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Categorification of Verma modules

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Contents

Introduction

The birth of knot theory can be traced back up to the 18th century with the work of A. T. Vandermonde and C. F. Gauss (see [72] for a nice exposition of the history of knot theory). Knots attracted the interest of many physicists during the second half of the 19th century, thanks to W. Thomson Lord Kelvin's vortex theory of the atoms. His idea was that atoms would be knotted vortices floating in aether. In particular, he suggested that the topological nature of the knot would explain the physical and chemical properties of the atom. Inspired by this hypothesis, P. G. Tait started classifying knots in order to make a periodic table of the elements. He also made some important conjectures involving knots. However, when Kelvin's theory became obsolete, the general interest for knot theory dropped. The apparent lack of tools to address the problem of classifying knots at that time consequently impacted any further progresses in that direction until the 20th century.

Alongside the development of topology, J. W. Alexander made an interesting breakthrough during the 1920s with the discovery of the first polynomial knot invariant: the *Alexander polynomial* [2]. The construction goes through an intricate topological machinery applied to the knot complement. The topological nature of the construction, while being truly beautiful, makes it also hard to handle and to generalize. Still, as remarked by J. Conway [16], the Alexander polynomial can curiously be computed using a very efficient combinatorial rule called *skein relation*. Anyhow, the lack of connection to other fields seems to have put aside these discoveries for fifty years, until the 80s when a second polynomial invariant was revealed. This one was fortuitously discovered by V. Jones around 1984 [31] while he was investigating von Neumann algebras. The invariant, called the *Jones polynomial*, is constructed by taking a trace of a representation of the braid group on some finite-dimensional algebra related to statistical physics. The polynomial can also be computed very efficiently by a skein relation. Moreover, one of Tait's conjectures about alternating knots was surprisingly quickly proven using the Jones polynomial [34, 62, 87]. Soon after, E. Witten related the Jones polynomial to quantum field theory [96]. These connections between link invariants, representation theory and mathematical physics drastically changed the status of knot theory at the eyes of both the community of mathematicians and the one of physicists. It is worth mentioning that both Jones and Witten obtained the Fields Medal in 1990 for these results.

Quantum groups

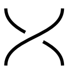
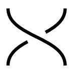
In parallel with the discovery of the Jones polynomial, V. Drinfeld [20] and independently M. Jimbo [30] introduced the notion of a *quantum group*. It is a 1-parameter deformation of a semi-simple complex Lie algebra, or more generally, of a Kac–Moody algebra. Concretely, each generator H_i of the Cartan part is replaced by a quantum variant $K_i = q^{H_i}$, where q is the deformation variable. The notation q^{H_i} can be made formal by interpreting q as a formal power series $q = \exp(\nu)$. Then, for example, the main $\mathfrak{sl}_2$ relation is replaced by its quantized version:

$$\big([E,F] =\big) EF - FE = H \qquad \rightsquigarrow \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1}$$

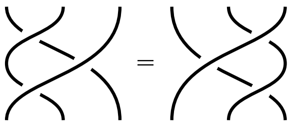
Quantum groups were first developed with the aim of constructing non-trivial solutions to the Yang–Baxter equation:

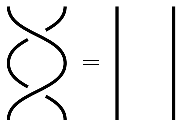
$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R), \tag{2}$$

where R is an invertible morphism $U \otimes U \to U \otimes U$ called the *R-matrix*, and U is the quantum group. By non-trivial we mean $R \neq R^{-1}$. Writing R and its inverse R^{-1} as braid-like diagrams,

$R \mapsto$  $\qquad R^{-1} \mapsto$ 

then the equations (2) and $R^{-1}R = 1$ take the following suggestive form:





Topologically, these correspond to the Reidemeister braid moves. Taking the non-quantized version of Lie algebras yields a trivial R-matrix $R = R^{-1}$, which loses all the information about the depth of the braiding. This justifies the usefulness of going to the quantum world for knot theory. Furthermore, quantum groups gained a broad recognition with applications in representation theory, non-commutative geometry, mathematical physics and low-dimensional topology. In particular, they yield a supply of non-commutative, non-cocommutative (quasitriangular) Hopf algebras [52].

Partially inspired by Witten's quantum field approach for the Jones polynomial [96], N. Reshetikhin and V. Turaev [80] built a machinery that takes representations of ribbon

Hopf algebras and link diagrams as input, and produces link invariants in the form of 1-variable Laurent polynomials as output. In particular, by feeding it with the fundamental representation of quantum $\mathfrak{sl}_2$, the machinery gives back the Jones polynomial. This resulted in the construction of a flood of infinitely many new polynomial link invariants called *quantum link invariants*. But more than that, WRT's machinery produces invariants of 3-manifolds when fed with modular categories (e.g. representations of a modular Hopf algebra, such as a quantum group with the deformation variable being a root of unity), taking the form a 3D topological quantum field theory (TQFT) [95].

Categorification

The next major advancement in the theory of quantum link invariants was achieved by the means of *categorification*. This notion was introduced by L. Crane and I. Frenkel [18] while being motivated by physics [17]. In short, the philosophy behind categorification is to lift classical objects to the level of categories, turning set elements into objects, functions into functors, equalities into isomorphisms, and revealing a whole new layer of structure achieving these isomorphisms.

In order to make this more concrete, let us explain some simple, but fundamental examples. We start with a categorification of the semiring of natural numbers $\mathbb{N}$. It is given by the monoidal category $\Bbbk$-vect of finite dimensional $\Bbbk$-vector spaces. Indeed, if we look at vector spaces up to isomorphism and forget the higher structure (i.e. the morphisms), then the only remaining information is the dimension of the vector spaces, which is a natural number. One calls this procedure *decategorification*, and one often formalizes it as taking the Grothendieck (semi-)group. Moreover, the addition and multiplication operations are categorified through direct sum and tensor product, since they respectively sum and multiply the dimensions of the spaces:

$$\dim(A \oplus B) = \dim(A) + \dim(B), \qquad \dim(A \otimes B) = \dim(A) \times \dim(B).$$

Note that categorification is not unique: for instance, we could also categorify $\mathbb{N}$ by the category of finite sets, and categorify addition and multiplication by coproduct and cartesian product respectively.

Adding a bit of homological algebra to the recipe, one obtains a categorification of the ring $\mathbb{Z}$. In particular, we consider the homotopy category $Kom(\Bbbk\text{-vect})$ of bounded complexes of finite dimensional vector spaces. In this context, decategorification ends up being equivalent to computing the graded dimension of the complex:

$$\operatorname{gdim} A_\bullet := \sum_i (-1)^i \dim A_i.$$

Therefore, the subtraction operation is categorified in this context by the procedure of taking a mapping cone:

$$\operatorname{gdim} \operatorname{Cone}(A_\bullet \xrightarrow{f} B_\bullet) = \operatorname{gdim}(B_\bullet) - \operatorname{gdim}(A_\bullet).$$

It is worth mentioning that while the addition and multiplication are canonical (taking a direct sum and tensor product are universal procedures), the subtraction requires to make a choice: the map f over which we take the mapping cone.

To illustrate how this construction is relevant to topology, it is common to consider the map χ : Top $\to \mathbb{Z}$ that takes a finite CW-complex $M \in$ Top, and assigns to it its (generalized) Euler characteristic $\chi(M)$. Then we consider the functor C_* : Top $\to$ $Kom(\Bbbk\text{-vect})$ that yields the cellular chain complexes of M. We can interpret it as a categorification of χ since there is a commutative diagram:

$$\begin{tikzcd} & Kom(\Bbbk\text{-vect}) \arrow[d, dashed, "\text{decategorification}"] \\ \text{Top} \arrow[ur, "C_*"] \arrow[r, "\chi"'] & \mathbb{Z} \end{tikzcd}$$

where the decategorification arrow computes the graded dimension of the cellular homology groups. While being a stronger invariant than Euler characteristic, taking cellular homology is more importantly a functorial operation. This is one example of what we mean by ‘categorification reveals a new layer of structure’. Thus, one of the upshots of categorification is that it might lead to richer theories, as in this example.

Keeping this idea in mind, it is time to focus again on low-dimensional topology. As explained in [39], alongside the combinatorially defined invariants of links and 3-manifolds given for example by the Alexander polynomial and WRT, there exists a handful of geometrically defined invariants of 3- and 4-manifolds. We can cite Floer homology of homology 3-spheres, Seiberg–Witten homology for 3-manifolds and Donaldson–Seiberg–Witten invariants of 4-manifolds [19]. While invariants of the first class can be algorithmically computed, the others can be incredibly hard to calculate, requiring to consider each case independently. Curiously, the Euler characteristic of Floer homology corresponds to the combinatorially defined Casson invariant [23], which is closely related to the Alexander polynomial and WRT. Similarly, the Milnor torsion, algorithmically computable, can be related to Seiberg–Witten through decategorification [59]. It appears that WRT quantum invariants also carry nice integral properties, suggesting they could exist as Euler characteristic of some homology theories. Inspired by these observations, Crane and Frenkel's original motivation was to look for a combinatorial categorification of WRT's 3D TQFT, with the extra layer of structure hypothetically turning the construction into a non-trivial

4D TQFT. Hopefully, it would correspond somehow to one of the geometric invariants already known, and help understand and compute it.

Since then, many progresses where made in several directions, with much too many results to be all cited here. Still, we note the groundbreaking work of M. Khovanov, who built a categorification of the Jones polynomial [39]. This means that the Jones polynomial can be interpreted as a graded Euler characteristic of a graded homology theory of links, where the deformation variable q is encoded by the grading. Khovanov homology is both a stronger invariant than Jones's one, and functorial [11]. Again, it shows the upshot of going through categorification. Furthermore, Khovanov homology detects 4D information [77], which is completely unseen at the polynomial level. However, Khovanov homology was originally built by categorifying a construction of the Jones polynomial different from WRT: Khovanov categorified the Kauffman bracket [34]. Thus it still remains in the story to find a representation theoretic explanation for the existence of Khovanov homology. For this, one would need a lifted version of the (representations of) quantum groups. Before going any further in this direction, let us mention that the Alexander polynomial has also been categorified. This was done by P. Osvath and Z. Szabo [70] and independently by J. Rasmussen [78], using Heegard–Floer homology of links.

Not long after the work of Khovanov, the first categorifications of representations of (quantum) $\mathfrak{sl}_2$ were built by J. Chuang and R. Rouquier [15] and independently by Frenkel, Khovanov and C. Stroppel [24]. These categorifications take the form of graded additive categories where the action of q is replaced by the grading shift functor, and the action of the quantum group is replaced by (additive) endofunctors, respecting the defining relations of the quantum group up to isomorphism. For example, the relation

$$(EF-FE)v_k = \frac{K-K^{-1}}{q-q^{-1}}v_k = \frac{q^k-q^{-k}}{q-q^{-1}}v_k = \begin{cases} \sum\limits_{\ell=0}^{k-1} q^{k-1-2\ell}v_k, & \text{if } k \geqslant 0, \\ \sum\limits_{\ell=0}^{-k-1} -q^{-k-1-2\ell}v_k, & \text{if } k \leqslant 0, \end{cases}$$

is replaced by natural isomorphisms

$$\begin{aligned} \mathcal{E}\mathcal{F}\,\mathrm{Id}_k &\cong \mathcal{F}\mathcal{E}\,\mathrm{Id}_k \oplus \bigoplus_{\ell=0}^{k-1} \mathrm{Id}_k\langle k-1-2\ell\rangle, && \text{if } k \geqslant 0, \\ \mathcal{F}\mathcal{E}\,\mathrm{Id}_k &\cong \mathcal{E}\mathcal{F}\,\mathrm{Id}_k \oplus \bigoplus_{\ell=0}^{-k-1} \mathrm{Id}_k\langle -k-1-2\ell\rangle, && \text{if } k \leqslant 0, \end{aligned} \tag{3}$$

where $\langle - \rangle$ is the grading shift functor, and Id_k is the identity functor on the category that contains the categorification of v_k.

The construction was extended to categorify all integrable representations of all quantum Kac–Moody algebras [33, 42, 94]. Some integral versions of the quantum groups themselves were categorified by A. Lauda [47] and Khovanov–Lauda [43], and independently by Rouquier [82]. An impressive application of this theory is given by the proof of Broué's abelian defect group conjecture for symmetric groups in the original paper of Chuang–Rouquier [15]. The connection between categorified quantum groups and Khovanov homology can be realized in several ways (see for examples [40] or [49]), with the work of B. Webster [94] giving a categorification of WRT's machinery for links. This produces homology theories for all WRT's quantum link invariants. For the case of 3-manifolds, WRT's construction requires to work at root of unity. To address this problem, Y. Qi and Khovanov developed a framework [41, 74], inspired by homological algebra and representation theory of Hopf algebras, that allows to categorify representations of quantum groups at root of unity [21, 38]. The construction of a 4D TQFT is still an open problem today.

Higher representation theory

Categorification of representations of quantum groups led to the birth of *higher representation theory*. This young field of mathematics sits at the intersection of representation theory and (higher) category theory, and has deep connections to geometric representation theory [93]. It has already found applications in low-dimensional topology [94], combinatorics [22] and algebraic K-theory [15]. It can be approached from different directions.

Classically, a representation of a group G is the datum of a $\Bbbk$-vector space V and a homomorphism $G \to \mathrm{GL}(V) \subset \mathrm{End}_{\Bbbk}(V)$. One can restate this using the language of categories. Let $\mathbf{B}G$ be the delooping of G, that is the category with a single abstract object $\star \in \mathbf{B}G$ and $\mathrm{Hom}_{\mathbf{B}G}(\star, \star) := G$. Then a representation of G is a functor

$$\mathbf{B}G \to \Bbbk\text{-vect}.$$

In order to categorify this picture, we first lift the category $\Bbbk$-vect. For this we consider the 2-category $\mathfrak{L}\mathrm{in}_{\Bbbk}$, where the objects are the $\Bbbk$-linear categories, 1-morphisms being (linear) functors, and 2-morphisms given by natural transformations. Then, a *weak categorical action* of G on $\mathscr{V} \in \mathfrak{L}\mathrm{in}_{\Bbbk}$ is the datum of a collection of endofunctors of $\mathscr{V}$, one for each generator of G, that respect the defining relations of G up to isomorphism. In particular, they descend to an action of G on the decategorification V of $\mathscr{V}$:

$$\begin{tikzcd}
 & \mathscr{V} \arrow[d, dashed, "\text{decategorification}"] \\
G \arrow[ur, "\text{cat. action}"] \arrow[r, "\text{action}"'] & V
\end{tikzcd}$$

When G has the structure of an R-algebra, with all its structure constants in R, we usually also require R to admit a categorification by a monoidal category $\mathscr{R}$ (like $\mathbb{Z}$ above) and $\mathscr{V}$ to be a module category over $\mathscr{R}$. This means that there is a bifunctor $\mathscr{R} \times \mathscr{V} \xrightarrow{\triangleright} \mathscr{V}$ compatible with the monoidal structure.

Yet, the definition of a weak categorical action is in general too loose, and we want to fix the natural transformations realizing the isomorphisms of the defining relations. This leads to the notion of a *categorical action*. A famous example of a categorical action is given by the braid group on n strands that acts on the derived category of modules over a polynomial ring in n variables [81]. The action is realized by tensoring with complexes of Soergel bimodules [85]. This construction actually categorifies an action of the braid group on the Hecke algebra.

As it turns out, fixing the higher structure allows sometimes to categorify $\mathbf{B}G$ itself, giving a 2-category $\mathfrak{B}G$. In that situation, a categorical action of G gives rise to a 2-functor

$$\mathfrak{B}G \to \mathfrak{Lin}_{\Bbbk}.$$

This leads to the idea of *2-representation theory*, which is about looking at precise 2-categories and their 2-functors into $\mathfrak{Lin}_{\Bbbk}$. Often, the objects of a 2-representation are given by categories of modules and 1-morphisms are functors of tensoring with bimodules (or derived equivalents). Then the 2-morphisms can simply be described by bimodule maps. A program to study 2-representations in a systematic approach was started by V. Mazorchuk and V. Miemietz [57, 58]. They construct notions of simple 2-representations and give a Jordan–Hölder-type theorem. They work especially on problems of classifying 2-representations (see [56] for a survey on the subject). However, it is worth mentioning that they only focus, until now, on the case of additive categories having quite strong finiteness conditions, called *finitary categories*. This is in some way the equivalent of studying representations of (finite dimensional) algebras where all the structure constants are natural numbers.

The categorifications of the integral versions of quantum groups from [43, 47, 82] yield examples of 2-categories $\mathscr{U}$, and their categorified representations from [33, 42, 94] can be lifted to 2-representations of $\mathscr{U}$. To sum up, we picture this as a diagram:

$$\begin{tikzcd}
\mathscr{U} \arrow[r, "\text{2-action}"] \arrow[d, dashed] & \mathscr{V} \arrow[d, dashed, "\text{decategorification}"] \\
U \arrow[r, "\text{action}"'] \arrow[ur, "\text{cat. action}"] & V
\end{tikzcd}$$

As already mentioned, geometry is often intimately related to higher representation theory. More precisely, a non negligible amount of representations can be realized via categories of sheaves on algebraic objects, which can then be lifted to the level of categorical actions [93]. In particular, a fundamental role is played by the geometry of Grassmannian varieties (and other flag-like objects) in the construction of 2-representations of the categorified quantum groups. Indeed, it appears that they control most of the higher structure of the construction. Concretely, from this geometry one can obtain an algebra called the *nilHecke algebra*, that always acts on (some of) the 2-hom spaces of any 2-representation of $\mathcal{U}$. Actually, this nilHecke algebra can be thought of as associated to $\mathfrak{sl}_2$, and it admits generalizations to all types. These generalizations are called the *KLR algebras* for Khovanov–Lauda–Rouquier [42, 44, 82], and are related to the geometry of quiver varieties [83]. The KLR algebras are the main ingredients of the categorifications of the quantum groups and their integrable representations. Moreover, Rouquier showed that given a categorical action of U on $\mathcal{V}$, with the Chevalley generators E_i, F_i of U acting by adjoint endofunctors of $\mathcal{V}$, and carrying an action of the KLR algebra on the space of natural transformations on the compositions of F_i's, then the categorification is essentially unique. In particular, he showed that under these conditions the construction always lifts to a 2-representation of $\mathcal{U}$ [82].

Summary of results

All the above-mentioned categorifications related to quantum groups share one common feature: they only categorify representations that are locally finite-dimensional when restricted to a simple root. This is forced by the additive nature of the categories used in these constructions. Indeed, based on the results in [82], it can be shown that a construction of a non-integrable weight representation is not possible within the usual additive framework. Because of these restrictions, it was not commonly believed that a categorification of such representations should exist.

In this thesis, we prove the opposite, and we make a step towards a categorification of infinite-dimensional and non-integrable representations of quantum Kac–Moody algebras. We start with the simplest case by proposing a categorification of the universal Verma modules for quantum $\mathfrak{sl}_2$, using homological algebra and infinite dimensional dg-algebras. In particular, the defining relation (1) of the quantum group is categorified through a quasi-isomorphism of mapping cones:

$$\mathrm{Cone}(\mathcal{F}\mathcal{E} \to \mathcal{E}\mathcal{F}) \xrightarrow{\simeq} \mathrm{Cone}(\mathbb{Q}\mathcal{K} \to \mathbb{Q}\mathcal{K}^{-1}), \tag{4}$$

where $\mathbb{Q}$ is a categorification of $\frac{1}{q^{-1}-q}$.

Then we extend our construction to all (parabolic) Verma modules of quantum Kac–Moody algebras. We would like to point out that categorified Verma modules are not 2-representations of the 2-categories defined in [43, 47, 82]. In particular, they do not fit into the finitary, additive framework of [57]. Still, they carry a similar looking higher structure, suggesting the existence of a more general categorification of the quantum groups, in a homological algebra setting. Constructing such a categorification is still an open problem, that might bring new perspectives for 2-representation theory.

Verma modules play a major role in the representation theory of quantum groups. Indeed, any (cyclic) highest weight representation can be recovered as a quotient of a Verma module. We recover this property at the categorical level by equipping our categorification of Verma modules with a family of differentials. These induce derived equivalences with all the categorified integrable modules of [33], giving a new insight into these constructions.

Our construction relies on a dg-enhancement of the categorification of integrable modules from [15, 24]. More precisely, they categorify the $(N-2k)$-weight space of the $(N+1)$-dimensional irreducible module of $\mathfrak{sl}_2$ by a category of modules over the cohomology ring $H(G_k^N)$ of the Grassmannian variety of k-planes in $\mathbb{C}^N$. The two-steps flag variety $G_{k,k+1}^N$ of k-spaces in $k+1$-spaces in $\mathbb{C}^N$ admits forgetful maps to G_k^N and G_{k+1}^N. This endows the cohomology $H(G_{k,k+1}^N)$ with an $H(G_k^N)$-$H(G_{k+1}^N)$-bimodule structure:

$$\begin{tikzcd} & G_{k,k+1}^N \arrow[dl] \arrow[dr] & \\ G_k^N & & G_{k+1}^N \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & H(G_{k,k+1}^N) & \\ H(G_k^N) \arrow[ur] & & H(G_{k+1}^N) \arrow[ul] \end{tikzcd}$$

In particular, it gives rise to two functors $\mathscr{E} : H(G_{k+1}^N)\text{-mod} \to H(G_k^N)\text{-mod}$ and $\mathscr{F} : H(G_k^N)\text{-mod} \to H(G_{k+1}^N)\text{-mod}$ by tensoring with these bimodules. These functors allow to jump from one categorified weight space to the other. It is shown in [15, 24] that the pair $\mathscr{E}, \mathscr{F}$ respects the defining relations of $\mathfrak{sl}_2$ up to isomorphism, with the action of the Cartan generator K categorified by a degree shift. However, this construction depends on the fact that all the weights that appear are integers. In that case, the main $\mathfrak{sl}_2$ relation (1) yields only rational fractions like $(q^{N-2k}-q^{2k-N})/(q-q^{-1})$, which can be simplified into finite polynomials $q^{N-2k-1}+q^{N-2k-3}+\cdots+q^{1-N+2k}$. In particular, they categorify the commutator relations as direct sum decompositions such as in (3).

We have observed that $H(G_k^N)$ admits a finite free resolution over the cohomology ring of Grassmannian varieties in the infinite space $\mathbb{C}^\infty$, giving rise to infinite dimensional (graded) dg-algebras (Ω_k, d_N). Similarly, the functors induced by the two-steps flag varieties lift to the dg-world. This means we can recover the categorification of integrable modules from [15, 24] using derived categories of dg-modules over (Ω_k, d_N). One of the upshots of going to the dg-world is that (the dg-version of) the functors $\mathscr{E}, \mathscr{F}$ are now related to each other through a quasi-isomorphism of cones as in (4). This gives a categorification of the relation (1) without needing the rational fraction to simplify. The simplification process can still be categorified as going through the quasi-isomorphism $(\Omega_k, d_N) \xrightarrow{\simeq} (H(G_k^N), 0)$. Finally, the categorification of the Verma module appears naturally from this construction by annihilating the differential, thus using (derived) categories of dg-modules over $(\Omega_k, 0)$. Similarly, the categorified parabolic Verma modules are obtained from a dg-enhancement of the KLR algebras and of the functors used for the categorification of integrable modules in [33, 42, 94].

One technical difficulty in the categorification of Verma modules is that some structure constants take the form of rational fractions like

$$\frac{\lambda - \lambda^{-1}}{q - q^{-1}},$$

where λ, q are formal invertible variables. Therefore, we needed to develop a framework for categorifying such rational fractions. The idea, which is not completely new (see for example [1] or [46]), is to reinterpret the fraction as a formal Laurent series:

$$\frac{\lambda - \lambda^{-1}}{q - q^{-1}} = q(\lambda^{-1} - \lambda)(1 + q^2 + q^4 + \cdots).$$

We can then categorify it by different means depending on the context: an infinite coproduct, an infinite composition series, or an infinite iterated extension, all these infinities being controlled in a certain way. Quotienting the usual Grothendieck group by the relations induced by such infinite constructions leads to the notion of *topological Grothendieck group* (with a similar flavor as in [1]). We explore these different possibilities and give examples where the topological Grothendieck group can be explicitly computed. In particular, we categorify the ring of formal Laurent series $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$ (as constructed by A. Aparicio-Monforte and M. Kauers [4]). It takes the form of a thick subcategory of the derived category of multigraded dg-modules over the dg-algebra $(\Bbbk, 0)$.

To return one more time to low-dimensional topology, we also showed in [64] how to use parabolic Verma modules of quantum $\mathfrak{sl}_n$ to construct the homfly-pt link polynomial invariant [25, 73], inspired by the construction of H. Queffelec and A. Sartori [75]. Moreover, we showed that the construction can be lifted to the categorical setting using our categorified Verma modules. It yields a triply-graded link homology equivalent to Khovanov–Rozansky link homology [45].

Plan of the thesis

- Chapter 1: we recall the basics about quantum groups and their representations, with a special focus on the case of $\mathfrak{sl}_2$.
- Chapter 2: we explain the classical notion of Grothendieck group which is essential to the understanding of categorification. Indeed, we use it as a definition of decategorification. In particular we quickly (re-)explore the case of Grothendieck groups appearing in additive, abelian and triangulated categories. We also recall the usual tools of homological algebra that we will need in the subsequent chapters.
- Chapter 3: our contributions start here. We develop different notions of topological Grothendieck groups and we categorify the ring of formal Laurent series.
- Chapter 4: it is the heart of the thesis. We construct a categorification of the universal Verma module of $\mathfrak{sl}_2$ using the geometry of Grassmannian varieties, revisiting our paper [65]. In particular, we show how the construction arises naturally from a dg-enhancement of the original construction of categorified integrable modules from [15, 24]. This also immediately gives the existence of the differentials that yield the derived equivalences with the categorifications of integrable modules. We would like to point out that this approach is slightly different than in the paper [65]. Indeed, in the reference, Ω_k is obtained by taking the Hochschild homology of the cohomology ring of Grassmannian varieties in $\mathbb{C}^\infty$.
- Chapter 5: we give another construction for the categorification of the $\mathfrak{sl}_2$-Verma module by using a dg-enhanced version of the (cyclotomic) nilHecke algebra. Moreover, we construct several bases of this dg-enhanced nilHecke algebra, and explain how to derive it from an action of the symmetric group on a supercommutative ring. We also explain how it is related to the geometry of the Grassmannian varieties, and to the construction in Chapter 4. In particular,we show that the two constructions of categorified Verma modules are equivalent, but naturally categorify different bases. This is based on our second paper [66].
- Chapter 6: we extend the construction of the dg-enhanced nilHecke algebra to all symmetrizable types, giving dg-enhanced KLR algebras. Then we use them to categorify parabolic Verma modules of all quantum Kac–Moody algebras, following our preprint [63].

Notations

We have gathered here some of the notations used in this document and give some references for definitions.

- $\mathrm{Cone}(f)$ or $\mathrm{Cone}(X \xrightarrow{f} Y)$ is the mapping cone of f (Eq. (2.1)) ;
- $\mathcal{D}(A, d)$ is the derived category of (A, d) (§2.2.2) ;
- $\mathcal{D}^c(A, d)$ is the compact derived category of (A, d) (§2.2.3) ;
- $\mathcal{D}^{lc}(A, d)$ is the subcategory of $\mathcal{D}(A, d)$ generated by objects quasi-isomorphic to c.b.l. compact modules (Definition 3.4.7) ;
- $e_i(z_1, \dots, z_n)$ is the i-th elementary symmetric polynomial (Definition 5.1.2) ;
- $G_0(\mathcal{C})$ is the abelian Grothendieck group (§2.1.2) ;
- $\boldsymbol{G}_0(\mathcal{C})$ is the topological abelian Grothendieck group (Definition 3.3.2) ;
- $h_i(z_1, \dots, z_n)$ is the i-th complete homogeneous symmetric polynomial (Definition 5.1.2) ;
- $\Bbbk[x_1, \dots, x_n]$ is the $\Bbbk$-algebra of polynomials ;
- $\Bbbk(x_1, \dots, x_n)$ is the $\Bbbk$-algebra of rational fractions ;
- $\Bbbk(\!(x_1, \dots, x_n)\!)$ is the $\Bbbk$-algebra of formal Laurent series (§3.1) ;
- $K_0^{\oplus}(\mathcal{C})$ is the split Grothendieck group of $\mathcal{C}$ (§2.1.1) ;
- $K_0^{\Delta}(\mathcal{C})$ is the triangulated Grothendieck group (§2.1.3) ;
- $\boldsymbol{K}_0^{\Delta}(\mathcal{C})$ is the topological triangulated Grothendieck group (Definition 3.4.4) ;
- MColim is the Milnor colimit (§3.4) ;
- $\mathbb{N} := \{0, 1, \dots\}$ is the semiring of non-negative integers ;
- $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$ is the quantum integer (§1.1.1) ;
- $\mathrm{RHOM}(-,-)$ is the derived hom functor (§2.2.2) ;
- S^m is the symmetric group on m elements ;
- $- \otimes^{\mathrm{L}} -$ is the derived tensor product functor (§2.2.2) ;

Chapter 1

Quantum groups and their representation theory

Quantum groups were introduced by Vladimir Drinfeld [20] and Michio Jimbo [30] as a 1-variable deformation of (enveloping algebras of) classical, finite dimensional Lie algebras, and, more generally, of Kac–Moody algebras. Our presentation is close to [29] and [51], where the proofs of all the claims we make can be found. References for classical results about (non-quantum, generalized) Verma modules are [55] and [28], see [3] for the quantum case.

1.1 Quantum $\mathfrak{sl}_2$

Classically, $\mathfrak{sl}_2$ is the Lie algebra given by two by two matrices with trace zero. It is generated by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Its enveloping algebra $U(\mathfrak{sl}_2)$ is the $\mathbb{Q}$-algebra generated by elements e, f, h with Lie bracket appearing as commutator. Thus, it admits a presentation by generators e, f, h and relations

$$ef - fe = h, \qquad he = e(h+2), \qquad hf = f(h-2).$$

1.1.1 Quantum deformation

Taking a q-deformation of a $\Bbbk$-algebra A is a procedure which consists in building a new $\Bbbk(q)$-algebra A_q on the ring of rational functions $\Bbbk(q)$, such that taking the limit of $q \to 1$ gives back the initial algebra A in some sense. One easy example is given by the quantum plane, which is a q-deformation of the polynomial ring $\Bbbk[x, y]$ in two variables. It is given by the $\Bbbk(q)$-algebra generated by the symbols x and y with relation $yx = qxy$. Simply putting $q = 1$ gives back $\Bbbk[x, y]$.

For $U(\mathfrak{sl}_2)$, one achieves it by informally replacing the commutator relation $ef - fe = h$ by its quantized variant $ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}$. Since q^h has no formal meaning, we replace it by a new invertible generator $K \sim q^h$.

Definition 1.1.1. *$U_q(\mathfrak{sl}_2)$ is the unital, associative $\mathbb{Q}(q)$-algebra generated by K, K^{-1}, E and F with relations*

$$KE = q^2 EK, \qquad KK^{-1} = 1 = K^{-1}K,$$
$$KF = q^{-2} FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Remark 1.1.2. *The procedure of replacing h by K can be made formal by working in a slightly different context where we interpret q as a formal power series $q = \exp(\nu)$, and thus $K = \exp(\nu h)$, as done originally by Drinfeld [20]. The operation of replacing q by 1 is then simply given by setting $\nu = 0$.*

The *quantum number* $[n]_q \in \mathbb{Z}(q)$ is the q-deformation of the integer $n \in \mathbb{Z}$, and is given by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} + q^{3-n} + \cdots + q^{n-3} + q^{n-1}.$$

Note that $[0]_q = 0$ and $[1]_q = 1$. Similarly, there are quantum versions of the factorial and the binomial

$$[a]_q! := [a]_q \left([a-1]_q!\right), \qquad \begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{[N]_q!}{[k]_q![N-k]_q!},$$

which can both be simplified into polynomials in $\mathbb{Z}[q, q^{-1}]$.

Symmetries

Quantum $\mathfrak{sl}_2$ has multiple important symmetries, coming in the form of algebra (anti)automorphisms. First, we consider the $\mathbb{Q}$-linear involution $\bar{\ }$ given by $\bar{q} = q^{-1}$. Then we define a $\mathbb{Q}(q)$-linear algebra anti-involution $\rho : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\mathrm{op}}$ by setting

$$\rho(E) := q^{-1}K^{-1}F, \qquad \rho(F) := q^{-1}KE, \qquad \rho(K) := K. \tag{1.1}$$

We also define a $\mathbb{Q}(q)$-antilinear anti-automorphism $\tau : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\mathrm{op}}$ by

$$\tau(E) := q^{-1}K^{-1}F, \qquad \tau(F) := q^{-1}KE, \qquad \tau(K) := K^{-1}, \tag{1.2}$$

and $\tau(fu) = \bar{f}\tau(u)$ for $f \in \mathbb{Q}(q)$ and $u \in U_q(\mathfrak{sl}_2)$.

1.1.2 Weight modules

Let M be an $U_q(\mathfrak{sl}_2)$-module and let λ be an invertible scalar in the ground ring $R \supset \mathbb{Q}(q)$ of M. Then we write

$$M_\lambda := \{v \in M | Kv = \lambda v\}.$$

We have $EM_\lambda \subset M_{q^2\lambda}$ and $FM_\lambda \subset M_{q^{-2}\lambda}$. We say that $v \in M_\lambda$ is a *weight vector* of weight λ. We say that M is a *weight module* if it decomposes as

$$M \cong \bigoplus_{\lambda \in R^\times} M_\lambda.$$

A *highest weight module* is a cyclic module over $U_q(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} R$, generated by $v_\lambda \in M_\lambda$ such that $Ev_\lambda = 0$. We call v_λ a *highest weight vector*. It is a well-known fact that any finite dimensional module is a weight module. If in addition it is irreducible, then it is a highest weight module with $\lambda = \pm q^N$ for some $N \in \mathbb{N}$. When K acts with positive eigenvalues on the module, then we say it is a *type 1 module*. We restrict to this case in this thesis.

Shapovalov form

Let M be a highest weight module with highest weight vector v_λ. One defines the *Shapovalov form*

$$(-,-) : M \times M \to R, \tag{1.3}$$

as the unique R-bilinear form respecting:

- $(v_\lambda, v_\lambda) = 1$;
- $(uv, v') = (v, \rho(u)v')$ where ρ is defined in (1.1) ;
- $f(v, v') = (fv, v') = (v, fv')$;

for all $v, v' \in M$, $u \in U_q(\mathfrak{sl}_2)$ and $f \in R$.

Remark 1.1.3. *Since M is a highest weight module, the second item in the definition of the Shapovalov form is equivalent to $(Fv, v') = (v, \rho(F)v')$ and $(Kv, v') = (v, Kv')$ for all $v, v' \in M$.*

Verma modules

The (standard) *Borel subalgebra* $U_q(\mathfrak{b})$ of $U_q(\mathfrak{sl}_2)$ is the algebra generated by E and K. It corresponds with the quantum deformation of the subalgebra of $\mathfrak{sl}_2$ given by upper triangular matrices.

Definition 1.1.4. *The* Verma module *of highest weight* $\lambda \in R^\times$ *over the ground ring* $R \supset \mathbb{Q}(q)$ *is the induced module*

$$M(\lambda) := U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} Rv_\lambda,$$

where $Ev_\lambda := 0$ *and* $Kv_\lambda := \lambda v_\lambda$.

The Verma module is an infinite dimensional highest weight module. It admits an induced basis $\{v_{q^{-2k}\lambda}\}_{k\in\mathbb{N}}$ where $v_{q^{-2k}\lambda} = F^k v_\lambda$. Then the action of $U_q(\mathfrak{sl}_2)$ is explicitly given by

$$\begin{aligned} Kv_{q^{-2k}\lambda} &= q^{-2k}\lambda v_{q^{-2k}\lambda}, \\ Fv_{q^{-2k}\lambda} &= v_{q^{-2(k+1)}\lambda}, \\ Ev_{q^{-2k}\lambda} &= [k]_q \frac{q^{1-k}\lambda - q^{k-1}\lambda^{-1}}{q-q^{-1}} v_{q^{-2(k-1)}\lambda}. \end{aligned}$$

We picture $M(\lambda)$ as

$$\begin{tikzcd} \cdots \arrow[r, bend left, "E"] & v_{q^{-6}\lambda} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow["K"', loop below] & v_{q^{-4}\lambda} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow["K"', loop below] & v_{q^{-2}\lambda} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow["K"', loop below] & v_\lambda \arrow[l, bend left, "F"] \arrow[r, bend left, dashed, "0"] \arrow["K"', loop below] & 0. \end{tikzcd} \tag{1.4}$$

From now on, we restrict to the case where λ is either a formal parameter that we identify with q^β for a formal β, or $\lambda = q^N$ for some $N \in \mathbb{Z}$. In order to encompass both cases, we write $\lambda = q^\mu$ for $\mu \in \mathbb{Z} \sqcup \{\beta\}$. Then we fix the ground ring $R = \mathbb{Q}(q,\lambda)$ if $\mu = \beta$, or $R = \mathbb{Q}(q)$ if $\mu = N$. We also introduce the notation

$$[\mu + k]_q := \frac{q^{\mu+k} - q^{-\mu-k}}{q - q^{-1}} = \frac{\lambda q^k - \lambda^{-1} q^{-k}}{q - q^{-1}}, \tag{1.5}$$

which agrees with the definition of quantum integer when $\mu \in \mathbb{Z}$.

Using these notations and plugging in coefficients in (1.4) we obtain

$$\begin{tikzcd} \cdots \arrow[r, bend left, "{[4]_q[\mu-3]_q}"] & v_{\mu-6} \arrow[l, bend left, "1"] \arrow[r, bend left, "{[3]_q[\mu-2]_q}"] \arrow["q^{\mu-6}"', loop below] & v_{\mu-4} \arrow[l, bend left, "1"] \arrow[r, bend left, "{[2]_q[\mu-1]_q}"] \arrow["q^{\mu-4}"', loop below] & v_{\mu-2} \arrow[l, bend left, "1"] \arrow[r, bend left, "{[1]_q[\mu]_q}"] \arrow["q^{\mu-2}"', loop below] & v_\mu \arrow[l, bend left, "1"] \arrow[r, bend left, dashed, "0"] \arrow["q^{\mu}"', loop below] & 0. \end{tikzcd} \tag{1.6}$$

Replacing F^k by the *divided powers*

$$F^{(k)} := F^k/[k]_q!,$$

we obtain another basis that we refer to as *canonical basis*. We write it $\{v'_{\mu-2k}\}_{k\in\mathbb{N}}$. It is explicitly given by taking $v'_{\mu-2k} = v_{\mu-2k}/[k]_q! \in M_{q^{\mu-2k}}$. With this basis, the Verma module looks like

$$\begin{tikzcd} \cdots \arrow[r, bend left, "{[\mu-3]_q}"] & v'_{\mu-6} \arrow[l, bend left, "{[4]_q}"] \arrow[r, bend left, "{[\mu-2]_q}"] & v'_{\mu-4} \arrow[l, bend left, "{[3]_q}"] \arrow[r, bend left, "{[\mu-1]_q}"] & v'_{\mu-2} \arrow[l, bend left, "{[2]_q}"] \arrow[r, bend left, "{[\mu]_q}"] & v'_{\mu} \arrow[l, bend left, "{[1]_q}"] \arrow[r, bend left, dashed, "0"] & 0, \end{tikzcd} \tag{1.7}$$

where we omit the action of K since it is clear from the weights.

Whenever $\mu = \beta$ or $\mu = -N$ for $N \in \mathbb{N}$, then $M(\lambda)$ is irreducible. In this case, the Shapovalov form (1.3) is non-degenerate. Then, the *dual canonical basis* $\{v^{\mu-2k}\}_{k\in\mathbb{N}}$ is defined as the unique one satisfying

$$(v'_{\mu-2j}, v^{\mu-2i}) = \delta_{ij}, \tag{1.8}$$

for all $i, j \in \mathbb{N}$. The change of basis is explicitly given by

$$v^{\mu-2k} = q^{k(k-\mu)} \frac{[k]_q!}{[\mu-k+1]_q[\mu-k+2]_q\cdots[\mu]_q} v'_{\mu-2k}. \tag{1.9}$$

The action of quantum $\mathfrak{sl}_2$ on this basis looks like

$$\begin{tikzcd} \cdots \arrow[r, bend left, "{q^{7-\mu}[4]_q}"] & v^{\mu-6} \arrow[l, bend left, "{q^{\mu-7}[\mu-3]_q}"] \arrow[r, bend left, "{q^{5-\mu}[3]_q}"] & v^{\mu-4} \arrow[l, bend left, "{q^{\mu-5}[\mu-2]_q}"] \arrow[r, bend left, "{q^{3-\mu}[2]_q}"] & v^{\mu-2} \arrow[l, bend left, "{q^{\mu-3}[\mu-1]_q}"] \arrow[r, bend left, "{q^{1-\mu}[1]_q}"] & v^{\mu} \arrow[l, bend left, "{q^{\mu-1}[\mu]_q}"] \arrow[r, bend left, dashed, "0"] & 0. \end{tikzcd} \tag{1.10}$$

<u>Finite dimensional modules</u>

Whenever $\mu = N \in \mathbb{N}$, then (1.7) takes the form

$$\begin{tikzcd} \cdots \arrow[r, bend left, "{[-1]_q}"] & v_{-N-2} \arrow[l, bend left, "{[N+2]_q}"] \arrow[r, bend left, dashed, "0"] & v_{-N} \arrow[l, bend left, "{[N+1]_q}"] \arrow[r, bend left, "{[1]_q}"] & \cdots \arrow[l, bend left, "{[N]_q}"] \arrow[r, bend left, "{[N-1]_q}"] & v_{N-2} \arrow[l, bend left, "{[2]_q}"] \arrow[r, bend left, "{[N]_q}"] & v_N \arrow[l, bend left, "{[1]_q}"] \arrow[r, bend left, dashed, "0"] & 0. \end{tikzcd} \tag{1.11}$$

Since $E(v_{-N-2}) = 0$, $M(q^N)$ admits a submodule. This submodule is isomorphic to $M(q^{-N-2})$, and the quotient is a finite dimensional module

$$V(N) := M(q^N)/M(q^{-N-2}), \tag{1.12}$$

which is irreducible, and has dimension $N + 1$. The short exact sequence

$$0 \to M(q^{-N-2}) \to M(q^N) \to V(N) \to 0,$$

is known in the literature as the BGG resolution, for Bernstein–Gelfand–Gelfand. The Shapovalov form restricts on $V(N)$ and this restriction is non-degenerate. Thus we have a dual canonical basis for $V(N)$, given by taking $\mu = N$ in (1.9) for $k \leqslant N$.

Moreover, there is a surjective map

$$ev_N : M(q^\beta) \twoheadrightarrow M(q^N),$$

given by evaluating $\beta = N$ (or $\lambda = q^N$). Since any (type 1) irreducible, finite dimensional $U_q(\mathfrak{sl}_2)$-module is isomorphic to $V(N)$ for some N, we can recover them from the Verma module $M(q^\beta)$. Thus, we call it the *universal Verma module*.

1.2 Quantum groups

Kac–Moody algebras were independently introduced by Victor Kac [32] and Robert Moody [60]. They take the form of combinatorially constructed Lie algebras that are piecewise-defined as copies of finite dimensional, semisimple Lie algebras.

1.2.1 Quantum Kac–Moody algebras

A *generalized Cartan matrix* is a finite dimensional square matrix $A = \{a_{ij}\}_{i,j\in I} \in \mathbb{Z}^{|I|\times|I|}$ such that

- $a_{ii} = 2$ and $a_{ij} \leqslant 0$ for all $i \neq j \in I$;
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

We say that A is symmetrizable if there exists a diagonal matrix D with positive entries $d_i \in \mathbb{Z}_{>0}$ for all $i \in I$, such that DA is symmetric. A *Cartan datum* consists of

- a symmetrizable generalized Cartan matrix A ;
- a free abelian group Y called the *weight lattice* ;
- a set of linearly independent elements $\Pi = \{\alpha_i\}_{i\in I} \subset Y$ called *simple roots* ;
- a *dual weight lattice* $Y^\vee := \mathrm{Hom}(Y, \mathbb{Z})$;

- a set of *simple coroots* $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I} \subset Y^\vee$;

such that

- $\alpha_i^\vee(\alpha_j) = a_{ij}$;
- for each $i \in I$ there is a *fundamental weight* $\Lambda_i \in Y$ such that $\alpha_j^\vee(\Lambda_i) = \delta_{ij}$ for all $j \in I$.

The abelian subgroup $X = \bigoplus_i \mathbb{Z}\alpha_i \subset Y$ is called the *root lattice*. We also write $X^+ := \bigoplus_i \mathbb{N}\alpha_i \subset X$ for the *positive roots*. Given a Cartan datum, since A is symmetrizable with $d_i a_{ij} = d_j a_{ji}$, one can construct a symmetric bilinear form

$$(-|-) : Y \times Y \to \mathbb{Z},$$

respecting

- $(\alpha_i|\alpha_i) = 2d_i \in \{2, 4, \dots\}$;
- $(\alpha_i|\alpha_j) = d_i a_{ij} \in \{0, -1, -2, \dots\}$ for all $i \neq j$;
- $\alpha_i^\vee(y) = 2\frac{(\alpha_i|y)}{(\alpha_i|\alpha_i)}$ for all $y \in Y$.

In the end, a Cartan datum is completely determined by $(I, X, Y, (-|-))$.

Definition 1.2.1. *The* quantum Kac–Moody algebra $U_q(\mathfrak{g})$ *associated to a Cartan datum* $(I, X, Y, (-|-))$ *is the associative, unital* $\mathbb{Q}(q)$*-algebra generated by the set of elements* E_i, F_i *and* K_γ *for all* $i \in I$ *and* $\gamma \in Y^\vee$*, with relations for all* $i \in I$ *and* $\gamma, \gamma' \in Y^\vee$*:*

$$K_0 = 1, \qquad K_\gamma K_{\gamma'} = K_{\gamma+\gamma'},$$
$$K_\gamma E_i = q^{\gamma(\alpha_i)} E_i K_\gamma, \qquad K_\gamma F_i = q^{-\gamma(\alpha_i)} F_i K_\gamma,$$

One also imposes the $\mathfrak{sl}_2$-commutator relation *for all* $i, j \in I$*:*

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{1.13}$$

where $q_i = q^{d_i}$ *and* $K_i = K_{\alpha_i^\vee}$.
Finally, there are the Serre relations *for* $i \neq j \in I$*:*

$$\sum_{r+s=1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^r E_j E_i^s = 0, \tag{1.14}$$

$$\sum_{r+s=1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^r F_j F_i^s = 0. \tag{1.15}$$

Given a sequence $\boldsymbol{i} = i_1 \cdots i_m$ of elements in I, we write $F_{\boldsymbol{i}} := F_{i_1} \cdots F_{i_m}$ and $E_{\boldsymbol{i}} := E_{i_1} \cdots E_{i_m}$. We write $\mathrm{Seq}(I)$ for the set of such sequences. Any element of $U_q(\mathfrak{g})$ decomposes as a sum of elements $F_{\boldsymbol{i}} K_\gamma E_{\boldsymbol{j}}$ with $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(I)$.

The *half quantum group* $U_q^-(\boldsymbol{g})$ of $U_q(\boldsymbol{g})$ is the subalgebra generated by the elements $\{F_i\}_{i \in I}$. As a $\mathbb{Q}(q)$-vector space, it admits a basis given by a subset of $\{F_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathrm{Seq}(I)}$ (we take only a subset because of the Serre relation (1.15)).

