid (A_\sigma, B_\sigma, C_\sigma, D_\sigma) \in \mathbf{A}\},$$

to compute $\mathcal{B}_v^\perp$.

Given a system, we compute the subspace $\mathcal{B}_S$ and $\mathcal{C}_s$. Then, if at a node s, $\mathcal{B}_s \neq \mathbb{R}^{n_s}$ for example we can actually restrict the state space at that node to be $\mathcal{B}_s$ by projection (same with $\mathcal{C}_s^\perp$).

Proposition 4.15. *Given a rectangular system $(\mathbf{G} = (S, E), \mathbf{\Sigma}, \mathbf{n})$, let n_s be the dimension of the node $s \in S$ and $m_s \leq n_s$ be the dimension of $\mathcal{B}_s$. For $s \in S$, let $L_s \in \mathbb{R}^{n_s\times m_s}$ be an orthogonal basis of $\mathcal{B}_s$. The rectangular system $(\mathbf{G} = (S, E), \bar{\mathbf{\Sigma}}, \bar{\mathbf{n}})$ on the set*

$$\bar{\mathbf{\Sigma}} = \left\{(L_b^\top A_\sigma L_a,\ L_b^\top B_\sigma,\ C_\sigma L_a,\ D_\sigma), (a, b, \sigma) \in E\right\},$$

has $\mathcal{B}_s = \mathbb{R}^{m_s}$, $\bar{\mathbf{n}} = \{m_s, s \in S\}$, and $\gamma_p(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n}) = \gamma_p(\mathbf{G}, \bar{\mathbf{\Sigma}}, \bar{\mathbf{n}})$.

Proof. Observe that the subspaces $\mathcal{B}_s$ are invariant in the sense that

$$(a, b, \sigma) \in E, \mathcal{B}_b \supseteq A_\sigma \mathcal{B}_a + \mathrm{im}(B_\sigma). \tag{4.17}$$

Consider a switching path visiting the nodes $s_0, s_1, \ldots$. Since $x(0) = \mathbf{0}$, we have $x(0) \in \mathcal{B}_{s_0}$. By invariance, after a path of t steps, if the next edge is (a, b, σ), we get both $x(t) \in \mathcal{B}_a$ and $x(t+1) \in \mathcal{B}_b$. Hence, the dynamics are equivalent to the following system:

$$\begin{aligned} L_{s_t}^\top L_{s_t} x(t+1) &= A_\sigma L_{s_{t-1}}^\top L_{s_{t-1}} x(t) + B_{\sigma(t)} v(t), \\ z(t) &= C_\sigma L_{s_{t-1}}^\top L_{s_{t-1}} x(t) + D_{\sigma(t)} v(t), \\ &\{(s_{t-1}, s_t, \sigma(t)), t \in \mathbb{N}\} \in \mathscr{P}(\mathbf{G}),\ x(0) = \mathbf{0}. \end{aligned}$$

The system on the parameters $\bar{\mathbf{\Sigma}}$ is then obtained by multiplying the first equation on the left by L_b (since $L_b L_b^\top L_b = L_b$), and then doing the following change of variables: at each node $s \in S$, $y = L_s^\top x$ (this is allowed since we deal with rectangular systems). ■

Proposition 4.16. *Given a rectangular system $(\mathbf{G} = (S, E), \mathbf{\Sigma}, \mathbf{n})$, let n_s be the dimension of the node $s \in S$ and $m_s \leq n_s$ be the dimension of $\mathcal{C}_s^\perp$. Let $K_s \in \mathbb{R}^{n_s \times m_s}$ be an orthogonal basis of $\mathcal{C}_s^\perp$. The rectangular system $(\mathbf{G} = (S, E), \bar{\mathbf{\Sigma}}, \bar{\mathbf{n}})$ on the set*

$$\bar{\mathbf{A}} = \left\{(K_b^\top A_\sigma K_a,\ K_b^\top B_\sigma,\ C_\sigma K_a,\ D_\sigma), (a, b, \sigma) \in E\right\},$$

has $\mathcal{C}_s^\perp = \mathbb{R}^{m_s}$, $\bar{\mathbf{n}} = \{m_s, s \in S\}$, and $\gamma_p(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n}) = \gamma_p(\mathbf{G}, \bar{\mathbf{\Sigma}}, \bar{\mathbf{n}})$.

Proof. The proof is similar to that of Proposition 4.16. Observe that for any $(a, b, \sigma) \in E$, $\mathcal{C}_b \supseteq A_\sigma \mathcal{C}_a$. What we do then is to project the state onto $\mathcal{C}_s^\perp$ at every step. Doing so does not affect the gain since the part of the state in $\mathcal{C}$ does not affect the output at any time, before and after the projection. ■

Theorem 4.17. *Given a rectangular system $(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n} = \{n_s, s \in S\})$, consider the following iteration procedure. Let $\boldsymbol{\mathcal{S}}_0 = (\mathbf{G}, \mathbf{\Sigma}, \mathbf{n})$, and for $k = 1, 2, \ldots$, let $\boldsymbol{\mathcal{S}}_k$ be the system obtained by applying first Proposition 4.15 then Proposition 4.16 to $\boldsymbol{\mathcal{S}}_{k-1}$. Then $\boldsymbol{\mathcal{S}}_N$ is minimal for $N = \sum_{s \in S} n_s$, and $\gamma_p(\mathbf{G}, \mathbf{\Sigma}, \mathbf{n}) = \gamma_p(\mathbf{G}, \bar{\mathbf{\Sigma}}, \bar{\mathbf{n}})$.*

Proof. The fact that the gain is preserved along the iterations is granted from Proposition 4.15 and Proposition 4.16. The number of iterations needed to converge is actually conservative, but is easily shown. If the system is non-minimal, then at least one node will have a reduced dimension after an iteration. We can register a decrease at at least one node at most $\sum_s n_s$ times. ■

As a conclusion, even when given a non-minimal system $(\mathbf{G}, \mathbf{\Sigma})$, we may always construct a minimal *rectangular* system $(\mathbf{G}, \bar{\mathbf{\Sigma}}, \{(n_s)_{s \in S}\})$ with the same $\mathcal{L}_p$-gain. Theorem 4.6 is easily translated for rectangular systems, by defining a multinorm $\{V_s(\cdot), s \in S\}$ as a set of norms, where $V_s(\cdot)$ is a norm on the space $\mathbb{R}^{n_s}$. The storage functions of Theorem 4.8 remain well defined, with $F_{s,\gamma,p}$ now taking values in $\mathbb{R}^{n_s}$.

Remark 4.18. *There are some pathological cases where the dimension at a node $s \in S$ reduces to 0. This would correspond to a situation where, initially, $\mathcal{B}_s \subseteq \mathcal{C}_s$.*

4.2.3 Stability from an Initial Condition

In this subsection, we make a brief detour from input-output stability analysis to explain how minimal realizations can help to know when a system is stable from a given *initial condition*.

Definition 4.19. *We say that a switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is exponentially stable for initial conditions in $\mathcal{X}$ if there is $\rho < 1$ and $K \geq 1$ such that*

$$\forall x_0 \in \mathcal{X}, \forall w \in \mathscr{L}(\mathbf{G}) : \|x(x_0, w)\| \leq K\rho^{|w|}\|x_0\|. \tag{4.18}$$

The ideas are better illustrated through an example.

Example 4.20. *We seek to know whether or not the system on two modes (4.1, 4.2) and with the graph* **G** *of Figure 4.1 is stable when we have the following knowledge on the initial condition:* From time t=0, all switching sequences must correspond to paths in **G** starting at mode a, and the initial state is of the form

$$\tilde{x}(0) = B_0 x_0 = \begin{pmatrix} I \\ K \end{pmatrix} x_0,\ x_0 \in \mathbb{R}^n.$$

To answer this question, consider a system with input and outputs *on the graph* $\mathbf{G}'$ *of Figure 4.3. This graph is peculiar as it is not strongly connected - this is*

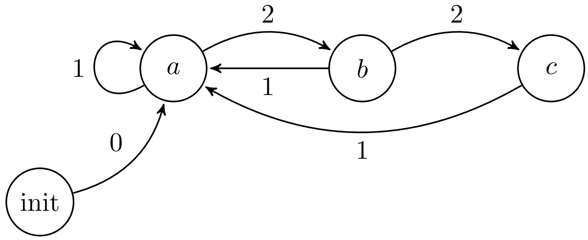


Figure 4.3: Graph $\mathbf{G}'$, to be used to decide the stability of an initial condition.

by design, and is suitable for our purposes.

There are labels of two sort. The first sort - labels 1 and 2 - are the labels of the original systems, and correspond to modes

$$\{A_\sigma, \mathbf{0}, I, \mathbf{0}\}$$

in our system with inputs and outputs.
The second sort - the label 0 - is used to push an initial condition into the

system. It corresponds to a mode

$$\left\{\mathbf{0}, B_0 = \begin{pmatrix} I \\ K \end{pmatrix}, I, \mathbf{0}\right\}.$$

If we compute a minimal realization for this system, what we obtain is a rectangular system where, at each node, the state space corresponds exactly to the span of all the reachable states from initial conditions in the image of the matrix B_0*.*

We can show that the original system is stable from the desired initial condition if and only if the rectangular system is itself internally stable. To verify this in practice, one may rely on the tools of Chapter 4. The constrained joint spectral radius of a rectangular system can be defined similarly to that of a regular constrained switching system (see Definition 2.5). This quantity can in turn be approximated by a straightforward adaptation of the tools of Section 2.3, where now at each node $s \in S$*, we look for a quadratic form* $Q_s \succ 0$ *in* $\mathbb{R}^{n_s \times n_s}$*,* n_s *being the dimension of the node.*

In the construction presented in Example 4.20, we build a system with input and outputs from a constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$. Initial conditions can be injected into the system at only one location. For checking stability from a union of such initial conditions (e.g. different subspaces from different nodes), we may just repeat the process for each one of them.

Remark that, if an initial condition generates a rectangular system with at least one node $s \in S$ of dimension $n_s < n$, then from (4.17) in the proof of Proposition 4.15, we need to have

$$\forall (a, b, \sigma) \in E,\, \mathcal{B}_b \supset A_\sigma \mathcal{B}_a.$$

This implies that the system $\boldsymbol{\mathcal{S}}$ is *graph-reducible* in the sense of Definition 3.9. Hence, if the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ is irreducible, the choice initial conditions do not matter for the question of stability.

Remark 4.21. *The construction presented here suggests a gateway to enable the tools developed throughout this thesis to be applied to systems with constraints generated by more general automata (see Remark 1.20), where switching paths must start at an initial state.*
Through the construction above, we can first compute the reachable states space at every node of the automata. Then, we can use our tools on the linearly connected components *(Definition 3.17) of the resulting system. Further examination is left for future work.*

4.2.4 Storage Functions for the $\mathcal{L}_2$-gain

We now aim at computing the extremal storage functions $F_{s,\gamma,2}(x)$ (see Theorem 4.8) of a system for computing its $\mathcal{L}_2$-gain. We no longer consider rectangular systems here, but the generalization of the results is straightforward. The Assumptions 4.3 and 4.5 still hold.

In approximating the function $F_{s,\gamma,2}(x)$, two questions arise. First, given a value of γ, how can we get a good approximation of $F_{s,\gamma,2}(x)$? Second, and more importantly, how can we obtain obtain a good estimate of the $\mathcal{L}_2$-gain? We propose a first method, based on dynamic programming and asymptotically tight under-approximations of the $\mathcal{L}_2$-gain, and a second method, based on the path-dependent Lyapunov function framework of [ELD14, LD06b], for obtaining over-approximations of the $\mathcal{L}_2$-gain. Both are based on the following observation. Given a system $(\mathbf{G} = (S, E), \boldsymbol{\Sigma})$, we can write

$$F_{s,\gamma,p}(x) = \sup_{\pi \in \mathscr{P}_s^{\infty}(\mathbf{G})} \check{F}_{w(\pi),\gamma,p}(x),$$

where we define for every word w (of bounded or infinite length)

$$\check{F}_{w,\gamma,p}(x) := \sup_{\mathbf{v} \in \mathcal{L}_p} \|C_w x + D_w \mathbf{v}\|_p^p - \gamma^p \|\mathbf{v}\|_p^p. \tag{4.19}$$

Note that this function is well-defined as long as $\gamma \geq \|D_w\|_p$.

Lower bounds on the gain

A first approximation of $F_{s,\gamma,p}$ is obtained by restricting the length of the paths to some $K \geq 1$. We let

$$\tilde{F}_{s,\gamma,p,K} = \max_{\pi \in \mathscr{P}_s^{K}(\mathbf{G})} \check{F}_{w(\pi),\gamma,p}(x) \tag{4.20}$$

Proposition 4.22. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$, an integer $K \geq 1$ and $p \geq 1$, let*

$$\tilde{\gamma}_{K,p}(\boldsymbol{\mathcal{S}}) = \max_{\pi \in \mathscr{P}^{K}(\mathbf{G})} \|D_{w(\pi)}\|_p.$$

For any $\gamma \geq \check{\gamma}_{K,p}(\boldsymbol{\mathcal{S}})$, $K \geq n|S|$ and for any $s \in S$, the function $\check{F}^{1/p}_{s,\gamma,p,K}$ is a norm.

Proof. If $\gamma \geq \check{\gamma}_{K,p}$, then by (4.19 - 4.20), the function $\check{F}_{s,\gamma,p,K}$ is well defined. Then, by Assumption 4.5 and Proposition 4.14, for $K \geq n|S|$, for all $x \in \mathbb{R}^n$, there is a path π starting at node s such that $C_{w(\pi)} x \neq \mathbf{0}$. Following the ideas behind the proof of Theorem 4.8, we can then conclude that $\check{F}^{1/p}$ is a norm. ■

Proposition 4.23. *Given a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}(S, E), \boldsymbol{\Sigma})$, we have for any $k \geq 1$, $\check{\gamma}_{k,p}(\boldsymbol{\mathcal{S}}) \leq \gamma_p(\boldsymbol{\mathcal{S}})$, and moreover $\lim_{k \to \infty} \check{\gamma}_{k,p} = \gamma_p(\boldsymbol{\mathcal{S}})$.*

The approach above allows to get asymptotically tight lower-bounds on the gain. The following allows us to obtain upper bounds.

Upper bounds and Quadratic Storage Functions

In order to get upper bounds, we *approximate* storage functions by quadratic norms using the Horizon-Dependent Lyapunov functions of [ELD14]. Before presenting the method, let us discuss how, in general, we can approximate storage functions with quadratic functions - using LMIs alike that of Proposition 2.20.

Consider a graph $\mathbf{G} = (S, E)$ (it does not need to be in *expanded form*). Then, the program of Theorem 4.6 for the case $p = 2$ and using quadratic functions V_s is as follows

$$\begin{aligned}
\gamma_{p,Q}(\boldsymbol{\mathcal{S}}) = &\inf_{\gamma,\, \{Q_s \in \mathbb{R}^{n\times n} : s \in S\}} \gamma \qquad s.t. \\
&\forall (a,b,w) \in E : \\
&\begin{pmatrix} A_w & B_w \\ C_w & D_w \end{pmatrix}^\top \begin{pmatrix} Q_b & \mathbf{0} \\ \mathbf{0} & I_{|w|m} \end{pmatrix} \begin{pmatrix} A_w & B_w \\ C_w & D_w \end{pmatrix} - \begin{pmatrix} Q_a & \mathbf{0} \\ \mathbf{0} & \gamma^2 I_{|w|d} \end{pmatrix} \preceq \mathbf{0}, \\
&\forall s \in S : \, Q_s \succ \mathbf{0},
\end{aligned} \tag{4.21}$$

where for an integer k, I_k is the identity matrix in $\mathbb{R}^{k\times k}$, and A_w, B_w, C_w and D_w are as defined in (4.8,4.9,4.10,4.11). From there, we can consider using that program alongside the T-product lift and T-path-dependent lift (Section 2.4) for example, to get (hopefully) better upper bounds on the gain. Path-Dependent methods have been proven effective there, allowing to provide arbitrarily good estimates [LD06a]. The following depicts the *horizon-dependent* storage functions of [ELD14], and reformulates the result of [ELD14, Theorem 2.6] within our context. It is an *horizon-dependent* version of [LD06a, Theorem 3.3].

Proposition 4.24 ([ELD14]). *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$ with $\mathbf{G}$ in expanded form. Given an integer $K \geq 1$, consider the following program:*

$$\begin{aligned}
&\hat{\gamma}_K(\boldsymbol{\mathcal{S}}) = \inf_{\gamma,\, \{Q_\pi \in \mathbb{R}^{n\times n} : \pi \in \mathscr{P}^K(\mathbf{G})\}} \gamma \qquad s.t. \\
&\forall \pi_1, \pi_2 \in \mathscr{P}^K(\mathbf{G}), \; \textit{with } \pi_1 = e_1 e_2 \ldots e_K, \pi_2 = e_2 e_3 \ldots e_{K+1}, \\
&\qquad e_i \in E \textit{ and } e_1 = (a, b, \sigma) : \\
&\begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix}^\top \begin{pmatrix} Q_{\pi_2} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix} - \begin{pmatrix} Q_{\pi_1} & \mathbf{0} \\ \mathbf{0} & \gamma^2 I \end{pmatrix} \preceq \mathbf{0}, \\
&\forall \pi \in \mathscr{P}^K(\mathbf{G}) : \, Q_\pi \succ \mathbf{0}.
\end{aligned}$$

Then, for all $K \geq 1$, $\hat{\gamma}_K(\boldsymbol{\mathcal{S}}) \geq \gamma_2(\boldsymbol{\mathcal{S}})$ and for any $\gamma > \gamma_2(\boldsymbol{\mathcal{S}})$, there is $K \geq 1$ such that $\hat{\gamma}_K(\boldsymbol{\mathcal{S}}) \leq \gamma$.

Remark 4.25. *Horizon-Dependent Lyapunov functions are part of the broader* path-dependent *methods discussed in Subsection 2.4.2, and can similarly be defined through a lifting of the graph* $\mathbf{G}$*. Alike the path-dependent lifting of*

Definition 2.40, we build a graph with one node per path of length K, and there is still an edge between two nodes if and only if the $K-1$ last edges of the path for the source node correspond to the $K-1$ first edges of the path for the destination node. The difference now is the the label of the edge is that of the first edge of the path for the source node. For the path-dependent lifting, it is the label of the last edge of the destination node.

Remark 4.26. *There is an interesting interpretation of Horizon-Dependent Lyapunov functions as approximations of the functions $F_{s,\gamma,p}$. For a number $\gamma > 0$, an integer p, and a path $\pi \in \mathscr{P}(\mathbf{G})$ with length K, consider the function*

$$\hat{F}_{\pi,\gamma,p}(x) = \sup_{\substack{\phi \in \mathscr{P}^{\infty}(\mathbf{G}),\\ \phi \text{ and } \pi \text{ have the same first } K \text{ edges}}} \check{F}_{w(\phi),\gamma,p}(x).$$

We can see that for all K,

$$F_{s,\gamma,p}(x) = \max_{\pi \in \mathscr{P}_s^K(\mathbf{G})} \hat{F}_{\pi,\gamma,p}(x).$$

By definition, if we take two paths π_1 and π_2 of length K such that $\pi_1 = e_1 e_2 \dots e_K$, and $\pi_2 = e_2 e_3 \dots e_{K+1}$, with $e_1 = (a, b, \sigma)$, we can then verify that for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^d$,

$$\hat{F}_{\pi_2,\gamma,p}(A_\sigma x + B_\sigma v) + \|C_\sigma x + D_\sigma v\|_p^p \leq \hat{F}_{\pi_1,\gamma,p}(x) + \gamma^p \|v\|_p^p.$$

For $p = 2$, assuming the functions $\hat{F}$ to be quadratic, the inequalities above are equivalent to the LMIs of Proposition 4.24.

Note that, together, Propositions 4.23 and 4.24 enable us to approximate the $\mathcal{L}_2$-gain of a system, and its storage functions, in an arbitrarily accurate manner.

In view of the similarities with Sections 2.3 and 2.4, we now investigate converse result for the existence of quadratic storage functions.

Theorem 4.27. *Given a constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$, consider the system $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \boldsymbol{\Sigma}')$ with*

$$\boldsymbol{\Sigma}' = \{(\sqrt{n}A_\sigma,\, \sqrt{n}B_\sigma,\, C_\sigma, D_\sigma) :\, (A_\sigma,\, B_\sigma,\, C_\sigma, D_\sigma) \in \boldsymbol{\Sigma}\}.$$

If $\gamma_2(\boldsymbol{\mathcal{S}}') \leq 1$, then $\gamma_2(\boldsymbol{\mathcal{S}}) \leq 1$ and there is a set $\{|\cdot|_{Q,s}, s \in S\}$ of quadratic *norms such that for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^d$ and $(a, b, \sigma) \in E$,*

$$|A_\sigma x + B_\sigma v|_{Q,b}^2 + \|C_\sigma x + D_\sigma v\|_2^2 \leq |x|_{Q,a}^2 + \|w\|_2^2.$$

Proof. The result relies on Theorem 4.6, and similarly to Theorem 2.21, exploits the fact that the functions $F_{s,\gamma,2}^{1/2}$ are norms. If $\gamma_2(\boldsymbol{\mathcal{S}}') < 1$, then there are norms $\{|\cdot|_s, s \in S\}$ such that

$$\forall (a, b, \sigma) \in E,\; n|A_\sigma x + B_\sigma v|_b^2 + \|C_\sigma x + D_\sigma v\|_2^2 \leq |x|_a^2 + \|w\|_2^2.$$

By John's ellipsoid theorem, we can approximate these norms with quadratic norms $\{|\cdot|_{Q,s}, s \in S\}$ such that

$$\forall s \in S : |x|_s \leq |x|_{Q,s} \leq \sqrt{n}|x|_s.$$

Taking quadratic norms satisfying John's inequalities, we have

$$\begin{aligned}|A_\sigma x + B_\sigma v|^2_{Q,b} + \|C_\sigma x + D_\sigma v\|^2_2 &\leq n|A_\sigma x + B_\sigma v|^2_b + \|C_\sigma x + D_\sigma v\|^2_2 \\ &\leq |x|^2_a + \|w\|^2_2 \\ &\leq |x|^2_{Q,a} + \|w\|^2_2,\end{aligned}$$

which concludes the proof. ■

Remark 4.28. *If $\gamma_2(\boldsymbol{\mathcal{S}}') \leq 1$ in the above, this means that the constrained joint spectral radius of the system on the set $\mathbf{A} = \{A_\sigma, \sigma \in \langle M \rangle\}$ is smaller than $1/\sqrt{n}$. Consequently, by Theorem 2.21, the system has a* quadratic multinorm *has a stability certificate. Also, note that the existence of a storage function as studied here is a proof of stability, hence a system can have a quadratic storage function for some value γ only if it has a quadratic multinorm.*

Remark 4.29. *There is an important difference between and Theorem 2.21, in the context of exponential stability, and Theorem 4.27 in the current context. The first converse result allows us to extract* bounds on the quantity of interest*, which is there the CJSR. This is not the case for the second converse result, as it is unclear how the gain of the scaled system on the set $\mathbf{\Sigma}'$ relates to that of the original system on the set $\mathbf{\Sigma}$.*

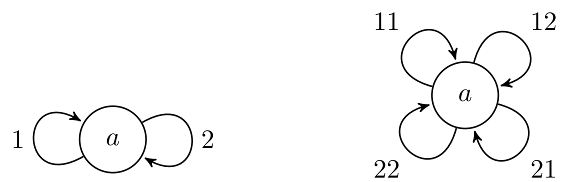
We conjecture that the following extension, which follows from the stability analysis case (Corollary 2.45), holds true for the performance analysis case:

Conjecture 4.30. *Given a constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \mathbf{\Sigma})$ and an integer $k \geq 1$, consider the system $\boldsymbol{\mathcal{S}}' = (\mathbf{G}, \mathbf{\Sigma}')$ with*

$$\mathbf{\Sigma}' = \{n^{\frac{1}{2k}} A_\sigma, n^{\frac{1}{2k}} B_\sigma, C_\sigma, D_\sigma) : (A_\sigma, B_\sigma, C_\sigma, D_\sigma) \in \mathbf{\Sigma}\}.$$

If $\gamma_2(\boldsymbol{\mathcal{S}}') \leq 1$, then $\boldsymbol{\mathcal{S}}$ has a Horizon-Dependent storage function (see Proposition 4.24) for $K = k - 1$.

So far we have not been able to prove or disprove the result. In fact, we can prove that we cannot simply follow the steps that lead us to Corollary 2.45 from Theorem 2.33 , using the T-product lift (Definition 2.30). In particular, if we can easily define the concept of the T-product lift for the input-output case, we can build a system such that using quadratic forms on T-product lifts, for any value of T, does not allow for a better approximation of the $\mathcal{L}_2$-gain.



(a) Graph $\mathbf{G}$.

(b) Graph $\mathbf{G}^T$, for $T = 2$. There is one edge per path of length T, with the same source, destination, and label as the path.

Figure 4.4: Graphs for Example 4.31.

Example 4.31. *Consider an* arbitrary switching system *on two modes, with*

$$\begin{aligned}
&A_1 = A_2 = \mathbf{0} \in \mathbb{R}^{2\times 2},\\
&B_1 = \begin{pmatrix}1\\1\end{pmatrix}, B_2 = \begin{pmatrix}1\\0\end{pmatrix},\\
&C_1 = \begin{pmatrix}1 & 0\end{pmatrix}, C_2 = \begin{pmatrix}0 & 1\end{pmatrix},\\
&D_1 = D_2 = 0.
\end{aligned}$$

The $\mathcal{L}_2$-gain of this system is

$$\gamma_2 = \max_{\sigma_1,\sigma_2} \|C_{\sigma_2} D_{\sigma_1}\|_2 = 1.$$

In this example, we are going to compute first the best approximation of the $\mathcal{L}_2$-gain, by solving (4.21), one can obtain using quadratic storage functions on the graph $\mathbf{G}$ at Figure 4.4a. Then, we show that taking the T-product lifts $\mathbf{G}^T$ of the graph $\mathbf{G}$ (Figure 4.4b for $T = 2$) and solving the program (4.21), we do not improve the approximation.

