INTERBANK CREDIT RISK MODELLING WITH SELF-EXCITING JUMP PROCESSES

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Abstract. The credit crunch of 2007 caused major changes in the market of interbank rates making the existing interest rate theory inconsistent. This article puts forward one way to reconcile practice and theory by modifying the arbitrage-free condition. In this framework, the forward Libor rate is no longer considered as a risk-free rate and the credit and liquidity risks within the interbank market partly drive its dynamics. In a similar manner to the multiple-curve approach, we model the evolution of default-free rates, assimilated to overnight interest swap rates, and the default times of an interbank market segment determined by its tenor. For each segment, we use the reduced form approach to model the arrival rate of defaults with a self-exciting jump-diffusion process. Then, we deduce the dynamics of the spot forward Libor rates and provide closed-form approximation pricing formulae for options on forward Libor rates and swap rates. Even in a context of negative interest rates and compared to other forms of intensity processes such as a CIR, the self-excitation property allows a better understanding of the spread OIS-IRS and provides information about the interbank credit risk. Furthermore, our framework enables to parse the impact of the interbank credit risk on forward Libor as well as on interest rates derivatives like Caps, Floors, and Swaptions.

keywords Interest rates; multiple-curves; self-exciting process; simple forward Libor rate.

1. Introduction

To the best of our knowledge, up to now there is not a standard theory on the market of interbank rates which explains the observed market rates quotes since the credit crunch of 2007. As the interest rate derivatives market is the largest derivatives market, modelling interbank rates with a realistic and analytically tractable approach is a great challenge for the financial industry. Most of the previous research focuses either on a way to make consistent the interest rate theory with market rates by the so-called multiple curve approach (see e.g. [Henrard(2014)], [Morini(2009)], [Grbac & Runggaldier(2010)]), or on the change on credit quality within the bank panel determining the interbank rate (see e.g. [Filipovic & Trolle(2012)]). However, as underlined by [Mercurio(2010)], model the default time of the interbank market remains a non-trivial task when it comes to a generic counterparty.

From the 1980s to nowadays, many integrated frameworks dedicated to interest rates modelling were proposed in the literature. The common aim of these frameworks is to explain changes in the behavior of market rates quotes, but also to capture emerging risks within the interest rates market. This article focuses on the modelling of forward Libor rates and proposes another framework. Before 2007, interest rate swaps (IRSs) were priced with a single interest rate curve that was totally independent from the frequency of floating payments. Libor was considered as a good proxy for the risk free rate. The deposit rates deduced from Libor, and Overnight indexed swap rates (OIS) for the same maturity chased each other with a small distance of few basis points. Earlier in 2000, authors like [Schonbucher(2000)] integrate the default risk in the Libor market model to care about the risk introduced by changes in the level of the default-free interest rates or in the market's assessment of the credit quality of the obligor. Although neglected, the presence of credit and liquidity risk within the interbank market seemed to be a good justification for the small distance observed between deposit rates from Libor and OIS. After the credit crunch, we witnessed a serious widening of the distance between equivalent rates, the explosion of basis swap rates based on Libor rates with different tenors and the divergence between forward rate agreements (FRAs) and corresponding forward rates implied by deposits. Moreover, FRAs were not necessary hedgeable with a deposit and a loan, leading traditional no arbitrage conditions unsatisfied in practice.

Consequently, the discount curve has to be modeled differently and Libor for different tenors must be analyzed separately. From a single interest rate curve framework we move to a multiple curves framework where the classical interest rate theory as detailed by [Brigo & Mercurio(2006)] can not be applied. In the literature two strands of solution have emerged. The first one, called multiple curves approach by practitioners consists in dealing with the discrepancies between rates by segmenting market rates and labeling them differently according to their application period. In this setting [Bianchetti(2010)] uses two yield curves for market coherent estimation of discount factors and forward rates with different underlying tenor. [Mercurio(2010)] follows the multiple curves approach, and proposes a new Libor market model based on the joint evolution of FRA rates and forward rates belonging to the discount curve. [Ametrano & Bianchetti(2013)] provide the theoretical background necessary for bootstrapping interest curves in the multiple curves framework, while [Fries(2016)] focuses on the interpolation method. [Hull & White(2015)] validates the use of multiple curves by showing that the risk-free rates when pricing derivatives should be bootstrapped from OIS instead of Libor swap rate. [Henrard(2014)] explains from a practical point of view, foundations and evolution of the multiple curves framework, and its implementation in finance. Then, to be consistent with the new feature of interest rates, [Eberlein *et al.*(2017)] model negative rates or extremely low rates in a multiple curve setup.

The second strand of solutions comes from researchers and consists in modelling credit and liquidity risks of panel banks using the counterparties credit risk theory. [Filipovic & Trolle(2012)] follow this approach and infer a term structure of the interbank risks from spreads between rates on IRS indexed to Libor and OIS. They model the time when a deterioration in the credit quality of a bank within the initial Libor banks panel occurs. Their model is fitted to CDS spread term structures for every individual banks panel. Even if this approach is realistic, it remains cumbersome as the composition of panel banks changes dynamically according to the selection criteria. On the other hand, since the credit crunch of 2007-2008 the feedback phenomena [Hurd(2018)] observed between bankruptcies have motivated the use of self-exciting processes in credit risk modelling.

The theory about self-exciting processes was introduced by [Hawkes(1971)]. In the financial literature, these processes were firstly used in high frequency trading around the years 2000s. In credit risk, [Giesecke & Goldberg(2010)] use a self-exciting process to model the cumulative losses of a credit portfolio taking into account the fact that when a loss occurs, the probability that another loss occurs, increases. [Dassios & Zhao(2017)] meanwhile introduce a new class of self-exciting processes that are driven by external factors . They use this process to calculate default probabilities and price zero coupon bonds with a high default probability. [Coonjobeharry *et al.*(2016)] also price convertible bonds with default risk using jump-diffusion processes. Moreover, in the literature, self-exciting point processes are used for the modelling of interest rates. [Hainaut(2016a)] models short term rates via a self-exciting jump process and reproduces the clustering of shocks on the Euro overnight index average. [Hainaut(2016b)] models again the short rate by taking into account the aggregate supply and demand for fixed income instruments modeled via a bi-variate self-exciting point process.

This paper explores another way to reconcile the classical interest rate theory and market quotes by combining a multiple curve approach with a counterparty credit risk model. In our framework, we segment the interbank market into sub-markets according to the tenor of the interbank rate applied. The interbank credit risk of a sub-market, is the risk that a loss occurs from the default of one counterparty in a transaction between two banks. Motivated by the possible succession of default losses due to interconnections between banks, as [Fontana & Runggaldier(2010)] we consider a reduced-form credit risk model where the arrival rate of credit events in a sub-market follows a self-exciting jump-diffusion process. The jump part in the intensity is a point process with positive jumps when a credit event occurs. This ensures that a default raises the probability of the arrival of the next credit event.

The rest of the paper is structured as follows: in Section (2) we introduce notations and definitions. Then, we explain our framework and motivate the choice of self-exciting jump diffusion processes for the default intensity. In Section (3) we derive the forward Libor dynamics under the risk neutral world and study through simulations the impact of jumps on our default intensity. In Section (4), we price options on forward Libor rates and swap rates. For this purpose, we describe the change of measure respectively towards the forward measure and the annuity measure. Then, we use a bivariate Inverse Fourier transform to infer under these measures the joint probability distribution of the short rate and of the intensity process. In Section (5), we consider a bicurves framework involving the discount curve and the six months forward Libor. We calibrate our intensity model and compare it to other models in terms of goodness of fit. Finally, we analyze the sensitivity of the survival probability of an interbank market segment to parameters defining the self-excitation.

2. Notation and basic definitions

Let consider a filtered probability space $(\Omega,\, \mathcal{G},\, \mathbb{Q})$ where the filtration $(\mathcal{G}_t)_{t\geq 0}$ satisfies the usual conditions and $\mathbb{Q}$ is a spot martingale measure. For sake of convenience, quantities that refer to defaultable bond prices or interest rates carry an index 'c'. We shall write $\mathbb{E}^{\mathbb{Q}}(\cdot) = \mathbb{E}(\cdot)$, and unless otherwise specified all expectations are determined under the risk neutral measure $\mathbb{Q}$. We assume that the instantaneous risk-free rate process is adapted to the subfiltration $(\mathcal{F}_t)_{t\geq 0}$ of $(\mathcal{G}_t)_{t\geq 0}$ and is ruled by the Hull and White model:

$$\mathrm{d}r_t = (\theta(t) - ar_t)\mathrm{d}t + \sigma_r \mathrm{d}W_t^r, \tag{2.1}$$

where W_t^r is a Brownian motion, $a \geq 0$, and $\sigma_r \geq 0$. $\theta(t)$ is a deterministic function of time t satisfying:

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma_r^2}{2a}\left(1 - e^{-2at}\right), \tag{2.2}$$

where $f^M(0,t) = -\frac{\partial \ln\left(P^M(0,t)\right)}{\partial t}$ is the instantaneous forward rate deduced from the discount curve observed. Furthermore, our framework may be extended to most of alternative interest rate models. Transactions in the interbank market are exposed to the risk of default of counterparties. Thus, we define the default time of the interbank market as a stopping time, denoted by τ. For instance, this default time may occur when a credit event hits one of the interbank market participants or after a succession of credit events hitting interbank market participants. Let $(\mathcal{T}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by τ and defined by:

$$\mathcal{T}_t = \sigma(\{\tau \leq u\} : u \leq t).$$

For a sake of convenience, we assume that τ satisfies the following distributional constraints:

$$\begin{cases} \mathbb{Q}(\tau < +\infty) = 1 \\ \mathbb{Q}(\tau = 0) = 0 \\ \mathbb{Q}(\tau \geq t) > 0 \text{ for every } t \in \mathbb{R}_+ \end{cases}.$$

The default of the interbank market is triggered by the first jump of a doubly stochastic process $(N_t^\tau)_{t\geq 0}$ that has the same intensity process $(\lambda_t)_{t\geq 0}$ of a self-exciting jump process $(N_t)_{t\geq 0}$. The point process $(N_t)_{t\geq 0}$ records the number of negative shocks that impact the interbank market. Whereas $(\lambda_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are dependent, $(\lambda_t)_{t\geq 0}$ and $(N_t^\tau)_{t\geq 0}$ are independent. This intensity process is adapted to the subfiltration $(\mathcal{H}_t)_{t\geq 0}$. The augmented filtration $(\mathcal{G}_t)_{t\geq 0}$ is defined by:

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \vee \mathcal{T}_t, \text{ for any } t \in \mathbb{R}_+.$$

Unlike the process N_t^τ, the process N_t is a self-exciting process which is not an inhomogeneous Poisson process conditionally to the filtration of the intensity up to time t. The information about default in the interbank market will be recovered from the term structures of swap rates. According to the definition of a doubly stochastic process, for all $s > t$,

$$\mathbb{Q}\left(N_s^\tau - N_t^\tau = 0 \mid \mathcal{G}_t \vee \mathcal{H}_s\right) = \exp\left(-\int_t^s \lambda_u \mathrm{d}u\right). \tag{2.3}$$

The conditional survival probability is defined by

$$\mathbb{Q}\left(\tau \geq T \mid \tau \geq t\,,\, \mathcal{H}_t\right) = \mathbb{E}\left(e^{-\int_t^T \lambda_u \mathrm{d}u} | \mathcal{H}_t\right). \tag{2.4}$$

Interconnections between participants in the interbank market can lead to a succession of credit events that are hard to explain if the intensity is modeled by a diffusion process. For this reason, we work with an intensity process ruled by the following stochastic differential equation:

$$\mathrm{d}\lambda_t \;=\; \kappa(\gamma - \lambda_t)\mathrm{d}t + \sigma_\lambda \sqrt{\lambda_t}\mathrm{d}W_t^\lambda + \eta \mathrm{d}N_t \tag{2.5}$$

where κ is the rate of mean reversion, γ is the long-run level, σ_λ is the volatility coefficient, η is the jump in the intensity when a credit event occurs and W_t^λ is a standard Brownian motion. The diffusion part of this process is the Cox Ingersoll Ross (CIR) model which ensures the non-negativity of the intensity. Since N_t has λ_t for intensity, a negative shock increases the arrival rate of the next negative shock proportionally to η. This intensity process is driven by the SDE (2.5):

$$\lambda_T = \gamma + (\lambda_t - \gamma)\, e^{-\kappa(T-t)} + \sigma_\lambda \int_t^T e^{-\kappa(T-u)} \sqrt{\lambda_u}\mathrm{d}W_u^\lambda + \eta \int_t^T e^{-\kappa(T-u)} \mathrm{d}N_u. \tag{2.6}$$