Example 1.2.2. *Consider the Cartan datum given by the* $(n-1) \times (n-1)$ *matrix*

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

with the set of roots $\{\alpha_1, \dots, \alpha_{n-1}\}$*, the set of coroots* $\{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\}$*, weight lattice* $Y = \bigoplus_{i=1}^{n-1} \mathbb{Z}\Lambda_i$ *with*

$$\alpha_i = \begin{cases} 2\Lambda_1 - \Lambda_2, & \text{if } i = 1, \\ 2\Lambda_i - \Lambda_{i+1} - \Lambda_{i-1}, & \text{if } 1 < i < n-1, \\ 2\Lambda_{n-1} - \Lambda_{n-2}, & \text{if } i = n-1, \end{cases}$$

and

$$(\alpha_i | \alpha_j) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The quantum group associated to this Cartan datum is called $U_q(\mathfrak{sl}_n)$*. Taking* $n = 2$ *gives back Definition 1.1.1. In general,* $U_q(\mathfrak{sl}_n)$ *is a quantum deformation of the enveloping algebra of* n *by* n *matrices with trace zero.*

1.2.2 Weight modules

Again, there is a notion of weight spaces and weight modules. Let M be an $U_q(\mathfrak{g})$-module with ground ring $R \supset \mathbb{Q}(q)$. Consider a $\mathbb{Z}$-linear functional

$$\lambda : Y^\vee \to R^\times,$$

where the group structure on $R^\times$ is the product. For each such λ and $y \in Y$, we call (λ, y)-*weight space* the set

$$M_{\lambda,y} := \{v \in M | K_\gamma v = \lambda(\gamma) q^{\gamma(y)} v \text{ for all } \gamma \in Y^\vee\}.$$

We have $E_i M_{\lambda,y} \subset M_{\lambda,y+\alpha_i}$ and $F_i M_{\lambda,y} \subset M_{\lambda,y-\alpha_i}$. Like before, a weight module is a module that decomposes as a direct sum of its weight spaces, and it is a highest weight module if it is generated by an element $v_\lambda \in M_{\lambda,0}$ such that $E_i v_\lambda = 0$ for all $i \in I$. In that case, we call λ the highest weight and we have

$$M \cong \bigoplus_{y \in X^+} M_{\lambda,-y}.$$

as R-module.

We say that a $U_q(\mathfrak{g})$-module M is *integrable* if for each $v \in M$ there exists $k \gg 0$ such that $E_i^k v = 0$ and $F_i^k v = 0$ for all $i \in I$. Any finite dimensional module is integrable, and any integrable module is a weight module with $\lambda(\Pi^\vee) \subset \mathbb{Z}[q]$. In this thesis, as for $\mathfrak{sl}_2$, we consider only type 1 modules , that is $\lambda(\Pi^\vee) \subset \mathbb{N}[q]$.

Let M be a highest weight module with highest weight vector $v_\lambda \in M_{\lambda,0}$. Then we set $\lambda_i = \lambda(\alpha_i^\vee)$ for each $i \in I$. We are interested in λ such that each λ_i is either $\lambda_i = q^{n_i}$ for some $n_i \in \mathbb{Z}$ or λ_i is formal. In that case, we write it $\lambda_i = q^{\beta_i}$.

Parabolic Verma modules

The (standard) *Borel subalgebra* $U_q(\mathfrak{b})$ of $U_q(\mathfrak{g})$ is generated by K_γ and E_i for all $\gamma \in Y^\vee$ and $i \in I$. A (standard) *parabolic subalgebra* of $U_q(\mathfrak{g})$ is a subalgebra that contains $U_q(\mathfrak{b})$. It is generated by K_γ, E_i and F_j for all $\gamma \in Y^\vee, i \in I$ and $j \in I_f$ for some fixed subset $I_f \subset I$. The part given by K_γ, E_j and F_j for $j \in I_f$ is called the *Levi factor* and written $U_q(\mathfrak{i})$. The *nilpotent radical* $U_q(\mathfrak{n})$ is generated by E_i for all $i \in I_r := I \backslash I_f$. There is a decomposition $U_q(\mathfrak{p}) = U_q(\mathfrak{i}) \oplus U_q(\mathfrak{n})$. Note that parabolic subalgebras are in bijection with partitions $I = I_f \sqcup I_r$.

Let $U_q(\mathfrak{p})$ be a parabolic subalgebra determined by $I = I_f \sqcup I_r$. For each $i \in I_f$, we choose a weight $n_i \in \mathbb{N}$. For each $j \in I_r$ we choose a weight $\lambda_j \in \{q^{\beta_j}, q^{n_j}\}$. We write $N = \{n_i\}_{i \in I_f}$ and $\Lambda = \{\lambda_j\}_{j \in I_r}$. Let $V(\Lambda, N)$ be the unique (type 1) integrable, irreducible representation of $U_q(\mathfrak{i})$ on the ground ring $R = \mathbb{Q}(q, \Lambda)$, and with highest weight λ determined by

$$\lambda(\alpha_k^\vee) = \begin{cases} q^{n_i}, & \text{if } k = i \in I_f, \\ \lambda_j, & \text{if } k = j \in I_r. \end{cases}$$

We extend it to a representation of $U_q(\mathfrak{p})$ by setting $U_q(\mathfrak{n}) V(\Lambda, N) = 0$.

Definition 1.2.3. *The* parabolic Verma module *of highest weight* (Λ, N) *associated to* $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ *is the induced module*

$$M^{\mathfrak{p}}(\Lambda, N) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V(\Lambda, N).$$

Whenever $U_q(\mathfrak{p}) \subsetneq U_q(\mathfrak{g})$, then $M^{\mathfrak{p}}(\Lambda, N)$ is an infinite dimensional module. Moreover, for all parabolic Verma modules, there is a $\mathbb{Q}(q)$-linear surjection

$$U_q^-(\boldsymbol{g}) \otimes_{\mathbb{Q}(q)} R \twoheadrightarrow M^{\mathfrak{p}}(\Lambda, N).$$

Example 1.2.4. *If* $U_q(\mathfrak{p}) = U_q(\mathfrak{b})$*, then* $N = \varnothing$*, and* $V(\Lambda, N) \cong \mathbb{Q}((q, \Lambda))v_\Lambda$ *is 1-dimensional, and such that*

$$E_i v_\Lambda = 0, \qquad\qquad K_\gamma v_\Lambda = \prod_{j \in I} \lambda_j^{\gamma(\Lambda_j)} v_\Lambda.$$

In that case, we simply call it Verma module*, and we write it* $M^{\mathfrak{b}}(\Lambda)$*. If* $\lambda_j = q^\beta$ *is formal for all* $j \in I_r$*, then we call it the* universal Verma module.

Example 1.2.5. *If* $U_q(\mathfrak{p}) = U_q(\mathfrak{g})$*, then* $\Lambda = \varnothing$ *and* $M^{\mathfrak{p}}(\Lambda, N) \cong V(N)$ *is an integrable, irreducible* $U_q(\mathfrak{g})$ *representation.*

Since q is a generic parameter (i.e. not a root of unity), we can apply Jantzen's criterion [28, Theorem 9.12], thanks to the results in [3]. From it, we obtain that $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible whenever $\lambda_j \notin \{q^n | n \in \mathbb{N}\}$ for all $j \in I_r$. If $\lambda_j = q^{n_j}$ for $n_j \in \mathbb{N}$, then $M^{\mathfrak{p}}(\Lambda, N)$ contains a non-trivial, proper submodule, which is isomorphic to $M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N)$ for $\Lambda^{n_j}_{-n_j-2}$ given by exchanging q^{n_j} with q^{-n_j-2} in Λ. Moreover, the quotient

$$\frac{M^{\mathfrak{p}}(\Lambda, N)}{M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N)} \cong M^{\mathfrak{p}+j}(\Lambda \backslash \{q^{n_j}\}, N \sqcup \{n_j\})$$

is isomorphic to the parabolic Verma module associated to the parabolic subalgebra $\mathfrak{p} + j$ given by adding j to I_f, that is generated by $\mathfrak{p}$ and F_j.

Furthermore, whenever $\lambda_j = q^{\beta_j}$ is formal, then there is a surjective map

$$ev_{n_j} : M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow M^{\mathfrak{p}}(\Lambda^{n_j}_{\beta_j}, N),$$

for all $n_j \in \mathbb{Z}$, given by evaluating $\beta_j = n_j$.

These two facts together allow us to define a partial order on the parabolic Verma modules. For this, we say that there is an arrow from $M^{\mathfrak{p}}(\Lambda, N)$ to $M^{\mathfrak{p}'}(\Lambda', N')$ if we have an evaluation map ev_{n_j} such that

$$ev_{n_j}(M^{\mathfrak{p}}(\Lambda, N)) \cong M^{\mathfrak{p}'}(\Lambda', N'),$$

or if there is a short exact sequence

$$0 \to M^{\mathfrak{p}}(\Lambda^{n_j}_{-n_j-2}, N) \to M^{\mathfrak{p}}(\Lambda, N) \to M^{\mathfrak{p}'}(\Lambda', N') \to 0.$$

Then we say that a parabolic Verma module M is bigger than another one M' if there is a chain of arrows from M to M'. In that case, there is an M'', which is either trivial or a parabolic Verma module, and a short exact sequence

$$0 \to M'' \to ev(M) \to M' \to 0,$$

where ev is a composition of evaluation maps ev_{n_j}. With this partial order, the universal Verma module is a maximal element and each integrable, irreducible module is a minimum. This also means that we can recover any parabolic Verma module from the universal one.

Shapovalov form

Let $\rho : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\mathrm{op}}$ be the $\mathbb{Q}(q)$-linear algebra anti-involution given by

$$\rho(E_i) := q_i^{-1} K_i^{-1} F_i, \qquad \rho(F_i) := q_i^{-1} K_i E_i, \qquad \rho(K_\gamma) := K_\gamma, \tag{1.16}$$

for all $i \in I$ and $\gamma \in Y^\vee$. Let also $\tau : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\mathrm{op}}$ be the $\mathbb{Q}(q)$-antilinear anti-automorphism given by

$$\tau(E_i) := q_i^{-1} K_i^{-1} F_i, \qquad \tau(F_i) := q_i^{-1} K_i E_i, \qquad \tau(K_\gamma) := K_\gamma^{-1}. \tag{1.17}$$

Definition 1.2.6. *The Shapovalov form*

$$(-,-) : M^{\mathfrak{p}}(\Lambda, N) \times M^{\mathfrak{p}}(\Lambda, N) \to \mathbb{Q}(q, \Lambda),$$

is the unique bilinear form respecting

- $(v_{\Lambda,N}, v_{\Lambda,N}) = 1$, *for* $v_{\Lambda,N}$ *the highest weight vector,*
- $(uv, v') = (v, \rho(u)v')$ *where* ρ *is defined in* (1.16)*,*
- $f(v, v') = (fv, v') = (v, fv')$*,*

for all $v, v' \in M^{\mathfrak{p}}(\Lambda, N), u \in U_q(\mathfrak{g})$ *and* $f \in \mathbb{Q}(q, \Lambda)$.

Basis

Since parabolic Verma modules are highest weight modules, they admit at least one basis given by elements of the form $F_{\boldsymbol{i}} v_{\Lambda,N}$ for $\boldsymbol{i} \in \mathrm{Seq}(I)$, and where $v_{\Lambda,N}$ is a highest weight vector. In particular, as R-modules they are all subquotients of

$U_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} R$, meaning that these basis lives in a subset of $\{F_{\boldsymbol{i}} v_{\Lambda,N} | \boldsymbol{i} \in \mathrm{Seq}(I)\}$ modded out by the Serre relations. We call such a basis an *induced basis* and write it $\{v_{\Lambda,N} = m_0, m_1, \dots\}$. Any element in such basis takes the form $F_{\boldsymbol{i}} = F_{i_r}^{b_r} \cdots F_{i_1}^{b_1}$ for some $i_1, \dots, i_r \in I$ and $b_1, \dots, b_r \in \mathbb{N}$, with $i_\ell \neq i_{\ell+1}$. Replacing each F_i^b by the *divided power* $F_i^{(b)} = F_i^b/([b]_{q_i}!)$ yields another basis $\{v_{\Lambda,N} = m'_0, m'_1, \dots\}$ that we refer as a *canonical basis*. Whenever $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible, the Shapovalov form (see Definition 1.2.6) is non-degenerate. Thus in this case, for each canonical basis there is a *dual canonical basis* uniquely determined by

$$(m'_i, m^j) = \delta_{ij}.$$

Example in $U_q(\mathfrak{sl}_3)$

We now consider $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_3)$, which is generated by E_1, E_2, F_1, F_2 and K_γ. Its universal Verma module $M^{\mathfrak{b}}(\{q^{\beta_1}, q^{\beta_2}\})$ looks like

$$\begin{tikzcd}[column sep=tiny]
 & & & & M_0 \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & & & \\
 & & & M_{-\alpha_1} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & M_{-\alpha_2} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & & \\
 & & M_{-2\alpha_1} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & M_{-\alpha_1-\alpha_2} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & M_{-2\alpha_2} \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & & \\
 & M_{-3\alpha_1} \arrow[dl] \arrow[dr] & & M_{-2\alpha_1-\alpha_2} \arrow[dl] \arrow[dr] & & M_{-\alpha_1-2\alpha_2} \arrow[dl] \arrow[dr] & & M_{-3\alpha_2} \arrow[dl] \arrow[dr] & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{tikzcd}$$

where the action of each F_i is non-zero. For example $M_{-\alpha_1-2\alpha_2}$ is 2-dimensional generated by $F_1F_2^2 v_{\Lambda,N}$ and $F_2^2 F_1 v_{\Lambda,N}$. Taking the highest weight $(\beta_1, 1)$ instead of (β_1, β_2) gives

$$\begin{tikzcd}[column sep=tiny]
 & & & & M_0 \arrow[dl] \arrow[dr] & & & & \\
 & & & M_{-\alpha_1} \arrow[dl] \arrow[dr] & & M_{-\alpha_2} \arrow[dl] \arrow[dr] & & & \\
 & & M_{-2\alpha_1} \arrow[dl] \arrow[dr] & & M_{-\alpha_1-\alpha_2} \arrow[dl] \arrow[dr] & & M_{-2\alpha_2} \arrow[dl] \arrow[dr] \arrow[ul, dashed, bend right, "0"'] & & \\
 & M_{-3\alpha_1} \arrow[dl] \arrow[dr] & & M_{-2\alpha_1-\alpha_2} \arrow[dl] \arrow[dr] & & M_{-\alpha_1-2\alpha_2} \arrow[dl] \arrow[dr] & & M_{-3\alpha_2} \arrow[dl] \arrow[dr] & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{tikzcd}$$

Note that $E_2 M_{-2\alpha_2} = 0$. Thus $M^{\mathfrak{b}}(\{q^{\beta_1}, q^1\})$ admits as submodule the Verma module of highest weight $(\beta_1 + 2, -3)$. The quotient is isomorphic to the parabolic Verma module $M^{\mathfrak{p}}(\{q^{\beta_1}\}, \{1\})$, for $\mathfrak{p}$ generated by F_2, E_1, E_2, K_γ. It looks like

$$
\begin{tikzcd}
 & & & M_0 \arrow[dl] \arrow[dr] & & & \\
 & & M_{-\alpha_1} \arrow[dl] \arrow[dr] & & M_{-\alpha_2} \arrow[dl] \arrow[dr, dashed] & & \\
 & M_{-2\alpha_1} \arrow[dl] \arrow[dr] & & M_{-\alpha_1-\alpha_2} \arrow[dl] \arrow[dr] & & 0 & \\
M_{-3\alpha_1} \arrow[d] \arrow[dr] & & M_{-2\alpha_1-\alpha_2} \arrow[dl] \arrow[dr] & & M_{-\alpha_1-2\alpha_2} \arrow[dl] \arrow[dr, dashed] & & \\
\dots & \dots & & \dots & & 0 &
\end{tikzcd}
$$

Note that $M_{-\alpha_1-2\alpha_2}$ is now 1-dimensional generated by $F_1F_2^2v_{\Lambda,N}$. In conclusion, one can think of parabolic Verma module as a mild mix between integrable modules and Verma modules, with the action of F_i, E_i being locally nilpotent for $i \in I_f$, and F_j^k being never 0 for all $k \geqslant 0$ and $j \in I_r$.

Chapter 2

Elements of categorification

The philosophy behind categorification is to interpret some mathematical objects as 'shadows' of higher structures, and to build these higher, richer objects. One of the most common examples is the Euler characteristic, which can be thought of as a shadow of the homology groups. In addition to being a stronger invariant of topological spaces, homology groups most importantly reveal a whole new layer of structure given by induced maps on homology. But of course, giving a precise definition of categorification is not possible, unless we start by defining the reverse operation: that is how we decategorify. And there exist several ways of doing it in the literature: Hall algebras [84], categorical traces [8], Grothendieck groups... Then given an algebraic object A, a category $\mathscr{C}$ and a fixed decategorification $[\mathscr{C}]$ of $\mathscr{C}$, we say that $\mathscr{C}$ is a *categorification* of A whenever $[\mathscr{C}] \cong A$.

For quantum groups, and by consequence in this thesis, we use the notion of Grothendieck group $K_0(\mathscr{C})$, which itself declines in several constructions, depending on the structure of the category $\mathscr{C}$ we want to decategorify.

Even given a way to decategorify and an object admitting a categorification, there is not much chance to have an algorithm that allow us to recover this higher structure (except for a very naive, uninteresting one). And there is no chance that the construction is unique. Therefore, to cite Ben Webster, we would say that:

'Categorification is an art, not a functor.'

2.1 Classical Grothendieck groups

The recipe for Grothendieck groups is always the same: we start with a pointed, small category $\mathscr{C}$ admitting some structure that comes with a collection of distinguished triples of objects $\langle X, Y, Z\rangle$ for some $X, Y, Z \in \mathscr{C}$. Moreover, these distinguished triples should have nice properties (e.g. $\langle X, Y, 0\rangle$ and $\langle 0, X, Y\rangle$ are distinguished triples whenever $X \cong Y$). Then we construct the free abelian group $F(\mathscr{C}) := \bigoplus_{X \in \mathscr{C}} [X]\mathbb{Z}$ over the set of objects of $\mathscr{C}$, and we define the Grothendieck group as

$$K_0(\mathscr{C}) := F(\mathscr{C})/T(\mathscr{C}),$$

where

$$T(\mathscr{C}) := \{[Y] - [X] - [Z] | \langle X, Y, Z\rangle \text{ is a distinguished triple}\}.$$

In particular, $[X] = [Y]$ whenever $X \cong Y$. Usually, having a distinguished triple $\langle X, Y, Z\rangle$ means that Y is 'built up' from X and Z, and thus the Grothendieck group extracts this information.

The Grothendieck group inherits other properties from the category. Indeed, suppose we have two categories $\mathscr{C}$ and $\mathscr{C}'$, both having distinguished triples, and a functor $F : \mathscr{C} \to \mathscr{C}'$ that preserves these triples. Then it induces a map

$$K_0(\mathscr{C}) \xrightarrow{[F]} K_0(\mathscr{C}').$$

From this, we can start playing around. If the category $\mathscr{C}$ has a monoidal structure and the bifunctor

$$- \otimes - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

preserves distinguished triples, then the Grothendieck group comes with a ring structure

$$[X].[Y] := [X \otimes Y].$$

Another example of interest for us is when $\mathscr{C}$ has an auto-equivalence (i.e. is $\mathbb{Z}$-graded)

$$\langle 1\rangle : \mathscr{C} \to \mathscr{C}$$

preserving distinguished triples. We write $\langle -1\rangle = \langle 1\rangle^{-1}$ and in general $\langle k\rangle = \langle 1\rangle^k$ for all $k \in \mathbb{Z}$. In that case, the Grothendieck group is a $\mathbb{Z}[q, q^{-1}]$-module with the action of q^k given by

$$q^k \cdot [X] = [X\langle k\rangle].$$

2.1.1 Additive categories

The first example of a categories that comes with distinguished triples are the additive categories, where there is a triple $\langle X, Y, Z\rangle$ whenever

$$Y \cong X \oplus Z.$$

For this particular case, we call $K_0^{\oplus}(\mathscr{C})$ the *split Grothendieck group* of $\mathscr{C}$.

Krull–Schmidt categories

Recall that a unital ring is a *local ring* if given any invertible finite sum of elements, then one of its summand is invertible. A *Krull–Schmidt category* is an additive category in which every object decomposes as a finite direct sum of objects having local endomorphism rings. The main property of such a category is that any object X decomposes as a finite direct sum of indecomposable objects (which agree with the objects having local endomorphism rings). Moreover, this decomposition is essentially unique. From this, we easily deduce the following classical result:

Proposition 2.1.1. *Let $\mathscr{C}$ be a Krull–Schmidt category with its collection of pairwise non-isomorphic indecomposable objects $\{X_i\}_{i\in I}$. Then its split Grothendieck group is a free $\mathbb{Z}$-module*

$$K_0^{\oplus}(\mathscr{C}) \cong \bigoplus_{i\in I} \mathbb{Z}[X_i].$$

Definition 2.1.2. *A category $\mathscr{C}$ is* strictly $\mathbb{Z}$-graded *if it possesses an auto-equivalence $\langle 1\rangle : \mathscr{C} \to \mathscr{C}$ such that*

$$X\langle k\rangle \ncong X$$

for all $k \in \mathbb{Z}\backslash\{0\}$ and non-zero object $X \in \mathscr{C}$. We say that a category is $\mathbb{Z}$-graded Krull–Schmidt *if it is strictly $\mathbb{Z}$-graded and Krull–Schmidt.*

We call $\langle 1\rangle$ the *grading shift* and we say two objects X and Y are *distinct* if $X \ncong Y\langle k\rangle$ for all $k \in \mathbb{Z}$.

Proposition 2.1.3. *Let $\mathscr{C}$ be a $\mathbb{Z}$-graded Krull–Schmidt category with collection of pairwise distinct indecomposable objects $\{X_i\}_{i\in I}$. Then its split Grothendieck group respects*

$$K_0^{\oplus}(\mathscr{C}) \cong \bigoplus_{i\in I} \mathbb{Z}[q,q^{-1}][X_i].$$

Example 2.1.4. *Consider the category $\Bbbk$-gmod of finite dimensional $\mathbb{Z}$-graded $\Bbbk$-vector spaces, with degree preserving maps. It is a graded Krull–Schmidt category with a unique indecomposable object, up to grading shift and isomorphism, given by $\Bbbk$. Thus its split Grothendieck group is*

$$K_0^{\oplus}(\Bbbk\text{-gmod}) \cong \mathbb{Z}[q,q^{-1}].$$

If we consider $\Bbbk$-gmod with its monoidal structure given by tensor product over $\Bbbk$, we get a categorification of $\mathbb{Z}[q,q^{-1}]$ as a ring.

Remark 2.1.5. *It is important to restrict to finite dimensional modules, and in general we need to avoid categories with all (countably) infinite coproducts. Indeed, otherwise, the Eilenberg swindle applies:*

$$\coprod_{i\in\mathbb{N}} X \cong X \oplus \coprod_{i\in\mathbb{N}} X \qquad \Rightarrow \qquad [\coprod_{i\in\mathbb{N}} X] = [X] + [\coprod_{i\in\mathbb{N}} X],$$

implying that $[X] = 0$ *for all* X, *and thus* $K_0(\mathscr{C}) \cong 0$.

We would like to point out that a module category $\mathscr{C}$ over $(\Bbbk\text{-gmod}, \otimes_{\Bbbk})$ is basically a $\mathbb{Z}$-graded, additive, $\Bbbk$-linear category. The $\mathbb{Z}$-action on $\mathscr{C}$ is induced by acting by shifts of the unit object of $(\Bbbk\text{-gmod}, \otimes_{\Bbbk})$. Then the monoidal action

$$\Bbbk\text{-gmod} \rhd \mathscr{C} \to \mathscr{C}$$

descends to the Grothendieck groups

$$K_0^{\oplus}(\Bbbk\text{-gmod}) \times K_0^{\oplus}(\mathscr{C}) \xrightarrow{[\rhd]} K_0^{\oplus}(\mathscr{C})$$

as the $\mathbb{Z}[q, q^{-1}]$-action on $K_0^{\oplus}(\mathscr{C})$, after identifying $K_0^{\oplus}(\Bbbk\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$. What is nice with this point of view is that the $\mathbb{Z}[q, q^{-1}]$-module structure of $K_0^{\oplus}(\mathscr{C})$ is not just a coincidence, but comes from a categorical version of it.

Within this framework, we can already obtain a categorification of (the multiplication by) the quantum number $[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{1-n} \in \mathbb{Z}[q, q^{-1}]$ as the endofunctor $\bigoplus_{\ell=0}^{n} \mathrm{Id}\langle n-1-2\ell\rangle$.

Remark 2.1.6. *Note that we do not truly gave a categorical interpretation of the minus sign, which only comes as a formal consequence of the fact that we work with abelian groups. Therefore, we should probably think of additive categories only as a categorification of* $\mathbb{N}$*-modules, and of* $\Bbbk\text{-gmod}$ *as a categorification of* $\mathbb{N}[q, q^{-1}]$.

2.1.2 Abelian categories

Another class of categories that have distinguished triples is given by abelian categories. In that situation, a distinguished triple $\langle X, Y, Z\rangle$ is a short exact sequence

$$0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0.$$

This generalizes to Quillen exact categories [76]. Since abelian categories are also additive, we write $G_0(\mathscr{C})$ for the Grothendieck group obtained from short exact sequences, so that we do not mistake it with the split Grothendieck group $K_0^{\oplus}(\mathscr{C})$.

Jordan–Hölder

If $\mathscr{C}$ is a finite length abelian category, then by the Jordan–Hölder theorem every object admits an essentially unique finite composition series. Since in the Grothendieck group, an object is equal to the sum of the composition factors of its composition series, we get the following well-known result:

Proposition 2.1.7. *Let $\mathscr{C}$ be a finite length category with collection of pairwise non-isomorphic simple objects $\{S_i\}_{i\in I}$. Then its Grothendieck group is*

$$G_0(\mathscr{C}) \cong \bigoplus_{i\in I} \mathbb{Z}[S_i].$$

The result also holds in the graded case.

Proposition 2.1.8. *Let $\mathscr{C}$ be a strictly $\mathbb{Z}$-graded, finite length category with collection of pairwise distinct simple objects $\{S_i\}_{i\in I}$. Then its Grothendieck group is*

$$G_0(\mathscr{C}) \cong \bigoplus_{i\in I} \mathbb{Z}[q, q^{-1}][S_i].$$

Example 2.1.9. *In $\Bbbk\text{-gmod}$, there is a unique simple object up to grading shift and isomorphism, given by $\Bbbk$. Therefore*

$$G_0(\Bbbk\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}].$$

In this particular case, it can be viewed also as a consequence of Example 2.1.4 since every short exact sequence in $\Bbbk\text{-gmod}$ splits.

2.1.3 Triangulated categories

The third class of categories we are interested in is given by triangulated categories. These kind of structures appear naturally, among others, when one looks at the structure inherited on the homotopy or derived category of an abelian category. They were developed by Verdier under the supervision of Grothendieck [90]. A triangulated category is the datum of an additive category $\mathscr{C}$, an automorphism $[1] : \mathscr{C} \to \mathscr{C}$ called the *translation functor* and a class of *distinguished triangles*:

$$X \to Y \to Z \to X[1],$$

respecting some axioms (see below). We call Y an *extension of X by Z*. In particular $\mathscr{C}$ comes equipped with a $\mathbb{Z}$-action such that $[1] \circ [-1] = \mathrm{Id} = [-1] \circ [1]$.

Remark 2.1.10. *In opposite of, for example, abelian categories, an additive category could admit multiple possible triangulated structures. It is because triangulated* *categories are a truncation of higher categories (e.g. stable $(\infty, 1)$-categories [50]). On this higher level, the triangulated structure is unique. This is similar to the phenomenon of categorification: when one goes one level lower, we loose information and thus a 'shadow' can have been projected from different objects.*

Axiomatic

Here is the (redundant) list of axioms a triangulated category must respect [69]:

1. every diagram $X \to Y \to Z \to X[1]$ isomorphic to a distinguished triangle is itself a distinguished triangle ;

2. the diagram $X \xrightarrow{\mathrm{Id}_X} X \to 0 \to X[1]$ is a distinguished triangle ;

3. for each map $X \xrightarrow{f} Y$ there is a *mapping cone* $\mathrm{Cone}(f)$ fitting in a distinguished triangle
$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1],$$

4. a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if
$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$
is also a distinguished triangle ;

5. given a commutative diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
and two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ then there is a (non-unique) map $Z \xrightarrow{\gamma} Z'$ such that the following diagram commutes:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

6. finally, there is a coherence axiom for taking extensions, called the *octahedral axiom*, stating that if there are three distinguished triangles

$$X \xrightarrow{f} Y \longrightarrow Y/X \longrightarrow X[1],$$
$$Y \xrightarrow{g} Z \longrightarrow Z/Y \longrightarrow Y[1],$$
$$X \xrightarrow{g\circ f} Z \longrightarrow Z/X \longrightarrow X[1],$$

then there is a distinguished triangle

$$Y/X \to Z/X \to Z/Y \to (Y/X)[1],$$

such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Z \arrow[rr] \arrow[dr] & & Z/Y \arrow[rr] \arrow[dr] & & (Y/X)[1] \\
& Y \arrow[ur] \arrow[dr] & & Z/X \arrow[ur] \arrow[dr] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[rr] \arrow[ur] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

Remark 2.1.11. *One of the caveat of triangulated categories is that the construction of the mapping cone (among other similar homotopical constructions) is not necessarily (and often not) functorial. This is because as mentioned in Remark 2.1.10, triangulated categories are somehow only the truncation of a higher structure. In this higher structure, taking mapping cones becomes functorial. However, these higher structures are in general much harder to handle, and thus we prefer to stick with triangulated categories for computational purposes.*

By definition, triangulated categories come with distinguished triples given by the distinguished triangles. Hence, we get a *triangulated Grothendieck group* $K_0^\Delta(\mathscr{C})$. One of the nice properties of the triangulated Grothendieck group is that the second and fourth axioms of triangulated categories tells us that

$$X \to 0 \to X[1] \to X[1]$$

is a distinguished triangle. Therefore, in the triangulated Grothendieck group, we have

$$0 = [X] + [X[1]],$$

meaning that the translation functor descends as the multiplication by -1. This means that we get a categorical interpretation of the minus sign. Moreover, taking a mapping cone $X \to Y$ can be interpreted as a categorical way of subtracting X from Y.

2.2 How to construct triangulated categories

As already mentioned, one way of producing triangulated structures is by looking at homotopy categories or derived categories. In this thesis, we will only consider triangulated categories obtained from dg-algebras. For this, we will mainly follow [35] (see also [10]) and borrow some ideas and notations from [74] (mostly for categorification purposes). A good survey for dg-categories is [36].

2.2.1 Dg-algebras

Definition 2.2.1. *A* dg-($\Bbbk$-)algebra (A, d) *is a unital, $\mathbb{Z}$-graded $\Bbbk$-algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ with a differential $d : A_i \to A_{i-1}$ of degree -1 such that $d^2 = 0$ and*

$$d(ab) = d(a)b + (-1)^{\deg(a)} a d(b),$$

for all homogeneous elements $a, b \in A$.

Remark 2.2.2. *In opposite to [35], we consider differentials of degree -1 instead of $+1$. Of course, all homological gradings can be switched, and one can take degree $+1$ instead. But then, the positive dg-algebras we will consider later on would correspond with negative dg-algebras.*

Definition 2.2.3. *A (left)* dg-module (M, d_M) *over a dg-algebra (A, d) is the datum of a graded (left) A-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and a differential d_M such that $d_M^2 = 0$ and*

$$d_M(a \cdot m) = d_A(a) \cdot m + (-1)^{\deg(a)} a \cdot d_M(m),$$

for all homogeneous elements $a \in A$ and $m \in M$. A morphism of dg-modules is a map of graded modules that preserves the gradings and commutes with the differentials. There are similar obvious definitions for right dg-modules and dg-bimodules.

Given a dg-algebra (A, d) we can construct its homology

$$H(A, d) := \frac{\ker d}{\operatorname{im} d},$$

which is a graded algebra. Similarly we construct the homology $H(M, d)$ of an (A, d)-module, and it is a graded $H(A, d)$-module. As usual in homological algebra, a morphism of dg-modules induces a morphism of graded modules in homology, and a short exact sequence of dg-modules gives rise to a long exact sequence in homology. Moreover, one says that a map $f : X \to Y$ (of dg-modules or of dg-algebras) is a *quasi-isomorphism* if it induces an isomorphism in homology $f_* : H(X) \xrightarrow{\simeq} H(Y)$. A dg-algebra is *formal* if it is quasi-isomorphic to its homology equipped with a trivial differential.

The category $(A,d)\text{-mod}$ of (left) dg-modules over a dg-algebra (A,d) is a $\mathbb{Z}$-graded abelian category with $[1]$ acting by:

- shifting the degree of all elements up by 1 ;
- switching the sign of the differential $d_{M[1]} := -d_M$;
- introducing a sign in the action $r \cdot m[1] := (-1)^{\deg r}(r \cdot m)[1]$.

Remark 2.2.4. *If we consider right dg-modules instead, then the translation functor does not change the sign in the action. In the case of dg-bimodules, it only twists the left-action.*

Let $f : (X, d_X) \to (Y, d_Y)$ be a morphism of dg-modules. Then one constructs the *mapping cone* of f as

$$\operatorname{Cone}(f) := (X[1] \oplus Y, d_C), \qquad d_C := \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}. \tag{2.1}$$

It is an (A,d)-module and it fits in a short exact sequence

$$0 \to Y \hookrightarrow \operatorname{Cone}(f) \twoheadrightarrow X[1] \to 0.$$

2.2.2 Derived category

Given two dg-modules $X, Y \in (A,d)\text{-mod}$, one constructs the graded enriched hom-space as

$$\operatorname{HOM}_{A\text{-mod}}(X,Y) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{A\text{-mod}}(X, Y[i]).$$

It is a graded space with $f : X \to Y[i]$ having degree $-i$. It can upgraded to a dg-hom space $(\operatorname{HOM}_{A\text{-mod}}(X,Y), d)$ with differential given by

$$d(f) = d_Y \circ f - (-1)^{\deg(f)} f \circ d_X.$$

Similarly, one define the tensor product of a right (A,d)-module X and a left (A,d)-module Y as

$$X \otimes_{(A,d)} Y := (X \otimes_A Y, d),$$

where $\deg(x \otimes y) := \deg(x) + \deg(y)$ and $d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y)$.

Then we consider the dg-category of (A,d)-modules, written $(A,d)\text{-dgmod}$, where the objects are the same as in $(A,d)\text{-mod}$ and the hom-spaces are the dg-hom spaces

$$\operatorname{Hom}_{(A,d)\text{-dgmod}}(X,Y) := (\operatorname{HOM}_{A\text{-mod}}(X,Y), d).$$

Note that it is not an abelian category. We also define $Z^0((A,d)\text{-dgmod})$ and $H^0((A,d)\text{-dgmod})$, which have again the same objects as $(A,d)\text{-mod}$ but with hom-spaces being

$$\mathrm{Hom}_{Z^0((A,d)\text{-dgmod})}(X,Y) := \ker(\mathrm{Hom}_A(X,Y[0]) \xrightarrow{d} \mathrm{Hom}_A(X,Y[1])),$$

$$\mathrm{Hom}_{H^0((A,d)\text{-dgmod})}(X,Y) := \frac{\ker(\mathrm{Hom}_A(X,Y[0]) \xrightarrow{d} \mathrm{Hom}_A(X,Y[1]))}{\mathrm{im}(\mathrm{Hom}_A(X,Y[-1]) \xrightarrow{d} \mathrm{Hom}_A(X,Y[0]))}.$$

Note that $Z^0((A,d)\text{-dgmod})$ is naturally equivalent to $(A,d)\text{-mod}$.

One calls $H^0((A,d)\text{-dgmod})$ the *homotopy category* of (A,d). It is a triangulated category with translation functor given by $[1]$ defined as in $(A,d)\text{-mod}$. The distinguished triangles are isomorphic to triangles $X \to Y \to Z \to X[1]$ obtained from short exact sequences $X \to Y \to Z$ in $Z^0((A,d)\text{-dgmod})$ that split in $A\text{-mod}$ (the splitting maps allow to construct $Z \to X[1]$). Mapping cones are constructed as in (2.1).

The *derived category* $\mathcal{D}(A,d)$ of $(A,d)\text{-dgmod}$ is given by localizing the homotopy category $H^0((A,d)\text{-dgmod})$ along quasi-isomorphisms (we recall that localizing is the procedure of adding formal inverses to a specified class of maps). Therefore, quasi-isomorphisms become isomorphisms in the derived category. It inherits a triangulated structure from the homotopy category. The distinguished triangles are now obtained from all short exact sequences in $Z^0((A,d)\text{-dgmod})$ (they do not need to split as graded modules anymore).

It is in general a hard problem to compute the hom-spaces in the derived category, except for a specific class of objects.

Definition 2.2.5. *An (A,d)-module X is a* relatively projective module *if it is a direct summand of a direct sum of shifted copies of free modules in* $(A,d)\text{-mod}$.

Definition 2.2.6. *An (A,d)-module Y* satisfies property (P) *if there is an exhaustive filtration of (A,d)-modules*

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset Y,$$

such that each F_{r+1}/F_r is isomorphic to a relatively projective module.

Definition 2.2.7. *An (A,d)-direct summand of a property (P) module is called* cofibrant.

A nice property of cofibrant modules is that their hom-spaces are much easier to compute. Indeed for P cofibrant and any $Y \in (A,d)\text{-mod}$, we have

$$\mathrm{Hom}_{\mathcal{D}(A,d)}(P,Y) \cong \mathrm{Hom}_{H^0((A,d)\text{-dgmod})}(P,Y).$$

Of course, $\mathrm{Hom}_{H^0((A,d)\text{-dgmod})}(P,Y)$ can be computed as $\mathrm{Hom}_{(A,d)\text{-mod}}(P,Y)/\sim$ where $f \sim g$ if there exists a morphism $h : P \to Y[1]$ such that

$$f - g = d(h) = d_Y \circ h + h \circ d_P.$$

Another nice property of cofibrant modules is that taking tensor product with such modules preserves quasi-isomorphisms:

$$X \xrightarrow[\sim]{f} Y \ \Rightarrow\ P \otimes_{(A,d)} X \xrightarrow[\sim]{1\otimes f} P \otimes_{(A,d)} Y,$$

for P cofibrant right (A,d)-module and X left (A,d)-module quasi-isomorphic to Y.

Fortunately, there are enough cofibrant modules thanks to following result:

Proposition 2.2.8 ([35, §3.1])**.** *For any (A,d)-module X there is an (A,d)-module satisfying property (P), denoted $\mathbf{p}(X)$, with a surjective quasi-isomorphism*

$$\mathbf{p}(X) \twoheadrightarrow X.$$

We call $\mathbf{p}(X)$ the bar resolution *of X, and the assignment $X \to \mathbf{p}(X)$ is functorial.*

This allows the construction of a *derived hom functor* for each $X \in (A,d)\text{-mod}$ as

$$\mathrm{RHOM}(-,X) : \mathcal{D}(A,d) \to \mathcal{D}(\Bbbk,0), \quad \mathrm{RHOM}(Y,X) := (\mathrm{HOM}(\mathbf{p}(Y),X),d).$$

There is a similar definition for $\mathrm{RHOM}(X,-)$ with

$$\mathrm{RHOM}(X,Y) = (\mathrm{HOM}(\mathbf{p}(X),Y),d).$$

In that case, if X carries the structure of an (A,d)-(B,d)-bimodule, then it can be upgraded into a functor

$$\mathrm{RHOM}(X,-) : \mathcal{D}(A,d) \to \mathcal{D}(B,d).$$

Moreover, whenever X is cofibrant as (A,d)-module, then it is exact in the sense that it preserves distinguished triangles (and commutes with the translation functors). The derived hom-functor admits a left adjoint functor called *derived tensor product functor*

$$X \otimes^{\mathrm{L}}_{(B,d)} - : \mathcal{D}(B,d) \to \mathcal{D}(A,d), \qquad X \otimes^{\mathrm{L}}_{(B,d)} Y := X \otimes_{(B,d)} \mathbf{p}(Y),$$

where $\mathbf{p}(Y)$ is the bar resolution of Y in $(B,d)\text{-mod}$. It is an exact functor whenever X is cofibrant as (right) (B,d)-module.

2.2.3 Back to Grothendieck groups

In order to avoid the Eilenberg swindle (cf. Remark 2.1.5), we need to restrict to a subcategory of the derived category. In this setting, as explained in [74] for example, there is a special class of objects that play the role of finitely generated objects.

Definition 2.2.9. *An object X in a category with (infinite) coproducts is called* compact *if for any family of objects $\{N_i\}_{i\in I}$ the natural map*

$$\coprod_{i\in I} \mathrm{Hom}(M, N_i) \xrightarrow{\simeq} \mathrm{Hom}(M, \coprod_{i\in I} N_i)$$

is an isomorphism.

The *compact derived category* is the full subcategory $\mathcal{D}^c(A,d) \subset \mathcal{D}(A,d)$ given by objects quasi-isomorphic to compact objects. It is a triangulated (thanks to the five lemma), idempotent complete subcategory of $\mathcal{D}(A,d)$. Luckily, in the setting of dg-categories, $\mathcal{D}(A,d)$ is compactly generated (see [67, 68, 79]) by the free module (A,d) and all its shifts $A[r]$, $r \in \mathbb{Z}$. Equivalently, it is compactly generated by finitely generated, relatively projective modules. Therefore, we can obtain all compact objects as direct summands of finite iterated extensions of the compact generators. For this reason, as in [74], we define the following:

Definition 2.2.10. *A* finite cell module *is an (A,d)-module satisfying property (P) and being finitely generated as A-module.*

This means that finite cell modules are modules admitting finite filtrations in (A,d)-mod where each composition factor is a finitely generated, relatively projective module. Thus, finite cell modules are exactly the finite iterated extensions we were looking for, and any compact object is isomorphic to a direct summand of a finite cell module. We say that a collection $\{K_i\}_{i\in I}$ is a *minimal compact generating set* if

- the collection $\{K_i[n]\}_{i\in I, n\in\mathbb{Z}}$ compactly generates $\mathcal{C}$;
- K_i cannot be obtain as an iterated extension of objects in $\{K_j[n]\}_{j\in I\setminus\{i\}, n\in\mathbb{Z}}$.

Note that the second condition implies in particular that K_i is distinct from K_j with respect to $[1]$ for all $i \neq j$. Then we have the following result:

Proposition 2.2.11. *Let $\mathcal{C}^c$ be the compact subcategory of a compactly generated triangulated category $\mathcal{C}$ with minimal compact generating set $\{K_i\}_{i\in I}$. Then its triangulated Grothendieck group respects*

$$K_0^\Delta(\mathcal{C}^c) \cong \bigoplus_{i\in I} \mathbb{Z}[K_i].$$

Note that it is in general a non-trivial problem to determine if a set of compact generators is a minimal compact generating one.

Example 2.2.12. *The triangulated Grothendieck group of* $(\Bbbk, 0)$ *is given by*

$$K_0^{\Delta}(\mathscr{D}^c(\Bbbk, 0)) \cong \mathbb{Z},$$

with minimal compact generating collection given by $\{(\Bbbk, 0)\}$.

2.2.4 Homological toolbox

We explain some well-known tools that will reveal useful later to compute homology of dg-modules.

Proposition 2.2.13. *Let* (A, d_A) *be a dg-algebra,* (M, d_M) *be a right* (A, d_A)*-module, and* (N, d_N) *a left one. If* (M, d_M) *is formal and* (N, d_N) *is cofibrant, then we have*

$$H\big((M, d_M) \otimes_{(A,d_A)} (N, d_N)\big) \cong H\big(H(M, d_M) \otimes_{(A,d_A)} (N, d_N)\big).$$

Proof. Tensoring with a cofibrant dg-module preserves quasi-isomorphisms. ■

The following notions were introduced by Moore [61], but we use the presentation given in [88].

Definition 2.2.14 ([88, Definition 8.5])**.** *Let* $(R, 0)$ *be a ring* R *viewed as a dg-*$\mathbb{Z}$*-algebra concentrated in degree zero. An* $(R, 0)$*-module* (Q, d_Q) *is* strongly projective *if* $H(Q, d_Q)$ *and* $\operatorname{im} d_Q$ *are both projective* R*-modules.*

Lemma 2.2.15 ([91, Theorem 9.3.2])**.** *Let* (P, d_P) *be a strongly projective right* $(R, 0)$*-module and* (N, d_N) *any left* $(R, 0)$*-module, then*

$$H\big((P, d_P) \otimes_{(R,0)} (N, d_N)\big) \cong H(P, d_P) \otimes_R H(N, d_N).$$

Definition 2.2.16 ([88, Definition 8.17])**.** *Let* (A, d_A) *be a dg-*R*-algebra. A left (resp. right)* (A, d_A)*-module* (P, d_P) *is* strongly projective *if it is a dg-direct summand of* $(A, d_A) \otimes_{(R,0)} (Q, d_Q)$ *(resp.* $(Q, d_Q) \otimes_{(R,0)} (A, d_A)$*) for some strongly projective* $(R, 0)$*-module* (Q, d_Q).

Proposition 2.2.17 ([88, Lemma 8.23])**.** *If* (P, d_P) *is a strongly projective right* (A, d_A)*-module and* (N, d_N) *is any left* (A, d_A)*-module, then*

$$H\big((P, d_P) \otimes_{(A,d_A)} (N, d_N)\big) \cong H(P, d_P) \otimes_{H(A,d_A)} H(N, d_N).$$

Note that if (P, d_P) is a strongly projective (A, d_A)-module, then $H(P, d_P)$ is a projective $H(A, d_A)$-module. Indeed, we can assume $(P, d_P) = (A, d_A) \otimes_{(R,0)} (Q, d_Q)$, and we have

$$H(P, d_P) \cong H(A, d_A) \otimes_R H(Q, d_Q).$$

Since $H(Q, d_Q)$ is a projective R-module, it is a direct summand of a free R-module F. Therefore $H(P, d_P)$ is a direct summand of $H(A, d_A) \otimes_R F$, which is a free $H(A, d_A)$-module.

Remark 2.2.18. *Of course, in general, this result does not hold. As a counter example we can take $(A, d) = (\mathbb{Q}[x, y], 0)$ and consider the dg-module (X, d_X) given by the mapping cone $\operatorname{Cone}(\mathbb{Q}[x, y] \xrightarrow{x-y} \mathbb{Q}[x, y])$. Then we obtain $H(X, d_X) \cong \mathbb{Q}[x, y]/(x - y)$ but $H((X, d_X) \otimes_{(A,d)} (X, d_X)) \cong \mathbb{Q}[x, y] \oplus \mathbb{Q}[x, y][1]$.*

We will also need the following result:

Proposition 2.2.19. *[86, Lemma 09IZ] Let (M, d_M) be a $(\mathbb{Z}, 0)$-module. Let*

$$\cdots \xrightarrow{f_3} (A_2, d_2) \xrightarrow{f_2} (A_1, d_1) \xrightarrow{f_1} (A_0, d_0) \to (M, d_M) \to 0,$$

be an exact sequence, such that

$$\cdots \to \ker(d_2) \to \ker(d_1) \to \ker(d_0) \to \ker(d_M) \to 0,$$

is exact as well. Then define the $(\mathbb{Z}, 0)$-module (T, d_T) by $T^h := \bigoplus_{(a+b=h)} A_a^b$ and $d_T(x) := f_a(x) + (-1)^a d_a(x)$ for $x \in A_a^b$ and $f_0 = 0$. There is a surjective quasi-isomorphism

$$(T, d_T) \xrightarrow{\simeq} (M, d_M),$$

induced by $(A_0, d_0) \twoheadrightarrow (M, d_m)$.