Writing (4.21) for the graph $\mathbf{G}$, we get

$$Q = \begin{pmatrix} q_1 & q_3\\ q_3 & q_2\end{pmatrix} \succ 0, \tag{4.22}$$

$$Q - C_1^\top C_1 = \begin{pmatrix} q_1 - 1 & q_3\\ q_3 & q_2\end{pmatrix} \succ 0, \tag{4.23}$$

$$Q - C_2^\top C_2 = \begin{pmatrix} q_1 & q_3\\ q_3 & q_2 - 1\end{pmatrix} \succ 0, \tag{4.24}$$

$$\gamma^2 - B_1^\top Q B_1 = \gamma^2 - (q_1 + q_2 + 2q_3) > 0, \tag{4.25}$$

$$\gamma^2 - B_2^\top Q B_2 = \gamma^2 - q_1 > 0. \tag{4.26}$$

For the sake of clarity, we let $\gamma^2 = 1 + \delta$ for $\delta > 0$.
We extract upper and lowers bounds on q_1 and q_2 from (4.23,4.24,4.25,4.26):

$$1 < q_1 < 1 + \delta,\ 1 < q_2 < 1 + \delta - q_1 - 2q_3.$$

The determinant of the matrix in (4.23) needs to be positive, hence $(q_1-1)q_2-q_3^2>0$*, which holds only if (upper bounding the expression)*

$$(1+\delta-1)(1+\delta-1-2q_3)-q_3^2=\delta(\delta-2q_3)-q_3^2>0,$$

which implies $q_3\in[\delta(-\sqrt{2}-1),\delta(\sqrt{2}-1)]$*, which goes to 0 as* δ *goes to 0.*

The determinant of the matrix in (4.24) needs to be positive as well, hence $q_1(q_2-1)-q_3^2>0$*, which holds only if*

$$(1+\delta)(1+\delta-1-2x_3-1)-q_3^2=(\delta+1)((\delta-1)-2q_3)-q_3^2>0,$$

which implies $q_3\in[-\sqrt{2(1+\delta)\delta}-(1+\delta),\sqrt{2(1+\delta)\delta}-(1+\delta)]$*, which goes to 1 as* $\delta\to 0$*.*
From these two equations, we see there must be $\delta>0$ *such that there are no solutions to the LMIs (as* q_3 *needs both be close to 0 and -1). Hence, using quadratic multinorms, we don't reach the true gain of the system (which is exactly 1, for* $\delta=0$*).*

Now, let us take an integer $T\geq 2$*, and see if we can improve upon this situation. After writing the LMIs for the graph* $\mathbf{G}^T$*, we get that it must hold that*

$$\begin{aligned}&\forall\sigma_1\dots\sigma_T\in\{1,2\}^T,\ \forall v(1),\dots,v(T)\in\mathbb{R},\forall x\in\mathbb{R}^2:\\ &|B_{\sigma_T}v(T)|_Q^2+\|C_{\sigma_1}x\|_2^2+\|\sum_{k=1}^{T-1}\|C_{\sigma_{k+1}}B_{\sigma_k}v(k)\|_2^2\leq|x|_Q^2+\gamma^2\sum_{k=1}^{T}\|v(k)\|_2^2,\end{aligned}$$

where $|x|_Q^2=x^\top Qx$*.*
This must hold true in particular when $v(1),\dots,v(T-1)=0$*, which corresponds to*

$$\begin{aligned}&\forall\sigma_1,\sigma_T\in\{1,2\},\ \forall v(T)\in\mathbb{R},\forall x\in\mathbb{R}^2,\\ &|B_{\sigma_T}v(T)|_Q^2+\|C_{\sigma_1}x\|_2^2\leq|x|_Q^2+\gamma^2\|v(T)\|_2^2,\end{aligned}$$

and when we take the case $\sigma_T=\sigma_1$*, we fall back to the constraints for the case* $T=1$*, which if the case where we use the graph* $\mathbf{G}$ *and not the* T*-product lift.*

We conclude that the T*-product lift does not allow to refine the estimation of the gain in this case.*

4.2.5 Numerical Example

We illustrate the results of this Section with the scenario introduced in Section 4.1. We take the same numerical values for the matrices as in Section 2.5, that is

$$A=\frac{1}{1.1}\begin{pmatrix}0.94&0.56\\0.14&0.46\end{pmatrix},B=\frac{1}{1.1}\begin{pmatrix}0\\1\end{pmatrix},K=\begin{pmatrix}-0.49&0.27\end{pmatrix},$$

and consider first the constrained switching system $\boldsymbol{\mathcal{S}}=(\mathbf{G},\mathbf{A})$, with modes $\mathbf{A}=\{A_1,A_2\}$ as in (4.1) and (4.2) , i.e.,

$$A_1=\begin{pmatrix}0.8545&0.5091&0\\-0.3182&0.6636&0\\-0.4455&0.2455&0\end{pmatrix},A_2=\begin{pmatrix}0.8545&0.5091&0\\0.1273&0.4182&1\\0&0&1\end{pmatrix},$$

and for the graph $\mathbf{G}$ at Figure 4.1.
At the moment, we don't consider input or outputs yet.

- Is the system stable?
  For this system, we can show that the constrained switching system is indeed stable. Using the tools of Section 2.20, we obtain an upper bound on the CJSR of the system $\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq 0.987$.

- What about initial conditions of the form
  $$\tilde{x}(0) = \begin{pmatrix} x_0 \\ Kx_0 \end{pmatrix} = B_0 x_0, x_0 \in \mathbb{R}^n,$$
  where we know switching sequences start at node a in $\mathbf{G}$? Are these initial conditions more stable than others?
  We follow the the reasoning of Example 4.20. In this case, and one can verify that
  $$\operatorname{im}\left(\begin{pmatrix} B_0 & A_2 A_1 B_0 \end{pmatrix} = \mathbb{R}^3, \right),$$
  and therefore we have $\mathcal{B}_a = \mathbb{R}^{3\times3}$. Moreover, since A_2 is full rank, the reachable spaces for nodes b and c are $\mathbb{R}^{3\times3}$ as well. In conclusion, the stability properties of the system for these initial conditions are the same than for the original system (same exponential growth rates).

- Can we estimate the $\mathcal{L}_2$-gain for the system with input and outputs $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$, with the modes
  $$\Sigma = \{(A_\sigma, B_\sigma, C_\sigma, D_\sigma), \sigma \in \{1, 2\}\},$$
  where A_1, A_2 are defined above, $B_1 = \mathbf{0}$, $B_2 = B$, $C_1 = C_2 = (I,\, 0)$, $I \in \mathbb{R}^2$, $D_1 = D_2 = 0$.

  As discussed in Example 4.13, the system is not in a minimal realization. In practice, this does not forbid us to approximate gains using e.g. Proposition 4.22 and 4.24. We might face numerical issues however.

  For our matrices, the rectangular system of Example 4.13 is a minimal realization. We obtain upper bounds on the gain using the H-Horizon-Dependent method of Proposition 4.24, for $H = 0, \ldots, 4$.

  Figure 4.5 show the evolution of the game estimates as h varies from 0 to 4. For $h = 0$, the approximation is
  $$\hat{\gamma}_0 \simeq 37.04,$$
  and goes down to
  $$\hat{\gamma}_4 \simeq 13.87,$$
  a significant improvement, for $H = 4$. The level sets of the pieces of the storage function for the three nodes are represented at Figure 4.6 for $H = 0$, and Figure 4.7 for $H = 4$. In this second case, they are reconstructed as

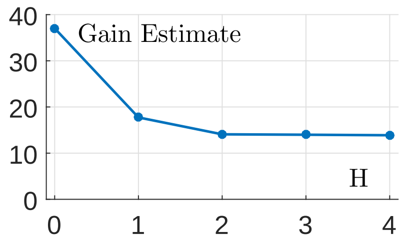
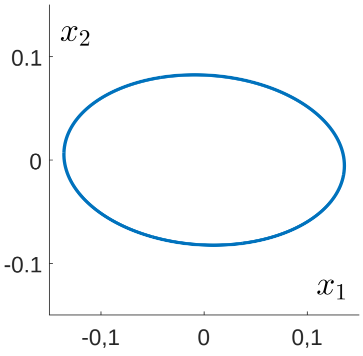
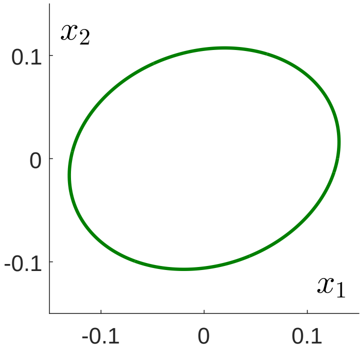


Figure 4.5: Upper bounds of the $\mathcal{L}_2$ gains. Range varies from approx. 37.05 (H=0) to approx. 13.87 (H=4).

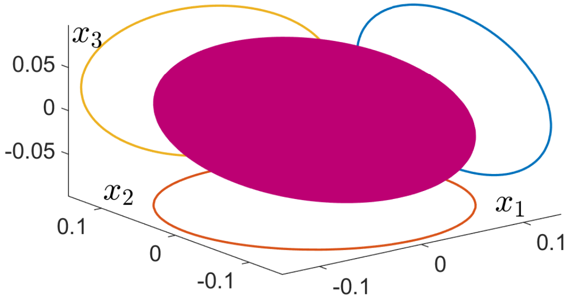
max-of-quadratics, using the pieces of the horizon-dependent Lyapunov function as explained in Remark 4.26. Recall that we are working here on the minimal realization, and the dimension of the state at node b is $n_b = 3$, where the dimension at the other nodes is 2.



(a) Level set of storage function at node a.

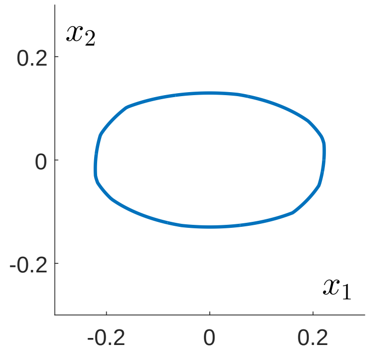


(b) Level set of storage function at node c.

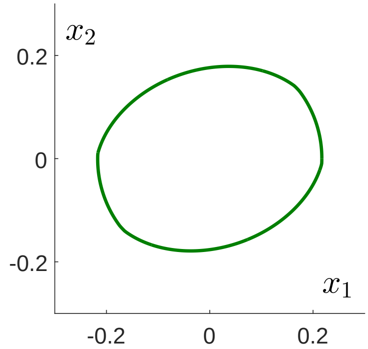


(c) Level set of storage function at node b

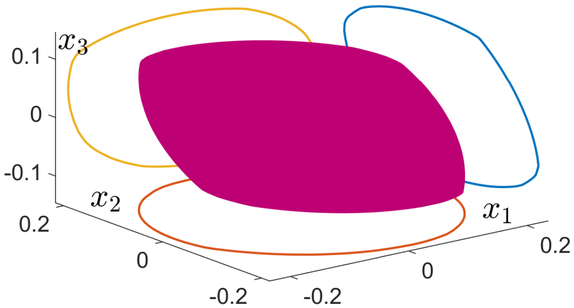
Figure 4.6: The case $H = 0$.



(a) Level set of storage function at node a.



(b) Level set of storage function at node c.



(c) Level set of storage function at node b.

Figure 4.7: The case $H = 4$.

4.3 Feedback Stabilization

We now consider systems of the form

$$\begin{aligned} x(t+1) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) \end{aligned} \tag{4.27}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^d$ is a *control* input, and $y(t) \in \mathbb{R}^m$ is the observed output of the system.
We are in a constrained switching setting, meaning that the switching sequences $\sigma(1), \sigma(2), \dots$ are words accepted by a graph $\mathbf{G} = (S, E)$, $E \subseteq S \times S \times \langle M \rangle$. *Throughout this section, we assume the graph* $\mathbf{G}$ *is in expanded form.* Given a graph $\mathbf{G}$ on $M \geq 1$ modes and a set of parameters

$$\mathbf{\Sigma} = \{(A_\sigma, B_\sigma, C_\sigma), \sigma \in \langle M \rangle\},$$

we let $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{\Sigma})$ denote the system (4.27) where the switching sequences are constrained to be in the language of $\mathbf{G}$.

The problem at hand is the synthesis of a controller stabilizing the system in the closed loop. In this section, we are particularly interested by the tools developed in [DRI02, LD06b, ELD14], where underlying techniques for synthesis there is reminiscent of those used in Section 2.3. Using these tools, the controller we obtain takes the form of a linear switching system, with the input y and the output u. The closed loop system is a linear constrained switching system as well, amenable to study using the framework of Chapter 2.
In this context, [DRI02] establishes sufficient conditions, under the form of LMIs, that allows to construct a controller of the form

$$u(t) = K_{\sigma(t)}y(t),$$

such that the closed loop system

$$x(t+1) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}C_{\sigma(t)})x(t) \tag{4.28}$$

is stable.
As an alternative to provide more flexibility in the design [LD06b, ELD14] and [LD06a] introduced path-dependent techniques, where the controller now not only depends on the current mode, but also a certain number of modes in the past and/or in the future [ELD14]. In [LD06a], the tool is adapted to tackle controller design with performance guarantees (in this context, the system has both control and disturbances input, and the controller's goal is both to stabilize the system and guarantee performances in term of the $\mathcal{L}_2$-gain).
We wish to embed these methods in the framework of Section 2.4.2. There, path-dependent methods are described multiple quadratic Lyapunov functions for a lifted graph (see Subsection 2.4.2).
In order to reproduce the approach here, and we first need to define such a

generic procedure in our context. This is presented in Subsection 4.3.1, and leads to controller that are, for a given graph, *edge-dependent*[4].

Definition 4.32. *Given a system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \boldsymbol{\Sigma})$*, with the state in* $\mathbb{R}^n$*, input in* $\mathbb{R}^d$ *and output in* $\mathbb{R}^m$*, an edge-dependent output-feedback controller of* order n_K *is defined as a set tuple*

$$\hat{\boldsymbol{\Sigma}} = \{(\hat{A}_e, \hat{B}_e, \hat{C}_e, \hat{D}_e), e \in E.\},$$

with $\hat{A}_e \in \mathbb{R}^{n_K \times n_K}$, $\hat{B}_e \in \mathbb{R}^{n_K \times m}$, $\hat{C}_e \in \mathbb{R}^{d \times n_K}$ *and* $\hat{D}_e \in \mathbb{R}^{d \times m}$*, and we call the matrices*

$$K_e = \begin{pmatrix} \hat{A}_e & \hat{B}_e \\ \hat{C}_e & \hat{D}_e \end{pmatrix} \in \mathbb{R}^{n_K + d \times n_K + m}, e \in E$$

the gains of the controller.

Given such a controller, the closed loop system *is given by*

$$\boldsymbol{\mathcal{S}}_{closed} = (\mathbf{G}, \boldsymbol{\Sigma}, \hat{\boldsymbol{\Sigma}})$$

with the dynamics

$$\tilde{x}(t+1) = \left(\begin{pmatrix} A_{\sigma(t)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & B_{\sigma(t)} \\ I & \mathbf{0} \end{pmatrix} K_{e(t)} \begin{pmatrix} \mathbf{0} & I \\ C_{\sigma(t)} & \mathbf{0} \end{pmatrix} \right) \tilde{x}(t),$$
$$\tilde{x}(t+1) = \left(\tilde{A}_{\sigma(t)} + \tilde{B}_{\sigma(t)} K_{e(t)} \tilde{C}_{\sigma(t)} \right) \tilde{x}(t),$$

where $\tilde{x}(t) \in \mathbb{R}^{n+n_K}$ *is the closed loop state,* $e(0)e(1)\ldots$ *is a path in* $\mathbf{G}$ *and* $\sigma(0)\sigma(1)\ldots$ *is the corresponding sequence of modes.*

Remark 4.33. *Depending on the graph considered, edge-dependent controllers can sometimes be understood as* mode-dependent *controllers, or path-dependent controllers in the sense of [LD06b], or other. For example,*

- *For the graph* $\mathbf{G}$ *at Figure 4.4a, there is one edge per mode, hence the controller is mode-dependent.*

- *For the graph* $\mathbf{G}$ *at Figure 4.1, each edge is uniquely defined by the current mode of the system and the number of successive occurrences of mode 2 right before the current time. For example, if the current mode is* 2 *and the mode prior to this was* 1*, we are at the edge* $(a, b, 2)$*.*

[4] In the context of the path-dependent methods, this in fact corresponds to the 1-path-dependent controllers defined in [LD06b, Definition 15], see Remarks 2.38 and 2.39. The *mode-dependent* controllers considered in [LD06b] are those described in Example 4.33, where the closed loop is stable under arbitrary switching.

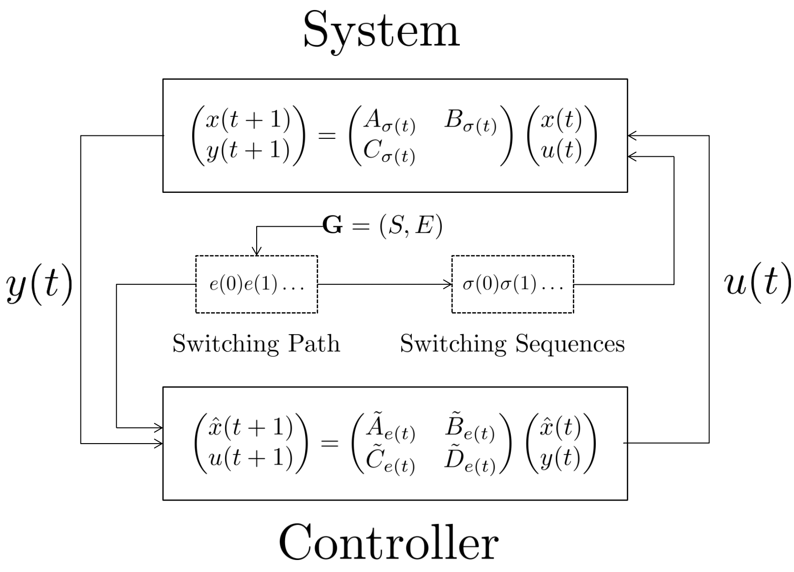


Figure 4.8: The controller architecture. The top side represents a constrained switching system, with graph $\mathbf{G}$, input u, and output y. The switching sequences of this system are defined in the usual way, as sequences of modes on a path of $\mathbf{G}$. We close the loop by constructing another constrained switching system (bottom side), taking y as input and u as output, that switches according to the sequence of edges taken in $\mathbf{G}$.

The closed loop system is represented at Figure 4.8 and results from the composition of the system (4.27) with the system

$$\begin{aligned} \hat{x}(t+1) &= \hat{A}_{e(t)}x(t) + \hat{B}_{e(t)}y(t) \\ u(t) &= \hat{C}_{e(t)}x(t) + \hat{D}_{e(t)}y(t), \end{aligned} \tag{4.29}$$

where $\hat{x}(t) \in \mathbb{R}^{n_K}$ is the internal state of the controller. We stress that this system associates a different set of parameters per *edge* of $\mathbf{G}$.

Notice that, even if $\mathbf{G} = (S, E)$ allows for arbitrary switching, the closed loop system is by definition a constrained switching system with exactly $|E|$ modes, and its stability follows from the results of Chapter 2. In particular, the following quantity allows us the characterize the performance and the existence of stabilizing edge-dependent controllers:

Definition 4.34. *Consider a system $\boldsymbol{\mathcal{S}} = (\mathbf{G} = (S, E), \boldsymbol{\Sigma})$ and an integer n_K. The optimal closed loop decay rate of the system with edge-dependent output* *feedback control is defined as*

$$\hat{\rho}_{n_K,\,\mathrm{closed}}(\mathbf{G},\mathbf{\Sigma}) = \inf_{\hat{\mathbf{\Sigma}} \text{ of order } n_K} \hat{\rho}((\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}})),$$

where for a controller $\hat{\mathbf{\Sigma}}$, $(\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}})$ is the closed loop system (Definition 4.32), and $\hat{\rho}((\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}}))$ is the constrained joint spectral radius of that system (Definition 2.5).

It is immediate to state that one can stabilize a system with edge-dependent control if and only if $\hat{\rho}_{n_K,\,\mathrm{closed}}(\mathbf{G},\mathbf{\Sigma}) < 1$.
With this in mind, we first discuss how we can synthesize controllers using LMIs, by seeking controllers that provide a stability certificate under the form of a *quadratic multinorm* (see Section 2.3). We then analyze the performances of this class of controllers. In particular, we provide bounds on their performances relative to the optimal closed loop decay rate.

4.3.1 Multinorms and Edge-Dependent Controllers

Since a stable closed loop system is a *stable* constrained switching system, we know from Chapter 2, Theorem 2.13, that it should have a contractive multinorm.

Lemma 4.35. *Consider a switching system $(\mathbf{G} = (S,E),\mathbf{\Sigma})$. Given an edge-dependent controller $\hat{\mathbf{\Sigma}}$ of order n_K and a multinorm $\mathcal{H} = \{\|\cdot\|_s, s \in S\}$ on $\mathbb{R}^{n+n_K}$ (Definition 2.11), we let*

$$\begin{aligned}\gamma^*(\mathcal{H},(\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}})) = \inf_{\gamma}\{\gamma :\\ \forall e = (a,b,\sigma) \in E, \forall \tilde{x} \in \mathbb{R}^{n+n_K} : \|(\tilde{A}_\sigma + \tilde{B}_\sigma K_e \tilde{C}_\sigma)\tilde{x}\|_b \leq \gamma \|\tilde{x}\|_a\}.\end{aligned} \tag{4.30}$$

We have

$$\hat{\rho}_{n_K,\,\mathrm{closed}}(\mathbf{G},\mathbf{\Sigma}) = \inf_{\hat{\mathbf{\Sigma}} \text{ of order } n_K} \inf_{\mathcal{H}} \gamma^*(\mathcal{H},(\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}})).$$

Proof. This follows from Definition 4.34 and Proposition 2.12, which characterizes the CJSR as an infimum over multinorms. ■

When restricting ourselves to using *quadratic multinorms* (see Section 2.3), the program (4.30) is equivalent to seeking the smallest number γ that satisfies

$$\forall e = (a,b,\sigma) \in E : \, (\tilde{A}_\sigma + \tilde{B}_\sigma K_e \tilde{C}_\sigma)^\top Q_b (\tilde{A}_\sigma + \tilde{B}_\sigma K_e \tilde{C}_\sigma) - \gamma^2 Q_a \preceq 0. \tag{4.31}$$

Since a priori the quadratic forms and control gains are unknown, these inequalities are nonlinear in the variables Q_b and K_e. In order to synthesize a controller, we follow the steps of [LD06b, Algorithm 1]. The general methodology has been introduced in [LD06b, ELD14] and relies on the techniques of Gahinet and Apkarian [GA94] and Packard [Pac94].

First, using a Schur complete, we rewrite (4.31) in the equivalent form

$$\begin{pmatrix} -Q_b^{-1} & \tilde{A}_\sigma + \tilde{B}_\sigma K_e \tilde{C}_\sigma \\ (\tilde{A}_\sigma + \tilde{B}_\sigma K_e \tilde{C}_\sigma)^\top & -\gamma^2 Q_a \end{pmatrix} \preceq 0. \tag{4.32}$$

This is linear in K_e, but not in Q_b.
Then, for each node $s \in S$, we write Q_s as

$$Q_s = \begin{pmatrix} R_s & T_s \\ T_s^\top & V_s \end{pmatrix} = \begin{pmatrix} P_s & U_s \\ U_s^\top & W_s \end{pmatrix}^{-1}, \tag{4.33}$$

with R_s, $P_s \in \mathbb{R}^{n\times n}$, $V_s, W_s \in \mathbb{R}^{n_K \times n_K}$. The following holds true (see [LD06b, Lemma 18], [Pac94]): we can write $Q_s \succ 0$ as in (4.33) if and only if

$$\forall s \in S : \begin{pmatrix} R_s & I \\ I & P_s \end{pmatrix} \succeq 0 \text{ and } \operatorname{rank} \begin{pmatrix} R_s & I \\ I & P_s \end{pmatrix} \leq n + n_K. \tag{4.34}$$

The rank constraint is non-trivial to deal with in general, but is immediately satisfied if $n_K \geq n$.
Finally, for each $\sigma \in \langle M \rangle$, pick two matrices $\bar{B}_\sigma$ and $\bar{C}_\sigma$ such that

$$\operatorname{im}(\bar{B}_\sigma) = \ker B_\sigma^\top,\ \operatorname{im}(\bar{C}_\sigma) = \ker C_\sigma.$$

Then, we can solve (4.32) if and only if we can solve the following (see e.g. [LD06b, Theorem 20]):

$$\forall (a,b,\sigma) \in E : \begin{cases} (\bar{B}_\sigma)^\top \left(A_\sigma R_a A_\sigma^\top - \gamma^2 R_b\right)(\bar{B}_\sigma) \preceq 0, \\ (\bar{C}_\sigma)^\top \left(A_\sigma^\top P_b A_\sigma - \gamma^2 P_a\right)(\bar{C}_\sigma) \preceq 0, \end{cases} \tag{4.35}$$

and for all $s \in S$ (4.34) holds.

When $C_\sigma = I \in \mathbb{R}^{n\times n}$ for all $\sigma \in \langle M \rangle$, then we must take $\bar{C}_\sigma = \mathbf{0}$ by definition. The second inequality in (4.35) becomes void, so we have no constraints on the matrices P_s, $s \in S$, so we can take w.l.o.g. $P_s = R_s^{-1}$ whenever R_s satisfies (4.35). This implies that (4.34) is trivially satisfied, even in the case $n_K = 0$, when $C_\sigma = I$.

Overall, this gives us a way to synthesize edge-dependent controllers. Assume that $n \geq n_K$. There is a solution to (4.34) and (4.35) if and only if there are matrices Q_s, $s \in S$, such that (4.32) is feasible for some gains K_e, $e \in E$. The matrices Q_s can be computed from the matrices R_s and P_S in (4.34) and (4.35), by taking e.g. $U_s = T_s = \mathbf{0}$, $V_s = W_s = I$ for $s \in S$ in (4.33). Having done so, we can solve (4.32) to recover the terms K_e.
In the case $C_\sigma = I$, we can take $n_K = 0$ without loss of generality, and take $Q_s = R_s$.

Remark 4.36. *The method above is only guaranteed to produce edge-dependent controller. In fact, as pointed out in [ELD14, Remark 16], we can solve inequalities of the form (4.31) using the technique presented above if and only if the*

controller variables K appear at most once per inequality. In our case, it means we need one control gain per edge.
If we want to obtain e.g. a mode-dependent *controller, we need to solve (4.31) with the added constraint that*

$$\forall e = (a, b, \sigma) \in E, K_e = K_\sigma.$$

These constraints are not carried through (4.34) and (4.35).
For mode-dependent controllers with quadratic multinorms, the work Daafouz et al. [DRI02] provides sufficient conditions under the form of LMIs for their existence.

