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Abstract

This paper purposes to formalize the three business models dominating peer-to-peer (P2P) property and casualty insurance: the self-governing model, the broker model and the carrier model. The former one develops outside the insurance market whereas the latter ones may originate from the insurance industry, by partnering with an existing company or by issuing a new generation of participating insurance policies where part of the risk is shared within a community and higher losses, exceeding the community's risk-bearing capacity are covered by an insurance or reinsurance company. The present paper proposes an actuarial modeling based on conditional mean risk sharing, to support the development of this new P2P insurance offer under each of the three business models. In addition, several specific questions are also addressed in the self-governing model. Considering an economic agent who has to select the optimal pool for a risk to be shared with other participants, it is shown that uniform comparison of the Lorenz or concentration curves associated to the respective total losses of the pools under consideration allows the agent to decide which pool is preferable. The monotonicity of the respective contributions of the participants is established with respect to the convex order, showing that increasing the number of participants is always beneficial under conditional mean risk sharing.

Keywords: Risk pooling, conditional mean risk sharing, Lorenz curve, concentration curve, convex order.

1 Introduction and motivation

Feeling to belong to a community and to serve societal needs has become nowadays important for many individuals. In that respect, commercial insurance does not seem to be very appealing. This may be attributed to the progressive de-mutualization of the insurance sector, which has broken the link with the ancestral compensation mechanism consisting in using the contributions of the many to balance the misfortunes of the few. Some policyholders even believe they should get their premium back in case no claims have been filed against the company by the end of coverage period. This demonstrates that the mutual aid principle at the heart of insurance has sometimes totally been forgotten. In order to remedy this situation, an offer has been developed in line with the sharing economy by organizing peer-to-peer (P2P) insurance communities. This paper focuses on property and casualty insurance and does not consider recent developments in relation with life and health covers.

The sharing economy is an economic model defined as a P2P-based activity of acquiring, providing, or sharing access to goods and services. It has rapidly grown and evolved in the recent past thanks to technological advances. Community-based online platforms acting as market places, connecting buyers and sellers are often the basic building blocks of this new economic model. Well-known examples of P2P business include Airbnb and eBay. Despite its success, the sharing economy also faces significant challenges in the form of regulatory uncertainty and serious concerns about abuses.

Several Insurtechs have tried to implement this approach in the insurance sector, with varying degrees of success. According to the classification proposed by Rego and Carvalho (2020), the dominant type of P2P insurance is the "P2P broker model". The main distinctive features of this model are as follows. Participants form groups (mainly online) and a part of the insurance premiums they pay goes into a common fund, while the remaining part is paid to an insurance or reinsurance company. If the common fund is insufficient to pay for the claims then the (re-)insurance carrier pays the excess over retained premiums. Conversely, if the pool has few claims then the surplus is given back to the participants or to a cause the pool members care about. The prominent example of this model is certainly Friendsurance launched in Germany in 2010. Inspeer operating in France since 2015 also falls under this model despite it works with only one insurance carrier.

Another type of P2P insurance that develops entirely outside the traditional insurance sector is the "self-governing model", or "no-insurance-company model". Such P2P organizations are made up of self-governing user communities consisting of peers who collectively manage all insurance functions. Communities can be created on the basis of peers' likeness or insured object: kind of insured object (e.g., car, house, health), kind of insured event (e.g., collision, damage to third party), social or professional affinity, or home/work location (e.g., living in the same town, working in the same office building), for instance.

Launched in China in 2015, TongJuBao belongs to this second type of P2P insurance. It provides an example working without any insurance company involved, showing the possible degree of disintermediation. Instead of partnering with, or intermediating for insurers, TongJuBao is a match-maker for people willing to join a common risk pool under an arrangement governed solely by civil law contracts. It can be seen as a pure mutual approach, where risks are shared among participants without any transfer to an insurance company.

Another example is Teambrella founded by Russian developers in 2015. Participating peers are organized into self-regulating, self-governing groups, which means that the exact rules that apply to each of them vary to some extent. All payments are made using bitcoins. Each member's exposure equals the sum available in its digital wallet. A number of co-signatures by other group members is necessary to withdraw the coins, bringing it closer to an escrow-type account.

A third type of P2P insurance is known as the "carrier model". Lemonade, a licensed US insurance company founded in 2015, is the prototype example of this approach. Compared to traditional insurers, Lemonade offers a fully digital, impressively effective customer relationship. Another prominent example of this model is certainly Alan, a licensed health insurer based in France operating as an insurance carrier since 2016.

The carrier model easily fits in the established insurance sector, taking advantage of its risk-bearing capacity and expertise in claim settlement procedures. Instead of simply transferring the risk to the insurance company (totally, or partly by applying individual deductibles for instance), customers are allowed to take part of the aggregate risk of the group they chose to adhere to, and take back a fraction of the end-of-period surplus. This mechanism is applied in the life insurance industry for many years: policyholders are granted bonuses, or dividends in relation with the performances of the financial market (sometimes, mortality credits are also awarded, but this is less common). Here, this idea is extended to the lower layer of total losses experienced by a community of policyholders.

The present paper purposes to formalize actuarial aspects of these new forms of insurance, providing insurers with an appropriate modeling to develop a new collaborative insurance offer. To this end, we use the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012) and apply it to each of the three business models discussed previously. Starting from the self-governing model where conditional mean risk sharing directly applies, we supplement it with guarantees to progressively move to the broker and carrier models. Precisely, we review the methodological results obtained in our preceding papers and use them to formalize the three dominant types of P2P insurance. Our analysis builds on the conditional mean risk sharing rule but also applies to other allocation rules among participants, such as those proposed by Clemente and Marano (2020), Abdikerimova and Feng (2019) and Feng et al. (2020), for instance. We do not consider legal issues related to the practical implementation of peer-to-peer insurance models. We refer the interested reader to relevant documentation produced by insurance regulators, such as EIOPA (2019) for instance.

The present paper also contains original methodological results. The following new research questions are addressed specifically in the self-governing model. First, we establish that increasing the number of participants is always beneficial under conditional mean risk sharing. This comes from the fact that each participant's contribution forms a backward, or reverse martingale when the size of the pool increases. As a consequence, there exists a limiting contribution when the number of participants tends to infinity.

Then, we discuss the choice of the optimal pool for a given loss. Precisely, assume that an economic agent must select among two pools where to allocate his or her own loss. It is shown that the comparison of the concentration curves of the individual loss with respect to the total losses of the two pools under consideration is consistent with second-order stochastic dominance, reflecting the common preferences for all risk-averse decision makers. When conditional expectations involved in the sharing rule are linear functions of the total loss experienced by the entire pool, in turns out that the choice is based on the Lorenz curves of the respective total losses, that is, the choice agrees with the Lorenz dominance relation.

The remainder of this paper is organized as follows. Section 2 is devoted to the self-governing model where participants share total losses, without transfer outside of the community. In Section 3, we move to the broker model in which risk sharing is restricted to the lower layer of total losses. The system proposed by Denuit (2020) replaces unlimited ex-post contributions under the self-governing model with upper-bounded ex-ante provisions combined with an ex-post bonus mechanism restoring fairness. In Section 4, a participating insurance policy falling under the carrier model is designed following Denuit and Robert (2021b). Compared to the broker model, the analysis starts from the insurance premium (so that the participating policy can be presented just as an option to any existing contract). The first step consists in identifying the lower layer subject to participation. As in the broker model, the upper layer is covered by the insurer and bonuses are granted as long as total losses do not exhaust the lower layer. The final Section 5 briefly discusses the results and concludes.

2 Self-governing, or no-insurance-company model

2.1 Business model

This technology-based model allows participants to pool their risks, self-organize and self-administer. In principle, the entirety of the risk is borne by the peers themselves, without transfer to an existing insurance or reinsurance company. For centuries, this has been done by mutual insurers and reciprocal insurance exchanges. This model thus builds on the fundamental principle at the origin of insurance without the equity buffer provided by insurer's capital. In line with the sharing economy, the self-governing model exploits technological advances in social networking to implement the mutual aid principle at the heart of insurance. Even if the P2P insurance community can only cope with relatively cheap sums insured because of its generally limited risk-bearing capacity, it is interesting to formalize the self-governing model to make explicitly the move to the other two cases (broker and carrier models). This move can be seen as pure risk sharing progressively supplemented with guarantees.

Let us now make the problem more formal. Consider n individuals, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing the monetary loss caused by some insurable peril over one period. The losses $X_1, X_2, \ldots, X_n$ obey some zero-augmented probability distributions, that is, $\mathrm{P}[X_i = 0] > 0$ for each i. This corresponds to the representation of insurance risks within the individual model of risk theory that applies very generally in property and casualty covers.