2.3 Multigradings

We say that a category $\mathscr{C}$ is $\mathbb{Z}^n$-(multi)graded if there is a collection of auto-equivalences $\langle 1 \rangle_i$ for $1 \leqslant i \leqslant n$, strictly commuting with each others. If $\mathscr{C}$ is triangulated, then we say it is a $\mathbb{Z}^n$- graded triangulated categories if $\langle 1 \rangle_i$ strictly commutes with the translation functor $[1]$ for each i (making it a $\mathbb{Z}^{n+1}$-graded category), and $\langle 1 \rangle_i$ preserves distinguishes triangles. The Grothendieck groups of a $\mathbb{Z}^n$-graded category is a $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$-module. In order to simplify notations, we write a shift in the $\mathbb{Z}^n$-multigrading as $x^{\boldsymbol{g}}$ for $\boldsymbol{g} = (g_1, \dots, g_n) \in \mathbb{Z}^n$, or as monomial

$x_1^{g_1} \cdots x_n^{g_n} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, so that we have

$$x^{\boldsymbol{g}} X := x_1^{g_1} \cdots x_n^{g_n} X := X\langle g_1 \rangle \cdots \langle g_n \rangle,$$

which is well defined since the shifts strictly commute with each others. We keep the notation $[1]$ for the shift in the homological grading. We say that $\mathscr{C}$ is *strictly* $\mathbb{Z}^n$-graded if it is $\mathbb{Z}^n$ graded and if

$$x^{\boldsymbol{g}} X \ncong X$$

for all $x \in \mathbb{Z}^n \backslash \{0\}$, and non-zero object $X \in \mathscr{C}$.

Remark 2.3.1. *Note that a category admitting arbitrary coproducts cannot be strictly $\mathbb{Z}^n$-graded.*

A $\mathbb{Z}^n$-graded dg-structure is a structure that is both $\mathbb{Z}^n$-graded and dg, and with the differential preserving the $\mathbb{Z}^n$-grading. In particular, a $\mathbb{Z}^n$-graded dg-algebra (A, d) decomposes as vector space as a direct sum

$$A \cong \bigoplus_{(\boldsymbol{g},h) \in \mathbb{Z}^n \oplus \mathbb{Z}} A_{\boldsymbol{g}}^h,$$

with the multiplication and differential respecting

$$A_{\boldsymbol{g}}^h \cdot A_{\boldsymbol{g}'}^{h'} \to A_{\boldsymbol{g}+\boldsymbol{g}'}^{h+h'}, \qquad\qquad A_{\boldsymbol{g}}^h \xrightarrow{d} A_{\boldsymbol{g}}^{h-1}.$$

Everything said in §2.1.3 generalizes to multigraded dg-structures. In this case, we let $(A, d)\text{-mod}$ to be the abelian category of $\mathbb{Z}^n$-graded dg-modules with maps that preserve all gradings. Note that in this context the enriched graded hom-spaces are given by

$$\mathrm{HOM}_{A\text{-mod}}(X, Y) := \bigoplus_{(h,\boldsymbol{g}) \in \mathbb{Z} \oplus \mathbb{Z}^n} \mathrm{Hom}_{A\text{-mod}}(X, x^{\boldsymbol{g}} Y[h]),$$

where a map $f : X \to x^{\boldsymbol{g}} Y[h]$ has degree $(-h, -\boldsymbol{g})$.

Example 2.3.2. *We consider $(\Bbbk, 0)$ as a $\mathbb{Z}^n$-graded dg-algebra concentrated in degree 0. Then its derived category is $\mathbb{Z}^n$-graded triangulated. Its triangulated Grothendieck group is given by*

$$K_0^{\Delta}(\mathscr{D}^c(\Bbbk, 0)) \cong \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Chapter 3

Topological Grothendieck groups

One limitation of the framework explained in Chapter 2 is that we always need to work with finitely generated objects. Therefore we end up with Grothendieck groups that are modules only over a polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. This is actually enough for example for constructing categorifications of integrable modules of quantum groups, as in [33]. This is because all coefficients appearing in (1.11) are quantum integers (the situation is similar in other types), and thus can be interpreted as polynomials in $\mathbb{Z}[q, q^{-1}]$.

However, if we look at the structure constants appearing in the universal Verma module of $U_q(\mathfrak{sl}_2)$, we see that we need to be able to categorify the rational fraction

$$\frac{\lambda q^k - \lambda q^{-k}}{q - q^{-1}}.$$

We already know thanks to §2.1.3 and §2.3 how to deal with the λ (we can simply add an extra grading) and with the minus sign (we can use a triangulated structure). However, we still need to figure out a way to interpret the fraction $\frac{1}{q-q^{-1}}$. Luckily, it is already common knowledge in categorification theory how one can solve this problem (see for example [46]). The idea is to rewrite it as a formal Laurent series

$$\frac{1}{q - q^{-1}} = -q\frac{1}{1 - q^2} = -q(1 + q^2 + q^4 + \dots).$$

Then we can categorify the fraction using objects that are locally finite (i.e. finite in each degree). In particular here, we can take an infinite coproduct $\coprod_{\ell \geqslant 0} q^{1+2\ell} \operatorname{Id}[1]$.

The procedure of embedding $\mathbb{Q}(q) \hookrightarrow \mathbb{Q}(\!(q)\!) \cong \mathbb{Q}[\![q]\!][q^{-1}]$ can be made formal and extended to $\mathbb{Q}(x_1, \dots, x_n) \hookrightarrow \mathbb{Q}(\!(x_1, \dots, x_n)\!)$ using the work of Aparicio-Monforte and Kauers [4]. Then we need to construct a bigger category than the compact derived category, but not too big so that we do not end up with Eilenberg swindle-like situations (see Remark 2.1.5). Finally, we need to slightly modify the notion of Grothendieck group in order to introduce a topology on it, so that formal Laurent series converge. We call these *topological Grothendieck groups*.

We proposed in [65, Appendix] a solution for Krull–Schmidt-like categories and for Jordan–Hölder-like categories, in order to work with locally finite direct sums and locally finite composition series respectively. Here, we develop a framework for the dg-setting, taking inspiration from [1]. In the reference, they construct a topological Grothendieck group of 'locally finite' derived categories of (nice enough) finite length abelian categories. It was mainly made to accommodate with (infinite) projective resolutions that appear in the categorification of the integrable $U_q(\mathfrak{g})$-modules. Unfortunately, it cannot be used directly for our case. Indeed, on one hand we need to directly work in a dg-setting, and on the other hand the basic objects we will manipulate in the next chapters are not finite (this is needed since the structure constants in a Verma module are not necessarily finite).

This chapter is a bit more technical than the others and the reader can easily skip it in a first reading. The most important things to retain here (for the rest of the thesis) is that there exists a class of well-behaved $\mathbb{Z}^n$-graded dg-algebras (A, d) (Definition 3.4.5). Moreover, these dg-algebras admit some subcategories $\mathcal{D}^{lc}(A, d)$ of their derived category $\mathcal{D}(A, d)$ having controlled infinite iterated extensions of objects (Corollary 3.4.17). This allows us to compute their topological Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A, d))$ (Definition 3.4.4). In that situation, the topological Grothendieck group is over $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$, and is generated either by the symbols of the projective A-modules (Theorem 3.4.26) or the simple A-modules (Corollary 3.4.28).

3.1 Ring of formal Laurent series

We follow the description given in [4]. We choose an arbitrary additive total order $\prec$ on $\mathbb{Z}^n$. That is, $a \prec b$ implies $a + c \prec b + c$, for every $a, b, c \in \mathbb{Z}^n$.

Definition 3.1.1. *A* cone *is a subset $C \subset \mathbb{R}^n$ such that*

$$C = \{\alpha_1 v_1 + \cdots + \alpha_p v_n | \alpha_i \in \mathbb{R}_{\geqslant 0}\},$$

for some generating elements $v_1, \dots, v_n \in \mathbb{Z}^n$. Moreover C is compatible with the order $\prec$ *if $0 \prec v_i$ for all $i \in \{1, \dots, n\}$.*

Remark 3.1.2. *An important fact to note here is that given a cone C compatible with $\prec$, then we can write $C \cap \mathbb{Z}^n = \{0 = \boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \dots\}$ with $\boldsymbol{c}_i \prec \boldsymbol{c}_{i+1}$ for all $i \in \mathbb{N}$. In particular given an element $\boldsymbol{z} \in C \cap \mathbb{Z}^n$, then there is only finitely many $\boldsymbol{c} \prec \boldsymbol{z}$ in $C \cap \mathbb{Z}^n$.*

Example 3.1.3. *If $n = 1$, then there are only two possible orders:*

- *if $0 \prec 1$, and there is only one (non-zero) compatible cone given by $[0, \infty[$,*
- *if $0 \prec -1$, and the only compatible cone is $]-\infty, 0]$.*

Let $\mathbb{k}$ be a unital, commutative ring. Recall that for $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we write $x^{\boldsymbol{k}} := x_1^{k_1} \dots x_n^{k_n}$.

Let $C \subset \mathbb{R}^n$ be a cone compatible with $\prec$ and define

$$\mathbb{k}_C[\![x_1, \dots, x_n]\!] := \left\{ \sum_{\boldsymbol{k} \in \mathbb{Z}^n} a_{\boldsymbol{k}} x^{\boldsymbol{k}} | a_{\boldsymbol{k}} \in \mathbb{k}, a_{\boldsymbol{k}} = 0 \text{ if } \boldsymbol{k} \notin C \right\}.$$

Proposition 3.1.4 ([4, Theorem 10])**.** *The set $\mathbb{k}_C[\![x_1, \dots, x_n]\!]$ together with the point-wise addition and usual multiplication forms a unital, commutative ring.*
Moreover, for $f(\boldsymbol{x}) \in \mathbb{k}_C[\![x_1, \dots, x_n]\!]$ with $f(0) = a_0 \in \mathbb{k}^\times$ invertible, then there exists $f^{-1}(\boldsymbol{x}) \in \mathbb{k}_C[\![x_1, \dots, x_n]\!]$ such that $f(\boldsymbol{x})f^{-1}(\boldsymbol{x}) = 1$.

Definition 3.1.5 ([4, Definition 14])**.** *We put*

$$\mathbb{k}_\prec[\![x_1, \dots, x_n]\!] := \bigcup_C \mathbb{k}_C[\![x_1, \dots, x_n]\!],$$

where the union is over all cones compatible with $\prec$. The ring of formal Laurent series *is*

$$\mathbb{k}_\prec(\!(x_1, \dots, x_n)\!) := \bigcup_{\boldsymbol{e} \in \mathbb{Z}^n} x^{\boldsymbol{e}} \mathbb{k}_\prec[\![x_1, \dots, x_n]\!].$$

Theorem 3.1.6 ([4, Theorem 15])**.** *The set $\mathbb{k}_\prec(\!(x_1, \dots, x_n)\!)$ together with the usual addition and multiplication forms a ring. Whenever $\mathbb{k}$ is a field, then $\mathbb{k}_\prec(\!(x_1, \dots, x_n)\!)$ is a field.*

When $\mathbb{k}$ is a field, there is a canonical inclusion $\mathbb{k}(x_1, \dots, x_n) \hookrightarrow \mathbb{k}_\prec(\!(x_1, \dots, x_n)\!)$ which consists in first writing $f(\boldsymbol{x}) \in \mathbb{k}(x_1, \dots, x_n)$ as $f(\boldsymbol{x}) = g(\boldsymbol{x})/h(\boldsymbol{x})$ with $g(\boldsymbol{x}), h(\boldsymbol{x}) \in \mathbb{k}[x_1, \dots, x_n]$. Then we consider $\tilde{g}(\boldsymbol{x}), \tilde{h}(\boldsymbol{x})$ given by the canonical inclusion $\mathbb{k}[x_1, \dots, x_n] \hookrightarrow \mathbb{k}_\prec(\!(x_1, \dots, x_n)\!)$, and finally we take $\tilde{g}(\boldsymbol{x})\tilde{h}^{-1}(\boldsymbol{x})$.

Example 3.1.7. *Again, if $n = 1$, we have two possible ways to construct $\mathbb{k}(\!(x)\!)$:*

- *if $0 \prec 1$, then we get $\mathbb{k}_\prec(\!(x)\!) = \mathbb{k}[\![x]\!][x^{-1}]$ and $\frac{1}{x - x^{-1}} = -x(1 + x^2 + \dots)$,*
- *if $0 \prec -1$, then $\mathbb{k}_\prec(\!(x)\!) = \mathbb{k}[\![x^{-1}]\!][x]$ and $\frac{1}{x - x^{-1}} = x^{-1}(1 + x^{-2} + \dots)$.*

Example 3.1.8. *Take $n = 2$, then there are six prdeers with which to construct $\Bbbk_{<}(\!(x_1, x_2)\!)$. In this case we abuse the notation and say, for example, that we choose the order $0 \prec x_1 \prec x_2$ for the order induced by the choice $(0,0) \prec (1,0) \prec (0,1)$, where we suppose $(1,0)$ corresponds to x_1 and $(0,1)$ to x_2. Then we get*

$$\frac{1}{1 - x_1^{-2}x_2^2} = (1 + x_1^{-2}x_2^2 + x_1^{-4}x_2^4 + \dots).$$

3.1.1 Topological structure on the ring of Laurent series

The ring of formal Laurent series $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ comes with a natural topology, which we refer as $(x_1, \dots, x_n)$*-adic topology*. It is defined by using

$$\{V_{\boldsymbol{e}} := x^{\boldsymbol{e}}\Bbbk_{<}[\![x_1, \dots, x_n]\!] | \boldsymbol{e} \in \mathbb{Z}^n\}, \tag{3.1}$$

as neighborhood basis of zero, extending so that $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ becomes a topological group under addition. Note that $V_{\boldsymbol{e}} \subset V_{\boldsymbol{f}}$ whenever $\boldsymbol{e} > \boldsymbol{f}$.

Remark 3.1.9. *We would like to point out that the subgroup $V_{\boldsymbol{e}}$ is not an ideal of the ring $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$, and thus the $(x_1, \dots, x_n)$-adic topology is not an adic topology in the proper sense.*

Proposition 3.1.10. *The ring of formal Laurent series $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ equipped with the $(x_1, \dots, x_n)$-adic topology is a Hausdorff topological ring. If $\Bbbk$ is a field, then $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ is a topological field.*

Proof. We recall that a subset $U \subset \Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ is open if and only if for each $f(\boldsymbol{x}) \in U$ there exists a neighborhood $V_{\boldsymbol{e}}$ such that $f(\boldsymbol{x}) + V_{\boldsymbol{e}} \subset U$. We only need to show that the multiplication map is continuous. Therefore we want to show that

$$U := \{(f(\boldsymbol{x}), g(\boldsymbol{x})) | f(\boldsymbol{x})g(\boldsymbol{x}) \in V_{\boldsymbol{e}}\},$$

for any $\boldsymbol{e} \in \mathbb{Z}^n$ fixed is open. We take any $(f(\boldsymbol{x}), g(\boldsymbol{x})) \in U$. It means we can find $\boldsymbol{f}$ and $\boldsymbol{g}$ such that $f(\boldsymbol{x}) \in V_{\boldsymbol{f}}$ and $g(\boldsymbol{x}) \in V_{\boldsymbol{g}}$ with $\boldsymbol{f} + \boldsymbol{g} \geq \boldsymbol{e}$. Then we put $\boldsymbol{r} = \max\{\boldsymbol{e}, \boldsymbol{e} - \boldsymbol{f}\}$ and $\boldsymbol{s} = \max\{\boldsymbol{e}, \boldsymbol{e} - \boldsymbol{g}\}$ so that

$$(f(\boldsymbol{x}), g(\boldsymbol{x})) + (V_{\boldsymbol{s}}, V_{\boldsymbol{r}}) \subset U.$$

This shows $\Bbbk_{<}(\!(x_1, \dots, x_n)\!)$ is a topological ring.

Moreover, we clearly have

$$\bigcap_{\boldsymbol{e} \in \mathbb{Z}^n} x^{\boldsymbol{e}}\Bbbk_{<}[\![x_1, \dots, x_n]\!] = \{0\},$$

and thus, the $(x_1, \dots, x_n)$-adic topology is Hausdorff.

If $\Bbbk$ is a field, then we write $U := \{f(\boldsymbol{x}) | f^{-1}(\boldsymbol{x}) \in V_{\boldsymbol{e}}\}$. We have $f(\boldsymbol{x}) \in U$ if and only if $f(\boldsymbol{x}) = x^{\boldsymbol{f}} h(\boldsymbol{x})$ for some $h(\boldsymbol{x})$ such that $h(0) \neq 0$, and $\boldsymbol{f} \leq -\boldsymbol{e}$. Then we obtain

$$f(\boldsymbol{x}) + V_{\boldsymbol{f}} \subset U,$$

which concludes the proof. ∎

3.2 C.b.l. additive categories

We review and slightly improve the notions presented in [65, Appendix]. We fix an arbitrary additive total order $\prec$ on $\mathbb{Z}^n$.

Definition 3.2.1. *Let $\{X_1, \ldots, X_m\}$ be a finite collection of objects in a stricly $\mathbb{Z}^n$-graded category. Then we consider a coproduct of the form*

$$\coprod_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} (X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}}),$$

with each $k_{j,\boldsymbol{g}} \in \mathbb{N}$. We call such coproducts locally finite. *We call it* c.b.l.f. (cone bounded, locally finite) *if in addition, there exists a cone C compatible with $\prec$, and $\boldsymbol{e} \in \mathbb{Z}^n$ such that we have $k_{j,\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C$.*

One of the nice properties of c.b.l.f. coproducts is that taking a c.b.l.f. coproduct of c.b.l.f. coproducts yields again a c.b.l.f. coproduct.

If $\mathscr{C}$ is a category admitting zero morphisms (e.g. a preadditive category) and $Y_j \xrightarrow{\imath_j} \coprod_i Y_i$ is a coproduct in $\mathscr{C}$, then we have unique maps $p_j : \coprod_i Y_i \to Y_j$ given by universal property of the coproduct making the following diagram commutes

$$\begin{tikzcd} Y_k \arrow[r, "\imath_k"] \arrow[dr, "\delta jk"'] & \coprod_i Y_i \arrow[d, dashed, "\exists! p_j"] \\ & Y_j \end{tikzcd} \tag{3.2}$$

for all j, k. Then by universal property of the product $\prod_i \mathrm{Hom}(-, Y_i) \xrightarrow{\pi_j} \mathrm{Hom}(-, Y_j)$, we get in turn a map $r : \mathrm{Hom}(-, \coprod_i Y_i) \to \prod_i \mathrm{Hom}(-, Y_i)$ making the diagram

$$\begin{tikzcd} \mathrm{Hom}(-, \coprod_i Y_i) \arrow[d, dashed, "\exists! r"'] \arrow[dr, "p_j \circ -"] & \\ \prod_i \mathrm{Hom}(-, Y_i) \arrow[r, "\pi_j"'] & \mathrm{Hom}(-, Y_j) \end{tikzcd} \tag{3.3}$$

commute for all j.

Definition 3.2.2. *We say that a coproduct $\coprod_i Y_i$ in a category with zero morphisms is a* biproduct *if the natural transformation*

$$\operatorname{Hom}(-, \coprod_i Y_i) \xrightarrow{r} \prod_i \operatorname{Hom}(-, Y_i)$$

is an isomorphism.

Proposition 3.2.3. *Let $\mathscr{C}$ be a category admitting zero morphisms. If $\coprod_i Y_i$ is a biproduct, then it is a product as well. Moreover, if $p_j : \coprod_i Y_i \to Y_j$ and $\imath_j : Y_i \to \coprod_i Y_i$ are the projection and injection maps respectively, then $\prod_j (\imath_j \circ p_j) = \mathrm{Id}$.*

Proof. Let $Z \xrightarrow{f_i} Y_i$ be a collection of arrows in $\mathscr{C}$. Consider the maps $p_j : \coprod_i Y_i \to Y_j$ as in (3.2), and the maps $r : \operatorname{Hom}(-, \coprod_i Y_i) \xrightarrow{r} \prod_i \operatorname{Hom}(-, Y_i)$ and $\pi_j : \prod_i \operatorname{Hom}(-, Y_i) \to \operatorname{Hom}(-, Y_j)$ as in (3.3). Then we obtain $p_j \circ r^{-1}(\prod_i f_i) = \pi_j(\prod_i f_i) = f_j$, and by consequence we get a commutative diagram

$$\begin{tikzcd} Z \arrow[d, "r^{-1}(\prod f_i)"'] \arrow[dr, "f_j"] & \\ \coprod_i Y_i \arrow[r, "p_j"'] & Y_j. \end{tikzcd}$$

This means $\coprod_i Y_i$ is a product with projection maps given by p_j. The second claim follows by universal property of the product. ∎

Remark 3.2.4. *The reciprocal of Proposition 3.2.3 is also true: if the maps p_j turn a coproduct $\coprod Y_i$ into a product, then it is a biproduct. This follows directly from the fact that in general $\operatorname{Hom}(-, \prod_i X_i) \cong \prod_i \operatorname{Hom}(-, X_i)$.*

Definition 3.2.5. *An Ab-enriched, strictly $\mathbb{Z}^n$-graded category admitting all c.b.l.f. coproducts is called* c.b.l. additive *if all its c.b.l.f. coproducts are biproducts. In that case, we write c.b.l.f. coproducts with a $\bigoplus$ sign, and refer to them as* c.b.l.f. direct sums.

Because a c.b.l. additive category $\mathscr{C}$ admits c.b.l.f. direct sums, its Grothendieck group $K_0(\mathscr{C})$ is naturally a $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$-module with the action of $\sum_{\boldsymbol{g} \in C} a_{\boldsymbol{g}} x^{\boldsymbol{e}+\boldsymbol{g}} \in \mathbb{Z}(\!(x_1, \dots, x_n)\!)$ being

$$\sum_{\boldsymbol{g} \in C} a_{\boldsymbol{g}} x^{\boldsymbol{e}+\boldsymbol{g}} [X] := [\bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{g}+\boldsymbol{e}} X^{\oplus a_{\boldsymbol{g}}}]. \tag{3.4}$$

Note that when we write $\boldsymbol{g} \in C$ in this context, we implicitly suppose we are considering $\boldsymbol{g} \in C \cap \mathbb{Z}^n$ and C compatible with $\prec$. We will abuse of this notation whenever it is clear from the context.

Proposition 3.2.6. *Let $\mathscr{C}$ be a c.b.l. additive category. Suppose $M \in \mathscr{C}$ is isomorphic to a c.b.l.f. direct sum*

$$M \cong \bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{e}+\boldsymbol{g}} \left(X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}} \right).$$

Then we have

$$[M] - \sum_{\boldsymbol{g} \in C} x^{\boldsymbol{e}+\boldsymbol{g}} \left(k_{1,\boldsymbol{g}}[X_1] + \cdots + k_{m,\boldsymbol{g}}[X_m] \right),$$

in $K_0^{\oplus}(\mathscr{C})$.

Proof. Straightforward since by definition

$$\sum_{\boldsymbol{g} \in C} x^{\boldsymbol{e}+\boldsymbol{g}} \left(k_{1,\boldsymbol{g}}[X_1] + \cdots + k_{m,\boldsymbol{g}}[X_m] \right) = [\bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{e}+\boldsymbol{g}} \left(X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}} \right)],$$

and isomorphisms become equalities in the Grothendieck group. ■

3.2.1 C.b.l. Krull–Schmidt categories

We recall that an object S in a category $\mathscr{C}$ is compact (also called small in the literature) if the natural map

$$\coprod_{i \in I} \operatorname{Hom}(S, Y_i) \to \operatorname{Hom}(S, \coprod_{i \in I} Y_i)$$

is an equivalence for all coproducts in $\mathscr{C}$. In other words, every map $f : S \to \coprod_{i \in I} Y_i$ factors through $\coprod_{j \in J} Y_j$ for a finite subset $J \subset I$. A typical example is given by finitely generated modules in a category of modules. We also recall that a ring R is called *local* if for every finite sum that is invertible, then (at least) one of the summands is invertible. An object with local endomorphism ring is indecomposable, since whenever e is an idempotent, then either e or $1 - e$ is the identity.

Definition 3.2.7. *We say a that a c.b.l. additive category $\mathscr{C}$ is* c.b.l. Krull–Schmidt *if every object decomposes into a c.b.l.f. direct sum of compact objects having local endormophism rings.*

Clearly, a c.b.l. Krull–Schmidt category is idempotent complete. Moreover, as it turns out, the compactness condition allows us to mimic the classical proof (see for example [92, Theorem 1]) of the Krull–Schmidt property in a Krull–Schmidt category: that is every object decomposes into an essentially unique direct sum of indecomposable objects. For this, we will need the following lemma:

Lemma 3.2.8 ([6, Lemma 3.3]). *Let A, B, C be objects in an additive category such that*

$$A \oplus B \xrightarrow[\sim]{f=\begin{pmatrix} f_{AA} & f_{AB} \\ f_{CA} & f_{CB} \end{pmatrix}} A \oplus C$$

is an isomorphism. If A has a local endormorphism ring and f_{AA} is invertible, then $B \cong C$.

Recall that two objects X and Y are *distinct* if $X \not\cong x^{\boldsymbol{g}} Y$ for all $\boldsymbol{g} \in \mathbb{Z}^n$.

Theorem 3.2.9. *Consider an isomorphism between c.b.l.f. direct sums in a c.b.l. Krull–Schmidt category*

$$\bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \left(X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}} \right) \cong \bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \left(Y_1^{\oplus k'_{1,\boldsymbol{g}}} \oplus \cdots \oplus Y_{m'}^{\oplus k'_{m',\boldsymbol{g}}} \right),$$

where each X_i, X_j (resp. Y_i, Y_j) are pairwise distinct indecomposable objects. Then $m = m'$ and there is a permutation $\sigma \in S^m$ and elements $\boldsymbol{g}_i \in \mathbb{Z}^n$ such that $X_i \cong x^{\boldsymbol{g}_i} Y_{\sigma(i)}$ for all $i \in \{1, 2, \ldots, m\}$, and $k_{i,\boldsymbol{g}} = k'_{\sigma(i), \boldsymbol{g}+\boldsymbol{g}_i}$ for all $\boldsymbol{g} \in \mathbb{Z}^n$.

Proof. Write $M = \bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \left(X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}} \right)$. We consider for $k \in \{1, \ldots, k'_{i,\boldsymbol{g}}\}$,

$$f_{i,\boldsymbol{g}}^k : x^{\boldsymbol{g}} Y_i \hookrightarrow M, \qquad\qquad q_{i,\boldsymbol{g}}^k : M \twoheadrightarrow x^{\boldsymbol{g}} Y_i,$$

the injection and projection for each copy of $x^{\boldsymbol{g}} Y_i$ in $x^{\boldsymbol{g}} Y_i^{\oplus k'_{i,\boldsymbol{g}}}$, given by the universal properties of the direct sum. We choose $i_0 \in \{1, \ldots, m\}$ and $\boldsymbol{g}_0 \in \mathbb{Z}^n$, and write X_0 for one of the copies of $x^{\boldsymbol{g}_0} X_{i_0}$ in $x^{\boldsymbol{g}_0} X_{i_0}^{\oplus k_{i_0,\boldsymbol{g}_0}}$. We consider the corresponding injection $\imath_0 : X_0 \hookrightarrow M$ and projection $\pi_0 : M \twoheadrightarrow X_0$.

By Proposition 3.2.3, we have $\mathrm{Id}_M = \prod_{(i,\boldsymbol{g},k)} f_{i,\boldsymbol{g}}^k \circ q_{i,\boldsymbol{g}}^k$ and thus $\mathrm{Id}_{X_0} = \pi_0 \left(\prod_{(i,\boldsymbol{g},k)} f_{i,\boldsymbol{g}}^k \circ q_{i,\boldsymbol{g}}^k \right) \imath_0$. Since X_0 is compact, the product restricts to a finite sum

$$\mathrm{Id}_{X_0} = \pi_0 \left(\sum_{(i,\boldsymbol{g},k)} f_{i,\boldsymbol{g}}^k \circ q_{i,\boldsymbol{g}}^k \right) \imath_0.$$

Then because $\mathrm{End}(X_0)$ is a local ring, there is a triple $(i_1, \boldsymbol{g}_1, k_1)$ such that $x = \pi_0 f_1 q_1 \imath$ is invertible for $f_1 = f_{i_1,\boldsymbol{g}_1}^{k_1}$ and $q_1 = q_{i_1,\boldsymbol{g}_1}^{k_1}$. We write $Y_1 = x^{\boldsymbol{g}_1} Y_{i_1}$. We take $e = q_1 \imath_0 x^{-1} \pi_0 f_1 \in \mathrm{End}(Y_1)$, which is an idempotent and therefore 0 or Id_{Y_1}. Since e factors through Id_{A_0}, it cannot be zero. Thus, $q_1 \imath_0 x^{-1}$ is an isomorphism with inverse $\pi_0 f_1$, such that $X_0 \cong Y_1$. Hence

$$X_{i_0} \cong x^{\boldsymbol{g}_1 - \boldsymbol{g}_0} Y_{i_1}.$$

Since i_0 is arbitrary and everything is symmetric, we obtain $m = m'$. Then we use Lemma 3.2.8 to cancel each $x^{\boldsymbol{g}_0} X_{i_0}$ with $x^{\boldsymbol{g}_1} Y_{i_1}$ and conversely (we can do it by induction since everything is locally finite), and thus $k_{i_0,\boldsymbol{g}_0} = k'_{i_1,\boldsymbol{g}_1}$. In conclusion, we can define $\sigma(i_0) = i_1$ and $\boldsymbol{g}_{i_0} = \boldsymbol{g}_1 - \boldsymbol{g}_0$. ■

Remark 3.2.10. *Note that if $\mathscr{C}$ is an $\mathrm{Ab}\,5$-category (see for example [71]), then Theorem 3.2.9 follows from the Krull–Schmidt–Remak–Azumaya theorem [5], in which case the indecomposable objects does not need to be compact and the direct sum can be arbitrary. There exist in the literature different versions of such theorems, for different contexts. However, we are interested in a quite particular case for which we did not find any version that exactly matches our needs.*

Corollary 3.2.11. *A c.b.l. Krull–Schmidt category possesses the cancelation property: for each triple A, B, C such that $A \oplus B \cong A \oplus C$ then $B \cong C$.*

From all of this, we easily obtain the following, which is a 'c.b.l.f. version' of Proposition 2.1.3.

Proposition 3.2.12. *Let $\mathscr{C}$ be a c.b.l. Krull–Schmidt category with collection of pairwise distinct indecomposable objects $\{X_i\}_{i\in I}$. Then its split Grothendieck group respects*

$$K_0^{\oplus}(\mathscr{C}) \cong \bigoplus_{i\in I} \mathbb{Z}(\!(x_1,\dots,x_n)\!)[X_i].$$

Consider a c.b.l. Krull–Schmidt category $\mathscr{C}$ with collection of pairwise distinct indecomposable objects $\{X_i\}_{i\in I}$. We define

$$K_{\boldsymbol{e}} := \bigoplus_{i\in I} V_{\boldsymbol{e}}[X_i],$$

for each $\boldsymbol{e} \in \mathbb{Z}^n$ and where $V_{\boldsymbol{e}} \subset \mathbb{Z}(\!(x_1,\dots,x_n)\!)$ is as in (3.1). Then we endow $K_0^{\oplus}(\mathscr{C})$ with a topology by using $\{K_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathbb{Z}^n}$ as basis of neighborhood of zero. We also refer to this topology as the $(x_1,\dots,x_n)$-adic topology. Note that the choice of X_i does not matter for defining the topology.

Proposition 3.2.13. *The split Grothendieck group $K_0^{\oplus}(\mathscr{C})$ of a c.b.l. Krull–Schmidt category $\mathscr{C}$ endowed with the $(x_1,\dots,x_n)$-adic topology is a Hausdorff topological module over $\mathbb{Z}(\!(x_1,\dots,x_n)\!)$.*

Proof. It follows from similar arguments as in the proof of Proposition 3.1.10. We leave the details to the reader. ■

We also obtain the following nice property:

Proposition 3.2.14. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a $\mathbb{Z}^n$-graded, additive functor between c.b.l. Krull–Schmidt categories. Then it induces a continuous map*

$$K_0^{\oplus}(\mathscr{C}) \xrightarrow{[F]} K_0^{\oplus}(\mathscr{C}').$$

Proof. Suppose we have in $\mathscr{C}$ a c.b.l.f. direct sum

$$M \cong \bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{g}+\boldsymbol{e}} X_{\boldsymbol{g}}(M),$$

where $X_{\boldsymbol{g}}(M) = X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}}$ for $\{X_1, \dots, X_m\}$ distinct indecomposable objects of $\mathscr{C}$. Then for each $\boldsymbol{f} \in C$ we have

$$M \cong \left(\bigoplus_{0 < \boldsymbol{g} \preceq \boldsymbol{f}} x^{\boldsymbol{g}+\boldsymbol{e}} X_{\boldsymbol{g}}(M) \right) \oplus \left(\bigoplus_{\boldsymbol{g} > \boldsymbol{f}} x^{\boldsymbol{g}+\boldsymbol{e}} X_{\boldsymbol{g}}(M) \right),$$

where $\boldsymbol{g} \in C$. Because F is additive and commutes with the grading shift, and because there is only finitely many $\boldsymbol{g} \in C$ such that $0 < \boldsymbol{g} \preceq \boldsymbol{f}$, we obtain

$$FM \cong \left(\bigoplus_{0 < \boldsymbol{g} \preceq \boldsymbol{f}} x^{\boldsymbol{g}+\boldsymbol{e}} F X_{\boldsymbol{g}}(M) \right) \oplus F \left(\bigoplus_{\boldsymbol{g} > \boldsymbol{f}} x^{\boldsymbol{g}+\boldsymbol{e}} X_{\boldsymbol{g}}(M) \right). \tag{3.5}$$

We put $F(M_{>\boldsymbol{f}}) := F\left(\bigoplus_{\boldsymbol{g} > \boldsymbol{f}} x^{\boldsymbol{g}+\boldsymbol{e}} X_{\boldsymbol{g}}(M)\right)$. Moreover, let $\{X_i'\}_I$ be a complete collection of distinct indecomposable objects of $\mathscr{C}'$. Then we must have

$$FX_j \cong \bigoplus_{\boldsymbol{g} \in C_j} x^{\boldsymbol{g}+\boldsymbol{e}_j} X'_{\boldsymbol{g}}(FX_j)$$

for each $j \in \{1, \dots, m\}$. Similarly, FM and $F(M_{>\boldsymbol{f}})$ decompose as c.b.l.f. direct sums of elements in $\{X_i'\}$. Then by cancellation property and working inductively on the grading, since (3.5) holds for any $\boldsymbol{f} \in C$, we must have

$$FM \cong \bigoplus_{\boldsymbol{g} \in C} x^{\boldsymbol{g}+\boldsymbol{e}} F X_{\boldsymbol{g}}(M).$$

In particular $[FM] = \bigoplus_{\boldsymbol{g} \geq 0} x^{\boldsymbol{g}+\boldsymbol{e}} [F X_{\boldsymbol{g}}(M)]$. ∎

An example with modules

Here $\Bbbk$ is a field. We say that a $\mathbb{Z}^n$-graded $\Bbbk$-vector space $M = \bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} M_{\boldsymbol{g}}$ is *c.b.l.f. dimensional* if $\dim M_{\boldsymbol{g}} < \infty$ for all $\boldsymbol{g}$, and if there exists a cone C_M compatible with $<$, and $\boldsymbol{e} \in \mathbb{Z}^n$, such that $M_{\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C_M$. We call such C_M a *grading cone of* M and $\boldsymbol{e}$ is the *minimal degree* of M. Similarly, a $\mathbb{Z}^n$-graded

R-module M is *c.b.l.f. generated* (cone bounded, locally finitely generated) if there is a collection of homogeneous elements $\{x_i\}_{i\in I} \subset M$ generating $M = \sum_{i\in I} Rx_i$ such that $d_{\boldsymbol{g}} := \#\{x_i | \deg(x_i) = \boldsymbol{g}\} < \infty$ and $d_{\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C_M$.

Let R be a c.b.l.f. dimensional, $\mathbb{Z}^n$-graded $\Bbbk$-algebra. Consider the category $R\text{-mod}$ of $\mathbb{Z}^n$-graded R-modules with degree zero maps. Then let $R\text{-mod}_{\text{lf}}$ be the full subcategory of $R\text{-mod}$ consisting of c.b.l.f. dimensional modules. Similarly, $R\text{-pmod}_{\text{lfg}}$ is the full subcategory of c.b.l.f. generated, projective R-modules. There are embeddings of categories:

$$R\text{-pmod}_{\text{lfg}} \subset R\text{-mod}_{\text{lf}} \subset R\text{-mod}\,.$$

Remark 3.2.15. *We would like to point out that $R\text{-pmod}_{\text{lfg}}$ is not abelian (it does not have kernels), and thus does not fit in the framework of Azumaya [5], nor [92].*

Proposition 3.2.16. *Both $R\text{-pmod}_{\text{lfg}}$ and $R\text{-mod}_{\text{lf}}$ are c.b.l. additive categories. In addition, $R\text{-pmod}_{\text{lfg}}$ is c.b.l. Krull–Schmidt.*

Proof. They clearly both have all c.b.l.f. coproducts. For the first claim, we can suppose without loosing generality that we are in $R\text{-mod}_{\text{lf}}$. Let

$$M = \bigoplus_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}} \left(X_1^{\oplus k_{1,\boldsymbol{g}}} \oplus \cdots \oplus X_m^{\oplus k_{m,\boldsymbol{g}}} \right)$$

be a c.b.l.f. coproduct. We will show it is a biproduct in $R\text{-mod}_{\text{lf}}$ (and actually even in $R\text{-mod}$). Let Z be an object in $R\text{-mod}_{\text{lf}}$ with arrows $f_{i,\boldsymbol{g}} : Z \to x^{\boldsymbol{g}} X_i^{\oplus k_{i,\boldsymbol{g}}}$. If the number of non-zero arrows is finite, then we are in the classical case of additive categories and we are done. Therefore, we suppose that for each $\boldsymbol{g}$ there is $\boldsymbol{g}' > \boldsymbol{g}$ and i' such that $f_{i',\boldsymbol{g}'} \neq 0$. Choose any homogeneous element $z \in Z$. We claim that for almost all $(i, \boldsymbol{g})$ then $f_{i,\boldsymbol{g}}(z) = 0$. Indeed, suppose that it does not hold. Then for each $\boldsymbol{g}$ we can find $0 \neq f_{i',\boldsymbol{g}'}(z) \in X_{i'}$ with $\boldsymbol{g}' > \boldsymbol{g}$. But $\deg(f_{i',\boldsymbol{g}'}(z)) = \deg(z) - \boldsymbol{g}'$ which is absurd since $X_{i'}$ is cone bounded. Therefore $\prod_{(i,\boldsymbol{g})} f_{i,\boldsymbol{g}}$ defines a map $Z \to \coprod_i Y_i$, and we get a bijection

$$\mathrm{Hom}(Z, \coprod_i Y_i) \xrightarrow{\cong} \prod_i \mathrm{Hom}(Z, Y_i).$$

Since Z is arbitrary, this concludes the proof of the first claim.

For the second claim, we only need to show that endomorphism rings of indecomposable modules are local. Let $P = Re$ be an indecomposable projective module given by projection with a primitive idempotent e. Then $\mathrm{End}(P) = R_0 e$ is an indecomposable, finite-dimensional R_0-module. Take $f \in R_0 e$. By Fitting's lemma, f is either e or 0, and therefore either f or $f - e$ is invertible, implying that $\mathrm{End}(P)$ is local. Moreover, since R_0 is finite-dimensional, there is only finitely many distinct indecomposable objects. ■

Corollary 3.2.17. *Let R be as above. We extract from the collection of primitive idempotents $\{e_i\}_{i\in I}$ a sub-collection $\{e_j\}_{j\in J\subset I} \subset \{e_i\}_{i\in I}$ of non-equivalent idempotents. Then we obtain*

$$K_0^{\oplus}(R\text{-pmod}_{\text{lfg}}) \cong \bigoplus_{j\in J} \mathbb{Z}(\!(x_1,\dots,x_n)\!)[Re_j].$$

Remark 3.2.18. *In general, one can construct products in a category of graded modules by taking the direct sum of the products degree-wise: let $\{M_i\}_{i\in I}$ be a collection of $\mathbb{Z}^n$-graded R-modules then we can take*

$$\prod_{i\in I} M_i := \bigoplus_{\boldsymbol{g}\in\mathbb{Z}^n} \prod_{i\in I} (M_i)_{\boldsymbol{g}}.$$

If we take a c.b.l.f. product of c.b.l.f. modules, then each homogeneous part of the product is a finite product. In particular, the product coincides with the coproduct.

3.3 Topological Grothendieck group

We continue to revisit the notions presented in [65, Appendix]. Let X be an object in an abelian category $\mathscr{C}$. We call $\mathbb{N}$-filtration a sequence of sub-objects $X_i \hookrightarrow X$ for all $i \in \mathbb{N}$ such that $X_{i+1} \hookrightarrow X_i$ and $X_0 = X$. We write it as

$$\cdots \subset X_r \subset X_{r-1} \subset \cdots \subset X_1 \subset X_0 = X.$$

Such a filtration is called *Hausdorff* if the inverse limit $\varprojlim_i X_i \cong 0$. We say it has simple quotients if all quotients X_i/X_{i+1} are either trivial or simple. A Hausdorff, $\mathbb{N}$-filtration with simple quotients is called an $\mathbb{N}$-*composition series*, and we refer to the non-trivial quotient objects as *composition factors*.

Mimicking the definition of a c.b.l.f. coproduct in a $\mathbb{Z}^n$-graded category, there is an obvious notion of c.b.l.f. filtration.

Definition 3.3.1. *Let $\{S_1,\dots,S_m\}$ be a finite collection of objects in a $\mathbb{Z}^n$-graded abelian category $\mathscr{C}$. Consider an $\mathbb{N}$-filtration*

$$\cdots \subset X_r \subset X_{r-1} \subset \cdots \subset X_1 \subset X_0 = X,$$

such that each $X_r/X_{r+1} \cong x^{\boldsymbol{g}} S_i$ for some $i \in \{1,\dots,m\}$ and $\boldsymbol{g} \in \mathbb{Z}^n$. If $\kappa_{i,\boldsymbol{g}} := \#\{r \in \mathbb{N} | X_r/X_{r+1} \cong x^{\boldsymbol{g}} S_i\} < \infty$, then we say that the filtration is locally finite, *and we refer to $\kappa_{i,\boldsymbol{g}}$ as the* degree $\boldsymbol{g}$ multiplicity *of S_i. We call such filtration a* c.b.l.f.

filtration *if in addition, there exists a cone C compatible with $\prec$, and $\boldsymbol{e} \in \mathbb{Z}^n$ such that $\kappa_{i,\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C$.*

Let $\mathscr{C}$ be both a strictly $\mathbb{Z}^n$-graded abelian category and a c.b.l. additive category. Its Grothendieck group $G_0(\mathscr{C})$ is a $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$-module through (3.4) again. We would like to find hypothesis that allow to equip it with a topology compatible with the $(x_1, \dots, x_n)$-adic topology of $\mathbb{Z}(\!(x_1, \dots, x_n)\!)$. However, even if we were able to construct a compatible topology on $G_0(\mathscr{C})$, it would (in general) not be Hausdorff since we could have that M admits a Hausdorff c.b.l.f. filtration with quotient objects being $\{S_1, \dots, S_m\}$ and multiplicities $\kappa_{i,\boldsymbol{g}} \in \mathbb{N}$ but

$$[M] \neq \sum_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \left(\kappa_{1,\boldsymbol{g}}[S_1] + \cdots + \kappa_{m,\boldsymbol{g}}[S_m]\right).$$

For this reason, we define a "topological" version of the Grothendieck group of $\mathscr{C}$.

Definition 3.3.2. *The* topological Grothendieck group *of $\mathscr{C}$ is*

$$\boldsymbol{G}_0(\mathscr{C}) := G_0(\mathscr{C})/J(\mathscr{C}),$$

where $J(\mathscr{C})$ is generated by the elements

$$[M] - \sum_{\boldsymbol{g} \in \mathbb{Z}^n} x^{\boldsymbol{g}} \left(\kappa_{1,\boldsymbol{g}}[S_1] + \cdots + \kappa_{m,\boldsymbol{g}}[S_m]\right),$$

whenever M admits a Hausdorff c.b.l.f. filtration with quotient objects being elements in $\{S_1, \dots, S_m\}$ and multiplicities $\kappa_{i,\boldsymbol{g}} \in \mathbb{N}$.

Like before, an exact, graded functor that preserves c.b.l.f. filtrations induces a map on the topological Grothendieck groups.

3.3.1 C.b.l. Jordan–Hölder categories

Clearly, in general in an abelian category, we cannot hope to always have $\mathbb{N}$-composition series. Moreover, even if we have such infinite composition series, they do not have to be essentially unique. We propose to look into one condition that ensures the uniqueness of such filtrations.

Definition 3.3.3. *We say that an object X in an abelian category $\mathscr{C}$ is* stable for the filtrations *if for all pairs of $\mathbb{N}$-composition series*

$$0 \subset \cdots \subset X_1 \subset X_0 = X,$$
$$0 \subset \cdots \subset X'_1 \subset X'_0 = X,$$

then for each $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $X'_k \subset X_i$.

Remark 3.3.4. *It is important to note here that the condition compares only $\mathbb{N}$-composition series and not arbitrary Hausdorff $\mathbb{N}$-filtrations. For example, as a $\Bbbk$-vector space, $\Bbbk[x,y]$ is stable for the filtrations, but if we try to compare the filtrations:*

$$0 \subset \cdots \subset x^2\Bbbk[x] \subset x\Bbbk[x] \subset \Bbbk[x] \subset \Bbbk[x,y],$$
$$0 \subset \cdots \subset y^2\Bbbk[y] \subset y\Bbbk[y] \subset \Bbbk[y] \subset \Bbbk[x,y],$$

they would not respect the condition. Of course, it is not a problem since $\Bbbk[x,y]/\Bbbk[x]$ is not simple.

We will now focus on proving the following theorem:

Theorem 3.3.5. *Let X be an object stable for the filtrations. Suppose X admits two locally finite, $\mathbb{N}$-composition series*

$$0 \subset \cdots \subset X_1 \subset X_0 = X,$$
$$0 \subset \cdots \subset X_1' \subset X_0' = X,$$

with respective multiplicities $\kappa_{i,\boldsymbol{g}}$ and $\kappa'_{i,\boldsymbol{g}}$, and pairwise distinct simple quotients $\{S_1,\dots,S_m\}$ and $\{S_1',\dots,S_{m'}'\}$. Then $m = m'$, and there exists a permutation $\sigma \in S^m$ such that $S_i \cong x^{\boldsymbol{g}_i} S'_{\sigma(i)}$ with $\boldsymbol{g}_i \in \mathbb{Z}^n$. Moreover, $\kappa_{i,\boldsymbol{g}} = \kappa_{\sigma(i),\boldsymbol{g}+\boldsymbol{g}_i}$.