4.3.2 Bounds and Guarantees on Edge-Dependent Controllers

From Lemma 4.35, we know that for any $\delta > 0$, there is a controller of order n_K such that the CJSR of the closed loop system is at most $\hat{\rho}_{n_K,\text{closed}}(\mathbf{G}, \mathbf{\Sigma}) + \delta$. However, there are no guarantees that such a good controller can be obtained by satisfying the LMIs of Subsection 4.3.1. We are now in a setting similar to that of Section 2.3, and we trade optimality of the controller with respect to the achieved decay rate for tractability.
We would like to understand exactly how much is lost in terms of performances. In the following, we provide a first converse result to that end.

Definition 4.37. *Consider a switching system $(\mathbf{G}, \mathbf{\Sigma})$. For an integer n_K, we let*

$$\hat{\rho}_{n_K,\text{ quad}}(\mathbf{G}, \mathbf{\Sigma}) = \inf_{\hat{\mathbf{\Sigma}} \textit{ of order } n_K} \inf_{\mathcal{H} \textit{ quadratic}} \gamma^*(\mathcal{H}, (\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})).$$

By definition $\hat{\rho}_{n_K,\text{ quad}}(\mathbf{G}, \mathbf{\Sigma})$ is the infimum over γ such that (4.31) holds. It is the best guarantee on the closed loop decay rate we can obtain if we consider closed loop systems with quadratic multinorms, allowing controller synthesis as in Subsection 4.3.1.

Theorem 4.38. *Consider a switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{\Sigma})$ and an integer $n_K \geq 0$. The following holds:*

$$\hat{\rho}_{n_K,\text{ closed}}(\mathbf{G}, \mathbf{\Sigma}) \leq \hat{\rho}_{n_K,\text{ quad}}(\mathbf{G}, \mathbf{\Sigma}) \leq \sqrt{(n + n_K)}\hat{\rho}_{n_K,\text{ closed}}(\mathbf{G}, \mathbf{\Sigma}).$$

Proof. Take any controller $\hat{\mathbf{\Sigma}}$ with order n_K. By Theorem 2.21, we can write

$$\begin{aligned}\hat{\rho}((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})) &\leq \inf_{\mathcal{H} \text{ is a quadratic multinorm}} \gamma^*(\mathcal{H}, (\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})),\\ &\leq \sqrt{(n + n_K)}\hat{\rho}((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})),\end{aligned}$$

where $\hat{\rho}((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}}))$ is the CJSR of the closed loop system. Also, for all $\delta > 0$, there is a controller $\hat{\mathbf{\Sigma}}_\delta$ satisfying

$$\hat{\rho}_{n_K,\text{ closed}}(\mathbf{G}, \mathbf{\Sigma}) \leq \hat{\rho}((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}}_\delta)) \leq \hat{\rho}_{n_K,\text{ closed}}(\mathbf{G}, \mathbf{\Sigma}) + \delta.$$

Therefore, for such controllers, we have

$$\begin{aligned}\hat{\rho}_{n_K,\,\text{closed}}(\mathbf{G},\mathbf{\Sigma}) &\leq \inf_{\mathcal{H}\text{ is a quadratic multinorm}} \gamma^*(\mathcal{H},(\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}}_\delta)),\\ &\leq \sqrt{(n+n_K)}\,(\hat{\rho}_{n_K,\text{closed}}(\mathbf{G},\mathbf{\Sigma})+\delta)\,.\end{aligned}$$

Lemma 4.35 and the definition of $\hat{\rho}_{n_K,\,\text{quad}}(\mathbf{G},\mathbf{\Sigma})$ allow us to write

$$\hat{\rho}_{n_K,\,\text{closed}}(\mathbf{G},\mathbf{\Sigma}) \leq \hat{\rho}_{n_K,\,\text{quad}}(\mathbf{G},\mathbf{\Sigma}) \leq \inf_{\mathcal{H}\text{: quadratic multinorm}} \gamma^*(\mathcal{H},(\mathbf{G},\mathbf{\Sigma},\hat{\mathbf{\Sigma}}_\delta)).$$

This allows to conclude the proof by taking $\delta \to 0$. ■

The result above can be used to provide crude bounds on $\hat{\rho}_{n_K,\,\text{closed}}(\mathbf{G},\mathbf{\Sigma})$.

Remark 4.39. *We may not directly rely on the tools of Section 2.4 to improve upon these bounds. In our context, the T-product lift (Subsection 2.4.1) and [d]-lift (Subsection 2.4.3) lead us to non-linear polynomial inequalities, requiring tools beyond LMIs to solve. For example, for a 2-product lift, the multinorms inequalities would be as follows:* $\forall e_1=(s,d,\sigma), e_2=(d,d',\sigma')\in E$:

$$\forall x\in\mathbb{R}^n : \|(A_{\sigma'}+B_{\sigma'}K_{e_2}C_{\sigma'})(A_\sigma+B_\sigma K_{e_1}C_\sigma)x\|_{d'} \leq \|x\|_s.$$

When taking quadratic norms, these equations do not form LMIs. Nevertheless, these inequalities are systems of polynomial inequalities. We can therefore decide their feasibility using quantifier elimination techniques [CJ12]. This shows that we can approximate the quantity $\hat{\rho}_{n_K,\,\text{closed}}$ with arbitrary accuracy. It remains unclear whether algorithms with complexity comparable to those of Chapter 2 exist for estimating $\hat{\rho}_{n_K,\,\text{closed}}$ with arbitrary accuracy.

The path-dependent techniques of Subsection 2.4.2, however, provide LMIs from which we can extract controllers that are *path-dependent* relative to the original graph $\mathbf{G}$. As it turns out, we can use *upper bounds* on the ideal performances of these controllers to extract *lower bounds* on the optimal decay rate for an edge-dependent controller. To do this, given a graph $\mathbf{G}$ and an integer $T\geq 0$, we let $\mathbf{G}_T$ be the T-path-dependent lift of $\mathbf{G}$ as defined in Definition 2.40.
Following our notations, the quantity $\hat{\rho}_{n_K,\,\text{closed}}(\mathbf{G}_T,\mathbf{\Sigma})$ represents the smallest decay rate achievable by using a path-dependent controller of the form[5]

$$\hat{\mathbf{\Sigma}} = \{(\hat{A}_p,\hat{B}_p,\hat{C}_p,\hat{D}_p), p \text{ is a path in } \mathbf{G} \text{ with length } T+1\}.$$

Similarly, $\hat{\rho}_{n_K,\,\text{quad}}(\mathbf{G}_T,\mathbf{\Sigma})$ is best guaranteed decay rate achievable if we consider paths-dependent controllers as in the above for which the closed loop system has a quadratic multinorm (see Theorem 4.38).

[5] Effectively this corresponds to one set of control gains per edge of $\mathbf{G}_T$, which are paths of length $T+1$ in $\mathbf{G}$.

Theorem 4.40. *Consider a switching system* $(\mathbf{G}, \mathbf{\Sigma})$ *and an integer* T*. The following holds:*

$$\hat{\rho}_{n_K,\text{ quad}}\left(\mathbf{G}_T, \mathbf{\Sigma}\right) \leq (n+n_K)^{1/2(T+1)}\hat{\rho}_{n_K,\text{ closed}}(\mathbf{G}, \mathbf{\Sigma}).$$

Proof. The proof follows the same idea to that of Theorem 4.38. Consider first an edge-dependent controller $\hat{\mathbf{\Sigma}}$. For a given $T \geq 0$, let $(\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})_T$ be the *T-path-dependent lift* of the closed loop system, and let $\gamma((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})_T)$ be the best CJSR estimate of that system when using quadratic multinorms. We know from Corollary 2.45 that

$$\gamma((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})_T) \leq (n+n_K)^{1/(2(T+1))}\hat{\rho}(\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}}).$$

Following the arguments of the proof of Theorem 4.38, we have that

$$\inf_{\hat{\mathbf{\Sigma}}}\gamma((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})_T) \leq (n+n_K)^{1/(2(T+1))}\hat{\rho}_{n_K,\text{closed}}(\mathbf{G}, \mathbf{\Sigma}),$$

where $\hat{\mathbf{\Sigma}}$ is an *edge-dependent* controller. Finally, observe that

$$\hat{\rho}_{n_K,\text{ quad}}\left(\mathbf{G}_T, \mathbf{\Sigma}\right) \leq \inf_{\hat{\mathbf{\Sigma}}}\gamma((\mathbf{G}, \mathbf{\Sigma}, \hat{\mathbf{\Sigma}})_T).$$

Indeed, the left hand side quantity is obtained by taking the infemum over γ such that (4.31) holds for the system $(\mathbf{G}_T, \mathbf{\Sigma})$. This effectively corresponds to using path-dependent controllers for the original system. The right hand side is obtained by solving the same inequalities, with the additional constraints that all gain parameters for paths ending with the same edge should be equal. Hence, we have an inequality there. This concludes the proof. ■

In practice, the path-dependent methods provide the user with a hierarchy of more and more complex controllers to stabilize a system. If one can not solve the LMIs (4.31) for a path-dependent controller with parameter $T \geq 0$, he still obtains information about the performances of controllers with parameters $K \leq T$. However, as represented in Figure 4.9, the bounds obtained are not necessarily tight.

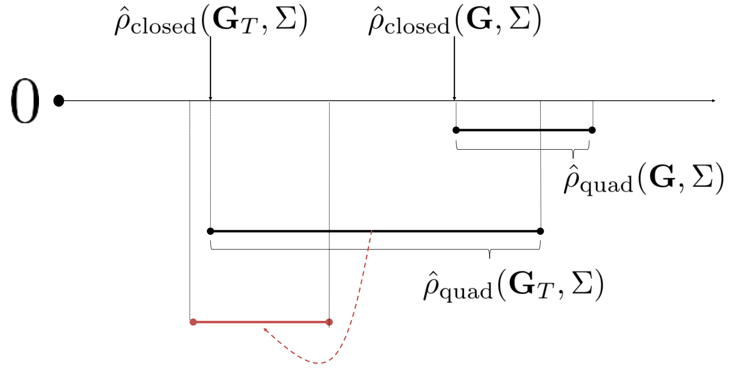


Figure 4.9: How the bounds behave, and what we known about them (we omit n_K for simplicity). We know $\hat{\rho}_{\text{quad}}(\mathbf{G}, \mathbf{\Sigma})$ lies in a close interval upper bounding $\hat{\rho}_{\text{closed}}(\mathbf{G}, \mathbf{\Sigma})$. In relation, $\hat{\rho}_{\text{quad}}(\mathbf{G}_T, \mathbf{\Sigma})$ could a priori lie anywhere within $\hat{\rho}_{\text{closed}}(\mathbf{G}_T, \mathbf{\Sigma})$ and $\hat{\rho}_{\text{quad}}(\mathbf{G}, \mathbf{\Sigma})$.
We can extract lowers bounds on $\hat{\rho}_{\text{closed}}(\mathbf{G}, \mathbf{\Sigma})$ from $\hat{\rho}_{\text{quad}}(\mathbf{G}_T, \mathbf{\Sigma})$, by dividing this estimate by $(n + n_K)^{1/(2(T+1))}$ (in red). This quantity is not guaranteed to be a lower bound on $\hat{\rho}_{\text{closed}}(\mathbf{G}_T, \mathbf{\Sigma})$.

4.4 Summary

In this chapter we considered constrained switching systems with inputs, and approached two main problems: input-output stability and output feedback controller design.

Section 4.1: Systems with Input and Output.

In this section, we introduce with a small examples three problems to be addressed: deciding the exponential stability of an initial condition, the study of input-output stability, and the design of output-feedback controllers.

Section 4.2: Input-Output Stability.

We characterize the $\mathcal{L}_p$-gains of constrained switching systems using multiple storage functions with one piece per node of the switching graph (Theorems 4.6 and 4.8). We assume internal stability and minimality - both assumptions are discussed in Subsection 4.2.2. Minimal realizations can be used to analyze stability from an initial condition (Subsection 4.2.3). We then focus on the $\mathcal{L}_2$-gain and provide asymptotically tight lower and upper bounds (Propositions 4.23 and 4.24). Finally, we provide a converse theorem for the existence of quadratic multiple storage functions (Theorem 4.27).

Section 4.3: Feedback Stabilization.

We study the performances of *edge-dependent* output-feedback controllers, with one control gain per edge in the graph (Definition 4.32). We discuss their synthesis (Subsection 4.3.1), and provide lower and upper bounds on the best achievable growth rate for such controllers (Theorems 4.38 and 4.40).

There remains plenty of challenges to solve regarding both the input-output stability of constrained switching system and their control. Relative to our work, we highlight the following for further study:

Question 4.1. *Can we deal with more general constrained switching systems through minimal realization techniques?*

It has been noticed in Remark 4.21 that the constructions in Subsection 4.2.3 should allow to specify initial states and nodes in a constrained switching systems, and perform stability analysis there.

This is interesting in the fact that this pre-processing of a system would allow us to deal with richer classes of constraints on the switching sequences.

Question 4.2. *Can we approximate accurately the $\mathcal{L}_2$-gain of (internally stable)* arbitrary *switching systems in a time polynomial in the size of the system (for a fixed accuracy, and fixed number of modes)?*

To the best of our knowledge, no answers to this question have been provided. An argument in favor of a positive answer would be the fact that this question, and the tools used to approximate the $\mathcal{L}_2$-gain, share a lot of similarities with the question and tools related to the approximation of the CJSR (there, the answer is positive [BN05]). An argument against is that the $\mathcal{L}_2$-gain behaves differently than the CJSR, and in particular is not homogeneous in the dynamics. It may therefore be the case that it presents challenges nonexistent when considering the CJSR.

Question 4.3. *Given an exponentially stable constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, let us define the* overshoot *as follows:*

$$\inf_{K \geq 1} : \forall x_0 \in \mathbb{R}^n, \forall t \in \mathbb{N}, \forall w \in \mathscr{L}(\mathbf{G}) : \|x(t, x_0, w)\|_2 \leq K \|x_0\|_2.$$

Can we compute/approximate to any accuracy that quantity?

The question is of course related to the material of Chapter 2. The proof of Proposition 4.9 relies on making a link between that overshoot of a system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$ and the $\mathcal{L}_2$-gain of a constrained switching system with input and outputs constructed from $\boldsymbol{\mathcal{S}}$.

Question 4.4. *Can we approximate accurately the quantity $\hat{\rho}_{n_K,\,\mathrm{closed}}(\mathbf{G}, \mathbf{\Sigma})$ of Definition 4.34 (smallest decay rate achievable by edge-dependent controllers) of constrained switching systems in a time polynomial in the size of the system (for a fixed accuracy, and fixed graph)?*

Here, there is a difference with the case of the $\mathcal{L}_2$ gain. More preciesly, we know we can approximate $\hat{\rho}_{n_K,\,\mathrm{closed}}$ arbitrarily accurately in finite time (see Remark 4.39).

Part II

Path-Complete Methods

5 | The Flexibility of Path-Complete Methods

The object of this chapter is an analysis of a wide class of Lyapunov methods for the stability analysis of switching systems known as *path-complete methods*. They were initially introduced for the analysis of *arbitrary switching systems* in [AJPR14], and are naturally extended by our work in Part I. The chapter reports the results of [AAJP17a] and is organized as follows.

- Section 5.1 begins with a general introduction to path-complete methods, and then discusses their applications within the scopes of Chapters 2, 3 and 4. Path-complete methods formulate stability certificates under the form of a multiple Lyapunov function, where the Lyapunov inequalities between a finite set of Lyapunov functions candidates $\mathcal{V}$ is given by a directed and labeled graph.

- Section 5.2 we highlight a fundamental property of path-complete methods. In the case of arbitrary switching systems, Path-Complete methods provide us multiple Lyapunov functions from which we can *always* extract a *common Lyapunov function*, expressed as a finite composition of minima and maxima of the pieces of the multiple Lyapunov function. This bridges an existing gap between methods using complex common Lyapunov functions, and multiple Lyapunov functions.

5.1 Graphs and (Multiple) Lyapunov Functions

We discuss here stability certificates for general, non-linear, switching systems - with minimal assumptions on the dynamics, in order to put our focus on the stability certificates themselves.

The systems under consideration here take the form

$$x(t+1) = f_{\sigma(t)}(x(t)), \tag{5.1}$$

with $x(t) \in \mathbb{R}^n$. Similarly to what we did so far, we let M be the number of modes of the system, hence $\sigma(t) \in \langle M \rangle$ and each mode is represented by a map $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$. We denote the set of modes by $\mathbf{F} = \{f_\sigma, \sigma \in \langle M \rangle\}$.
We make two assumptions on the dynamics of each mode.

Assumption 5.1. *The dynamics have a unique equilibrium at the origin*

$$\forall \sigma \in \langle M \rangle : f_\sigma(x) = 0 \Leftrightarrow x = \mathbf{0}.$$

Assumption 5.2. *The dynamics are bounded in the following sense*[1]*:*

$$\exists \delta \in \mathcal{K}_\infty, \forall x \in \mathbb{R}^n, \forall \sigma \in \langle M \rangle : \ \|f_\sigma(x)\| \leq \delta(\|x\|). \tag{5.2}$$

When considering *linear dynamics*, with a set of M matrices $\mathbf{A}$, we can pick

$$\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \delta(r) = \left(\max_{A \in \mathbf{A}} \|A\| \right) r.$$

The notations used in the linear case naturally carry over to the non-linear case. Given a graph $\mathbf{G}$, the constrained switching system on the dynamics (5.1) is denoted by $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{F})$. For a finite word $w = \sigma_1 \ldots \sigma_k \in \langle M \rangle^*$, we let

$$f_w = f_{\sigma_k} \circ \ldots \circ f_{\sigma_1},$$

and given a word $w \in \langle M \rangle^*$, an initial condition $x_0 \in \mathbb{R}^n$,

$$x(x_0, w) = f_w(x_0).$$

Throughout this chapter when not specified otherwise, we focus on stability in the following sense (which is direct from Definition 1.3).

Definition 5.3. *The system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{F})$ *is* globally uniformly stable *if there is a* $\mathcal{K}_\infty$*-function* $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ *such that for all* $x_0 \in \mathbb{R}^n$ *and all* $w \in \mathscr{L}(\mathbf{G})$

$$\|x(x_0, w)\| \leq \alpha(\|x_0\|).$$

[1] This is known as $\mathcal{K}$-boundedness in [GGLW14].

To put it simply, the framework was introduced in [AJPR14] in order to answer the question:

> *What are the sets of Lyapunov inequalities such that, when satisfied by Lyapunov function candidates, forms a stability certificate for an arbitrary switching system?*

In order to answer this question, the authors rely on *directed and labeled graphs*, such as the ones we use to encoded rules on switching sequences. More precisely, given a set of functions $V_1, V_2, \dots, V_N$, we build a graph on N nodes, each node being associated to one function. Then, For a given mode $\sigma \in \langle M \rangle$, if the Lyapunov inequality

$$\forall x \in \mathbb{R}^n : V_j(f_\sigma(x)) \leq V_i(x)$$

holds, then we add the edge (i, j, σ) in this graph.

The next subsections aim at presenting the path-complete framework, and at establishing the links between this framework and the problems tackled in Part I of this thesis.

5.1.1 Stability under Arbitrary Switching

A standard approach to certify the stability of a system is to search for a Lyapunov function [Kha02]. A well known stability certificate is the existence of a *common quadratic Lyapunov function* (CQLF) [LA09, Section II-A], which we already discussed in Chapter 2. The existence of such a certificate is only a sufficient condition for stability - not all stable systems have a CQLF. Nevertheless converse results such as the ones of [AS98] allow us to better understand, and *quantify*, how conservative such a certificate is. For quadratic Lyapunov functions and linear switching systems, a CQLF exists if the *joint-spectral radius* of the system is bounded above by $1/\sqrt{n}$ (see also Section 2.3) - in other words, a CQLF is guaranteed to exist if the system converges to the origin at an exponential rate smaller than $1/\sqrt{n}$. This condition becomes harder to satisfy as n increases.
Less conservative criteria can be obtained at the cost of increased computational complexity. For example, we can rely on sum-of-squares polynomials [PJ08], max-of-quadratics [GHT06, MP89] or polytopic Lyapunov functions [BM08]. In the case of sum-of-squares polynomials, a converse result similar to the one mentioned the CQLF exist (see [PJ08], and Subsection 2.4.3 herein). Roughly speaking, there is a function $K : \mathbb{N} \to [0, 1[$, strictly increasing with $\lim_{d \to \infty} K(d) = 1$, such that, given an integer d, if the JSR of the set of matrices is below $K(d) < 1$, the system has a sum-of-square Lyapunov function with degree $2d$. Therefore, as long as a linear switching system is exponentially stable, we can find some sum-of-square Lyapunov function for it. This provides us with *semi-algorithms* for deciding exponential stability: if we know that the JSR is below 1, we can try all values of d until we find a SOS Lyapunov function.

Multiple Lyapunov functions [Bra98, SWM+07, JR98, LD06b, BFT03, DRI02], that are composed of several pieces whose *interaction* forms a stability certificate are also an attractive alternative to common Lyapunov functions. There again, there are converse results, see e.g., [LD06b], that induce *semi–algorithms*, using hierarchies of less and less conservative classes of Lyapunov functions [AL14a]. In particular, the *path-dependent Lyapunov functions* [LD06b,ELD14, PEDJ16b] are multiple Lyapunov functions that are guaranteed to eventually provide a stability certificate when a system is stable, and present converse results similar to those guaranteeing the existence of a CQLF or a SOS Lyapunov function (see [AJPR14] where such result has been proven in the arbitrary switching case).

The main reason for the existence of so many different tools is that they provide stability certificates that are only conservative, and sometimes one tool may be more efficient than another. Non-conservative procedures rely on hierarchies of techniques, and provide only asymptotic guarantees. In view of this, it is crucial to understand which performances can *a priori* be expected from a given criterion.

In an effort to unify and generalize many of the existing techniques for discrete-time switching systems, Ahmadi et al. introduced the *Path-Complete Framework* [AJPR14]. The main idea is to describe a stability certificate given by a Lyapunov function (common or multiple) by its *fundamental constitutive elements*. These are:

1. One or more functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, that need to satisfy a set of properties (such as *Lyapunov function candidates* (Definition 1.11), i.e. positive definite and radially unbounded functions). We may also state the *nature/parametrization* of the function(s) (quadratic or sum-of-squares or polytopic, etc...).

2. A set of Lyapunov inequalities that these functions need to satisfy. These describe the evolution of the values taken by function(s) described above along the trajectories of the system.

The path-complete framework defines and studies *graph-Lyapunov* functions (see [AJPR14, Definition 2.3]), where both items above are inscribed in a graph $\mathbf{G} = (S, E)$. This is illustrated in Example 5.6.

Definition 5.4 (Graph-Lyapunov Function)**.** *Consider a graph* $\mathbf{G} = (S, E)$ *on an alphabet* $\langle M \rangle$*, a set of maps* $\mathbf{F} = \{f_\sigma, \sigma \in \langle M \rangle\}$ *on* $\mathbb{R}^n$*, and a set of* $|S|$ Lyapunov function candidates $\mathcal{V} = \{V_s, s \in S\}$ *on* $\mathbb{R}^n$*. We say that* $\mathcal{V}$ *is a solution to* $\mathbf{G}$ *for* $\mathbf{F}$*, written*

$$glf(\mathbf{G}, \mathcal{V}, \mathbf{F}),$$

if

$$\forall (s, d, w) \in E, \forall x \in \mathbb{R}^n : V_d(f_w(x)) \leq V_s(x). \tag{5.3}$$

In this case, we call the pair $(\mathbf{G}, \mathcal{V})$ *a* graph-Lyapunov function *for the dynamics* $\mathbf{F}$.

Remark 5.5. *The terminology "$\mathcal{V}$ is a solution to $\mathbf{G}$" is natural with Section 2.3 in mind. From an optimization theory point of view, a graph $\mathbf{G}$ describes a set of inequalities (constraints) that need to be satisfied by some functions (variable). Finding a graph-Lyapunov function amounts to satisfying that set of constraints with an appropriate set of functions.*

A graph-Lyapunov function is defined, roughly speaking, by a set of inequalities involving Lyapunov functions and dynamics. In order for a graph-Lyapunov function to be a *stability certificate*, we need more information on the graph itself.

Example 5.6. *Consider the linear switching system*

$$x(t+1) = A_{\sigma(t)}x(t)$$

on two modes, with

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 2 \\ 1/4 & 0 \end{pmatrix}.$$

That system is unstable under arbitrary switching. It suffices to take the sequence $\sigma(\cdot) = 12121212\ldots$ to generate an unstable trajectory from the initial point $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$. Consider the graph $\mathbf{G}$ at Figure 5.1, and the function

$$V_a(x) = \|x\|^2,\ V_b(x) = \|A_2^{-1}x\|.$$

We can verify that $(\mathbf{G}, \{V_a, V_b\})$ is a graph-Lyapunov function for the system.

Remark 5.7. *While the graph-Lyapunov function $(\mathbf{G}, \mathcal{V})$ in Example 5.6 is not a stability certificate under arbitrary switching, it is a stability certificate for the constrained switching system on the same modes where the switching sequences belong to $\mathscr{L}(\mathbf{G})$. This is a connection with the multinorm framework of Chapter 2, which is discussed in Subsection 5.1.2.*

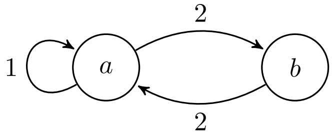
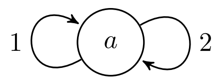


Figure 5.1: Graph for Example 5.1.

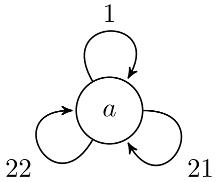
A necessary and sufficient condition under which a graph-Lyapunov function proves stability under arbitrary switching is that the graph $\mathbf{G}$ is *path-complete* (illustrated at Figure 5.2).

Definition 5.8. *A graph $\mathbf{G} = (S, E)$ on the alphabet $\langle M \rangle$ is* path-complete *if $\mathscr{L}(\mathbf{G}) = \langle M \rangle^*$.*

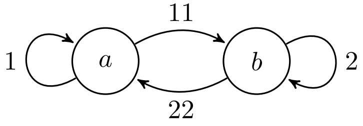
Remark 5.9. *If* $\mathbf{G}$ *is a* path-complete, *a constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{F})$ *is in fact an arbitrary switching system.*



(a) Simple path-complete graph.



(b) Path-complete graph, not in expanded form.



(c) Not a path-complete graph.

Figure 5.2: The graphs on Figure 5.2a and 5.2b are both path-complete, but the graph on Figure 5.2c is not as there are no paths containing the finite sequence 1212.

The next theorem characterizes all graph-Lyapunov functions that provide stability certificates for arbitrary switching systems. The following is taken from [AJPR14, Theorem 2.4] and [JAPR17].