In our context, X_i corresponds to the amount of losses pooled within the P2P insurance community, not necessarily to the whole loss amount. Collaborative insurance splits the risks into several layers, each managed in a specific way (individually by each participant, collectively at the community's level or transferred to an insurer or reinsurer, for example). To make this clear, let us represent the loss X_i for participant i to the insurance pool as a compound Poisson sum: each of the N_i events affecting participant i produces a loss amount

C_{ik} called severity, $k = 1, \ldots, N_i$. Formally,

$$X_i = \sum_{k=1}^{N_i} C_{ik} \text{ with } N_i \sim \text{Poisson}(\lambda_i),\ \ i = 1, 2, \ldots, \tag{2.1}$$

where the severities C_{ik} are positive, all these random variables being independent. A first layer may be retained by participants, individually by introducing a deductible δ_i. The loss pooled is then limited to $(C_{ik} - \delta_i)_+ = \max\{C_{ik} - \delta_i, 0\}$ and the amount $\min\{C_{ik}, \delta_i\}$ is retained by participant i. This case is easily accounted for by modifying the Poisson parameter λ_i and considering claim severities distributed as $C_{ik} - \delta_i$ given $C_{ik} > \delta_i$ so that we do not need to discuss this case separately. The introduction of a deductible δ_i of course reduces the contribution by participant i to the total loss experienced by the pool.

In the remainder of this paper, we assume that $X_1, X_2, \ldots, X_n$ are independent. This assumption requires a careful examination given the nature of collaborative insurance. Since P2P insurance communities typically gather individuals sharing some familial, social or entrepreneurial relationship, we can expect that members of the same group have similar risk profiles, to some extent, because they live in the same area or belong to the same socio-economic class, for instance. Including these characteristics in the risk classification plan, that is, letting the distribution of X_i depend on these risk factors through an appropriate regression model, we can end up with independent losses $X_1, \ldots, X_n$ given the risk factors. This is the case for motor or home insurance, for instance. However, independence is sometimes unrealistic so that the formulas presented in this paper do not apply. This is for instance the case for the cover against natural catastrophes where geographic proximity rules out independence. To sum up, the independence assumption is not too restrictive in the one-period setting considered here and allows us to keep the mathematical aspects to a reasonable level, but it rules out the case of natural catastrophes and other major events impacting on all losses X_i, for instance.

Henceforth, we use the notation $S_n = \sum_{i=1}^n X_i$ for the total losses of the pool, making explicit the dependence on the number n of participants. In the self-governing model, participants contribute ex-post to the total losses experienced by the community. Therefore, we need a risk sharing rule to distribute total losses S_n among participants, in a fair and transparent way.

2.2 Homogeneous losses

Let us start with the most elementary case: Pooling independent and identically distributed losses. If $X_1, X_2, \ldots$ are independent and identically distributed then the total loss S_n is distributed uniformly among the n participants so that participant i is invited to pay

$$\overline{X}_n = \frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{2.2}$$

ex-post, instead of X_i. It can be shown that for every concave utility function u_i and initial wealth w_i,

$$\text{E}[u_i(w_i - \overline{X}_n)] \geq \text{E}[u_i(w_i - X_i)] \text{ for all } i \in \{1, 2, \ldots, n\}. \tag{2.3}$$

See e.g. Example 3.A.29 in Shaked and Shanthikumar (2007). In words, inequality (2.3) means that every risk-averse economic agent prefers joining the pool over the stand-alone position. Moreover,

$$\mathrm{E}[u_i(w_i - \overline{X}_{n+1})] \geq \mathrm{E}[u_i(w_i - \overline{X}_n)] \text{ for all } n \text{ and } i \in \{1, 2, \dots, n\}, \tag{2.4}$$

that is, risk-averse economic agents value the inclusion of new members in the pool, as long as the losses they bring to the pool are independent and distributed as those already comprised in the pool. Stated otherwise, the expected utility $\mathrm{E}[u_i(w_i - \overline{X}_n)]$ is an increasing function of the number n of participants to the pooling arrangement. To sum up, risk-averse economic agents consider pooling as beneficial and even more so if the number of participants in the pooling arrangement increases. On average, pooling has no effect but it "reduces risk" in the sense that the pooling arrangement moves probability mass from the tails towards the center of the distribution. We refer the reader to Albrecht and Huggenberger (2017) for a thorough investigation of the homogeneous case.

The law of large numbers and central-limit theorem even show that the risk disappears at the limit, in relative terms. Precisely, denoting as μ and σ^2 the common mean and variance of $X_1, X_2, \dots$, the law of large numbers ensures that

$$\lim_{n\to\infty} \overline{X}_n = \mu \text{ with probability } 1 \tag{2.5}$$

where the mathematical expectation μ is referred to as the pure premium, or actuarially fair price. The risk per participant thus disappears at the limit. The central-limit theorem then assesses the fluctuation of the individual contributions $\overline{X}_n$ around the pure premium μ, in the sense that

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \overset{\mathcal{D}}{\to} \text{Normal}\,(0,1) \tag{2.6}$$

where $\overset{\mathcal{D}}{\to}$ refers to convergence in distribution and Normal(0,1) is the standard Normal distribution, with zero mean and unit variance. Precisely, (2.6) means that

$$\mathrm{P}\left[\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leq x\right] \to \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\mathrm{d}t$$

for all x when $n \to \infty$. Here, $\Phi(\cdot)$ is the distribution function of the Normal(0,1) law.

2.3 Extension to heterogeneous losses

2.3.1 Questions to be addressed

When losses are independent and identically distributed, pooling is thus extremely easy to operate as every participant must contribute the same amount $\overline{X}_n$ to the total losses S_n. However, homogeneity is more the exception than the rule for insurance risks and equally sharing heterogeneous losses among participants may well appear to be unfair. To be successful, P2P insurance schemes require an appropriate risk sharing mechanism recognizing the different distributions of the risks brought to the pool.

We aim to answer the following questions:

Question 1 is there a way to distribute the total loss S_n among the n participants so that every risk-averse economic agent considers that enlarging the pool is always beneficial, whatever the distribution of the losses $X_1, X_2, \ldots, X_n$?

Question 2 does the risk per participant still disappear as $n \to \infty$?

Question 3 is there a central-limit theorem assessing the fluctuations of the participant's contribution around the actuarially fair price for large n?

Moving to heterogeneous risks, denote the mean and variance of X_i as

$$\mu_i = \mathrm{E}[X_i] > 0 \text{ and } \sigma_i^2 = \mathrm{Var}[X_i] > 0.$$

Denuit (2019) proposed a simple and mathematically correct allocation within P2P insurance communities based on the conditional mean risk sharing rule introduced by Denuit and Dhaene (2012).

2.3.2 Conditional mean risk sharing rule

A risk sharing rule is a way to distribute the total losses among participants. Formally, let x_i be the realization of the loss X_i (known ex-post) and denote as $s = \sum_{i=1}^n x_i$ the corresponding realization of S_n. Under the risk sharing rule described by the functions $h_1, h_2, \ldots, h_n$, participant i contributes ex-post an amount $h_{i,n}(s)$ such that

$$\sum_{i=1}^{n} h_{i,n}(s) = s.$$

Here, the function defining the sharing rule is doubly indexed because it is specific to each participant i but also changes with the size n of the pool. The uniform allocation (2.2) is an example of risk sharing rule, with $h_{i,n}(s) = s/n$ for $i = 1, 2, \ldots, n$.

Fairness is an important aspect of collaborative insurance. By fairness, it is meant that the expected contribution must be equal to the expected loss transferred to the pool. Precisely, a fair sharing rule does not induce any transfer between participants, on average, that is

$$\mathrm{E}[h_{i,n}(S_n)] = \mu_i \text{ for } i = 1, 2, \ldots, n.$$

Apart from social insurance, where transfers are sometimes considered as desirable and explicitly organized (with premiums based on salaries and not reflecting risk, for instance), fairness is generally considered as an important requirement in insurance. When losses are heterogenous, the uniform allocation (2.2) generally fails to be fair.

The conditional mean risk sharing rule proposed by Denuit and Dhaene (2012) appears to be particularly attractive for P2P insurance schemes (as it will become clear from the material discussed in the remainder of this section). Under the conditional mean risk sharing rule, participant i contributes

$$h^{\star}_{i,n}(S_n) = \mathrm{E}[X_i|S_n] \text{ to the total loss } S_n = \sum_{i=1}^{n} X_i.$$

In words, each participant contributes the expected value of the risk he or she brings to the pool, given the total loss experienced by the entire group. Stated differently, the contribution paid by each participant is the average part of the total loss that can be attributed to the risk he or she brings to the pool. In particular,

$$X_1, X_2, \ldots, X_n \text{ identically distributed} \Rightarrow h^{\star}_{i,n}(S_n) = \mathrm{E}[X_i|S_n] = \overline{X}_n$$

so that we recover the uniform allocation (2.2). The conditional mean risk sharing rule thus extends the uniform allocation applying in the homogeneous case to heterogeneous losses. Since participants can be informed when they enter the pool about the amount they will have to contribute as a function of the total realized loss (the function $s \mapsto \mathrm{E}[X_i|S_n = s]$ can be communicated to participant i beforehand), this approach ensures full transparency.