Let us recall some results in abelian categories.

Lemma 3.3.6. *Let M, N be sub-objects of X such that $M + N \not\subset N$. If both X/M and X/N are simple, then*

$$M/(M \cap N) \cong X/N.$$

Proof. It follows directly from the second isomorphism theorem together with the fact that $M + N \cong X$. ■

Lemma 3.3.7. *Let M and N be sub-objects of some X such that N admits an $\mathbb{N}$-composition series*

$$0 \subset \cdots \subset N_1 \subset N_0 = N,$$

then

$$0 \subset \cdots \subset N_1 \cap M \subset N_0 \cap M = N \cap M,$$

is an $\mathbb{N}$-composition series of $N \cap M$, with each composition factor being a composition factor of the original filtration $N_\bullet$ on N, with multiplicities less or equal.

Proof. For all $r \in \mathbb{N}$ we have by the second isomorphism theorem

$$\frac{N_r \cap M}{N_{r+1} \cap M} \cong \frac{N_r \cap M}{N_{r+1} \cap (N_r \cap M)} \cong \frac{N_{r+1} + (N_r \cap M)}{N_{r+1}}.$$

If $N_r \cap M \subset N_{r+1}$ then the quotient is zero. Otherwise, we have $N_{r+1} \subsetneq N_{r+1} + (N_r \cap M)$, thus $N_r \cong N_{r+1} + (N_r \cap M)$. Therefore

$$\frac{N_r \cap M}{N_{r+1} \cap M} \cong N_r/N_{r+1},$$

concluding the proof. ∎

Proof of Theorem 3.3.5. We choose $i \in \{1, \dots, m\}$ and $\boldsymbol{g} \in \mathbb{Z}^n$. The condition of being locally finite implies that we can reach all X_j such that $X_j/X_{j+1} \cong x^{\boldsymbol{g}} S_i$ in a finite number of steps. Hence we can take a minimal sub-filtration

$$X_m \subset \cdots \subset X_1 \subset X_0 = X,$$

containing all occurrences of $x^{\boldsymbol{g}} S_i$. The idea is to prove by induction on m that $\kappa'_{i',\boldsymbol{g}'} \geqslant \kappa_{i,\boldsymbol{g}}$ for some $i' \in \{1, \dots, m'\}$ and $\boldsymbol{g}' \in \mathbb{Z}^n$ such that $x^{\boldsymbol{g}} S_i \cong x^{\boldsymbol{g}'} S'_{i'}$. Then by symmetry of the argument, we will obtain the equality, and we can define $\sigma(i) = i'$ and $\boldsymbol{g}_i = \boldsymbol{g}' - \boldsymbol{g}$.

Because X is stable for the filtrations, there exists $t \in \mathbb{N}$ such that $X'_t \subset X_1$. We can suppose t minimal so that $X'_{t-1} \not\subset X_1$. We consider the filtration

$$0 \subset \cdots \subset X'_{t+1} \subset X'_t \subset X_1 \cap X'_{t-1} \subset \cdots \subset X_1 \cap X'_2 \subset X_1 \cap X'_1 \subset X_1 \cap X'_0 = X_1. \tag{3.6}$$

We claim that $X'_{t-1}/X'_t \cong X/X_1$, and that the filtration (3.6) is a $\mathbb{N}$-composition series with composition factors

$$\dots, X'_t/X'_{t+1}, 0, X'_{t-2}/X'_{t-1}, \dots, X'_1/X'_2, X/X'_1.$$

Therefore, it can be rewritten as

$$0 \subset \cdots \subset X'_{t+1} \subset X_1 \cap X'_{t-1} \subset \cdots \subset X_1 \cap X'_2 \subset X_1 \cap X'_1 \subset X_1.$$

We can then finish the proof by applying the same argument recursively, working with the filtration above and

$$X_m \subset \cdots \subset X_2 \subset X_1,$$

which is of length $m - 1$.

We now prove our claim. First observe that if $X_1 \cap X'_{t-1} \neq X'_t$ then $X'_t \subset X_1 \cap X'_{t-1} \subset X'_{t-1}$ are strict inclusions, implying that X'_{t-1}/X'_t is not simple, which is absurd. Hence we get $(X_1 \cap X'_{t-1})/X'_t = 0$. We can suppose $X_1 \neq X'_1$, and therefore by Lemma 3.3.6 we get

$$\frac{X_1}{X_1 \cap X'_1} \cong \frac{X}{X'_1}.$$

Again, if $X_1 \cap X'_1 \neq X'_2$ (if not, then $t = 2$ are we are finished), the lemma gives

$$\frac{X_1 \cap X'_1}{X_1 \cap X'_2} = \frac{X_1 \cap X'_1}{(X_1 \cap X'_1) \cap X'_2} \cong \frac{X'_1}{X'_2}.$$

Suppose that $X'_{i-1}/(X_1 \cap X'_{i-1}) \cong X/X_1$, which is simple. We have for $i < t$,

$$\frac{X'_i}{X_1 \cap X'_i} \cong \frac{X'_{i-1}}{X_i \cap X'_{i-1}}.$$

Then in particular, since $X'_t = X_1 \cap X'_{t-1}$, we obtain

$$\frac{X'_{t-1}}{X'_t} \cong \frac{X}{X_1}.$$

Moreover, in general for $i < t-1$ we get

$$\frac{X_1 \cap X'_i}{X_1 \cap X'_{i+1}} \cong \frac{X'_i}{X'_{i+1}}.$$

This concludes the proof. ■

Remark 3.3.8. *As already mentioned, it is important to take an hypothesis like stable for the filtration in order to obtain Theorem 3.3.5. As a counter example, we can consider $\mathbb{Z}$ viewed as a module over itself. Then we consider a sequence of non-equal prime numbers $\{p_0, p_1, p_2, \dots\}$. We construct two Hausdorff filtrations*

$$0 \subset \cdots \subset \cdots \subset p_{2r} \cdots p_0\mathbb{Z} \subset \cdots \subset p_4 p_2 p_0 \mathbb{Z} \subset p_2 p_0 \mathbb{Z} \subset p_0\mathbb{Z} \subset \mathbb{Z},$$
$$0 \subset \cdots \subset \cdots \subset p_{2r+1} \cdots p_1\mathbb{Z} \subset \cdots \subset p_5 p_3 p_1 \mathbb{Z} \subset p_3 p_1 \mathbb{Z} \subset p_1\mathbb{Z} \subset \mathbb{Z}.$$

Both filtrations have simple quotients, given by $\{\mathbb{Z}/p_{2r}\mathbb{Z}\}$ and $\{\mathbb{Z}/p_{2r+1}\mathbb{Z}\}$ respectively, and these are not equivalent. It is clear that $p_{2r+1} \cdots p_1\mathbb{Z} \not\subset p_0\mathbb{Z}$ for all $r \geqslant 0$, and thus $\mathbb{Z}$ is, as expected, not stable for the filtrations.

Definition 3.3.9. *An object X admits a* c.b.l.f. composition series *if it admits an $\mathbb{N}$-composition series which is a c.b.l.f. filtration. An abelian category $\mathscr{C}$ being c.b.l.*

additive is called c.b.l. Jordan–Hölder *if each object is stable for the filtrations and admits a c.b.l.f. composition series.*

As a direct consequence of Theorem 3.3.5 we obtain the following:

Corollary 3.3.10. *The topological Grothendieck group of a c.b.l. Jordan–Hölder category $\mathscr{C}$ with collection of pairwise distinct simple objects $\{S_j\}_{j\in J}$ respects*

$$\boldsymbol{G}_0(\mathscr{C}) \cong \bigoplus_{j\in J} \mathbb{Z}(\!(x_1,\dots,x_n)\!)[S_j].$$

Moreover, under the hypothesis of Corollary 3.3.10, we endow $\boldsymbol{G}_0(\mathscr{C})$ with an $(x_1,\dots,x_n)$-adic topology, turning it into a Hausdorff topological $\mathbb{Z}(\!(x_1,\dots,x_n)\!)$-module, using

$$G_{\boldsymbol{e}} := \bigoplus_{j\in J} V_{\boldsymbol{e}}[S_j],$$

as neighborhood basis of zero. As before, we would like to say that a (graded) exact functor induces a continuous map on the topological Grothendieck groups. For this, we will add an hypothesis, similar to the definition of mixed abelian category (see for example [7]).

Definition 3.3.11. *We say that a c.b.l. Jordan–Hölder category is* strongly c.b.l. Jordan–Hölder *if there is only a finite number of distinct simple objects $\{S_1,\dots,S_m\}$, and they respect $\operatorname{Ext}^1_{\mathscr{C}}(S_i, x^{\boldsymbol{g}}S_j) = 0$ whenever $\boldsymbol{g} \le 0$. We refer to these objects as* primitive simples.

We recall that $\operatorname{Ext}^1_{\mathscr{C}}(A,B)$ counts the number of non-trivial extensions of A by B, that is short exact sequences $B \to E \to A$, up to equivalence and where $E \not\cong A \oplus B$.

Let $\mathscr{C}$ be a strongly c.b.l. Jordan–Hölder category with fixed set of primitive simples $\{S_1,\dots,S_m\}$. Therefore any $X \in \mathscr{C}$ admits a c.b.l.f. composition series with composition factors given by the primitive simples. We call such a filtration a *primitive composition series*. Moreover, each object X admits a unique maximal element $\deg(X) \in \mathbb{Z}^n$ such that X admits a primitive composition series with multiplicities $\kappa_{i,\boldsymbol{g}+\deg(X)} = 0$ whenever $\boldsymbol{g} \prec 0$.

Proposition 3.3.12. *Let $\mathscr{C}$ be a strongly c.b.l. Jordan–Hölder category. Let X be an object in $\mathscr{C}$, with primitive composition series contained in a cone $C = \{\deg(X) = \boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \dots\}$ with $\boldsymbol{c}_i \prec \boldsymbol{c}_{i+1}$. Then X admits an ordered primitive composition series*

$$0 \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X$$

where $X_r/X_{r+1} \cong x^{\boldsymbol{c}_r}\left(S_1^{\oplus\kappa_{1,c_r}} \oplus \cdots \oplus S_m^{\oplus\kappa_{m,c_r}}\right)$ is a finite direct sum of the primitive simples shifted by $\boldsymbol{c}_r$. Moreover, this composition series is unique when C is fixed.

In order to prove this proposition, we will need the following lemma, which is most-certainly a well-known result.

Lemma 3.3.13. *A filtration $X_3 \subset X_2 \subset X_1$ with $X_1/X_2 \cong A$ and $X_2/X_3 \cong B$ gives rise to a commutative diagram*

$$\begin{tikzcd}
X_3 \arrow[r, hook] \arrow[d, equal] & X_2 \arrow[r, two heads] \arrow[d, hook] & B \arrow[d, hook] \\
X_3 \arrow[r, hook] & X_1 \arrow[r, two heads] \arrow[d, two heads] & E \arrow[d, two heads] \\
& A \arrow[r, equal] & A
\end{tikzcd}$$

where each row and column are short exact sequences. Moreover, if the sequence $B \to E \to A$ splits, then there exists X_2' such that $X_3 \subset X_2' \subset X_1$, and $X_1/X_2' \cong B$ and $X_2'/X_3 \cong A$.

Proof. Composing the map $X_1 \twoheadrightarrow E$ with the splitting map $E \to B$ yields an epimorphism $p : X_1 \twoheadrightarrow B$. We put $X_2' = \ker p$. Then the universal property of the kernel induces a monomorphism $X_3 \hookrightarrow \ker p$ such that $\ker p/X_3 \cong A$. ■

Proof of Proposition 3.3.12. We take a primitive composition series $Y_\bullet$ of X and consider $\boldsymbol{c}_r$ for some $r \geqslant 0$. Since the filtration is c.b.l.f., we can extract a sub filtration of $Y_\bullet$ containing all occurrences of the primitive simples in degree $\preceq \boldsymbol{c}_r$. Then, thanks to Lemma 3.3.13 we can apply a bubble sorting algorithm to bring all of these primitive simples to the beginning of the filtration, exchanging $x^{\boldsymbol{g}} S_i \cong Y_s/Y_{s+1}$ with $x^{\boldsymbol{g}'} S_{i'} \cong Y_{s+1}/Y_{s+2}$ whenever $\boldsymbol{g}' \prec \boldsymbol{g}$. Therefore we obtain a filtration

$$X_{\preceq \boldsymbol{c}_r} \subset X,$$

where $X/X_{\preceq \boldsymbol{c}_r}$ is finite length and contains all composition factors of degree $\preceq \boldsymbol{c}_r$. We conclude by taking the filtration given by $X_r := X_{\preceq \boldsymbol{c}_r}$ for each $r \in \mathbb{N}$. ■

Proposition 3.3.14. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a $\mathbb{Z}^n$-graded, exact functor between strongly c.b.l. Jordan–Hölder categories. Then F induces a continuous map*

$$\boldsymbol{G}_0(\mathscr{C}) \xrightarrow{[F]} \boldsymbol{G}_0(\mathscr{C}').$$

Sketch of proof. The proof is similar to the one of Proposition 3.2.14, but where we truncate the ordered primitive composition series instead of the direct sum decomposition. We also use the fact that if there is a short exact sequence $M \to X \to X/M$ then the multiplicities of the primitive composition factors of X are the sum of the multiplicities in M plus the ones in X/M. ■

An example with modules

Consider as before a c.b.l.f. dimensional, $\mathbb{Z}^n$-graded $\Bbbk$-algebra R, where $\Bbbk$ is a field.

Proposition 3.3.15. *If R is positive, that is $R = \bigoplus_{\boldsymbol{g} \geq 0} R_{\boldsymbol{g}}$, and R_0 is semi-simple, then $R\text{-}\mathrm{mod}_{\mathrm{lf}}$ is strongly c.b.l. Jordan–Hölder.*

Proof. Since $\mathrm{End}(R) = R_0$ is finite dimensional, there are only finitely many projective indecomposable modules $\{P_i = Re_i\}_{i \in I}$, given by primitive idempotents $\{e_i \in R_0\}_{i \in I}$. Because R is positive, then $R_{>0} := \bigoplus_{\boldsymbol{g}>0} R_{\boldsymbol{g}}$ is a c.b.l.f. dimensional submodule of R, and $R_0 \cong R/R_{>0}$ is finite-dimensional. Since R_0 is semi-simple and $S_i := P_i/R_{>0}P_i$ is R_0-indecomposable, S_i is a finite-dimensional simple module. Furthermore, any simple R-module is isomorphic (up to grading shift) to S_i for some i. Since R is positive, we have $\mathrm{Ext}^1_R(S_i, x^{\boldsymbol{g}} S_j) = 0$ whenever $\boldsymbol{g} \prec 0$.

In addition, for any c.b.l.f. module $M = \bigoplus_{\boldsymbol{g} \in \mathbb{Z}^n} M_{\boldsymbol{g}}$ and $\boldsymbol{e} \in \mathbb{Z}^n$ then $M_{\geq \boldsymbol{e}} := \bigoplus_{\boldsymbol{g} \geq \boldsymbol{e}} M_{\boldsymbol{g}}$ is a submodule. In particular we get a short exact sequence

$$M_{>\boldsymbol{e}} \hookrightarrow M_{\geq \boldsymbol{e}} \twoheadrightarrow M_{\geq \boldsymbol{e}}/M_{>\boldsymbol{e}},$$

where $M_{\geq \boldsymbol{e}}/M_{>\boldsymbol{e}}$ is isomorphic to a finite direct sum of elements in $\{S_i\}_{i \in I}$. We obtain by the classical Jordan–Hölder theorem a finite filtration with simple quotients

$$M_{>\boldsymbol{e}} = M^0_{\boldsymbol{e}} \subset M^1_{\boldsymbol{e}} \subset \cdots \subset M^{r-1}_{\boldsymbol{e}} \subset M^r_{\boldsymbol{e}} = M_{\geq \boldsymbol{e}}.$$

Let $C_M = \{\deg(M) = \boldsymbol{c}_0 \prec \boldsymbol{c}_1 \prec \boldsymbol{c}_2 \prec \cdots\}$ be a grading cone of M. Applying the same reasoning on each $\boldsymbol{c}_i$ with $\boldsymbol{e} = \boldsymbol{c}_{i+1}$, we obtain by concatening all these filtrations a c.b.l.f. composition series of M.

We now prove M is stable for the filtrations. Suppose we have two composition series $X_\bullet$ and $X'_\bullet$ of M. Suppose by contradiction that $X'_k \not\subset X_i$ for all $k \in \mathbb{N}$. Then $X_i \subsetneq X_i + X'_k$. Moreover, we know that M/X_i is a finite dimensional $\Bbbk$-vector space and we get an infinite filtration

$$X_i \subset \cdots \subset X_i + X'_2 \subset X_i + X'_1 \subset X.$$

As vector spaces we have $X'_j \cong X'_{j+1} \oplus H_j$ for some H_j. We can find $H_j \not\subset X_i$ for arbitrary big j since $X'_k \cong \bigoplus_{j \geqslant k} H_k$, and thus otherwise we would have $X'_k \subset X_i$. Then we write $\tilde{H}_j = H_j/H_j \cap X_i$ and $\bigoplus_j \tilde{H}_j$ is infinite dimensional. Then we have $M \cong X_i \oplus \bigoplus_{j \geqslant 0} \tilde{H}_j$, which contradicts the fact that M/X_i is finite dimensional. ∎

Corollary 3.3.16. *Let $R \cong \bigoplus_{i\in I} Re_i$ be as in Proposition 3.3.15 and take a collection $\{e_j\}_{j\in J\subset I}$ of non-equivalent idempotents. Then we have*

$$\boldsymbol{G}_0(R\text{-mod}_{\mathrm{lf}}) \cong \bigoplus_j \mathbb{Z}(\!(x_1, \dots, x_n)\!)[S_j],$$

with $S_j := Re_j/R_{>0}e_j$.

Since $[P_i]$ can be written as $f_i(\boldsymbol{x})[S_i]$ for some $f_i(\boldsymbol{x}) \in \mathbb{Z}_C[\![x_1, \dots, x_n]\!]$ with $f_i(0) = 1$, we also have $[S_i] = f_i^{-1}(\boldsymbol{x})[P_i]$. Therefore

$$\boldsymbol{G}_0(R\text{-mod}_{\mathrm{lf}}) \cong \bigoplus_j \mathbb{Z}(\!(x_1, \dots, x_n)\!)[P_j]. \tag{3.7}$$

However, with the current framework, this isomorphism is purely formal, and has no categorical meaning (e.g. it involves minus signs). In the dg-setting, we will be able to reinterpret it using the projective resolution of S_i.

3.4 Triangulated topological Grothendieck group

In a $\mathbb{Z}^n$-graded triangulated category $\mathcal{T}$, we define the notion of *c.b.l.f. coproduct* as follows:

- take a a finite collection of objects $\{K_1, \dots, K_m\}$ in $\mathcal{T}$;
- consider a coproduct of the form

$$\coprod_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}}(K_{1,\boldsymbol{g}} \oplus \cdots \oplus K_{m,\boldsymbol{g}}), \qquad \text{with} \qquad K_{i,\boldsymbol{g}} = \bigoplus_{j=1}^{k_{i,\boldsymbol{g}}} K_i[h_{i,j,\boldsymbol{g}}],$$

  where $k_{i,\boldsymbol{g}} \in \mathbb{N}$ and $h_{i,j,\boldsymbol{g}} \in \mathbb{Z}$;
- there exists a cone C compatible with $\prec$, and $\boldsymbol{e} \in \mathbb{Z}^n$ such that for all j we have $k_{j,\boldsymbol{g}} = 0$ whenever $\boldsymbol{g} - \boldsymbol{e} \notin C$;
- there exists $h \in \mathbb{Z}$ such that $h_{i,j,\boldsymbol{g}} \geqslant h$ for all $i, j, \boldsymbol{g}$.

Remark 3.4.1. *This definition ensures that if we forget the homological grading (i.e. $\mathbb{Z}$-grading coming from the translation functor), then a c.b.l.f. coproduct remains c.b.l.f. for the $\mathbb{Z}^n$-grading. Moreover, the homological degree is bounded from below.*

In this section, we will assume we work with a triangulated category $\mathcal{C}$ that is a full subcategory of some triangulated category $\mathcal{T} \supset \mathcal{C}$ admitting arbitrary (i.e. infinite) coproducts. We also suppose that $\mathcal{C}$ is c.b.l. additive (for c.b.l.f. coproducts defined as above).

Following the terminology of [37], the *Milnor colimit* $\mathrm{MColim}_{r\geqslant 0}(f_r)$ of a collection of arrows $\{X_r \xrightarrow{f_r} X_{r+1}\}_{r\in\mathbb{N}}$ in $\mathcal{T}$ is the mapping cone fitting inside the following distinguished triangle

$$\coprod_{r\in\mathbb{N}} X_r \xrightarrow{1-f_\bullet} \coprod_{r\in\mathbb{N}} X_r \to \mathrm{MColim}_{r\geqslant 0}(f_r) \to \coprod_{r\in\mathbb{N}} X_r[1],$$

where the left arrow is given by the infinite matrix

$$1 - f_\bullet := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -f_0 & 1 & 0 & 0 & \cdots \\ 0 & -f_1 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Remark 3.4.2. *Sometimes the Milnor colimit is also called 'homotopy colimit' in the literature (e.g. in [74]).*

Given a collection of arrows $\{X_r \xrightarrow{f_r} X_{r+1}\}_{r\in\mathbb{N}}$ in $\mathcal{T}$ and $m > 0$, we obtain by the axiom (5.) of triangulated category a commutative diagram

$$\begin{tikzcd}
\coprod_{r=0}^{m-1} X_r \arrow[r] \arrow[d] & \coprod_{r=0}^{m} X_m \arrow[r] \arrow[d] & X_r \arrow[r] \arrow[d] & \coprod_{r=0}^{m-1} X_r[1] \arrow[d] \\
\coprod_{r\in\mathbb{N}} X_r \arrow[r, "1-f_\bullet"] & \coprod_{r\in\mathbb{N}} X_r \arrow[r] & \mathrm{MColim}_{r\geqslant 0}(f_r) \arrow[r] & \coprod_{r\in\mathbb{N}} X_r[1],
\end{tikzcd}$$

where the rows are distinguished triangles, and the two vertical arrows on the left are canonical inclusions. For each $r \geqslant m$, we write X_r/X_m for the mapping cone $\mathrm{Cone}(X_m \to X_r)$ of the map $f_{r-1} \circ \cdots \circ f_m$. Then we get by the 3x3 lemma [54, Lemma 2.6] a commutative diagram

$$\begin{tikzcd}
X_m \arrow[r] \arrow[d] & X_m \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_m[1] \arrow[d] \\
X_r \arrow[r] \arrow[d] & X_{r+1} \arrow[r] \arrow[d] & X_{r+1}/X_r \arrow[r] \arrow[d] & X_r[1] \arrow[d] \\
X_r/X_m \arrow[r, "f_r^m"] \arrow[d] & X_{r+1}/X_m \arrow[r] \arrow[d] & X_{r+1}/X_r \arrow[r] \arrow[d] & X_r/X_m[1] \arrow[d] \\
X_m[1] \arrow[r] & X_m[1] \arrow[r] & 0 \arrow[r] & X_m[2],
\end{tikzcd}$$

where each row and column is a distinguished triangle.

In the end, we obtain a commutative diagram

$$\begin{tikzcd}
\coprod_{r>m} X_m \arrow[r, "1-1_\bullet"] \arrow[d] & \coprod_{r>m} X_m \arrow[r] \arrow[d] & X_m \arrow[r] \arrow[d] & \coprod_{r\geqslant 0} X_r[1] \arrow[d] \\
\coprod_{r>m} X_r \arrow[r, "1-f_\bullet"] \arrow[d] & \coprod_{r>m} X_r \arrow[r] \arrow[d] & \mathrm{MColim}_{r>m}(f_r) \arrow[r] \arrow[d] & \coprod_{r>m} X_r[1] \arrow[d] \\
\coprod_{r>m} X_r/X_m \arrow[r, "1-f_\bullet^m"] \arrow[d] & \coprod_{r>m} X_r/X_m \arrow[r] \arrow[d] & \mathrm{MColim}_{r>m}(f_r^m) \arrow[r] \arrow[d] & \coprod_{r>m} X_r/X_m[1] \arrow[d] \\
\coprod_{r>m} X_m[1] \arrow[r] & \coprod_{r>m} X_m[1] \arrow[r] & X_m[1] \arrow[r] & \coprod_{r\geqslant 0} X_r[2],
\end{tikzcd}$$

where each row and columns is a distinguished triangle.

In particular, we have a distinguished triangle

$$X_m \to \mathrm{MColim}_{r\geqslant 0}(f_r) \to \mathrm{MColim}_{r>m}(f_r^m) \to X_m[1],$$

and X_m is a finite iterated extension of X_{r+1}/X_r for $r \leqslant m$:

$$\begin{tikzcd}
X_0 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & X_2 \arrow[rr] \arrow[dl] & & \dots \arrow[rr] \arrow[dl] & & X_m. \arrow[dl] \\
& X_1/X_0 \arrow[ul, "{[1]}"] & & X_2/X_1 \arrow[ul, "{[1]}"] & & X_3/X_2 \arrow[ul, "{[1]}"] & & X_m/X_{m-1} \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

As a consequence, in the triangulated Grothendieck group of $\mathcal{T}$ we have

$$[X_m] = [X_0] + \sum_{r=0}^{m-1} [X_{r+1}/X_r],$$

and thus, since $\mathrm{MColim}_{r\geqslant 0}(f_r) \cong \mathrm{MColim}_{r>m}(f_r)$, we also get,

$$[\mathrm{MColim}_{r\geqslant 0}(f_r)] = [\mathrm{MColim}_{r>m}(f_r^m)] + [X_0] + \sum_{r=0}^{m-1} [X_{r+1}/X_r],$$

for all $m \geqslant 0$.

Definition 3.4.3. *Let $\{K_1, \dots, K_m\}$ be a finite collection of objects in $\mathscr{C}$, and let $\{E_r\}_{r\in\mathbb{N}}$ be a family of direct sums of $\{K_1, \dots, K_m\}$ such that $\bigoplus_{r\in\mathbb{N}} E_r$ is a c.b.l.f. direct sum of $\{K_1, \dots, K_m\}$. Let $\{M_r\}_{r\in\mathbb{N}}$ be a collection of objects in $\mathscr{C}$ with $M_0 = 0$, such that they fit in distinguished triangles*

$$M_r \xrightarrow{f_r} M_{r+1} \to E_r \to M_r[1].$$

Then we say that an object $M \in \mathscr{C}$ such that $M \cong_{\mathscr{T}} \mathrm{MColim}_{r\geqslant 0}(f_r)$ in $\mathscr{T}$ is a c.b.l.f. iterated extension of $\{K_1, \dots, K_m\}$*. The sequence $t_{i,\boldsymbol{g}} = \{h_{i,1,\boldsymbol{g}}, \dots, h_{i,k_{i,\boldsymbol{g}},\boldsymbol{g}}\}$ of occurences of $x^{\boldsymbol{g}} K_i[h_{i,j,\boldsymbol{g}}]$ as a direct summand in $\bigoplus_{r\in\mathbb{N}} E_r$ is called the* degree $\boldsymbol{g}$ multiplicity *of K_i.*

Definition 3.4.4. *The* topological triangulated Grothendieck group *of $\mathscr{C} \subset \mathscr{T}$ is given by*

$$\boldsymbol{K}_0^{\Delta}(\mathscr{C}) := K_0^{\Delta}(\mathscr{C})/T(\mathscr{C}),$$

where $T(\mathscr{C})$ is generated by elements

$$[M] - \sum_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}} \left([K_{1,\boldsymbol{g}}] + \cdots + [K_{m,\boldsymbol{g}}]\right), \quad \textit{where} \quad [K_{i,\boldsymbol{g}}] = \sum_{j=1}^{k_{i,\boldsymbol{g}}} (-1)^{h_{i,j,\boldsymbol{g}}} [K_i],$$

whenever M is a c.b.l.f. iterated extension of $\{K_1, \dots, K_m\}$ with K_i appearing with degree $\boldsymbol{g}$ multiplicity $t_{i,\boldsymbol{g}} = \{h_{i,1,\boldsymbol{g}}, \dots, h_{i,k_{i,\boldsymbol{g}},\boldsymbol{g}}\}$.

To keep notations simple, we write $[t_{i,\boldsymbol{g}}] := \sum_{j=1}^{k_{i,\boldsymbol{g}}} (-1)^{h_{i,j,\boldsymbol{g}}} \in \mathbb{Z}$, so that

$$[M] = \sum_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}} \left([t_{1,\boldsymbol{g}}][K_1] + \cdots + [t_{m,\boldsymbol{g}}][K_m]\right),$$

in $\boldsymbol{K}_0^{\Delta}(\mathscr{C})$.

3.4.1 Positive c.b.l.f. dg-algebras

For this section, we restrict to derived categories that appear from a particular class of $\mathbb{Z}^n$-graded dg-$\Bbbk$-algebras (A, d), where $A = \bigoplus_{(h,\boldsymbol{g})\in\mathbb{Z}\oplus\mathbb{Z}^n} A^h_{\boldsymbol{g}}$, and $\Bbbk$ is a field. Thus we will consider c.b.l. additive subcategories of $\mathscr{T} := \mathscr{D}(A, d)$.

Definition 3.4.5. *We say that (A, d) is a* positive c.b.l.f. dg-algebra *if*

1. *A is positive for the homological degree: $A^h_{\boldsymbol{g}} = 0$ whenever $h < 0$;*
2. *A is positive, c.b.l.f. dimensional for the $\mathbb{Z}^n$-grading: $\bigoplus_{h\in\mathbb{N}} A^h_{\boldsymbol{g}}$ is finite dimensional for each $\boldsymbol{g} \in \mathbb{Z}^n$, and $\sum_{(h,\boldsymbol{g})} \dim(A^h_{\boldsymbol{g}}) x^{\boldsymbol{g}} \in \mathbb{Z}_{C_A}[\![x_1, \dots, x_n]\!]$ for some cone C_A compatible with $\prec$;*
3. *A^0_0 is semi-simple as module over itself ;*
4. *$A^h_0 = 0$ whenever $h > 0$.*

Similarly, we say that a dg-module is c.b.l.f. dimensional *(resp.* c.b.l.f. generated*) if it is c.b.l.f. dimensional (resp. c.b.l.f. generated) for the $\mathbb{Z}^n$-grading, and bounded from below for the homological grading.*

We write $A_0 := A_0^0$ and $A_{>0} := \bigoplus_{\{(h,\boldsymbol{g})|\boldsymbol{g}>0\}} A_{\boldsymbol{g}}^h$. Since $d(A_0) = 0$, indecomposable relatively projective (A, d)-modules are in bijection with indecomposable projective A-modules. Let $\{P_i := Ae_i\}_{i\in I}$ be the finite set of indecomposable relatively projective modules, up to shift. For a shifted copy $P_i' = x^{\boldsymbol{g}} P_i[h]$ of P_i we write $\deg_x(P_i') := \boldsymbol{g}$ and $\deg_h(P_i') := h$. Note that $\deg_x(P_i')$ coincides with the minimal $\mathbb{Z}^n$-degree of P_i'.

Remark 3.4.6. *We would like to point out that the condition $A_0^h = 0$ whenever $h > 0$ ensures the graded Euler characteristic of each P_i is non-zero. Also, each P_i is non-acyclic.*

We see that $A_{>0}e_i$ is a sub-dg-module of P_i, since $A_0^1 = 0$. Then, there is a simple dg-module $S_i := P_i/A_{>0}e_i$. We write $\deg_*(S_i') := \deg_*(P_i')$ whenever $S_i' \cong P_i'/A_{>0}P_i'$.

Definition 3.4.7. *An (A, d)-module is called* c.b.l.f. cell module *if it satisfies property (P) and if it is, as graded A-module, at the same time c.b.l.f. generated for the $\mathbb{Z}^n$-grading and bounded from below for the homological grading.*
A c.b.l. compact module *is a direct summand of a c.b.l.f. cell module.*

We write $\mathcal{D}^{lc}(A, d)$ for the strictly full subcategory of $\mathcal{D}(A, d)$ generated by objects (quasi-)isomorphic to c.b.l. compact modules. Our goal now will be to prove that any object of $\mathcal{D}^{lc}(A, d)$ is quasi-isomorphic to a c.b.l.f. cell module.

Proposition 3.4.8. *Let (M, d_M) be an (A, d)-module which is projective c.b.l.f. generated as A-module. Then (M, d_M) is a c.b.l.f. cell module.*

Proof. As A-module, (M, d_M) decomposes as a c.b.l.f. direct sum of shifted elements in $\{P_i\}_{i\in I}$. The generators of M are contained in $\boldsymbol{e} + C_M$ for some $\boldsymbol{e} \in \mathbb{Z}^n$ and cone $C_M = \{0 = \boldsymbol{c}_0 \prec \boldsymbol{c}_1 \prec \cdots\}$ compatible with $\prec$. Moreover, the differential d_M when restricted to $x^{\boldsymbol{g}} P_i[h] \to x^{\boldsymbol{g}'} P_j[h']$ can be non zero only if $h > h'$ and $\boldsymbol{g} \geq \boldsymbol{g}'$. Then we can construct M_r as the sub-dg-module of (M, d_M) with underlying A-module given by the direct sum of $x^{\boldsymbol{g}} P_i[h]$ with $\boldsymbol{g} \prec \boldsymbol{e} + \boldsymbol{c}_r$. It yields an exhaustive filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r \subset M_{r+1} \subset \cdots \subset M,$$

of (M, d_M) such that M_{r+1}/M_r is, as A-module, a finite direct sum of $x^{\boldsymbol{e}+\boldsymbol{c}_r} P_i[h]$. Since the sum is finite, we can further refine the filtration by filtering on the homological degree. ∎

Because any c.b.l.f. cell module admits as underlying graded module a projective c.b.l.f. generated A-module, we also obtain the following result from the proof of Proposition 3.4.8.

Proposition 3.4.9. *Let M be a property (P) module in $\mathcal{D}^{lc}(A,d)$, and $C_M = \{0 = \boldsymbol{c}_0 \prec \boldsymbol{c}_1 \prec \boldsymbol{c}_2 \prec \cdots\}$ be a $\mathbb{Z}^n$-grading cone of M, and let $\boldsymbol{e}$ be the minimal $\mathbb{Z}^n$-degree of M. There exists a filtration*

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset M$$

such that F_{r+1}/F_r is isomorphic (as A-module) to a finite direct sum of indecomposable relatively projective $\{P_i'\}$ with $\deg_x(P_i') = \boldsymbol{c}_r + \boldsymbol{e}$. This filtration is unique in $(A,d)\text{-mod}$ whenever C_M is fixed.

Example 3.4.10. *In order to make things clearer, we will try to get an insight of the typical look of a c.b.l.f. cell module M when there is only one extra grading (thus $n = 1$), and only one indecomposable relatively projective module P_0 of degree 0. We suppose that P_0 appears with minimal x-degree $g \in \mathbb{Z}^n$ and homological degree $h \in \mathbb{Z}$ in the filtration of M. Write $P := x^g P_0[h]$. Then we can visualize M as:*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \iddots \\
\oplus_{\alpha_{2,0}} P[2] \arrow[d] & \oplus_{\alpha_{2,1}} xP[2] \arrow[dl] \arrow[d] & \oplus_{\alpha_{2,2}} x^2P[2] \arrow[dll] \arrow[dl] \arrow[d] & \cdots \\
\oplus_{\alpha_{1,0}} P[1] \arrow[d] & \oplus_{\alpha_{1,1}} xP[1] \arrow[dl] \arrow[d] & \oplus_{\alpha_{1,2}} x^2P[1] \arrow[dll] \arrow[dl] \arrow[d] & \cdots \\
\oplus_{\alpha_{0,0}} P & \oplus_{\alpha_{0,1}} xP & \oplus_{\alpha_{0,2}} x^2P & \cdots
\end{tikzcd}$$

where each $\alpha_{i,j} \in \mathbb{N}$. For j fixed, we have $\alpha_{k,j} = 0$ for $k \gg 0$. Note that for i fixed, then we can have $\alpha_{i,j} > 0$ for all $j \geqslant n_0$. A composition of arrows in the diagram represents where the differential can be non zero. Then, the ordered filtration of Proposition 3.4.9 takes bigger and bigger vertical slices of this diagram, starting from the left. Since the differential only goes to bottom/left, it is easy to see from here that these vertical slices are sub-dg modules. Moreover, because of $\alpha_{k,j} = 0$ for $k \gg 0$, the quotient of a slice by another one is a finite direct sum of shifted copies of P (as graded module).

The following lemma was inspired by [86, Lemma 09KN].

Lemma 3.4.11. *Let (M, d_M) be a c.b.l.f. dimensional (A,d)-module with grading cone C_M and minimal $\mathbb{Z}^n$-degree $\boldsymbol{e}$, and let C_A be a grading cone of (A,d). There exists a c.b.l.f. cell module (P, d_P) with grading cone $C_A + C_M$ such that:*

1. *there is an epimorphism $\pi : (P, d_P) \to (M, d_M)$ in $(A,d)\text{-mod}$;*

2. $\ker(d_P) \to \ker(d_M)$ *is surjective ;*
3. $\deg_{\boldsymbol{x}}(\ker(\pi)) \succ \deg_{\boldsymbol{x}}((M, d_M))$ *or* $\deg_h(\ker(\pi)) > \deg_h((M, d_M))$ *;*
4. $\dim \ker(\pi)^h_{\boldsymbol{e}} \leqslant \sum_{(a+b=h)} \dim M^a_{\boldsymbol{e}} \dim A^b_0$.

Proof. Take $t \in \mathbb{Z}$ such that $M = \bigoplus_{\{(\boldsymbol{g},h) | \boldsymbol{g} \succeq \boldsymbol{e} \vee h \geqslant t\}} M^h_{\boldsymbol{g}}$ and $M^t_{\boldsymbol{e}} \neq 0$. We also have $\ker(d_M) = \bigoplus_{\{(\boldsymbol{g},h) | \boldsymbol{g} \succeq \boldsymbol{e} \vee h \geqslant t\}} \ker(d_M)^h_{\boldsymbol{g}}$.

For each $(\boldsymbol{g}, h)$ we can choose a $\Bbbk$-basis $\{m^j_{\boldsymbol{g},h}\}_{j \in J_{\boldsymbol{g},h}}$ of $M^h_{\boldsymbol{g}}$ such that there is $k_j \in I$ such that $e_{k_j} m^j_{\boldsymbol{g},h} = m^j_{\boldsymbol{g},h}$. This is possible since given any basis $\{m_j\}$ of $M_{\boldsymbol{g},h}$ then $\langle e_k m_j \rangle_{k \in I, j \in J_{\boldsymbol{g},h}}$ is a generating family of $M_{\boldsymbol{g},h}$. Note that $e_i m^j_{\boldsymbol{g},h} = 0$ whenever $i \neq k_j$. We choose a similar basis $\{x^j_{\boldsymbol{g},h}\}_{j \in K_{\boldsymbol{g}}}$ for $\ker(d_M)_{\boldsymbol{g},h}$.

Moreover, the kernel of the surjection $A_0 e_{k_j} \twoheadrightarrow A_0 e_{k_j} m^j_{\boldsymbol{e},t} \subset M_{\boldsymbol{e},t} m^j_{\boldsymbol{e},t}$ must be zero by semi-simplicity of A_0. Therefore, it is an isomorphism. Then, since $M^t_{\boldsymbol{e}}$ is finitely generated, it is semi-simple over A_0. Thus, we can extract a subset $J'_{\boldsymbol{e},t} \subset J_{\boldsymbol{e},t}$ such that $\bigoplus_{j \in J'_{\boldsymbol{e},t}} A_0 e_{k_j} m^j_{\boldsymbol{e},t} \cong M_{\boldsymbol{e},t}$. We replace $J_{\boldsymbol{e},t}$ by $J'_{\boldsymbol{e},t}$ for the remaining of the proof. This will ensure that condition (3.) is respected.

We construct

$$P' := \bigoplus_{\substack{(\boldsymbol{g},h) \in \mathbb{Z}^n \times \mathbb{Z} \\ j \in J_{\boldsymbol{g},h}}} P_{k_j} \cdot m^j_{\boldsymbol{g},h},$$

where $P_{k_j} \cdot m^j_{\boldsymbol{g},h}$ is the relatively projective module given by P_{k_j} shifted in x-degree so that $\deg_x(P_{k_j} \cdot m^j_{\boldsymbol{g},h}) = \deg_x(m^j_{\boldsymbol{g},h})$. We set

$$d_{P'}(a \cdot m^j_{\boldsymbol{g},h}) := d_A(a) \cdot m^j_{\boldsymbol{g},h} + (-1)^{\deg_h(a)} a \cdot d_M(m^j_{\boldsymbol{g},h}),$$

for all $a \in P_{k_j}$. Then, since M is c.b.l.f. dimensional, by Proposition 3.4.8 $(P', d_{P'})$ is a c.b.l.f. cell module. The obvious surjection $P' \twoheadrightarrow M$ gives point (1.). Point (4.) is clearly respected by construction, and point (3.) comes from the fact that the surjection restricts to an isomorphism $(P')^t_{\boldsymbol{e}} \cong M^t_{\boldsymbol{e}}$. In order to also respect point (2.) we add

$$P := P' \oplus \bigoplus_{\boldsymbol{g} \succ \boldsymbol{e} \vee h > t} P_{k_j} \cdot x^j_{\boldsymbol{g},h},$$

with d_P induced by $d_{P'}$ and $d_P(a \cdot x^j_{\boldsymbol{g},h}) = d_A(a) \cdot x^j_{\boldsymbol{g},h}$. Then (P, d_P) respects all the requirements, concluding the proof. ■

Theorem 3.4.12. *Any c.b.l.f. dimensional (A, d)-module (M, d_M) admits a quasi-isomorphic cover by a c.b.l.f. cell module (P, d_P).*

Proof. The proof is similar to the one of [86, Lemma 09KP]. We set $(M_0, d_{M_0}) := (M, d_M)$. Then we construct inductively for each $i > 0$ short exact sequences

$$0 \to (M_i, d_{M_i}) \to (P_i, d_{P_i}) \xrightarrow{\pi_i} (M_{i-1}, d_{M_{i-1}}) \to 0,$$

with $(M_i, d_{M_i}) := \ker(\pi_i)$, using Lemma 3.4.11. Thus we get a resolution

$$\begin{tikzcd}
\cdots \arrow[rr, "f_3"] \arrow[dr, two heads, "\pi_3"'] & & P_2 \arrow[rr, "f_2"] \arrow[dr, two heads, "\pi_2"'] & & P_1 \arrow[rr, "f_1"] \arrow[dr, two heads, "\pi_1"'] & & P_0 \arrow[r, two heads, "\pi_0"] & M \arrow[r] & 0. \\
& \ker(\pi_2) \arrow[ur, hook] & & \ker(\pi_1) \arrow[ur, hook] & & \ker(\pi_0) \arrow[ur, hook] & & &
\end{tikzcd} \tag{3.8}$$

Then we put

$$P := \bigoplus_{i \geqslant 0} P_i,$$

with $\boldsymbol{x}$-grading induced by P_i and homological grading given by $P^h = \oplus_{(a+b=h)} P_a^b$. It is a c.b.l.f. generated projective A module thanks to points (3.) and (4.) from Lemma 3.4.11, and the fact that $A_0^h = 0$ whenever $h > 0$. We equip P with a differential d_P by setting

$$d_P(x) = f_a(x) + (-1)^a d_{P_a}(x),$$

for each $x \in P_a^b$. Therefore, thanks to Proposition 3.4.8, (P, d_P) is a c.b.l.f. cell module, coming with an epimorphism $\overline{\pi} : (P, d_P) \to (M, d_M)$ induced by $(M_0, d_{M_0}) \twoheadrightarrow (M, d_M)$.

Moreover, point (2.) of Lemma 3.4.11 tells us that $\pi_{i+1}(\ker d_{P_{i+1}}) = \ker(d_{P_i}) \cap \ker(\pi_i)$. Thus we have an exact sequence

$$\cdots \to \ker(d_{P_2}) \to \ker(d_{P_1}) \to \ker(d_{P_0}) \to \ker(d_M) \to 0.$$

Then, using also the exactness of (3.8), we conclude by Proposition 2.2.19 that $\overline{\pi}$ is a quasi-isomorphism. ■

Corollary 3.4.13. *Any c.b.l. compact (A, d)-modules M is quasi-isomorphic to a c.b.l.f. cell module.*

Proof. Suppose $P \cong M \oplus N$ in $\mathcal{D}(A, d)$ with P being a c.b.l.f. cell module. Thus we have a distinguished triangle

$$N \to P \xrightarrow{\pi_M} M \to N[1],$$

where π_M lifts to a map of (A, d)-modules. There is a map $\xi_M \in \mathrm{Hom}_{\mathcal{D}(A,d)}(P, P)$, given by composing projection and then injection of M in P. Again, since P is cofibrant, we can lift it to $(A, d)\text{-mod}$. In particular, $M' := \mathrm{im}(\xi_M)$ is a sub-(A, d)-module of P, and is quasi-isomorphic to M through the map $\mathrm{im}(\xi_M) \hookrightarrow P \xrightarrow{\pi_M} M$. Since P is c.b.l.f. dimensional, so is M'. Therefore we can apply Theorem 3.4.12. ■

Lemma 3.4.14. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a distinguished triangle in $\mathcal{D}(A,d)$. If two out of three terms in {X,Y,Z} are c.b.l.f. cell modules, then the third is quasi-isomorphic to a c.b.l.f. cell module.

Proof. We have three cases to consider:

- If X and Y are c.b.l.f. cell modules, then so is Z. Indeed, since X is cofibrant, f lifts to a map of (A,d)-modules. Then, Z is quasi-isomorphic to the mapping cone $\mathrm{Cone}(X \xrightarrow{f} Y)$. As A-module, the mapping cone decomposes as a direct sum of two projective c.b.l.f. generated A-modules, and therefore is projective c.b.l.f. generated itself. We conclude by applying Proposition 3.4.8.
- If X and Z are c.b.l.f. cell modules, then so is Y: by rotating the triangle we have a triangle $Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{Z}$. Then we can apply the same argument as before to show that Y is isomorphic to a c.b.l.f. cell module, since $Z[-1]$ is c.b.l.f. cell module whenever Z is.
- If Y and Z are c.b.l.f. cell modules, then so is X: we apply the same argument as above to show that $X[1]$ is isomorphic to a c.b.l.f. cell module. Applying the shift functor show that it is also the case for X.