Theorem 5.10. *A graph-Lyapunov function* $(\mathbf{G}, \mathcal{V})$ *for the system (5.1) is a stability certificate under arbitrary switching if and only if* $\mathbf{G}$ *is path-complete. In this case,* $(\mathbf{G}, \mathcal{V})$ *is called a* path-complete *Lyapunov function.*

In the next subsection, we provide a generalization of the result to constrained switching systems, and discuss proof details.

5.1.2 Stability under Constrained Switching

Stability certificates under constrained switching have been investigated in the past (see e.g. [AL14b,LD06b,LD06a,LD07,BFT03], and the references in Chapter 2). The path-complete framework introduced in [AJPR14] allows us to understand which multiple Lyapunov functions (more precisely, which graph-Lyapunov functions) provide stability certificates for *arbitrary switching systems.* These are exactly the graph-Lyapunov functions with a path-complete graph, i.e., whose language is $\langle M \rangle^*$.

Remark 5.11. *Given a system (5.1), a stability certificate under arbitrary switching is also a stability certificate under constrained switching (if all possible switching sequences lead to bounded trajectories, those that belong to a particular language as well).*

In Chapter 5, we develop stability certificates for linear constrained switching systems that were *actually* graph-Lyapunov functions where the pieces are norms, and where the graph of Lyapunov inequalities is the same graph as the one describing the constraints on switching sequences.

Theorem 5.12. *Consider a constrained switching system* $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{F})$*, with* $\mathbf{G}$ *defined on the alphabet* $\langle M \rangle$*. A graph-Lyapunov function* $(\mathbf{G}', \mathcal{V})$*, with* $\mathbf{G}'$ *defined on* $\langle M \rangle$*, is a stability certificate for this system if* $\mathscr{L}(\mathbf{G}) \subseteq \mathscr{L}(\mathbf{G}')$*.*

Remark 5.13. *We expect the condition stated in the above to be necessary and sufficient. The proof should follow from the construction used in the proof of [JAPR17, Theorem 3]. We prefer to emphasize the sufficiency, and provide its proof in a self-contained manner.*

Proof of sufficiency. We have a constrained switching system on M modes, whose switching sequences belong to a language $\mathscr{L}(\mathbf{G})$ generated by a graph $\mathbf{G}$. Assume we have a graph-Lyapunov function $(\mathbf{G}' = (S', E'), \mathcal{V})$, where $\mathscr{L}(\mathbf{G}) \subseteq \mathscr{L}(\mathbf{G}')$. Our aim is to provide a $\mathcal{K}_\infty$ function α such that for any word $w \in \mathscr{L}(\mathbf{G})$, and any $x \in \mathbb{R}^n$, $\|f_w(x)\| \leq \alpha(\|x\|)$. To do so, we rely on upper bounds relative to the functions in $\mathcal{V}$, and to the dynamics (using Assumption 5.2).

Since all functions in $\mathcal{V}$ are Lyapunov function candidates, there are two functions $\alpha_{\mathcal{V}}$ and $\beta_{\mathcal{V}}$ of class $\mathcal{K}_\infty$ such that

$$\forall x \in \mathbb{R}^n, \forall V_s \in \mathcal{V} : \, \alpha_{\mathcal{V}}(\|x\|) \leq V_s(x) \leq \beta_{\mathcal{V}}(\|x\|).$$

Additionally, we let $\delta \in \mathcal{K}_\infty$ be defined as in (5.2). For any $w \in \langle M \rangle^*$, it satisfies

$$\forall x \in \mathbb{R}^n : \, \|f_w(x)\| \leq \delta^{|w|}(\|x\|),$$

where $\delta^{|w|}$ is δ composed with itself $|w|$ times. Without loss of generality, we can take δ satisfying $\forall r \in \mathbb{R}_{\geq 0} : \, \delta(r) \geq r$.
Let K' be the length of the longest word that is a label in $\mathbf{G}'$, i.e.

$$K' = \max_{(s,d,w) \in E'} |w|.$$

Note that K', for any word $w \in \mathscr{L}(\mathbf{G})$ with $|w| \leq K'$, $\|f_w(x)\| \leq \delta^{K'}(\|x\|)$.

Take any word $w \in \mathscr{L}(\mathbf{G})$. Since $\mathscr{L}(\mathbf{G}) \subseteq \mathscr{L}(\mathbf{G}')$, we can factorize w as $w = uw'v$, where the prefix and suffix u and v have a length bounded by K', and where there is a path $p' \in \mathscr{P}(\mathbf{G}')$ such that $w' = w(p')$. Hence, we have for any $x \in \mathbb{R}^n$

$$\begin{aligned}\|f_w(x)\| &= \|f_v \circ f_{w'} \circ f_u(x)\| \\ &\leq \delta^{K'}(\|f_{w'} \circ f_u(x)\|),\end{aligned}$$

since $\|f_v(x)\| \leq \delta^{K'}(\|x\|)$ for any $x \in \mathbb{R}^n$.

For the next step, we take the path p in $\mathbf{G}'$ such that $w' = w(p)$. Let s and d be respectively the source and destination nodes of p'. Since $(\mathbf{G}', \mathcal{V})$ is a graph-Lyapunov function, we deduce from (5.3) that

$$\forall x \in \mathbb{R}^n : \alpha_{\mathcal{V}}(\|f_{w'}(x)\|) \leq V_d(f_{w'}(x)) \leq V_s(x) \leq \beta_{\mathcal{V}}(\|x\|).$$

Because $\mathcal{K}_\infty$ functions are invertible, with their inverse being themselves of class $\mathcal{K}_\infty$, we conclude that

$$\forall x \in \mathbb{R}^n : \|f_{w'}(x)\| \leq \alpha_{\mathcal{V}}^{-1} \circ \beta_{\mathcal{V}}(\|x\|).$$

From there, we can conclude for the word uw' that

$$\begin{aligned}\forall x \in \mathbb{R}^n : \|f_{w'} \circ f_u(x)\| &\leq \alpha_{\mathcal{V}}^{-1} \circ \beta_{\mathcal{V}}(\|f_u(x)\|) \\ &\leq \alpha_{\mathcal{V}}^{-1} \circ \beta_{\mathcal{V}} \circ \delta^{K'}(\|x\|),\end{aligned}$$

because $\mathcal{K}_\infty$ functions are strictly increasing and $\delta(r) \geq r$ by assumption. Therefore, we have

$$\forall x \in \mathbb{R}^n, \forall w \in \mathscr{L}(\mathbf{G}) : \|f_w(x)\| \leq \delta^{K'} \circ \alpha_{\mathcal{V}}^{-1} \circ \beta_{\mathcal{V}} \circ \delta^{K'}(\|x\|).$$

Since the composition of $\mathcal{K}_\infty$ functions is itself $\mathcal{K}_\infty$, this concludes the proof, and the function α from Definition 5.3 can be taken as

$$\alpha = \delta^{K'} \circ \alpha_{\mathcal{V}}^{-1} \circ \beta_{\mathcal{V}} \circ \delta^{K'},$$

where all parameters depend only on the graph-Lyapunov function $(\mathbf{G}', \mathcal{V})$.

■

5.1.3 Reducibility

The techniques developed in (see Chapter 3, Corollary 3.15) show how the broad ideas behind path-complete Lyapunov functions can be used to decide *structural properties* of a system, in particular, the existence of invariant subspaces.

The concept involved here is similar to that of a graph Lyapunov function. On the one hand, we have a graph $\mathbf{G} = (S, E)$ that encodes inequalities. On the other hand, we have a set of *semi-norms* $\mathcal{V} = \{V_s, s \in S\}$. Edges in E encode the same inequalities as in (5.3), and impose invariance properties for the sets

$$\mathcal{X}_s = \{x : V_s(x) = 0\}.$$

More precisely, (5.3) implies the following invariance property

$$\forall (s, d, w) \in E, f_w(\mathcal{X}_s) \subseteq \mathcal{X}_d,$$

where for a set $\mathcal{X}$,

$$f_w(\mathcal{X}) = \{y : \exists (x \in \mathcal{X}), y = f_w(x)\}.$$

5.1.4 Input-Output Stability

In Chapter 4, Section 4.2, we consider systems with input and outputs and are interested by characterizing their input-output gains. The general path-complete framework is once again handy for this task.

Consider a system

$$x(t+1) = f_{\sigma(t)}(x(t), v(t)),\ z(t) = g_{\sigma(t)}(x(t), v(t)),$$

where $x(t) \in \mathbb{R}^n$ is the state, $v(t) \in \mathbb{R}^d$ a disturbance, and $z(t) \in \mathbb{R}^m$ a performance output. The switching sequence is constrained by a graph $\mathbf{G} = (S, E)$ in expanded form. Then, the $\mathcal{L}_p$-gain of the system is smaller than γ if there are Lyapunov function candidates[2] $\mathcal{V} = \{V_s, s \in S\}$ such that

$$\forall (a, b, \sigma) \in E,\ \forall x \in \mathbb{R}^n, v \in \mathbb{R}^d, V_b(f_\sigma(x, v)) + \|g_\sigma(x, v)\|_p^p \leq V_a(x) + \gamma^p \|v\|_p^p.$$

The above is reminiscent of several classical results related to dissipativity and storage functions [Son08, Wil76]. In Chapter 4, Section 4.2, we use these inequalities to characterize gains *exactly* for linear constrained switching systems.

5.1.5 Stabilization

In Chapter 4, Section 4.3, we consider systems with input and outputs and are interested by computing edge-dependent stabilizing controllers. In general, our approach can be summarized as follows.

Consider a system

$$x(t+1) = f_{\sigma(t)}(x(t), u(t)),\ y(t) = g_{\sigma(t)}(x(t)),$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^d$ a control input, and $y(t) \in \mathbb{R}^m$ an observed output. The switching sequence is constrained by a graph $\mathbf{G} = (S, E)$ in expanded form. We can stabilize the system with output-feedback if there exists a function $k : \mathbb{R}^m \to \mathbb{R}^d$ and Lyapunov function candidates $\mathcal{V} = \{V_s, s \in S\}$ such that

$$\forall (a, b, \sigma) \in E, \forall x \in \mathbb{R}^n,\ V_b(f_\sigma(x, k(g_{\sigma(t)}(x))) \leq V_a(x)$$

In Section 4.3, the controller is defined as a constrained switching system which is interconnected with the system to stabilize. The particularity is that the controller is able to track switching sequences on its own graph, knowing at each time step which edge to follow in order to generate the sequence. This allows to define *edge-dependent* controllers. For linear systems, if the order of the controller is high enough, and when using quadratic Lyapunov function candidates, one can compute the controller using LMIs and convex optimization.

[2] Remark that, in Chapter 4, we raise the functions V_s to the power p - this is to highlight the fact that for linear systems these storage functions can be expressed in terms of norms.

5.2 From Path-Complete To Common Lyapunov Functions

As established in the previous section, path-complete methods are a flexible tool through with we may address several stability analysis problem. We focus here on path-complete methods for the deciding the stability of a system. In general, these tools present us with multiple Lyapunov functions as a proof of stability, with exactly one piece per node of the graph $\mathbf{G}$ on which the method is built.

It is well known, however, that for *linear arbitrary switching systems*, the system is stable if and only if it has a common Lyapunov function, which is a norm (see e.g. [Jun09, Proposition 1.4]). More generally, as shown in Section 2.2, a linear constrained switching system is stable if and only if, for any graph generating its switching sequences, there is a multiple Lyapunov function with exactly one norm per node. It is natural to ask ourselves whether these constructions can be related to one another.

In this section, we reveal links between different graph-Lyapunov functions, focusing on arbitrary switching systems. We show that from *any* path-complete-Lyapunov function, one can extract a *common Lyapunov function.*

Example 5.14. *We consider the following* linear *switching system consisting of $M = 2$ modes: $x(t+1) = f_{\sigma(t)}x(t) = A_{\sigma(t)}x(t)$, $\sigma(t) \in \{1, 2\}$, with*

$$A_1 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \alpha \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \alpha = 0.9. \tag{5.4}$$

It is known that this system does not have a common quadratic Lyapunov function [PJ08, AAJP17a]. A graph-Lyapunov function for this system is given by $(\mathbf{G}, \mathcal{V})$, where $\mathbf{G}$ is represented at Figure 5.3, and $\mathcal{V}$ is given by

$$\mathcal{V} = \left\{ \begin{aligned} V_a\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= 5x_1^2 + x_2^2, \\ V_b\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= x_1^2 + 5x_2^2 \end{aligned} \right\}. \tag{5.5}$$

Since $\mathbf{G}$ is path-complete, the graph-Lyapunov function is a stability certificate. This graph-Lyapunov function is represented at Figure 5.4, along with a trajectory. Observe that the yellow sets, which are the level set of the function

$$V(x) = \max\{V_a(x), V_b(x)\},$$

s invariant here.

More precisely, we show that we can always extract a Lyapunov function which is of the form

$$V(x) = \min_{S_1,\dots,S_k \subseteq S} \left(\max_{s \in S_i} V_s(x) \right), \tag{5.6}$$

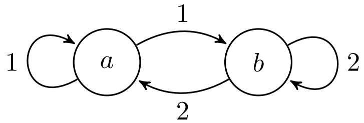


Figure 5.3: A path-complete graph leading to a path-complete Lyapunov function for Example 5.14.

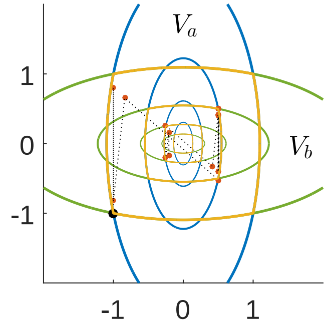


Figure 5.4: Geometric representation of the path-complete stability criterion using the graph G_2 in Example 5.14. A trajectory for the system of Example 5.14, with the black point as initial condition is given. One can show that the intersection (in yellow) of the level sets of the functions V_{a_2} and V_{b_2} (resp. in blue and red) is the level set of a common Lyapunov function for that system [AAJP17a].

for some finite integer k. Our proof is constructive and makes use of a classical tool from automata theory, namely the *observer automaton*, to form subsets of nodes in $\mathbf{G}$ that interact in a well defined manner.

Remark 5.15. *In e.g. [JAPR17, Theorem 5], the authors provided a way to extract common Lyapunov functions from path-complete Lyapunov functions but assumed* exponential stability*. Our tools apply there as well, but do not require the system to be exponentially stable.*

Throughout this section, we are going to focus on graphs $\mathbf{G}$ in *expanded-form*, where the label on each edge is of length 1 (which is not the case, e.g. for the graph of Figure 5.2c). It is easy to extend our results to the more general case (see Remark 5.16).

Remark 5.16 (Dealing with graphs that are not in expanded form)**.** *Given a graph-Lyapunov function for a graph* $\mathbf{G}$*, we can always construct a graph-Lyapunov function for its* expanded form *(see Definition 1.14). To do so, we first construct the expanded form* $\mathbf{G}_e = (S_e, E_e)$ *of* $\mathbf{G}$*, following the technique of Chapter 1, Remark 1.15. There, an edge* $(s, d, w) \in E$*, with* $w = \sigma_1\sigma_2 \dots \sigma_{|w|}$

is split into $|w|$ *edges* (s_i, s_{i+1}, σ_i), $1 \leq i \leq |w|$, *where* $s_1 = s$, $s_{|w|+1} = d$, *and the nodes* s_2 *to* $s_{|w|}$ *are added to the graph (see Figure 5.5).*

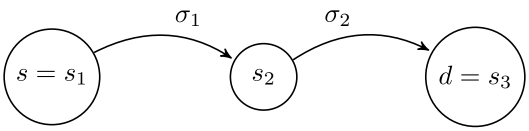


Figure 5.5: An edge in $\mathbf{G}$, $(s, d, \sigma_1\sigma_2)$ (with label of length 2), is replaced by a path of length 2 in the expanded form.

If a set of Lyapunov function candidates $\mathcal{V}$ *is solution to a graph* $\mathbf{G} = (S, E)$ *for the dynamics* $\mathbf{F}$, *we can always construct a set of functions* $\mathcal{W} = \{W_r, r \in S_e\}$ *solution to the expanded form of* $\mathbf{G}$ *for the same dynamics.*
For the path considered above, we would have $W_{s_1} = V_s$, $W_{s_{|w|+1}} = V_d$, *and*

$$W_{s_i} = V_d(f_{\sigma_{|w|}} \circ \ldots \circ f_{\sigma_i}(x)).$$

In Figure 5.5, we would have $W_{s_2}(x) = V_j(f_{\sigma_2}(x))$.

5.2.1 Observer Graph and Induced Lyapunov Functions

In this section, we show that we can always extract from a Path-Complete Lyapunov function an *induced common Lyapunov function* $V(x)$ for the system, that satisfies

$$\forall x \in \mathbb{R}^n, \forall \sigma \in \langle M \rangle, V(f_\sigma(x)) \leq V(x).$$

Similar results have been established in [AJPR14] for special cases where $\mathbf{G}$ is either *complete* or *co-complete*.

Definition 5.17 ((Co)-Complete Graph)**.**
A graph $\mathbf{G} = (S, E)$ *is* complete *if for all* $s \in S$, *for all* $\sigma \in \langle M \rangle$ *there exists* at least one *edge* $(s, d, \sigma) \in E$.
The graph is co-complete *if for all* $d \in S$, *for all* $\sigma \in \langle M \rangle$, *there exists at least one edge* $(s, d, \sigma) \in E$.

Note that the graph of Example 5.14 is a *co-complete* graph.

Lemma 5.18 ([AJPR14, Corollaries 3.4 and 3.5])**.** *Let* $(\mathbf{G}, \mathcal{V})$ *be a graph-Lyapunov function for the system (5.1).*

1. *If* $\mathbf{G}$ *is* complete, *then the function*

$$V(x) = \min_{s \in S} V_s(x)$$

is a common Lyapunov function *for the system.*

2. *If* $\mathbf{G}$ *is* co-complete*, then the function*

$$V(x) = \max_{s \in S} V_s(x)$$

is a common Lyapunov function *for the system.*

Remark 5.19. *The results above apply directly to* path-dependent *Lyapunov functions, see [LD06b, ELD14] and Chapter 2, Section 2.4.2. In particular, if we take* $\mathbf{G}$ *as the path-complete graph with one node and one self-loop per mode, the path-dependent methods of [LD06b] construct* complete graphs*. An horizon-dependent (for [ELD14]) method provides sets of inequalities forming a co-complete graph.*

In order to obtain more general results, holding true for *all path-complete graphs*, we use the concept of *observer automaton* [CL09, Section 2.3.4]. It is adapted from general automata to directed and labeled graphs (see Remark 5.22). The *observer graph* is defined as follows, and its construction is illustrated in Example 5.21.

Definition 5.20 (Observer Graph)**.** *Consider a graph* $\mathbf{G} = (S, E)$*. The* observer *graph* $O(\mathbf{G}) = (S_O, E_O)$ *is a graph where each state corresponds to a subset of* S*, i.e.* $S_O \subseteq 2^S$*, and is constructed as follows:*

1. *Let* $S_O := \{S\}$ *and* $E_O := \emptyset$*.*
2. *Let* $X := \emptyset$*. For each pair* $(P, \sigma) \in S_O \times \langle M \rangle$*:*
   (a) *Compute* $Q := \bigcup_{p \in P} \{q | \, (p, q, \sigma) \in E\}$*.*
   (b) *If* $Q \neq \emptyset$*, set* $E_O := E_O \cup \{(P, Q, \sigma)\}$ *then* $X := X \cup Q$*.*
3. *If* $X \subseteq S_O$*, then the observer is given by* $O(\mathbf{G}) = (S_O, E_O)$*. Else, let* $S_O := S_O \cup X$ *and go to step 2.*

We stress that the nodes of the observer graph $O(\mathbf{G})$ correspond to *sets of nodes* of the graph $\mathbf{G}$.

Example 5.21. *Consider the graph* $\mathbf{G}$ *of Figure 5.6. The observer graph* $O(\mathbf{G})$ *is given on Figure 5.7. The first run through step 2 in Definition 5.20 is as follows. We have* $P = S$*. For* $\sigma = 1$ *the set* Q *is again* S *itself: indeed, each node* $s \in S$ *has at least one inbound edge with the label* 1*. For* $\sigma = 2$*, the nodes* a, c *and* d *have an inbound edge, but not node* b*. Hence, we get* $Q = \{a, c, d\}$*. This set is then added to* S_O *in step 3, and the algorithm repeats step 2 with the updated* S_O*.*

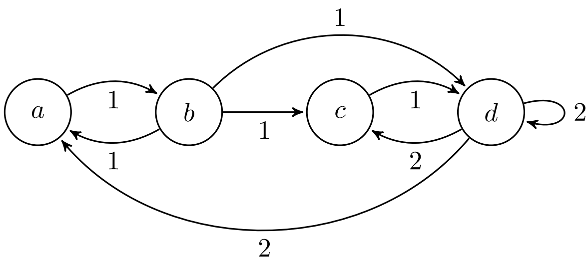


Figure 5.6: Graph for Example 5.21.

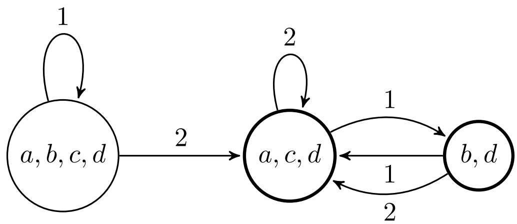


Figure 5.7: Observer graph constructed from the graph $\mathbf{G}$ on Figure 5.6. Each node of the observer $O(\mathbf{G})$ is associated to a set of nodes of $\mathbf{G}$. Notice that the subgraph on the nodes $\{a, c, d\}$, $\{b, d\}$ is itself a *complete* graph.

Remark 5.22. *The* observer automaton *is presented in [CL09, Section 2.3.4]. Generally, an automaton is represented by a directed labeled graph with a* start *state and one or more* accepting *states. The graphs considered here can be easily transformed into non-deterministic automata by using the so-called ϵ-transitions (see [CL09, Section 2.2.4] for definitions). Given a graph $\mathbf{G} = (S, E)$, one can add ϵ-transitions from a new node* "a" *to all node in S and from all nodes in S to a new node* "b"*. The automaton we obtain has the node* "a" *as the start state and the node* "b" *as the (single) accepting state. If we then construct the observer automaton, we obtain the observer graph of Definition 5.20, where all states are accepting, and $S \in S_O$ is the start state.*

One of the main usage of the observer graph is has an intermediate step for constructing the *minimal deterministic automaton* (see [CL09]) that generates a given language.

Definition 5.23 ((Co)-Deterministic Graph)**.**
A graph $\mathbf{G} = (S, E)$ is deterministic *if for all $s \in S$, and all $\sigma \in \langle M \rangle$, there is* at most one *edge $(s, q, \sigma) \in E$.*
The graph is co-deterministic *if for all $q \in S$, and all $\sigma \in \langle M \rangle$, there is at most one edge $(s, q, \sigma) \in E$.*

As discussed in Remark 5.22 our graphs can be seen as non-deterministic automata, and given a language, there are infinitely many of those that can generate it. There is, however, a unique deterministic automaton with the minimal number of states (see e.g. [HMU06, Section 4.4.3]). For generating the full language $\langle M \rangle^*$, that unique automaton is nothing but the one with one node, and a self loop for each mode.

Observe that in Figure 5.7 the subgraph of $O(\mathbf{G})$ with two nodes $\{a, c, d\}$ and $\{b, d\}$ is complete and strongly connected. This is due to a key property of the observer graph.

Lemma 5.24. *The observer graph $O(\mathbf{G}) = (S_O, E_O)$ of any path-complete graph $\mathbf{G} = (S, E)$ contains a* unique *sub-graph $O^\star(\mathbf{G}) = (S_O^\star, E_O^\star)$ which is* strongly connected, deterministic and complete*.*

Proof. The fact that the observer automaton has a complete, deterministic, connected component is well known [CL09, p.90]. From Remark 5.22, the result extends as well to the observer graph.
We prove that this component is unique. For the sake of contradiction, we assume that the observer graph has two complete and deterministic connected components $\mathbf{G}_1 = (S_{O,1}, E_{O,1})$ and $\mathbf{G}_2 = (S_{O,2}, E_{O,2})$. Each component is itself a path-complete graph. Moreover, since they are deterministic and complete, there can never be a path from one component to another.

For word w in $\langle M \rangle^*$, there exists a *unique* path in $O(\mathbf{G})$ with source $S \in S_O$ and label w. Since $\mathbf{G}_1$, $\mathbf{G}_2$ are in $O(\mathbf{G})$, then by construction, there exist two sequences w_1 and w_2 such that there is a path from $S \in S_O$ with label w_1 that ends in a node in $\mathbf{G}_1$ and a path with label w_2 that ends in a node in $\mathbf{G}_2$.

We now consider two paths of infinite length which start from $S \in S_O$. The first has the label $w_1 w_2 w_1 \ldots$, illustrated below,

$$S \rightarrow_{w_1} P^1 \rightarrow_{w_2} Q^1 \rightarrow_{w_1} P^2 \rightarrow_{w_2} Q^2 \cdots$$

and visits the nodes $P^i \in S_{O,1}$ and $Q^i \in S_{O,1}$ after the *ith* occurrence of the sequences w_1 and w_2 respectively. The second path has the label $w_2 w_1 w_2 \ldots$, illustrated below,

$$S \rightarrow_{w_2} R^1 \rightarrow_{w_1} T^1 \rightarrow_{w_2} R^2 \rightarrow_{w_2} T^2 \cdots$$

and visits $R^i \in S_{O,2}$ and $T^i \in S_{O,2}$ after the i-th occurrence of the word w_2 and w_1 respectively.

Since $\mathbf{G}_1$ and $\mathbf{G}_2$ are disconnected, we know that $S \neq P^i \neq T^i$ and $S \neq Q^i \neq R^i$. Thus, $P^1 \subset S$. By construction, this implies $Q^1 \subset R^1$, since these are the set of nodes reachable respectively from P^1 and S with the word w_1. By iteration, we have for all i, it holds that $Q^i \subset R^i$ and $P^{i+1} \subset T^i$. By symmetry, we have that $T^i \subset P^i$ and $R^{i+1} \subset Q^i$ for all i as well. Consequently, we observe that $|P^{i+1}| \leq |P^i| - 2$, thus, necessarily, $|P^{|S|-1}| = 0$, which is a contradiction since $O(\mathbf{G})$ by construction cannot have empty nodes. Thus, $O(\mathbf{G})$ has a unique, strongly connected, deterministic and complete sub-graph. ∎

We are now in position to introduce the main result of this chapter.