The analytical tractability of this model allows us to deduce its first and second order moments by solving the partial differential equations derived from the Dynkin formula (see DH for details),

$$\mathbb{E}[\lambda_T \mid \mathcal{G}_t] = -\frac{\kappa\gamma}{\eta - \kappa} + \left(\lambda_t + \frac{\kappa\gamma}{\eta-\kappa}\right) e^{(\eta-\kappa)(T-t)}, \tag{2.7}$$

$$\begin{aligned}\mathbb{E}[\lambda_T^2 \mid \mathcal{G}_t] &= \lambda_t^2 e^{-a_1(T-t)} + \frac{a_2 \lambda_\infty}{a_1}\left(1 - e^{-a_1(T-t)}\right) \\ &\quad + \frac{a_2\left(\lambda_t - \lambda_\infty\right)}{3\eta + \kappa}\left(e^{(\eta-\kappa)(T-t)} - e^{-a_1(T-t)}\right),\end{aligned} \tag{2.8}$$

where $a_1 = 2\left(\eta + \kappa\right)$ and $a_2 = 2\kappa\gamma + \sigma_\lambda^2 + \eta^2$. Equation (2.7) involves to choose $\eta \leq \kappa$, in order to ensure that the expected arrival rate of credit events denoted by λ_∞ is finite when $T \to \infty$. This long term mean λ_∞ is equal to $-\frac{\kappa\gamma}{\eta-\kappa}$. The figure (2.1) shows the simulated path of processes $(\lambda_t)_{t\geq 0}\,, (N_t)_{t\geq 0}$ and $(N_t^\tau)_{t\geq 0}$ for $\kappa = 2$, $\gamma = 1.5$, $\sigma = 1$, $\eta = 1$ and $\lambda_0 = 0.75$. From this graph, we see that the default of the interbank market is not directly linked to the default of one of its members. Moreover, a default of an interbank member raises the probability of the arrival of the next default among members as well as the default of interbank market himself. The next proposition presents the moment generating function of the cumulated arrival rate of credit events over a time interval called the compensator and denoted by $\int_t^T \lambda_s \mathrm{d}s$.

Proposition 1. *For $\omega \in \mathbb{C}$, let $\psi(t, \lambda_t, N_t) = \mathbb{E}\left(e^{\omega \int_t^T \lambda_s ds} | \mathcal{F}_t\right)$ be the moment generating function of $\int_t^T \lambda_s ds$. There exist two multivariate functions C, and D such that*

$$\psi(t, \lambda_t, N_t) = \exp\left(C(t,T) + D(t,T)\lambda_t\right) \tag{2.9}$$

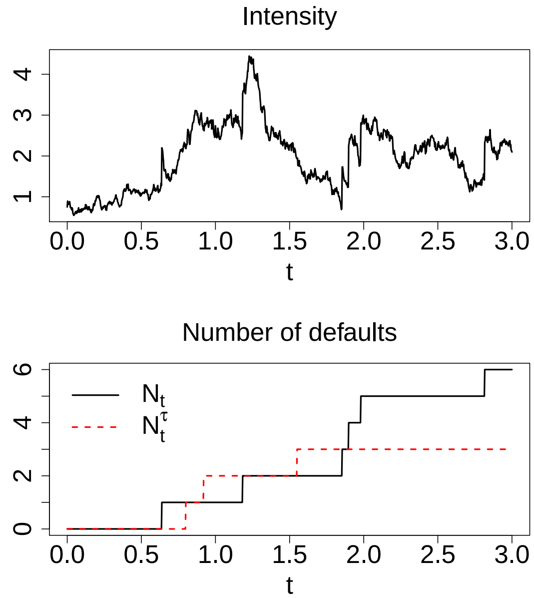

FIGURE 2.1. Simulated path of the intensity process and the counting processes over a period three years. The intensity parameters considered are $\kappa = 2$, $\gamma = 1.5$, $\sigma = 1$, $\eta = 1$ and $\lambda_0 = 0.75$. The arrival rate of credit events increases with an amplitude of 1 at each jump of the process N_t.

where C and D are solutions of the following partial differential equations (PDE)

$$\begin{cases} \partial_t C(t,T) + \kappa\gamma D(t,T) & = 0 \\ \partial_t D(t,T) + \omega - \kappa D(t,T) + \frac{1}{2}\sigma_\lambda^2 (D(t,T))^2 + \exp(D(t,T)\eta) - 1 & = 0 \end{cases}, \tag{2.10}$$

with the terminal conditions $C(T,T) = 0$ and $D(T,T) = 0$.

Proof. Let us define

$$Y_t \;:=\; \mathbb{E}\left(e^{\omega\int_t^T \lambda_s \mathrm{ds}} | \mathcal{H}_t\right) \tag{2.11}$$

As $\mathcal{H}_t \subset \mathcal{H}_u$ for any $u \geq t$, applying the rule of conditional expectations leads to:

$$\begin{aligned} Y_t &= \mathbb{E}\left(e^{\omega\int_t^u \lambda_s \mathrm{ds}} \mathbb{E}\left(e^{\omega\int_u^T \lambda_s \mathrm{ds}} \,|\, \mathcal{H}_u\right) \,|\, \mathcal{H}_t\right) \\ &= \mathbb{E}\left(e^{\omega\int_t^u \lambda_s \mathrm{ds}} Y_u \,|\, \mathcal{H}_t\right). \end{aligned} \tag{2.12}$$

Then, the following limit converges to zero:

$$\lim_{u\to t} \frac{\mathbb{E}\left(e^{\omega\int_t^u \lambda_s \mathrm{ds}} Y_u \,|\, \mathcal{H}_t\right) - Y_t}{u - t} \;=\; 0. \tag{2.13}$$

By assuming enough regularity and using the first order Taylor approximation of the exponential, this limit is rewritten as follows:

$$\lim_{u\to t} \frac{\mathbb{E}\left(Y_u \,|\, \mathcal{H}_t\right) - Y_t}{u - t} = -\omega\lambda_t Y_t\,, \tag{2.14}$$

in which the left term is the infinitesimal generator of the moment generating function. So as to lighten future calculations, let us denote $f(t,\lambda_t,N_t):=Y_t$ and f_t , f_λ, $f_{\lambda\lambda}$ the partial derivatives of f with respect to time, intensity and intensity twice. The Equation (2.14) is equal to $\mathcal{A}f=\omega\lambda_t f$, or after developments to:

$$\begin{aligned}\mathcal{A}f &= f_t+\kappa(\gamma-\lambda_t)f_\lambda+\frac{1}{2}\sigma_\lambda^2\lambda_t f_{\lambda\lambda}\\ &\quad +\left(f\left(t,\lambda_t+\eta,N_t+1\right)-f(t,\lambda_t,N_t)\right)\lambda_t,\end{aligned} \tag{2.15}$$

with terminal condition

$$f(T,\lambda_T,N_T)=1.$$

We must then solve the next SDE

$$\begin{aligned}-\omega\lambda_t f &= f_t+\kappa(\gamma-\lambda_t)f_\lambda+\frac{1}{2}\sigma_\lambda^2\lambda_t f_{\lambda\lambda}\\ &\quad +\left(f\left(t,\lambda_t+\eta,N_t+1\right)-f(t,\lambda_t,N_t)\right)\lambda_t.\end{aligned} \tag{2.16}$$

We do the assumption that $f=\exp\left(C(t,T)+D(t,T)\lambda_t+H(t,T)N_t\right)$. Then we rewrite Equation (2.16) as follows

$$\begin{aligned}0 &= \left(\partial_t C\left(t,T\right)+\partial_t D\left(t,T\right)\lambda_t+\partial_t H\left(t,T\right)N_t\right)+\omega\lambda_t+\kappa(\gamma-\lambda_t)D\left(t,T\right)\\ &\quad +\frac{1}{2}\sigma_\lambda^2\lambda_t D\left(t,T\right)^2+\left(\exp\left(D(t,T)\eta+H(t,T)\right)-1\right)\lambda_t,\end{aligned} \tag{2.17}$$

equivalent to the following PDE system:

$$\begin{cases}\partial_t C\left(t,T\right)+\kappa\gamma D\left(t,T\right) & =0\\ \partial_t D\left(t,T\right)+\omega-\kappa D\left(t,T\right)+\frac{1}{2}\sigma_\lambda^2 D\left(t,T\right)^2+\left(\exp\left(D(t,T)\eta+H(t,T)\right)-1\right) & =0\\ \partial_t H\left(t,T\right) & =0\end{cases}.$$

Given that $\partial_t H\left(t,T\right)=0$, and $f(T,\lambda_T,N_T)=1$ then $H=0$. □

Hereafter, we define the different type of bond prices used in the sequel. Let $\mathcal{L}^F=\{T_{\alpha_1},T_{\alpha_1+1},\cdots,T_{\beta_1}\}$ and $\mathcal{L}^S=\{S_{\alpha_2},S_{\alpha_2+1},\cdots,S_{\beta_2}\}$ be two collections of time. The distances between two successive times are denoted respectively by δ^F and δ^S for the time structures $\mathcal{L}^F$ and $\mathcal{L}^S$. Let $R\in[0,\,1]$ be the recovery rate, which is the fraction of the bond's face value repaid to the bondholder in case of default. The price of a default risk-free discount bond at time t with maturity T_k is defined by:

$$P(t,T_k)=\mathbb{E}\left(e^{-\int_t^{T_k}r_s\mathrm{d}s}|\mathcal{F}_t\right). \tag{2.18}$$

From [Brigo & Mercurio(2006)] part (3.3.2), we know that:

$$P(t,T_k)=\exp\left(A(t,T_k)-B(t,T_k)r_t\right), \tag{2.19}$$

where the functions A and B are given by:

$$A = \ln\left(\frac{P^M(0,T_k)}{P^M(0,t)}\right) + Bf^M(0,t) - \frac{\sigma_r^2\left(1-e^{-2at}\right)}{4a}B^2, \tag{2.20}$$

$$B = \frac{1-\exp(-a(T_k-t))}{a}. \tag{2.21}$$

In practice, the prices of default risk-free discount bonds are obtained by bootstrapping the OIS swap curve, as done by [Smith(2013)] and implemented in the last section of this article.