This concludes the proof. ■

Corollary 3.4.15. *$\mathcal{D}^{lc}(A,d)$ is a triangulated subcategory of $\mathcal{D}(A,d)$.*

Proof. It is a direct consequence of Lemma 3.4.14 and Corollary 3.4.13. ■

Lemma 3.4.16. *Let (M,d_M) be a c.b.l.f. iterated extension of some c.b.l.f. cell modules $\{(K_1,d_1),\dots,(K_m,d_m)\}$ where K_i appears with degree $\boldsymbol{g}$ multiplicity $t_{i,\boldsymbol{g}} = \{h_{i,1,\boldsymbol{g},\dots,h_{i,k_{i,\boldsymbol{g}},\boldsymbol{g}}}\}$. Then (M,d_M) is quasi-isomorphic to a c.b.l.f. cell module $(M',d_{M'})$, having $\bigoplus_{i,\boldsymbol{g}} x^{\boldsymbol{g}} \bigoplus_{s=0}^{k_{i,\boldsymbol{g}}} K_i[h_{i,j,\boldsymbol{g}}]$ as underlying graded A-module.*

Proof. Let $\{K_1,\dots,K_m\}$ be a finite set of c.b.l.f. cell modules, and $\bigoplus_{r\in\mathbb{N}} E_r$ be a c.b.l.f. direct sum of elements in $\{K_1,\dots,K_m\}$. Consider a collection of iterated extensions $\{M_r\}_{r\in\mathbb{N}}$, $M_0 = 0$, given by distinguished triangles

$$M_r \xrightarrow{f_r} M_{r+1} \to E_r \to M_r[1].$$

Take $M := \mathrm{MColim}_{r\geqslant 0}(f_r)$. We will prove that M is quasi-isomorphic to a c.b.l.f. cell module. We can replace inductively each M_r by a quasi-isomorphic $M'_{r+1} := \mathrm{Cone}(E_r[-1] \to M'_r)$. The module M'_{r+1} decomposes as

$$M'_{r+1} \cong \underset{\circlearrowleft}{E_r} \overset{\curvearrowright}{\oplus} \underset{\circlearrowleft}{E_{r-1}} \overset{\curvearrowright}{\oplus} \dots \overset{\curvearrowright}{\oplus} \underset{\circlearrowleft}{E_0}$$

where the arrows represent the possible action of the differential on M'_{r+1}. Thus we have an inclusion $M'_r \subset M'_{r+1}$ as (A,d)-modules. Taking the direct limit of this filtration in $(A,d)\text{-mod}$ yields an (A,d)-module $(M', d_{M'})$, which decomposes as $M' \cong \bigoplus_{r\in\mathbb{N}} E_r$ as graded A-module, and which is quasi-isomorphic to (M, d_M). Then M' decomposes as A-module as a c.b.l.f. direct sum of projective c.b.l.f. generated A-modules, and thus is projective c.b.l.f. generated itself. By Proposition 3.4.8, $(M', d_{M'})$ is a c.b.l.f. cell module. ∎

Corollary 3.4.17. *$\mathcal{D}^{lc}(A,d)$ is equivalent to the full subcategory of $\mathcal{D}(A,d)$ given by object quasi-isomorphic to c.b.l.f. iterated extensions of compact objects.*

Proof. It is a direct consequence of Lemma 3.4.16 and Corollary 3.4.13. ∎

This result allows us to extend the definition of $\mathcal{D}^{lc}(A,d)$ to any $\mathbb{Z}^n$-graded dg-algebra, as being the triangulated subcategory of $\mathcal{D}(A,d)$ generated by c.b.l.f. iterated extensions of compact objects in $\mathcal{D}(A,d)$.

Corollary 3.4.18. *Let $f : (A,d) \to (B,d')$ be a quasi-isomorphism between $\mathbb{Z}^n$-graded dg-algebras. There is an equivalence of triangulated categories*

$$\mathcal{D}^{lc}(A,d) \cong \mathcal{D}^{lc}(B,d').$$

Proof. It is a classical fact that a quasi-isomorphism induces an equivalence of derived categories. The characterization of compact objects being categorical, we obtain an equivalence between compact derived categories. The same goes for c.b.l. iterated extensions. ∎

3.4.2 Back to Grothendieck groups

Our goal for this section is to introduce a topological Grothendieck group à la Achar–Stroppel [1] for $\mathcal{D}^{lc}(A,d)$, and describe it. We will then show it coincides with our notion of triangulated topological Grothendieck group.

Let M be a c.b.l.f. cell module. Given $\boldsymbol{g} \in \mathbb{Z}^n$, we pick a maximal $r \in \mathbb{Z}$ in Proposition 3.4.9 such that $\boldsymbol{c}_r \le \boldsymbol{g}$. We write $\beta_{\le \boldsymbol{g}}(M) := F_r$. Therefore, $\beta_{\le \boldsymbol{g}}(M)$ contains all copies of P'_i in the filtration of M with $\deg_x(P'_i) \le \boldsymbol{g}$. Also, if we consider the collection of maps $\{\imath_r : \beta_{\le \boldsymbol{c}_r} M \to \beta_{\le \boldsymbol{c}_{r+1}} M\}_{r\in\mathbb{N}}$, we obtain

$$M \cong \operatorname{MColim}_{r\geqslant 0}(\imath_r). \tag{3.9}$$

We would like to extend $\beta_{\le \boldsymbol{g}}$ to a functor $\beta_{\le \boldsymbol{g}} : \mathcal{D}^{lc}(A,d) \to \mathcal{D}^{lc}(A,d)$.

Lemma 3.4.19. *Let $f : X \to Y$ be a map between c.b.l.f. cell modules. Then we have an induced map $\beta_{\le \boldsymbol{g}} f : \beta_{\le \boldsymbol{g}} X \to \beta_{\le \boldsymbol{g}} Y$. This operation sends identity map to identity map and preserves compositions.*

Proof. There is an induced map $f_0 : X \to Y$ of graded A-modules. In this context, X and Y are both c.b.l.f. direct sums of $\{P_i\}_i$. Since $\mathrm{Hom}(P'_i, P'_j) = 0$ whenever $\deg_x(P'_i) \prec \deg_x(P'_j)$, we can restrict the map to $\bar{f}_0 : X_{\leq \boldsymbol{g}} \to Y_{\leq \boldsymbol{g}}$, where $X_{\leq \boldsymbol{g}}$ is the finite sub-direct sum of X given by all P'_i with $\deg_x(P'_i) \leq \boldsymbol{g}$ (and the same for Y). Therefore, we have that $\beta_{\leq \boldsymbol{g}} X \cong X_{\leq \boldsymbol{g}}$ as graded A-modules. We take $\beta_{\leq \boldsymbol{g}} f := \bar{f}_0$. ■

Proposition 3.4.20. *Let $M \in \mathscr{D}^{lc}(A,d)$. If X and Y are two c.b.l.f. cell modules, both quasi-isomorphic to M, then $\beta_{\leq x} X \cong \beta_{\leq x} Y$.*

Proof. Since X and Y are cofibrant and quasi-isomorphic, we get a commutative diagram in (A,d)-mod

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \beta_{\leq x} X \arrow[u, hook, "\imath_X"] \arrow[r, "\beta_{\leq x} f"'] & \beta_{\leq x} Y \arrow[u, hook, "\imath_Y"'] \end{tikzcd}$$

where f is a quasi-isomorphism, and both $\imath_X$ and $\imath_Y$ are monomorphisms of (A,d)-modules. We clearly have $\mathrm{Cone}(\beta_{\leq x} f) \cong \beta_{\leq x} \mathrm{Cone}(f)$, and therefore we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r] & \mathrm{Cone}(f) \\ \beta_{\leq x} X \arrow[u, hook, "\imath_X"] \arrow[r, "\beta_{\leq x} f"'] & \beta_{\leq x} Y \arrow[u, hook, "\imath_Y"] \arrow[r] & \beta_{\leq x} \mathrm{Cone}(f) \arrow[u, hook, "\imath_C"] \end{tikzcd}$$

Since $\mathrm{Cone}(f)$ is both cofibrant and acyclic, it is contractible. Thus there exists $h : \mathrm{Cone}(f) \to \mathrm{Cone}(f)[1]$ such that $d \circ h + h \circ d = \mathrm{Id}_{\mathrm{Cone}(f)}$. Then we get a map $\beta_{\leq x} h : \beta_{\leq x} \mathrm{Cone}(f) \to \beta_{\leq x} \mathrm{Cone}(f)[1]$ such that

$$\begin{aligned} \imath_C \circ (d \circ \beta_{\leq x} h + \beta_{\leq x} h \circ d) &= (d \circ h + h \circ d) \circ \imath_C \\ &= \imath_C. \end{aligned}$$

Since $\imath_C$ is a monomorphism, we obtain

$$d \circ \beta_{\leq x} h + \beta_{\leq x} h \circ d = \mathrm{Id}_{\beta_{\leq x} \mathrm{Cone}(f)}.$$

In conclusion, $\beta_{\leq x} \mathrm{Cone}(f) \cong \mathrm{Cone}(\beta_{\leq x} f)$ is contractible, and $\beta_{\leq x} f$ is a quasi-isomorphism. ■

Using Corollary 3.4.13, we extend $\beta_{\leq \boldsymbol{g}}$ to a well-defined functor

$$\beta_{\leq \boldsymbol{g}} : \mathscr{D}^{lc}(A,d) \to \mathscr{D}^{lc}(A,d).$$

In addition, since $\beta_{\leq \boldsymbol{g}}(M)$ is a finitely generated graded A-module for each $M \in \mathcal{D}^{lc}(A,d)$, the functor factorizes through

$$\beta^c_{\leq \boldsymbol{g}} : \mathcal{D}^{lc}(A,d) \to \mathcal{D}^c(A,d).$$

Proposition 3.4.21. *The functors $\beta_{\leq \boldsymbol{g}}$ and $\beta^c_{\leq \boldsymbol{g}}$ are exact.*

Proof. They clearly commute with the translation functor. Let

$$X \to Y \to Z \to X[1]$$

be a distinguished triangle in $\mathcal{D}^{lc}(A,d)$. Without loosing generality, we can replace each object by a c.b.l.f. cell module. Consider the union of grading cones $C_X + C_Y = \{\boldsymbol{c}_0 \prec \boldsymbol{c}_1 \prec \cdots\}$. Since the set $\{\mathrm{Cone}(\beta_{\leq \boldsymbol{c}_i} X \to \beta_{\leq \boldsymbol{c}_i} Y)\}_{i \in \mathbb{N}}$ gives an ordered filtration of $\mathrm{Cone}(X \to Y) \cong Z$, we have by Proposition 3.4.20 that

$$\beta_{\leq \boldsymbol{g}} Z \cong \beta_{\leq \boldsymbol{g}} \mathrm{Cone}(X \to Y) \cong \mathrm{Cone}(\beta_{\leq \boldsymbol{g}} X \to \beta_{\leq \boldsymbol{g}} X).$$

Therefore

$$\beta_{\leq \boldsymbol{g}} X \to \beta_{\leq \boldsymbol{g}} Y \to \beta_{\leq \boldsymbol{g}} Z \to \beta_{\leq \boldsymbol{g}} X[1]$$

is a distinguished triangle. ■

Mimicking the construction from [1], the *Achar–Stroppel topological Grothendieck group* of $\mathcal{D}^{lc}(A,d)$ is defined as

$$\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d)) := K_0^{\Delta}/T_A,$$

where

$$T_A := \{f \in K_0^{\Delta}(\mathcal{D}^{lc}(A,d)) | [\beta_{\leq \boldsymbol{g}}](f) = 0 \text{ for all } \boldsymbol{g} \in \mathbb{Z}^n\}. \tag{3.10}$$

It is also a $\mathbb{Z}(\!(x_1,\dots,x_n)\!)$-module. Thanks to (3.9), we know that if $[\beta^c_{\leq \boldsymbol{g}} M] = 0$ for all $\boldsymbol{g}$, then $[M] = 0$ in $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lf}(A,d))$. Thus we obtain a surjective map of $\mathbb{Z}(\!(x_1,\dots,x_n)\!)$-modules

$$\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d)) \twoheadrightarrow \boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d)).$$

Proposition 3.4.22. *There is an isomorphism*

$$\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d)) \cong \boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d)).$$

Proof. The idea of the proof is that any relation in $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d))$ can be rewritten as relations involving only symbols of relatively projective modules P_i. Then the relation can be obtained from an ordered iterated extension of relatively projectives, and thus holds in $\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d))$.

Let M be isomorphic to a c.b.l.f. iterated extension of $\{K_1,\dots,K_m\}$, such that

$$[M] - \sum_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}}\left([t_{1,\boldsymbol{g}}][K_1] + \cdots + [t_{m,\boldsymbol{g}}][K_m]]\right) = 0, \tag{3.11}$$

in $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d))$. Since K_i is a c.b.l.f. cell module, it can be itself rewritten as a c.b.l.f. iterated extension of indecomposable relatively projective modules $\{P_1,\dots,P_s\}$ such that

$$[K_i] - \sum_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}}\left([t^i_{1,\boldsymbol{g}}][P_1] + \cdots + [t^i_{s,\boldsymbol{g}}][P_s]\right) = 0,$$

in $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d))$. This also holds in $\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d))$ since we can use Proposition 3.4.9 to reorder the filtration. Then (3.11) is equivalent to some equation

$$[M] - \sum_{\boldsymbol{g}\in\mathbb{Z}^n} x^{\boldsymbol{g}}\left([t'_{1,\boldsymbol{g}}][P_1] + \cdots + [t'_{s,\boldsymbol{g}}][P_s]]\right) = 0. \tag{3.12}$$

Moreover, we obtain that M is isomorphic to a c.b.l.f. cell module with filtration giving (3.12), by Lemma 3.4.16. We conclude by using again Proposition 3.4.9, showing that (3.12) holds in $\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d))$, and therefore so does (3.11). ■

For M a c.b.l.f. cell module, taking the quotient filtration on $M/\beta_{\leq\boldsymbol{g}}(M)$ shows that it is also a c.b.l.f. cell module. Therefore we define the functor

$$\beta_{>\boldsymbol{g}} : \mathcal{D}^{lc}(A,d) \to \mathcal{D}^{lc}(A,d),$$

by setting $\beta_{>\boldsymbol{g}}(M) := M/\beta_{\leq\boldsymbol{g}}(M)$. Then, we get a functorial distinguished triangle

$$\beta_{\leq\boldsymbol{g}}M \to M \to \beta_{>\boldsymbol{g}}M \to \beta_{\leq\boldsymbol{g}}M[1].$$

Moreover, by similar arguments as in the proof of Proposition 3.4.21, we obtain that $\beta_{>\boldsymbol{g}}$ is also exact. Note that $\beta_{\leq\boldsymbol{g}} \circ \beta_{\leq\boldsymbol{g}} \cong \beta_{\leq\boldsymbol{g}}$. We write $\beta_{\leq\boldsymbol{g}}\mathcal{D}^{lc}(A,d)$ for the essential image of $\beta_{\leq\boldsymbol{g}}$ in $\mathcal{D}^{lc}(A,d)$. The same goes for $\beta_{>\boldsymbol{g}}$.

Inspired by [1, Definition 2.3], we define the following:

Definition 3.4.23. *Let (A,d_A) and (B,d_B) be c.b.l.f. positive dg-algebras. We say that a functor $F : \mathcal{D}(A,d_A) \to \mathcal{D}(B,d_B)$ that preserves c.b.l. compact modules and commutes with the $\mathbb{Z}^n$-grading shifts has* finite amplitude *if there exists $|F| \in \mathbb{Z}^n$ such that for all c.b.l.f. cell module $M \in \mathcal{D}(A,d_A)$ and $\boldsymbol{g} \in \mathbb{Z}^n$ we have*

$$F\beta_{>\boldsymbol{g}}M \cong \beta_{>\boldsymbol{g}+|F|}FM.$$

Lemma 3.4.24. *There is an isomorphism of abelian groups*

$$K_0^\Delta(\mathcal{D}^{lc}(A,d)) \cong K_0^\Delta(\beta_{\leq \boldsymbol{g}}\mathcal{D}^{lc}(A,d)) \oplus K_0^\Delta(\beta_{>\boldsymbol{g}}\mathcal{D}^{lc}(A,d)).$$

Proof. Straightforward as in proof of [1, Lemma 3.1]. ∎

Proposition 3.4.25. *Let $F : \mathcal{D}(A,d_A) \to \mathcal{D}(B,d_B)$ be an exact functor with finite amplitude. There is an induced map*

$$[F] : \boldsymbol{K}_0^\Delta(A,d_A) \to \boldsymbol{K}_0^\Delta(B,d_B).$$

Proof. We will show that there is an induced map $[F] : \boldsymbol{K}_0^{AS}(A,d_A) \to \boldsymbol{K}_0^{AS}(B,d_B)$, and use Proposition 3.4.22. By exactness we have a map

$$[F] : K_0^\Delta(A,d_A) \to \boldsymbol{K}_0^{AS}(B,d_B),$$

and thus we only need to prove $F(T_A) \subset T_B$, for T_A as in (3.10).

Take $f \in T_A$ and $\boldsymbol{g} \in \mathbb{Z}^n$. By Lemma 3.4.24 we obtain

$$f = [\beta_{\leq \boldsymbol{g}}]f + [\beta_{>\boldsymbol{g}}]f, \qquad [F]f = [\beta_{\leq \boldsymbol{g}+|F|}][F]f + [\beta_{>\boldsymbol{g}+|F|}][F]f.$$

We have $[\beta_{\leq \boldsymbol{g}}]f = 0$ and $[\beta_{>\boldsymbol{g}+|F|}][F]f = [F][\beta_{>\boldsymbol{g}}]f$. Therefore we deduce that $[F]f = [\beta_{>\boldsymbol{g}+|F|}][F]f$ and $[\beta_{\leq \boldsymbol{g}+|F|}][F]f = 0$. Since $\boldsymbol{g}$ is arbitrary, we conclude that $[F]f \in T_B$. ∎

The functor $\beta_{>\boldsymbol{e}} : \mathcal{D}^{lc}(A,d) \to \mathcal{D}^{lc}(A,d)$ is exact with finite amplitude, thus we have $\boldsymbol{K}_0^{AS}(\beta_{>\boldsymbol{e}}\mathcal{D}^{lc}(A,d)) \subset \boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d))$. We endow $\boldsymbol{K}_0^{AS}(\mathcal{D}^{lc}(A,d))$ with a $(x_1,\dots,x_n)$-adic topology using

$$\boldsymbol{T}_{\boldsymbol{e}} := \boldsymbol{K}_0^{AS}(\beta_{>\boldsymbol{e}}\mathcal{D}^{lc}(A,d)),$$

as neighborhood basis of zero. Then it becomes a Hausforff topological $\mathbb{Z}(\!(x_1,\dots,x_n)\!)$-module.

Theorem 3.4.26. *There is an isomorphism*

$$\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(A,d)) \cong \bigoplus_{i\in I} \mathbb{Z}(\!(x_1,\dots,x_\ell)\!)[P_i],$$

where the sum is over all distinct, indecomposable relatively projective P_i.

Proof. By construction, c.b.l.f. cell modules are spanned by the symbols of the $\{P_i\}_{i\in I}$ in the Grothendieck group. Thus there is a surjection

$$\bigoplus_{i\in I}\mathbb{Z}(\!(x_1,\dots,x_\ell)\!)[P_i] \twoheadrightarrow \boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(A,d)).$$

For a c.b.l.f. cell module M and a shifted simple S'_j we have that $\mathrm{HOM}_{(A,d)\text{-mod}}(M,S'_j)$ is a c.b.l.f. dimensional $\Bbbk$-vector space. Therefore, as in [74, Corollary 10.12], for each (non-acyclic) shifted simple S'_j the derived hom-functor

$$\mathrm{RHOM}(-,S'_j) : \mathcal{D}^{lc}(A,d) \to \mathcal{D}^{lc}(\Bbbk,0)$$

induces a map on the Grothendieck groups

$$[\mathrm{RHOM}(-,S'_j)] : \boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(A,d)) \to \boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(\Bbbk,0)).$$

In particular, we have

$$\mathrm{RHOM}(P'_i,S'_j) = \begin{cases}\mathrm{End}(S'_j), & \text{if } i=j \text{ and } \deg(P'_i)=\deg(S'_j),\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, $[\mathrm{End}(S'_j)] = \dim(\mathrm{End}(S'_j))[\Bbbk]$ in $K_0^\Delta(\mathcal{D}^{lc}(\Bbbk,0))$. For $(\Bbbk,0)$, we have that $\mathcal{D}^{lc}(\Bbbk,0)$ corresponds with the category defined in [1, §2.4], slightly generalized to encompass multigradings. Then our definition of topological Grothendieck group matches with the one in the reference. By the results in [1] we get that $\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(\Bbbk,0))$ is the completion of $K_0(\Bbbk\text{-gmod}) \cong \mathbb{Z}^n[x_1^{\pm1},\dots,x_n^{\pm1}]$ in $\mathbb{Z}(\!(x_1,\dots,x_\ell)\!)$. Therefore, $\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(\Bbbk,0))$ is the free $\mathbb{Z}(\!(x_1,\dots,x_\ell)\!)$-module generated by $[\Bbbk]$. Then any relation between the symbols $[P_i]$ in $\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(A,d))$ would imply that there is $\mathbb{Z}$-torsion in $\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(\Bbbk,0))$, which is a contradiction. ■

Proposition 3.4.27. *Let (M,d_M) be a c.b.l.f. cell module admitting a Hausdorff c.b.l.f. filtration*

$$0 \subset \cdots \subset M_2 \subset M_1 \subset M_0 = M,$$

with quotient objects $\{X_1,\dots,X_m\}$ and degree $(\boldsymbol{g},h)$ multiplicities $\kappa_{i,\boldsymbol{g},h}$ for all $i\in I$ and $(\boldsymbol{g},h)\in\mathbb{Z}^n\times\mathbb{Z}$. Then we have

$$[M] = \sum_{i,\boldsymbol{g},h}(-1)^h x^{\boldsymbol{g}}[X_i],$$

in $\boldsymbol{K}_0^\Delta(\mathcal{D}^{lc}(A,d))$.

Proof. We can suppose without loosing generality that each X_i is a simple module. Let $C_M = \{0 = \boldsymbol{c}_0 \prec \boldsymbol{c}_1 \prec \cdots\}$ be a $\mathbb{Z}^n$-grading cone for M and let $\boldsymbol{e}$ be the minimal $\mathbb{Z}^n$-degree of M. Then we can construct an ordered filtration (for the x-degree)

$$0 \subset \cdots \subset F_2 \subset F_1 \subset F_0 = M$$

using Proposition 3.3.15 where each F_{r+1}/F_r is a finite iterated extension of $x^{\boldsymbol{e}+\boldsymbol{c}_r} X_i[h]$ for different $h \in \mathbb{Z}$. Then M/F_r is a finite iterated extension in $\mathcal{D}^{lc}(A,d)$ of X_i with x-degree $\prec \boldsymbol{c}_r + \boldsymbol{e}$. Thus we have

$$[M] = [F_r] + [M/F_r] = [F_r] + \sum_{\substack{i,h \\ \boldsymbol{g} \prec \boldsymbol{e}+\boldsymbol{c}_r}} (-1)^h x^{\boldsymbol{g}} [X_i].$$

Then we use Theorem 3.4.12 to replace F_r by a c.b.l.f. cell module containing copies of P_i in x-degree $\geq \boldsymbol{e} + \boldsymbol{c}_r$. Therefore,

$$[\beta_{\prec \boldsymbol{e}+\boldsymbol{c}_r} M] = \sum_{\substack{i,h \\ \boldsymbol{g} \prec \boldsymbol{e}+\boldsymbol{c}_r}} (-1)^h x^{\boldsymbol{g}} [\beta_{\prec \boldsymbol{e}+\boldsymbol{c}_r} X_i].$$

Since $\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d))$ is Hausdorff, this concludes the proof. ■

Corollary 3.4.28. *There is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d)) \cong \bigoplus_{i \in I} \mathbb{Z}(\!(x_1, \dots, x_\ell)\!)[S_i],$$

where the direct sum is over all distinct simple modules S_i.

Proof. By Proposition 3.4.27, we have $[P_i] = f_i(\boldsymbol{x})[S_i]$ for some series $f_i(\boldsymbol{x}) \in \mathbb{Z}_C[\![x_1, \dots, x_n]\!]$ such that $f_i(0) = 1$. ■

Remark 3.4.29. *Note that any c.b.l.f. dimensional module admits an ordered c.b.l.f. composition series by first filtering on the x-degree in increasing order, and then refining the filtration by the homological degree in decreasing order.*

We now suppose (A,d) does not necessarily respect point (4.) of Definition 3.4.5.

Lemma 3.4.30. *Any primitive idempotent of $H(A,d)$ comes from a primitive idempotent of A through the projection*

$$A_0 \to A_0/d(A_0^1) \cong H(A,d)_0.$$

As a consequence, the set of indecomposable $H(A,d)$-modules is a truncation of the set of indecomposable (A,d)-modules.

Proof. Let $e \in H(A,d)_0$ be an indempotent. Thus $e^2 - e = d(h)$ in A for some $h \in A_0^1$. Since A has locally finite dimension, A_0 is a finite length subalgebra of A. By Fitting's lemma, there exists $k > 0$ such that

$$A_0 = \ker d(h)^k \oplus \operatorname{im} d(h)^k.$$

Then, as in [74, Lemma 9.6], by Newton's method [9, Theorem 1.7.3] we can find an idempotent f of A such that $f = e$ in $H(A,d)$. This proves the first claim.

The second claim follows from the first one and the fact that direct sum decompositions of A obtained from idempotents descend in the homology since $d(Ae) = d(A)e$. Also, the quotient by $d(A_0^1)$ could not identify orthogonal idempotents together. Indeed if we have $e_1 + e_2 = d(h)$ then $e_1^2 + e_1e_2 = e_1 d(h)$ and thus $e_1 = d(e_1 h)$, meaning that both e_1 and e_2 are killed in $H(A,d)$. ■

Note that any indecomposable (A,d)-module that is killed in $H(A,d)\text{-mod}$ in Lemma 3.4.30 will be also killed in $D(A,d)$ since it is acyclic. Indeed suppose we have $e = d(h)$ for some idempotent $e \in A_0$ and $h \in A_0^1$. Then we get $d(he) = e^2 = e$. From this, we obtain the following:

Proposition 3.4.31. *Let (A,d) be a $\mathbb{Z}^n$-graded dg-algebra respecting the points (1.),(2.) and (3.) of Definition 3.4.5. If $H(A,d)$ is a c.b.l.f. positive dg-algebra, then there are isomorphisms*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(A,d)) \cong \boldsymbol{K}_0^{\Delta}(\mathcal{D}^{lc}(H(A,d),0)) \cong \boldsymbol{G}_0(H(A,d)).$$

Let R be as in Proposition 3.3.15. Then, by going to $\mathcal{D}^{lc}(R,0)$, we obtain a categorical interpretation of (3.7), where the bar resolution of the simple S_j constructed in Theorem 3.4.12 coincides with its projective resolution viewed as an $(R,0)$-module.

Remark 3.4.32. *We would like to point out that all results leading to Theorem 3.4.26 and its corollary generalize to the case where (A,d) is not necessarily positive for the $\mathbb{Z}^n$-grading, but only c.b.l.f. for the $\mathbb{Z}^n$-grading and positive for the homological grading up to adding some hypothesis: it is enough that $A \cong \bigoplus P_i'$ where each P_i' is isomorphic (up to shift) to some relatively projective module $P_i = Ae_i$, and P_i respects the points (1.), (2.) and (4.) of Definition 3.4.5. This ensures that $\operatorname{Hom}(P_j', P_i) = 0$ whenever $\deg_x(P_j') < \deg_x(P_i)$ or $\deg_h(P_j') < \deg_h(P_i)$. It also allows to have Theorem 3.4.12.*

Chapter 4

Geometric categorification of $\mathfrak{sl}_2$ Verma modules

We construct a categorification of the universal Verma module $M(\lambda)$ for quantum $\mathfrak{sl}_2$, using the geometry of the Grassmannian varieties, as done in [65]. We start by recalling the categorification of the finite dimensional module $V(N)$ of Frenkel–Khovanov–Stroppel [24, §6], following the description in [47] (see also [15] for the non graded case). We then explain how to dg-enhance this construction and use it to construct a categorification of $M(\lambda)$. This approach is actually slightly different than the one presented in [65], where we work with Hochschild homology and supermodules instead of dg-enhancements and dg-modules.

Before delving into the details of the construction, what should we expect from a categorification of a highest weight module ? It should be a $\mathbb{Z}$-graded (or $\mathbb{Z}^2$-graded if the highest weight is formal) $\Bbbk$-linear category that decomposes as a direct sum $\mathcal{M} \cong \bigoplus_{\lambda \in R^\times} \mathcal{M}_\lambda$, where each $\mathcal{M}_\lambda$ is the categorification of the λ-weight space. Moreover it should come with exact functors

$$\mathcal{E} : \mathcal{M}_\lambda \to \mathcal{M}_{q^2\lambda}, \qquad \mathcal{F} : \mathcal{M}_\lambda \to \mathcal{M}_{q^{-2}\lambda},$$

such that at least in the Grothendieck group we have

$$[\mathcal{E}\mathcal{F}][X_\lambda] - [\mathcal{F}\mathcal{E}][X_\lambda] = \frac{\lambda - \lambda^{-1}}{q - q^{-1}}[X_\lambda], \tag{4.1}$$

for each object $X_\lambda \in \mathcal{M}_\lambda$. We would call such a structure a *weak categorification* (of a highest weight module). Of course, we would like that (4.1) is realized by a categorical relation between $\mathcal{E}$ and $\mathcal{F}$, and fix the natural transformations realizing this relation. In that case, it would be a *(strong) categorification* of the highest weight module.

4.1 Categorification of the finite dimensional modules

All coefficients appearing in (1.11) are positive quantum integers $[k]_q$ for $k \in \mathbb{N}$, and the commutator relation can be rewritten as

$$\begin{aligned} EFv_{N-2k} &= (FE + [N-2k]_q)v_{N-2k}, && \text{if } N-2k \geqslant 0, \\ FEv_{N-2k} &= (EF + [2k-N]_q)v_{N-2k}, && \text{if } N-2k \leqslant 0, \end{aligned}$$

where again all coefficients are positive. Therefore, one can expect to categorify the $\mathfrak{sl}_2$-commutator relation using a direct sum decomposition of functors. In particular we look for a category $\mathcal{V}(N) \cong \bigoplus_{k=0}^{N} \mathcal{V}_{N-2k}$ where each category $\mathcal{V}_{N-2k}$ is a categorification the weight space $V(N)_{N-2k}$. We look for endofunctors $\mathcal{E}, \mathcal{F}$ of $\mathcal{V}(N)$ such that

$$\begin{aligned} \mathcal{E}\mathcal{F}\,\mathrm{Id}_{N-2k} &\cong \mathcal{F}\mathcal{E}\,\mathrm{Id}_{N-2k} \oplus \bigoplus_{\ell=0}^{N-2k} q^{N-2k-1+2\ell}\,\mathrm{Id}_{N-2k}, \quad \text{if } N-2k \geqslant 0, \\ \mathcal{F}\mathcal{E}\,\mathrm{Id}_{N-2k} &\cong \mathcal{E}\mathcal{F}\,\mathrm{Id}_{N-2k} \oplus \bigoplus_{\ell=0}^{2k-N} q^{2k-N-1+2\ell}\,\mathrm{Id}_{N-2k}, \quad \text{if } N-2k \leqslant 0, \end{aligned} \tag{4.2}$$

where q denotes the shift the $\mathbb{Z}$-grading, and Id_{N-2k} is the identity functor on $\mathcal{V}_{N-2k}$. In order to keep notations short, we introduce the following

$$\oplus_{[k]_q}(-) := \bigoplus_{\ell=0}^{k-1} q^{1-k+2\ell}(-).$$

4.1.1 Geometry of the Grassmannian varieties

We follow the presentation given in [47, §6], but classical references for this are [27, Chapter 3] and [26, Chapter 10]. Let $G_k^N := \{V_k \in \mathbb{C}^N \mid \dim(V_k) = k\}$ be the Grassmannian manifold of k-planes in $\mathbb{C}^N$. Its cohomology ring $H(G_k^N)$ (over $\mathbb{Q}$) is generated by the Chern classes $X_{1,k}, \dots, X_{k,k}$ and $Y_{1,k}, \dots, Y_{N-k,k}$, modded out by the Whitney sum formula. Explicitly, we have

$$H(G_k^N) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}, Y_{1,k}, \dots, Y_{N-k,k}]/I_k^N,$$

where I_k^N is the ideal generated by all homogeneous terms in the equation

$$(1 + tX_{1,k} + t^2X_{2,k} + \cdots + t^kX_{k,k})(1 + tY_{1,n} + \cdots + t^{N-k}Y_{N-k,k}) = 1.$$

The cohomology ring is naturally $\mathbb{Z}$-graded with

$$\deg(X_{i,k}) = 2i, \qquad \deg(Y_{i,k}) = 2i.$$

The same applies for the two-step flag variety

$$G^N_{k,k+1} := \{V_k \subset V_{k+1} \subset \mathbb{C}^N |\, \dim(V_k) = k, \dim(V_{k+1}) = k+1\},$$

with Chern classes $X_{1,k}, \dots, X_{k,k}, Y_{1,k+1}, \dots, Y_{N-k-1,k+1}$ and ξ_{k+1}, resulting in

$$H(G^N_{k,k+1}) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}, \xi_{k+1}, Y_{1,k+1}, \dots, Y_{N-k-1,k+1}]/I^N_{k,k+1},$$

where $I^N_{k,k+1}$ is given by

$$(1+tX_{1,k}+\dots+t^kX_{k,k})(1+t\xi_{k+1})(1+tY_{1,k+1}+\dots+t^{N-k-1}Y_{N-k-1,k+1}) = 1.$$

Since ξ_{k+1} corresponds with the line bundle over V_{k+1}/V_k, we have $\deg(\xi_{k+1}) = 2$.

The forgetful maps induce maps on cohomology

$$\begin{tikzcd}[column sep=small] & G^N_{k,k+1} \arrow[dl] \arrow[dr] & \\ G^N_k & & G^N_{k+1} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd}[column sep=small] & H(G^N_{k,k+1}) & \\ H(G^N_k) \arrow[ur, "\varphi^N_k"] & & H(G^N_{k+1}) \arrow[ul, "\psi^N_{k+1}"'] \end{tikzcd} \tag{4.3}$$

turning $H(G^N_{k,k+1})$ into an $H(G^N_k)$-$H(G^N_{k+1})$-bimodule. The action is given by

$$\varphi^N_k : \begin{cases} X_{i,k} & \mapsto X_{i,k}, \\ Y_{i,k} & \mapsto Y_{i,k+1} + \xi_{k+1}Y_{i-1,k+1}, \end{cases}$$
$$\psi^N_{k+1} : \begin{cases} X_{i,k+1} & \mapsto X_{i,k} + \xi_{k+1}X_{i-1,k}, \\ Y_{i,k+1} & \mapsto Y_{i,k+1}, \end{cases}$$

with the understanding that $Y_{0,k} := X_{0,k} := 1$ and $X_{k+1,k} := Y_{N-k,k+1} := 0$. Since everything is commutative, we can also turn $H(G^N_{k,k+1})$ into a $H(G^N_{k+1})$-$H(G^N_k)$-bimodule, which we write $H(G^N_{k+1,k})$.

4.1.2 The categorification theorem

We reinterpret the results of [24] in the framework of §3.4, obtaining similar results as in [1]. In this context, $V(N)$ is constructed with ground ring $\mathbb{Q}(\!(q)\!) \supset \mathbb{Q}(q)$, with choice of additive order induced by $q \succ 0$. We view $H(G^N_k)$ as a positive, c.b.l.f. dimensional, $\mathbb{Z}$-graded dg-algebra concentrated in homological degree 0. We put

$$\mathcal{V}(N) := \bigoplus_{k=0}^N \mathcal{V}_{N-2k}, \qquad \mathcal{V}_{N-2k} := \mathcal{D}^{lc}(H(G^N_k), 0).$$

The bimodule $H(G^N_{k,k+1})$ is biprojective (i.e. projective as a left module and projective as a right module), and finitely generated as left and as right module. In particular, there are decompositions

$$q^{-N+k+1}H(G^N_{k,k+1}) \cong \oplus_{[N-k]_q} H(G^N_k), \qquad \text{as } H(G^N_k)\text{-left-module}, \quad (4.4)$$
$$q^{-k}H(G^N_{k,k+1}) \cong \oplus_{[k+1]_q} H(G^N_{k+1}), \qquad \text{as } H(G^N_{k+1})\text{-right-module}. \quad (4.5)$$

Therefore the functors

$$\mathscr{E}^N_k : H(G^N_{k+1})\text{-mod} \to H(G^N_k)\text{-mod},$$
$$\mathscr{F}^N_k : H(G^N_k)\text{-mod} \to H(G^N_{k+1})\text{-mod},$$

defined by

$$\mathscr{E}^N_k(-) := q^{k+1-N}H(G^N_{k,k+1}) \otimes_{H(G^N_{k+1})} (-),$$
$$\mathscr{F}^N_k(-) := q^{-k}H(G^N_{k+1,k}) \otimes_{H(G^N_k)} (-).$$

are both exact and preserve c.b.l. compact modules. Moreover, one of the results of [24] is that there are natural isomorphisms

$$\mathscr{E}^N_k\mathscr{F}^N_k \cong \mathscr{F}^N_{k-1}\mathscr{E}^N_{k-1} \oplus_{[N-2k]_q} \mathrm{Id}_k, \qquad \text{if } N-2k \geqslant 0,$$
$$\mathscr{F}^N_{k-1}\mathscr{E}^N_{k-1} \cong \mathscr{E}^N_k\mathscr{F}^N_k \oplus_{[2k-N]_q} \mathrm{Id}_k, \qquad \text{if } N-2k \leqslant 0.$$

Then $\mathscr{E}^N := \bigoplus_{k=0}^N \mathscr{E}^N_k$ and $\mathscr{F}^N := \bigoplus_{k=0}^N \mathscr{F}^N_k$ are exact and preserve c.b.l. compact modules. Therefore, they induce endofunctors on $\mathscr{V}(N)$, which have finite amplitude. Thus, the Grothendieck group of $\mathscr{V}(N)$ equipped with $[\mathscr{E}^N]$ and $[\mathscr{F}^N]$ is an $U_q(\mathfrak{sl}_2)$ weight module.

The algebra $H(G^N_k)$ admits a unique projective indecomposable module $P_k := H(G^N_k)$ (up to shift and isomorphism), and a unique simple module

$$S_k := P_k/H(G^N_k)_{>0}P_k \cong \mathbb{Q}.$$

Then Theorem 3.4.26 and Corollary 3.4.28 tells us that we have

$$\boldsymbol{K}^{\Delta}_0(\mathscr{V}_{N-2k}) \cong \mathbb{Z}(\!(q)\!),$$

and is generated either by $[P_k]$ or $[S_k]$.

An important property of this construction is that $\mathscr{E}^N_k$ and $\mathscr{F}^N_k$ are biadjoint (up to a grading shift), meaning that $\mathscr{E}^N_k$ is both the right adjoint and the left adjoint of $\mathscr{F}^N_k$. This symmetry can be interpreted as a categorical version of the symmetry induced by ρ in (1.1). Indeed, the right adjoint of $\mathcal{F}_k^N$ descends in the Grothendieck group as $\rho([\mathcal{F}_k^N])$, and the same for $\mathcal{E}_k^N$. Because of this, we write $\rho(\mathcal{F}_k^N)$ for the shifted copy of $\mathcal{E}_k^N$ that is right adjoint of $\mathcal{F}_k^N$. More explicitly we put

$$\rho(\mathcal{F}_k^N) := q^{N-2k-1}\mathcal{E}_k^N, \qquad\qquad \rho(\mathcal{E}_k^N) := q^{1+2k-N}\mathcal{F}_k^N.$$

Since $H(G_k^N)$ is commutative, any left module M can be viewed as right module $\bar{M}$. Then, we define the bifunctor

$$(-,-) : \mathcal{V}(N) \otimes_{\mathbb{Q}} \mathcal{V}(N) \to \mathcal{D}^{lc}(\mathbb{Q},0), \quad (M, M') := \bar{M} \otimes^{\mathrm{L}}_{(\oplus_k H(G_k^N))} M'.$$

It respects for all $M, M' \in \mathcal{V}(N)$,

- $(P_0, P_0) \cong (\mathbb{Q}, 0)$;
- $(\mathcal{F}_k^N M, M') \cong (M, \rho(\mathcal{F}_k^N)N)$ and $(\mathcal{E}_k^N M, M') \cong (M, \rho(\mathcal{E}_k^N)M')$;
- $\oplus_f(M, M') \cong (\oplus_f M, M') \cong (M, \oplus_f M')$ for any $f \in \mathbb{Z}(\!(q)\!)$,

where we identify $P_0 \cong \mathbb{Q}$ with its dg-version $(P_0, 0)$. As a consequence, it descends on $\boldsymbol{K}_0^{\Delta}(\mathcal{V})$ as the Shapovalov form (1.3). Moreover, we have

$$(P_k, S_{k'}) \cong \begin{cases} (\mathbb{Q}, 0), & \text{if } k = k', \\ 0, & \text{otherwise,} \end{cases}$$

which is a categorification of (1.8).

Theorem 4.1.1 ([24] revisited)**.** *The Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathcal{V}(N))$ is a $U_q(\mathfrak{sl}_2)$-weight module with action of E, F given by $[\mathcal{E}^N], [\mathcal{F}^N]$. Moreover, there is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{V}(N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong V(N),$$

sending $[P_k]$ to the canonical basis element v'_{N-2k}, and $[S_k]$ to the dual canonical basis element v^{N-2k}.

4.2 Categorification of the Verma module

Because of the close relationship between integrable modules and Verma modules (1.12), it is natural to construct a categorification of $M(\lambda)$ using similar tools: that is the geometry of the Grassmannian varieties. Moreover, since the weight spaces are indexed by the dimension of the planes inside the Grassmannians, and the Verma module having non-trivial weight space for arbitrarily low weight, it is also natural to consider the Grassmannian of k-planes inside the infinite space $\mathbb{C}^{\infty}$.

4.2.1 Geometry of the infinite Grassmannian varieties

The good news is that the cohomology ring of

$$G_k^\infty := \{V_k \subset \mathbb{C}^\infty | \dim(V_k) = k\},$$

is given by the inductive limit of $H(G_k^N)$ for $N \to \infty$. More explicitly, it is generated by the Chern classes $X_{1,k}, \dots, X_{k,k}$ and infinitely many $Y_{1,k}, \dots, Y_{i,k}, \dots$ with degrees $\deg(X_{i,k}) = \deg(Y_{i,k}) = 2i$, modded out again by the Whitney sum formula. Explicitly, we obtain

$$H(G_k^\infty) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}, Y_{1,k}, \dots, Y_{i,k}, \dots]/I_k^\infty,$$

where I_k^∞ is the ideal generated by the homogeneous components in t satisfying the equation

$$(1 + X_{1,k}t + \cdots + X_{k,k}t^k)(1 + Y_{1,k}t + \cdots + Y_{i,k}t^i + \cdots) = 1.$$

From this equation, we derive the following recursive formulas

$$Y_{0,k} = 1, \qquad Y_{i,k} = -\sum_{\ell=1}^{i} X_{\ell,k}Y_{i-\ell,k}, \tag{4.6}$$

where we put $X_{j,k} := 0$ whenever $j > k$. Therefore, $H(G_k^\infty)$ is isomorphic to the graded polynomial ring

$$H(G_k^\infty) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}], \qquad \deg(X_{i,k}) = 2i.$$

There are similar results for the 2-steps flag variety

$$G_{k,k+1}^\infty := \{V_k \subset V_{k+1} \subset \mathbb{C}^\infty | \dim(V_k) = k, \dim(V_{k+1}) = k+1\}.$$

Its cohomology ring respects

$$H(G_{k,k+1}^\infty) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}, \xi_{k+1}, Y_{1,k+1}, \dots, Y_{i,k+1}, \dots]/I_{k,k+1}^\infty,$$

with $\deg(X_{i,k}) = \deg(Y_{i,k+1}) = 2$ and $\deg(\xi_{k+1}) = 2$, and where $I_{k,k+1}^\infty$ is given by

$$(1 + X_{1,k}t + \cdots + X_{k,k}t^k)(1 + \xi_{k+1}t)(1 + Y_{1,k+1}t + \cdots + Y_{i,k+1}t^i + \cdots) = 1.$$

Again, we can rewrite recursively each $Y_{i,k+1}$ in terms of $X_{i,k+1}$ and ξ_{k+1}, so that

$$H(G_{k,k+1}^\infty) \cong \mathbb{Q}[X_{1,k}, \dots, X_{k,k}, \xi_{k+1}], \quad \deg(X_{i,k}) = 2i, \quad \deg(\xi_{k+1}) = 2.$$

As for the finite case, the forgetful maps induce maps on the cohomology

$$
\begin{tikzcd}
 & G^{\infty}_{k,k+1} \arrow[dl] \arrow[dr] & \\
G^{\infty}_{k} & & G^{\infty}_{k+1}
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & H(G^{\infty}_{k,k+1}) & \\
H(G^{\infty}_{k}) \arrow[ur, "\varphi_k"] & & H(G^{\infty}_{k+1}) \arrow[ul, "\psi_{k+1}"']
\end{tikzcd}
$$

explicitly given by

$$
\varphi_k : \begin{cases} X_{i,k} & \mapsto X_{i,k}, \\ Y_{i,k} & \mapsto Y_{i,k+1} + \xi_{k+1} Y_{i-1,k+1}, \end{cases}
$$
$$
\psi_{k+1} : \begin{cases} X_{i,k+1} & \mapsto X_{i,k} + \xi_{k+1} X_{i-1,k}, \\ Y_{i,k+1} & \mapsto Y_{i,k+1}, \end{cases}
$$

with the understanding that $Y_{0,k} := X_{0,k} := 1$ and $X_{k+1,k} := 0$.

A quick verification shows that

$$
H(G^{\infty}_{0,1}) \otimes_{H(G^{\infty}_{1})} H(G^{\infty}_{1,0}) \cong \mathbb{Q}[\xi],
$$

so that if we try to define $\mathscr{E}_0, \mathscr{F}_0$ as tensoring with the bimodule $H(G^{\infty}_{0,1})$, we would obtain $\mathscr{E}_0\mathscr{F}_0 \cong \oplus_{\ell \in \mathbb{N}} q^{2\ell} \operatorname{Id}_0$. It decategorifies as

$$
\sum_{\ell \in \mathbb{N}} q^{2\ell} = 1 + q^2 + q^4 + \cdots = \frac{1}{1-q^2}.
$$

Going to the dg-world and shifting the degree of $\mathbb{Q}[\xi]$ by $q[1]$, we obtain $-q(1 + q^2 + q^4 + \cdots)$, which coincides with $\frac{1}{q-q^{-1}}$. However, it still does not look quite like (4.1), since there are no λ appearing in the equation.

4.2.2 Dg-enhancement

Up to shift, we want to categorify $(1-\lambda^2)/(1-q^2)$. Evaluating $\lambda = q^N$ yields

$$
\frac{1-\lambda^2}{1-q^2} = \frac{1-q^{2N}}{1-q^2} = 1 + q^2 + \cdots + q^{2N-2}.
$$

We known the quantum polynomial $1 + q^2 + \cdots + q^{2-2N}$ is morally categorified by $\mathbb{Q}[\xi]/(\xi^N)$, and the fraction $1/(1-q^2)$ by $\mathbb{Q}[\xi]$. Therefore, we would like to obtain $\mathbb{Q}[\xi]/(\xi^N)$ as a difference (i.e. mapping cone) of two copies of $\mathbb{Q}[\xi]$, with one shifted in q-degree $2N$.