Theorem 5.25 (Induced Common Lyapunov Function)**.** *Consider a path-complete Lyapunov function* $(\mathbf{G} = (S, E), \mathcal{V} = \{V_s,\, s \in S\})$ *for the system (5.1). Let* $O^\star(\mathbf{G}) = (S_O^\star, E_O^\star)$ *be the complete and connected sub-graph of the observer* $O(\mathbf{G})$*. Then, the function*

$$V(x) = \min_{Q \in S_O^\star} \left(\max_{s \in Q} V_s(x) \right) \tag{5.7}$$

is a Common Lyapunov function *for the system* (5.1)*.*

The result is illustrated in the following example, and its proof is provided in Subsection 5.2.2.

Example 5.26. *Consider the graph* $\mathbf{G}$ *of Figure 5.6 and its observer graph in Figure 5.7. For this observer graph, the unique, strongly connected, deterministic and complete component* $O^\star(\mathbf{G}) = (S_O^\star, E_O^\star)$ *has* $S_O^\star = \{\{a, c, d\},\, \{b, d\}\}$*. Thus, if* $\mathbf{G}$ *is feasible for a set of functions* $\mathcal{V} = \{V_a, V_b, V_c, V_d\}$*, from Theorem 5.25, we conclude that*

$$V(x) = \min\left\{\max\left(V_a(x), V_c(x), V_d(x)\right),\, \max\left(V_b(x), V_d(x)\right)\right\} \tag{5.8}$$

is a common Lyapunov function. Figure 5.8a presents an illustration of the level sets of a function of the form (5.8)*. Note that this level set is not necessarily convex, which shows the expressive power of path-complete criteria.*
For illustration purposes, Figure 5.8b shows how the Lyapunov inequalities inferred by the graph $\mathbf{G}$ *are involved in certifying that the function above is indeed a common Lyapunov function*[3]*.*

The next subsection is devoted to the proof (and the presentation of the tools for the proof) of Theorem 5.25.

5.2.2 Lyapunov Inequalities Between Sets of Pieces

In order to extract a common Lyapunov function from a Path-Complete one, we need to unfold new Lyapunov valid inequalities implied by the graph. The following results expose relations between subsets of states of a graph $\mathbf{G} = (S, E)$ that lead to Lyapunov inequalities between the corresponding subsets of pieces of a Path-Complete Lyapunov function. These intermediate results are central to the proof of Theorem 5.25.

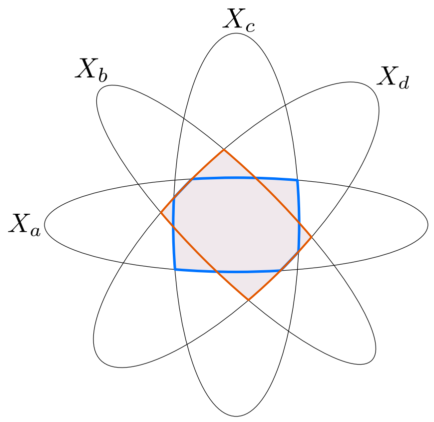
Proposition 5.27. *Consider the system* (5.1) *and a graph-Lyapunov function* $(\mathbf{G} = (S, E), \mathcal{V})$ *for the system. Take two subsets* P *and* Q *of* S*. If there is a label* σ *such that*

$$\forall p \in P,\, \exists q \in Q : \, (p, q, \sigma) \in E, \tag{5.9}$$

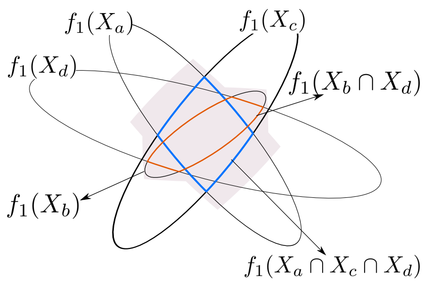
then

$$\min_{q \in Q} V_q(f_\sigma(x)) \leq \min_{p \in P} V_p(x).$$

[3]The figure only provides a geometric interpretation - it does not need to correspond to some real dynamics.



(a) A graphical illustration of the level set of V at Eq. (5.8) in Example 5.26. The unit sublevel sets X_s, $s \in \{a, b, c, d\}$ of the functions $(V_s)_{s\in S}$ are ellipsoids. The level set of $V(x)$ is the union of two sets: the set $X_a \cap X_c \cap X_d$ (in blue) and the set $X_b \cap X_d$ (in orange).



(b) A graphical illustration of the Lyapunov inequalities for Example 5.26. Let $(X_s)_{s\in S}$, the level sets of the functions $(V_s)_{s\in S}$ be such as in Figure 5.8a. The image of the set X_s set through f_1 is $f_1(X_s) = \{f_1(x),\, x \in X_s\}$. From an edge $(s, d, 1) \in E$ of the graph $\mathbf{G}$, we can infer that $f_1(X_s) \subseteq X_d$ since $V_s(x) \geq V_d(f_1(x))$. We can infer more refined relations by taking several edges. For example, from the edges $(b, a, 1)$, $(b, c, 1)$, and $(b, d, 1)$, we infer that $f_1(X_b) \subseteq X_a \cap X_c \cap X_d$. Taking all edges of the form $(s, d, 1)$ into account, we observe that the level set of $V(x)$ (in gray) at Eq. (5.6) is mapped into itself through f_1.

Figure 5.8: Illustrations for Example 5.26.

Proof. Take any $x \in \mathbb{R}^n$. There exists a node $p^\star \in P$ such that

$$\min_{p\in P} V_p(x) = V_{p^\star}(x).$$

Also, there is at least one edge $(p^\star, q^\star, \sigma) \in E$, with $q^\star \in Q$. Thus, $V_{q^\star}(f_\sigma(x)) \leq$

$V_{p^\star}(x)$ and taking into account that $\min_{q\in Q} V_q(f_\sigma(x)) \leq V_{q^\star}(f_\sigma(x))$ the result follows. ■

Proposition 5.27 provides the proof for the first part (complete graphs) in Lemma 5.18.

Proposition 5.28. *Consider the system* (5.1) *and a graph-Lyapunov function* $(\mathbf{G} = (S, E), \mathcal{V})$ *for the system. Take two sets of nodes* P *and* Q*. If there is a label* σ *such that,*

$$\forall q \in Q, \exists p \in P : \ (p, q, \sigma) \in E, \tag{5.10}$$

then

$$\max_{q\in Q} V_q(f_\sigma(x)) \leq \max_{p\in P} V_p(x).$$

Proof. Take any $x \in \mathbb{R}^n$. There exists a node $q^\star \in Q$ such that

$$\max_{q\in Q} V_q(f_\sigma(x)) = V_{q^\star}(f_\sigma(x)).$$

Also, since there exists a node $p^\star \in P$ such that $(p^\star, q^\star, \sigma) \in E$, it holds that $V_{q^\star}(f_\sigma(x)) \leq V_{p^\star}(x) \leq \max_{p\in P} V_p(x)$ and the result follows. ■

Proposition 5.28 provides the proof for the second part (co-complete graphs) in Lemma 5.18.

Proof of Theorem 5.25. Take a path-complete Lyapunov function with a graph $\mathbf{G} = (S, E)$ and pieces $\{V_s, s \in S\}$. Then, construct the *observer graph* $O(\mathbf{G}) = (S_O, E_O)$. By definition, there is an edge $(P, Q, \sigma) \in E_O$ if and only if $Q = \cup_{p\in P}\{q | \, (p, q, \sigma) \in E\}$, and therefore, the following property holds for such edges: $\forall q \in Q$, $\exists p \in P$ such that $(p, q, \sigma) \in E$. Consequently, from Proposition 5.28, we have that

$$\forall x \in \mathbb{R}^n, \forall (P, Q, \sigma) \in E_O : \max_{q\in Q} V_q(f_\sigma(x)) \leq \max_{p\in P} V_p(x).$$

Therefore the set of functions $\mathcal{W} = \{W_P(x), P \in S_O\}$, where

$$\forall P \in S_0 : W_P(x) := \max_{p\in P} V_p(x),$$

if a solution for the graph $O(\mathbf{G})$.
From Lemma 5.24, there exists a sub-graph $O^\star(\mathbf{G}) = (S_O^\star, E_O^\star)$ of $O(\mathbf{G})$ (with $S_O^\star \subseteq S_O$) which is complete and strongly connected. Since $O(\mathbf{G})$ is feasible for $\mathcal{W}$, its subgraph $O^\star(\mathbf{G})$ is feasible for the $\{W_P\}_{P\in S_O^\star}$. Finally, since by Lemma 5.24 $O^\star(\mathbf{G})$ is complete, we apply Corollary 5.27 and deduce that the function $W(x) = \min_{P\in S_O^\star} W_P(x)$ is a common Lyapunov function for the system. ■

5.2.3 A Dual Approach: The Co-Observer Graph

The proof of Theorem 5.25 relies on both the concepts of completeness and co-completeness. Co-completeness, through Proposition 5.28, allows to form the pieces of the PCLF using the observer graph, and completeness, through Lemma 5.18, allow to aggregate these pieces into a common Lyapunov function.

Using the same ideas, we can propose another way to construct a common Lyapunov function from a path-complete Lyapunov function. There, we use completeness and Proposition 5.27 to form the pieces of a PCLF, and aggregate them using the result about co-complete graphs from Lemma 5.18.

In order to do so, we rely on a dual construction to the observer graph we name the *co*-observer graph:

Definition 5.29 (Co-Observer Graph)**.** *Consider a graph* $\mathbf{G} = (S, E)$*. The* co-observer *graph* $cO(\mathbf{G}) = (S_C, E_C)$ *is a graph where each state corresponds to a subset of* S*, i.e.* $S_C \subseteq 2^S$*, and is constructed as follows:*

1. $S_C := \{S\}$ *and* $E_C := \emptyset$.
2. $X := \emptyset$*. For each pair* $(Q, \sigma) \in S_C \times \langle M \rangle$*:*
   (a) *Compute* $P := \bigcup_{q \in Q} \{p : (p, q, \sigma) \in E\}$.
   (b) *If* $P \neq \emptyset$*, set* $E_C := E_C \cup \{(P, Q, \sigma)\}$ *then* $X := X \cup Q$.
3. *If* $X \subseteq S_C$*, then the co-observer is given by* $cO(\mathbf{G}) = (S_C, E_C)$*. Else, set* $S_C := S_C \cup X$ *and go to step 2.*

Example 5.30. *The co-observer of the graph* $\mathbf{G}$ *of Example 5.21 is given in Fig. 5.9.*

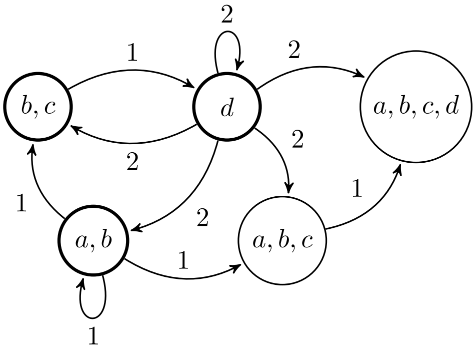


Figure 5.9: Co-Observer graph of the graph on Figure 5.6

We begin with a few elementary facts about co-observers.

Definition 5.31. *Given a graph $\mathbf{G} = (S, E)$, we let $\mathbf{G}^\top = (S, E^\top)$ where*

$$E^\top = \{(d, s, \sigma) \mid (s, d, \sigma) \in E\}.$$

that is, $\mathbf{G}^\top$ is $\mathbf{G}$ with the edges reversed.

Lemma 5.32. *The co-observer of a graph $\mathbf{G} = (S, E)$ is given by*

$$cO(\mathbf{G}) = O(\mathbf{G}^\top)^\top.$$

Proof. When constructing the graph $O(\mathbf{G}^\top)$, at every iteration, we obtain the same sets P and Q as those constructed for the graph $cO(\mathbf{G})$. The only difference is that the edges would be reversed in the end, and so $O(\mathbf{G}^\top)^\top = cO(\mathbf{G})$. ■

Lemma 5.33. *The co-observer graph $cO(\mathbf{G}) = (S_C, E_C)$ of any path-complete graph $\mathbf{G} = (S, E)$ contains a* unique *sub-graph $cO^\star(\mathbf{G}) = (S_C^\star, E_C^\star)$ which is* strongly connected, co-deterministic and co-complete*.*

Proof. Given a graph $\mathbf{G}$, we know that $O(\mathbf{G}^\top)$ has a unique deterministic and complete component (Lemma 5.24). By reversing the edges, we get that $cO(\mathbf{G}) = O(\mathbf{G}^\top)^\top$ has a unique co-deterministic and co-complete component. ■

We are now in position to establish a dual version of Theorem 5.25.

Corollary 5.34. *Consider a path-complete Lyapunov function $(\mathbf{G} = (S, E), \mathcal{V})$ for the system* (5.1)*. Let $cO^* = (S_C^*, E_C^*)$ be the co-complete and connected sub-graph of the co-observer $cO(\mathbf{G})$. Then, the function*

$$V(x) = \max_{P \in S_C^*} \min_{s \in P} V_s(x)$$

is a common Lyapunov function for the system.

Proof. The proof of this result can be obtained by repeating the steps leading to that of Theorem 5.25, but using Proposition 5.27 and the co-complete part of Lemma 5.18 instead of Proposition 5.28 and the complete part of Lemma 5.18. ■

The Lyapunov functions provided by both Theorem 5.25 and Corollary 5.34 share the same form, a combination of min and max of the pieces of the CLF, and one can go from one form to the other by simple algebra[4]. It is thus natural to question whether or not both results actually provide the same function. The answer is negative.

[4]Recall that the min and max operations are distributive with each other: $\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$ and $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$.

Example 5.35. *For the graph of Figure 5.6, the Lyapunov functions provided by both Theorem 5.25 and Corollary 5.34 are equal. Indeed, using the co-observer Figure 5.9, the CLF provided by Corollary 5.34 is given by*

$$\begin{aligned} V(x) &= \max_{P \in \{\{a,b\},\{b,c\},\{d\}\}} \min_{s \in P} V_s(x), \\ &= \max(V_d, \min(V_b, \max(V_a, V_c))), \\ &= \min(\max(V_b, V_d), \max(V_a, V_c, V_d)), \end{aligned}$$

which is equal to that produced by Theorem 5.25, see Example 5.26. which can be written as $\max(V_d, \min(V_c, \max(V_a, V_b)))$.

Example 5.36. *Consider the graph of Figure 5.10a. Let V_{obs} and V_{cobs} be respectively Lyapunov functions induced from the observer and co-observer of this graph (see Figures 5.10b and 5.10c). Then, we have $V_{cobs} \leq V_{obs}$ point-wise, since one can verify that*

$$\begin{aligned} V_{obs} &= \min\left(\max(V_a, V_b, V_c), \max(V_a, V_d, V_e)\right) \\ &\geq \max\left(\min(V_a, V_b, V_d), \min(V_a, V_c, V_e)\right) = V_{cobs}, \end{aligned}$$

which is represented graphically on Figure 5.11.

In the following, we prove that given a graph $\mathbf{G} = (S, E)$, if V_{obs} is the Lyapunov function produced by the observer graph, and V_{cobs} that produced by the co-observer graph, then necessarily

$$V_{cobs}(x) \leq V_{obs}(x), \; \forall x \in \mathbb{R}^n.$$

The proof is based on the following Lemma, which highlights a relation between nodes of the observer and those of the co-observer.

Lemma 5.37. *Consider a path-complete graph $\mathbf{G} = (S, E)$. For all sets $P \subset S$ associated to a node in the observer graph of $\mathbf{G}$ and all sets $Q \subset S$ associated to a node in its co-observer graph,*

$$P \cap Q \neq \emptyset.$$

Proof. First, recall that if $\mathbf{G} = (S, E)$ is path-complete, then for any word w, there exists a unique path in the observer (resp. co-observer) starting at the node S (resp. ending at S) carrying the word in the observer (resp. co-observer).

Take P, a node in the observer graph and let

$$W_P^O = \{w \in \langle M \rangle^* \mid w = w(p), \; p \text{ is a path from } S \text{ to } P \text{ in } O(\mathbf{G})\}.$$

The following must hold: a node s is in P if and only if, for all $w \in W_P^O$, there is a path p in $\mathbf{G}$ with $w(p) = w$ and p ends at s.

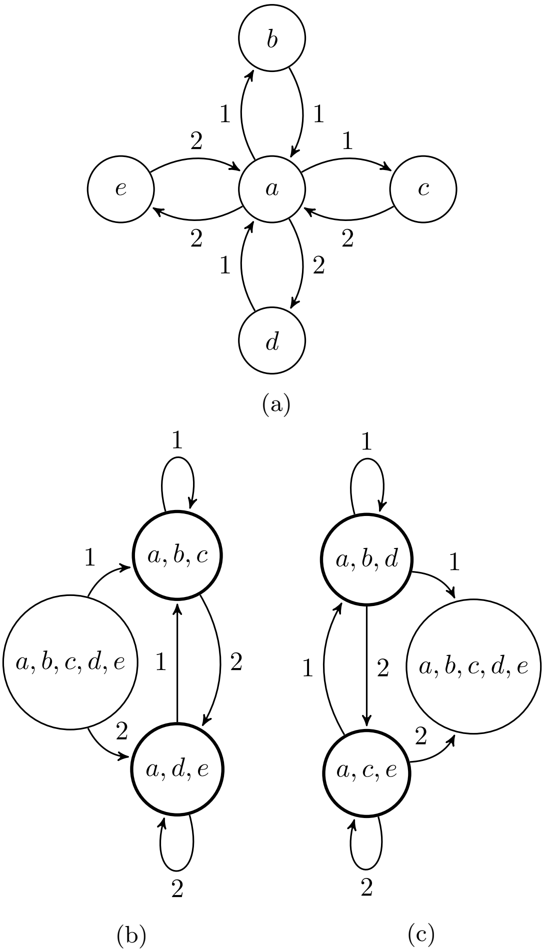


Figure 5.10: A graph (5.10a), its observer (5.10b) and co-observer (5.10c), for Example 5.36.

We do the same for the co-observer. Let Q be a node of the co-observer graph. Let

$$W_Q^C = \{w \in \langle M \rangle^* \mid w = w(p),\ \ p \text{ is a path from } Q \text{ to } S \text{ in } cO(\mathbf{G})\}.$$

Then, $s \in Q$ if and only if $\forall w \in W_Q^C$, there is a path p in $\mathbf{G}$ with $w = w(p)$ and p starts at s.

Pick a node P in the observer graph, and a node Q in the co-observer graph. Pick any word $w_O \in W_P^O$, and any word $w_C \in W_Q^C$. Consider the word $w_O w_C$. Any path in $\mathbf{G}$ carrying w_O ends at some node in P, and any path carrying w_C starts at some node in Q. Therefore, any path carrying $w_O w_C$ has to visit,

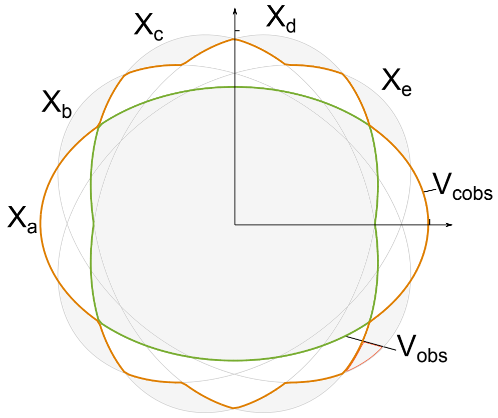


Figure 5.11: Graphical illustration of the level sets of V_{cobs} (orange) and V_{obs} (green).

right after generating the word w_O, a node which must be inside both P and Q. Consequently, since $\mathbf{G}$ is path-complete, and thus must contain a path with the label $w_O w_C$, it must be so that $P \cap Q \neq \emptyset$. Repeating the process for any nodes P and Q concludes the proof.

■

Proposition 5.38. *Consider a PCLF* $(\mathbf{G} = (S, E), \mathcal{V})$*, and let* V_{obs} *and* V_{cobs} *be the common Lyapunov functions obtained by applying Theorem 5.25 and Corollary 5.34 respectively. Then the following holds :*

$$\forall x \in \mathbb{R}^n : V_{obs}(x) \geq V_{cobs}(x).$$

Proof. For the sake of clarity, in this proof, we will describe the Lyapunov functions V_{obs} and V_{cobs} in terms of unions and intersections of levels lets of the functions $\mathcal{V} = \{V_s, s \in S\}$. We let $X_s = \{x \mid V_s(x) \leq 1\}$ be the one-level set of V_s.
Take any set $\mathbf{S} = \{S_i, i = 1, \ldots, k\}$, where each element of $\mathbf{S}$ is a subset $S_i \subset S$. We define the two following sets for each of such set:

$$K(\mathbf{S}) = \cup_{S_i \in \mathbf{S}} \cap_{s \in S_i} X_s,$$

and

$$L(\mathbf{S}) = \cap_{S_i \in \mathbf{S}} \cup_{s \in S_i} X_s.$$

If $\mathbf{S}$ is the set of nodes of an observer (or co-observer), then $K(\mathbf{S})$ is the level set of the function V_{obs} (and $L(\mathbf{S})$ that of V_{cobs}).

We seek to show that, if $\mathbf{P} = \{P_i, i = 1, \ldots, k\}$ is the set of nodes of the observer and $\mathbf{Q} = \{Q_j, j = 1, \ldots, \ell\}$ that of the co-observer, then $K(\mathbf{P}) \subseteq L(\mathbf{Q})$. This is true if and only if

$$\forall i = 1, \ldots, k : \cap_{s \in P_i} X_s \subseteq \cap_{j=1,\ldots,\ell} \cup_{s \in Q_j} X_s.$$

From Lemma 5.37, we know that for each pair P_i, Q_j, the intersection of the two sets is non-empty. We pick a node $s_{i,j} \in P_i \cap Q_j$ for each of these pairs (it does not matter which node, as long as it is in the intersection).

For a fixed P_i, we can write

$$\cap_{s \in P_i} X_s \subseteq \cap_{j=1,\dots,\ell} X_{s_{i,j}} \subseteq \cap_{j=1,\dots,\ell} \cup_{s \in Q_j} X_s,$$

where the first inclusion is due to the fact that $P_i \supseteq \cup_{j=1,\dots,\ell} s_{i,j}$, and the second to the fact that $s_{i,j} \in Q_j$. From there, considering the union of all P_i and the inclusions above, we conclude the proof. ■

5.2.4 Limitations of Path-Complete Lyapunov Functions

In this subsection we investigate whether or not *any* Lyapunov function of the form (5.6) can be induced from a path-complete graph with as many nodes as the number of pieces of the function itself. To do so, we first focus on common Lyapunov functions of the form $V(x) = \max_{1 \leq i \leq N} V_i(x)$, for linear systems and with quadratic pieces. Necessary and sufficient conditions for the existence of such functions exist, relying on *Lyapunov-Metlzer* inequalities (see [GHT06], and [HKD16] for more on these inequalities), that are bilinear matrix inequalities.

Not all Lyapunov functions that can be written as a max of pieces correspond to path-complete Lyapunov functions. We provide a counter-example from [GHT06, Example 11]. Consider the discrete-time linear switching system on two modes $x(t+1) = A_{\sigma(t)} x(t)$ with

$$A_1 = \begin{pmatrix} 0.3 & 1 & 0 \\ 0 & 0.6 & 1 \\ 0 & 0 & 0.7 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0.3 & 0 & 0 \\ -0.5 & 0.7 & 0 \\ -0.2 & -0.5 & 0.7 \end{pmatrix}. \tag{5.11}$$

The system has a max-of-quadratics Lyapunov function

$$V(x) = \max\{V_1(x), V_2(x)\},$$

with $V_i(x) = \left(x^\top Q_i x\right)$, $Q_i \succ 0$. An explicit Lyapunov function is given by[5]

$$Q_1 = \begin{pmatrix} 36.95 & -36.91 & -5.58 \\ \cdot & 84.11 & -38.47 \\ \cdot & \cdot & 49.32 \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} 13.80 & -6.69 & 4.80 \\ \cdot & 21.87 & 10.11 \\ \cdot & \cdot & 82.74 \end{pmatrix}.$$

We first observe that these quadratic functions cannot be the solution of a path-complete stability criterion for our example. Indeed, let us draw the

[5] Such a function can be found numerically by solving the Lyapunov-Metzler inequalities of [GHT06, Section 5] for a choice of $\lambda_{:,:,1} = \left(\begin{smallmatrix} .627 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

graph of all the valid Lyapunov inequalities. More precisely, we define the graph $\mathbf{G} = (\{1,2\}, E)$ with two nodes and

$$(i, j, \sigma) \in E \Leftrightarrow A_\sigma^\top Q_j A_\sigma - Q_i \preceq 0, \tag{5.12}$$

i.e. the matrix $A_\sigma^\top Q_j A_\sigma - Q_i$ is negative semi-definite. The graph obtained is presented on Figure 5.12. This graph is not path-complete, and thus we cannot form a Common Lyapunov Function, as done in the previous section, with these two particular pieces. However, we can go further and investigate

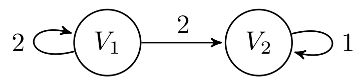


Figure 5.12: The valid Lyapunov inequalities for the quadratic functions for the system and the Lyapunov function in [GHT06, Example 11] are represented by the graph above. The graph is not path complete.

whether *another* pair of quadratic functions would exist, which we could find by solving a path-complete criterion, and such that their maximum would be a valid CLF. Recall that co-complete graphs induce Lyapunov functions of the form $\max_{v \in S} V_v(x)$ (see Lemma 5.18).

Proposition 5.39. *Consider the discrete-time linear system with two modes (5.14). The system does not have a path-complete Lyapunov function with quadratic pieces defined on co-complete graphs with 2 nodes.*

Proof. From Definition 5.17, there is a total of 16 graphs that are co-complete and consist of two nodes and four edges (1 edge per mode and per state). We do not examine co-complete graphs with more than four edges since satisfaction of the Lyapunov conditions for these graphs would imply that of the conditions for at least one graph with four edges.
For each graph, the existence of a feasible set of quadratic functions can be tested by solving the LMIs (5.12).
For the system under consideration, none of the 16 sets of LMIs have a solution. Thus, no induced Lyapunov function of the type $\max\{V_1(x), V_2(x)\}$ exists. ∎

Remark 5.40. *One can show that among the 16 graphs involved in the proof of Proposition 5.39, only 3 are relevant (a solution to one of the 13 others would implie a solution for one of the 3). These are the following:*

$$\begin{aligned}
G_1 &= (\{a, b\}, \{(a, a, 1), (a, b, 1), (b, a, 2), (b, b, 2)\}), \\
G_2 &= (\{a, b\}, \{(a, a, 1), (a, b, 1), (a, b, 2), (b, a, 2)\}), \\
G_3 &= (\{a, b\}, \{(a, a, 2), (a, b, 1), (a, b, 2), (b, a, 1)\}).
\end{aligned}$$

Remark 5.41. *For linear systems and for the assessment of asymptotic stability, path-complete Lyapunov functions have been shown to be* universal*. In*

particular, the path-dependent Lyapunov functions introduced in [LD06b] are path-complete Lyapunov functions with a particular choice of complete graphs, specifically, the so-called De Bruijn *graphs.*
The system involved in Proposition 5.39 is asymptotically stable (see [GHT06]). The interest of Proposition 5.39 lies in the fact that there do not exist necessarily path-complete Lyapunov functions with the same number of pieces *as a* max-*type common Lyapunov function. This is a limitation imposed by the combinatorial structure of the Path-Complete Lyapunov function.*

The proof of Proposition 5.39 highlights an interesting fact. Several different path-complete graphs may induce common Lyapunov function with the same form (5.6). However, the strength of the stability certificate they provide may differ. This has a practical implication: if we are given a system of the form (5.1), it is unclear which graph $\mathbf{G}$ we should use to form a Path-Complete Lyapunov function for some number of pieces satisfying a given template (e.g., quadratic functions). This matter is further discussed in Chapter 6.