The key point to explain to participants is that the realization of X_i is mostly due to chance, and only partly reveals the underlying magnitude of the risk brought to the community. Over one period, the risk X_i is decomposed into a random part $X_i - \mathrm{E}[X_i|S_n]$ shared among participants by virtue of the mutuality (or random solidarity) principle and a structural part $\mathrm{E}[X_i|S_n]$ to be supported by participant i, individually. Formally, the risk X_i brought by participant i is decomposed into

$$X_i = \underbrace{\mathrm{E}[X_i|S_n]}_{\text{=structural part}} + \underbrace{X_i - \mathrm{E}[X_i|S_n]}_{\text{=random departure } \mathcal{E}_i}. \tag{2.7}$$

Both terms entering this split are uncorrelated and random departures $\mathcal{E}_i$ have zero means and always sum to 0. With P2P insurance, each participant must contribute ex-post the amount $\mathrm{E}[X_i|S_n]$ to the pool whereas $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are re-allocated among them according to the mutuality principle.

Clearly,

$$\sum_{i=1}^{n} h^{\star}_{i,n}(S_n) = \sum_{i=1}^{n} \mathrm{E}[X_i|S_n] = S_n$$

so that $h^{\star}_{i,n}$ allocates the total loss among participants. Furthermore, the risk sharing rule $h^{\star}_{i,n}$ is fair since

$$\mathrm{E}\big[h^{\star}_{i,n}(S_n)\big] = \mathrm{E}\big[\mathrm{E}[X_i|S_n]\big] = \mu_i \text{ for } i = 1, 2, \ldots, n,$$

as a direct application of the tower rule for conditional expectations.

From a practical point of view, the calculation of $h^{\star}_{i,n}(s)$ can be performed in a direct way or by exploiting representations in terms of size-biasing derived by Denuit (2019). Numerical approximations using orthogonal polynomials are also available.

2.3.3 Question 1: risk reduction by pooling?

The conditional mean risk allocation is regarded as beneficial by all risk-averse economic agents, because

$$\mathrm{E}[u_i(w_i - X_i)] \leq \mathrm{E}[u_i(w_i - h^{\star}_{i,n}(S_n))] \text{ for all } i \in \{1, 2, \ldots, n\}, \tag{2.8}$$

whatever the concave utility function u_i and the initial wealth level w_i. This is a direct application of Jensen inequality for conditional expectations. Inequality (2.8) extends (2.3) to the heterogeneous case. It shows that provided total losses are distributed among participants according to the conditional mean risk sharing rule, every risk-averse agent prefers joining the pool and paying $h^\star_{i,n}(S_n)$ over the stand-alone position, keeping the individual loss X_i. Interestingly, this is true whatever the distribution of the other losses X_j, $j \neq i$, comprised in the pool.

We know from (2.4) that with identically distributed losses X_i, enlarging the pool is always beneficial when the total loss S_n is distributed uniformly among the n participants. Let us now prove that this remains true with heterogeneous losses provided S_n is allocated among participants according to the conditional mean risk sharing rule $h^\star_{i,n}$. Precisely, for any independent losses X_i, we want to establish that

$$\mathrm{E}[u_i(w_i - h^\star_{i,n}(S_n))] \leq \mathrm{E}[u_i(w_i - h^\star_{i,n+1}(S_{n+1}))] \text{ for all } n \text{ and } i \in \{1, 2, \ldots, n\}, \tag{2.9}$$

for any concave utility function u_i and initial wealth w_i.

To this end, we need a tool expressing the common preferences for all risk-averse economic agents. This probabilistic tool is called the convex order, henceforth denoted as $\preceq_{\text{convex}}$. Precisely, given two losses X and Y, X is said to be smaller than Y in the convex order, henceforth denoted by $X \preceq_{\text{convex}} Y$ if

$$\mathrm{E}[u(w - X)] \geq \mathrm{E}[u(w - Y)] \text{ for all concave utility function } u \text{ and wealth level } w,$$

provided the expectations exist. In particular, $X \preceq_{\text{convex}} Y \Rightarrow \mathrm{E}[X] = \mathrm{E}[Y]$ so that $\preceq_{\text{convex}}$ only applies to losses with the same expected value. Hence, $\preceq_{\text{convex}}$ corresponds to the mean-preserving increase in risk, or second-order stochastic dominance relation restricted to random variables with equal expected values in economics when gains are replaced with losses.

It can be shown that $X \preceq_{\text{convex}} Y$ holds if, and ony if, $\mathrm{E}[X] = \mathrm{E}[Y]$ and $\mathrm{E}[(X - d)_+] \leq \mathrm{E}[(Y - d)_+]$ for all $d \geq 0$. Thus, the respective stop-loss premiums are ordered for all possible levels d of the deductible. The term "convex" is used since $X \preceq_{\text{convex}} Y \Leftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all convex functions g for which the expectations exist. The stochastic inequality $X \preceq_{\text{convex}} Y$ intuitively means that X and Y have the same "size" (as $\mathrm{E}[X] = \mathrm{E}[Y]$ holds) but that Y is "more variable" than X. For instance, the variance of Y is larger than the variance of X. The next property will be useful in the remainder of this text:

$$X \preceq_{\text{convex}} Y \Leftrightarrow aX + b \preceq_{\text{convex}} aY + b \text{ for all } a,\, b \in \mathbb{R}. \tag{2.10}$$

Notice that a may be negative in (2.10) since the convex order focuses on variability (this comes from the fact that if $x \mapsto g(x)$ is convex then $x \mapsto g(-x)$ also is). For a thorough description of the convex order and its applications in an actuarial context, we refer the reader to Denuit et al. (2005).

We are now ready to state the main result of this section.

Proposition 2.1. *Consider independent losses $X_1, X_2, \ldots$ with partial sums $S_n = X_1 + \ldots + X_n$. Then, the stochastic inequality*

$$h^\star_{i,n+1}(S_{n+1}) = \mathrm{E}[X_i | S_{n+1}] \preceq_{\text{convex}} \mathrm{E}[X_i | S_n] = h^\star_{i,n}(S_n)$$

holds for every integer n and $i \in \{1, 2, \ldots, n\}$.

Proof. Without loss of generality, let us establish the result for $i = 1$. Since the random variables $X_1, X_2, \ldots$ are mutually independent, we can write

$$\begin{aligned} \mathrm{E}[X_1|S_n] &= \mathrm{E}[X_1|S_n, X_{n+1}, X_{n+2}, \ldots] \\ &= \mathrm{E}[X_1|S_n, S_{n+1}, S_{n+2}, \ldots] \end{aligned}$$

because the sigma-algebra $\sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$ generated by $S_n, X_{n+1}, X_{n+2}, \ldots$ coincides with $\sigma(S_n, S_{n+1}, S_{n+2}, \ldots)$. Similarly, we get

$$\mathrm{E}[X_1|S_{n+1}] = \mathrm{E}[X_1|S_{n+1}, S_{n+2}, S_{n+3}, \ldots].$$

Thus, we see that

$$\begin{aligned} \mathrm{E}[X_1|S_{n+1}] &= \mathrm{E}\big[\mathrm{E}[X_1|S_n, S_{n+1}, S_{n+2}, \ldots]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big] \\ &= \mathrm{E}\big[\mathrm{E}[X_1|S_n]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big] \end{aligned}$$

for every integer n. Considering a convex function g, Jensen's inequality allows us to write

$$\begin{aligned} \mathrm{E}\Big[g\big(\mathrm{E}[X_1|S_n]\big)\Big] &= \mathrm{E}\Big[\mathrm{E}\big[g\big(\mathrm{E}[X_1|S_n]\big)\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big]\Big] \\ &\geq \mathrm{E}\Big[g\big(\mathrm{E}\big[\mathrm{E}[X_1|S_n]\big|S_{n+1}, S_{n+2}, S_{n+3}, \ldots\big]\big)\Big] \\ &= \mathrm{E}\Big[g\big(\mathrm{E}[X_1|S_{n+1}]\big)\Big]. \end{aligned}$$

This ends the proof. $\square$

Proposition 2.1 shows that $\mathrm{Cov}[X_i, h^\star_{i,n}(S_n)] = \mathrm{Var}[h^\star_{i,n}(S_n)]$ decreases with the number n of participants. Let us also mention that Proposition 2.1 is related to peacock theory. A peacock is a stochastic process with marginals that are monotonic in the sense of the convex order. We refer the reader to Hirsch et al. (2011) for a detailed account.

In the expected utility setting, Proposition 2.1 shows that increasing the number of participants is always regarded as beneficial by all risk-averse economic agents, whatever the distribution of the losses they bring to the pool as long as these losses are mutually independent and the sharing is operated according to the conditional mean risk allocation rule. The conditional mean risk sharing rule thus provides the appropriate extension to simple averaging in the homogeneous case.

2.3.4 Question 2: risk elimination at the limit?

From the proof of Proposition 2.1, we see that the sequence $\{h^\star_{1,n}(S_n),\ n = 1, 2, \ldots\}$ forms a backward, or reverse martingale. As a consequence, $h^\star_{1,n}(S_n)$ converges to the conditional expectation $\mathrm{E}[X_1|\mathcal{T}]$ where $\mathcal{T}$ is the tail sigma-field

$$\mathcal{T} = \cap_{n=1}^{\infty}\sigma(S_n, S_{n+1}, S_{n+2}, \ldots).$$

If $\mathcal{T}$ is a trivial sigma-field then $\mathrm{E}[X_1|\mathcal{T}] = \mathrm{E}[X_1]$ holds with probability 1. This is the case for instance when the random variables X_i are independent and identically distributed and corresponds to Proposition 1.1 in Zabell (1980).