The obvious projection

$$\pi : \mathbb{Q}[\xi] \twoheadrightarrow \mathbb{Q}[\xi]/(\xi^N),$$

turns $\mathbb{Q}[\xi]/(\xi^N)$ into a $\mathbb{Q}[\xi]$-module. Then, it admits a free resolution

$$0 \to q^{2N}\mathbb{Q}[\xi] \xrightarrow{\xi^N} \mathbb{Q}[\xi] \twoheadrightarrow \mathbb{Q}[\xi]/(\xi^N) \to 0.$$

The graded Euler characteristic of this resolution matches with $(1 - q^{2N})(1 - q^2)$, so that we think of it as a categorification of this fraction. From the free resolution, we obtain a dg-algebra by taking the mapping cone

$$\mathrm{Cone}(q^{2N}\mathbb{Q}[\xi] \xrightarrow{\xi^N} \mathbb{Q}[\xi]) \cong (\mathbb{Q}[\xi] \otimes \textstyle\bigwedge^\bullet(\omega), d_N),$$

with differential

$$d_N : \mathbb{Q}[\xi] \otimes \textstyle\bigwedge^\bullet(\omega) \to \mathbb{Q}[\xi] \otimes \textstyle\bigwedge^\bullet(\omega), \qquad \begin{cases} \xi & \mapsto 0, \\ \omega & \mapsto \xi^N, \end{cases}$$

such that

$$H(\mathbb{Q}[\xi] \otimes \textstyle\bigwedge^\bullet(\omega), d_N) \cong \mathbb{Q}[\xi]/(\xi^N).$$

The last step consists in replacing the differential d_N by a trivial one, and introducing a λ-grading, so that we obtain a $\mathbb{Z}^2$-graded dg-algebra

$$(\mathbb{Q}[\xi] \otimes \textstyle\bigwedge^\bullet(\omega), 0),$$

with $\deg_q(\xi) = 2$, $\deg_q(\omega) = 0$ and $\deg_\lambda(\xi) = 0$, $\deg_\lambda(\omega) = 2$. Thus, its graded dimension is $(1 - \lambda^2)/(1 - q^2)$. Because of (1.11), it is natural to look for a similar pattern on §4.1.1 and §4.2.1.

Dg-enhancement of $H(G_k^N)$

The natural inclusion $G_k^N \to G_k^\infty$ induces a surjective map

$$\pi_k : H(G_k^\infty) \twoheadrightarrow H(G_k^N),$$

sending $X_{i,k} \in H(G_k^\infty)$ to $X_{i,k} \in H(G_k^N)$ (and the same for $Y_{i,k}$).

Lemma 4.2.1. *The kernel* $\ker \pi_k$ *is generated by the collection of the* k *elements* $Y_{N-k+1,k}, Y_{N-k+2,k}, \ldots, Y_N$.

Proof. Clearly, $\ker \pi_k$ is generated by the infinite collection of elements

$$\{Y_{N-k+1,k}, Y_{N-k+2,k}, \dots, Y_{N-k+i,k}, \dots\}.$$

Using (4.6), one can show by induction that if $\langle Y_{N-k+1,k}, Y_{N-k+2,k}, \dots, Y_N \rangle = 0$, then $Y_{N+\ell,k} = 0$ for all $\ell > 0$. ■

This means $H(G_k^\infty)/(Y_{N-k+1,k}, Y_{N-k+2,k}, \dots, Y_N) \cong H(G_k^N)$. Furthermore, the sequence $Y_{N-k+1,k}, Y_{N-k+2,k}, \dots, Y_N$ is a regular sequence of $H(G_k^\infty)$. Thus we get a finite free resolution of $H(G_k^N)$ as $H(G_k^\infty)$-module

$$0 \to Y_{N-k+1,k} Y_{N-k+2,k} \cdots Y_N H(G_k^\infty) \to \cdots$$
$$\cdots \to \bigoplus_{\ell=1}^{k} Y_{N-k+\ell} H(G_k^\infty) \to H(G_k^\infty) \twoheadrightarrow H(G_k^N) \to 0.$$

Turning this into the dg-language, we obtain a $\mathbb{Z}$-graded dg-algebra by taking iterated mapping cones over Y_{N-k+i}, yielding

$$(H(G_k^\infty) \otimes \textstyle\bigwedge^\bullet(\omega_{1,k}, \dots, \omega_{k,k}), d_N), \tag{4.7}$$

with

$$d_N(X_{i,k}) := 0, \qquad d_N(\omega_{i,k}) := Y_{N-i+1,k}.$$

Since it is built from a free resolution of $H(G_k^N)$, we immediately obtain that its homology is isomorphic to $H(G_k^N)$.

Setting the differential to zero and introducing a λ-grading, we obtain a $\mathbb{Z}^2$-graded dg-algebra $(\Omega_k, 0)$ with

$$\Omega_k := \mathbb{Q}[X_{1,k}, \dots, X_{k,k}] \otimes \textstyle\bigwedge^\bullet(\omega_{1,k}, \dots, \omega_{k,k})$$

and $\mathbb{Z}^2$-degrees

$$\deg_{q,\lambda}(X_{i,j}) := (2i, 0), \qquad \deg_{q,\lambda}(\omega_{i,k}) := (2-2i, 2),$$

and homological degrees

$$\deg_h(X_{i,j}) := 0, \qquad \deg_h(\omega_{i,k}) := 1.$$

We refer to the $\mathbb{Z}^2$-degrees (q, λ) as *quantum degrees*. Then, we rewrite (4.7) as (Ω_k, d_N), where we assume that the λ-grading is specialized to $\lambda = q^N$.

Playing the same game with $G^{\infty}_{k,k+1}$ and $G^{N}_{k,k+1}$, we get

$$\Omega_{k,k+1} := \mathbb{Q}[X_{1,k},\dots,X_{k,k},\xi_{k+1}] \otimes \textstyle\bigwedge^{\bullet}(\omega_{1,k+1},\dots,\omega_{k+1,k+1}),$$

with

$$\begin{aligned} &\deg_{q,\lambda}(X_{i,j}) := (2i,0), \quad &&\deg_{q,\lambda}(\xi_{k+1}) := (2,0), \quad &&\deg_{q,\lambda}(\omega_{i,k}) := (2-2i,2),\\ &\deg_h(X_{i,j}) := 0, &&\deg_h(\xi_{k+1}) := 0, &&\deg_h(\omega_{i,k}) := 1. \end{aligned}$$

It can be equipped with a differential d_N defined by

$$d_N(X_{i,k}) := 0, \qquad d_N(\xi_{k+1}) := 0, \qquad d_N(\omega_{i,k+1}) := Y_{N-i+1,k+1}. \tag{4.8}$$

Proposition 4.2.2. *The dg-algebras (Ω_k, d_N) and $(\Omega_{k,k+1}, d_N)$ are formal with homology*

$$H(\Omega_k, d_N) \cong H(G^N_k), \qquad\qquad H(\Omega_{k,k+1}, d_N) \cong H(G^N_{k,k+1}),$$

and are acyclic whenever $k > N$ or $k+1 > N$ respectively.

Proof. The homology of (Ω_k, d_N) is computed above. Moreover, the obvious projection map $(\Omega_k, d_N) \twoheadrightarrow (H(G^N_k), 0)$ is a quasi-isomorphism. If $k > N$, then $d_N(\omega_{N+1,k}) = Y_{0,k} = 1$, implying $H(\Omega_k, d_N) \cong 0$. The proof is similar for $H(\Omega_{k,k+1}, d_N)$. ∎

Bimodule structure of $\Omega_{k,k+1}$

The next step is to lift the maps φ^N_k and ψ^N_{k+1} of (4.3) to dg-maps

$$\begin{tikzcd} & (\Omega_{k,k+1}, d_N) & \\ (\Omega_k, d_N) \arrow[ur, "\tilde\varphi^N_k"] & & (\Omega_{k+1}, d_N). \arrow[ul, "\tilde\psi^N_{k+1}"'] \end{tikzcd}$$

The first observation we make is that the diagram

$$\begin{tikzcd} H(G^\infty_k) \arrow[r, "\varphi_k"] \arrow[d, "\pi_k"] & H(G^\infty_{k,k+1}) \arrow[d, "\pi_{k,k+1}"] & H(G^\infty_{k+1}) \arrow[l, "\psi_k"'] \arrow[d, "\pi_{k+1}"] \\ H(G^N_k) \arrow[r, "\varphi^N_k"'] & H(G^N_{k,k+1}) & H(G^N_{k+1}) \arrow[l, "\psi^N_k"] \end{tikzcd}$$

commutes.

Thus we put as before

$$\widetilde{\varphi}_k^N(X_{i,k}) := X_{i,k}, \qquad \widetilde{\psi}_{k+1}^N(X_{i,k+1}) := X_{i,k} + \xi_{k+1}X_{i-1,k}.$$

Moreover, since $\psi_{k+1}(Y_{i,k+1}) = Y_{i,k+1}$ and $\varphi_k(Y_{i,k}) = Y_{i,k+1} + \xi_{k+1}Y_{i-1,k}$, we define

$$\widetilde{\psi}_{k+1}^N(\omega_{i,k+1}) := \omega_{i,k+1}, \qquad \widetilde{\varphi}_k^N(\omega_{i,k}) := \omega_{i,k+1} + \xi_{k+1}\omega_{i+1,k+1},$$

so that

$$\begin{tikzcd}
\Omega_k \arrow[r, "\widetilde{\varphi}_k^N"] \arrow[d, "d_N"] & \Omega_{k,k+1} \arrow[d, "d_N"] & \Omega_{k+1} \arrow[l, "\widetilde{\psi}_{k+1}^N"'] \arrow[d, "d_N"] \\
\Omega_k \arrow[r, "\widetilde{\varphi}_k^N"'] & \Omega_{k,k+1} & \Omega_{k+1}, \arrow[l, "\widetilde{\psi}_{k+1}^N"]
\end{tikzcd}$$

commutes. Using this, we turn $(\Omega_{k,k+1}, d_N)$ into a (Ω_k, d_N)-(Ω_{k+1}, d_N)-bimodule, lifting the whole construction of §4.1.1 to the dg-world.

Finally, taking a trivial differential instead of d_N, we still get maps

$$\begin{tikzcd}
 & (\Omega_{k,k+1}, 0) & \\
(\Omega_k, 0) \arrow[ur, "\widetilde{\varphi}_k"] & & (\Omega_{k+1}, 0), \arrow[ul, "\widetilde{\psi}_{k+1}"']
\end{tikzcd}$$

explicitly given by

$$\widetilde{\varphi}_k : \begin{cases} X_{i,k} & \mapsto X_{i,k}, \\ Y_{i,k} & \mapsto Y_{i,k+1} + \xi_{k+1}Y_{i-1,k+1}, \\ \omega_{i,k} & \mapsto \omega_{i,k+1} + \xi_{k+1}\omega_{i+1,k+1}, \end{cases} \tag{4.9}$$

$$\widetilde{\psi}_{k+1} : \begin{cases} X_{i,k+1} & \mapsto X_{i,k} + \xi_{k+1}X_{i-1,k}, \\ Y_{i,k+1} & \mapsto Y_{i,k+1}, \\ \omega_{i,k+1} & \mapsto \omega_{i,k+1}. \end{cases} \tag{4.10}$$

They preserve the λ-grading, turning $\Omega_{k,k+1}$ into an Ω_k-Ω_{k+1}-bimodule.

4.3 Categorical action

We introduce the notation

$$\oplus_{[\beta+k]_q}(-) := \bigoplus_{\ell\in\mathbb{N}} \big(q^{1-k+2\ell}\lambda^{-1}(-) \oplus q^{1+k+2\ell}\lambda(-)[1]\big).$$

It is a c.b.l.f. direct sum that decategorifies as $[\beta + k]_q$ (see (1.5)).

Proposition 4.3.1. *We have decompositions*

$$q^{-k}\Omega_{k,k+1} \cong \oplus_{[k+1]_q}\Omega_{k+1}, \qquad \text{as } \Omega_{k+1}\text{-module,}$$
$$\lambda^{-1}q^{k+1}\Omega_{k,k+1} \cong \oplus_{[\beta-k]_q}\Omega_k, \qquad \text{as } \Omega_k\text{-module.}$$

Proof. The first claim is a direct consequence of the fact that

$$q^{-k}H(G^{\infty}_{k,k+1}) \cong \oplus_{[k+1]_q}H(G^{\infty}_{k+1}),$$

for similar reasons as in (4.5), and the action of the exterior part of Ω_{k+1} on the exterior part of $\Omega_{k,k+1}$ is the identity.

For the second claim, we first observe that

$$H(G^{\infty}_{k,k+1}) \cong H(G^{\infty}_k) \otimes \mathbb{Q}[\xi_{k+1}].$$

Then we construct a surjective map

$$\begin{aligned}\big(\textstyle\bigwedge^\bullet(\omega_{1,k},\dots,\omega_{k,k}) \otimes \mathbb{Q}[\xi_{k+1}]\big) \oplus \lambda^2q^{2k}\big(\textstyle\bigwedge^\bullet(\omega_{1,k},\dots,\omega_{k,k}) \otimes \mathbb{Q}[\xi_{k+1}]\big)[1] \\ \longrightarrow \big(\textstyle\bigwedge^\bullet(\omega_{1,k+1},\dots,\omega_{k+1,k+1}) \otimes \mathbb{Q}[\xi_{k+1}]\big),\end{aligned}$$

induced by $\widetilde{\varphi}_k + \omega_{k+1,k+1}\widetilde{\varphi}_k$. By comparing the graded dimensions, we conclude that it is in fact an isomorphism. Taking the grading shift in consideration finishes the proof. ■

We define

$$\mathcal{M}(\lambda) := \bigoplus_{k\in\mathbb{N}} \mathcal{M}_{\beta-2k}, \qquad \mathcal{M}_{\beta-2k} := \mathcal{D}^{lc}(\Omega_k, 0).$$

Using Proposition 4.3.1, we can already define functors

$$\mathcal{F}_k : \Omega_k\text{-mod} \to \Omega_{k+1}\text{-mod}, \qquad \mathcal{F}_k(-) := q^{-k}\Omega_{k+1,k} \otimes_{\Omega_k} (-),$$
$$\mathcal{E}_k : \Omega_{k+1}\text{-mod} \to \Omega_k\text{-mod}, \qquad \mathcal{E}_k(-) := \lambda^{-1}q^{k+1}\Omega_{k,k+1} \otimes_{\Omega_{k+1}} (-).$$

By Proposition 4.3.1, we know they each have finite amplitude, preserve c.b.l. compact modules and are exact. Thus $\mathcal{E} := \oplus_{k\geqslant 0}\mathcal{E}_k$ and $\mathcal{F} := \oplus_{k\geqslant 0}\mathcal{F}_k$ induce maps on $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\lambda))$ that agree with (1.7). Hence, we can already conclude that $\mathcal{M}(\lambda)$ is a weak categorification of $M(\lambda)$. However, as mentioned before, we would like that $\mathcal{E}_k$ and $\mathcal{F}_k$ are related through a categorical version of (4.1). We would also like to see them related with a similar flavor as the biadjunction in 4.1.2.

Remark 4.3.2. *We cannot hope for $\mathscr{E}_k$ and $\mathscr{F}_k$ to form a biadjoint pair. Indeed, in that case it would be possible to show that if $\mathscr{E}_0$ kill a highest weight space, then $\mathscr{F}_\ell$ kills a lowest weight space for some ℓ, by the results in [82]. This would imply that we would be categorifying a finite dimensional module and not a Verma module. Furthermore, we should not hope to have an adjoint pair and a direct sum decomposition of the form*

$$\mathscr{E}_k\mathscr{F}_k \cong \mathscr{F}_{k-1}\mathscr{E}_{k-1} \oplus_{[\beta-2k]} \mathrm{Id},$$

since it would imply that $\mathscr{F}_k$ and $\mathscr{E}_k$ are biadjoint by [13]. In conclusion, the best case scenario would be that $\mathscr{E}_k$ and $\mathscr{F}_k$ form an adjoint pair, and $\mathscr{E}_k\mathscr{F}_k$ is related to $\mathscr{F}_{k-1}\mathscr{E}_{k-1}$ through a short exact sequence.

4.3.1 A short exact sequence

We have two goals for this section. The first one is to prove $\mathscr{E}_k$ and $\mathscr{F}_k$ form an adjoint pair. The second one is to construct a short exact sequence

$$0 \to \mathscr{F}_{k-1}\mathscr{E}_{k-1} \hookrightarrow \mathscr{E}_k\mathscr{F}_k \twoheadrightarrow \oplus_{[\beta-2k]} \mathrm{Id}_{\Omega_k} \to 0.$$

In order to simplify notations, we will write $X_{i,k}$ (resp $X_{i,k+1}$) for the image of $X_{i,k}$ (resp. $X_{i,k+1}$) in $\Omega_{k,k+1}$ through $\widetilde{\varphi}_k$ (resp. $\widetilde{\psi}_{k+1}$). The same applies for all the $Y_{*,*}$ and $\omega_{*,*}$. We also write $\otimes_k$ for the tensor product over Ω_k. Since we want to relate $\mathscr{E}_k\mathscr{F}_k$ to $\mathscr{F}_{k-1}\mathscr{E}_{k-1}$, we write

$$\Omega_{k(k+1)k} := \Omega_{k,k+1} \otimes_{\Omega_{k+1}} \Omega_{k+1,k}, \qquad \Omega_{k(k-1)k} := \Omega_{k,k-1} \otimes_{\Omega_{k-1}} \Omega_{k-1,k}.$$

Finally, we write Id_k, $\mathrm{Id}_{k,k\pm1}$ and $\mathrm{Id}_{k(k\pm1)k}$ for the identity maps on the different bimodules $\Omega_k, \Omega_{k,k\pm1}$ and $\Omega_{k(k\pm1)k}$.

Lemma 4.3.3. *In $\Omega_{k,k+1}$, the following identities*

$$X_{i,k} = \sum_{\ell=0}^{i} (-1)^\ell X_{i-\ell,k+1}\xi_{k+1}^\ell, \tag{4.11}$$

$$Y_{i,k+1} = \sum_{\ell=0}^{i} (-1)^\ell Y_{i-\ell,k}\xi_{k+1}^\ell, \tag{4.12}$$

$$\xi_{k+1}^i = (-1)^i \sum_{\ell=0}^{i} X_{\ell,k}Y_{i-\ell,k+1}, \tag{4.13}$$

$$\omega_{i,k+1} = (-1)^{k+1-i}\xi_{k+1}^{k+1-i}\omega_{k+1,k+1} + \sum_{\ell=0}^{k-i}(-1)^\ell \omega_{i+\ell,k}\xi_{k+1}^\ell, \tag{4.14}$$

hold for all $i \geqslant 0$.

Proof. The four equations are obtained by induction using (4.9) and (4.10). We leave the details to the reader. ■

Lemma 4.3.4. *In $\Omega_{k(k+1)k}$, the following identity holds*

$$\sum_{\ell=0}^{k}(-1)^k X_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell}=\sum_{\ell=0}^{k}(-1)^\ell\xi_{k+1}^{k-\ell}\otimes_{k+1}X_{\ell,k}.$$

Moreover, ξ_{k+1} slides over this sum

$$\xi_{k+1}\Big(\sum_{\ell=0}^{k}(-1)^k X_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell}\Big)=\Big(\sum_{\ell=0}^{k}(-1)^k X_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell}\Big)\xi_{k+1}. \tag{4.15}$$

Proof. This follows from [48, §3.2] since the part in homological degree zero of $\Omega_{k(k+1)k}$ is the cohomology of the two-step flag variety. ■

As in [48], using Lemma 4.3.4, we construct an injective map of bimodules

$$\eta_k:\Omega_k\hookrightarrow\Omega_{k(k+1)k},\qquad 1\mapsto\sum_{\ell=0}^{k}(-1)^\ell X_{\ell,k}\otimes_{k+1}\xi_{k+1}^{k-\ell},$$

of q-degree $(2k,0)$. We also define the bimodule map

$$\varepsilon_{k-1}:\Omega_{k(k-1)k}\twoheadrightarrow\Omega_k,\qquad \xi_k^i\otimes_{k-1}\xi_k^j\mapsto(-1)^{i+j-k+1}Y_{i+j-k+1,k},$$

of degree $(2-2k,0)$.

Proposition 4.3.5. *We have*

$$(\varepsilon_k\otimes_{k+1}\mathrm{Id}_{k+1,k})\circ(\mathrm{Id}_{k+1,k}\otimes_k\eta_k)=\mathrm{Id}_{k+1,k},$$
$$(\mathrm{Id}_{k,k+1}\otimes_{k+1}\varepsilon_k)\circ(\eta_k\otimes_k\mathrm{Id}_{k,k+1})=\mathrm{Id}_{k,k+1}\,.$$

Proof. It follows from similar computations as in [48, Lemma 4.5]. ■

As a direct consequence, we obtain the following theorem.

Proposition 4.3.6. *The functor $\mathcal{F}_k$ is left adjoint to $\rho(\mathcal{F}_k):=\lambda q^{-2k-1}\mathcal{E}_k$.*

Therefore, we have already achieved one of our two goals. We can now focus on constructing the short exact sequence. We put

$$\Omega_k^\xi:=\Omega_k\otimes_{\mathbb{Q}}\mathbb{Q}[\xi]\cong\bigoplus_{\ell\in\mathbb{N}}q^{2\ell}\Omega_k,$$

as Ω_k-Ω_k-bimodule.

We write $Y_{i,k+1} \in \Omega_k^\xi$ for the element recursively defined as

$$Y_{0,k+1} := 0, \qquad Y_{i,k+1} := -\sum_{\ell=1}^{i} (X_{\ell,k} + \xi X_{\ell-1,k})\, Y_{i-\ell,k+1}.$$

It corresponds with the element given by taking $Y_{i,k+1} \in H(G^\infty_{k,k+1})$ and passing it through the isomorphism of $H(G^\infty_k)$-modules $H(G^\infty_{k,k+1}) \cong H(G^\infty_k) \otimes \mathbb{Q}[\xi]$.

Proposition 4.3.7. *The map of bimodules*

$$\imath : \Omega_k^\xi \hookrightarrow \Omega_{k(k+1)k}, \qquad x\xi^p \mapsto \eta_k(x)\xi^p_{k+1},$$

for $x \in \Omega_k$ and $p \in \mathbb{N}$ admits a left inverse

$$\pi : \Omega_{k(k+1)k} \twoheadrightarrow \Omega_k^\xi, \qquad \begin{cases} \xi^i_{k+1} \otimes_{k+1} \xi^j_{k+1} & \mapsto (-1)^{i+j-k} Y_{i+j-k,k+1}, \\ \xi^i_{k+1} \otimes_{k+1} \xi^j_{k+1}\omega_{k+1,k+1} & \mapsto 0. \end{cases}$$

Proof. First we observe that $\imath$ is indeed a map of bimodules thanks to (4.15). Then, we compute

$$\begin{aligned}(\pi \circ \imath)(\xi^p) &= \pi\left(\xi^p_{k+1} \sum_{\ell=0}^{k} (-1)^\ell \xi^{k-\ell}_{k+1} \otimes_{k+1} X_{\ell,k}\right) \\ &= \sum_{\ell=0}^{k} (-1)^p X_{\ell,k} Y_{i-\ell,k+1} \overset{(4.13)}{=} \xi^p,\end{aligned}$$

for all $p \geqslant 0$. ∎

We define yet another bimodule map

$$\mu : \Omega_{k(k+1)k} \twoheadrightarrow \Omega_k^\xi[1], \qquad \begin{cases} \xi^i_{k+1} \otimes_{k+1} \xi^j_{k+1} & \mapsto 0, \\ \xi^i_{k+1} \otimes_{k+1} \xi^j_{k+1}\omega_{k+1,k+1} & \mapsto (-1)^{i+j} Y_{i+j,k+1}. \end{cases}$$

of degree $(2k+2,-2)$. It is a map of bimodules since we recall that the translation functor $[1]$ twists the left action (see §2.2.1). Moreover, as in [48], we define the *nilCoxeter maps* by setting

$$\begin{aligned}\Psi_\uparrow :&\Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k+2} \to \Omega_{k,k+1} \otimes_{k+1} \Omega_{k+1,k+2}, \\ &\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2} \mapsto \sum_{\ell=0}^{i-1} \xi^{i+j-1-\ell}_{k+1} \otimes_{k+1} \xi^\ell_{k+2} - \sum_{\ell=0}^{j-1} \xi^{i+j-1-\ell}_{k+1} \otimes_{k+1} \xi^\ell_{k+2},\end{aligned}$$

and

$$
\begin{aligned}
\Psi_\downarrow :&\Omega_{k+2,k+1} \otimes_{k+1} \Omega_{k+1,k} \to \Omega_{k+2,k+1} \otimes_{k+1} \Omega_{k+1,k}, \\
&\xi_{k+2}^i \otimes_{k+1} \xi_{k+1}^j \mapsto \sum_{\ell=0}^{j-1} \xi_{k+2}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+1}^\ell - \sum_{\ell=0}^{i-1} \xi_{k+2}^{i+j-1-\ell} \otimes_{k+1} \xi_{k+1}^\ell,
\end{aligned}
$$

extending using the Ω_{k+2}-module structure. They are both of degree $(-2, 0)$. We will now prove they define maps of bimodules.

Lemma 4.3.8. *For all $i, j \geqslant 0$ and $\updownarrow \in \{\uparrow, \downarrow\}$ we have*

$$
\begin{aligned}
\Psi_\updownarrow(\xi^{i+1} \otimes_{k+1} \xi^j) - \Psi_\updownarrow(\xi^i \otimes_{k+1} \xi^j)\xi &= \xi^i \otimes_{k+1} \xi^j, \\
\xi\Psi_\updownarrow(\xi^i \otimes_{k+1} \xi^j) - \Psi_\updownarrow(\xi^i \otimes_{k+1} \xi^{j+1}) &= \xi^i \otimes_{k+1} \xi^j,
\end{aligned}
\tag{4.16}
$$

and thus

$$
\xi\Psi_\updownarrow(\xi^i \otimes_{k+1} \xi^j)\xi = \Psi_\updownarrow(\xi^{i+1} \otimes_{k+1} \xi^{j+1}).
$$

Proof. The proof is a straightforward computation, which is identical to what is done in [48, Lemma 7.10]. ■

Proposition 4.3.9. *The maps $\Psi_\uparrow$ and $\Psi_\downarrow$ are both maps of Ω_k-Ω_{k+2}-bimodules.*

Proof. We prove it for $\Psi_\uparrow$, the proof being similar for $\Psi_\downarrow$. By construction, it is a map of Ω_{k+2}-module, thus we only need to prove it is also an Ω_k-module morphism. Therefore, using (4.11), we compute

$$
\begin{aligned}
\Psi_\uparrow(X_{t,k}\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) &= \sum_{\ell=0}^{t} (-1)^\ell \Psi_\uparrow(\xi_{k+1}^{i+\ell} \otimes_{k+1} \xi_{k+2}^j X_{t-\ell,k+1}) \\
&= \sum_{\ell=0}^{t}\sum_{p=0}^{t-\ell} (-1)^{\ell+p} \Psi_\uparrow(\xi_{k+1}^{i+\ell} \otimes_{k+1} \xi_{k+2}^{j+p}) X_{t-\ell-p,k+2},
\end{aligned}
$$

and

$$
\begin{aligned}
&X_{t,k}\Psi_\uparrow(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) \\
&\qquad = \sum_{\ell=0}^{t} (-1)^\ell \xi_{k+1}^\ell \Psi_\uparrow(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) X_{t-\ell,k+1} \\
&\qquad = \sum_{\ell=0}^{t}\sum_{p=0}^{t-\ell} (-1)^{\ell+p} \xi_{k+1}^\ell \Psi_\uparrow(\xi_{k+1}^i \otimes_{k+1} \xi_{k+2}^j) \xi_{k+2}^p X_{t-\ell-p,k+2},
\end{aligned}
$$

which are the same for all $i, j \geqslant 0$ and $0 \leqslant t \leqslant k$, thanks to Lemma 4.3.8.

Then we compute

$$\begin{aligned}
&\Psi_\uparrow(\omega_{t,k}\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2}) \\
&\overset{(4.9)}{=} \Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})\omega_{t,k+2} + \Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^{j+1}_{k+2})\omega_{t+1,k+2} \\
&\quad + \Psi_\uparrow(\xi^{i+1}_{k+1} \otimes_{k+1} \xi^j_{k+2})\omega_{t+1,k+2} + \Psi_\uparrow(\xi^{i+1}_{k+1} \otimes_{k+1} \xi^{j+1}_{k+2})\omega_{t+2,k+2} \\
&\overset{(4.16)}{=} \Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})\omega_{t,k+2} + \Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})\xi_{k+2}\omega_{t+1,k+2} \\
&\quad \xi_{k+1}\Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})\omega_{t+1,k+2} + \xi_{k+1}\Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})\xi_{k+2}\omega_{t+2,k+2} \\
&\overset{(4.9)}{=} \omega_{t,k}\Psi_\uparrow(\xi^i_{k+1} \otimes_{k+1} \xi^j_{k+2})
\end{aligned}$$

which concludes the proof. ■

Proposition 4.3.10. *There is an injection of Ω_k-Ω_k-bimodules*

$$u : \Omega_{k(k-1)k} \hookrightarrow \Omega_{k(k+1)k},$$

preserving all degrees and given by

$$\begin{aligned}
u :&= (\varepsilon_{k-1} \otimes_k \mathrm{Id}_{k(k+1)k}) \circ (\mathrm{Id}_{k,k-1} \otimes_{k-1} \Psi_\uparrow \otimes_{k+1} \mathrm{Id}_{k+1,k}) \circ (\mathrm{Id}_{k(k-1)k} \otimes_k \eta_k) \\
&= (\mathrm{Id}_{k(k+1)k} \otimes_k \varepsilon_{k-1}) \circ (\mathrm{Id}_{k,k+1} \otimes_{k+1} \Psi_\downarrow \otimes_{k-1} \mathrm{Id}_{k,k-1}) \circ (\eta_k \otimes_k \mathrm{Id}_{k(k-1)k}).
\end{aligned}$$

Moreover, this map takes the form

$$u(\xi^i_k \otimes_{k-1} \xi^j_k) = -\xi^j_{k+1} \otimes_{k+1} \xi^i_{k+1}, \tag{4.17}$$

for all $i + j < k$.

Proof. First we suppose that

$$u = (\varepsilon_{k-1} \otimes_k \mathrm{Id}_{k(k+1)k}) \circ (\mathrm{Id}_{k,k-1} \otimes_{k-1} \Psi_\uparrow \otimes_{k+1} \mathrm{Id}_{k+1,k}) \circ (\mathrm{Id}_{k(k-1)k} \otimes_k \eta_k).$$

We compute for $i < k$,

$$u(\xi^i_k \otimes_{k-1} 1) = -\sum_{r+s+t=i} (-1)^{i-s} Y_{r,k}\xi^s_{k+1} \otimes_{k+1} X_{t,k}.$$

Moreover, by applying (4.13) followed by (4.12) we obtain

$$1 \otimes_{k+1} \xi^i_{k+1} = (-1)^i \sum_{r+s+t=i} (-1)^s Y_{r,k}\xi^s_{k+1} \otimes_{k+1} X_{t,k},$$

and thus $u(\xi^i_k \otimes_{k-1} 1) = 1 \otimes_{k+1} \xi^i_{k+1}$.

Then, using again (4.13) and (4.11), we compute

$$\xi_k^i \otimes_{k-1} \xi_k^j = (-1)^j \sum_{r+s+t=j} (-1)^s X_{r,k} \xi_k^{i+s} \otimes_{k-1} Y_{t,k},$$
$$u(\xi_k^i \otimes_{k-1} \xi_k^j) = -(-1)^j \sum_{r+s+t=j} (-1)^s X_{r,k} \otimes_{k+1} \xi_{k+1}^{i+s} Y_{t,k},$$

and also using (4.13) and (4.12), we get

$$\xi_{k+1}^j \otimes_{k+1} \xi_{k+1}^i = (-1)^j \sum_{r+s+t=j} (-1)^s X_{r,k} \otimes_{k+1} \xi_{k+1}^{i+s} Y_{t,k}.$$

This proves that $u(\xi_k^i \otimes_{k-1} \xi_k^j) = -\xi_{k+1}^j \otimes_{k+1} \xi_{k+1}^i$, for all $i + j < k$. Similar computations do the trick for the case

$$u = (\mathrm{Id}_{k(k+1)k} \otimes_k \varepsilon_{k-1}) \circ (\mathrm{Id}_{k,k+1} \otimes_{k+1} \Psi_\downarrow \otimes_{k-1} \mathrm{Id}_{k,k-1}) \circ (\eta_k \otimes_k \mathrm{Id}_{k(k-1)k}).$$

Then the injectivity follows from Proposition 4.3.1. ■

Lemma 4.3.11. *The bimodule $\Omega_{k(k-1)k}$ is generated by $\sum_{\ell=0}^{k-1} \Omega_k \xi_k^\ell \otimes_{k-1} \Omega_k$.*

Proof. By (4.13), we obtain that $\Omega_{k-1,k}$ is generated by $\sum_{\ell=0}^{k-1} x_{\ell,k-1}\Omega_k$. Thus sliding $x_{\ell,k-1}$ over the tensor product in $\Omega_{k(k-1)k}$, we get that $\Omega_{k(k-1)k}$ is generated by $\Omega_{k,k-1} \otimes_{k-1} \Omega_k$. By (4.11), $\Omega_{k,k-1}$ is generated by $\sum_{\ell=0}^{k-1} \Omega_k \xi_k^\ell$. Putting these together concludes the proof. ■

Lemma 4.3.12. *There is an equality of graded dimensions*

$$\operatorname{gdim} \Omega_{k(k+1)k} - \operatorname{gdim} \Omega_{k(k-1)k} = (q^{2k} + \lambda^2 q^{-2k} h) \operatorname{gdim} \Omega_k^\xi.$$

Proof. As left-Ω_k-module, by Proposition 4.3.1, we have

$$\begin{aligned}\Omega_{k(k-1)k} &\cong \bigoplus_{\ell \geqslant 0} \Omega_{k,k-1} \otimes_{k-1} \xi_k^\ell \oplus \Omega_{k,k-1} \otimes_{k-1} \xi_k^\ell \omega_{k,k} \\ &\cong \bigoplus_{\ell \geqslant 0} \bigoplus_{r=0}^{k-1} \Omega_k \xi_k^r \otimes_{k-1} \xi_k^\ell \oplus \Omega_k \xi_k^r \otimes_{k-1} \xi_k^\ell \omega_{k,k}.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\Omega_{k(k+1)k} &\cong \bigoplus_{r=0}^{k} \Omega_{k,k+1} \otimes_{k+1} \xi_{k+1}^r \\ &\cong \bigoplus_{r=0}^{k} \bigoplus_{\ell \geqslant 0} \Omega_k \xi_{k+1}^\ell \otimes_{k+1} \xi_{k+1}^r \oplus \Omega_k \xi_{k+1}^\ell \omega_{k+1,k+1} \otimes_{k+1} \xi_{k+1}^r.\end{aligned}$$

Since $\deg(\xi_{k+1}\omega_{i,k+1}) = \deg(\omega_{i-1,k})$, we obtain

$$\begin{aligned}\operatorname{gdim}\Omega_{k(k+1)k} - \operatorname{gdim}\Omega_{k(k-1)k} =& \\ \deg(\xi_{k+1}^k)\sum_{\ell\geqslant 0} q^{2\ell}\operatorname{gdim}\Omega_k &+ \deg(\omega_{k+1,k+1})\sum_{\ell\geqslant 0} q^{2\ell}\operatorname{gdim}\Omega_k,\end{aligned}$$

which concludes the proof. ■

Theorem 4.3.13. *There is a short exact sequence of Ω_k-Ω_k-bimodules*

$$0 \to \Omega_{k(k-1)k} \xrightarrow{u} \Omega_{k(k+1)k} \xrightarrow{\pi\oplus\mu} q^{2k}\Omega_k^\xi \oplus \lambda^2 q^{-2k}\Omega_k^\xi[1] \to 0, \tag{4.18}$$

with degree zero maps.

Proof. We have

$$(\pi\oplus\mu)\circ u(\xi_k^\ell \otimes_{k-1} 1) = (\pi\oplus\mu)(-1\otimes_{k-1}\xi_k^\ell) = 0$$

for $0 \leqslant \ell \leqslant k-1$. Thus by Lemma 4.3.11, the composition $(\pi\oplus\mu)\circ u = 0$ is zero. Since we already know that u is injective and π and μ are surjective, the exactness follows from a dimensional argument. Therefore we conclude the proof by using Lemma 4.3.12. ■

Corollary 4.3.14. *There is a natural short exact sequence*

$$0 \to \mathscr{F}_{k-1}\mathscr{E}_{k-1} \hookrightarrow \mathscr{E}_k\mathscr{F}_k \twoheadrightarrow \oplus_{[\beta-2k]}\operatorname{Id}_k \to 0.$$

By Toen's results [89] (see also [36]), we can interpret the spaces of (quasi-)functors between the categorified weight spaces $\mathscr{M}_{\beta-2k}$ as (subcategories of) the categories of dg-bimodules. In particular we obtain:

Corollary 4.3.15. *There is a quasi-isomorphism*

$$\operatorname{Cone}(\mathscr{F}_{k-1}\mathscr{E}_{k-1} \xrightarrow{u} \mathscr{E}_k\mathscr{F}_k) \xrightarrow{\simeq} \oplus_{[\beta-2k]}\operatorname{Id}_k .$$

4.3.2 A long exact sequence

We write $(\Omega_{k(k-1)k}, d_N)$ for the dg-bimodule $(\Omega_{k,k-1}, d_N)\otimes_{(\Omega_{k-1},d_N)}(\Omega_{k-1,k}, d_N)$. The same goes for $(\Omega_{k(k+1)k}, d_N)$ and (Ω_k^ξ, d_N).

Lemma 4.3.16. *The map $u : \Omega_{k(k-1)k} \to \Omega_{k(k+1)k}$ lifts to a map of dg-bimodules*

$$u : (\Omega_{k(k-1)k}, d_N) \to (\Omega_{k(k+1)k}, d_N).$$

Proof. All the following maps obvously lift to the dg-setting: ε_k, η_k and $\Psi_\uparrow$. ■

The next step is to lift the map $\pi \oplus \mu$. However, the map π alone does not commute with d_N. Thus we need to replace the direct sum in the cokernel of (4.18) by something else (like a mapping cone), with the same underlying graded space. Therefore, we want to compute

$$\begin{aligned}
\pi \circ d_N \circ \mu^{-1}(Y_{i,k+1}) &= \pi \circ d_N((-1)^i \xi_{k+1}^i \otimes_{k+1} \omega_{k+1,k+1}) \\
&\overset{(4.8)}{=} \pi((-1)^i \xi_{k+1}^i \otimes_{k+1} Y_{N-k,k+1}) \\
&\overset{(4.12)}{=} \sum_{\ell=0}^{N-k} (-1)^{i+\ell} \pi(\xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^{\ell}) Y_{N-k-\ell,k} \\
&= (-1)^k \sum_{\ell=0}^{N-k} Y_{i+\ell-k,k+1} Y_{N-k-\ell,k},
\end{aligned}$$

where $\mu^{-1}(Y_{i,k+1})$ only means an arbitrary choice of inverse element. From this we define the map

$$y_N : \Omega_k^{\xi} \to \Omega_k^{\xi}, \quad Y_{i,k+1} \mapsto (-1)^k \sum_{\ell=0}^{N-k} Y_{i+\ell-k,k+1} Y_{N-k-\ell,k},$$

of q-degree $2(N-2k)$, and we consider the dg-bimodule given by the mapping cone

$$(q^{2k}\Omega_k^{\xi} \oplus q^{2(N-k)}\Omega_k^{\xi}[1], d_N) := \mathrm{Cone}(q^{2N-2k}(\Omega_k^{\xi}, d_N) \xrightarrow{y_N} q^{2k}(\Omega_k^{\xi}, d_N)).$$

Thus, we obtain the following lemma and theorem:

Lemma 4.3.17. *The map $\pi \oplus \mu$ lifts to a map of dg-bimodules*

$$(\Omega_{k(k+1)k}, d_N) \to (q^{2k}\Omega_k^{\xi} \oplus q^{2(N-k)}\Omega_k^{\xi}[1], d_N).$$

Theorem 4.3.18. *There is a short exact sequence of dg-bimodules*

$$0 \to (\Omega_{k(k-1)k}, d_N) \to (\Omega_{k(k+1)k}, d_N) \to (q^{2k}\Omega_k^{\xi} \oplus q^{2(N-k)}\Omega_k^{\xi}[1], d_N) \to 0.$$

From the short exact sequence, we obtain by the snake lemma a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^1(\Omega_{k(k+1)k}, d_N) \arrow[r] & H^1(q^{2k}\Omega_k^{\xi} \oplus q^{2(N-k)}\Omega_k^{\xi}[1], d_N) \arrow[dll] \\
H^0(\Omega_{k(k-1)k}, d_N) \arrow[rr] & & H^0(\Omega_{k(k+1)k}, d_N) \arrow[r] & H^0(q^{2k}\Omega_k^{\xi} \oplus q^{2(N-k)}\Omega_k^{\xi}[1], d_N) \arrow[dll] \\
 & 0 & &
\end{tikzcd}$$

Our goal now is to compute the different homology spaces involved in this exact sequence.

Proposition 4.3.19. *We have*

$$\begin{aligned} H(\Omega_{k(k-1)k}, d_N) &\cong H(G^N_{k,k-1}) \otimes_{H(G^N_{k-1})} H(G^N_{k-1,k}), \\ H(\Omega_{k(k+1)k}, d_N) &\cong H(G^N_{k,k+1}) \otimes_{H(G^N_{k+1})} H(G^N_{k+1,k}). \end{aligned}$$

Proof. We want to apply Proposition 2.2.17 together with Proposition 4.2.2. Thus we need to show that $(\Omega_{k,k-1}, d_N)$ is a strongly projective (Ω_{k-1}, d_N)-module, and that $(\Omega_{k,k+1}, d_N)$ is a strongly projective (Ω_{k+1}, d_N)-module. From Proposition 4.3.1 we deduce that $(\Omega_{k,k-1}, d_N)$ decomposes as a mapping cone

$$(\Omega_{k,k-1}, d_N) \cong \mathrm{Cone}\big(\oplus_{\ell \geqslant 0} \xi_k^\ell (\Omega_{k-1}, d_N) \xrightarrow{Y_{N-k+1,k}} \oplus_{\ell \geqslant 0} \xi_k^\ell (\Omega_{k-1}, d_N)\big).$$

By (4.12), we obtain a triangular change of basis, turning ξ_k^ℓ into $Y_{\ell,k}$, so that

$$(\Omega_{k,k-1}, d_N) \cong \mathrm{Cone}\big(\oplus_{\ell \geqslant 0} Y_{\ell,k} (\Omega_{k-1}, d_N) \xrightarrow{f} \oplus_{\ell \geqslant 0} Y_{\ell,k} (\Omega_{k-1}, d_N)\big),$$

where $f(Y_{\ell,k}) := Y_{\ell+N-k+1,k}$. Therefore, identifying $Y_{\ell,k}$ with y^ℓ we obtain

$$(\Omega_{k,k-1}, d_N) \cong (\mathbb{Q}[y,\theta]/(\theta^2), d) \otimes_{\mathbb{Q}} (\Omega_{k-1}, d_N),$$

where $d(y) := 0$ and $d(\theta) := y^{N-k+1}$, proving that $(\Omega_{k,k-1}, d_N)$ is strongly projective. Then again from Proposition 4.3.1 we obtain

$$(\Omega_{k,k+1}, d_N) \cong \bigoplus_{\ell=0}^{k} (\Omega_{k+1}, d_N),$$

and thus $(\Omega_{k,k+1}, d_N)$ is strongly projective, as (Ω_{k+1}, d_N)-module. ∎

Proposition 4.3.20. *We have*

$$(q^{2k}\Omega_k^\xi \oplus q^{2(N-k)}\Omega_k^\xi[1], d_N) \cong \begin{cases} \oplus_{\ell=0}^{N-2k-1} q^{2k+2\ell} H(G_k^N), & \text{if } N-2k \geqslant 0, \\ \oplus_{\ell=0}^{2k-N-1} q^{2k+2\ell} H(G_k^N)[1], & \text{if } N-2k \leqslant 0. \end{cases}$$

Proof. We observe that if $N-2k \geqslant 0$ then $y_N(\xi^i)$ is a polynomial with dominant monomial ξ_{k+1}^{N-2k+i} (up to sign). If $N-2k \leqslant 0$, then $d_N(\xi^i) = 0$ for $i < 2k-N$ and $y_N(Y_{2k-N,k+1}) = 1$, and $y_N(\xi^i)$ is polynomial with dominant monomial ξ_{k+1}^{N-2k+i} for $i > 2k-N$. Putting all this together concludes the proof. ∎

Plugging the results of Proposition 4.3.19 and Proposition 4.3.20 into the long exact sequence we obtain an exact sequence

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[r] & H^1(q^{2k}\Omega_k^\xi \oplus q^{2(N-k)}\Omega_k^\xi[1], d_N) \arrow[dl, "\delta_{k-1}^N"'] \\
H^0(\Omega_{k(k-1)k}, d_N) \arrow[r] & H^0(\Omega_{k(k+1)k}, d_N) \arrow[r] & H^0(q^{2k}\Omega_k^\xi \oplus q^{2(N-k)}\Omega_k^\xi[1], d_N) \arrow[dll] \\
0 & &
\end{tikzcd}$$

If $N-2k \geqslant 0$, then it truncates to a short exact sequence

$$\begin{aligned}
&0 \to H(G_{k,k-1}^N) \otimes_{H(G_{k-1}^N)} H(G_{k-1,k}^N) \to H(G_{k,k+1}^N) \otimes_{H(G_{k+1}^N)} H(G_{k+1,k}^N) \\
&\to \oplus_{\ell=0}^{N-2k-1} q^{2k+2\ell} H(G_k^N) \to 0
\end{aligned}$$

which splits with splitting morphism induced by the lift of $\imath$ from Proposition 4.3.7. If $N-2k \leqslant 0$, then we obtain a short exact sequence

$$\begin{aligned}
0 \to \oplus_{\ell=0}^{2k-N-1} q^{2k+2\ell} H(G_k^N) \xrightarrow{\delta_{k-1}^N} H(G_{k,k-1}^N) \otimes_{H(G_{k-1}^N)} H(G_{k-1,k}^N)& \\
\to H(G_{k,k+1}^N) \otimes_{H(G_{k+1}^N)} H(G_{k+1,k}^N) \to 0.&
\end{aligned}$$

To show that it also splits, for $i < 2k-N$, we compute the connecting homomorphism

$$\begin{aligned}
\delta_{k-1}^N(Y_{i,k+1}) &= u^{-1}(d_N(\mu^{-1}(Y_{i,k+1}))) \\
&= (-1)^i u^{-1}(d_N(\xi_{k+1}^i \otimes_{k+1} \omega_{k+1,k+1})) \\
&\overset{(4.8)}{=} (-1)^i u^{-1}(\xi_{k+1}^i \otimes_{k+1} Y_{N-k,k+1}) \\
&\overset{(4.12)}{=} \sum_{\ell=0}^{N-k} (-1)^{i+\ell} u^{-1}(\xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^\ell) Y_{N-k-\ell,k} \\
&\overset{(4.17)}{=} -\sum_{\ell=0}^{N-k} (-1)^{i+\ell} \xi_k^\ell \otimes_{k-1} \xi_k^i Y_{N-k-\ell,k}.
\end{aligned}$$

Note that for $i=0$, it yields exactly the same map as the counit of the second adjunction in [47]. We already know it can not lift to a map of (dg-)bimodules. However, it admits a left inverse that comes from the map of (dg-)bimodules

$$(\delta_{k-1}^N)^{-1} : \Omega_{k(k-1)k} \to \Omega_k^\xi, \quad \xi_k^i \otimes_{k-1} \xi_k^j \mapsto (-1)^{N-k+1} \xi^{i+j-N+k}.$$

In conclusion, we obtain two direct sum decompositions that match with the results of [24] (as presented in §4.1.2).