5.2.5 Numerical Example

In this section, we provide two numerical examples where we construct a common Lyapunov function for an linear switching system with arbitrary switching.

Example 5.42. *We begin by constructing a set of three matrices. First we let*

$$A = \frac{1}{\gamma}\begin{pmatrix} 0.970 & 0.464 \\ 0.213 & 0.5 \end{pmatrix}, B = \frac{1}{\gamma}\begin{pmatrix} 0.4 \\ 0.85 \end{pmatrix}, K = \begin{pmatrix} -0.687 & 0.240 \end{pmatrix},$$

with the parameter[6] $\gamma = 0.917$. *The three modes are defined as*

$$\mathbf{A} = \{A_1, A_2, A_3\},\ A_i = A^{(i-1)}(A+BK)^{(3-(i-1))}.$$

These matrices tell a story similar to the example in Section 2.1. This time, at every three consecutive time steps in a controlled system, either the controller functions properly, or it fails at the third step, or at the second and third steps. We seek to compute a path-complete *Lyapunov function with quadratic pieces for this system, and extract a common Lyapunov function out of it. We attempt this on the graph* $\mathbf{G}$ *on 4 nodes at figure 5.13.*

In order to do so, we first solve the optimization program of Remark 2.15 for the system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$*, with the graph and matrices defined above. This gives us a path-complete Lyapunov functions, with pieces* V_a*,* V_b*,* V_c *and* V_d*at Figure 5.15, that certifies exponential stability with*

$$\hat{\rho}(\boldsymbol{\mathcal{S}}) \leq 0.99.$$

We then construct the observer graph for $\mathbf{G}$*, represented in Figure 5.14. We*

[6]This choice makes the switching system we are building barely stable.

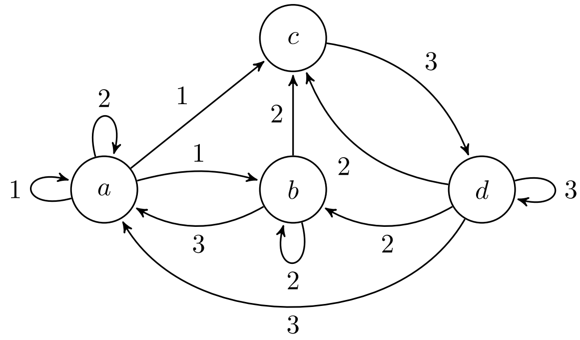


Figure 5.13: Graph **G**.

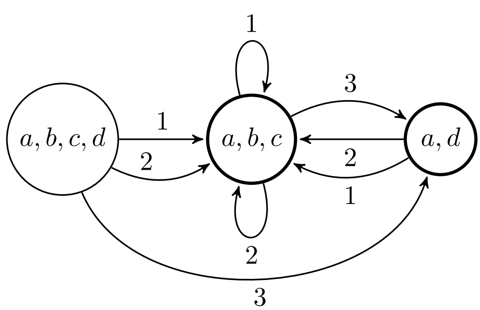


Figure 5.14: Example 5.42. Observer of the graph **G** in Figure 5.13.

compute the two pieces of the observer, i.e. $\max\{V_a, V_b, V_c\}$ *and* $\max\{V_a, V_d\}$*. Their level sets are the* intersection *of those of their respective pieces, represented in Figure 5.16. Finally, the pointwise minimum of the two functions hands us a common Lyapunov function for the system. This is represented at Figure 5.17.*

Example 5.43. *If a switching system has a* Path-Complete Lyapunov function *on the graph* $\mathbf{G} = (S, E)$ *of Figure 5.6 with pieces* $\{V_s, s \in S\}$*, then by inspecting the observer graph* $O(\mathbf{G})$ *at Figure 5.7 we conclude that*

$$\min\{\max(V_a(x), V_c(x), V_d(x)),\ \max(V_b(x), V_d(x))\} \tag{5.13}$$

is a common Lyapunov function for the system*.*
As an illustration, consider the following set of matrices taken from [GHT06, Example 11]:

$$\mathbf{A} = \left\{ \alpha \begin{pmatrix} 0.3 & 1 & 0 \\ 0 & 0.6 & 1 \\ 0 & 0 & 0.7 \end{pmatrix}, \alpha \begin{pmatrix} 0.3 & 0 & 0 \\ -0.5 & 0.7 & 0 \\ -0.2 & -0.5 & 0.7 \end{pmatrix} \right\} \tag{5.14}$$

with the choice[7] *of* $\alpha = 1.03$*. We can show numerically that a* quadratic Path-Complete Lyapunov function $\{V_s, s \in S\}$ *for the graph* $\mathbf{G}$*.*
Figure 5.18 then shows the evolution in time of these functions, and of the common Lyapunov function (5.13)*, from the initial condition* $x(0) = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^\top$*, and for the* periodic *switching sequence repeating the pattern*[8] 21111 *.*

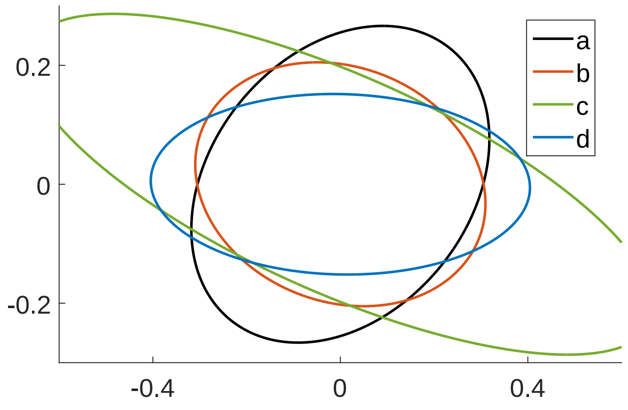


Figure 5.15: Example 5.42. Level sets of the pieces of the path-complete Lyapunov function.

[7] Chosen for cosmetic reasons here.
[8] We computed this using the JSR toolbox [VHJ14] and Gripenberg's algorithm [**?**].

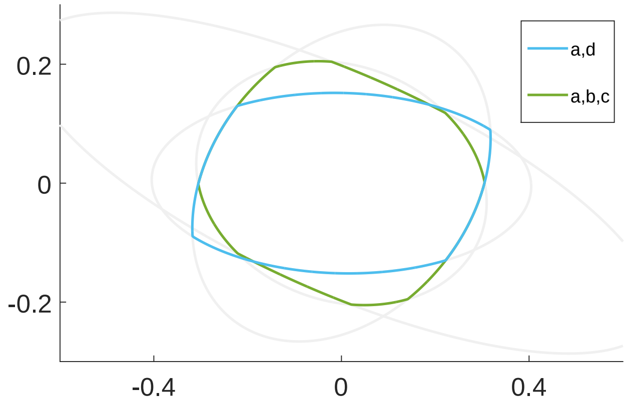


Figure 5.16: Example 5.42. Level sets of the pieces for the observer graph.

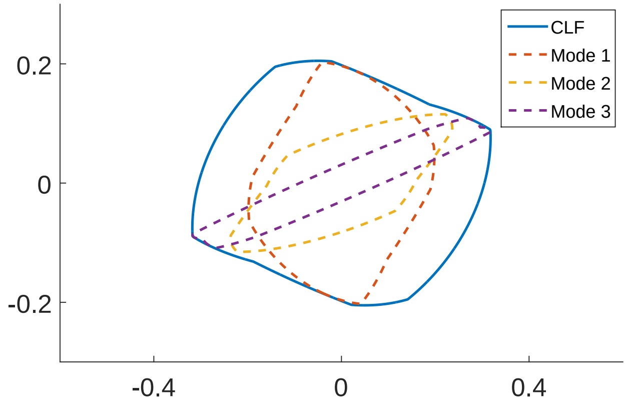


Figure 5.17: Example 5.42. Level sets of the common Lyapunov extracted from the PCLF, and its image through each mode.

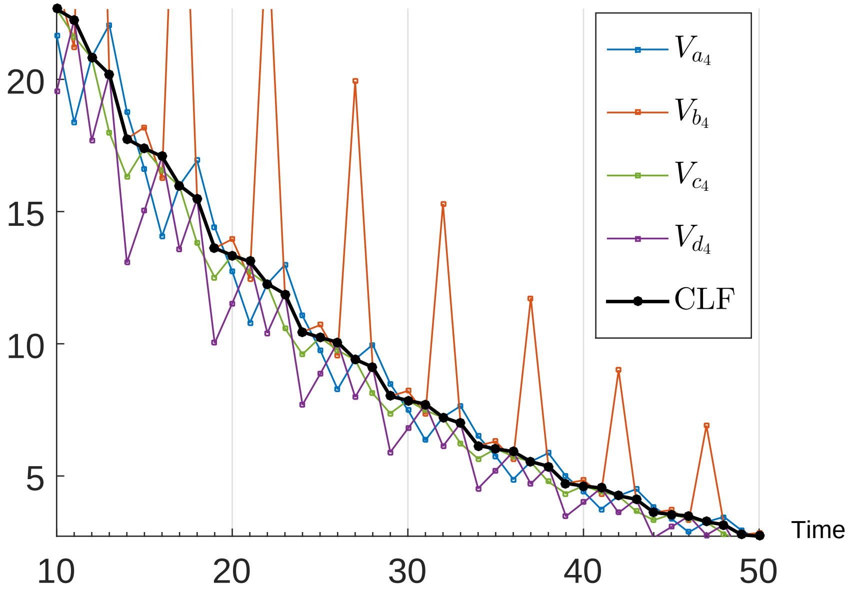


Figure 5.18: Example 5.43. Evolution of a PCLF for the system in Example 5.43 for the graph **G** at Figure 5.6. Observe that none of the pieces of the PCLF decreases monotonically. However, the common Lyapunov function (5.13) (in black) does decrease monotonically.

5.3 Summary

In this chapter, our goal is twofolds, to introduce the reader to the rich path-complete framework, by putting it in the context of Part I, and to further discuss path-complete Lyapunov functions in relation with common Lyapunov functions in the context of arbitrary switching systems.

Section 5.1: Graphs and (Multiple) Lyapunov Functions.

We introduce the reader to the general path-complete framework. We present the fundamental concepts of graph-Lyapunov functions 5.4 and path-complete Lyapunov functions 5.12. These define Lyapunov inequalities using directed and labeled graph of the same sort than the ones used to define admissible switching sequences in a constrained switching system. We then revisit the results of Chapters 2, 3 and 4 in this context, showing the flexibility of the path-complete framework.

Section 5.2: From Path-Complete To Common Lyapunov Functions.

We then focus on path-complete Lyapunov functions, stability certificates for arbitrary switching systems. By exhibiting structures in the path-complete graphs allowing us to extract non-trivial Lyapunov inequalities (Propositions 5.27 and 5.28), we are able to construct, from any given path-complete Lyapunov function, a common Lyapunov function for the corresponding system 5.25. The result relies on the construction of the so-called Observer graph Definition 5.20. A dual class of common Lyapunov functions can be constructed using the so-called co-Observer graph (Subsection 5.2.3).
Finally, we investigate the generality of the obtained common Lyapunov functions (Subsection 5.2.4).

For the link between path-complete and common Lyapunov functions, we highlight the following questions.

Question 5.1. *What are exactly the piecewise quadratic Lyapunov functions that can be induced by path-complete Lyapunov functions? How are they related to those of [GHT06]?*

In Subsection 5.2.4, we discussed the fact that not all max-of-2-quadratic Lyapunov functions can be induced by path-dependent Lyapunov functions on two nodes. To show this, we construct a max-of-quadratics using the tools of [GHT06], relying on Lyapunov-Metzler inequalities to compute these Lyapunov functions. These techniques are, to our knowledge, only conservative for constructing such Lyapunov functions. Here, we are interested in knowing first whether or not the max-of-2-quadatrics obtained using path-complete Lyapunov functions are a special case of those of [GHT06]. It it it the case, then [HKD16] recently exhibited the link between Lyapunov-Metlzer inequalities and the celebrated S-procedure, and this should lead us to novel interpretations regarding path-complete Lyapunov certificates.

Question 5.2. *Given two graphs* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *such that* $\mathscr{L}(\mathbf{G}_1) = \mathscr{L}(\mathbf{G}_2)$, *and a solution to* $\mathcal{V}$ *to* $\mathbf{G}_1$, *how can we construct, in a systematic way, a solution to* $\mathbf{G}_2$ *independently of the dynamics considered?*

Section 5.2 answers this question in the arbitrary switching case. First, using Theorem 5.25, we construct a common Lyapunov function, and we can use that single function to satisfy the constraints of any path-complete graph. It remains to be answered how we can to this in general, for any language.

6 Comparison of Path-Complete Methods

Path-complete methods encompass many stability analysis algorithms. Among these, some produce less conservative stability certificates than others, often at the cost of requiring more computations. However, it is not because a multiple Lyapunov function is harder to compute (because e.g. it has more pieces) than another that it is necessarily less conservative - this depends on factors including the nature of these functions and the sets of Lyapunov inequalities, encoded in graphs here, they need to satisfy.
In this chapter, we seek to understand when, given two families of graph-Lyapunov functions with pieces of similar nature, but different graphs, the first graph-Lyapunov function is less conservative than the other. More precisely, given two graphs, we provide tools that allow to construct a graph-Lyapunov function for the second graph given a graph-Lyapunov function for the first graph.
The results presented here are based of those of [AAJP17a, AAJP17b]. The chapter is organized as follows.

- In Section 6.1, we introduce the problem of *ordering graphs* accordingly to the relative conservatism of the associated graph Lyapunov functions.
- In Section 6.2, we present a combinatorial criterion. Given two graphs, we highlight relations between the two which are sufficient to state that whenever we can build a graph-Lyapunov function for the first graph, we can do so for the other one, by reusing the pieces of the first graph-Lyapunov function.
- In Section 6.3, we take an algebraic route and answer the following question: *Given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, when is it true that whenever one can find a solution $\mathcal{V}$ to the first graph, one can find a solution $\mathcal{U}$ to the second where each function is a combination of the functions in $\mathcal{V}$?* We show that the above has a "*yes*" answer if and only if a simple linear program, involving only the graphs, has a solution.

6.1 Ordering Path-Complete Methods

The path-complete framework provides an elegant and general way to encode multiple Lyapunov functions for switching systems (see Chapter 5). They are used to tackle the very difficult question of knowing whether or not a switching system is stable (see Definition 1.3), which is undecidable in general [BT00a]. In this chapter, our interest lies in providing tools to *order* such stability certificate relative to how conservative they are. Let us introduce the problem through an example.

Example 6.1. *We consider the system of Example 5.14. It is a linear arbitrary switching system $x(t+1) = A_{\sigma(t)}x(t)$ with two modes:*

$$A_1 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \alpha \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \ \alpha = 0.9. \tag{6.1}$$

Let us test four different stability certificates: two are common Lyapunov functions (quadratic and sum-of-squares), two are multiple quadratic Lyapunov functions, and all are path-complete methods.

1. It does not have a common quadratic Lyapunov function (CQLF). *In the path complete framework, a CQLF is a graph-Lyapunov function (see Definition 5.4) for the graph at Figure 6.1 where the unique Lyapunov function candidate involved is a quadratic function.*

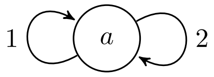


Figure 6.1: Path-complete graph for common Lyapunov functions.

2. It has a common sum-of-squares Lyapunov function with degree 4. *By taking the same graph as before (Figure 6.1) and allowing the Lyapunov function candidate to be more complex, we obtain a stability certificate (see Figure 6.2).*

3. It does not have a multiple quadratic Lyapunov function for the graph on three nodes of Figure 6.3b. *We can verify whether or not such a multiple Lyapunov function exists using LMIs (cfr Section 2.3).*

4. It has a multiple quadratic Lyapunov function for the graph on two nodes of Figure 6.3d. *This certificate is given in Example 5.14.*

It is not that surprising that using a sum-of-square Lyapunov function could provide a stability certificate where using a simple quadratic does not. It is however curious that a multiple-Lyapunov function with two pieces can provide a stability certificate where a multiple Lyapunov function on two pieces does not.

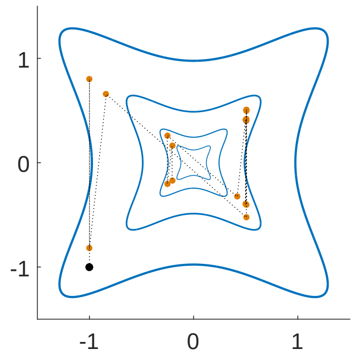


Figure 6.2: Level set of common sum-of-squares Lyapunov function, with an example trajectory (the black dot is the initial condition.

In practice, for some switching systems, it can be straigthforward to obtain a *stability certificate* (proof of stability) using simple Lyapunov functions (such as the celebrated common quadratic Lyapunov function). For others, this can be very difficult. This is well illustrated in Subsection 2.5, where to compute the CJSR of a system in dimension 2, which amounts at deciding stability, we needed a *graph-Lyapunov function* with hundreds of pieces. For the case mentioned above, perhaps there was a more efficient tool, providing us with a stability certificate but requiring less computations, to decide stability.
However, to the best of our knowledge, there is no a priori way to know whether this is the case or not. In our opinion, this justifies as well the existence of the many techniques (e.g. [Bra98, SWM$^+$07, JR98, LD06b, BFT03, DRI02, BM08, PJ08, ELD14], which is non-exhaustive), which triggered the interest in path-complete methods in the first place, as a common tool to unify and study these techniques.
The tools developed in this chapters aim at establishing whether or not, given two classes of graph-Lyapunov functions, with different graphs, and same type of pieces (e.g., quadratic...), one class provides *strictly more conservative* stability certificates[1] than the other. Effectively, we aim at *ordering* graph-Lyapunov functions according to how well they perform, in general, as stability certificates. In this context we know ordering for some particular *hierachies* of multiple Lyapunov functions. For example, the path-dependent hierarchy of [LD06b, ELD14] is ordered, and the T-product lifts of Subsection 2.4.1 are ordered as well[2]. Ordering can also be established across those hierarchies (see [AJPR14, Section 6], or Theorem 2.44 above). To our knowledge, the

[1] Bianchini and Miani in [BM08, Section 1.2.3] propose a very interesting discussion about the notion of *conservativity*.

[2] If we obtain a stability certificate for some value of the parameter T, we have one for all multiples of T.

general problem of ordering path-complete Lyapunov functions as been first approached in similar terms in [AJPR14, Section 4.2], where the ordering was established for several path-complete graphs on one or two nodes. However, to the best of our knowledge, there lacks general criteria that, when satisfied, imply ordering.

We first address the meaning of *a graph-Lyapunov function* is *less-conservative than another.* In their original exposition in [AJPR14], path-complete Lyapunov functions - which are graph-Lyapunov functions on path-complete graphs - with quadratic pieces are used for the *approximation of the joint spectral radius of linear, arbitrary switching systems.* This is extended naturally in Section 2.3 to the estimation of the CJSR of linear constrained switching systems.

In this context, we can relate the conservativeness of a graph-Lyapunov function by the quality of the CJSR estimation it provides. More precisely, for each graph $\mathbf{G} = (S, E)$ on the alphabet $\langle M \rangle$, and for any set of M matrices $\mathbf{A}$ we let

$$\begin{aligned} \gamma_Q(\mathbf{G}, \mathbf{A}) = \inf_{Q,\gamma} & \ \gamma \\ & \forall (s,d,w) \in E : A_w^\top Q_d A_w \preceq \gamma^{2|w|} Q_s, \\ & \forall s \in S : Q_s \succ 0, \end{aligned} \tag{6.2}$$

be the best CJSR estimation attained by a graph-Lyapunov function $(\mathbf{G}, \mathcal{V})$ for the constrained switching system $\boldsymbol{\mathcal{S}} = (\mathbf{G}, \mathbf{A})$, when $\mathcal{V}$ is a set of quadratic functions (see Remark 2.22).

For two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathscr{L}(\mathbf{G}_1) = \mathscr{L}(\mathbf{G}_2)$, we write down the following *ordering relation*:

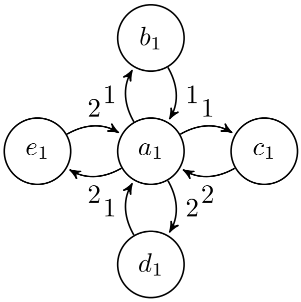
$$\mathbf{G}_1 \leq \mathbf{G}_2 \text{ if } \forall n \in \mathbb{N}, \forall \mathbf{A} \subset \mathbb{R}^{n \times n} : \gamma_Q(\mathbf{G}_1, \mathbf{A}) \geq \gamma_Q(\mathbf{G}_2, \mathbf{A}). \tag{6.3}$$

Remark 6.2. *We are especially interested in graphs sharing a same language, but we can define the ordering for any pair of graphs on a same alphabet. The issue then is that it becomes impossible to discuss ordering in terms of performances as stability certificates.*

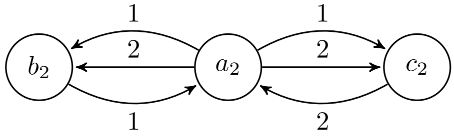
The ordering is defined such as a minimal element provides the *worst* CJSR estimations. For that ordering relation, we may furthermore assume that the matrices in $\mathbf{A}$ are always *invertible* (see [AJPR14, Remark 2.1]).

In Example 6.3, we present the ordering relations between the five *path-complete* graphs, $\mathbf{G}_1$ to $\mathbf{G}_5$, represented at Figure 6.3. The construction we use are inspired from [AJPR14, Sections 4 and 5][3]. These proofs are presented in the more general setup of Chapter 5, with non-linear systems and general pieces for the Lyapunov functions. As a results, the ordering relations are presented in Figure 6.4.

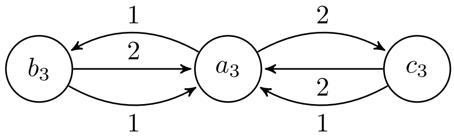
Example 6.3 (Proving an ordering relation)**.** *Consider the graphs* $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3,$ $\mathbf{G}_4$ *and* $\mathbf{G}_5$ *at Figure 6.3.*



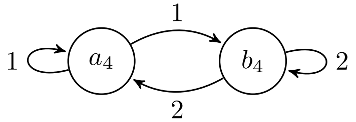
(a) $\mathbf{G}_1 = (S^1, E^1)$.



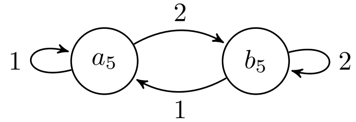
(b) $\mathbf{G}_2 = (S^2, E^2)$.



(c) $\mathbf{G}_3 = (S^3, E^3)$.



(d) $\mathbf{G}_4 = (S^4, E^4)$.



(e) $\mathbf{G}_5 = (S^5, E^5)$.

Figure 6.3: The 5 path-complete graphs ordered in Example 6.3.

1: The CJSR estimation of $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ are equal.

Consider first a solution $\mathcal{V} = \{V_s, s \in S^1)$ to $\mathbf{G}_1$.

- *A solution to $\mathbf{G}_2$ is given by*

$$\mathcal{U} = \{U_{a_2} = V_{a_1}, U_{b_2} = V_{a_1} \circ f_1, U_{c_2} = V_{a_2} \circ f_2\}. \tag{6.4}$$

- *A solution to $\mathbf{G}_3$ is given by*

$$\mathcal{U} = \{U_{a_3} = V_{a_1}, U_{b_3} = V_{a_1} \circ f_1^{-1}, U_{c_3} = V_{a_1} \circ f_2^{-1}\}.$$

[3] In fact, our construction that $\mathbf{G}_1 \leq \mathbf{G}_2$ is part of the proof [AJPR14, Theorem 5.4], where the authors establish that $\mathbf{G}_1 \leq \mathbf{G}_4$.

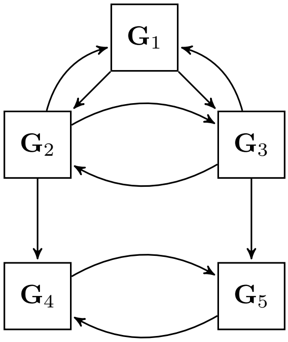


Figure 6.4: Summary of ordering relations between the graphs in Figure 6.3. An arrow between two graphs means that the existence of a solution for the first implies that of a solution for the second.

Hence, if the functions f_i are linear and invertible, whenever there is are quadratic functions solution to $\mathbf{G}_1$, there are quadratic functions[4] *solution to $\mathbf{G}_2$ and functions solutions to $\mathbf{G}_3$.*

Also, for any solution $\mathcal{U} = \{U_{a_2}, U_{b_2}, U_{c_2}\}$ to $\mathbf{G}_2$, we can construct a solution to $\mathbf{G}_1$ as follows:

$$\mathcal{V} = \{V_{a_1} = U_{a_2}, V_{b_1} = V_{d_1} = U_{a_2} \circ f_1, V_{c_1} = V_{e_1} = U_{a_2} \circ f_2\}.$$

A similar approach allows to construct solutions to $\mathbf{G}_1$ from those of $\mathbf{G}_3$. From the above, we can conclude that

$$\forall n \in \mathbb{N}, \forall \mathbf{A} \subset \mathbb{R}^{n \times n} : \gamma_Q(\mathbf{G}_1, \mathbf{A}) = \gamma_Q(\mathbf{G}_2, \mathbf{A}) = \gamma_Q(\mathbf{G}_3, \mathbf{A}).$$

Hence, in terms of ordering, we have for all $1 \leq i, j \leq 3$, $\mathbf{G}_i \leq \mathbf{G}_j$.

2: The graphs $\mathbf{G}_4$ and $\mathbf{G}_5$ provides better estimates than the graphs $\mathbf{G}_2$ and $\mathbf{G}_3$, respectively.