Denuit and Robert (2021a) demonstrated that the pure premium can be obtained at the limit when the number of participants tends to infinity provided mild technical conditions are fulfilled. Precisely, consider independent losses $X_1, X_2, \dots$ of the form (2.1), obeying compound Poisson distributions with absolutely continuous severities. Subject to some technical conditions (including the existence of moments up to order 4),

$$\lim_{n\to\infty} h^{\star}_{i,n}(S_n) = \mu_i \text{ with probability 1.} \tag{2.11}$$

The limit in (2.11) extends the classical law of large numbers (2.5). The technical conditions stated in Denuit and Robert (2021a) for (2.11) to hold help to identify situations where risk can be fully eliminated in an infinitely large pool. The results obtained so far thus show that expected utility $\mathrm{E}[u_i(w_i - h^{\star}_{i,n}(S_n))]$ increases in n when u_i is concave, tending to $u_i(w_i - \mu_i)$ under (2.11).

Considering the split in equation (2.7), we can further expand it thanks to (2.11) as

$$X_i = \underbrace{h^{\star}_{i,n}(S_n)}_{=\text{structural part}} + \underbrace{X_i - \mu_i}_{=\text{limit random departure}} + \underbrace{\mu_i - h^{\star}_{i,n}(S_n)}_{=\text{imperfect diversification}}$$

where the last term accounts for the limited size of the pool, preventing perfect risk diversifcation. In commercial insurance, the variations in S_n are absorbed by insurer's capital and individual i must contribute a fixed amount p_i to compensate the cost of insured events. The premium p_i is equal to the pure premium $\mu_i = \mathrm{E}[X_i]$ supplemented with shareholder's profit for providing the risk-bearing capital. Hence, the commercial insurance premium p_i is loaded: $\mathrm{p}_i > \mu_i$.

Considering the homogeneous case where losses X_i are independent and identically distributed, with common mean μ, the premium p is then equal for all individuals. Assume that participants are expected-utility maximizers and let w_i stand for the initial wealth for participant i, with utility function u_i. The expected utility $\mathrm{E}[u_i(w_i - \overline{X}_n)]$ increases in n when u_i is concave (expressing risk aversion), tending to $u_i(w_i - \mu)$. Hence, for some n_i, it must exceed $u_i(w_i - \mathrm{p})$. Thus, there must be a critical size $n = \max_i n_i$ such that every agent joins the pool instead of buying insurance.

Now, let us consider the more general setting where losses X_i are mutually independent but not necessarily identically distributed. Since we known from Proposition 2.1 that $h^{\star}_{i,n}(S_n)$ decreases with n in the sense of $\preceq_{\text{convex}}$, a similar reasoning holds provided $\overline{X}_n$ is replaced with $h^{\star}_{i,n}(S_n)$. If (2.11) holds true then the expected utility $\mathrm{E}[u_i(w_i - h^{\star}_{i,n}(S_n))]$ thus increases in n when u_i is concave, tending to $u_i(w_i - \mu_i)$. As in the simple case of independent and identically distributed risks, there must be some number n_i of participants such that it exceeds $u_i(w_i - \mathrm{p}_i)$. The expected utility when joining the pool thus becomes larger than the expected utility associated to buying insurance for a loaded premium for some sufficiently large n, precisely $n \geq \max_i n_i$.

2.3.5 Question 3: is there a central-limit theorem for $h^{\star}_{i,n}(S_n)$?

Central-limit theorems for participants' contributions have also been established by Denuit and Robert (2021a), to investigate how the individual contribution fluctuates around the pure premium when the size of the group is large enough. A linear approximation for $h^\star_{i,n}(S_n)$ in large pools is also derived there, that is recalled next and considerably eases calculations.

Define the linear regression rule as

$$h^{\text{reg}}_{i,n}(S_n) = \mu_i + \frac{\sigma_i^2}{\sum_{j=1}^n \sigma_j^2}\left(S_n - \mathrm{E}[S_n]\right), \;\; i = 1, 2, \ldots, n.$$

Under $h^{\text{reg}}_{i,n}$ participants agree to pay the pure premium and to divide the deviations of S_n from its expected value according to the volatility of the risks brought to the pool. The linear regression rule is intimately related to the conditional mean risk sharing rule. Indeed, the conditional expectation $h^\star_{i,n}(S_n)$ is the function of S_n that is the closest to the individual loss X_i, in the sense of mean squared error. The same holds for $h^{\text{reg}}_{i,n}(S_n)$ provided the search is restricted to linear functions of S_n. Hence, $h^{\text{reg}}_{i,n}(S_n)$ can be seen as a linear version of $h^\star_{i,n}(S_n)$.

Assume that $X_1, X_2, \ldots$ are absolutely continuous or obey compound Poisson distributions with absolutely continuous severities and denote

$$s_n^2 = \mathrm{Var}[S_n] = \sum_{i=1}^n \sigma_i^2.$$

Subject to some technical conditions stated in Denuit and Robert (2021a),

$$\frac{s_n}{\sigma_i^2}\begin{pmatrix} h^{\text{reg}}_{i,n}(S_n) - \mu_i \\ h^\star_{i,n}(S_n) - \mu_i \end{pmatrix} \xrightarrow{\mathcal{D}} \text{Normal}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right).$$

This shows that there is a linear approximation to the conditional mean risk sharing rule when the number of participants gets large.

2.4 Comparing pools

2.4.1 Criterion

Assume that an economic agent must select a pool where to allocate his or her own loss X. There are two possibilities: pool 1 with total losses $S_n = X + Y_1$ and pool 2 with total losses $T_m = X + Y_2$. Here, Y_k stands for the risks in pool k before agent's entrance, that is,

$$Y_1 = \sum_{i=1}^{n-1} X_j \text{ and } Y_2 = \sum_{j=1}^{m-1} Z_j,$$

say. As an application, we consider the case where $Y_2 = \sum_{j=1}^n X_j$ to compare the impact of the size of the pool on its attractiveness.

Since the convex order $\preceq_{\text{convex}}$ reflects the preferences about monetary losses shared by all risk-averse agents in the expected utility setting, we take as criterion

$$\text{pool 1 is preferred over pool 2 } \Leftrightarrow \mathrm{E}[X|S_n] \preceq_{\text{convex}} \mathrm{E}[X|T_m].$$

Denuit (2010) established conditions under which conditional expectations are ranked in the sense of the convex order $\preceq_{\text{convex}}$ when S_n and T_m are identically distributed. Here, we adapt the reasoning to the present setting, removing the constraint of identical distribution for S_n and T_m that appears to be unrealistic in our context. Before discussing the general case, we start with a particular situation of interest.

2.4.2 Losses supporting a proportional risk allocation

Consider the pair of random variables (X, Y). Furman et al. (2018) characterized the situation where the conditional expectation $\mathrm{E}[X|S]$ is proportional to $S = X + Y$. Let $\mathcal{L}(u,v) = \mathrm{E}[\exp(-uX - vY)]$ be the Laplace transform of the random couple (X, Y). Then,

$$\mathrm{E}[X|S] = \frac{\mathrm{E}[X]}{\mathrm{E}[S]} S \Leftrightarrow \left.\frac{\frac{\partial}{\partial u}\mathcal{L}(u,v)}{\frac{\partial}{\partial v}\mathcal{L}(u,v)}\right|_{(u,v)=(t,t)} = \frac{\mathrm{E}[X]}{\mathrm{E}[Y]} \text{ for all } t. \tag{2.12}$$

If X and Y are mutually independent then the requirement in (2.12) is equivalent to the constantness of the function $t \mapsto \ln \mathcal{L}_X(t)/\ln \mathcal{L}_Y(t)$ over $[0,\infty)$, where $\mathcal{L}_X(t) = \mathrm{E}[\exp(-tX)]$ and $\mathcal{L}_Y(t) = \mathrm{E}[\exp(-tY)]$ are the respective Laplace transforms of X and Y. The constant must be $\mathrm{E}[X]/\mathrm{E}[Y]$.

In essence, (2.12) means that all risk-averse agents recognize that there is some diversification between X and Y, in the sense that the sum S is smaller compared to both X and Y in the Lorenz order defined by means of pointwise comparison of the Lorenz curves. These curves are widely used in economics to measure income inequality. Precisely, the Lorenz curve L_X associated with the non-negative random variable X is defined at probability level $p \in (0,1)$ by

$$L_X(p) = \frac{\int_0^p F_X^{-1}(\xi)\mathrm{d}\xi}{\int_0^1 F_X^{-1}(\xi)\mathrm{d}\xi} = \frac{\mathrm{E}\left[X\mathrm{I}\left[X \leq F_X^{-1}(p)\right]\right]}{\mathrm{E}[X]}. \tag{2.13}$$

where $\mathrm{I}[\cdot]$ is the indicator function (equal to 1 if the condition appearing within the brackets is fulfilled and to 0 otherwise). We refer the interested reader to Arnold and Sarabia (2018) for an extensive presentation of Lorenz curves and to Denuit and Vermandele (1999) for applications to insurance.