The price of a defaultable discount bond at time t with maturity T_k is denoted by $P^c(t,T_k)$. Under the assumption that the time of default τ and the risk free rate process r_t are independent, this price is equal to the following product:

$$P^c(t,T_k) = P(t,T_k)\left(R + 1_{\tau\geq t}(1-R)\,\mathbb{Q}\left(\tau \geq T_k \,|\, \tau \geq t\,,\, \mathcal{H}_t\right)\right). \tag{2.22}$$

We now define forward rates and swap rates. The default-free simple forward Libor rate over $[T_k, T_{k+1}]$ as seen from time t is the rate of interest that must apply between times T_k and T_{k+1} so that the price at T_k of a discount bond maturing at T_{k+1} be equal to the forward price. This default-free simple forward Libor is:

$$F(t,T_k,T_{k+1}) \;=\; \frac{1}{\delta_k^F}\left(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1\right), \tag{2.23}$$

whereas the defaultable simple forward Libor rate over $[T_k, T_{k+1}]$ seen from time t is:

$$F^c(t,T_k,T_{k+1}) = \frac{1}{\delta_k^F}\left(\frac{P^c(t,T_k)}{P^c(t,T_{k+1})} - 1\right). \tag{2.24}$$

The swap rate associates to an interest rate swap defined on the time structure $\mathcal{L}^F$ and $\mathcal{L}^S$ is the value of the fixed rate such that the corresponding swap value is zero. This swap rate is given by:

$$S_{\alpha,\beta}(t) \;=\; \frac{\sum_{i=\alpha_1+1}^{\beta_1}\delta^F\,P(t,T_i)\,F^c(t,T_{i-1},T_i)}{\sum_{j=\alpha_2+1}^{\beta_2}P(t,S_j)}. \tag{2.25}$$

We end up this section by a brief review of the modification of the arbitrage-free condition in presence of counterparty risk. A good no-arbitrage relation takes into account the fact that transactions in the interbank market are not riskless. Under the assumption of no-arbitrage, the unitary FRA payoff can be replicated at time t with the following operations: the payer of this FRA borrows $\left(1+K^{FRA}(T_k-T_{k+1})\right)P^c(t,T_{k+1})$ till T_{k+1}, lends $P^c(t,T_k)$ till T_k and reinvests from T_k to T_{k+1} at the defaultable simple forward Libor. Then, the price of FRA is then

$$FRA(t;T_k;T_{k+1};K) = P^c(t,T_k) - \left(1+K^{FRA}(T_k-T_{k+1})\right)P^c(t,T_{k+1}), \tag{2.26}$$

where the rate K^{FRA} that cancels the FRA at the inception t is given by

$$K^{FRA} = F^c(t,T_k,T_{k+1}) = \frac{1}{\delta_k^F}\left(\frac{P^c(t,T_k)}{P^c(t,T_{k+1})} - 1\right). \tag{2.27}$$

Therefore, it follows that

$$F^c(t,T_k,T_{k+1}) \quad = \quad \frac{1}{\delta_k^F}\left(\frac{P(t,T_k)\,(R+1_{\tau\geq t}(1-R)\,\mathbb{Q}\,(\tau\geq T_k\,|\,\tau\geq t\,,\,\mathcal{H}_t))}{P(t,T_{k+1})\,(R+1_{\tau\geq t}(1-R)\,\mathbb{Q}\,(\tau\geq T_{k+1}\,|\,\tau\geq t\,,\,\mathcal{H}_t))}-1\right).$$

In the rest of this paper, we assume that the recovery rate is null, meaning that transactions are totally non collateralized. Since

$$\frac{\mathbb{Q}\,(\tau\geq T_{k+1}\,|\,\tau\geq t\,,\,\mathcal{H}_t)}{\mathbb{Q}\,(\tau\geq T_k\,|\,\tau\geq t\,,\,\mathcal{H}_t)}=\mathbb{Q}\,(\tau\geq T_{k+1}\,|\,\tau\geq T_k\,,\,\mathcal{H}_t)\,, \tag{2.28}$$

it follows that

$$F^c(t,T_k,T_{k+1})=\frac{1}{\delta_k^F}\left(\frac{P(t,T_k)}{P(t,T_{k+1})}\frac{1}{\mathbb{Q}\,(\tau\geq T_{k+1}\,|\,\tau\geq T_k\,,\,\mathcal{H}_t)}-1\right). \tag{2.29}$$

If market FRA rates and theoretical FRA rates are respectively denoted by $F^c(t,T_k,T_{k+1})$ and $F(t,T_k,T_{k+1})$, we observe that $F^c(t,T_k,T_{k+1})\geq F(t,T_k,T_{k+1})$ since $\frac{1}{\mathbb{Q}(\tau\geq T_{k+1}\,|\,\tau\geq T_k\,,\,\mathcal{H}_t)}\geq 1$. With a proof by contradiction, we deduce as [Bianchetti(2010)] that an equality between our market FRA rates and the theoretical FRA rates leads to an arbitrage. This equality holds only for a transaction with a collateral or an overnight tenor. In the sequel, under the assumption that there is no collateral we work with F^c. Since, FRA contracts and OIS bonds or default risk-free discount bonds form the basic traded assets within the interbank market, the new no arbitrage condition is such that under the risk neutral measure $\mathbb{Q}$ the discounted prices of OIS bonds and FRA contracts are martingales. As we set our models under the martingale measure $\mathbb{Q}$, discounted OIS bond prices are martingales under $\mathbb{Q}$. Furthermore, show that the discounted FRA contract prices are martingales under $\mathbb{Q}$ is equivalent to show that FRA contract prices are martingales under forward measures. Let consider a FRA contract concluded at time t over $[T_k,T_{k+1}]$ where the nominal is equal to one, the fix rate is K^{FRA} and the floating rate is $L^c\,(T_k,T_{k+1})$. The price of this FRA contract is given by :

$$\begin{aligned} FRA\left(t;T_k;T_{k+1};K^{FRA}\right) \quad &= \quad \delta_k^F P\,(t,T_{k+1}) \\ &\quad \times\mathbb{E}^{\mathbb{Q}_{T_{k+1}}}\left[L^c\,(T_k,T_{k+1})-K^{FRA}\mid\mathcal{G}_t\right], \end{aligned} \tag{2.30}$$

where $\mathbb{Q}_{T_{k+1}}$ is the forward measure. The fix rate K^{FRA} that cancels the FRA at the inception t is then:

$$F^c\,(t,T_k,T_{k+1})=\mathbb{E}^{\mathbb{Q}_{T_{k+1}}}\left[L^c\,(T_k,T_{k+1})\mid\mathcal{G}_t\right]. \tag{2.31}$$

From the two last equations the interbank market is arbitrage-free if the ratio $\frac{P^c(t,T_k)}{P^c(t,T_{k+1})}$ is martingale under $\mathbb{Q}_{T_{k+1}}$. This, yields to the following condition:

$$\exp\left(-\eta\Delta_k D(t)\right)-1=\frac{-\frac{1}{2}\Delta_k D(t)^2\eta^2+\sigma_\lambda^2 D(t,T_k)\Delta_k D(t)\lambda_t}{(\lambda_t e^{\eta D(t,T_{k+1})}+1)}, \tag{2.32}$$

where $\Delta_k D(t)=D(t,T_{k+1})-D(t,T_k)$. In appendix (7.1) we detail how we derive this arbitrage-free condition.

3. Forward Libor dynamics

The forward Libor rate differs from one tenor to another. This is due to the difference in exposure times to default risk. But also by the demand for loans of each tenor. Then, we subdivide the interbank market into sub-markets according to the tenor of the transactions. The forward Libor in equation (2.29) depends on the tenor representing an interbank market segment. Instead of a multiplicative spread between forward and discounting curves as proposed by [Henrard(2014)], we have in Equation (2.29) an additional term defined as follows:

$$\beta(t,T_k,T_{k+1}) = \frac{1}{\mathbb{Q}\left(\tau \geq T_{k+1} \,|\, \tau \geq T_k\,,\, \mathcal{H}_t\right)}, \tag{3.1}$$

that is always greater or equal to one. From equations (3.1) and (2.29), when the probability of default over a time period $[T_k, T_{k+1}]$ rises, both the factor β and the forward Libor rate increases. In the sequel of this paper, the term structure of the interbank credit risk for a tenor δ is defined by $\{T \longmapsto \beta(t,T,T+\delta),\ T>t\}$. Hence, at time t, the measure of the credit risk within the interbank market segment associated to the tenor δ for a time period $[T, T+\delta]$ is given by:

$$\begin{aligned}
\beta(t,T,T+\delta) &= \frac{1}{\mathbb{Q}\left(\tau \geq T+\delta \,|\, \tau \geq T\,,\, \mathcal{H}_t\right)} \\
&= \frac{\mathbb{Q}\left(\tau \geq T \,|\, \tau \geq t\,,\, \mathcal{H}_t\right)}{\mathbb{Q}\left(\tau \geq T+\delta \,|\, \tau \geq t\,,\, \mathcal{H}_t\right)} \\
&= \frac{\mathbb{E}\left(e^{-\int_t^T \lambda_u \mathrm{d}u} | \mathcal{H}_t\right)}{\mathbb{E}\left(e^{-\int_t^{T+\delta} \lambda_u \mathrm{d}u} | \mathcal{H}_t\right)}.
\end{aligned}$$

From equation (2.9), we infer that

$$\beta(t,T,T+\delta) = \exp\left(C(t,T) - C(t,T+\delta) + \left(D(t,T) - D(t,T+\delta)\right)\lambda_t\right). \tag{3.2}$$

Thus, the term structure of the interbank risk is fully determined by the bivariates functions C and D. The simple[1] spot forward Libor for $[t,T]$ is given by:

$$\begin{aligned}
L^c(t,T) &= F^c(t,t,T) \\
&= \frac{1}{\delta(t,T)}\left[\frac{1}{P^c(t,T)} - 1\right],
\end{aligned}$$

where $P^c(t,T)$ is the defaultable bond price:

$$\begin{aligned}
P^c(t,T) &= 1_{\tau \geq t} P(t,T)\, \mathbb{Q}\left(\tau \geq T \,|\, \tau \geq t\,,\, \mathcal{H}_t\right) \\
&= 1_{\tau \geq t} \mathbb{E}\left(e^{-\int_t^T r_s \mathrm{d}s} | \mathcal{F}_t\right)\, \mathbb{E}\left(e^{-\int_t^T \lambda_u \mathrm{d}u} | \mathcal{H}_t\right) \\
&= 1_{\tau \geq t} \exp\left(A(t,T) - B(t,T) r_t + C(t,T) + D(t,T)\lambda_t\right).
\end{aligned} \tag{3.3}$$

[1] We work with simple rate because markets use the simple capitalization.

Hence, if $\tau > t$, the Libor rate is

$$L^c(t,T) = \frac{1}{\delta\left(t,T\right)}\left[\exp\left(-A(t,T) - C(t,T) + B(t,T)r_t - D(t,T)\lambda_t\right) - 1\right]. \tag{3.4}$$

Whereas the simple forward Libor is given by:

$$\begin{aligned} F^c(t,T,S) = & \left[\exp\left(A(t,T) - A(t,S) + C(t,T) - C(t,S) + \left(B(t,S) - B(t,T)\right)r_t\right)\right. \\ & \left.\times \exp\left(-\left(D(t,S) - D(t,T)\right)\lambda_t\right) - 1\right] \times \frac{1}{\delta\left(T,S\right)}. \end{aligned} \tag{3.5}$$

In equations (3.4) and (3.5) we note that if D is a bivariate function with negatives values, a jump in the intensity process increases both the simple spot forward Libor and the simple forward Libor. Applying the Itô's lemma for jump processes to equations (3.4) and (3.5) allows us to establish the dynamics of these rates in the next proposition.