If $\mathcal{V} = \{V_{a_2}, V_{b_2}, V_{c_2}\}$ is a solution for $\mathbf{G}_2$, then

$$\mathcal{U} = \{U_{a_4} = V_{a_2} + V_{b_2}, U_{b_4} = V_{a_2} + V_{c_2}\}$$

is a solution to $\mathbf{G}_4$. This is sufficient to prove that

$$\forall n \in \mathbb{N}, \forall \mathbf{A} \subset \mathbb{R}^{n \times n} : \gamma_Q(\mathbf{G}_4, \mathbf{A}) \leq \gamma_Q(\mathbf{G}_2, \mathbf{A}),$$

hence $\mathbf{G}_4 \geq \mathbf{G}_2$.
In view of (6.4), this also implies $\mathbf{G}_4 \geq \mathbf{G}_1$.
Regarding $\mathbf{G}_3$ and $\mathbf{G}_5$, a similar argument can be used to show $\mathbf{G}_5 \geq \mathbf{G}_3 \geq \mathbf{G}_1$.

[4] If $V(x) = x^\top Q x$, $Q \succ 0$, then $V(Ax) = x^\top (A^\top Q A) x$.

3: The graphs $\mathbf{G}_4$ and $\mathbf{G}_5$ provide the same CJSR estimations

We can form a solution $\mathcal{U}$ to $\mathbf{G}_5$ from a solution $\mathcal{V}$ to $\mathbf{G}_4$ as follows:

$$\mathcal{U} = \{U_{a_5} = V_{a_4} \circ f_1^{-1}, U_{b_5} = V_{b_4} \circ f_2^{-1}\}, \tag{6.5}$$

and a solution $\mathcal{V}$ to $\mathbf{G}_5$ is constructed from a solution $\mathcal{U}$ to $\mathbf{G}_4$ as follows

$$\mathcal{V} = \{V_{a_5} = U_{a_4} \circ f_1, V_{b_5} = U_{b_4} \circ f_2\}. \tag{6.6}$$

Interestingly, while this is a very limited number of graphs, the proofs of the ordering relations are representative of existing results. More precisely, they proceed by *constructing the solution to a graph, from the solution to another, by taking conic combinations of its pieces (perhaps composed with the dynamics and/or their inverses)*. In this chapter we take an first step towards establishing necessary and sufficient conditions under which such a construction exist given a pair of graphs. More precisely, we highlight conditions such that *given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ and a set of functions $\mathcal{V}$ solution to $\mathbf{G}_2$, we can construct a solution $\mathcal{U}$ with combinations of the functions in $\mathcal{V}$ only (no compositions).* Since these combinations *do not involve the dynamics*, we consider here the problem taking into account general non-linear switching systems.

We present two conditions for when this is possible. The first (Section 6.2) is a sufficient[5] condition, is a combinatorial criterion (Theorem 6.6) involving relations between sets of nodes of the two graphs. It provides intuitive understanding of the mechanism allowing for such constructions. The second (Section 6.3) is obtained by following an algebraic route, and provides us with necessary and sufficient conditions under the form of the existence of a solution to a linear program (Theorem 6.20).

Without loss of generality, we are here going to consider graphs that are *path-complete*, so that the corresponding graph-Lyapunov functions, called path-complete Lyapunov functions, are indeed stability certificates (see Remark 6.2). All the path-complete Lyapunov functions defined on pieces of the same type (e.g. quadratic pieces) have the property that they all are at least as conservative as a common Lyapunov function of this type. For a system on M modes, this correspond to the graph

$$\mathbf{G}^* = (a, \{(a, a, \sigma), \sigma \in \{M\}\}). \tag{6.7}$$

6.2 Combinatorial Criterion

Intuitively, since we seek to find general criteria independent of the dynamics and the nature of the Lyapunov function themselves, we need to focus our study one the combinatorial objects that are the graphs of path-complete Lyapunov functions.

[5]We actually conjecture that it is necessary as well, left as future work.

The criterion developed herein generalizes those presented in [AAJP17a, Section 4]. The next definition is inspired by the concept of simulation between two automata [CL09, pp. 91-92].

Definition 6.4. *Consider two path-complete graphs* $\mathbf{G}_1 = (S^1, E^1)$ *and* $\mathbf{G}_2 = (S^2, E^2)$ *with a same labels* $\langle M \rangle$*. We say that* $\mathbf{G}_1$ simulates $\mathbf{G}_2$ *if there exists a function* $R : S^2 \to 2^{S^1}$ *such that for any edge* $(s, d, \sigma) \in E^2$*, there exists a subset* $E' \subseteq E^1$ *such that*

$$\begin{aligned} &\forall p \in R(s),\, \exists!\, q \in R(d) : (p, q, \sigma) \in E', \\ &\forall q \in R(d),\, \exists!\, p \in R(s) : (p, q, \sigma) \in E', \end{aligned} \tag{6.8}$$

or in other words, each node in $R(s)$ *can be matched to exactly one node in* $R(d)$ *using edges in* E^1 *with label* σ*.*

Remark 6.5. *The function* $R : S^2 \to 2^{S^1}$ *is such that for any edge* $(s, d, \sigma) \in E^2$*, we must be able to extract edges in* E^1 *that form a* bijection *between the two sets* $R(s)$ *and* $R(d)$*.*
As a consequence of this definition, if $\mathbf{G}_1 = (S^1, E^1)$ *simulates* $\mathbf{G}_2 = (S^2, E^2)$ *through the function* R *and* $\mathbf{G}_2$ *is strongly connected, then for any* $s, d \in S^2$, $|R(s)| = |R(d)|$*. In that, we differ from the classical definition of simulation between automata (see [CL09, pp. 91-92]), where a* relation *is used instead of a function.*

Theorem 6.6. *Consider two graphs* $\mathbf{G}_1 = (S^1, E^1)$ *and* $\mathbf{G}_2 = (S^2, E^2)$*. Let* $\mathbf{G}_1$ *simulate* $\mathbf{G}_2$ *through the function* $R : S^2 \to 2^{S^1}$*. Then, if* $\mathcal{V} = \{V_s, s \in S^1\}$ *is a solution for* $\mathbf{G}_1$*, the set*

$$\mathcal{U} = \left\{ U_r = \sum_{s \in R(r)} V_s, r \in S^2 \right\}$$

is a solution to $\mathbf{G}_2$*.*

Proof. Take any edge $(s, d, \sigma) \in E^2$. There is a set of edges $E' \in E^1$, one per node in $R(s)$ such that (6.8) holds. Hence, we can write the $|E'|$ inequalities that take the form

$$\forall x \in \mathbb{R}^n, \forall (s', d', \sigma) \in E' : V_{d'}(f_\sigma(x)) \leq V_{s'}(x)$$

and by summing them up, we have that

$$\forall x \in \mathbb{R}^n : \sum_{d' \in R(d)} V_{d'}(f_\sigma(x)) \leq \sum_{s' \in R(s)} V_{s'}(x).$$

Since this holds for any edge in E^2, we have that $\mathcal{U}$ is a solution to $\mathbf{G}_2$. ∎

Example 6.7. *Consider the graphs $\mathbf{G}_2 = (S^2, E^2)$ and $\mathbf{G}_4 = (S^4, E^4)$ of Figures 6.3b and 6.3d respectively.*
For these graphs, $\mathbf{G}_2$ simulates $\mathbf{G}_4$*. The function R is given by*

$$R(a_4) = \{a_2, b_2\},\ R(b_4) = \{a_2, c_2\}.$$

We conclude that if $\{V_{a_2}, V_{b_2}, V_{c_2}\}$ is a solution for $\mathbf{G}_2$, then the set $\mathcal{U} = \{V_{a_2} + V_{b_2}, V_{a_2} + V_{c_2}\}$ is a solution for $\mathbf{G}_4$, as shown in Example 6.3.

Corollary 6.8. *Given two path-complete graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$, if $\mathbf{G}_1$ simulates $\mathbf{G}_2$ through the function $R : S^2 \to 2^{S^1}$ and that, moreover,*

$$\forall (s \in S^2), R(s) = S^1,$$

then for any system, if $\mathcal{V} = \{V_s, s \in S^1\}$ solution to $\mathbf{G}_1$, the function $\sum_{s \in S^1} V_s$ is a common Lyapunov function.

Example 6.9. *Consider the graph $\mathbf{G} = (S, E)$ of Figure 6.5 on four nodes and two modes. The graph $\mathbf{G}$ simulates the graph $\mathbf{G}^\star = (S^\star, E^\star)$ of eq. (6.7), with for the unique node $s \in S^\star$, $R(s) = S$. If $\mathbf{G}$ is feasible for a set $\{V_a, V_b, V_c, V_d\}$, then the system has a common Lyapunov function given by $V_a + V_b + V_c + V_d$.*

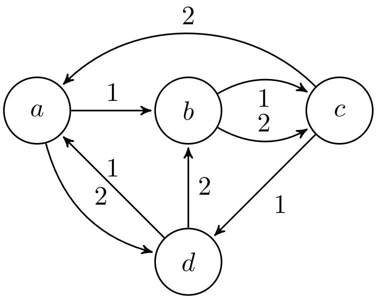
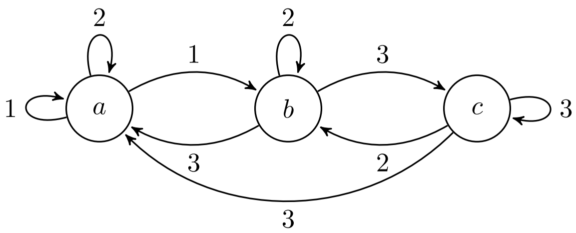


Figure 6.5: A graph $\mathbf{G}$ whose feasibility implies that of $\mathbf{G}^\star$, see (6.7).

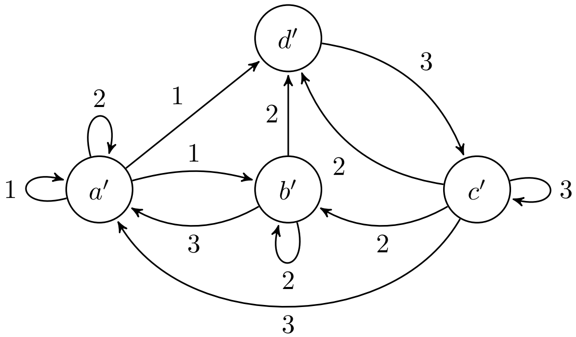
A special case of the simulation criterion is when for all node s of the graph $\mathbf{G}_2$, the $R(s)$ is a singleton. Then Theorem 6.6 simply associates to each node s of the graph $\mathbf{G}_2$ one of the pieces of the set $\mathcal{V}$, solution to the graph $\mathbf{G}_1$.

Example 6.10. *Consider the graphs on three modes $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ on three modes with the first depicted on Fig. 6.6a and the second on Fig. 6.6b. We have that $\mathbf{G}_1$ simulates $\mathbf{G}_2$ for*

$$R(a') = \{a\},\ R(b') = \{b\},\ R(c') = \{c\},\ \textit{and } R(d') = \{b\}.$$



(a) Graph $\mathbf{G}_1$ for Example 6.10.



(b) Graph $\mathbf{G}_2$ for Example 6.10.

Figure 6.6: $\mathbf{G}_1$ simulates $\mathbf{G}_2$.

Simulation between graphs is a combinatorial criterion that *provides us with proofs of ordering*, constructions that map the solution to one graph to a solution to another. In particular, here, the solution to the graph $\mathbf{G}_2$ is obtained as a sum of the elements of the solution to the graph $\mathbf{G}_1$. In the next section, we provide necessary and sufficient conditions for when such construction is possible. Note that we have shown that simulation is a sufficient condition - we believe it to be necessary as well, the proof of which is left at future work.

6.3 Ordering using Conic Combination of Pieces

In this section, we develop algorithms to establish proofs of ordering alike those presented in Example 6.3. More precisely, we aim to decide whether or not, given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, for any switching system and any solution $\mathcal{V}$ to $\mathbf{G}_1$, one can construct a solution $\mathcal{U}$ to $\mathbf{G}_2$ where each function in $\mathcal{U}$ is a *conic combination of the functions of* $\mathcal{V}$ For the sake of exposition, we assume graphs are here in expanded form.

Assumption 6.11. *Throughout this section, all graphs involved are in expanded form.*

6.3.1 Problem Formulation

We begin with conventions. For any given discrete set Z, we assume there is an arbitrary *ordering* function $\mathrm{ord} : Z \mapsto \{1, \dots, |Z|\}$ of its elements. This allows us, given a vector $v \in \mathbb{R}^{|Z|}$ and an element $z \in Z$, to write the shortcut notation $(v)_z := (v)_{\mathrm{ord(z)}}$. These ordering functions are considered implicit throughout, and are made explicit only when needed.

In order to formulate and solve the problem we tackle, it is convenient to see Lyapunov function candidates and related Lyapunov inequalities in terms of matrix/vectors.

Definition 6.12 (VLFC). *A* Vector Lyapunov function candidate (VLFC) *is a vector function* $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^N_{\geq 0}$*, where each element* $(\mathbf{V})_i : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$*,* $i \in \{1, \dots, N\}$*, is a Lyapunov function candidate (Definition 1.11).*

Given a VLFC $\mathbf{V}$ and a map $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $\mathbf{V} \circ f : \mathbb{R}^n \to \mathbb{R}^N_{\geq 0}$ denotes the vector function where

$$(\mathbf{V} \circ f)_i = (\mathbf{V})_i \circ f.$$

Given two vector functions $\mathbf{V}$ and $\mathbf{U}$ from $\mathbb{R}^n$ to $\mathbb{R}^N$, we write

$$\mathbf{V} \leq \mathbf{U} \Leftrightarrow \forall (x \in \mathbb{R}^n), \forall (1 \leq i \leq N), (\mathbf{V})_i(x) \leq (\mathbf{U})_i(x),$$

that is the inequality holds point-wise across all components.

Definition 6.13. *Consider a graph* $\mathbf{G} = (S, E)$ *in expanded form on the alphabet* $\langle M \rangle$*. For any* $\sigma \in \langle M \rangle$*, let* $E_\sigma = \{(s, d, \sigma') \in E : \sigma' = \sigma\}$ *be the set of edges with label* σ*. We define the two matrices* $\mathbf{S}_\sigma(\mathbf{G}) \in \{0,1\}^{|E_\sigma| \times |S|}$ *and* $\mathbf{D}_\sigma(\mathbf{G}) \in \{0,1\}^{|E_\sigma| \times |S|}$ *as follows:*

$$\begin{aligned} (\mathbf{S}_\sigma)_{e,s} = 1 \Leftrightarrow \exists d \in S : e = (s, d, \sigma) \in E, \\ (\mathbf{D}_\sigma)_{e,d} = 1 \Leftrightarrow \exists s \in S : e = (s, d, \sigma) \in E. \end{aligned} \tag{6.9}$$

From the definition above, the matrices $\mathbf{S}$ and $\mathbf{D}$ are not unique - they depend on the way one orders edges and nodes in a graph.

Example 6.14. *We construct the matrices* $\mathbf{S}_1$*,* $\mathbf{S}_2$*,* $\mathbf{D}_1, \mathbf{D}_2$ *for the graphs* $\mathbf{G}_2$ *and* $\mathbf{G}_4$ *of Figures 6.3b and 6.3d. We start from* $\mathbf{G}_1$ *and let* a_2*,* b_2 *and* c_2 *be the first, second and third node of the graph respectively. We take the following ordering for the edges:* $(a_2, b_2, 1) < (a_2, c_2, 1) < (a_2, b_2, 2) < (a_2, c_2, 2) < (b_2, a_2, 1) < (c_2, a_2, 2)$*. With these conventions, we can write*

$$\mathbf{S}_1(\mathbf{G}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{D}_1(\mathbf{G}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.10}$$

The first row of $\mathbf{S}_1(\mathbf{G}_1)$ *and* $\mathbf{D}_1(\mathbf{G}_1)$ *corresponds to an edge with label* 1 *whose source is the first node (*a_2*) and destination the second node (*b_2*), that is, the* *edge $(a_2, b_2, 1)$. With the same conventions, we have*

$$\mathbf{S}_2(\mathbf{G}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{D}_2(\mathbf{G}_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.11}$$

For $\mathbf{G}_4$, we take a_4 as the first node and b_4 as the second. Edges are ordered as follows: $(a_4, a_4, 1) < (a_4, b_4, 1) < (b_4, a_4, 2) < (b_4, b_4, 2)$. We get the following matrices

$$\mathbf{S}_1(\mathbf{G}_4) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{D}_1(\mathbf{G}_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.12}$$

$$\mathbf{S}_2(\mathbf{G}_4) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{D}_2(\mathbf{G}_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.13}$$

Using these matrices, we can now represent the Lyapunov inequalities encoded into graphs in an easy to handle matrix/vector form. Recall that we say that $\mathcal{V}$ is a solution to a graph $\mathbf{G} = (S, E)$ if

$$\forall((s, d, \sigma) \in E), \forall x \in \mathbb{R}^n,\ V_d(f_\sigma(x)) \leq V_s(x). \tag{6.14}$$

The inequalities (6.14) translate as follows.

Definition 6.15. *Consider a graph $\mathbf{G} = (S, E)$ in expanded form on the alphabet $\langle M \rangle$, dynamics $\mathbf{F} = \{f_1, \ldots, f_M\}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, and a VLFC $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S|}_{\geq 0}$. We say that $\mathbf{V}$ is a solution to $\mathbf{G}$ for $\mathbf{F}$, written*

$$\mathit{glf}(\mathbf{G}, \mathbf{V}, \mathbf{F}),$$

if for all $\sigma \in \langle M \rangle$,

$$\mathbf{D}_\sigma(\mathbf{G})(\mathbf{V} \circ f_\sigma) \leq \mathbf{S}_\sigma(\mathbf{G})\mathbf{V}. \tag{6.15}$$

Definition 6.16. *Given a VLFC $\mathbf{V} : \mathbb{R}^n \mapsto \mathbb{R}^{N_1}_{\geq 0}$ and maps $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$, a VLFC $\mathbf{U} : \mathbb{R}^n \mapsto \mathbb{R}^{N_2}_{\geq 0}$ is said to be a* conic-combination *of $\mathbf{V}$ if it can be written as*

$$\mathbf{U} = C\mathbf{V},$$

for a conic combination matrix *$C \in \mathbb{R}^{|N_2| \times |N_1|}$, with*

$$\forall 1 \leq i \leq N_2 : \sum_{j=1}^{N_1} (C)_{i,j} \geq 1. \tag{6.16}$$

Example 6.17. *Consider the two graphs $\mathbf{G}_2 = (S^2, E^2)$ and $\mathbf{G}_4 = (S^4, E^4)$ at Figure 6.3. We discuss graph-Lyapunov functions on these graphs in terms of VLFCs. To refer to their entries, we provide the following ordering for the nodes: $a_2 \leq b_2 \leq c_2$ in $\mathbf{G}_2$ and $a_4 < b_4$ in $\mathbf{G}_4$. If $\mathbf{V}$ is a VLCF solution for $\mathbf{G}_2$, then we can obtain a VLFC $\mathbf{U}$ solution to $\mathbf{G}_4$ as follows (see Example 6.3, Eq. (6.5)).*

$$\begin{aligned}\mathbf{U} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{V} \\ &= C\mathbf{V}.\end{aligned} \tag{6.17}$$

We are now in position to state the precise property we seek to decide.

Definition 6.18. *Given two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ on an alphabet $\langle M \rangle$ and an integer $k \geq 0$, we write*

$$\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$$

if the following holds: there is a matrix $C \in \mathbb{R}_{\geq 0}^{|S^2| \times |S^1|}$ satisfying (6.16) such that

$$\begin{aligned}&\forall\, n \in \mathbb{N},\, \forall\, \mathbf{F} = \{f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \sigma \in \langle M \rangle\},\, \forall \text{ VLFC } (\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S^1|}) : \\ &\qquad \mathit{glf}(\mathbf{G}_1, \mathbf{V}, \mathbf{F}) \Rightarrow \mathit{glf}(\mathbf{G}_2, C\mathbf{V}, \mathbf{F}),\end{aligned} \tag{6.18}$$

with glf *is defined in Definition 6.15.*
Given a matrix C such that (6.18) is satisfied, we say that $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ through *C.*

The next theorem translates the relation $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ into conditions that are much easier to handle algebraically.

Theorem 6.19. *Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ on an alphabet $\langle M \rangle$, and a conic combination matrix $C \in \mathbb{R}^{|S_2| \times |S_1|}$ satisfying (6.16). The following statements are equivalent.*

(1.) *$\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ through C.*

(2.) *The following implications hold*

$$\begin{aligned}&\forall\, n \in \mathbb{N},\, \forall\, \mathbf{F} = \{f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \sigma \in \langle M \rangle\},\, \forall \text{ VLFC } (\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^{|S^1|}), \\ &\forall x \in \mathbb{R}^n, \forall \sigma \in \langle M \rangle,\end{aligned} \tag{6.19}$$

the following implication holds

$$\mathbf{D}_\sigma(\mathbf{G}_1)(\mathbf{V} \circ f_\sigma)(x) \leq \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) \Rightarrow \tag{6.20}$$
$$\mathbf{D}_\sigma(\mathbf{G}_2)C(\mathbf{V} \circ f_\sigma)(x) \leq \mathbf{S}_\sigma(\mathbf{G}_2)C\mathbf{V}(x). \tag{6.21}$$

Proof. The main difference between **(1.)** and **(2.)** is that, fixing the dimension n, dynamics $\mathbf{F}$, and VLFC $\mathbf{V}$, **(1.)** states (from Definition 6.18)

> "*if (6.20) holds for all $x \in \mathbb{R}^n$ and all $\sigma \in \langle M \rangle$, then (6.21) holds as well for all $x \in \mathbb{R}^n$ and all $\sigma \in \langle M \rangle$*",

whereas **(2.)** states

> "*for all $x \in \mathbb{R}^n$ and all $\sigma \in \langle M \rangle$, if (6.20) holds, then (6.21) holds as well*".

From there, we can directly conclude that **(2.)** $\Rightarrow$ **(1.)**.

Let us now show that **(1.)** $\Rightarrow$ **(2.)** by contraposition. In specific, for a choice of $\mathbf{F}$ and $\mathbf{V}$, assume **(2.)** does not hold: there is a vector $\hat{x} \in \mathbb{R}^n$ and a $\sigma \in \langle M \rangle$ such that (6.20) is satisfied but not (6.21). We show then that we can build dynamics $\mathbf{F}'$ such that (6.20) holds for all $x \in \mathbb{R}^n$ and $\sigma \in \langle M \rangle$, but not (6.21). Hence, if **(2.)** does not hold, **(1.)** can not hold either.

To this purpose, we define the function for all $\sigma \in \langle M \rangle$,

$$\omega_\sigma(x) := \min_\alpha \alpha : \{\alpha \geq 1, \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) - \mathbf{D}_\sigma(\mathbf{G}_1)\mathbf{V}(f_\sigma(x)/\alpha) \geq \mathbf{0}\}.$$

A pair $\sigma \in \langle M \rangle$, $x \in \mathbb{R}^n$ satisfies (6.20) if and only if $\omega_\sigma(x) = 1$. If a point does not satisfy (6.20), then $x \neq 0$ and we can provide an *upper bound* on $\omega_\sigma(x)$ using the properties of Lyapunov function candidates: there are two functions α and β of class $\mathcal{K}_\infty$ such that,

$$\forall s \in S^1, \forall x \in \mathbb{R}^n : \alpha(\|x\|) \leq (\mathbf{V})_s(x) \leq \beta(\|x\|).$$

Therefore, if $x \neq 0$, for any node $s \in S^1$, there is $\delta > 0$ such that $(\mathbf{V})_s(x) \geq \alpha(\|x\|) > \delta$. Consequently, we can pick ω_σ large enough such that

$$\forall (s, d, \sigma) \in E^1,\ (\mathbf{V})_d(f_\sigma(x)/\omega_\sigma(x)) \leq \beta(\|f_\sigma(x)\|/\omega_\sigma(x)) < \delta < (\mathbf{V})_s(x).$$

In summary, for the dynamics $\mathbf{F}' = \{f'_\sigma = (f_\sigma/\omega_\sigma), f_\sigma \in \mathbf{F}\}$, and the VLFC $\mathbf{V}$, (6.20) must hold for all $x \in \mathbb{R}^n$ and all $\sigma \in \langle M \rangle$ or in other words, $\mathrm{glf}(\mathbf{G}_1, \mathbf{V}, \mathbf{F}')$ is true. However, for the pair σ, $\hat{x}$ that satisfied (6.20) initially for the dynamics $\mathbf{F}$, we have $\omega_\sigma(\hat{x}) = 1$, and so $f'_\sigma(\hat{x}) = f_\sigma(x)$ for all $\sigma \in \langle M \rangle$. If that point did not satisfy (6.21) for $\mathbf{F}$, it does not satisfy it for $\mathbf{F}'$. Therefore, we have constructed dynamics $\mathbf{F}'$ such that $\mathrm{glf}(\mathbf{G}_1, \mathbf{V}, \mathbf{F}')$ holds, but where $\mathrm{glf}(\mathbf{G}_2, C\mathbf{V}, \mathbf{F}')$ does not due to $\hat{x}$. We conclude that if **(2.)** does not hold, **(1.)** does not hold. ■

In the next subsections, our goal is to translate the intricate conditions of Definition 6.18 in more and more tractable conditions, and we eventually show how $\mathbf{G}_1 \leq_k \mathbf{G}_2$ is equivalent to the *feasibility of a linear program*

6.3.2 An LP-Formulation for Conic Comparisons

This subsection aims at proving our main result for this chapter:

Theorem 6.20. *Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$. The following are equivalent.*

(1.) $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$.

(2.) There exists a matrix $C \in \mathbb{R}^{|S^2|\times|S^1|}$ satisfying (6.16), and for each mode $\sigma \in \langle M \rangle$, a matrix

$$K_\sigma \in \mathbb{R}_{\geq 0}^{|E^2_\sigma|\times|E^1_\sigma|},$$

such that for all $\sigma \in \langle M \rangle$,

$$\mathbf{S}_\sigma(\mathbf{G}_2)C \geq K_\sigma \mathbf{S}_\sigma(\mathbf{G}_1), \tag{6.22}$$

$$\mathbf{D}_\sigma(\mathbf{G}_2)C \leq K_\sigma \mathbf{D}_\sigma(\mathbf{G}_1), \tag{6.23}$$

It shows that given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ can be decided efficiently, and the corresponding conic combination matrix C can be obtained through linear programming.