In addition, we recover the biadjunction by taking as unit the map induced by

$$\Omega_{k(k+1)k} \to \Omega_k, \begin{cases} \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j & \mapsto (-1)^{i+j-N+k} X_{i+j-N+k,k}, \\ \xi_{k+1}^i \otimes_{k+1} \xi_{k+1}^j \omega_{k+1,k+1} & \mapsto 0. \end{cases}$$

and as counit the restriction of δ_{k-1}^N to the summand $i = 0$.

Therefore we define

$$\mathscr{V}^{\Omega}(N) := \bigoplus_{k=0}^{N} \mathscr{V}_{N-2k}^{\Omega}, \qquad \mathscr{V}_{N-2k}^{\Omega} := \mathscr{D}^{lc}(\Omega_k, d_N),$$

and the functors

$$\begin{aligned} \mathscr{F}_k^N &: \mathscr{D}^{lc}(\Omega_k, d_N) \to \mathscr{D}^{lc}(\Omega_{k+1}, d_N), \\ &\mathscr{F}_k(-) := q^{-k}(\Omega_{k+1,k}, d_N) \otimes^{\mathrm{L}}_{(\Omega_k, d_N)} (-), \\ \mathscr{E}_k^N &: \mathscr{D}^{lc}(\Omega_{k+1}, d_N) \to \mathscr{D}^{lc}(\Omega_k, d_N), \\ &\mathscr{E}_k(-) := q^{k+1-N}(\Omega_{k,k+1}, d_N) \otimes^{\mathrm{L}}_{(\Omega_{k+1}, d_N)} (-). \end{aligned}$$

They are biadjoint (up to degree shift) and respect

$$\begin{aligned} \mathscr{E}_k^N \mathscr{F}_k^N &\cong \mathscr{F}_{k-1}^N \mathscr{E}_{k-1}^N \oplus_{[N-2k]_q} \mathrm{Id}_k, && \text{if } N - 2k \geqslant 0, \\ \mathscr{F}_{k-1}^N \mathscr{E}_{k-1}^N &\cong \mathscr{E}_k^N \mathscr{F}_k^N \oplus_{[2k-N]_q} \mathrm{Id}_k, && \text{if } N - 2k \leqslant 0. \end{aligned}$$

4.4 The categorification theorems

We choose the additive order $\lambda \succ q \succ 0$ so that $\lambda q^{-2k} \succ 0$ for all $k \in \mathbb{Z}$. We consider the universal Verma module $M(\lambda)$ with ground ring $\mathbb{Q}(\!(q,\lambda)\!)$. Then $(\Omega_k, 0)$ is a positive c.b.l.f. dg-algebra, with trivial differential. It admits a single (up to shift and isomorphism) relatively projective module given by Ω_k, and a single simple module $S_k := \Omega_k/(\Omega_k)_{>0} \cong \mathbb{Q}$. Thus we have

$$\boldsymbol{K}_0^{\Delta}(\mathscr{M}_{\beta-2k}) \cong \mathbb{Z}(\!(q,\lambda)\!).$$

Since Ω_k is (super)commutative, any left Ω_k-module M can be viewed as a right module $\bar{M}$ by twisting the action

$$m \cdot x := (-1)^{\deg_h(m)\deg_h(x)} x \cdot m$$

for homogeneous $m \in M$ and $x \in \Omega_k$. Then we define a bifunctor

$$(-,-) : \mathcal{M}(\lambda) \otimes_{\mathbb{Q}} \mathcal{M}(\lambda) \to \mathcal{D}^{lc}(\mathbb{Q},0), \quad (M,M') := \bar{M} \otimes^{\mathrm{L}}_{\oplus_k \Omega_k} M'.$$

It respects for all $M, M' \in \mathcal{M}(\lambda)$,

- $(\Omega_0, \Omega_0) \cong (\mathbb{Q}, 0)$;
- $(\mathcal{F}_k M, M') \cong (M, \rho(\mathcal{F}_k)N)$;
- $\oplus_f (M, M') \cong (\oplus_f M, M') \cong (M, \oplus_f M')$ for any $f \in \mathbb{Z}(\!(q,\lambda)\!)$.

Since $M(\lambda)$ is generated by $F^k v_\lambda$ for all $k \geqslant 0$, this is enough to show that it descends on $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\lambda))$ as the Shapovalov form. We also have

$$(S_k, \Omega_k) \cong \begin{cases} (\mathbb{Q}, 0), & \text{if } k = k', \\ 0, & \text{otherwise,} \end{cases}$$

which again categorifies (1.8). Moreover, taking the c.b.l.f. composition series of Ω_k or the c.b.l.f. cell module replacement of S_k categorifies the change of basis (1.9).

Theorem 4.4.1. *The Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\lambda))$ is a $U_q(\mathfrak{sl}_2)$-weight module with the action of E, F given by $[\mathcal{E}], [\mathcal{F}]$. Moreover, there is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\lambda)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M(\lambda)$$

sending $[(\Omega_k, 0)]$ to the canonical basis element $v'_{\lambda-2k}$, and $[(S_k, 0)]$ to the dual canonical basis element $v^{\lambda-2k}$.

Using the results of §4.3.2, we also obtain the following:

Theorem 4.4.2. *The Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathcal{V}^{\Omega}(N))$ is a $U_q(\mathfrak{sl}_2)$-weight module with the action of E, F given by $[\mathcal{E}^N], [\mathcal{F}^N]$. Moreover, there is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{V}^{\Omega}(N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong V(N)$$

sending $[(\Omega_k, d_N)]$ to the canonical basis element v'_{N-2k}, and $[(S_k, d_N)]$ to the dual canonical basis element v^{N-2k}.

Note that Theorem 4.1.1 can be reinterpreted as a consequence of Theorem 4.4.2 together with the equivalence of categories

$$\mathcal{D}^{lc}(\Omega_k, d_N) \cong \mathcal{D}^{lc}(H(G_k^N), 0),$$

induced by the quasi-isomorphism $(\Omega_k, d_N) \xrightarrow{\simeq} (H(G_k^N), 0)$, and with the proof of Proposition 4.3.19.

Chapter 5

Dg-enhanced nilHecke algebra

The most important features of a categorification lie in the higher structure. In terms of categorical action, it means we should study the space of natural transformations between the endofunctors realizing the action on the category. In particular for the case of $\mathfrak{sl}_2$, lies inside this higher structure an important ingredient: the nilHecke algebra. It arises naturally over $\mathbb{Q}$ from the geometry of the flag varieties as

$$\mathrm{NH}_n \cong \mathrm{End}_{H(G_n^\infty)}(H(G_{0,1,\dots,n}^\infty)),$$

where

$$G_{0,1,\dots,n}^\infty := \{0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^\infty | \dim V_i = i\}$$

is the full flag variety, and the action is induced by the forgetful map $G_{0,1,\dots,n}^\infty \to G_n^\infty$.

By a remarkable result [82], given a categorical action of $\mathfrak{sl}_2$ on some category $\mathcal{V}'$ respecting (4.2) and categorifying the integrable module $V(N)$ by adjoint functors $\mathcal{E}, \mathcal{F}$, then asking that NH_n acts on $\mathrm{End}(\mathcal{F}^n)$ ensures that $\mathcal{V}'$ is morally equivalent to a category of modules over the cohomology rings of Grassmannians in $\mathbb{C}^N$. In other words, the nilHecke algebra captures almost all the higher structure in the categorification of $V(N)$.

5.1 NilHecke algebra

We fix $\Bbbk$ a unital commutative ring. The nilHecke algebra admits a presentation by generators and relations (see for example [82, §3] where it appears under the name 'nil affine Hecke algebra').

Definition 5.1.1. *The* nilHecke algebra NH_n *is the $\mathbb{Z}$-graded $\Bbbk$-algebra generated by $\tau_1, \tau_2, \dots, \tau_{n-1}$ and $x_1, x_2, \dots, x_n$ with relations*

$$\begin{aligned}
\tau_i^2 &= 0, & \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1}, & & \\
\tau_i\tau_j &= \tau_j\tau_i, & x_ix_j &= x_jx_i, & &\text{for } |i-j| > 1, \\
\tau_i x_j &= x_j\tau_i, & & & &\text{for } j \neq i, i+1, \\
\tau_i x_i &= x_{i+1}\tau_i + 1, & x_i\tau_i &= \tau_i x_{i+1} + 1, & &
\end{aligned}$$

with $\deg(\tau_i) = -2$ *and* $\deg(x_i) = 2$.

5.1.1 From the symmetric group to the nilHecke algebra

Using an approach of Borel [12] (see also [27]), we describe the cohomology ring of the flag varieties using symmetric polynomials. We consider the $\mathbb{Z}$-graded polynomial ring $Q_N := \mathbb{Q}[z_1, \dots, z_N]$ with $\deg(z_i) = 2$. We also recall some classical definitions of symmetric polynomials.

Definition 5.1.2. *The i-th complete homogeneous symmetric polynomial in the variables $z_1, \dots, z_N$ is*

$$\mathscr{h}_i(z_1, \dots, z_N) := \sum_{1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_i \leqslant N} z_{j_1} z_{j_2} \cdots z_{j_i}.$$

The i-th elementary symmetric polynomial in the variables $z_1, \dots, z_N$ is

$$\mathscr{e}_i(z_1, \dots, z_N) := \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant N} z_{j_1} z_{j_2} \cdots z_{j_i},$$

with $\mathscr{e}_0(z_1, \dots, z_N) := 1$ *and* $\mathscr{e}_i(z_1, \dots, z_N) := 0$ *for* $i > N$.

Complete homogeneous symmetric polynomials can be computed recursively as

$$\mathscr{h}_i(z_1, \dots, z_N) = \sum_{\ell=0}^{i} \mathscr{h}_{i-\ell}(z_1, \dots, z_{N-1}) z_N^{\ell}.$$

Moreover, these polynomials respect the equation

$$\sum_{\ell=0}^{i} (-1)^{\ell} \mathscr{e}_{\ell}(z_1, \dots, z_N) \mathscr{h}_{i-\ell}(z_1, \dots, z_N) = 0, \tag{5.1}$$

for all $i \geqslant 0$.

The polynomial ring Q_N comes with an action of the symmetric group S_N by permutation $w(z_i) := z_{w(i)}$ for all $w \in S_N$. Let $C_N := Q_N/(e_j(z_1, \dots, z_N))_{j>0}$

denote the coinvariant algebra. It inherits an S_N-action from Q_N by permutations. Then we consider the ring of invariants C_N^k (resp. $C_N^{k,k+1}$) for the action on C_N given by restriction to $S_k \times S_{N-k} \subset S_N$ (resp. to $S_k \times S_1 \times_{N-k-1} \subset S_N$). Clearly we have

$$\begin{aligned} C_N^k \cong \mathbb{Q}[&e_1(z_1,\dots,z_k),\dots,e_k(z_1,\dots,z_k),\\ &e_1(z_{k+1},\dots,z_N),\dots,e_{N-k}(z_{k+1},\dots,z_N)]/J, \end{aligned}$$

where J is the Q_N ideal generated by $e_j(z_1,\dots,z_N)$ for all $j>0$, and

$$\begin{aligned} C_N^{k,k+1} \cong \mathbb{Q}[&e_1(z_1,\dots,z_k),\dots,e_k(z_1,\dots,z_k),z_{k+1}\\ &e_1(z_{k+2},\dots,z_N),\dots,e_{N-k-1}(z_{k+2},\dots,z_N)]/J. \end{aligned}$$

Then one can show that

$$e_i(z_{k+1},\dots,z_N) \equiv (-1)^i \hbar_i(z_1,\dots,z_k) + J.$$

From this, one obtains an isomorphism $H(G_k^N) \cong C_N^k$ by sending

$$X_{i,k} \mapsto e_i(z_1,\dots,z_k), \quad Y_{i,k} \mapsto e_i(z_{k+1},\dots,z_N) \equiv (-1)^i \hbar_i(z_1,\dots,z_k), \tag{5.2}$$

and an isomorphism $H(G_{k,k+1}^N) \cong C_N^{k,k+1}$ with

$$X_{i,k} \mapsto e_i(z_1,\dots,z_k), \qquad \xi_{k+1} \mapsto z_{k+1}, \qquad Y_{i,k+1} \mapsto e_i(z_{k+2},\dots,z_N).$$

Moreover, the $H(G_k^N)$-$H(G_{k+1}^N)$-bimodule structure of $H(G_{k,k+1}^N)$ is recovered by letting C_N^k and C_N^{k+1} act by polynomial multiplication on $C_N^{k,+1}$, since

$$\begin{aligned} e_i(z_1,\dots,z_{k+1}) &= e_i(z_1,\dots,z_k) + z_{k+1}e_{i-1}(z_1,\dots,z_k),\\ \hbar_i(z_1,\dots,z_k) &= \hbar_i(z_1,\dots,z_{k+1}) - z_{k+1}\hbar_{i-1}(z_1,\dots,z_{k+1}). \end{aligned}$$

Taking the limit $N \to \infty$ ends up being equivalent to consider the ring of invariants $Q_k^{S_k}$ and $Q_{k+1}^{S_k \times S_1}$, so that

$$H(G_k^\infty) \cong Q_k^{S_k}, \qquad\qquad H(G_{k,k+1}^\infty) \cong Q_{k+1}^{S_k \times S_1}.$$

Thus the nilHecke algebra is isomorphic to

$$\mathrm{NH}_n \cong \mathrm{End}_{Q_n^{S_n}}(Q_n).$$

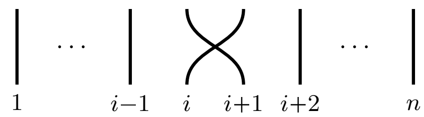
From the perspective of Definition 5.1.1, this isomorphism is given by sending x_i to the multiplication by z_i, and τ_i to the *divided difference operator*

$$\partial_i : Q_n \to Q_n, \quad f \mapsto \frac{f - \sigma_i(f)}{z_i - z_{i+1}},$$

where $\sigma_i \in S_n$ is the simple transposition $(i, i+1)$. Note that the operator ∂_i decreases the degree by 2.

5.1.2 Diagrammatic description

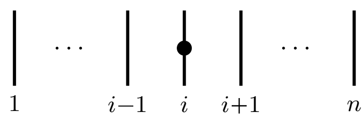
The symmetric group S_n admits a diagrammatic description by braid-like diagram on n strands. The simple transposition $\sigma_i \in S_n$ is drawn as a crossing

$\sigma_i \sim$ 

between the ith and $(i+1)$th strands. Composition in S_n is then given by stacking diagrams on top of each other. We read diagrams from bottom to top, so that the composition $w_1 w_2$ is given by putting the diagram of w_1 on top of the one of w_2. Then the diagrams are subject to the local relations

which correspond with the relations $\sigma_i \sigma_i = 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ in the presentation by generators and relations of the symmetric group. We consider these diagrams up to regular isotopy, which coincides with the relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$.

Then we look at the subalgebra of $\mathrm{End}_{\mathbb{Q}}(Q_n)$ generated by action of $\sigma_1, \dots, \sigma_{n-1}$, together with multiplication on the left by $z_1, \dots, z_n$. It can also be described using ($\mathbb{Q}$-linear combinations of) braid-like diagrams, where we add a dot on the ith strand for the multiplication by z_i,

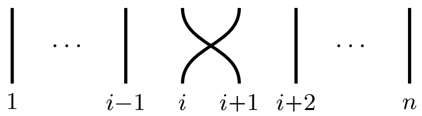
$z_i =$ 

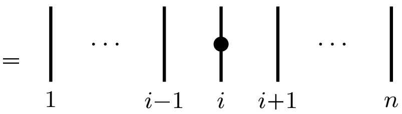
Again, distant permutations $\sigma_i z_j = z_j \sigma_i$ for $j - i \neq 0, 1$ are given by regular isotopy, and we get $\sigma_i z_j = z_{j+1}\sigma_i$ and $z_j \sigma_i = \sigma_i z_{j+1}$ as local relations

Putting all these things together gives us a diagrammatic description of the above-mentioned algebra.

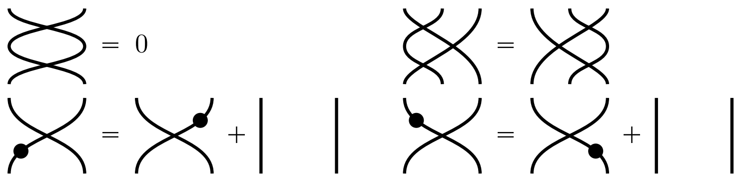
Since the nilHecke algebra has a similar-looking presentation by generators and relations, it naturally gives NH_n a diagrammatic description, with braid-like diagram on n strands.

Proposition 5.1.3. *The nilHecke algebra* NH_n *admits a presentation by* $\Bbbk$*-linear combinations of braid-like diagrams generated by*

$\tau_i =$ 

$x_j =$ 

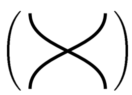
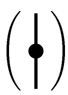
for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$*, up to regular isotopy, and with local relations*



(5.3)

(5.4)

and $\mathbb{Z}$*-grading*

$\deg_q$  $= -2,$ $\qquad$ $\deg_q$  $= 2.$

Recall the functors $\mathcal{F}_k$ of §4.3 given by tensoring with the bimodule $\Omega_{k+1,k}$. Also recall the nilCoxeter map $\Psi_\downarrow$ from §4.3.1.

Proposition 5.1.4. *There is a injective map of algebras*

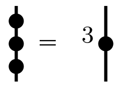
$$\mathrm{NH}_n \hookrightarrow \mathrm{End}(\mathcal{F}_{k+n-1} \cdots \mathcal{F}_{k+1}\mathcal{F}_k)$$

for all $k \geqslant 0$, explicitly given by

$$\tau_i \mapsto \mathrm{Id}_{\mathscr{F}_{k+n-1}\mathscr{F}_{k+i+1}} \otimes \Psi_\downarrow \otimes \mathrm{Id}_{\mathscr{F}_{k+i-2}\cdots\mathscr{F}_k},$$
$$x_j \mapsto \mathrm{Id}_{\mathscr{F}_{k+n-1}\mathscr{F}_{k+j}} \otimes \xi_{k+j-1} \otimes \mathrm{Id}_{\mathscr{F}_{k+j-2}\cdots\mathscr{F}_k}.$$

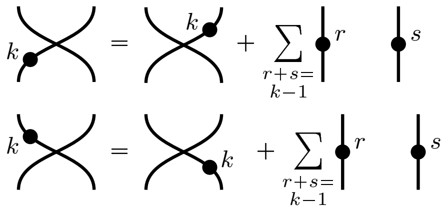
Proof. Thanks to Lemma 4.3.8, we can apply the exact same computations as in [48, Lemma 4.10], which concludes the proof. ■

When there is k dots on a strand, we usually draw it as a single dot with a label k next to it. For example, we draw



We now describe some useful, classical relations in the nilHecke algebra.

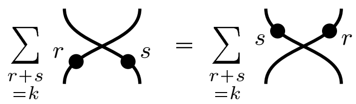
Lemma 5.1.5. *In* NH_n *we have*



for all $k \geqslant 0$.

Proof. Both equations are obtained by using recursively (5.4). ■

Lemma 5.1.6. *In* NH_n *we have*



(5.5)

for all $k \geqslant 0$.

Proof. It follows from Lemma 5.1.5. ■

5.2 Cyclotomic quotient and categorification

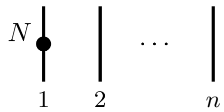
The nilHecke algebra is given by using full flags in $\mathbb{C}^\infty$. Working in $\mathbb{C}^N$ instead yields the *N-cyclotomic quotient* [42, §3.4] of the nilHecke algebra:

$$\mathrm{NH}_n^N \cong \mathrm{End}_{H(G_n^N)}(H(G_{1,2,\ldots,n}^N)).$$

The good news is that it also admits a nice diagrammatic description. As a matter of facts, it can be described directly as a quotient of the nilHecke algebra

$$\mathrm{NH}_n^N := \mathrm{NH}_n /(x_1^N).$$

This means we kill all the diagrams that admit, at some height, N dots on their leftmost strand



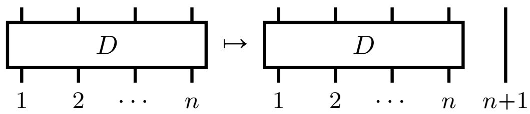
$$= 0. \tag{5.6}$$

It yields a finite-dimensional algebra, and is zero whenever $n > N$.

Furthermore, there is a Morita equivalence (i.e. an equivalence of categories of representations) between NH_n^N and $H(G_n^N)$. Therefore, one can construct a categorification of the integrable $U_q(\mathfrak{sl}_2)$-module $V(N)$ with NH_n^N. As a matter of fact, it gives naturally a categorification of the induced basis (1.6). Since it will be useful later on, we now proceed to quickly describe the categorical action of $U_q(\mathfrak{sl}_2)$ on NH_n^N-mod.

5.2.1 Categorical action

There is an inclusion of algebras $j_n : \mathrm{NH}_n^N \hookrightarrow \mathrm{NH}_{n+1}^N$ that consists in adding a vertical strand to the right of a diagram $D \in \mathrm{NH}_n^N$,



$\in \mathrm{NH}_{n+1}^N$.

From this map, we obtain induction and restriction functors on the categories of modules

$$\mathrm{Ind}_n^{n+1} : \mathrm{NH}_n^N\text{-mod} \to \mathrm{NH}_{n+1}^N\text{-mod},$$
$$\mathrm{Res}_n^{n+1} : \mathrm{NH}_{n+1}^N\text{-mod} \to \mathrm{NH}_n^N\text{-mod},$$

which form an adjoint pair. In terms of bimodules, we can describe them as

$$\mathrm{Ind}_n^{n+1}(-) = \big(\mathrm{NH}_{n+1}^N\big)_{(n)} \otimes_{\mathrm{NH}_n^N} -,$$
$$\mathrm{Res}_n^{n+1}(-) \cong {}_{(n)}\big(\mathrm{NH}_{n+1}^N\big) \otimes_{\mathrm{NH}_{n+1}^N} -,$$

where $\big(\mathrm{NH}_{n+1}^N\big)_{(n)}$ (resp. ${}_{(n)}\big(\mathrm{NH}_{n+1}^N\big)$) is given by looking at NH_{n+1}^N as an NH_{n+1}^N-NH_n^N-bimodule (resp NH_n^N-NH_{n+1}^N-bimodule).

We cite the following results, which can be found as a particular case of [33] for example. We will also recover it as a consequence of our work later.

Proposition 5.2.1. *The* NH_n^N*-module* $\left(\mathrm{NH}_{n+1}^N\right)_{(n)}$ *decomposes as a direct sum*

$$\left(\mathrm{NH}_{n+1}^N\right)_{(n)} \cong \bigoplus_{\ell=0}^{N-n-1} \bigoplus_{a=0}^{n} q^{2\ell-2a}\, \mathrm{NH}_n^N .$$

Therefore we define the exact functors

$$\mathscr{F}_n^N := \mathrm{Ind}_n^{n+1}, \qquad \mathscr{E}_n^N := q^{2n-N}\,\mathrm{Res}_n^{n+1},$$

such that in the Grothendieck group, $[\mathrm{NH}_n^N]$ will descend to the induced basis element v_{N-2n}, with $\mathscr{F}_n^N$ and $\mathscr{E}_n^N$ giving the coefficients appearing in (1.6). They respect direct sum decompositions similar the ones in §4.1.2.

Theorem 5.2.2. *There are decompositions of bimodules*

$$\begin{aligned} \mathscr{E}_n^N\mathscr{F}_n^N &\cong \mathscr{F}_{n-1}^N\mathscr{E}_{n-1}^N \oplus_{[N-2n]_q} \mathrm{Id}_n && \text{if } N-2n \geqslant 0, \\ \mathscr{F}_{n-1}^N\mathscr{E}_{n-1}^N &\cong \mathscr{E}_n^N\mathscr{F}_n^N \oplus_{[2n-N]_q} \mathrm{Id}_n && \text{if } N-2n \leqslant 0. \end{aligned}$$

Moreover, there is a biadjunction $\mathscr{F}_n^N \dashv q^{N-2n}\mathscr{E}_n^N$ *and* $\mathscr{E}_n^N \dashv q^{2n-N}\mathscr{F}_n^N$.

For a permutation $w \in S_n$ with minimal presentation $w = \sigma_{i_r}\cdots\sigma_{i_2}\sigma_{i_1}$, we write $\tau_w := \tau_{\sigma_{i_r}}\cdots\tau_{\sigma_{i_2}}\tau_{\sigma_{i_1}}$. It is well defined thanks to (5.3). Let ϑ_n denotes the longest word in S_n, and define

$$e_n := \tau_{\vartheta_n} x_1^{n-1}x_2^{n-2}\cdots x_{n-1} \in \mathrm{NH}_n . \tag{5.7}$$

It is a well known fact that it is an idempotent of NH_n (see for example [42, Section 2.2]), and therefore also of NH_n^N. Then, as in [42] we put

$$P_n^N := \mathrm{NH}_n^N\, e_n,$$

which is the unique indecomposable projective module of NH_n^N, up to shift and isomorphism. It is concentrated only in positive degrees. Moreover, there is a decomposition

$$\mathrm{NH}_n^N \cong \oplus_{([n]_q!)} q^{n(n-1)/2} P_n^N . \tag{5.8}$$

We write $\tilde{P}_n^N := q^{n(n-1)/2}P_n^N$, so that in the Grothendieck group $[\tilde{P}_n^N]$ will correspond with the canonical basis element v'_{N-2n}. In addition, P_n^N admits a unique simple quotient

$$S_n^N := P_n^N / \mathrm{Sym}^+(\mathrm{NH}_n^N),$$

where $\mathrm{Sym}^+(\mathrm{NH}_n^N)$ is the subalgebra given by symmetric polynomials in the variables $x_1, \dots, x_n \in \mathrm{NH}_n^N$ of non-zero degree. We also write $\tilde{S}_n^N := q^{n(n-1)/2} S_n^N$.

We note that NH_n^N equipped with a trivial differential does not form a positive c.b.l.f. $\mathbb{Z}$-graded dg-algebra, since it is not concentrated in positive q-degree. Fortunately enough, it respects all the conditions of Remark 3.4.32 so that we can still apply the results of §3.4. Thus we put

$$\mathcal{V}^{\mathrm{NH}}(N) := \bigoplus_{n=0}^{N} \mathcal{V}_{N-2n}^{\mathrm{NH}}, \qquad \mathcal{V}_{N-2n}^{\mathrm{NH}} := \mathcal{D}^{lc}(\mathrm{NH}_n^N, 0).$$

We obtain that

$$\boldsymbol{K}_0^{\Delta}(\mathcal{V}_{N-2n}^{\mathrm{NH}}) \cong \mathbb{Z}(\!(q)\!),$$

is generated by either $[\mathrm{NH}_n^N]$, $[\tilde{P}_n^N]$ or $[\tilde{S}_n^N]$. We put $\mathcal{E}^N := \oplus_{n=0}^N \mathcal{E}_n^N$ and $\mathcal{F}^N := \oplus_{n=0}^N \mathcal{F}_n^N$. By Proposition 5.2.1 we know that $\mathcal{F}^N$ and $\mathcal{E}^N$ are exact, preserve c.b.l. compact modules and are of finite amplitude. Thus $\boldsymbol{K}_0^{\Delta}(\mathcal{V}^{\mathrm{NH}}(N))$ is an $U_q(\mathfrak{sl}_2)$-module, with action of E, F given by $[\mathcal{E}^N]$, $[\mathcal{F}^N]$.

We consider the map $\psi : \mathrm{NH}_n^N \to \left(\mathrm{NH}_n^N\right)^{\mathrm{op}}$ that takes the mirror image of a diagram along the horizontal axis. From it, given a left NH_n^N-module M we obtain a right NH_n^N-module M^ψ. Then, we define the bifunctor

$$(-,-) : \mathcal{V}^{\mathrm{NH}}(N) \otimes_{\Bbbk} \mathcal{V}^{\mathrm{NH}}(N) \to \mathcal{D}^{lc}(\Bbbk, 0), \quad (M, M') := M^\psi \otimes^{\mathrm{L}}_{(\oplus_n \mathrm{NH}_n^N)} M'.$$

It respects again all the requirement for a categorical version of the Shapovalov form, and we obtain that $\tilde{S}_n^N$ is the dual of $\tilde{P}_n^N$. Therefore, $[\tilde{S}_n^N]$ corresponds in the Grothendieck group to the dual canonical basis element v^{N-2n}. In conclusion, we obtain the following theorem:

Theorem 5.2.3. *The Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathcal{V}^{\mathrm{NH}}(N))$ is an $U_q(\mathfrak{sl}_2)$-module, with action of E, F given by $[\mathcal{E}^N]$, $[\mathcal{F}^N]$. Moreover, there is an isomorphism*

$$\boldsymbol{K}_0^{\Delta}(\mathcal{V}^{\mathrm{NH}}(N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong V(N),$$

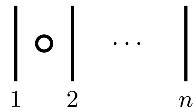
sending $[\mathrm{NH}_n^N]$ to the induced basis element v_{N-2n}, $[\tilde{P}_n^N]$ to the canonical basis element v'_{N-2n}, and $[\tilde{S}_n^N]$ to the dual canonical basis element v^{N-2n}.

5.3 Dg-enhancement

To go from the categorification of $V(N)$ to a Verma module $M(\lambda)$, we constructed a dg-enhancement of $H(G_k^N)$. We now do the same for NH_n^N. Actually, we have three equivalent ways to obtain a dg-enhancement of the cyclotomic nilHecke algebra, each of them bringing their own advantages.

5.3.1 Primitivating a quotient relation

The first approach is similar to what we did to go from $H(G_k^N)$ to Ω_k passing by $H(G_k^\infty)$, but where we replace $H(G_k^N)$ with NH_n^N and $H(G_k^\infty)$ with NH_n. Since NH_n^N is obtained as a quotient of NH_n by the cyclotomic relation (5.6), we introduce a new generator

$\omega :=$ 

in homological degree 1, that will play the role of a primitivation of (5.6).

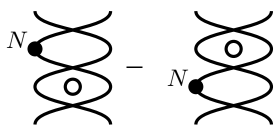
Therefore, we consider the algebra $(\mathrm{NH}_n \star \mathbb{Q}[\omega], d_N)$, where $\mathrm{NH}_n \star \mathbb{Q}[\omega]$ is the free product of algebras, with differential

$$d_N(x_i) := d_N(\tau_i) := 0, \qquad\qquad d_N(\omega) := x_1^N.$$

Clearly in homological degree zero we obtain $H^0(\mathrm{NH}_n \star \mathbb{Q}[\omega], d_N) \cong \mathrm{NH}_n^N$. However, in non-zero homological degree, $H(\mathrm{NH}_n \star \mathbb{Q}[\omega], d_N)$ also contains non-trivial elements, since there is no relations between ω and the generators of NH_n. For example

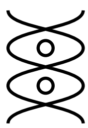
$$d_N(\omega x_i - x_i\omega) = 0, \qquad\qquad \omega x_i - x_i\omega \notin \mathrm{im}(d_N).$$

Therefore we need to introduce relations between ω and the generators of NH_n. Intuitively the idea is to look for the relations that x_1^N respects for all $N \geqslant 0$ and lift them to ω. We call this procedure a *primitivization of the relation* $x_1^N = 0$. For example, we obtain that ω must commute with all x_i and τ_j for $1 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant n-1$. Also we must have $\omega^2 = 0$. A little bit less trivial is the relation



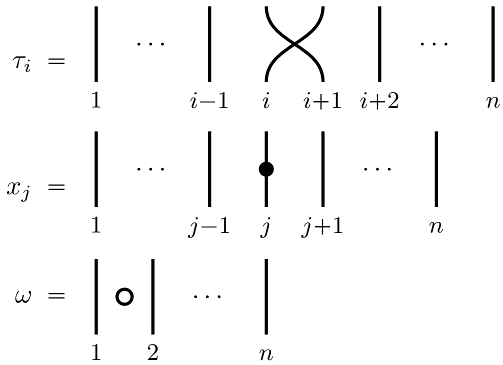
$= 0.$

which is obtained by applying Lemma 5.1.5 followed by (5.3). It implies that we need to impose

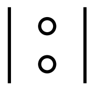
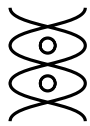
 $:= 0.$

As we will prove later, it appears that these are enough. Forgetting the differential and introducing a λ-grading (as we did in §4.2.1), we get the algebra we were looking for.

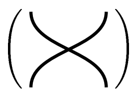
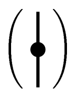
Definition 5.3.1. *The dg-enhanced nilHecke algebra A_n is the diagrammatic $\Bbbk$-algebra generated by*

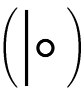


for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, up to regular isotopy, and with the nilHecke relations (5.3-5.4), and the new relations

 $= 0,$  $= 0.$ (5.9)

It is $\mathbb{Z}^2 \times \mathbb{Z}$-graded with

$\deg_{q,\lambda,h}$  $:= (-2,0,0),$ $\deg_{q,\lambda,h}$  $:= (2,0,0),$

$\deg_{q,\lambda,h}$  $:= (0,2,1).$

We will prove the following proposition later on:

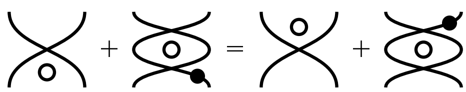
Proposition 5.3.2. *The dg-algebra (A_n, d_N) is formal with homology*

$$H(A_n, d_N) \cong \mathrm{NH}_n^N,$$

for $n \leqslant N$, and is acyclic whenever $n > N$.

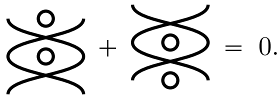
We also collect some results that will reveal to be useful.

Lemma 5.3.3. *In A_n we have*



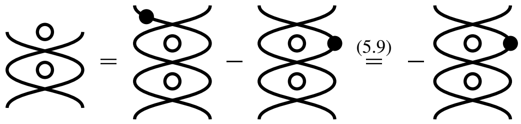
Proof. We apply (5.4) on both sides. ∎

Lemma 5.3.4. *In A_n we have*

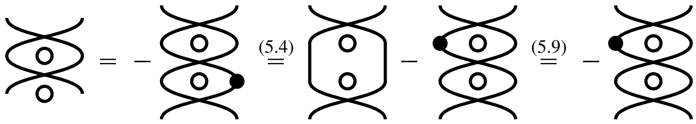


(5.10)

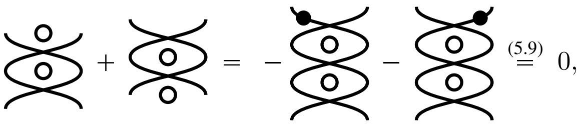
Proof. Using (5.4), we obtain



Similarly, we obtain



Then by (5.5) we get



concluding the proof. ■

5.3.2 From Ω_k to A_n

A second approach consists in looking at the endomorphism algebra

$$A'_n := \mathrm{End}_{\Omega_n}(\Omega_{n,n-1} \otimes_{n-1} \cdots \otimes_2 \Omega_{2,1} \otimes_1 \Omega_{1,0}).$$

By Proposition 5.1.4, we already know that there is an inclusion $\mathrm{NH}_n \hookrightarrow A'_n$.

Proposition 5.3.5. *We have*

$$\mathrm{gdim}\, A'_n = \big(\mathrm{gdim}\, \mathrm{NH}_n\big)\big(\mathrm{gdim} \textstyle\bigwedge^\bullet(\omega_{1,n}, \dots, \omega_{n,n})\big).$$

Proof. From Proposition 4.3.5, we obtain

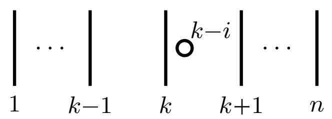
$$\begin{aligned}
&\mathrm{End}_{\Omega_n}(\Omega_{n,n-1} \otimes_{n-1} \cdots \otimes_2 \Omega_{2,1} \otimes_1 \Omega_{1,0}) \\
&\qquad \cong \mathrm{Hom}_{\Omega_n}(\mathbb{Q}, \Omega_{0,1} \otimes_1 \cdots \otimes_{n-1} \Omega_{n-1,n} \otimes_n \Omega_{n,n-1} \otimes_{n-1} \cdots \otimes_1 \Omega_{1,0}),
\end{aligned}$$

then we use (4.18) recursively to compute the graded dimension. ■

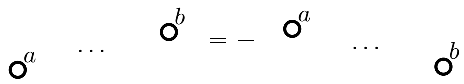
Living inside A'_n also lies the multiplication map by $\omega_{i,k} \in \Omega_{k,k-1}$ for all $1 \leqslant k \leqslant n$ and $1 \leqslant i \leqslant k$. Since the elements $\omega_{i,n}$ are linearly independent for all $1 \leqslant i \leqslant n$ we obtain that $A'_n \cong \big(\mathrm{NH}_n \star \bigoplus_{k=1}^{n} \bigwedge(\omega_{1,k}, \dots, \omega_{k,k})\big)/\sim$ where $\sim$ is given by the kernel of the map

$$\mathrm{NH}_n \star \bigoplus_{k=1}^{n} \bigwedge(\omega_{1,k}, \dots, \omega_{k,k}) \twoheadrightarrow A'_n, \tag{5.11}$$

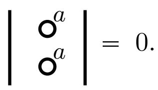
sending $\omega_{i,k}$ to the multiplication by $\omega_{i,k} \in \Omega_{k,k-1}$. In order to get a diagrammatic description of A'_n we use the nilHecke diagrams together with the new generator

$\omega_{i,k} :=$ 

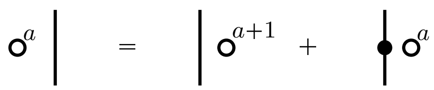
that we refer as a *floating dot with label* $k - i$. We do not allow two floating dots to be put at the same height. Moreover, they anticommute with each other,



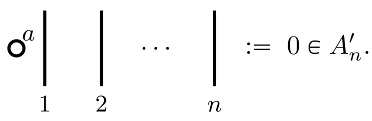
for all possible labels a, b. In particular, two floating dots with the same label in the same region is zero



Now it only remains to compute the kernel of the map (5.11). We obtain easily that $\omega_{i,k}$ commutes with τ_j for $j \neq k$, and commutes with x_j for all j. This coincides with the relations given by regular isotopy between floating dots and dots or distant crossings. From (4.9) we get the local relation

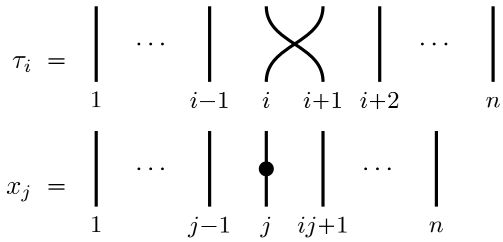


where we interpret a floating dot in the leftmost region of a diagram as zero

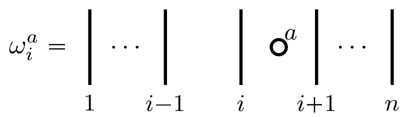


Since these relations make sense for any $a \geqslant 0$, from now on we allow floating dots with label a for all $a \in \mathbb{N}$.

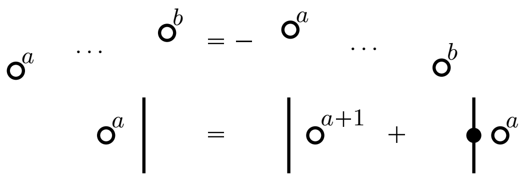
Proposition 5.3.6. *The algebra A'_n admits a diagrammatic presentation with generators given by the nilHecke elements*



for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, and the floating dots



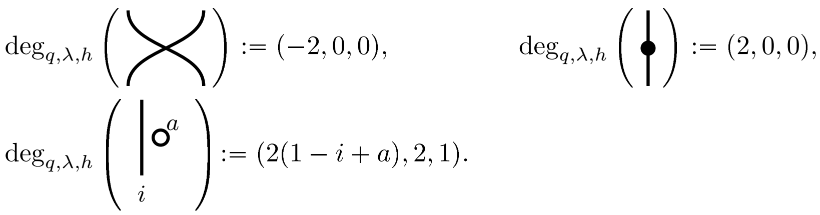
for $1 \leqslant i \leqslant n$ and $a \geqslant 0$, up to regular isotopy that preserves the relative height of the floating dot, subject to the nilHecke relations (5.3-5.4), and the floating dots relations



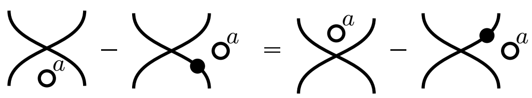
(5.12)

(5.13)

with $\mathbb{Z}^2 \times \mathbb{Z}$-grading

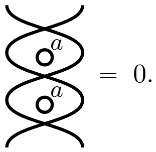


Lemma 5.3.7. *In A'_n we have*

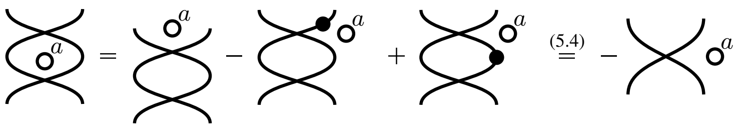


Proof. It follows directly from (5.13) and commutation of ω_{i+1}^a with τ_i. ∎

Lemma 5.3.8. *In A'_n we have*

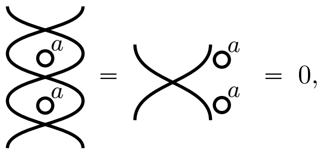


Proof. First we compute using Lemma 5.3.7



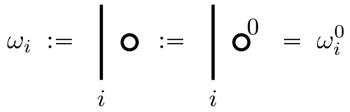
(5.14)

Then we obtain

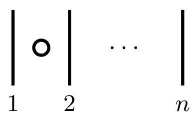


which concludes the proof. ■

To simplify notations, we do not write the label of a floating dot whenever it is zero

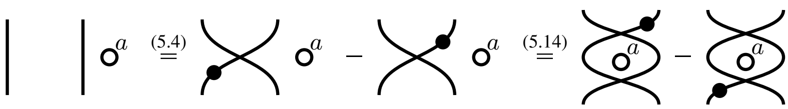


Moreover, we say that a floating dot placed directly at the right of the leftmost strand and with label 0 is a *tight floating dot*



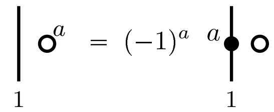
Proposition 5.3.9. *The algebra A'_n is generated by nilHecke elements and tight floating dots.*

Proof. First, we obtain that all floating dots can be brought to be next to the leftmost strand by applying recursively



(5.15)

Then, since a floating dot in the leftmost region is zero, we obtain for a floating dot next to the leftmost strand



by applying recursively (5.13). ∎

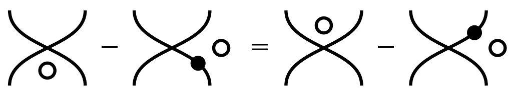
Lemma 5.3.10. *There is an isomorphism of algebras*

$$A_n \xrightarrow{\simeq} A'_n,$$

preserving nilHecke element and sending ω to the tight floating dot ω_1.

Proof. Lemma 5.3.8 shows the map is well-defined, and Proposition 5.3.9 gives the surjectivity. For the injectivity, on can construct an inverse map by introducing elements $\varpi_i^a \in A_n$ defined recursively from ω using a formula similar to (5.15) and verifying that it respects all relations of ω_i^a in A'_n. We leave the details to the reader. ∎

Remark 5.3.11. *Note that alternatively we can present A'_n using only floating dots with label* 0 *and imposing the relation*



Then we recover the floating dots with label greater than zero by defining them recursively using (5.13).

Using the isomorphism $A'_n \xrightarrow{\simeq} A_n$, we can compute the value of d_N for all floating dots. Doing so, we obtain

$$d_N(\omega_i^a) = (-1)^{N+a-i+1}\hbar_{N+a-i+1}(x_1, \ldots, x_i), \tag{5.16}$$

which coincides with (4.8) and (5.2).

5.3.3 From symmetric group to A_n

The third way to obtain A_n is by 'dg-enhancing' the action of the symmetric group S_n on Q_n. For this we consider the $\mathbb{Z}^2$-graded ring

$$Q_n^\wedge := \mathbb{Q}[x_1, \ldots, x_n] \otimes_{\mathbb{Q}} \Lambda^\bullet(\omega_1, \ldots, \omega_n),$$

with $\deg_{q,\lambda}(x_i) = (2,0)$ and $\deg_{q,\lambda}(\omega_i) = (2-2i,2)$. We think of ω_i as a primitivization of the relation

$$(-1)^{N-i+1}\hbar_{N-i+1}(x_1,\dots,x_i) = 0,$$

for all $N \in \mathbb{N}$. Therefore we compute

$$\begin{aligned}
&\sigma_i(\hbar_{N-j+1}(x_1,\dots,x_j)) \\
&\quad= \begin{cases} \hbar_{N-j+1}(x_1,\dots,x_j) + (x_{i+1}-x_i)\hbar_{N-j}(x_1,\dots,x_{j+1}), & \text{if } i=j, \\ \hbar_{N-j+1}(x_1,\dots,x_j), & \text{if } i \neq j. \end{cases}
\end{aligned}$$

for $\sigma_i \in S_n$ the simple transposition. From it, we deduce an S_n action on $Q_n^\wedge$.

Proposition 5.3.12. *There is an action of S_n on $Q_n^\wedge$ given by setting*

$$\begin{aligned}
\sigma_i(x_j) &= x_{\sigma_i(j)}, \\
\sigma_i(\omega_j) &= \omega_j + \delta_{i,j}(x_i - x_{i+1})\omega_{i+1},
\end{aligned}$$

and extending using $\sigma_i(fg) = \sigma_i(f)\sigma_i(g)$ for all $f, g \in Q_n^\wedge$.

Proof. The proof is a straightfoward computation checking that $\sigma_i\sigma_{i+1}\sigma_i(\omega_j) = \sigma_{i+1}\sigma_i\sigma_{i+1}(\omega_j)$ and $\sigma_i^2(\omega_j) = \omega_j$. We leave the details to the reader. ∎

Then we consider the divided difference operator on $Q_n^\wedge$

$$\partial_i(f) := \frac{f - \sigma_i(f)}{x_i - x_{i+1}},$$

for all $1 \leqslant i \leqslant n-1$, and $f \in Q_n^\wedge$.

Proposition 5.3.13. *The action of ∂_i on $Q_n^\wedge$ satisfies the Leibniz rule*

$$\partial_i(fg) = \partial_i(f)g + \sigma_i(f)\partial_i(g),$$

for all $f, g \in Q_n^\wedge$, and the relations

$$\begin{aligned}
\partial_i^2(f) &= 0, & \partial_i\partial_{i+1}\partial_i(f) &= \partial_{i+1}\partial_i\partial_{i+1}(f), \\
\partial_i(x_i f) &= x_{i+1}\partial_i(f) + f, & x_i\partial_i(f) &= \partial_i(x_{i+1}f) + f, \\
\partial_i\partial_j(f) &= \partial_j\partial_i(f),
\end{aligned}$$

for $|i-j|>1$, and

$$\partial_i(\omega_j f) = \omega_j \partial_i(f),$$
$$\partial_i((\omega_i - x_{i+1}\omega_{i+1})f) = (\omega_i - x_{i+1}\omega_{i+1})\partial_i(f),$$

for $i \neq j$.