Proposition 2. *Under the risk neutral measure,*

(i) the stochastic differential equation of a simple spot Libor is given by

$$\begin{aligned} dL^c(t,T) = & \left(\frac{1}{\delta\left(t,T\right)} + L^c(t,T)\right)\left[\left(H_1\left(t,T\right) + H_2\left(t,T\right)r_t + H_3\left(t,T\right)\lambda_t\right)dt\right. \\ & + B\left(t,T\right)\sigma_r dW_t^r - \sigma_\lambda D\left(t,T\right)\sqrt{\lambda_t}dW_t^\lambda \\ & \left. + \eta D\left(t,T\right)\left(\frac{1}{2}\eta D\left(t,T\right) - 1\right)dN_t\right] + \left(L^c\left(t,T\right) - L^c(t^-,T)\right)dN_t\,, \end{aligned} \tag{3.6}$$

where H_1, H_2 and H_3 are solutions of the following PDE system:

$$\begin{cases} H_1\left(t,T\right) = -\partial_t A\left(t,T\right) - \partial_t C\left(t,T\right) + \theta\left(t\right)B\left(t,T\right) + \frac{1}{2}\sigma_r^2 B^2\left(t,T\right) \\ \qquad\qquad -\kappa\gamma D\left(t,T\right), \\ H_2\left(t,T\right) = \partial_t B\left(t,T\right) - aB\left(t,T\right), \\ H_3\left(t,T\right) = \frac{1}{2}\sigma_\lambda^2 D^2\left(t,T\right) - \partial_t D\left(t,T\right) + \kappa D\left(t,T\right). \end{cases} \tag{3.7}$$

(ii) the defaultable forward Libor dynamics is given by

$$\begin{aligned} dF^c(t,T,S) = & \left[\left(R_1\left(t,T,S\right) + R_2\left(t,T,S\right)r_t + R_3\left(t,T,S\right)\cdot\lambda_t\right)dt\right. \\ & + \left(B(t,S) - B(t,T)\right)\sigma_r dW_t^r - \sigma_\lambda\left(D(t,S) - D(t,T)\right)\sqrt{\lambda_t}dW_t^\lambda \\ & \left. + \eta\left(D(t,S) - D(t,T)\right)\left(\frac{1}{2}\eta\left(D(t,S) - D(t,T)\right) - 1\right)dN_t\right] \\ & \times\left(\frac{1}{\delta} + F^c(t,T,S)\right) + \left(F^c(t,T,S) - F^c(t^-,T,S)\right)dN_t\,, \end{aligned} \tag{3.8}$$

where R_1, R_2 and R_3 are solutions of the following PDE system

$$
\begin{cases}
R_1(t,T,S) = \partial_t A(t,T) - \partial_t A(t,S) + \partial_t C(t,T) - \partial_t C(t,S) \\
\qquad + \theta(t)\left(B(t,S) - B(t,T)\right) + \frac{1}{2}\sigma_r^2 \left(B(t,S) - B(t,T)\right)^2 \\
\qquad - \kappa\gamma \left(D(t,S) - D(t,T)\right), \\
R_2(t,T,S) = \left(\partial_t B(t,S) - \partial_t B(t,T)\right) - a\left(B(t,S) - B(t,T)\right), \\
R_3(t,T,S) = \frac{1}{2}\sigma_\lambda^2 \left(D(t,S) - D(t,T)\right)^2 - \left(\partial_t D(t,S) - \partial_t D(t,T)\right) \\
\qquad + \kappa\left(D(t,S) - D(t,T)\right).
\end{cases}
\tag{3.9}
$$

Proof. (i) We define the function $g \in \mathcal{C}^{1,2,2}$ such as

$$
g(t,\, y,\, z) = \frac{1}{\delta(t,T)}\left[\exp\left(-A(t,T) - C(t,T) + B(t,T)y - D(t,T)z\right) - 1\right], \tag{3.10}
$$

by multidimensional Ito's lemma we can write

$$
\begin{aligned}
\mathrm{d}L^c(t,T) = {} & \frac{\partial g}{\partial t}(t,r_t,\lambda_t)\mathrm{d}t + \frac{\partial g}{\partial y}(t,r_t,\lambda_t)\mathrm{d}r_t + \frac{\partial g}{\partial z}(t,r_t,\lambda_t)\mathrm{d}\lambda_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,r_t,\lambda_t)\mathrm{d}\left[r_t,r_t\right]^c \\
& + \frac{1}{2}\frac{\partial^2 g}{\partial z^2}\mathrm{d}\left[\lambda_t,\lambda_t\right]^c + \left(L^c(t,T) - L^c(t^-,T)\right)\mathrm{d}N_t\,,
\end{aligned}
\tag{3.11}
$$

with

$$
\begin{aligned}
& \frac{\partial g}{\partial t}(t,r_t,\lambda_t) = \left(-A_t - C_t + B_t r_t - D_t\lambda_t\right)\left(\frac{1}{\delta} + L^c(t,T)\right), \\
& \frac{\partial g}{\partial y}(t,r_t,\lambda_t) = B\left(\frac{1}{\delta} + L^c(t,T)\right),\ \frac{\partial g}{\partial z}(t,r_t,\lambda_t) = -D\left(\frac{1}{\delta} + L^c(t,T)\right), \\
& \frac{\partial^2 g}{\partial y^2}(t,r_t,\lambda_t) = B^2\left(\frac{1}{\delta} + L^c(t,T)\right),\ \frac{\partial^2 g}{\partial z^2}(t,r_t,\lambda_t) = D^2\left(\frac{1}{\delta} + L^c(t,T)\right), \\
& \mathrm{d}\left[r_t,r_t\right]^c = \sigma_r^2\mathrm{d}t \ \text{ and } \mathrm{d}\left[\lambda_t,\lambda_t\right]^c = \sigma_\lambda^2\lambda_t\mathrm{d}t + \eta^2\mathrm{d}N_t.
\end{aligned}
$$

Then by replacing each term of equation (3.11) we retrieve the equation (3.6). Similarly, we can proof the point (ii). □

The results in proposition (2) is important for pricing of option on forward Libor. In the next, section we discuss in detail the pricing of some interest rates derivatives.

4. The pricing of option on spot forward Libor, and swap

This section illustrates how our model can be used for the pricing of interest rates derivatives under a forward measure or an annuity measure. Let us define by $\Pi\left(L^c(T,S)\right)$ the payoff paid at time $S \geq T$ by a European option written on $L^c(T,S)$, which is driven by Equation (3.4). The price of such an option under the risk neutral measure is given by:

$$
\mathrm{C}(t,r_t,\lambda_t) = \mathbb{E}\left[e^{-\int_t^S r_u \mathrm{d}u}\Pi\left(L^c(T,S)\right) \mid \mathcal{G}_t\right]. \tag{4.1}
$$

However, due to the dependence between the discount factor $e^{-\int_t^S r_u \mathrm{d}u}$ and the payoff $\Pi\left(L^c\left(T,S\right)\right)$, we change of numeraire. This technique allows us to switch to the forward measure and simplifies the calculation of the price. Since under the forward measure the price of any financial assets divided by the numeraire $P\left(\cdot,S\right)$ is a martingale, therefore the option price becomes

$$\begin{aligned} \mathrm{C}\left(t,r_t,\lambda_t\right) &= P\left(t,S\right)\mathbb{E}^{\mathbb{Q}_S}\left[\Pi\left(L^c\left(T,S\right)\right) \mid \mathcal{G}_t\right] \\ &= P\left(t,S\right)\int_0^{+\infty}\int_{-\infty}^{+\infty}\Pi\left(y_1,y_2\right)f_{r_T,\lambda_T}\left(y_1,y_2\right)\mathrm{d}y_1\mathrm{d}y_2 \end{aligned} \tag{4.2}$$

where f_{r_T,λ_T} is the joint density function of $r_T \mid \mathcal{G}_t$ and $\lambda_T \mid \mathcal{G}_t$ under the forward measure. The Radon-Nikodym density associated to this change of measure is given by:

$$\frac{\mathrm{d}\mathbb{Q}_S}{\mathrm{d}\mathbb{Q}} = e^{-\int_0^S r_u \mathrm{d}u}\frac{1}{P\left(0,S\right)},$$

where

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}_S}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_t\right] = e^{-\int_0^t r_u \mathrm{d}u}\frac{P\left(t,S\right)}{P\left(0,S\right)}.$$

The calculation of the expected payoff under $\mathbb{Q}_S$ requires to determine the joint density function f_{r_T,λ_T}. The joint density function f_{r_T,λ_T} does not admit any closed form expression. However, a numerical solution can be found with the discrete Fourier transform (DFT) method. To implement this method we need the moment generating function of $r_T + \lambda_T$ that is detailed in the next proposition.

Proposition 3. *Under the measure $\mathbb{Q}_S$ the moment generating function of $r_T + \lambda_T$ denoted by $\Phi_{T,S}^{r,\lambda}$ is equal to*

$$\begin{aligned} \Phi_{T,S}^{r,\lambda}\left(\omega_1,\omega_2\right) &= \mathbb{E}^{\mathbb{Q}_S}\left[e^{\omega_1 r_T + \omega_2\lambda_T} \mid \mathcal{G}_t\right] \\ &= e^{G_1(t,T)-A(t,S)+(G_2(t,T)+B(t,S))r_t+G_3(t,T)\lambda_t}, \end{aligned} \tag{4.3}$$

where $(\omega_1,\omega_2)\in\mathbb{C}$, the bivariate functions $A(\cdot,\cdot)$, $B\left(\cdot,\cdot\right)$are defined by Equations (2.21). Whereas $G_1(\cdot,\cdot)$, $G_2\left(\cdot,\cdot\right)$ and $G_3\left(\cdot,\cdot\right)$ are solution of the following PDE system:

$$\begin{cases} \partial_t G_1\left(t,T\right) + \theta\left(t\right)G_2\left(t,T\right) + \kappa\gamma G_3\left(t,T\right) + \frac{1}{2}\sigma_r^2\left(G_2\left(t,T\right)\right)^2 & = 0 \\ \partial_t G_2\left(t,T\right) - aG_2\left(t,T\right) & = 1 \\ \partial_t G_3\left(t,T\right) - \kappa G_3\left(t,T\right) + \frac{1}{2}\sigma_\lambda^2\left(G_3\left(t,T\right)\right)^2 + \left(e^{\eta G_3(t,T)}-1\right) & = 0 \end{cases}, \tag{4.4}$$

and satisfy the terminal conditions:

$$\begin{cases} G_1\left(T,T\right) & = A\left(T,S\right) \\ G_2\left(T,T\right) & = \omega_1 - B(T,S) \\ G_3\left(T,T\right) & = \omega_2 \end{cases}.$$

Proof. By definition of the forward measure and using the fact that $\mathcal{G}_t \subset \mathcal{G}_T$, the moment generating function of $r_T + \lambda_T$ is given by:

$$\phi^{r,\lambda}_{T,S}(\omega_1,\omega_2) = \frac{\mathbb{E}\left[\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}_S}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_T\right] e^{\omega_1 r_T + \omega_2 \lambda_T} \mid \mathcal{G}_t\right]}{\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}_S}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_t\right]}$$

$$= \frac{\mathbb{E}\left[e^{-\int_t^T r_u \mathrm{d}u + A(T,S) + (\omega_1 - B(T,S)) r_T + \omega_2 \lambda_T} \mid \mathcal{G}_t\right]}{e^{A(t,S) - B(t,S) r_t}}. \tag{4.5}$$

In the sequel of this proof, we focus on the evaluation of

$$Y_t = \mathbb{E}\left[e^{-\int_t^T r_u \mathrm{d}u + A(T,S) + (\omega_1 - B(T,S)) r_T + \omega \lambda_T} \mid \mathcal{G}_t\right], \tag{4.6}$$

and find its closed form thanks to the infinitesimal generator properties for Markov processes. As in the proposition (1) we can proof that for $Y_t = f(t, r_t, \lambda_t, N_t)$

$$\lim_{v \to t} \frac{\mathbb{E}\left(Y_v \mid \mathcal{G}_t\right) - Y_t}{v - t} = r_t Y_t = \mathcal{A} f.$$

Then, using the Itô Lemma the infinitesimal generator is

$$\mathcal{A} f = \partial_t f + (\theta(t) - a r_t)\partial_r f + \kappa(\gamma - \lambda_t)\partial_\lambda f + \frac{1}{2}\sigma_r^2 \partial_r^2 f + \frac{1}{2}\sigma_\lambda^2 \lambda_t \partial_\lambda^2 f$$
$$+ \lambda_t \left(f(t, r_t, \lambda_t + \eta, N_t + 1) - f(t, r_t, \lambda_t, N_t)\right). \tag{4.7}$$

Assuming that $f(t, r_t, \lambda_t, N_t) = e^{G_1(t,T) + G_2(t,T) r_t + G_3(t,T)\lambda_t}$ we deduce the following ODEs system

$$\begin{cases} \partial_t G_1(t,T) + \theta(t) G_2(t,T) + \kappa\gamma G_3(t,T) + \frac{1}{2}\sigma_r^2 \left(G_2(t,T)\right)^2 & = 0 \\ \partial_t G_2(t,T) - a G_2(t,T) & = 1 \\ \partial_t G_3(t,T) - \kappa G_3(t,T) + \frac{1}{2}\sigma_\lambda^2 \left(G_3(t,T)\right)^2 + \left(e^{\eta G_3(t,T)} - 1\right) & = 0 \end{cases},$$

with the following terminal conditions

$$\begin{cases} G_1(T,T) & = A(T,S) \\ G_2(T,T) & = \omega_1 - B(T,S) \\ G_3(T,T) & = \omega_2 \end{cases}.$$

It is sufficient to replace the value of Y_t in our last expression of this moment generating function to complete the proof. $\square$

The next proposition introduces the discretization framework used in numerical applications to construct the joint density function f_{r_T,λ_T} under the forward measure.