Given two risks X and Y, X is said to be smaller than Y in the Lorenz order, which is henceforth denoted as $X \preceq_{\text{Lorenz}} Y$ when the graph of the Lorenz curve for X lies above the graph of the Lorenz curve for Y. Formally,

$$X \preceq_{\text{Lorenz}} Y \Leftrightarrow L_X(p) \geq L_Y(p) \text{ for all } p \in [0,1]. \tag{2.14}$$

It is known from formula (3.A.33) in Shaked and Shanthikumar (2007) that given two risks X and Y,

$$X \preceq_{\text{Lorenz}} Y \Leftrightarrow \frac{X}{\mathrm{E}[X]} \preceq_{\text{convex}} \frac{Y}{\mathrm{E}[Y]}. \tag{2.15}$$

Considering $\mathrm{E}[X]$ and $\mathrm{E}[Y]$ as the pure premiums for X and Y, Lorenz order is thus equivalent to the comparison of loss ratios $\frac{X}{\mathrm{E}[X]}$ and $\frac{Y}{\mathrm{E}[Y]}$ with the help of convex order. Whereas only risks possessing the same means can be compared through $\preceq_{\text{convex}}$, ordering risks through $\preceq_{\text{Lorenz}}$ does not require that X and Y have the same first moment. Obviously, when $\mathrm{E}[X] = \mathrm{E}[Y]$, we then have $X \preceq_{\text{Lorenz}} Y \Leftrightarrow X \preceq_{\text{convex}} Y$ from (2.15) together with (2.10).

Coming back to (2.12), we see that it can be equivalently rewritten as

$$\mathrm{E}\left[\frac{X}{\mathrm{E}[X]}\Big|\frac{S}{\mathrm{E}[S]}\right] = \frac{S}{\mathrm{E}[S]} \text{ and } \mathrm{E}\left[\frac{Y}{\mathrm{E}[Y]}\Big|\frac{S}{\mathrm{E}[S]}\right] = \frac{S}{\mathrm{E}[S]}.$$

By Jensen inequality, this implies that

$$\frac{S}{\mathrm{E}[S]} \preceq_{\text{convex}} \frac{X}{\mathrm{E}[X]} \text{ and } \frac{S}{\mathrm{E}[S]} \preceq_{\text{convex}} \frac{Y}{\mathrm{E}[Y]}.$$

Hence, (2.12) ensures that the sum S is smaller compared to both X and Y in the Lorenz order. In other words, aggregating the risks X and Y is considered to be beneficial in terms of loss ratios by all risk-averse agents when (2.12) holds true.

The next result shows that agents prefer the pool with the smaller total loss in the sense of the Lorenz order, when conditional expectations involved in the risk sharing are proportional to total losses.

Proposition 2.2. *Assume that both* (X, Y_1) *and* (X, Y_2) *satisfy* (2.12)*. Then,*

$$\mathrm{E}[X|S_n] \preceq_{\text{convex}} \mathrm{E}[X|T_m] \Leftrightarrow S_n \preceq_{\text{Lorenz}} T_m.$$

Proof. We see that

$$\begin{aligned}\mathrm{E}[X|S_n] \preceq_{\text{convex}} \mathrm{E}[X|T_m] \quad &\Leftrightarrow \quad \frac{\mathrm{E}[X]}{\mathrm{E}[S_n]} S_n \preceq_{\text{convex}} \frac{\mathrm{E}[X]}{\mathrm{E}[T_m]} T_m \\ &\Leftrightarrow \quad \frac{S_n}{\mathrm{E}[S_n]} \preceq_{\text{convex}} \frac{T_m}{\mathrm{E}[T_m]} \text{ by (2.10).}\end{aligned}$$

It turns out from (2.15) that the latter condition corresponds to the Lorenz order. This ends the proof. □

Thus we see that under (2.12), pool 1 is preferred over pool 2 by all risk-averse agents if, and only if, $S_n \preceq_{\text{Lorenz}} T_m$.

Remark 2.3 (Recruiting new participants in the proportional case)**.** We know that (2.12) in essence ensures the existence of risk diversification by pooling. Indeed, assume that condition (2.12) holds true with $X = X_i$ and $Y = S_n - X_i$ for all n and $i \in \{1, 2, \ldots, n\}$. Thus,

$$\mathrm{E}[X_i|S_n] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n \text{ for all } n \text{ and } i \in \{1, 2, \ldots, n\},$$

which can be rewritten as

$$\mathrm{E}\left[\frac{X_i}{\mathrm{E}[X_i]}\Big|\frac{S_n}{\mathrm{E}[S_n]}\right] = \frac{S_n}{\mathrm{E}[S_n]} \quad \Rightarrow \quad \frac{S_n}{\mathrm{E}[S_n]} \preceq_{\text{convex}} \frac{X_i}{\mathrm{E}[X_i]} \Leftrightarrow S_n \preceq_{\text{Lorenz}} X_i.$$

Now, summing $\mathrm{E}[X_i|S_{n+1}] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_{n+1}]} S_{n+1}$ over $i \in \{1, 2, \ldots, n\}$, we also have

$$\sum_{i=1}^{n} \mathrm{E}[X_i|S_{n+1}] = \mathrm{E}[S_n|S_{n+1}] = \frac{\mathrm{E}[S_n]}{\mathrm{E}[S_{n+1}]} S_{n+1} \Rightarrow S_{n+1} \preceq_{\text{Lorenz}} S_n.$$

This means that recruiting a new member bringing the loss X_{n+1} is beneficial for all participants, in the sense that $\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{convex}} \mathrm{E}[X_i|S_n]$ for every $i = 1, 2, \ldots, n$. Whatever the distributions of the risks X_i provided condition (2.12) is fulfilled, the inclusion of new participants within the pool is always considered as beneficial by all risk-averse economic agents provided independent losses are allocated according to the conditional mean risk sharing. Proposition 2.1 shows that this result holds true beyond the proportional case (2.12).

2.4.3 General conditional mean risk sharing

Let us now move to the general case. To this end, we need to introduce concentration curves that are intimately related to Lorenz curves. The concentration curve of X with respect to S is defined as

$$C_{(X,S)}(p) = \frac{\mathrm{E}\big[X\mathrm{I}[S \leq F_S^{-1}(p)]\big]}{\mathrm{E}[X]}, \ \ p \in (0,1),$$

where F_S^{-1} is the quantile function corresponding to the distribution function F_S for S. For an exhaustive review of the properties of a concentration curve we refer the interested reader to the book by Yitzhaki and Schechtman (2013). For applications to insurance, see for instance Denuit et al. (2019).

The extension of the result derived for losses supporting a proporional risk allocation to general losses requires the assumption that the functions $h_{i,n}^\star$ are all continuously increasing. It is worth to stress that these functions are not necessarily increasing. Counterexamples are easy to build when the supports of the risks X_i differ or when some risks dominate the others, as established in Denuit and Robert (2020a). There are also numerous examples where the functions $h_{i,n}^\star$ are increasing. In the homogeneous case, the uniform allocation (2.2) increases linearly in S_n. This is also the case for risks supporting a proportional allocation, as discussed in the previous section. This includes the semi-homogeneous collective risk model, where $X_1, X_2, \ldots, X_n$ obey compound Poisson distributions with identically distributed severities. If all $h_{i,n}^\star(S_n)$ increase with S_n then they are said to be comonotonic.

Denuit and Robert (2020b) proved that the functions $h_{i,n}^\star$ become generally increasing when n gets large, in the sense that, under suitable technical conditions,

$$\lim_{n\to\infty} s_n \frac{\partial}{\partial\alpha}\mathrm{E}\left[X_i - \mu_i \middle| S_n = \sum_{i=1}^{n} \mu_i + s_n \Phi^{-1}(\alpha)\right] = \frac{\sigma_i^2}{\varphi\left(\Phi^{-1}(\alpha)\right)} \tag{2.16}$$

for any $\alpha \in (0,1)$, where $\varphi(\cdot)$ denotes the probability density function of the standard Normal distribution, that is, $\varphi = \Phi'$. Let us stress that no log-concavity assumption is needed for (2.16) to hold, in contrast with sums comprising a limited number of terms where restrictive conditions must be imposed on the distribution of each term.

The limit in (2.16) ensures that the conditional mean risk exchange is Pareto-optimal within large pools, as established by Denuit and Dhaene (2012). The increasingness of the risk sharing rule appears to be a reasonable requirement in P2P insurance since it guarantees that all participants have an interest to keep total losses S_n as small as possible. Formula (2.16) shows that this requirement is likely to be met within large pools.

The next result gives a simple criterion ensuring that $\mathrm{E}[X|S_1] \preceq_{\text{convex}} \mathrm{E}[X|S_2]$ holds true so that pool 1 is preferred over pool 2.