Example 6.21 (Example 6.17 cont.)**.** *Consider the graphs $\mathbf{G}_2$ and $\mathbf{G}_4$ at Figure 6.3. There, we have $\mathbf{G}_2 \leq_{\sum} \mathbf{G}_4$ with C defined in (6.17). For these two graphs, define the matrices $\mathbf{S}_\sigma$ and $\mathbf{D}_\sigma$ as in Example 6.14. Then, we satisfy (6.22) and (6.23) with the matrices*

$$K_1 = K_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

where the equality is here due to our choice of conventions when computing the matrices $\mathbf{S}_\sigma$ and $\mathbf{D}_\sigma$.

Both (6.20) and (6.21) can be seen as *linear inequalities* on the values taken by a *possibly non-linear vector function.* For example, for a system with 2 modes and a given $x \in \mathbb{R}^n$, we can write (6.20) as

$$\begin{pmatrix} \mathbf{S}_1(\mathbf{G}_1) & -\mathbf{D}_1(\mathbf{G}_1) & \mathbf{0} \\ \mathbf{S}_2(\mathbf{G}_1) & \mathbf{0} & -\mathbf{D}_2(\mathbf{G}_1) \end{pmatrix} \begin{pmatrix} \mathbf{V}(x) \\ \mathbf{V}(f_1(x)) \\ \mathbf{V}(f_2(x)) \end{pmatrix} \geq \mathbf{0}, \tag{6.24}$$

and (6.21) as

$$\begin{pmatrix} \mathbf{S}_1(\mathbf{G}_2)C & -\mathbf{D}_1(\mathbf{G}_2)C & \mathbf{0} \\ \mathbf{S}_2(\mathbf{G}_2)C & \mathbf{0} & -\mathbf{D}_2(\mathbf{G}_2)C \end{pmatrix} \begin{pmatrix} \mathbf{V}(x) \\ \mathbf{V}(f_1(x)) \\ \mathbf{V}(f_2(x)) \end{pmatrix} \geq \mathbf{0}. \tag{6.25}$$

We can then reformulate the fact that (6.20) $\Rightarrow$ (6.21) as follows:

For all vectors that can be written as $y = \left(\mathbf{V}(x)^\top \; \mathbf{V}(f_1(x))^\top \; \mathbf{V}(f_2(x))^\top \right)^\top$*, if that vector satisfies (6.24), it does satisfy (6.25).*

This motivates us to study the set of all such vectors.

Consider an integer $n \geq 1$, a VLFC $\mathbf{V} : \mathbb{R}^n \to \mathbb{R}^N_{\geq 0}$, a set of M maps $\mathbf{F} = \{f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \sigma \in \langle M \rangle\}$, and a point $x \in \mathbb{R}^n$. We define the vector

$$y(x, \mathbf{F}, \mathbf{V}) = \begin{pmatrix} [y]^0 \\ [y]^1 \\ \vdots \\ [y]^M \end{pmatrix} = \begin{pmatrix} \mathbf{V}(x) \\ \mathbf{V}(f_1(x)) \\ \vdots \\ \mathbf{V}(f_M(x)) \end{pmatrix} \in \mathbb{R}^{N(M+1)}_{\geq 0}, \tag{6.26}$$

where the blocs of y are $[y]^i \in \mathbb{R}^N_{\geq 0}$ for all $0 \leq i \leq M$. Additionally, we let

$$\mathbf{Y}_{n,M,N} = \begin{Bmatrix} y(x, \mathbf{F}, \mathbf{V}) : x \in \mathbb{R}^n, \\ \mathbf{F} = \{f_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \sigma \in \langle M \rangle\}, \\ \mathbf{V} \text{ is a VLFC on } \mathbb{R}^n \text{ with } N \text{ pieces} \end{Bmatrix},$$

as the set of all vectors $y(x, \mathbf{F}, \mathbf{V})$ for a fixed dimension n of the dynamics and state, and finally we let

$$\mathcal{Y}_{M,N} = \bigcup_{n=1}^{\infty} \mathbf{Y}_{n,M,N}, \tag{6.27}$$

which is the set of all possible vectors which we can write as in (6.26) for some x in some dimension n, sets $\mathbf{F}$ of M maps in $\mathbb{R}^n$, and VLFC on $\mathbb{R}^n$ with N pieces.

Lemma 6.22. *For any $M \geq 1$, $N \geq 1$, it holds that*

$$\mathbb{R}^{N(M+1)}_{\geq 0} \supset \mathcal{Y}_{M,N} \supset \mathbb{R}^{N(M+1)}_{>0}.$$

Proof. The left hand side inclusion holds by the definition of a VLFC. Note that $y(\mathbf{0}, \mathbf{F}, \mathbf{V}) = \mathbf{0} \in \mathcal{Y}_{M,N}$ for any choice of dynamics $\mathbf{F}$ and VLFC $\mathbf{V}$.

For proving the right hand side inclusion, our aim is *to pick, for any $\lambda \in \mathbb{R}^{N(M+1)}_{>0}$, a dimension n, a vector $\hat{x} \in \mathbb{R}^n$, a VLFC $\mathbf{V}$ and maps $\mathbf{F}$ such that*

$$\lambda = y(\hat{x}, \mathbf{F}, \mathbf{V}).$$

Similarly to y in (6.26), we define λ with $M+1$ blocs of size N, denoted $[\lambda]^i$, $0 \leq i \leq M$. We can thus write down our goal as follows: find $\mathbf{V}$, $\mathbf{F}$, and $\hat{x}$, such that

$$[\lambda]^0 = \mathbf{V}(\hat{x}), \text{ and } \forall 1 \leq \sigma \leq M : [\lambda]^\sigma = \mathbf{V}(f_\sigma(\hat{x})).$$

To do so, we pick $\hat{x}$ of dimension $n = N(M+1)$, the same as λ. Our vector $\hat{x}$ is defined as follows (again, defined block-wise):

$$[\hat{x}]^0 = \mathbf{1} \text{ and } \forall 1 \le \sigma \le M : [\hat{x}]^\sigma = \mathbf{0}.$$

The VLCF is then constructed to satisfy the following conditions. For any index $0 \le i \le M$ and for any vector $x \in \mathbb{R}^{N(M+1)}$ satisfying

$$[x]^i = \mathbf{1} \text{ and } \forall j \ne i : [x]^j = \mathbf{0},$$

we have $\mathbf{V}(x) = [\lambda]^i$. In particular, we want $\mathbf{V}(\hat{x}) = [\lambda]^0$.
To achieve this, we let $\underline{\lambda} = \min_i(\lambda)_i/2 > 0$, and define each entry of the VLFC $\mathbf{V}$ as follows: for the entry $1 \le \ell \le N$ of the VLCF,

$$(\mathbf{V})_\ell(x) = \frac{\underline{\lambda}}{N}\|x\|_2^2 + \sum_{i=0}^{M}(([\lambda]^i)_\ell - \underline{\lambda})([x]^i)_\ell^2.$$

For the vectors x with the property mentioned above (one component set to $\mathbf{1}$, the rest set to $\mathbf{0}$), the first term is always equal to $\underline{\lambda}/NN = \underline{\lambda}$. Among all of the others, there is only one term which is non-zero. It is precisely the one corresponding to the index i where $[x]^i = \mathbf{1}$. In this case, we have for all $1 \le \ell \le N$,

$$(\mathbf{V})_\ell(x) = \underline{\lambda} + (([\lambda]^i)_\ell - \underline{\lambda}) = ([\lambda]^i)_\ell,$$

and thus $\mathbf{V}(x) = [\lambda]^i$ by construction.

Finally, we pick M linear maps $\mathbf{F} = \{f_\sigma : \mathbb{R}^{N(M+1)} \to \mathbb{R}^{N(M+1)}, \sigma \in \langle M\rangle\}$ defined as follows:

$$\forall \sigma \in \langle M\rangle, \forall x \in \mathbb{R}^{N(M+1)} : \; f_\sigma \begin{pmatrix} [x]^0 \\ [x]^1 \\ \vdots \\ [x]^\sigma \\ \vdots \\ [x]^M \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ [x]^0 \\ \vdots \\ \mathbf{0} \end{pmatrix},$$

that is, f_σ shifts the block $[x]^0$ to the position $[x]^\sigma$, and sets all other blocs to $\mathbf{0}$.
By construction, with these choices, we have that $\mathbf{V}(\hat{x}) = [\lambda]^0$ and $\mathbf{V}(f_\sigma \hat{x}) = [\lambda]^\sigma$ for all $\sigma \in \langle M\rangle$. Since this can be achieved for any $\lambda \in \mathbb{R}_{>0}^{N(M+1)}$, we have that $\mathcal{Y}_{M,N} \supset \mathbb{R}_{>0}^{N(M+1)}$, which concludes the proof. ■

We are now in position to characterize the relation $\mathbf{G}_1 \le_{\sum} \mathbf{G}_2$ of Definition 6.18 without explicitely involving dynamics or Lyapunov functions.

Lemma 6.23. *Consider two graphs $\mathbf{G}_1 = (S^1, E^1)$ and $\mathbf{G}_2 = (S^2, E^2)$ with labels $\sigma \in \langle M\rangle$. Let $C \in \mathbb{R}_{\ge 0}^{|S^2|\times|S^1|}$ satisfy (6.16). The following are equivalent:*

(1.) $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ *through the combination matrix* C*.*

(2.) *For all* $y = \left([y]^{0\top} \; [y]^{1\top} \; \cdots \; [y]^{M\top} \right)^\top \in \mathbb{R}_{\geq 0}^{|S^1|(M+1)}$, $[y]^i \in \mathbb{R}^{|S^1|}$ *for* $0 \leq i \leq M$*, and all* $\sigma \in \langle M \rangle$*,*

$$\mathbf{D}_\sigma(\mathbf{G}_1)[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_1)[y]^0 \Rightarrow \tag{6.28}$$

$$\mathbf{D}_\sigma(\mathbf{G}_2)C[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_2)C[y]^0, \tag{6.29}$$

Proof. **(2.)** $\Rightarrow$ **(1.)**. In this case, the implication (6.28) $\Rightarrow$ (6.29) holds in particular for the non-negative vectors of the form (6.26). This means that the second part of Theorem 6.19 holds, which implies that $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$.
(1.) $\Rightarrow$ **(2.)**. For each $\sigma \in \langle M \rangle$, consider the the polyhedral sets

$$P_\sigma = \{y \in \mathbb{R}_{\geq 0}^{|S^1|(M+1)} \mid \mathbf{D}_\sigma(\mathbf{G}_1)[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_1)[y]^0\}, \tag{6.30}$$

$$Q_\sigma = \{y \in \mathbb{R}^{|S^1|(M+1)} \mid \mathbf{D}_\sigma(\mathbf{G}_2)C[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_2)C[y]^0\}. \tag{6.31}$$

Our goal is to show that when $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$, $P_\sigma \subseteq Q_\sigma$.
From Theorem 6.19, if $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$, we know that for the set

$$P'_\sigma = \{y \in \mathcal{Y}_{M,|S^1|} \mid \mathbf{D}_\sigma(\mathbf{G}_1)[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_1)[y]^0\},$$

we have $P'_\sigma \subseteq Q_\sigma$ by the definition of $\mathcal{Y}_{M,|S^1|}$ (see Eq. (6.27)).
From Lemma 6.22, the above implies that for the set

$$P''_\sigma = \{y \in \mathbb{R}_{>0}^{|S^1|(M+1)} \mid \mathbf{D}_\sigma(\mathbf{G}_1)[y]^\sigma \leq \mathbf{S}_\sigma(\mathbf{G}_1)[y]^0\},$$

we have $P''_\sigma \subseteq Q_\sigma$, since $\mathcal{Y}_{M,|S^1|} \supset \mathbb{R}_{>0}^{|S^1|(M+1)}$.
The above is the inclusion of an open polyhedral cone into a closed one. By a simple compacity argument, it hold that the closure of P''_σ, which is P_σ, is included in Q_σ. This concludes the proof. ■

Lemma 6.23 shows that the conic comparison formulated in Definition 6.18 is equivalent to verifying a set inclusion between two polyhedral sets. Such inclusions are well understood, and allows us to use and extended version of the so-called Farkas' Lemma, see e.g. [Hen95, Lemma II.2], to obtain algebraic relations enabling us to formulate and prove our main theorem in this chapter. Theorem 6.20 provides a linear program to decide, given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ whether or not $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$, and outputs a combination matrix C should it be the case.

Lemma 6.24 (**[Hen95]**). *Consider two matrices* $H \in \mathbb{R}^{p \times n}$ *and* $L \in \mathbb{R}^{q \times n}$*. The following are equivalent:*

$$\begin{aligned} &(\{y \in \mathbb{R}^n : Hy \geq \mathbf{0},\, y \geq \mathbf{0}\} \subseteq \{y \in \mathbb{R}^n : Ly \geq \mathbf{0}\}) \\ &\Leftrightarrow \exists K \in \mathbb{R}^{m \times p} : KH \leq L,\, K \geq \mathbf{0}. \end{aligned}$$

We are now in position to prove Theorem 6.20.

Proof of Theorem 6.20. **(1.)** $\Rightarrow$ **(2.)** By Lemma 6.23, we know that $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ implies

$$\forall \sigma \in \langle M \rangle,\, P_\sigma \subseteq Q_\sigma,$$

where P_σ is defined in (6.30) and Q_σ in (6.31).
For each $\sigma \in \langle M \rangle$, the set P_σ can be written as

$$P_\sigma = \{y \in \mathbb{R}^{|S^1|(M+1)} \mid H_\sigma y \geq \mathbf{0},\, y \geq \mathbf{0}\},$$

with $H_\sigma \in \mathbb{R}^{|E^1_\sigma| \times (M+1)|S^1|}$ defined as

$$H_\sigma = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & \sigma-1 & \sigma & \sigma+1 & \dots & M \end{array} \\ \left(\begin{array}{cccccccc} \mathbf{S}_\sigma(\mathbf{G}_1) & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{D}_\sigma(\mathbf{G}_1) & \mathbf{0} & \dots & \mathbf{0} \end{array} \right). \end{array}$$

Similarly, we can write Q_σ as

$$Q_\sigma = \{y \in \mathbb{R}^{|S^1|(M+1)} : L_\sigma y \geq \mathbf{0}\},$$

with $L_\sigma \in \mathbb{R}^{|E^2_\sigma| \times (M+1)|S^2|}$ defined as

$$L_\sigma = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & \sigma-1 & \sigma & \sigma+1 & \dots & M \end{array} \\ \left(\begin{array}{cccccccc} \mathbf{S}_\sigma(\mathbf{G}_2)C & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{D}_\sigma(\mathbf{G}_2)C & \mathbf{0} & \dots & \mathbf{0} \end{array} \right). \end{array}$$

By Lemma 6.24, $P_\sigma \subseteq Q_\sigma$ only if there exists a matrix $K_\sigma \in \mathbb{R}^{|E^2_\sigma| \times |E^1_\sigma|}_{\geq 0}$ such that

$$K_\sigma H_\sigma \leq L_\sigma,$$

which is equivalent to

$$K_\sigma \mathbf{S}_\sigma(\mathbf{G}_1) \leq \mathbf{S}_\sigma(G_2)C,$$
$$K_\sigma \mathbf{D}_\sigma(\mathbf{G}_1) \geq \mathbf{D}_\sigma(G_2)C,$$

and this concludes the first part of the proof.

(2.) $\Rightarrow$ **(1.)** This part is easier. Take a point x, dynamics $\mathbf{F}$ and a VLFC $\mathbf{V}$. We want to show that for any of such choices, if its satisfies (6.20), it satisfies (6.21), which is equivalent to $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ by Theorem 6.19.
Since each row of K_σ is non-negative, we have the following, starting from (6.20):

$$\mathbf{D}_\sigma(\mathbf{G}_1)(\mathbf{V} \circ f_\sigma)(x) \leq \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) \Rightarrow$$
$$K_\sigma \mathbf{D}_\sigma(\mathbf{G}_1)(\mathbf{V} \circ f_\sigma)(x) \leq K_\sigma \mathbf{S}_\sigma(\mathbf{G}_1)\mathbf{V}(x) \Rightarrow$$
$$\mathbf{D}_\sigma(\mathbf{G}_2)C(\mathbf{V} \circ f_\sigma)(x) \leq \mathbf{S}_\sigma(\mathbf{G}_2)C\mathbf{V}(x),$$

which is (6.21), and this concludes the proof. ∎

6.4 Summary

The final chapter of this thesis tackles the problem of comparing the conservativeness of different Lyapunov methods for the stability analysis of switching systems, and in particular, path-complete Lyapunov functions.
Inspired by existing proofs in the literature, we single out the problem of deciding when, given two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, if there exists a first path-complete Lyapunov function on the graph $\mathbf{G}_1$, we can construct a second path-complete Lyapunov function for the graph $\mathbf{G}_2$ where each piece is a conic combination of the pieces of the first.
Solving this problem provides a first, systematic way to compare the conservatism of Lyapunov methods.

Section 6.1: Ordering Path-Complete Methods.

The problem of the comparison of path-complete Lyapunov function is illustrated by the problem of estimating the CJSR (see Eq. (6.3)) of constrained switching systems. We present existing results for comparing and ordering path-complete methods (Example 6.3 - based on the developments of [AJPR14]).

Section 6.2: Combinatorial Criterion.

In this section, we introduce a combinatorial condition for when this is possible (Simulation, Definition 6.4, and Theorem 6.6). It provides intuition about what similarities need to exist in the structure of the graphs for when we can order them. So far, it has only been shown to be a sufficient condition.

Section 6.3: Ordering using Conic Combination of Pieces.

In this section we provide a necessary and sufficient condition, under the form of the existence of a solution to a linear program, for when for when we can construct a solution for one graph from conic combinations of the pieces of another (Theorem 6.20). This is done without any assumption on the nature of the Lyapunov functions or dynamics. This is achieved by following an algebraic route, and showing that the ordering relation we study (Definition 6.18) is equivalent to an inclusion of polyhedral sets expressed through the adjacency matrices of both graphs (Lemma 6.23).

For future work, we highlight the following questions.

Question 6.1. *Is it true that* $\mathbf{G}_1 \leq_{\sum} \mathbf{G}_2$ *as in Definition 6.18 if and only if* $\mathbf{G}_1$ *simulates* $\mathbf{G}_2$ *in the sense of Definition 6.4?*

The simulation criterion is valuable as it provides us with a structural characterization for when we can order graphs, whereas the linear-programming based formulation is easily checkable.

Question 6.2. *When are PCLFs on a graph* $\mathbf{G}_1$ probably less conservative *than PCLFs on a graph* $\mathbf{G}_2$*?*

Not all path-complete Lyapunov functions can be ordered: for some pairs, there are systems where the first is going to be more conservative that the other, and systems where the second is going to be more conservative.
Nevertheless, it has been reported in e.g. [AJPR14, Section 4] than, on random examples, some tend to produce stability certificates more often than others.
Given any system, one would be tempted a priori to seek stability certificates using graphs that tend to be less conservative, and such experiments are valuable for these reasons.
In this question, we ask whether or not we can go beyond numerical experiments, and compute or approximate the probability that, given any linear switching system, a graph is going to produce a better JSR estimation (using quadratic norms) than another.

Conclusions

In this thesis we provided novel tools for the study of complex dynamical systems, where automata and systems theory merge. We focused on the central question of the stability of a switching system under constrained switching.
The stability analysis of switching systems presents us with challenging mathematical problems and theoretical barriers. These challenges are clearly worth undertaking for two main reasons. First, because switching systems appear to be adequate models in many practical situations, and thus we need tools for checking their stability. Second, because we believe that through the study of this particular class of hybrid systems, we can generate ideas and intuitions to deal with more general problems in further works.

A short summary of the contributions

We began, in **Chapter 1**, by introducing several fundamental concepts for our analysis. In particular, we introduced *constrained switching systems* as switching systems whose *switching sequences* are generated by paths in *directed and labeled graphs*. In **Part I**, we provided theoretical characterizations and algorithmic tools for the stability analysis of constrained switching systems. After experiencing with the *Path-Complete Methods* at the core of our techniques in Part I, we analyzed these tools and their conservativeness in **Part II**.
Let us recall briefly our main results. For more details on a given chapter, we invite the reader to refer to the Summary section of that chapter.

Chapter 2: Given a constrained switching system, the work we conducted in this thesis *allowed to approximate its exponential decay rate, a.k.a.* constrained joint spectral radius (CJSR)*, with arbitrary accuracy in finite time.* To achieve this, we had to generalize the concept of the *contractive norm*, central for the stability analysis of *switching systems under arbitrary switching.* The fact that stability implies the existence of a contractive norm is true in the arbitrary switching case, *but it is generally not true in the constrained switching case.* This led us to propose the concept of *multinorm*, that defines a different norm for each node of the graph defining the switching sequences, and imposes Lyapunov inequalities along its edges. Using these, we provided an exact characterization of stability able to serve as a novel foundation for the stability analysis of constrained switching systems. In particular, we were then able to further study several stability analysis tools based on multiple Lyapunov functions. It is through this study and by the introduction of several *lifting* techniques - either transforming the graph or the state space of constrained switching systems - that we were able to formulate approximation schemes for the CJSR.

Chapter 3: The tools of Chapter 2 allowed us to *approximate* the constrained joint spectral radius (CJSR), and is in general hard to decide when the estimation is exact. Even worse, it is hard to know whether or not an exact estimation could ever be obtained through these techniques. Chapter 3 revolves around the concepts of *non-defectivity*, which is the existence of an *extremal multinorms*, that does achieve a decay rate equal to the CJSR. More precisely, we considered two situations.
First, in the case when CJSR = 1, *non-defectivity is equivalent to uniform stability.* There, we provided *decidable sufficient conditions for the non-defectivity of constrained switching systems.* In particular, we adapted the notion of *irreducibility*, i.e. the existence of invariant subspace, for the constrained switching setting by incorporating the graph of a constrained switching system in the notions of invariances.
The second case we studied *is when CJSR = 0.* There, we showed that even if extremal norms do not exist (except in trivial cases), we can decide CJSR = 0 in a time which is polynomial in the size of the system.

Chapter 4: In this chapter, we considered constrained switching systems with inputs and outputs. Several tools developed into the literature for (exponential) stability analysis and studied in Chapter 2 have counterparts for input-output stability analysis and stabilizability analysis. By studying these counterparts, we gain a better grasp on these important problems.
For the matter of input-output stability, *we provided an exact characterization of the $\mathcal{L}_p$-gains of constrained switching systems using multiple storage functions.* We then provided *a first converse theorem for the existence of quadratic storage functions for constrained switching systems.* Then, we exhibited a *numeral example highlighting a key challenge* for providing arbitrary accurate estimates of the gain in a manner similar to what we did for the CJSR. We ended that part with a conjecture about the validity of a hierarchy of converse theorems for approximating the $\mathcal{L}_2$-gains. Regarding the matter of the stabilization by feedback of constrained switching systems, *we introduced a class of feedback controllers encompassing several existing ones in the relevant literature.* There again, *we were able to provide converse theorem for the synthesis of such controllers when using LMIs and quadratic multiple control Lyapunov functions.*

Chapter 5: With this chapter begins the second part of the thesis. In the previous chapters, we tackled several challenging problems related to the stability analysis of constrained switching systems. At the core of our techniques is the merging between automata theory and Lyapunov methods following the ideas of the *Path-Complete Framework.* Path-Complete Methods were initially introduced in the arbitrary switching case, to generalize several techniques and to characterize the set of multiple Lyapunov functions that lead to stability certificates. In chapter 5, we first explained how our tools can be made part of an extended Path-Complete framework. Then, we provided a positive *answer to a fundamental question regarding these methods, namely whether or not it is possible to represent any Path-Complete Lyapunov function in the case of* *arbitrary switching systems as a common Lyapunov function.* In doing so, we bridged an existing gap between Path-Dependent methods and classical Lyapunov theory.

Chapter 6: Finally, in this last chapter, we looked at a question that transcends the whole thesis: *can we decide whether a Lyapunov method is more or less conservative than another?* In this context, we provided the first systematic approach to decide whether or not, given a Path-Complete Lyapunov function for a first graph, we can construct a Path-Complete Lyapunov function for another graph. Even if the algebraic formulation of this question is very complex, we provided an algorithm *that boils down to solving a simple linear program* involving only the two graphs characterizing the Path-Complete Lyapunov functions.

Perspectives

Overall, we believe this thesis opens several paths for further research. Several problems are discussed in the Summary sections of each chapter. Here, we focus on three topics: Applications where our tools seem promising, Extensions other classes of systems and finally, the central question of Path-Complete Methods about characterizing conservativeness.

Applications: In Section 2.5, we first apply our tools to the robust stability of *networked controlled systems* and then second to the computation of the convergence rate of a *consensus system*. There, we used our algorithms as black-boxes on instances of constrained switching systems arising as models for these situations.
Naturally, we believe that Path-Complete Methods can be used to further study such systems. Additionally, it seems that further applications are within the reach of our techniques. In particular, at the middle ground between consensus systems and networked control systems lie *decentralized control systems*. Switching systems can be used to model interactions between subsystems assigned to specific tasks. An application of particular interest here can be found in matters regarding *large-scale decentralized computation systems* [Tsi84].

Extension towards a wider class of hybrid systems: In an other direction, we believe that Path-Complete Methods, encompassing the tools within this thesis, should foster novel approaches for the study of other classes of systems. There, several directions can be investigated.
First, we may consider *switching systems in continuous time*, which also find wide areas of applications [LM99, The05]. Several challenges exist, even in the context of arbitrary switching systems in continuous time. An important first step for deploying path-complete methods in this context would be to develop hierarchies of LMIs providing semi-algorithms for deciding the stability of arbitrary switching systems in continuous time, in the image of the path-dependent methods [LD06b].
Another class of systems requiring attention is switching systems with state-dependent switching, e.g. [LDA13]. There, matters of *stabilization by control-*

ling the switching sequences appear of particular importance [KC15, KBC15, FJ14, HKD16].
Finally, one may also consider richer classes of constrained switching systems where the constraints on switching sequences can be richer than those investigated here. Section 4.2.3 provides intuition towards how to deal with classical automata instead of our directed and labeled graph. One may also consider *pushdown* automata and context-free grammars [HMU06] and devise tools and techniques in that more general case.

Conservativeness of Path-Complete methods: Of course, there are several remaining question regarding Path-Complete Methods in the context of this thesis. In our opinion, the key in further evolving these tools is to find a definitive answer to the question of *how conservative is a given Path-Complete Methods relative to the others?*. The question itself needs to be more precise as the notion of conservativeness can be interpreted in different ways. We propose to see conservativeness at this point as the ability to produce stability certificates when using quadratic pieces. The notion of a method being more conservative than another means that every time the first method produces a certificate, the second method will do so as well. Ideally, we would like to be able, given a constrained switching system and a desired accuracy, to select the most efficient path-complete method able to produce a CJSR estimate with the desired accuracy using, for example, quadratic pieces. Here, we propose to define the efficiency of a method through the size of its graph of Lyapunov inequalities.

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