Proposition 4. *Let M be the number of steps used in the two dimensional DFT and $\Delta_{y_l} = \frac{\left(1 + 1_{\{l=1\}}\right) y_l^{max}}{M-1}$ be this step of discretization. Let us denote $\Delta_{z_l} = \frac{2\pi}{M \Delta_{y_l}}$ and*

$$z_{j_1,j_2} = \left(-\frac{M}{2}\Delta_{z_1} + (j_1 - 1)\Delta_{z_1},\ -\frac{M}{2}\Delta_{z_2} + (j_2 - 1)\Delta_{z_2}\right),$$

for $j_l = 1...M$. Let $\Phi_{T,S}^{r,\lambda}$ be the moment generating function of $r_T + \lambda_T \mid \mathcal{G}_t$ under the measure $\mathbb{Q}_S$. Let defined by $\Pi\left(L^c\left(T,S\right)\right)$ the payoff paid at time $S \geq T$ by a European option written on $L^c\left(T,S\right)$. Let $C(t, r_t, \lambda_t)$ the price of this option. The values of $f_{r_T,\lambda_T}\left(\cdot,\cdot\right)$ at points $y_{k_1,k_2} = \left(-\frac{M}{2}\Delta_{y_1} + \left(k_1 - 1\right)\Delta_{y_1},\, -\frac{M}{2}\Delta_{y_2} + \left(k_2 - 1\right)\Delta_{y_2}\right)$ are approached by the sum:

$$
\begin{aligned}
f_{r_T,\lambda_T}\left(y_{k_1,k_2}\right) \approx\ & \frac{e^{-iM\pi + i\sum_{l=1}^{2}(k_l-1)\pi}}{M^2\Delta_{y_1}\Delta_{y_2}} \\
& \times \sum_{j_1=1}^{M}\sum_{j_2=1}^{M} \Upsilon_{j_1,j_2}\Phi_{T,S}^{r,\lambda}\left(iz_{j_1,j_2}\right)e^{i(j_1-1)\pi}e^{-i\sum_{l=1}^{2}(j_l-1)(k_l-1)\frac{2\pi}{M}},
\end{aligned}
\tag{4.8}
$$

where $\Upsilon_{j_1,j_2} = \left(\frac{1}{2}\right)^{\left(1_{\{j_1=1\}}+1_{\{j_2=1\}}\right)}\left(1 - 1_{\{j_1\neq 1, j_2\neq 1\}}\right) + 1_{\{j_1\neq 1, j_2\neq 1\}}$ and $\Phi_{T,S}^{r,\lambda}$ is defined in equation (4.3).

Proof. The joint probability density of $r_T \mid \mathcal{G}_t$ and $\lambda_T \mid \mathcal{G}_t$ is retrieved by calculating the Fourier Transform of the moment generating function $\Phi_{T,S}^{r,\lambda}$ as follows:

$$
\begin{aligned}
f_{r_T,\lambda_T}\left(y\right) &= \frac{1}{\left(2\pi\right)^2}\mathcal{F}\left[\Phi_{T,S}^{r,\lambda}\left(iz\right)\right]\left(y\right) \\
&= \frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_{T,S}^{r,\lambda}\left(iz\right)e^{-iz^t y}\mathrm{d}z_1\mathrm{d}z_2.
\end{aligned}
\tag{4.9}
$$

At points y_{k_1,k_2}, this last integral is approached with the trapezoid rule and leads to the following estimate for f_{r_T,λ_T}:

$$
f_{r_T,\lambda_T}\left(y_{k_1,k_2}\right) \approx \frac{1}{\left(2\pi\right)^2}\sum_{j_1=1}^{M}\sum_{j_2=1}^{M}\Upsilon_{j_1,j_2}\Phi_{T,S}^{r,\lambda}\left(iz_{j_1,j_2}\right)e^{-iz_{j_1,j_2}^t y_{k_1,k_2}}\Delta_{z_1}\Delta_{z_2},
\tag{4.10}
$$

and given that $\Delta_{z_l}\Delta_{y_l} = \frac{2\pi}{M}$

$$
\begin{aligned}
z_{j_1,j_2}^t y_{k_1,k_2} &= \sum_{l=1}^{2}\left(\frac{M^2}{4}\Delta_{z_l}\Delta_{y_l} - \frac{M}{2}\left(j_l-1\right)\Delta_{z_l}\Delta_{y_l} - \frac{M}{2}\left(k_l-1\right)\Delta_{z_l}\Delta_{y_l}\right) \\
&\quad + \sum_{l=1}^{2}\left(j_l-1\right)\left(k_l-1\right)\Delta_{z_l}\Delta_{y_l} \\
&= \sum_{l=1}^{2}\left(\frac{M}{2}\pi - \left(j_l-1\right)\pi - \left(k_l-1\right)\pi + \left(j_l-1\right)\left(k_l-1\right)\frac{2\pi}{M}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-iz_{j_1,j_2}^t y_{k_1,k_2}} &= e^{-iM\pi + i\sum_{l=1}^{2}(k_l-1)\pi}e^{i\sum_{l=1}^{2}(j_l-1)\pi} \\
&\quad \times e^{-i\sum_{l=1}^{2}(j_l-1)(k_l-1)\frac{2\pi}{M}}.
\end{aligned}
$$

□

In particular, based on the previous result we are able to compute the price of caplet. A caplet with price $\mathrm{capl}(t,T,S)$ purchased at date t with expiry date T, pays to the holder the positive difference between a simple spot forward Libor market rate $L^c(T,S)$ and the strike rate κ at settlement date S. If we denote by N the principal of this contract, the caplet price is given by

$$\begin{aligned} \operatorname{capl}(t,T,S) &= N\delta\left(T,S\right)P\left(t,S\right)\mathbb{E}^{\mathbb{Q}_S}\left[\left(L^c(T,S)-\kappa\right)^+ \mid \mathcal{G}_t\right], \\ &\approx N\delta\left(T,S\right)P\left(t,S\right)\sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\Pi\left(y_{k_1,k_2}\right)f_{r_T,\lambda_T}\left(y_{k_1,k_2}\right)\Delta_{y_1}\Delta_{y_2}. \end{aligned} \tag{4.11}$$

where

$$y_{k_1,k_2} = \left(-\frac{M}{2}\Delta_{y_1} + (k_1-1)\,\Delta_{y_1},\ -\frac{M}{2}\Delta_{y_2} + (k_2-1)\,\Delta_{y_2}\right),$$

$\Delta_{y_l} = \frac{\left(1+1_{\{l=1\}}\right)y_l^{max}}{M-1}$, $\Pi\left(r_T,\lambda_T\right) = \left(L^c(T,S)-\kappa\right)^+$ and $L^c\left(t,T\right)$ satisfies the equation (3.4). Apart from the round-off error, this approximation entails a truncation error and a sampling error which vanish when the number of steps M increases. Moreover, when we consider the following model parameters $a = 0.75$, $\sigma_r = 0.1$, $r_0 = -8.33\times10^{-5}$, $\kappa = 0.12$, $\gamma = \lambda_0 = 0.001$, $\sigma_\lambda = 6.9\times10^{-5}$ and $\eta = 0.13$, the figure (4.1) emphasizes that the approximated caplet price converges to a value around 0.0076 as from $M = 2^8$.

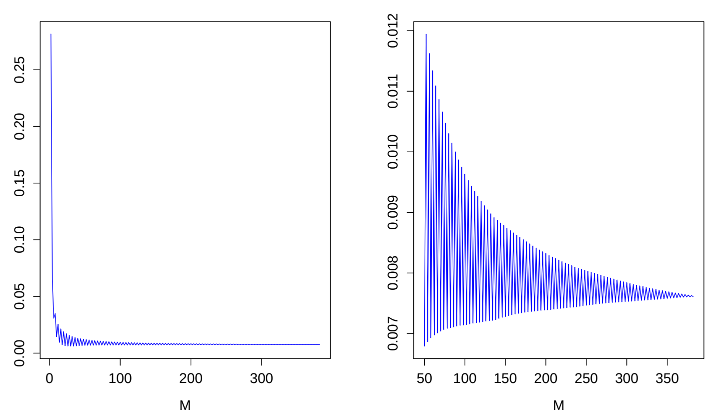

Figure 4.1. Convergence study of the caplet price approximation in equation (4.11). The number of steps M ranges from 2 to 382, and $y_1^{max} = y_2^{max} = 1$. The considered model parameters are $a = 0.75$, $\sigma_r = 0.1$, $r_0 = -8.33\times10^{-5}$, $\kappa = 0.12$, $\gamma = \lambda_0 = 0.001$, $\sigma_\lambda = 6.9\times10^{-5}$ and $\eta = 0.13$. The approximated caplet price converges to a value around 0.0076 as M increases.

We next derive a pricing formula for a cap, which is a portfolio of caplets. Let us consider a cap with nominal N, defined on a set of increasing dates $\{T_0, T_1, T_2, ..., T_n\}$. At each time T_i with $i = 1, ..., n$, the holder of a cap receives the following cash flow

$$N\delta\left(T_{i-1},T_i\right)\left(L^c(T_{i-1},T_i)-\kappa\right)^+.$$

Then, the price at time $t \leq T_0$ of a cap is given by the following sum

$$\begin{aligned}\operatorname{cap}(t) &= \sum_{i=1}^{n} \operatorname{capl}(t, T_{i-1}, T_i) \\ &= N \sum_{i=1}^{n} \delta\left(T_{i-1}, T_i\right) P\left(t, T_i\right) \mathbb{E}^{\mathbb{Q}_{T_i}}\left[\left(L^c(T_{i-1}, T_i) - \kappa\right)^+ \mid \mathcal{G}_t\right] \\ &\approx N \sum_{i=1}^{n} P\left(t, T_i\right) \delta\left(T_{i-1}, T_i\right) \sum_{k=1}^{M} \sum_{l=1}^{M} \Pi\left(x_k, y_l\right) f_{r_{T_{i-1}}, \lambda_{T_{i-1}}}\left(x_k, y_l\right) \Delta_x \Delta_y. \end{aligned} \tag{4.12}$$

Since interest rates may be negative, we use the Bachelier normal model for quoting interest rate caps in terms of implied volatility. Within this model the Euro cap price at time t is defined as follows:

$$\begin{aligned}\operatorname{cap}^{\text{Normal}}\left(t, \mathcal{L}^F, \mathcal{L}^S, \delta^F, N, K, \sigma_{0,n}\right) &= N \sum_{i=1}^{n} \delta_i^F P\left(t, T_i\right)\left(F^c\left(t, T_{i-1}, T_i\right) - K\right) \Phi\left(d\right) \\ &\quad + \sigma_{0,n} N \sum_{i=1}^{n} \delta_i^F P\left(t, T_i\right) \sqrt{T_{i-1}} \varphi\left(d\right) \end{aligned} \tag{4.13}$$

where $d = \frac{F^c(t, T_{i-1}, T_i) - K}{\sigma_{0,n}\sqrt{T_{i-1}}}$, and $\sigma_{0,n}$ is the implied volatility parameter.

In a similar way, we derive a pricing formula for a swaption. By definition, a swaption is an option that gives the right to enter into a payer or receiver swap at some future date. Let us consider an interest rate swap with m annual fixed payments $\{S_0, S_1, S_2, ..., S_m\}$ and the floating leg payments on set of time $\{T_0, T_1, T_2, ..., T_n\}$. The value of a payer swaption at time T_0 is given by

$$\begin{aligned}\mathrm{PS}_{0,n}\left(T_0\right) &= \max\left(\mathrm{IRS}\left(T_0\right), 0\right) \\ &= N \sum_{i=1}^{n} \delta^F\left(T_{i-1}, T_i\right) P\left(T_0, T_i\right)\left(S_{0,n}\left(T_0\right) - \kappa\right)^+, \end{aligned} \tag{4.14}$$

where the swap rate $S_{0,n}\left(t\right)$ is given by the equation (2.25). Then, at time $t \leq T_0$

$$\mathrm{PS}_{0,n}\left(t\right) = N \mathbb{E}^{\mathbb{Q}^{IRS}}\left[\left(S_{0,n}\left(T_0\right) - \kappa\right)^+ \mid \mathcal{G}_t\right] \sum_{i=1}^{n} \delta^F\left(T_{i-1}, T_i\right) P\left(t, T_i\right), \tag{4.15}$$

where $\mathbb{Q}^{IRS}$ is the equivalent martingale measure of $\mathbb{Q}$, obtained by choosing $\sum_{i=1}^{n} \delta^F\left(T_{i-1}, T_i\right) P\left(\cdot, T_i\right)$ as a numeraire. In order to calculate the expectation in the previous formula, we need to determine the swap rate dynamics under the measure $\mathbb{Q}^{IRS}$.