Proposition 2.4. *Assume that both $S_n = X + Y_1$ and $T_m = X + Y_2$ have a possible probability mass at the origin and a probability density function over $(0, \infty)$. Assume that $s \mapsto \mathrm{E}[X|S_n = s]$ and $s \mapsto \mathrm{E}[X|T_m = s]$ are continuously increasing. Then,*

$$\mathrm{E}[X|S_n] \preceq_{\text{convex}} \mathrm{E}[X|T_m] \Leftrightarrow C_{(X,S_n)}(p) \geq C_{(X,T_m)}(p) \text{ for all probability level } p.$$

Proof. Since $s \mapsto \mathrm{E}[X|S_n = s]$ is continuously increasing, the equivalence

$$\mathrm{E}[X|S_n] \geq F^{-1}_{\mathrm{E}[X|S_n]}(p) \Leftrightarrow S_n \geq F^{-1}_{S_n}(p)$$

holds for all probability level p, so that

$$\begin{aligned}\mathrm{E}\Big[\mathrm{E}[X|S_n]\Big|\mathrm{E}[X|S_n] \geq F^{-1}_{\mathrm{E}[X|S_n]}(p)\Big] &= \mathrm{E}\Big[\mathrm{E}[X|S_n]\Big|S_n \geq F^{-1}_{S_n}(p)\Big] \\ &= \mathrm{E}[X|S_n \geq F^{-1}_{S_n}(p)]. \end{aligned} \tag{2.17}$$

Considering formula (3.A.41) in Shaked and Shanthikumar (2007), we know that $Y \preceq_{\text{convex}} Z$ if, and only if, $\mathrm{E}[Y] = \mathrm{E}[Z]$ and $\mathrm{E}[Y|Y \geq F^{-1}_Y(p)] \leq \mathrm{E}[Z|Z \geq F^{-1}_Z(p)]$ for all $p \in [0, 1)$. Thus, (2.17) shows that $\mathrm{E}[X|S_n] \preceq_{\text{convex}} \mathrm{E}[X|T_n]$ holds true if, and only if, the inequality

$$\mathrm{E}[X|S_n \geq F^{-1}_{S_n}(p)] \leq \mathrm{E}[X|T_m \geq F^{-1}_{T_m}(p)] \text{ is valid for all } p \in (0, 1). \tag{2.18}$$

Inequality (2.18) corresponds to the announced uniform comparison of the respective concentration curves. This ends the proof. □

Notice that (2.18) can be equivalently rewritten as follows:

$$\begin{aligned}(2.18) \;\Leftrightarrow\;& \mathrm{E}\Big[X\mathrm{I}\big[S_n \geq F^{-1}_{S_n}(p)\big]\Big] \leq \mathrm{E}\Big[X\mathrm{I}\big[T_m \geq F^{-1}_{T_m}(p)\big]\Big] \text{ for all } p \in (0, 1) \\ \Leftrightarrow\;& \mathrm{Cov}\Big[X, \mathrm{I}\big[S_n \geq F^{-1}_{S_n}(p)\big]\Big] \leq \mathrm{Cov}\Big[X, \mathrm{I}\big[T_m \geq F^{-1}_{T_m}(p)\big]\Big] \text{ for all } p \in (0, 1).\end{aligned}$$

The latter inequality means that, to be preferred, pool 1 must be such that X is less correlated to the payoff $\mathrm{I}[S_n \geq F^{-1}_{S_n}(p)]$ of every digital option written on S_n (paying 1 if $S_n \geq F^{-1}_{S_n}(p)$ and nothing otherwise). In an insurance setting, we see that (2.18) ensures that the contribution of X to the total capital measured by the conditional tail expectation (CTE) is smaller for S_n compared to T_m.

3 Broker model

3.1 Business model

The approach described in the preceding section requires a high level of trust since the amounts $h^\star_{i,n}(S_n)$ to be paid at the end of the period remain unknown until then, and some participants may be unable, or unwilling to pay their contributions. Also, unless participants are ready to pay expensive contributions, the risk-bearing capacity of the community is limited. This is why an alternative approach may be interesting.

In the broker model, the organizer of the P2P insurance system acts as an intermediary and buys a protection on behalf of the participants whose liability is limited to the lower layer of the total loses. This has led Denuit (2020) to design a flexible system where the random, ex-post contribution is replaced with a fixed, ex-ante payment together with a cash-back mechanism and a reinsurance protection against adverse experience. Whereas pure self-governing model does not include any guarantee, the broker model combines risk sharing of the lower layer with risk transfer of the higher layer. By partnering with an insurance company, the community may also benefit from the claim settlement expertise developed by the insurer.

Henceforth, we assume that the conditional expectations $h^{\star}_{i,n}$ are continuously increasing for all i so that the functions $s \mapsto \mathrm{E}[X_i|S_n = s]$ are one-to-one. The one-to-one requirement is just made for mathematical convenience and we refer the interested reader to Denuit (2020) for more general formulas.

The ex-post contributions $h^{\star}_{i,n}(S_n)$ in the self-governing model are replaced in the broker model with a combination of

- an ex-ante provision $\pi_{i,n}$ depending on the distribution of both X_i and S_n

- and an ex-post cash-back $B_{i,n}$ in case of favorable experience.

In all cases, the maximal amount paid by participant i is $\pi_{i,n}$. To this end, the organizer acts as broker and buys a stop-loss protection transferring $(S_n - w_n)_+$ to a (re-)insurer. In other words, participants share total losses up to the retention w_n and the upper layer (w_n, ∞) is covered by the partnering insurer.

3.2 Retention level

The organizer of the P2P insurance scheme, acting as a broker, must first select the retention level w_n of the community. The value of w_n can be determined so that $F_{S_n}(w_n)$ is high enough. Since $F_{S_n}(w_n)$ is the probability that some surplus emerges at the end of the period, so that bonuses can be granted to participants, this ensures that a surplus can be distributed sufficiently often to make the system attractive.

Let β denote the target probability for distributing surpluses to participants. The retention level w_n corresponding to this probability β is then given by

$$\beta = F_{S_n}(w_n) \Leftrightarrow w_n = F^{-1}_{S_n}(\beta),$$

that is, w_n is the quantile of S_n corresponding to the probability level β.

3.3 Individual contributions

Once the organizer has determined the amount of retention w_n such that $0 < F_{S_n}(w_n) < 1$, define $w_{1,n}, \ldots, w_{n,n}$ as

$$w_{i,n} = F^{-1}_{h^{\star}_{i,n}(S_n)}\big(F_{S_n}(w_n)\big) = \mathrm{E}[X_i|S_n = w_n]. \tag{3.1}$$

It is easy to see that $\sum_{i=1}^{n} w_{i,n} = w_n$.

Because each $h^{\star}_{i,n}$ is continuously increasing, we have

$$S_n \geq w_n \Leftrightarrow h^{\star}_{i,n}(S_n) \geq w_{i,n} = F^{-1}_{h^{\star}_{i,n}(S_n)}(F_{S_n}(w_n)) \text{ for } i = 1, 2, \ldots, n,$$

so that

$$\begin{aligned}
(S_n - w_n)_+ &= \left(\sum_{i=1}^n \Big(h^\star_{i,n}(S_n) - F^{-1}_{h^\star_{i,n}(S_n)}(F_{S_n}(w_n))\Big)\right)_+ \\
&= \sum_{i=1}^n \Big(h^\star_{i,n}(S_n) - F^{-1}_{h^\star_{i,n}(S_n)}(F_{S_n}(w_n))\Big)_+ \\
&= \sum_{i=1}^n (h^\star_{i,n}(S_n) - w_{i,n})_+. \qquad (3.2)
\end{aligned}$$

The same reasoning applies to the decomposition of $(w_n - S)_+$ into the sum of the random variables $(w_{i,n} - h^\star_{i,n}(S_n))_+$.

This provides us with the right way to allocate the surplus $(w_n - S_n)_+$ and to compute the respective contributions for each participant to the retained loss $\min\{S_n, w_n\}$ since

$$\begin{aligned}
\min\{S_n, w_n\} &= w_n - (w_n - S_n)_+ \\
&= \sum_{i=1}^n \Big(w_{i,n} - (w_{i,n} - h^\star_{i,n}(S_n))_+\Big) \\
&= \sum_{i=1}^n \min\{h^\star_{i,n}(S_n), w_{i,n}\}. \qquad (3.3)
\end{aligned}$$

The loss $\min\{S_n, w_n\}$ retained by the pool is thus distributed among its members according to formula (3.3). Precisely, participant i contributes to the lower layer $(0, w_n)$ an amount $\min\{h^\star_{i,n}(S_n), w_{i,n}\}$ where the individual retention levels $w_{i,n}$ are given in (3.1).

3.4 Contribution to the upper layer

As explained previously, the P2P insurance community only retains the lower layer $(0, w_n)$ and buys a stop-loss protection from a (re-)insurer for the upper layer (w_n, ∞). The stop-loss premium for the upper layer must thus be allocated among participants. It could be shared equally among participants or according to their respective contribution to the upper layer of the losses. Here, we favor the second allocation to obtain a fair system.