Let $\Pi : (r_{T_0}, \lambda_{T_0}) \longmapsto \Pi(r_{T_0}, \lambda_{T_0})$ be a function such that:

$$\begin{aligned}
\Pi(r_{T_0}, \lambda_{T_0}) &= (S_{0,n}(T_0) - \kappa)^+ \\
&= \left(\frac{\sum_{i=1}^{n} \delta^F \, P(T_0, T_i) \; F^c(T_0, T_{i-1}, T_i)}{\sum_{j=1}^{m} P(T_0, S_j)} - \kappa \right)^+ \\
&= \left(\frac{\sum_{i=1}^{n} \left(P(T_0, T_{i-1}) \, \beta(T_0, T_{i-1}, T_i) - P(T_0, T_i) \right)}{\sum_{j=1}^{m} P(T_0, S_j)} - \kappa \right)^+ .
\end{aligned} \tag{4.16}$$

It follows that

$$\mathbb{E}^{\mathbb{Q}^{IRS}} \left[\Pi(r_{T_0}, \lambda_{T_0}) \mid \mathcal{G}_t \right] = \int_0^{+\infty} \int_{-\infty}^{+\infty} \Pi(x, y) \, f_{r_{T_0}, \lambda_{T_0}}(x, y) \, \mathrm{d}x \mathrm{d}y, \tag{4.17}$$

where $f_{r_{T_0}, \lambda_{T_0}}$ is the joint density function of $r_{T_0} \mid \mathcal{G}_t$ and $\lambda_{T_0} \mid \mathcal{G}_t$ under the measure $\mathbb{Q}^{IRS}$. Let $E_t = \sum_{i=1}^{n} \delta^F(T_{i-1}, T_i) \, P(t, T_i)$ be a numeraire chosen for defining the measure $\mathbb{Q}^{IRS}$. The Radon-Nykodym density associated is defined by:

$$\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}} = e^{-\int_0^{T_0} r_u \mathrm{d}u} \frac{E_{T_0}}{E_0},$$

where

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_t \right] = e^{-\int_0^{t} r_u \mathrm{d}u} \frac{E_t}{E_0}.$$

In the next proposition, we use this Radon-Nykodym density to find the moment generating function of $r_{T_0} + \lambda_{T_0}$ under the measure $\mathbb{Q}^{IRS}$.

Proposition 5. *Under the measure $\mathbb{Q}^{IRS}$ the moment generating function of $r_{T_0} + \lambda_{T_0}$ denoted by $\Phi_{T0}^{r,\lambda}$ is equal to*

$$\begin{aligned}
\Phi_{T0}^{r,\lambda}(\omega_1, \omega_2) = \mathbb{E}^{\mathbb{Q}^{IRS}} &\left[e^{\omega_1 r_{T_0} + \omega_2 \lambda_{T_0}} \mid \mathcal{G}_t \right] = \\
&\frac{1}{E_t} \sum_{i=1}^{n} \delta^F(T_{i-1}, T_i) \, e^{U_1^i(t,T_0) + U_2^i(t,T_0) r_t + U_3^i(t,T_0) \lambda_t},
\end{aligned} \tag{4.18}$$

where $U_j^i(\cdot, \cdot)$, for $j = 1, 2, 3$ are bivariate functions satisfying the following PDE system:

$$\begin{cases}
\partial_t U_1^i(t, T_0) + \theta(t) \, U_2^i(t, T_0) + \kappa\gamma U_3^i(t, T_0) + \frac{1}{2}\sigma_r^2 \left(U_2^i(t, T_0) \right)^2 & = 0 \\
\partial_t U_2^i(t, T_0) - a U_2^i(t, T_0) & = 1 \\
\partial_t U_3^i(t, T_0) - \kappa U_3^i(t, T_0) + \frac{1}{2}\sigma_\lambda^2 \left(U_3^i(t, T_0) \right)^2 + \left(e^{\eta U_3^i(t,T_0)} - 1 \right) & = 0
\end{cases} , \tag{4.19}$$

and satisfy the following terminal conditions:

$$\begin{cases}
U_1^i(T_0, T_0) & = A(T_0, T_i) \\
U_2^i(T_0, T_0) & = \omega_1 - B(T_0, T_i) \\
U_3^i(T_0, T_0) & = \omega_2
\end{cases} .$$

Proof. As we only know the dynamics of these processes under the risk neutral, we first rewrite the moment generating function as function of expectation under the risk neutral world. Indeed,

$$\begin{aligned}\Phi^{r,\lambda}_{T0}(\omega_1,\omega_2) &= \frac{\mathbb{E}\left[\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_T\right] e^{\omega_1 r_{T_0}+\omega_2\lambda_{T_0}} \mid \mathcal{G}_t\right]}{\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_t\right]}\\ &= \frac{1}{E_t}\sum_{i=1}^{n}\delta^F\left(T_{i-1},T_i\right)\mathbb{E}\left[e^{-\int_t^{T_0} r_u \mathrm{d}u + A(T_0,T_i)+(\omega_1-B(T_0,T_i))r_{T_0}+\omega_2\lambda_{T_0}} \mid \mathcal{G}_t\right].\end{aligned}$$

Next, we focus on the evaluation of

$$Y^i_t = \mathbb{E}\left[e^{-\int_t^{T_0} r_u \mathrm{d}u + A(T_0,T_i)+(\omega_1-B(T_0,T_i))r_{T_0}+\omega_2\lambda_{T_0}} \mid \mathcal{G}_t\right], \tag{4.20}$$

and find its closed form thanks to the infinitesimal generator properties for Markov processes. As in the proposition (1) we can proof that for $Y^i_t = f_i(t, r_t, \lambda_t, N_t)$,

$$\lim_{v\to t}\frac{\mathbb{E}\left(Y^i_v \,|\, \mathcal{G}_t\right) - Y^i_t}{v-t} = r_t Y^i_t = \mathcal{A} f_i.$$

Then, using Itô Lemma the infinitesimal generator is expressed as follows

$$\begin{aligned}\mathcal{A}f_i = \partial_t f_i + \left(\theta\left(t\right) - a r_t\right)\partial_r f_i + \kappa\left(\gamma - \lambda_t\right)\partial_\lambda f_i + \frac{1}{2}\sigma_r^2\partial_r^2 f_i + +\frac{1}{2}\sigma_\lambda^2\lambda_t\partial_\lambda^2 f_i\\ + \lambda_t\left(f_i\left(t,r_t,\lambda_t+\eta,N_t+1\right) - f_i\left(t,r_t,\lambda_t,N_t\right)\right).\end{aligned} \tag{4.21}$$

Assuming that $f_i(t, r_t, \lambda_t, N_t) = e^{U^i_1(t,T_0)+U^i_2(t,T_0)r_t+U^i_3(t,T_0)\lambda_t}$ we deduce the following ODEs system

$$\begin{cases}\partial_t U^i_1\left(t,T_0\right) + \theta\left(t\right)U^i_2\left(t,T_0\right) + \kappa\gamma U^i_3\left(t,T_0\right) + \frac{1}{2}\sigma_r^2\left(U^i_2\left(t,T_0\right)\right)^2 & = 0\\ \partial_t U^i_2\left(t,T_0\right) - aU^i_2\left(t,T_0\right) & = 1\\ \partial_t U^i_3\left(t,T_0\right) - \kappa U^i_3\left(t,T_0\right) + \frac{1}{2}\sigma_\lambda^2\left(U^i_3\left(t,T_0\right)\right)^2 + \left(e^{\eta U^i_3(t,T_0)} - 1\right) & = 0\end{cases},$$

with the following terminal conditions

$$\begin{cases}U^i_1\left(T_0,T_0\right) & = A\left(T_0,T_i\right)\\ U^i_2\left(T_0,T_0\right) & = \omega_1 - B(T_0,T_i)\\ U^i_3\left(T_0,T_0\right) & = \omega_2\end{cases}.$$

Thus,

$$\Phi^{r,\lambda}_{T0}(\omega_1,\omega_2) = \frac{1}{E_t}\sum_{i=1}^{n}\delta^F\left(T_{i-1},T_i\right)e^{U^i_1(t,T_0)+U^i_2(t,T_0)r_t+U^i_3(t,T_0)\lambda_t}. \tag{4.22}$$

□

Then, by a convenient discretization framework as in proposition (4) we find numerically the joint probability density $f_{r_{T_0},\lambda_{T_0}}(\cdot,\cdot)$ of $r_{T_0} \mid \mathcal{G}_t$ and $\lambda_{T_0} \mid \mathcal{G}_t$ under the measure $\mathbb{Q}^{IRS}$. Hence, equation (4.17) becomes

$$\mathbb{E}^{\mathbb{Q}^{IRS}}\left[\Pi\left(r_{T_0},\lambda_{T_0}\right)\mid\mathcal{G}_t\right]\approx\sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\Pi\left(y_{k_1,k_2}\right)f_{r_{T_0},\lambda_{T_0}}\left(y_{k_1,k_2}\right)\Delta_{y_1}\Delta_{y_2}. \tag{4.23}$$

From equations (4.15) and (4.23), the price of the European swaption defined above is approximated as follows

$$\begin{aligned}\mathrm{PS}_{0,n}\left(t\right)\approx N&\left(\sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\Pi\left(y_{k_1,k_2}\right)f_{r_{T_0},\lambda_{T_0}}\left(y_{k_1,k_2}\right)\Delta_{y_1}\Delta_{y_2}\right)\\&\times\sum_{i=1}^{n}\delta^F\left(T_{i-1},T_i\right)P\left(t,T_i\right).\end{aligned} \tag{4.24}$$

Then, based on Bachelier normal model, the implied volatility of this swaption satisfies the following equation $\mathrm{PS}^{\mathrm{Normal}}\left(t,\mathcal{L}^F,\mathcal{L}^S,\delta^F,N,K,\sigma_{0,n}\right)=\mathrm{PS}_{0,n}\left(t\right)$, where

$$\begin{aligned}\mathrm{PS}^{\mathrm{Normal}}\left(t,\mathcal{L}^F,\mathcal{L}^S,\delta^F,N,K,\sigma_{0,n}\right)=N&\left[\left(S_{0,n}\left(t\right)-K\right)\Phi\left(d\right)+\sigma_{0,n}\sqrt{T_0}\varphi\left(d\right)\right]\\&\times\sum_{i=1}^{n}\delta_i^F P\left(t,T_i\right),\end{aligned} \tag{4.25}$$

$d=\frac{S_{0,n}(t)-K}{\sigma_{0,n}\sqrt{T_0}}$ and $\sigma_{0,n}$ is the implied volatility parameter.

5. Calibration and illustration

In this section, we calibrate our model and illustrate the impact of parameters defining the level of self-excitation on the conditional survival probability and forward Libor rates. Considering a bi-curve framework formed by the discount rate and rates with six months tenor, estimated parameters found hereafter, come from a calibration process detailed throughout this section. Since the short rate is independent from the default intensity, short rate parameters are found in a way to fit at best the discount curve and the implied volatility surface of options with OIS discounting. The data treatment and the bootstraping procedure for interest rates in a multiple curve framework is based on the work of [Hull & White(2015)] and [Ametrano & Bianchetti(2013)]. For sake of convenience OISs rates of tenor up to a year, have a single fixed payment on the days after its maturity date. While, OISs of tenors greater than one year, have fixed payment annually. Then, the default free bond price with maturity date T_β is given by :

$$P\left(t,T_\beta\right)=\begin{cases}\frac{1}{1+\delta\left(t,T_\beta\right)S_{0,\beta}^{\mathrm{OIS}}(t)} & \text{if } t<T_\beta\leq t+1\\ \left(1-S_{0,\beta}^{\mathrm{OIS}}\left(t\right)\sum_{j=1}^{\beta-1}P(t,T_j)\right)\frac{1}{1+S_{0,\beta}^{\mathrm{OIS}}(t)} & \text{if } T_\beta>t+1\end{cases}, \tag{5.1}$$

where $S_{0,\beta}^{\mathrm{OIS}}\left(t\right)$ is the swap rate for collateralized transaction and using OIS discounting, with reset date T_0, maturity date T_β, and fixed payment times schedule $\{T_0,T_1,...,T_\beta\}$. The value of the discount factor for any other intermediate maturity is obtained by interpolation of zero coupon rates, with monotonic cubic spline method. This interpolation method leads to the following closed expressions of the observed discount factor and instantaneous forward rate for other maturity $T \in [T_i, T_{i+1}]$ with $i \in [0, \beta]$:

$$P^M(t,T) = a_i + b_i(T - T_i) + c_i(T - T_i)^2 + d_i(T - T_i)^3 \ , \tag{5.2}$$

$$f^M(t,T) = -\frac{b_i + 2c_i(T - T_i) + 3d_i(T - T_i)^2}{P^M(t,T)} \ , \tag{5.3}$$

where $a_i, b_i, c_i \in \mathbb{R}$. Notice that the function $\theta(\cdot)$ in equation (2.2) depend only on the parameter set $\{a, \sigma_r, r_0\}$ and fits the observed discount curve derived from the OISs rates curve. The last step of our short rate calibration, consists to estimate the parameter set $\{a, \sigma_r, r_0\}$ minimizing the sum of squared errors between observed implied volatility surface of swaptions using OIS discounting and those given by our model as described hereafter.

Swaptions for collateralized transaction and using OIS discounting refer to the single curve framework. Here we do not use the Libor rate instead OIS rate or mixed them in our calculation. The price of this swaption at the inception $t \geq 0$ is given by:

$$\begin{aligned}
\mathrm{PS}^{\mathrm{OIS}}_{\alpha,\beta}(t) &= N\mathbb{E}^{\mathbb{Q}^{IRS}}\left[\left(S^{\mathrm{OIS}}_{\alpha,\beta}(T_\alpha) - \kappa\right)^+ \mid \mathcal{G}_t\right] \sum_{i=\alpha+1}^{\beta} \delta^F(T_{i-1}, T_i) P(t, T_i) \\
&= N\left(\sum_{i=\alpha+1}^{\beta} \delta^F(T_{i-1}, T_i) P(t, T_i)\right) \int_{-\infty}^{+\infty} \Pi(x) f_{r_{T_\alpha}}(x)\,\mathrm{d}x,
\end{aligned} \tag{5.4}$$

where

$$\Pi: \ x \longmapsto \Pi(x) = \left(\frac{1 - e^{A\left(T_\alpha, T_\beta\right) - B\left(T_\alpha, T_\beta\right)x}}{\sum_{i=\alpha+1}^{\beta} \delta^F(T_{i-1}, T_i)\, e^{A(T_\alpha, T_i) - B(T_\alpha, T_i)x}} - \kappa\right)^+ ,$$

and $f_{r_{T_\alpha}}$ is the probability density function of r_{T_α} under measure $\mathbb{Q}^{\mathrm{IRS}}$. In appendix (7.2) we find a close-form approximation of this density function. Then, it follows that the estimated parameter set of $\{a, \sigma_r, r_0\}$ minimizes $\sum_{(\alpha,\beta)} \left(\sigma^{\mathrm{Market}}_{\alpha,\beta} - \sigma^{\mathrm{Model}}_{\alpha,\beta}\right)^2$, where $\sigma^{\mathrm{Model}}_{\alpha,\beta}$ satisfies the following equation:

$$\mathrm{PS}^{\mathrm{Normal}}\left(t, \mathcal{L}^F, \mathcal{L}^S, \delta^F, N, K, \sigma^{\mathrm{Model}}_{\alpha,\beta}\right) = \mathrm{PS}^{\mathrm{OIS}}_{\alpha,\beta}(t).$$

It remains to estimate the parameter set $\{\kappa, \gamma, \eta, \sigma_\lambda, \lambda_0\}$ of our intensity process. Under the constraint $\eta \leq \kappa$ that guarantees the stability of the intensity process, this parameter set is estimated by minimizing the sum of squared errors between the observed IRS six months curve and those given by our model. By doing this, we expect to capture the pattern of the spread OIS-IRS six months observed in the corresponding interbank market segment, or the credit risk within the interbank market where products with six months tenor are exchanged.

We complete our calibration process based on Bloomberg data. These data include the term structure of OIS rate, the implied volatility surface of swaptions with OIS discounting and collateral, and the term structure of IRS six months observed within the interbank market on 11 February 2019. The estimated parameters found for different models are in the following tables

	Short rate process		
Model	a	σ_r	r_0
HW	$7.487799e-01$	$9.851328e-02$	$-8.337071e-05$

TABLE 1. Estimated short rate parameters based on the term structure of OIS rate, and the implied volatility surface of option on OIS rates observed within the interbank market the 11 February 2019.

	Intensity process				
Model	κ	γ	σ_λ	η	λ_0
CIR	7.632%	0.177%	0.007%	0	0.151%
Hawkes	22.93%	0.119%	0	19.71%	0.123%
DMR	8.891%	0.389%	0	0	0
SEJCIR	22.24%	0.129%	6.639%	17.64%	0.122%

TABLE 2. Estimated intensity process parameters based on the term structure of OIS rate and IRS 6-months observed within the interbank market the 11 February 2019.

	CIR	Hawkes	DMR	SEJCIR
RSE	$3.896808e-04$	$4.943271e-05$	$2.07622e-02$	$4.347653e-05$

TABLE 3. Relative squared errors when the intensity model is either a CIR process, a Hawkes process, a deterministic mean reverting function or a SEJCIR model.

(1,2). Although, a weak value of λ_0 and σ_λ can lead to refute the presence of self-excitation, these values probably simply reflect the current state of the market or that the market does not include this risk in the pricing. In this context, compared to a CIR intensity process, a Hawkes intensity process with an exponential decay, or a deterministic mean-reverting (DMR[2]) function, the best fit of the IRS six months curve as shown in table (3) (see also figure (7.1)) is obtained with our self-exciting jump process denoted by SEJCIR[3] and the Hawkes process. In term of OIS-IRS 6 months spread, the SEJCIR model allows for a better fit of this spread than the other approaches. As shown in the figure (7.2), the SEJCIR captures global changes in the dynamic of this OIS-IRS six months spread with a mean squared errors below 0.1 basis points.

[2]Deterministic Mean reverting

[3]SEJCIR stands to our intensity process which is a CIR process plus a double stochastic jump process

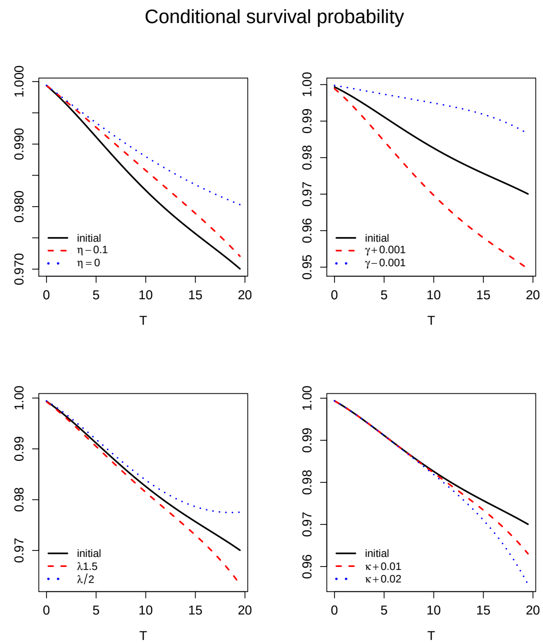

FIGURE 5.1. Conditional survival probability of the six months interbank market segment $\mathbb{Q}\left(\tau \geq T \mid \tau \geq 0,\ \mathcal{H}_0\right)$ for $T \in [0, 20]$. For each plot, we consider the initial set of SEJCIR estimated parameters and we vary one of them. This conditional survival probability decreases faster when the self-excitation parameter increases.

To wrap up this section, based on the estimated parameters of the SEJCIR model in table (2), we analyze the impact of the self-excitation parameter on default probabilities. For this purpose, we draw in figure (5.1) the conditional survival probability of six months interbank market segment. This probability decreases slowly and remains close to one for maturity up to 20 years. However, a change in the self-excitation parameter impacts the rate of decreasing. From the graph at the top left, the conditional survival probability rises for long term when the jump size decreases. This result comes from the fact that a credit event is less likely. A similar conclusion can be drawn from the graph at the top right and bottom left, a decrease in the long-run level γ or the initial value λ_t makes credit events less likely in this interbank market segment. On the contrary, the bottom right graph reveals that when the rate of mean reversion rises credit events are more frequent and hint a decrease of the conditional survival probability. Thus, the self-excitation property in our framework increases the conditional survival probability when a credit events occur.

In figure (5.2) we draw the six months forward rates for different reset dates, when changing self-excitation parameters. As the uncertainty rises with the reset date, the six months forward rates curve increases. From the graphs, when the jump size and the initial value of arrival rate of default rise the value of the six months forward rate increases. That is coherent with our model, since from equation (2.29), the forward rate increases when the survival default probability decrease. Then, the self-excitation property in our framework increases the forward Libor rate when credit events occur.

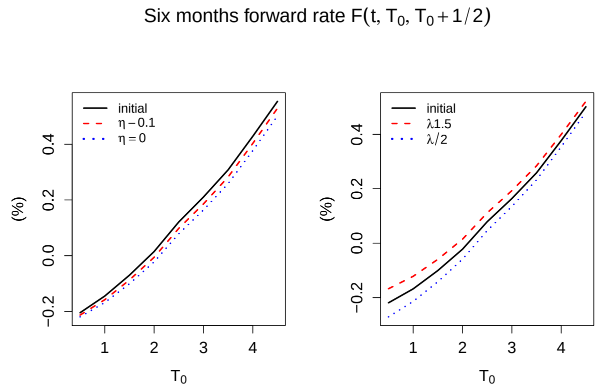

FIGURE 5.2. Six months forward rate sensitivity on self-excitation parameters η and λ_0. Given a reset date T_0, the forward Libor increases when the initial arrival rate of credit events λ_0 or the amplitude of credit events η increases.

6. Conclusions

In this paper, we have explored a new use of self-exciting processes to model the interbank credit risk in a multiple curve framework. In particular, we have segmented the interbank market according to tenor. Further, we have modeled the credit risk of each segment using an intensity-based approach, where the intensity is a self-exciting jump Cox Ingersoll Ross (SEJCIR) process. In a realistic way, we have used the self-exciting feature to introduce contagion between the interbank market participants.

In this setting, the forward Libor rate is a non-decreasing function of an interbank risk factor that modified the arbitrage-free condition. Thanks to the tractability and flexibility of the SEJCIR process, we have obtained the arbitrage-free condition, and we have derived closed-form approximation of pricing formulas for caplets, caps, as well as swaptions.

Numerical applications have shown that the SEJCIR process enables to reproduce the IRS curve and to explain the deviance between the IRS and OIS curve as a result of the interbank credit risk. The results obtained in the sensitivity analysis confirm the importance of the interbank credit risk to reflect the confidence level within the interbank market.