The price of the stop-loss protection is supposed to be of the form

$$\pi_{\mathrm{SL}} = (1 + \theta_{\mathrm{SL}})\mathrm{E}[(S_n - w_n)_+]$$

where θ_{SL} is the corresponding loading to obtain the premium charged by the (re-)insurer covering the upper layer (w_n, ∞). The identity (3.2) provides us with the appropriate allocation rule. The pure premium of the stop-loss cover is distributed among participants according to

$$\mathrm{E}[(S_n - w_n)_+] = \sum_{i=1}^n \mathrm{E}[(h^\star_{i,n}(S_n) - w_{i,n})_+]$$

and participant i pays

$$\pi^{\mathrm{SL}}_{i,n} = (1 + \theta_{\mathrm{SL}})\mathrm{E}[(h^\star_{i,n}(S_n) - w_{i,n})_+]$$

for the transfer of the upper layer (w_n, ∞) to the (re-)insurer. These amounts sum to the premium π_{SL} charged by the (re-)insurer, that is,

$$\sum_{i=1}^{n} \pi_{i,n}^{\text{SL}} = \pi_{\text{SL}}.$$

3.5 Ex-ante provisions

The amount of provision $\pi_{i,n}$ paid ex-ante by participant i is decomposed into

$$\pi_{i,n} = \pi_{i,n}^{\text{P2P}} + \pi_{i,n}^{\text{SL}} \text{ where } \pi_{i,n}^{\text{P2P}} = w_{i,n} = h_{i,n}^{\star}(w_n) = \mathrm{E}[X_i|S_n = w_n], \tag{3.4}$$

supplemented with running expenses. In (3.4), $\pi_{i,n}^{\text{P2P}}$ is that part of the total contribution paid by participant i covering the first layer shared within the P2P community, while $\pi_{i,n}^{\text{SL}}$ is the contribution to the price of the stop-loss protection. This ensures that the community has enough financial resources to cover the lower layer since

$$w_n = \sum_{i=1}^{n} \pi_{i,n}^{\text{P2P}}.$$

If $\pi_{i,n}^{\text{P2P}}$ appears to be too expensive, individual deductibles can be imposed to reduce participation costs.

3.6 Surplus

If $S_n \leq w_n$ then surplus can be shared among participants or given to a charity. Let $B_{i,n}$ denote the surplus that can be attributed to participant i, given by

$$B_{i,n} = \Big(w_{i,n} - h_{i,n}^{\star}(S_n)\Big)_+.$$

Defining $\pi_{i,n}^{\text{P2P}}$ according to (3.4) ensures that for any $s \leq w_n$, the contribution to be paid ex-post by participant i is always less than, or equal to $\pi_{i,n}^{\text{P2P}}$, that is,

$$h_{i,n}^{\star}(s) \leq \pi_{i,n}^{\text{P2P}} = h_{i,n}^{\star}(w_n) \text{ for all } s \leq w_n \Rightarrow B_{i,n} \geq 0 \text{ for all } i$$

and no additional contribution is required from participant i at the end of the period. The random variables $B_{i,n}$ obey zero-augmented distributions with $\mathrm{P}[B_{i,n} = 0] = \mathrm{P}[S_n \geq w_n]$. Furthermore, we have $\sum_{i=1}^{n} B_{i,n} = (w_n - S_n)_+$.

If $S_n < w_n$ then the surplus $(w_n - S_n)_+$ can be shared among participants or given to a charity. If the community decides to distribute $(w_n - S_n)_+$ then this amount must be split among participants. If $S_n < w_n$ then participant i receives a bonus equal to $B_{i,n} = w_{i,n} - h_{i,n}^{\star}(S_n)$. Each participant must thus pay ex-ante the maximal amount of contribution $w_{i,n}$ he or she is exposed to and the cash-back mechanism operates ex-post to distribute the surplus among members of the community, allocating $B_{i,n}$ to each of them, individually.

3.7 Fairness

Since the (re-)insurer charges a loading in addition to the pure premium for the stop-loss protection, the system cannot be fair as a whole but we want it to be fair if we exclude this loading. By fairness, we mean that the expected contribution paid by participant i is equal to the expected loss μ_i brought to the pool. The P2P insurance operation under the broker model can be decomposed into

- a deterministic, ex-ante payment $\pi_{i,n}^{\text{P2P}} = w_{i,n}$,
- a random, ex-post cash-back

$$B_{i,n} = \big(\pi_{i,n}^{\text{P2P}} - h_{i,n}^{\star}(S_n)\big)\mathrm{I}[S_n \leq w_n] = (w_{i,n} - h_{i,n}^{\star}(S_n))_+,$$

- a deterministic contribution $\mathrm{E}[(h_{i,n}^{\star}(S_n) - w_{i,n})_+]$ to the upper layer, excluding the loading,
- in exchange of the reimbursement of the loss X_i.

On average, we thus get

$$\mathrm{E}\big[X_i - w_{i,n} + (w_{i,n} - h_{i,n}^{\star}(S_n))_+ - \mathrm{E}[(h_{i,n}^{\star}(S_n) - w_{i,n})_+]\big] = \mathrm{E}\big[X_i - h_{i,n}^{\star}(S_n)\big] = 0$$

for $i = 1, 2, \ldots, n$. Here, we have worked under zero interest rate (as it is commonly the case for nonlife, short-term insurance). Apart from the contribution to the loading of the stop-loss cover, the P2P insurance scheme is thus fair.

4 Carrier model

4.1 Business model

In collaborative insurance, participants agree to pool the risks they face. The self-governing model builds on the fundamental principle of mutuality at the origin of insurance without the equity buffer provided by insurer's capital. This approach is however limited: unless the cover is restricted to relatively cheap items, it cannot work for coverages where sums at risk are large and/or claim settlement process is complicated and may extend over several years (like in third-party liability products). This is solved in the broker model by partnering with a traditional insurance company so that participants can access higher amounts of retention compared to deductibles included in standard insurance covers, thanks to the risk-reducing effect of pooling. The broker model essentially reduces to a shared, or pooled deductible where the lower layer $(0, w_n)$ is covered by the community whereas the upper layer is transferred to an insurer. The carrier model embeds this approach into a regular insurance policy including a participation to the surplus.

Given the high level of regulation within the insurance industry and the long settlement procedures, traditional insurance companies still have a strong competitive advantage in organizing a P2P-like insurance offer within the sector. The participating policy proposed under the carrier model somewhat reconciles insurance with the mutual aid principle at its origin. Sharing the first risk layer among policyholders is an effective way to access the best of the two worlds: protection by equity capital for the higher layer with mutualization for the lower layer, benefiting from the insurer's expertise in claim settlement. It is also important to adopt an appropriate pricing mechanism by charging insurer's fees for the lower layer equal to a percentage of the corresponding benefits paid to policyholders. In this way, interests of both parties are aligned and agency problems at the heart of traditional insurance are solved.

4.2 Participating layer

In the broker model, individual contributions derive from the total amount of retention w_n selected by the community. Denuit and Robert (2021b) make the system more flexible by starting from an amount π_i computed beforehand, according to a premium calculation principle. This means that π_i does not depend on the composition of the pool but only on the individual loss X_i. The subscript "n" can thus be omitted in this section. This approach allows us to design a participating contract falling into the carrier model. This also means that the P2P system can be added to any existing insurance product, as an alternative pricing structure. Besides the menu of deductibles offered to policyholders, the latter could also receive the option to enter the participating scheme (corresponding to a pooled deductible).

Compared to the broker model, the starting point is the actual contribution π_i paid ex-ante by participants in the carrier model, which is computed akin a regular insurance premium. The retention w_n of the pool can be obtained as the root of equation

$$\sum_{i=1}^{n} \pi_i = w + \pi_{\mathrm{SL}}(w)$$

in w, where $\pi_{\mathrm{SL}} = \pi_{\mathrm{SL}}(w) = (1+\theta_{\mathrm{SL}})\mathrm{E}[(S_n - w)_+]$. If the inequality $\pi_i > (1+\theta_{\mathrm{SL}})\mathrm{E}[X_i]$ is valid for all i then the solution w_n exists and is unique. This condition also ensures that each participant contributes to the lower layer. Precisely, define this contribution $\pi_{i,n}^{\mathrm{P2P}}$ as

$$\pi_{i,n}^{\mathrm{P2P}} = \pi_i - \pi_{i,n}^{\mathrm{SL}}$$

where $\pi_{i,n}^{\mathrm{SL}}$ is the individual contribution to the stop-loss protection given by

$$\pi_{i,n}^{\mathrm{SL}} = (1+\theta_{\mathrm{SL}})\mathrm{E}[(h_{i,n}^{\star}(S_n) - w_{i,n})_+] \text{ with } w_{i,n} = h_{i,n}^{\star}(w_n)$$

so that $\sum_{i=1}^{n} \pi_{i,n}^{\mathrm{SL}} = \pi_{\mathrm{SL}}$. We then see that

$$\begin{array}{rcl} \pi_i & > & (1+\theta_{\mathrm{SL}})\mu_i \\ & = & (1+\theta_{\mathrm{SL}})\mathrm{E}[h_{i,n}^{\star}(S_n)] \\ & > & (1+\theta_{\mathrm{SL}})\mathrm{E}[(h_{i,n}^{\star}(S_n) - w_{i,n})_+] \\ & = & \pi_{i,n}^{\mathrm{SL}} \end{array}$$

so that we can conclude that $\pi_{i,n}^{\mathrm{P2P}} > 0$ for all i.