7. Appendix

7.1. **Arbitrage free condition.** The arbitrage-free condition is such that the ratio $\frac{P^c(t,T_k)}{P^c(t,T_{k+1})}$ is martingale under $\mathbb{Q}_{T_{k+1}}$. The Radon-Nikodym density associated to the change of measure $\mathbb{Q}$ to $\mathbb{Q}_{T_{k+1}}$ is defined as follows:

$$\mathit{\Lambda}_t := \frac{\mathrm{d}\mathbb{Q}_{T_{k+1}}}{\mathrm{d}\mathbb{Q}} \mid \mathcal{G}_t = e^{-\int_0^t r_u \mathrm{d}u} \frac{P\left(t, T_{k+1}\right)}{P\left(0, T_{k+1}\right)}.$$

Then, the ratio $\frac{P^c(t,T_k)}{P^c(t,T_{k+1})}$ is martingale under $\mathbb{Q}_{T_{k+1}}$ if $\Lambda_t\frac{P^c(t,T_k)}{P^c(t,T_{k+1})}$ is a martingale under the risk neutral measure. We have

$$\begin{aligned}\mathrm{d}\left(\Lambda_t\frac{P^c(t,T_k)}{P^c(t,T_{k+1})}\right) &= \mathrm{d}\left(e^{-\int_0^t r_u \mathrm{d}u}\frac{P(t,T_k)}{P(0,T_{k+1})}\beta(t,T_k,T_{k+1})\right) \\ &= e^{-\int_0^t r_u \mathrm{d}u}\frac{P(t,T_k)}{P(0,T_{k+1})}\left(\mathrm{d}\beta(t,T_k,T_{k+1})\right. \\ &\quad \left.-\sigma_r B(t,T_k)\beta(t,T_k,T_{k+1})\mathrm{d}W_t^r\right), \end{aligned} \tag{7.1}$$

where the second line comes from the fact that $e^{-\int_0^t r_u \mathrm{d}u}P(t,T_k)$ is a martingale under $\mathbb{Q}$ and $\mathrm{d}[W^r,W^\lambda]_t=0$. Note that $\beta(t,T_k,T_{k+1})$ is defined in equation (3.2). Next, given the Ito differentiation rule, we obtain :

$$\begin{aligned}\frac{\mathrm{d}\beta(t,T_k,T_{k+1})}{\beta(t,T_k,T_{k+1})} &= \mu_t^\beta \mathrm{d}t+\sigma_\lambda\sqrt{\lambda_t}\Delta_k D(t)\mathrm{d}W_t^\lambda \\ &\quad +\left(\exp\left(-\eta\Delta_k D(t)\right)+\frac{1}{2}\Delta_k D(t)^2\eta^2-1\right) \\ &\quad \times\left(\mathrm{d}N_t-\lambda_t\mathrm{d}t\right), \end{aligned} \tag{7.2}$$

where $\Delta_k D(t)=D(t,T_{k+1})-D(t,T_k)$ and

$$\begin{aligned}\mu_t^\beta &= \left(e^{\eta D(t,T_k)}-e^{\eta D(t,T_{k+1})}-\sigma_\lambda^2 D(t,T_k)\Delta_k D(t)\right)\lambda_t \\ &\quad +\exp\left(-\eta\Delta_k D(t)\right)+\frac{1}{2}\Delta_k D(t)^2\eta^2-1. \end{aligned} \tag{7.3}$$

From equations (7.1,7.2) the market is arbitrage-free if the drift $\mu_t^\beta=0$. The arbitrage free condition can rewrite as condition on the change in the factor β when a credit event occurs:

$$\exp\left(-\eta\Delta_k D(t)\right)-1=\frac{-\frac{1}{2}\Delta_k D(t)^2\eta^2+\sigma_\lambda^2 D(t,T_k)\Delta_k D(t)\lambda_t}{\left(\lambda_t e^{\eta D(t,T_{k+1})}+1\right)}. \tag{7.4}$$

7.2. **Conditional density of the short rate under the annuity measure.** As we only know the dynamics of these processes under the risk neutral, we first rewrite the moment generating function as function of expectation under the risk neutral world. Indeed,

$$\begin{aligned}\phi_{T_\alpha}^r(\omega) &= \mathbb{E}^{\mathbb{Q}^{IRS}}\left[e^{\omega r_{T_\alpha}}\mid\mathcal{G}_t\right] \\ &= \frac{\mathbb{E}\left[\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}}\mid\mathcal{G}_T\right]e^{\omega r_{T_\alpha}}\mid\mathcal{G}_t\right]}{\mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}^{IRS}}{\mathrm{d}\mathbb{Q}}\mid\mathcal{G}_t\right]} \\ &= \frac{1}{E_t}\mathbb{E}\left[e^{-\int_t^{T_\alpha} r_u \mathrm{d}u}E_{T_\alpha}e^{\omega r_{T_\alpha}}\mid\mathcal{G}_t\right] \\ &= \frac{1}{E_t}\sum_{i=1}^n\delta^F\left(T_{i-1},T_i\right)\mathbb{E}\left[e^{-\int_t^{T_\alpha} r_u \mathrm{d}u+A(T_\alpha,T_i)+(\omega-B(T_\alpha,T_i))r_{T_\alpha}}\mid\mathcal{G}_t\right]. \end{aligned}$$

Next, we focus on the evaluation of

$$Y_t^i=\mathbb{E}\left[e^{-\int_t^{T_\alpha} r_u \mathrm{d}u+A(T_\alpha,T_i)+(\omega-B(T_\alpha,T_i))r_{T_\alpha}}\mid\mathcal{G}_t\right]$$

and find its closed form thanks to the infinitesimal generator properties for Markov processes. As in the proposition (1) we can proof that for $Y_t^i = f_i(t, r_t)$,

$$\lim_{v \to t} \frac{\mathbb{E}\left(Y_v^i \,|\, \mathcal{G}_t\right) - Y_t^i}{v - t} = r_t Y_t^i = \mathcal{A} f_i.$$

Then, using Itô Lemma the infinitesimal generator is

$$\mathcal{A} f_i = \partial_t f_i + \left(\theta\left(t\right) - a r_t\right) \partial_r f_i + \frac{1}{2} \sigma_r^2 \partial_r^2 f_i.$$

Assuming that $f_i(t, r_t, \lambda_t, N_t) = e^{V_1^i(t,T_\alpha) + V_2^i(t,T_\alpha) r_t}$ we deduce the following ODEs system

$$\begin{cases} \partial_t V_1^i\left(t, T_\alpha\right) + \theta\left(t\right) V_2^i\left(t, T_\alpha\right) + \frac{1}{2}\sigma_r^2 \left(V_2^i\left(t, T_\alpha\right)\right)^2 & = 0 \\ \partial_t V_2^i\left(t, T_\alpha\right) - a V_2^i\left(t, T_\alpha\right) & = 1 \end{cases},$$

with the following terminal conditions

$$\begin{cases} V_1^i\left(T_\alpha, T_\alpha\right) & = A\left(T_\alpha, T_i\right) \\ V_2^i\left(T_\alpha, T_\alpha\right) & = \omega - B(T_\alpha, T_i) \end{cases}.$$

Then, by solving this ODEs system for each i, the moment generating function of r_{T_α} is

$$\phi_{T_\alpha}^r\left(\omega\right) = \frac{1}{E_t} \sum_{i=\alpha+1}^{\beta} \delta^F\left(T_{i-1}, T_i\right) e^{V_1^i(t,T_\alpha) + V_2^i(t,T_\alpha) r_t},$$

where

$$\begin{aligned} V_1^i\left(t, T_\alpha\right) = {} & A\left(T_\alpha, T_i\right) + \mu \left[e^{au} f^M\left(0, u\right) + \left(\frac{\sigma_r}{a}\right)^2 \left(e^{-au} + e^{au}\right)\right]_{u=t}^{u=T_\alpha} \\ & - \frac{1}{a}\left[f^M\left(0, u\right) + a P^M\left(0, u\right) + \frac{\sigma_r^2}{a}\left(u + \frac{e^{-2au}}{2a}\right)\right]_{u=t}^{u=T_\alpha} \\ & + \left(\frac{\sigma_r}{2a}\right)^2 \left[a\mu^2 e^{2au} - 4\mu e^{au} + 2u\right]_{u=t}^{u=T_\alpha}, \end{aligned}$$

$$V_2^i\left(t, T_\alpha\right) = \mu e^{at} - \frac{1}{a},$$

with $\mu = \left(\omega + \frac{1}{a} - B(T_\alpha, T_i)\right) e^{-aT_\alpha}$. Then, using inverse DFT the values of $f_{r_{T_\alpha}}\left(\cdot\right)$ at points $x_k = -\frac{M}{2}\Delta_x + \left(k-1\right)\Delta_x$ are approached by the sum:

$$f_{r_{T_\alpha}}\left(x_k\right) \approx \frac{2}{M\Delta_x} Re\left(\sum_{j=1}^{M} \Upsilon_j \phi_{T_\alpha}^r\left(i z_j\right)\left(-1\right)^{(j-1)} e^{-i\frac{2\pi}{M}(j-1)(k-1)}\right),$$

where $\Upsilon_j = \frac{1}{2} 1_{\{j=1\}} + 1_{\{j \neq 1\}}$, M is the number of steps used in the DFT, $\Delta_x = \frac{2x_{max}}{M-1}$ is this step of discretization, and $\Delta_z = \frac{2\pi}{M\Delta_x}$ and $z_j = (j-1)\Delta_z$ for $j = 1 \ldots M$.

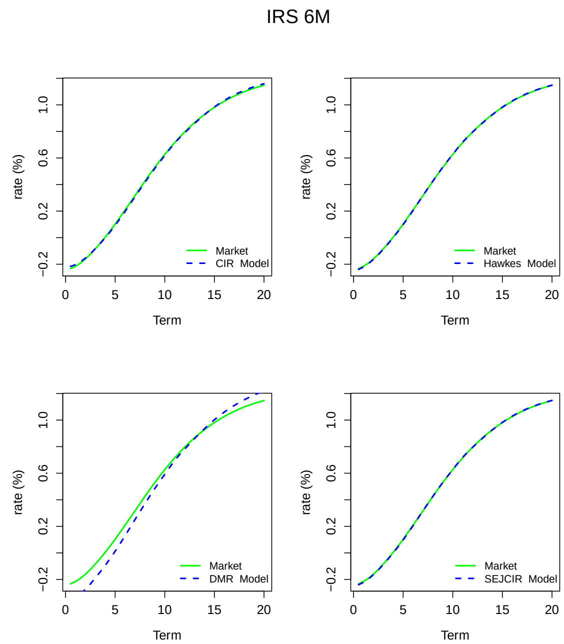

FIGURE 7.1. IRS curve with six months tenor for maturity from 6 months to 20 years observed in the market on 11/02/2019, and from the CIR model, the Hawkes model, the DMR model, and the SEJCIR with the estimated parameters of the tables (1,2).

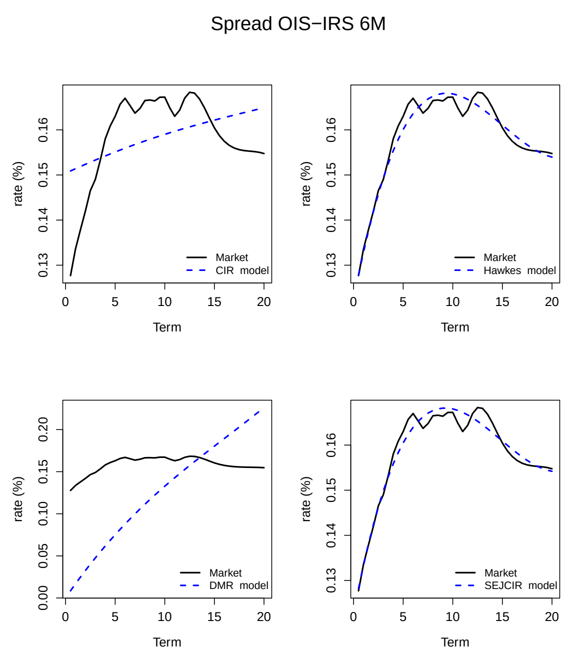

FIGURE 7.2. Spread between the OIS curve and the IRS curve with six months tenor for maturity from 6 months to 20 years observed in the market on 11/02/2019, and from the CIR model, the Hawkes model, the DMR model and the SEJCIR with the estimated parameters of the tables (1,2).

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