4.3 Fairness

The fairness of the system depends on the way premiums π_i are calculated. In that respect, the mean value principle is a natural choice to compute the premium of this participating contract. In this case

$$\pi_i = (1 + \theta_{\mathrm{P2P}})\mathrm{E}[X_i]. \tag{4.1}$$

With premiums of the form (4.1), the system is fair if, and only if, the losses X_i support a proportional allocation, that is, $h^{\star}_{i,n} = h^{\mathrm{prop}}_{i,n}$ where

$$h^{\mathrm{prop}}_{i,n}(S_n) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n.$$

Condition (2.12) identifies when $h^{\star}_{i,n} = h^{\mathrm{prop}}_{i,n}$ holds.

The individual retention levels $w_{i,n}$ can then be rewritten as

$$w_{i,n} = F^{-1}_{h^{\star}_{i,n}(S_n)}\big(F_{S_n}(w)\big) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} F^{-1}_{S_n}\big(F_{S_n}(w_n)\big) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} w_n \tag{4.2}$$

so that we obtain

$$\begin{aligned}
w_{i,n} &= \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]}\Big((1+\theta_{\mathrm{P2P}})\mathrm{E}[S_n] - (1+\theta_{\mathrm{SL}})\mathrm{E}[(S_n - w_n)_+]\Big) \\
&= \pi_i - (1+\theta_{\mathrm{SL}})\frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]}\mathrm{E}[(S_n - w_n)_+] \\
&= \pi_i - (1+\theta_{\mathrm{SL}})\mathrm{E}\Big[\big(h^{\star}_{i,n}(S_n) - w_{i,n}\big)_+\Big] \text{ by (4.2)} \\
&= \pi_i - \pi^{\mathrm{SL}}_{i,n} \\
&= \pi^{\mathrm{P2P}}_{i,n}
\end{aligned}$$

so that the identity $\pi^{\mathrm{P2P}}_{i,n} = w_{i,n}$ holds true for every participant. Hence, the provisions paid ex-ante by the participants match their individual retention level.

It is important to stress that, unless $h^{\star}_{i,n} = h^{\mathrm{prop}}_{i,n}$ the proposed system may induce some transfers between participants. The identity

$$\sum_{i=1}^{n} \pi^{\mathrm{P2P}}_{i,n} = \sum_{i=1}^{n} w_{i,n} = w_n$$

holds true, by construction. For risks supporting a proportional allocation and premiums (4.1), we have just seen that the identity $\pi^{\mathrm{P2P}}_{i,n} = w_{i,n}$ holds true for all i. However, this is generally not the case and $\pi^{\mathrm{P2P}}_{i,n}$ may depart from $w_{i,n}$ for some participants, inducing some transfers between participants.

In order to obtain equality, that is, to restore fairness, we can replace $h^{\star}_{i,n}$ with $h^{\mathrm{reg}}_{i,n}$ (which is perfectly reasonable in a sufficiently large pool) and compute π_i by the variance principle, that is,

$$\pi_i = \mathrm{E}[X_i] + \theta_{\mathrm{P2P}}\mathrm{Var}[X_i] = \mathrm{E}[X_i] + \theta_{\mathrm{P2P}}\mathrm{Cov}[X_i, S_n].$$

The retention of the pool is still obtained as the unique w_n satisfying $\sum_{i=1}^n \pi_i - \pi_{\mathrm{SL}}(w_n) = w_n$ with the same π_{SL} and $\pi_{i,n}^{\mathrm{SL}}$ as before. Let us now establish that the desired equality $\pi_{i,n}^{\mathrm{P2P}} = w_{i,n}$ holds for all i. Starting from

$$\begin{aligned}
w_{i,n} &= F^{-1}_{h^{\mathrm{reg}}_{i,n}(S_n)}\big(F_{S_n}(w_n)\big)\\
&= \mathrm{E}[X_i] + \frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}(w_n - \mathrm{E}[S_n])\\
&= \mathrm{E}[X_i] + \frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}\Big(\mathrm{E}[S_n] + \theta_{\mathrm{P2P}}\mathrm{Var}[S_n] - (1+\theta_{\mathrm{SL}})\mathrm{E}[(S_n - w_n)_+] - \mathrm{E}[S_n]\Big)\\
&= \mathrm{E}[X_i] + \theta_{\mathrm{P2P}}\mathrm{Var}[X_i] - (1+\theta_{\mathrm{SL}})\frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}\mathrm{E}[(S_n - w_n)_+]
\end{aligned}$$

and noting that

$$\frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}\mathrm{E}[(S_n - w)_+] = \mathrm{E}\left[\left(\frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}S_n - \frac{\mathrm{Var}[X_i]}{\mathrm{Var}[S_n]}w_n\right)_+\right] = \mathrm{E}\big[\big(h^{\mathrm{reg}}_{i,n}(S_n) - w_{i,n}\big)_+\big]$$

we get

$$\begin{aligned}
w_{i,n} &= \pi_i - (1+\theta_{\mathrm{SL}})\mathrm{E}\big[\big(h^{\mathrm{reg}}_{i,n}(S_n) - w_{i,n}\big)_+\big]\\
&= \pi_i - \pi_{i,n}^{\mathrm{SL}}\\
&= \pi_{i,n}^{\mathrm{P2P}}
\end{aligned}$$

so that we conclude that $\pi_{i,n}^{\mathrm{P2P}} = w_{i,n}$ holds true for every participant.

4.4 Transparency

Insurance is still very opaque whereas there is a global trend towards more transparency. In that respect, it appears to be of utmost importance to improve transparency and align interest of both parties. The new generation of participating contracts that could be launched should make explicit the split into taxes, intermediary's commission, claim costs (including settlement expenses, on an incurred basis), running expenses, and shareholders' profit better presented as cost of regulatory capital with calculations based on up-to-date actuarial modeling. Such a fully transparent account can then be communicated to policyholders, with a surplus distributed or allocated to a common project. This de-mystifies insurance (up to outputs of actuarial models calculating incurred losses or allocating capital) and revives the feeling to belong to a community, helping its unlucky members (like in old times).

The detailed account sent to participants could list the following elements:

- the number n of participants to the P2P scheme
- the retention w_n
- the contribution $\pi_{i,n}^{\mathrm{SL}}$ to the upper layer
- the individual retention level $w_{i,n} = \pi_{i,n}^{\mathrm{P2P}}$

- the actual claims X_i
- the individual fair contribution $\min\{h^\star_{i,n}(S_n), w_{i,n}\}$ to the lower layer
- the cash-back $B_{i,n}$
- the insurer's fees for the lower layer expressed as a percentage α of the total amount of benefits, that is
$$\text{insurer's fees} = \alpha \min\{S_n, w_n\}$$
so that
$$\text{individual contribution to insurer's fees} = \alpha \min\{h^\star_{i,n}(S_n), w_{i,n}\}.$$

Notice that the premium $\pi^{\text{SL}}_{i,n}$ is computed as a regular insurance premium, including cost of capital charges.

5 Discussion

Collaborative insurance schemes under the broker model and the carrier model involve a classical insurance company covering higher losses that cannot be retained within the community. Such schemes may well become an additional sales channel, complementing classical ones. The coverage can be sold as white-labelled products or the insurer's brand may be explicitly mentioned in the offer, depending on the targeted customers. This new channel thus addresses one of the key challenges faced by the insurance industry in the 21st century: offering an appropriate coverage to every customer, meeting his or her own personal aspirations.

The new participating contract is also expected to improve client experience. This is particularly important since customer relationship if often of poor quality in insurance, as reflected in the motto "Insurance is sold, not bought!". Many policyholders only have one contact a year with their insurance provider, when premium is invoiced (often with ever increasing amounts charged). There is thus an urgent need to re-invent insurance cover to make it appealing even to claim-free policyholders. Depending on the target, digitalization of the entire processes and product offering can be envisaged (for younger generations) but the approach equally applies to more conventional distribution channels. In both cases, the new participating contract is expected to maintain customer loyalty thanks to the feeling to belong to a community and to improve customer centricity.

This paper proposes an actuarial modeling to develop this new insurance offer within the sector, reviving the ancestral mutuality principle. The text only describes some examples of P2P insurance schemes, offering a starting point to design alternative P2P solutions, tailored to specific situations. Many variants are indeed possible and can be managed in the framework proposed in this paper. For instance, participants are often grouped in "teams" based on existing networks (family or relatives, for instance), common economic interests or geographic location. Denuit and Robert (2021b) show how to adapt the system to account for these two levels: the intermediate group level in addition to the individual one.

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