Proof. The proof consists in verifying that all relations hold. They all follow from the fact that there is an action of the symmetric group and ∂_i is a divided difference operator, implying that $\partial_i(f) = 0$ whenever f is invariant for the S_n-action, and that $\partial_i(f)$ is an invariant of the S_n-action for all f. We leave the details to the reader. ■

Definition 5.3.14. *We define A''_n as the subalgebra of $\mathrm{End}_{\mathbb{Q}}(Q_n^\wedge)$ generated by multiplication by x_i, multiplication by ω_i and by divided difference operator ∂_j for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$.*

Proposition 5.3.15. *There is a surjective map of algebras*

$$A'_n \twoheadrightarrow A''_n$$

given by sending x_i to multiplication by z_i, ω_i^0 to multiplication by ω_i, and τ_i to ∂_i.

Proof. It follows from Remark 5.3.11 and Proposition 5.3.13. ■

We will prove later that the epimorphism of Proposition 5.3.15 is actually an isomorphism.

5.3.4 Some basis for A_n

Recall that given $w \in S_n$, we write $\tau_w := \tau_{\sigma_{i_r}} \cdots \tau_{\sigma_{i_1}} \in A_n$ where $\sigma_{i_r} \cdots \sigma_{i_1} = w$ is a minimal presentation of w.

Proposition 5.3.16. *The collection of elements*

$$\{(\omega_1^0)^{\ell_1} \cdots (\omega_n^0)^{\ell_n} x_1^{k_1} \cdots x_n^{k_n} \tau_w | \ell_i \in \{0,1\}, k_i \in \mathbb{N}, w \in S_n\}, \tag{5.17}$$

forms a basis of $A'_n \cong A_n$.

Proof. We first show elements in (5.17) generate A'_n. Using (5.13), we can bring all floating dots to the right, then slide them to the top of the diagram and finally convert them back to floating dots with label 0, using again (5.13). Then using (5.4), we can bring all dots also to the top, doing an inductive argument on the number of crossings. Therefore, it remains only diagrams with crossings at the bottom, then dots, and finally floating dots. Since the crossing respects relations (5.3) similar to the symmetric group, we can apply a similar reduction algorithm to end up with crossings corresponding to minimal presentation of element of S_n.

To show they are linearly independent elements, we use Proposition 5.3.15. It is known that the action of the divided difference operators on the polynomial ring Q_n is faithful (see for example [53]). From it we get that different choices of $w \in S_n$ gives linearly independent elements in A''_n. Since the remaining part in (5.17) is only multiplication by polynomials, we conclude the action is faithful, and thus (5.17) is a basis. ■

Corollary 5.3.17. *The surjective map of Proposition 5.3.15 is an isomorphism.*

From this basis, we construct another one which has all its floating dots concentrated in the rightmost region.

Proposition 5.3.1. The collection of elements

$$\{(\omega_n^0)^{\ell_1} \cdots (\omega_n^{n-1})^{\ell_n} x_1^{k_1} \cdots x_n^{k_n} \tau_w | \ell_i \in \{0,1\}, k_i \in \mathbb{N}, w \in S_n\}, \tag{5.18}$$

forms a basis of A_n.

Proof. We have $\deg(\omega_i^0) = \deg(\omega_n^{n-i})$. Therefore, we only need to show that each ω_i can be rewritten as a combination of dots and floating dots in the rightmost region. This follows by applying (5.13) i times. Since the label of the floating dots increases by at most 1 for each strand crossed, the maximal label involved is $n-1$, obtained by sliding $\omega_1 = \omega_1^0$ to the far right. This concludes the proof. ■

From this basis, we obtain a proof of Proposition 5.3.2.

Proof of Proposition 5.3.2. We only need to show that $H(A_n, d_N)$ is concentrated in homological degree zero. By (5.16), we have

$$d_N(\omega_n^a) = (-1)^{N+a-n+1} \hbar_{N+a-n+1}(x_1, \dots, x_n),$$

for $0 \leqslant a \leqslant n-1$, which are in the center of $\mathrm{NH}_n \subset A_n$ thanks to (5.5). It is a well-known fact that there is an isomorphism between the center of NH_n and $H(G_n^\infty)$, where $(-1)^{N+a-n+1}\hbar_{N+a-n+1}(x_1, \dots, x_n)$ is identified with $Y_{N+a-n+1,n}$. Thus $d_N(\omega_n^0), \dots, d_N(\omega_n^{n-1})$ forms a regular sequence of the center $Z(A_n)$, and we deduce that $H(A_n, d_N)$ is concentrated in homological degree zero. ■

Tightened basis

We now describe yet another basis that will reveal to be important for categorification purposes. It only involves floating dots that are tight. We say that a reduced expression

$\sigma_{i_r} \cdots \sigma_{i_1}$ of $w \in S_n$ is *left-adjusted* if $i_r + \cdots + i_1$ is minimal. Equivalently, it is left-adjusted if and only if

$$\min_{t \in \{0,\dots,r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k) \leqslant \min_{t \in \{0,\dots,r\}} \sigma_{j_t} \cdots \sigma_{j_1}(k),$$

for all $k \in \{0, \dots, n\}$ and all other reduced expression $\sigma_{j_r} \cdots \sigma_{j_1} = w$. In this condition, we write

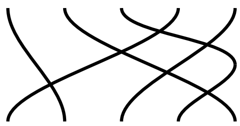
$$\min_w(k) := \min_{t \in \{0,\dots,r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k).$$

Note that a left adjusted expression always exists and is unique up to distant permutation ($\sigma_i \sigma_j \leftrightarrow \sigma_j \sigma_i$ for $|i - j| > 1$), so that $\min_w(k)$ is well-defined. In particular, one can obtain a left-adjusted reduced expression for any permutation by taking its representative in the coset decomposition

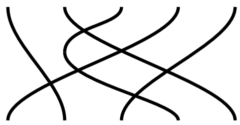
$$S_n = \bigsqcup_{a=1}^{n} S_{n-1} \sigma_{n-1} \cdots \sigma_a, \tag{5.19}$$

applied recursively. If we think of a reduced expression in terms of string diagrams, then it is left-adjusted if all strings are pulled as far as possible to the left.

Example 5.3.18. *The permutation* $(1\ 3\ 2\ 4) \in S_4$ *admits as left-adjusted reduced expression the word* $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$ *which comes from the summand* $S_2\sigma_3\sigma_2$ *in the first step of the recursive decomposition* (5.19). *Note that* $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ *is also left-adjusted while* $\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2$ *and* $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$ *are not. In terms of string diagrams, we consider as example the following reduced expression of the permutation* $w = (1\ 4\ 3\ 5\ 2) \in S_5$:



It is not left-adjusted since the 4*th strand (read at the bottom) can be pulled to the left. Hence we obtain the following left-adjusted minimal presentation:*

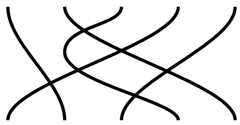


Suppose $\sigma_{i_r} \cdots \sigma_{i_1}$ is a left-adjusted reduced expression of w. Then we can choose for each $k \in \{1, \dots, n\}$ an index $t_k \in \{1, \dots, r\}$ such that

$$\sigma_{i_{t_k}} \cdots \sigma_{i_1}(k) = \min_w(k).$$

Clearly this choice is not necessarily unique and we can have $t_k = t_{k'}$ for $k \neq k'$. However, it defines a partial order $\prec$ on the set $\{1, \dots, n\}$ where $k \prec k'$ whenever $t_k \leqslant t_{k'}$. We extend this order arbitrarily and we write $<_t$ for it. There is a bijective map $s : \{1, \dots, n\} \to \{1, \dots, n\}$ which sends $k < k'$ to $s(k) <_t s(k')$, so that $t_{s(k)} \leqslant t_{s(k')}$. In terms of string diagrams, the map s tells us in which order the strands attain their (chosen) leftmost position while reading from bottom to top. In particular, $s(k)$ gives the starting point of the strand that attains its leftmost position in kth position.

Example 5.3.19. *Consider again the following left-adjusted string diagram:*



Both the 1*st and* 3*th strand attain their leftmost position at the bottom of the diagram, thus we can choose* $s(1) = 1$ *and* $s(2) = 3$*. Then both the* 2*nd and* 4*th strand attain their leftmost position, thus we can take* $s(3) = 4$ *and* $s(4) = 2$*. Finally, the* 5*th strand attains its leftmost position and we put* $s(5) = 5$*.*

For $k \in \{1, \dots, n+1\}$, we put

$$w^k := \sigma_{i_{t_{s(k)}}} \cdots \sigma_{i_{t_{s(k-1)}}},$$

where it is understood that $t_{s(0)} := 0$ and $t_{s(n+1)} := r$. It defines a partition of the reduced expression of $\sigma_{i_r} \cdots \sigma_{i_1} = w$. Moreover, it is constructed so that

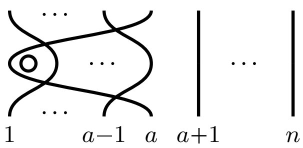
$$w^k \cdots w^1(s(k)) = \min_w(s(k)),$$

for all $1 \leqslant k \leqslant n$.

Example 5.3.20. *Consider again* $w = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$ *with* $i_1 = 2, i_2 = 3, i_3 = 1, i_4 = 2, i_5 = 1$*. We can choose for example* $t_1 = 0, t_2 = 0, t_3 = 3$ *and* $t_4 = 5$*.Then we can put* $s(1) = 1$ *(or* 2*),* $s(2) = 2$ *(or* 1*),* $s(3) = 3$ *and* $s(4) = 4$*, with* $w^1 = 1, w^2 = 1$*,* $w^3 = \sigma_1\sigma_3\sigma_2$ *and* $w^4 = \sigma_1\sigma_2$*.*

Now we define the element $\theta_a := \tau_{a-1} \cdots \tau_1 \omega \tau_1 \cdots \tau_{a-1}$, which gives

θ_a :=



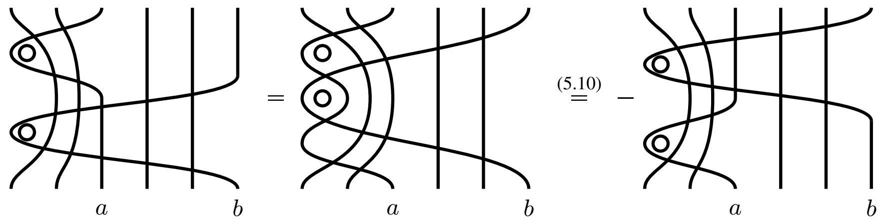
in terms of diagram. We call it a *tightened floating dot*.

Lemma 5.3.21. *Tightened floating dots anticommute with each others*

$$\theta_a\theta_b = -\theta_b\theta_a, \qquad\qquad (\theta_a)^2 = 0,$$

for all $1 \leqslant a, b \leqslant n$.

Proof. We have $\theta_1\theta_1 = 0$ since $\omega_1\omega_1 = 0$, and $\theta_a\theta_a = 0$ by (5.3) for $a > 0$. Also thanks to (5.3), we obtain



which ends the proof. ∎

We write $\theta_a^0 := \theta$ and $\theta_a^{-1} := 1$.

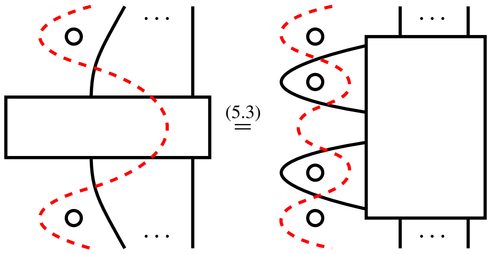
Theorem 5.3.22. *The collection of elements*

$$\begin{aligned}\{x_n^{k_n}\cdots x_1^{k_1}\tau_{w^{n+1}}\theta^{\ell_n}_{\min_w(s(n))}\tau_{w^n}\cdots\theta^{\ell_2}_{\min_w(s(2))}\tau_{w^2}\theta^{\ell_1}_{\min_w(s(1))}\tau_{w^1}|\\ k_i\in\mathbb{N}, \ell_i\in\{0,-1\}, w\in S_n\}\end{aligned} \tag{5.20}$$

for each choices involved, forms a basis for A_n.

Proof. A quick verification shows that there is degree-preserving bijection between the set (6.11) and the set (5.17), since $\deg(\omega_{k+1}) = \deg(\tau_k\cdots\tau_1\omega_1)$. Therefore, since (5.17) is already a basis, we only need to show that (6.11) generates A_n.

By the proof of Proposition 5.3.16 together with Proposition 5.3.9, we can assume A_n is generated by diagrams where all the dots are concentrated at the top and all the floating dots are tight. We claim that we can also assume there is at most one floating dot at the immediate right of each strand. Indeed, suppose there are two tight floating dots at the right of the same strand. Then we can apply a braid-isotopy using (5.3) to bring it as most as possible to the left, until it is possibly blocked by other tight floating dots. We depict it by the following picture:



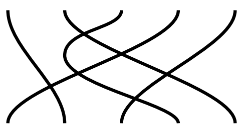
where the dashed strand in red represent the one we want to pull, and the boxes represent elements in A_n. If there is no floating dot in-between, then it is zero by (5.9). Otherwise, we apply (5.10) to jump the bottom floating dot over all the floating dots in-between, until we have two floating dots in the same region at the top, and thus zero again by (5.9). This proves the claim.

Finally, we observe that given a strand with a single tight floating dots, then we can tighten it using (5.3), until we end up with a tightened floating dot. Since tightened floating dots anticommute, this concludes the proof. ■

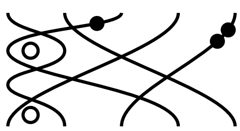
In terms of diagrams, one can build all elements of (6.11) using the following procedure:

- we take the string diagram corresponding with the minimal presentation of a permutation $w \in S_n$, and we make it left-adjusted by pulling all strands to the left (without making new intersection) ;
- for each strand, we choose whether we want to add a floating dot or not ; if yes, then we choose a position on the strand where it attains its left-most position (with respect to the other strands), and add a tightened floating dot there: that is, we pull the strand to the far left (making new intersections) and add a floating dot at its immediate right ;
- for each strand, we choose a non-negative number of dots to add at the top.

Example 5.3.23. *Consider the following left-adjusted reduced presentation*



Then we can add for example a floating dot on the 1*st and* 4*th strands,*

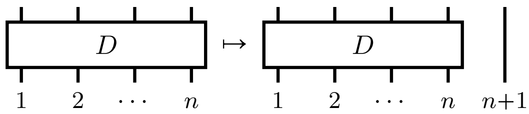


where we had to pull the 4th strand to the left in order to put a tight floating dot, the 1st one being already to the left. Finally, we add some dots at the top.

5.4 Categorical action

By [66, Proposition 2.44] we know that the center $Z(A_n)$ is isomorphic to Ω_n. Therefore, the results in [66, Corollary 2.32] tell us that A_n is Morita equivalent to Ω_n. Thus, A_n can also be used to construct a categorification of the Verma module $M(\lambda)$. We describe this construction.

As in §5.2.1, there is an injection of algebras $A_n \hookrightarrow A_{n+1}$, that consists in adding a vertical strand to the right of a diagram $D \in A_n$:



$\in A_{n+1}.$

From this map, we obtain induction and restriction functors

$$\operatorname{Ind}_n^{n+1} : A_n\text{-mod} \to A_{n+1}\text{-mod},$$
$$\operatorname{Res}_n^{n+1} : A_{n+1}\text{-mod} \to A_n\text{-mod},$$

which form an adjoint pair. Our goal is to prove they yield an exact sequence, similar to the one of §4.3.1. We write $\otimes_n$ for $\otimes_{A_n}$ and

$$A_n^\xi := A_n \otimes_{\mathbb{Q}} \mathbb{Q}[\xi] \cong \oplus_{\ell \geqslant 0} q^{2\ell} A_n.$$

We will prove the following theorem:

Theorem 5.4.1. *There is a short exact sequence*

$$0 \to q^{-2} A_n \otimes_{n-1} A_n \to A_{n+1} \to A_n^\xi \oplus q^{-4n}\lambda^2 A_n^\xi[1] \to 0$$

of A_n-A_n-bimodules, with degree-preserving maps.

Thus we define

$$\mathcal{F}_n := \operatorname{Ind}_n^{n+1}, \qquad \mathcal{F}_n(-) = A_{n+1(A_n)} \otimes_n -,$$
$$\mathcal{E}_n := q^{2n}\lambda^{-1} \operatorname{Res}_n^{n+1}, \qquad \mathcal{E}_n(-) \cong q^{2n}\lambda^{-1}{}_{(A_n)}A_{n+1} \otimes_{n+1} -,$$

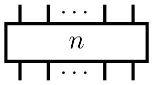
and we write $\mathrm{Id}_n := \mathrm{Id}_{A_n}$.

Corollary 5.4.2. *There is a natural short exact sequence*

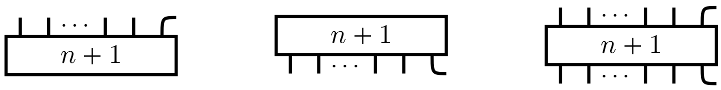
$$0 \to \mathscr{F}_{n-1}\mathscr{E}_{n-1} \to \mathscr{E}_n\mathscr{F}_n \to \oplus_{[\beta-2n]_q} \mathrm{Id}_n \to 0.$$

5.4.1 A short exact sequence

This section is dedicated to the proof of Theorem 5.4.1. Since we like pictures, we introduce some other diagrammatic notations. Firstly, we draw A_n (viewed as A_n-A_n-bimodule) as a box labeled by n

A_n = 

and $\otimes_n$ becomes stacking boxes on top of each other. Moreover, when A_{n+1} is viewed as a left A_n-module, as an right A_n-module or as an A_n-A_n-bimodule, we draw respectively



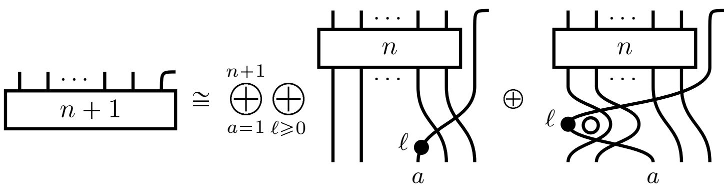
In order to prove Theorem 5.4.1, we need to first understand how A_{n+1} decomposes as module over A_n.

Lemma 5.4.3. *As a left A_n-module, A_{n+1} is free with decomposition given by*

$$A_{n+1} \cong \bigoplus_{a=1}^{n+1} \bigoplus_{\ell \geqslant 0} A_n\big(\tau_n \cdots \tau_a x_a^\ell \oplus A_n \tau_n \cdots \tau_a \theta_a^\ell\big),$$

where $\theta_a^\ell := \tau_{a-1} \cdots \tau_1 \omega x_1^\ell \tau_1 \cdots \tau_{a-1}$.

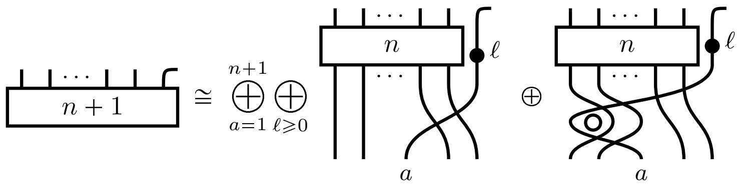
Using diagrams, we picture this decomposition as



Proof. Using the coset decomposition of S_n (5.19), we obtain left-adjusted minimal presentations of permutations, that allow to construct a basis through Theorem 5.3.22. From it, we obtain a decomposition

$$A_{n+1} \cong \bigoplus_{a=1}^{n+1} \bigoplus_{\ell \geqslant 0} A_n x_{n+1}^\ell \tau_n \cdots \tau_a \oplus A_n x_{n+1}^\ell \tau_n \cdots \tau_a \theta_a,$$

which diagrammatically looks like



We then apply (5.4) to bring all dots to the desired position. It is a triangular change of basis, concluding the proof. ■

By symmetry along the horizontal axis, we obtain a similar decomposition of A_{n+1} as right A_n-module. As expected (see Remark 4.3.2), the left and right decomposition are not compatible, and therefore we do not have a decomposition as A_n-A_n-bimodule. However, clearly the surjection $A_{n+1} \twoheadrightarrow q^{2\ell}A_n$ that projects on the summand $A_n x_{n+1}^{\ell}$, given by taking $a = n+1$ in Lemma 5.4.3, is a (left-invertible) map of bimodules. In addition, we also have the following less trivial result:

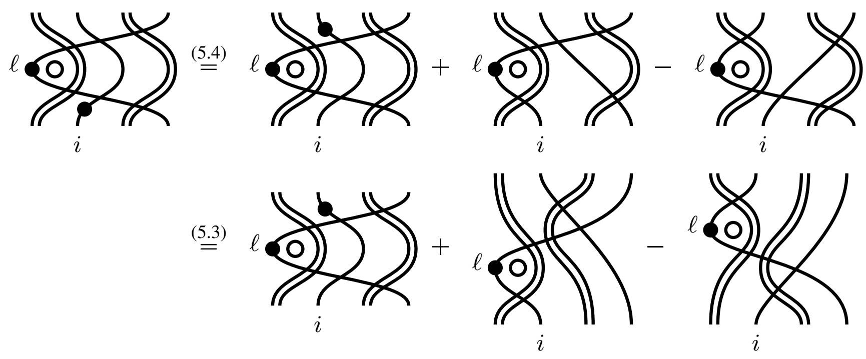
Lemma 5.4.4. *Let $\pi_L^{\ell} : A_{n+1} \twoheadrightarrow q^{2\ell-4n}\lambda^2 A_n[1]$ (resp. π_R^{ℓ}) be the projection map on the summand $A_n\theta_{n+1}^{\ell}$ (resp. on $\theta_{n+1}^{\ell}A_n$) in the left decomposition (resp. right decomposition) of A_{n+1} as A_n-module. We have*

$$\pi_L^{\ell}(y) = (-1)^{\deg_h(y)}\pi_R^{\ell}(y)$$

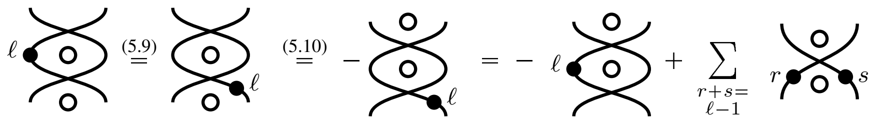
for all $y \in A_{n+1}$.

Proof. We can suppose $y = \theta_{n+1}^{\ell}y'$ with $y' \in A_n$. We want to prove that $y = (-1)^{\deg_h(y)}y'\theta_{n+1}^{\ell} + y_0$ for some $y_0 \notin A_n\theta_{n+1}^{\ell}$. For this, it is enough to show that $y_1\theta_{n+1}zy_2 = (-1)^{\deg_h(z)}y_1 z\theta_{n+1}y_2 + z_0$ where $y_1, y_2 \in A_n$, $z_0 \notin A_n\theta_{n+1}^{\ell}$ and z is any generator of A_n (i.e. crossing, dots or tight floating dot).

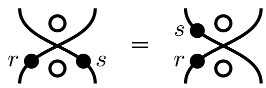
If $z = \tau_i$ is a crossing, then we obtain the desired property directly by (5.3). If $z = x_i$, then we compute



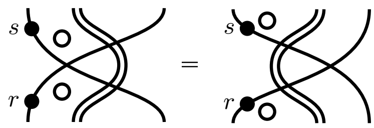
where the double strands represent multiple parallel strands (the number depending on n and a). Note that it is implicity supposed that there is the element y_1 at the top of each of these diagram, and y_2 at the bottom. Using Lemma 5.4.3, we can rewrite the composition of the last two terms in the equation above with y_2 as elements in $\oplus_{a=1}^{n} \oplus_{p\geqslant 0} A_{n-1}\tau_n\tau_{n-1}\cdots\tau_a x_a^p \notin A_n\theta_{n+1}^{\ell}$. Hence they form the term z_0. Finally if $z = \omega$, we compute using Lemma 5.1.5



Then for all $r, s \geqslant 0$ we compute using Lemma 5.1.5 again



Looking at these elements in the global picture yields



which is an element not contained in $A_n\theta_{n+1}^{\ell}$ for the same reasons as before. Thus, they all form together the element z_0, concluding the proof. ■

We now have all the ingredients we need to prove Theorem 5.4.1.

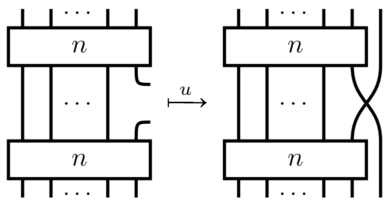
Proof of Theorem 5.4.1. We first construct an injective map

$$u : A_n \otimes_{n-1} A_n \hookrightarrow A_{n+1},$$

of A_n-A_n-bimodules, by setting (as in [33, Proposition 3.3] for cyclotomic nilHecke),

$$u(x \otimes_{n-1} y) := x\tau_n y.$$

In terms of diagrams, it consists in adding a crossing at the right



Then we construct a surjective map

$$A_{n+1} \twoheadrightarrow \bigoplus_{\ell\geqslant 0} q^{2\ell}A_n \oplus q^{2\ell-4n}\lambda^2 A_n[1]$$

by projecting onto the direct summands $\bigoplus_{\ell\geqslant 0} x_{n+1}^{\ell}A_n \oplus A_n\theta_{n+1}^{\ell}$ of the decomposition of A_{n+1} as A_n-right-module. By Lemma 5.4.4 we know it is a map of A_n-A_n-bimodules. Finally, exactness follows directly from Lemma 5.4.3, since

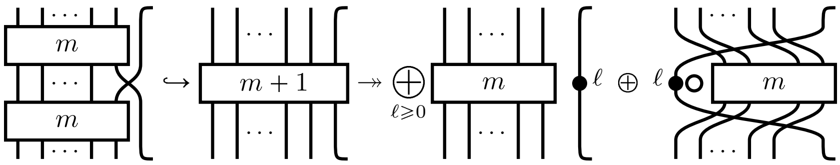
$$A_n \otimes_{n-1} A_n \cong A_n \otimes_{n-1} \big(\bigoplus_{a=1}^{n}\bigoplus_{\ell\geqslant 0} A_{n-1}\tau_{n-1}\cdots\tau_a x_a^{\ell} \oplus A_{n-1}\tau_{n-1}\cdots\tau_a\theta_a^{\ell}\big),$$

and thus

$$u(A_n \otimes_{n-1} A_n) \cong \bigoplus_{a=1}^{n}\bigoplus_{\ell\geqslant 0} A_n\tau_n\tau_{n-1}\cdots\tau_a x_a^{\ell} \oplus A_n\tau_n\tau_{n-1}\cdots\tau_a\theta_a^{\ell},$$

concluding the proof. ■

In terms of diagram, we like to view the short exact sequence of Theorem 5.4.1 as



where we think of the second summand in the cokernel not as the bimodule generated by these diagrams, but rather as the quotient of A_{n+1} corresponding to such diagrams.

Proposition 5.4.5. *The functors $\mathcal{F}_n$ and $\mathcal{E}_n$ are exact and send (c.b.l.f. generated) projective modules to (c.b.l.f. generated) projective modules.*

Proof. This is an immediate consequence of Lemma 5.4.3. ■

5.4.2 A long exact sequence

As in §4.3.2, we lift the short exact sequence to the dg-world of (A_n, d_N). Clearly, the injection $u : A_n \otimes_{n-1} A_n \hookrightarrow A_{n+1}$ lifts directly to a map of dg-bimodules. Then we want to compute the projection of the element $d_N(\theta_{n+1}^{\ell})$ in A_n^{ξ}.

Definition 5.4.6. *The i-th double complete homogeneous symmetric polynomial in the variables $x_1, \dots, x_n$ is*

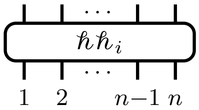
$$\hbar\hbar_i(x_1,\dots,x_n) := \sum_{r+s=i} \hbar_r(x_1,\dots,x_n)\hbar_s(x_1,\dots,x_n).$$

Note that we have

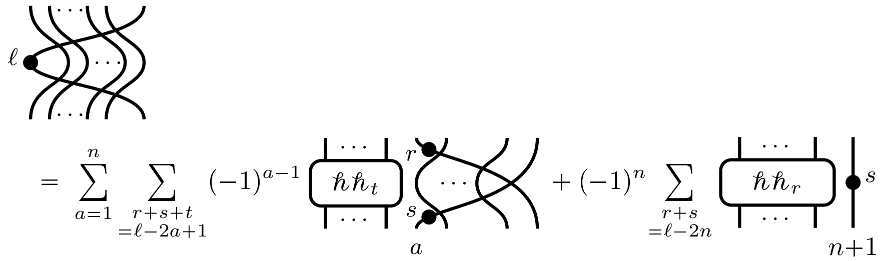
$$\hbar\hbar_i(x_1,\ldots,x_n) = \sum_{r+s=i} (r+1)x_1^r \hbar\hbar_s(x_2,\ldots,x_n). \tag{5.21}$$

We draw

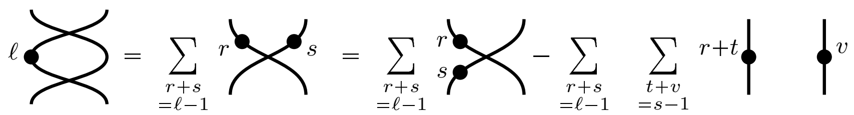
$$\hbar\hbar_i(x_1,\ldots,x_n) \; =:$$



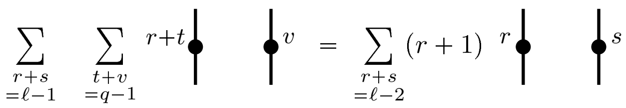
Lemma 5.4.7. *In A_{n+1} we have*



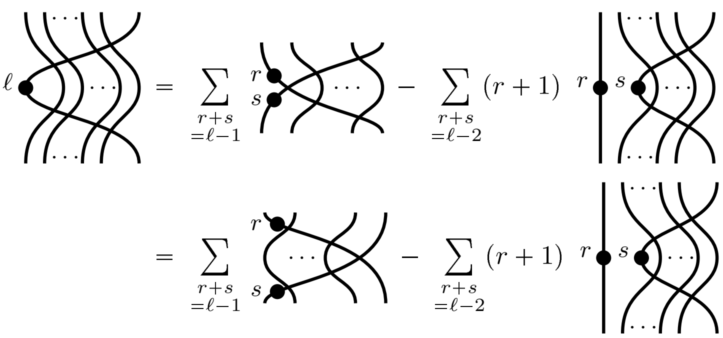
Proof. The proof goes by an induction on n. It is trivial for $n = 0$, so we suppose it is true for $n-1$. We compute using Lemma 5.1.5



and the last double sum can be rewritten as



which coincides with (5.21). Putting everything in the global picture, we get



using (5.3), so that we conclude by using the induction hypothesis. ∎

Therefore we consider the mapping cone

$$(A_n^\xi \oplus q^{2N-4n} A_n^\xi[1], d_N) := \mathrm{Cone}\big(q^{2N-4n}(A_n^\xi, d_N) \xrightarrow{y_N} (A_n^\xi, d_N)\big)$$

with the A_n-A_n-bimodule map y_N defined by

$$y_N(\xi^\ell) := (-1)^n \sum_{\substack{r+s\\ =N+\ell-2n}} \hbar\hbar_r(x_1, \dots, x_n)\xi^s.$$

Proposition 5.4.8. *There is a short exact sequence*

$$0 \to (q^{-2} A_n \otimes_{n-1} A_n, d_N) \to (A_{n+1}, d_N) \to (A_n^\xi \oplus q^{2N-4n} A_n^\xi[1], d_N) \to 0$$

of (A_n, d_N)*-*(A_n, d_N)*-bimodules.*

Proof. It follows from Theorem 5.4.1 and Lemma 5.4.7. ∎

As before, we want to study what are the consequences of this short exact sequence in homology.

Lemma 5.4.9 ([33, Lemma 4.18])**.** *Let A be a ring and consider the polynomial algebra $A[t]$. Let $p(t)$ be a monic polynomial in t with coefficients in the center $Z(A)$. Let M be an $A[t]$-module such that $p(t)M = 0$. If there is a short exact sequence*

$$0 \to P_1 \to P_0 \to M \to 0,$$

where P_0 and P_1 are both projective $A[t]$-modules, then M is a projective A-module.

Proposition 5.4.10. *As (A_n, d_N)-module (A_{n+1}, d_N) is strongly projective.*

Proof. From the basis (5.18), we obtain a decomposition

$$A_{n+1} \cong \bigoplus_{a=1}^{n+1} \big(A_n^\xi \tau_n \cdots \tau_a \oplus A_n^\xi \omega_{n+1}\tau_n \cdots \tau_a\big),$$

as left A_n-modules. We write $p(\xi) := d_N(\omega_{n+1})$ where we replace x_{n+1} by ξ. Thus we obtain

$$(A_{n+1}, d_N) \cong \mathrm{Cone}\big(\oplus_{a=1}^{n+1} q^{2N-2n} A_n^\xi \tau_n \cdots \tau_a \xrightarrow{p(\xi)} \oplus_{a=1}^{n+1} A_n^\xi \tau_n \cdots \tau_a\big),$$

as (A_n, d_N)-modules. Moreover, we have $A_n \cong A_n \otimes_{\mathrm{NH}_n} \mathrm{NH}_n$, so that we can write

$$(A_{n+1}, d_N) \cong (A_n, d_N) \otimes_{\mathrm{NH}_n} (Q, d_Q),$$

where

$$(Q, d_Q) := \mathrm{Cone}\big(\oplus_{a=1}^{n+1} q^{2N-2n}\,\mathrm{NH}_n^{\xi}\,\tau_n \cdots \tau_a \xrightarrow{p(\xi)} \oplus_{a=1}^{n+1}\,\mathrm{NH}_n^{\xi}\,\tau_n \cdots \tau_a\big),$$

and $\mathrm{NH}_n^{\xi} := \mathrm{NH}_n \otimes \mathbb{Q}[\xi]$. Clearly multiplication by $p(\xi)$ is an injective map. Moreover, $\mathrm{im}(d_Q) \cong \oplus_{a=1}^{n+1} q^{2N-2n}\,\mathrm{NH}_n^{\xi}\,\tau_n \cdots \tau_a$ is a projective NH_n-module. Also, we have $p(\xi)H(Q, d_Q) \cong 0$, so that by Lemma 5.4.9, $H(Q, d_Q)$ is a projective NH_n-module. ■

Note that in the proof of Proposition 5.4.10 lies also the proof of Proposition 5.2.1.

Proposition 5.4.11. *There are isomorphisms*

$$(A_n^{\xi} \oplus q^{2N-4n} A_n^{\xi}[1], d_N) \cong \begin{cases} \oplus_{i=0}^{N-2n-1} q^{2i}\,\mathrm{NH}_n^N, & \text{if } N - 2n \geqslant 0, \\ \oplus_{i=0}^{2n-N-1} q^{2N-4n+2i}\,\mathrm{NH}_n^N[1], & \text{if } 2n - N \leqslant 0, \end{cases}$$

of graded $\mathrm{NH}_n^N \cong H(A_n, d_N)$*-modules.*

Proof. Recall we have

$$y_N(\xi^{\ell}) = (-1)^N \sum_{\substack{r+s=\\ N+\ell-2n}} \hbar\hbar_r(x_1, \dots, x_n)\xi^s.$$

If $N - 2n \geqslant 0$, then $y_N(\xi^{\ell})$ is a monic polynomial in ξ with leading monomial $\xi^{N+\ell-2n}$ (up to sign). If $N - 2n \leqslant 0$, then $y_N(\xi^{\ell}) = 0$ for $\ell < N + \ell - 2n$, and it is a again a monic polynomial for $\ell \geqslant N + \ell - 2n$. In particular $y_N(\xi^{N+\ell-2n}) = (-1)^N$. ■

Therefore the whole story of §4.3.2 repeats, where we obtain short exact sequences by truncating the long exact sequence. Once again, one can show there are splitting maps, and recover Theorem 5.2.2. We put

$$\mathscr{V}^A(N) := \bigoplus_{n \geqslant 0} \mathscr{V}^A_{N-2n}, \qquad \mathscr{V}^A_{N-2n} := \mathscr{D}^{lc}(A_n, d_N),$$

and we define functors $\mathscr{E}_k^N$ and $\mathscr{F}_k^N$ as before. Note that $\mathscr{V}^A_{N-2n} \cong 0$ whenever $n > N$, since in that case (A_n, d_N) is acyclic.

5.5 The categorification theorems

The algebra $(A_n, 0)$ is a c.b.l.f. $\mathbb{Z}^2$-graded dg-algebra. The NH_n idempotent e_n (see (5.7)) yields an idempotent of A_n through the injection $\mathrm{NH}_n \hookrightarrow A_n$. Thus for the same reasons, A_n admits a unique projective indecomposable module (up to shift and isomorphism) given by

$$P_n := A_n e_n.$$

This module is concentrated only in positive degrees, and $(A_n, 0)$ respects the criteria of Remark 3.4.32. Moreover, by putting $\tilde{P}_n := q^{n(n-1)/2} P_n$, we obtain a decomposition

$$A_n \cong \oplus_{([n]_q!)} \tilde{P}_n. \tag{5.22}$$

There is an essentially unique simple module

$$S_n := P_n / \operatorname{Sym}^+(A_n),$$

where $\operatorname{Sym}^+(A_n)$ is the subalgebra of A_n given by $\operatorname{Sym}^+(\mathrm{NH}_n)$ together with $\omega_{n+1}^a \in A_n$ for all $0 \leqslant a \leqslant n$. Then we put $\tilde{S}_n := q^{n(n-1)/2} S_n$, and

$$\mathcal{M}^A(\lambda) := \bigoplus_{n \geqslant 0} \mathcal{M}^A_{\lambda-2n}, \qquad \mathcal{M}^A_{\lambda-2n} := \mathcal{D}^{lc}(A_n, 0).$$

We have

$$\boldsymbol{K}_0^\Delta(\mathcal{M}^A_{\lambda-2n}) \cong \mathbb{Z}(\!(q,\lambda)\!), \tag{5.23}$$

which is generated by either $[A_n]$, $[\tilde{P}_n]$ or $[\tilde{S}_n]$.

The functors $\mathcal{E}_n$ and $\mathcal{F}_n$ are both exact, preserve c.b.l. compact objects and are of finite amplitude thanks to Lemma 5.4.3. Therefore, $\mathcal{E} := \oplus_{n \geqslant 0} \mathcal{E}_n$ and $\mathcal{F} := \oplus_{n \geqslant 0} \mathcal{F}_n$ descend as endomorphisms of $\boldsymbol{K}_0^\Delta(\mathcal{M}^A(\lambda)) \cong \oplus_{n \geqslant 0} \boldsymbol{K}_0^\Delta(\mathcal{M}^A_{\lambda-2n})$.

We define the map $\psi : A_n \to {A_n}^{\mathrm{op}}$ by taking mirror image along the horizontal axis of diagrams in A_n. Then we consider the bifunctor

$$(-,-) : \mathcal{M}^A(\lambda) \times \mathcal{M}^A(\lambda) \to \mathcal{D}^{lc}(\Bbbk, 0), \quad (M, M') := M^\psi \otimes^{\mathrm{L}}_{\oplus_n A_n} M',$$

and we clearly have

$$(\operatorname{Ind}_n^{n+1} M, M') \cong (M, \operatorname{Res}_n^{n+1} M').$$

We deduce that this bifunctor is a lift of the Shapovalov form, making $\tilde{S}_n$ the dual of $\tilde{P}_n$.

Theorem 5.5.1. *The Grothendieck group $\boldsymbol{K}_0^\Delta(\mathcal{M}^A(\lambda))$ is a $U_q(\mathfrak{sl}_2)$-weight module, with the action of E, F given by $[\mathcal{E}], [\mathcal{F}]$. Moreover there is an isomorphism*

$$\boldsymbol{K}_0^\Delta(\mathcal{M}^A(\lambda)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M(\lambda),$$

sending $[(A_n, 0)]$ to the induced basis element $v_{\mu-2n}$, $[(\tilde{P}_n, 0)]$ to the canonical basis element $v'_{\mu-2n}$, and $[(\tilde{S}_n, 0)]$ to the dual canonical basis element $v'_{\mu-2n}$.

Proof. Both $\boldsymbol{K}_0^\Delta(\mathscr{M}^A(\lambda))$ and $M(\lambda)$ have the same dimension because of (5.23). We have $(A_n, 0) \cong \mathscr{F}^n(A_0, 0)$, and Lemma 5.4.3 tells us that $[\mathscr{E}][(A_n, 0)]$ can be rewritten as a sum of $[(A_{n-1}, 0)]$ with coefficients matching (1.6). This proves the statement about the induced basis elements. For the canonical basis elements it follows from (5.22). For dual canonical basis elements it comes from the fact that the bifunctor $(-,-)$ categorifies the Shapovalov form together with the fact that $\tilde{S}_n$ and $\tilde{P}_n$ are dual. ■

Introducing the differential d_N, we keep the decomposition

$$(A_n, d_N) \cong \oplus_{([n]_q!)}(\tilde{P}_n, d_N),$$

and the duality between $(\tilde{P}_n, d_N)$ and $(\tilde{S}_n, d_N)$ (after introducing d_N in the pairing $(-,-)$). Then we get for the same reasons as for Theorem 5.5.1:

Theorem 5.5.2. *The Grothendieck group $\boldsymbol{K}_0^\Delta(\mathscr{V}^A(N))$ is a $U_q(\mathfrak{sl}_2)$-weight module, with the action of E, F given by $[\mathscr{E}^N], [\mathscr{F}^N]$. There is an isomorphism*

$$\boldsymbol{K}_0^\Delta(\mathscr{V}^A(N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong V(N),$$

sending $[(A_n, d_N)]$ to the induced basis element v_{N-2n}, $[(\tilde{P}_n, d_N)]$ to the canonical basis element v'_{N-2n}, and $[(\tilde{S}_n, d_N)]$ to the dual canonical basis element v'_{N-2n}.

As already mentioned, we know by [66] that Ω_n is Morita equivalent to A_n and $Z(A_n) \cong \Omega_n$ (where $Z(A_n)$ is the supercenter with superdegree induced by the homological one). Therefore there is an equivalence of categories

$$\mathscr{D}^{lc}(\Omega_n, 0) \xrightarrow{\simeq} \mathscr{D}^{lc}(A_n, 0).$$

Moreover, as Ω_n-module, we have an isomorphism $\tilde{P}_n \cong \Omega_n$. Furthermore, the isomorphism $Z(A_n) \cong \Omega_n$ extends to $(Z(A_n), d_N) \cong (\Omega_n, d_N)$, and there is also a Morita equivalence between NH_n^N and $H(G_n^N)$ with $Z(\mathrm{NH}_n^N) \cong H(G_n^N)$. These imply dg-Morita equivalence between (Ω_n, d_N) and (A_n, d_N), and between $(H(G_n^N), 0)$ and $(\mathrm{NH}^N, 0)$. Therefore we get equivalences of categories

$$\begin{tikzcd}
\mathscr{D}^{lc}(\Omega_n, d_N) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathscr{D}^{lc}(H(G_n^N), 0) \arrow[d, "\simeq"] \\
\mathscr{D}^{lc}(A_n, d_N) \arrow[r, "\simeq"] & \mathscr{D}^{lc}(\mathrm{NH}^N, 0).
\end{tikzcd}$$

Moreover, all these equivalences commute up to quasi-isomorphism with $\mathscr{E}$ and $\mathscr{F}$, and they commute with the action of NH_k on $\mathscr{F}^k$ and on $\mathscr{E}^k$. It is obvious for the equivalences induced by quasi-isomorphism. For the (dg-)Morita equivalences, it comes from Proposition 4.3.1 and [66, Proposition 2.31]. The commutation with the action of NH_k comes from Lemma 5.3.10.

Chapter 6

Dg-enhanced KLR algebras

The nilHecke algebra, which is associated to $\mathfrak{sl}_2$, admits a generalization to other types. These algebras were introduced by Khovanov–Lauda in [42] and independently by Rouquier in [82], and thus are refered to as KLR algebras. They also appear in the literature as quiver Hecke algebras. We will follow the diagrammatic description of [42].

For the whole chapter, we fix a Cartan datum $(I, X, Y, (-|-))$ (see §1.2.1), and we put

$$d_{ij} := -\alpha_i^\vee(\alpha_j) \in \mathbb{N}.$$

For $\nu \in X^+$ we write

$$\nu = \sum_{i \in I} \nu_i \cdot \alpha_i, \quad \nu_i \in \mathbb{N},$$

and we also write $|\nu| := \sum_i \nu_i$, and $\mathrm{Supp}(\nu) := \{i|\nu_i \neq 0\}$.

We also fix a choice of scalars in a commutative, unital ring $\Bbbk$ as introduced in [83]. Following the conventions in [14], it consists of:

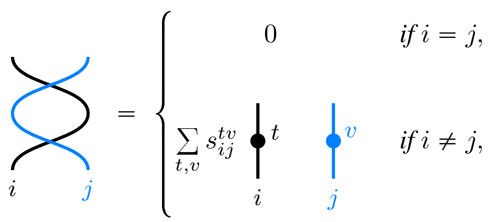
- $t_{ij} \in \Bbbk^\times$ for all $i, j \in I$,
- $s_{ij}^{tv} \in \Bbbk$ for $i \neq j$, $0 \leqslant t < d_{ij}$ and $0 \leqslant v < d_{ji}$,
- $r_i \in \Bbbk^\times$ for all $i \in I$,

respecting

- $t_{ii} = 1$,
- $t_{ij} = t_{ji}$ whenever $d_{ij} = 0$,
- $s_{ij}^{tv} = s_{ji}^{vt}$,
- $s_{ij}^{tv} = 0$ whenever $t(\alpha_i|\alpha_i) + v(\alpha_j|\alpha_j) \neq -2(\alpha_i|\alpha_j)$.

In addition, whenever $t < 0$ or $v < 0$, we put $s_{ij}^{tv} := 0$. Thus we have $s_{ij}^{pq} = 0$ for $p > d_{ij}$ or $q > d_{ji}$. We will also write $s_{ij}^{d_{ij}0} := t_{ij}$ and $s_{ij}^{0d_{ji}} := t_{ji}$. Hence if $(\alpha_i|\alpha_j) = 0$ we get $s_{ij}^{00} = s_{ji}^{00} = t_{ij} = t_{ji}$.

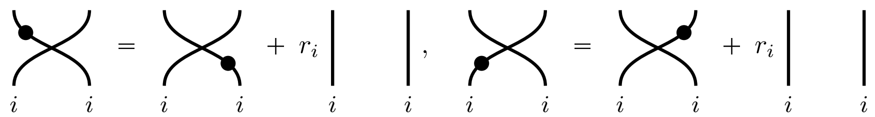
Definition 6.0.1. *The* Khovanov–Lauda–Rouquier (KLR) algebra $R(m)$ *is the diagrammatic* $\Bbbk$*-algebra on* m *strands, where the strands can be decorated by dots. Moreover, we require each strand to be labeled by a simple root, written* $i \in I$*, that we write at the bottom. The multiplication must preserve the labeling (i.e. connecting two strands with different labels gives zero). These diagrams are taken modulo the following local relations:*



(6.1)

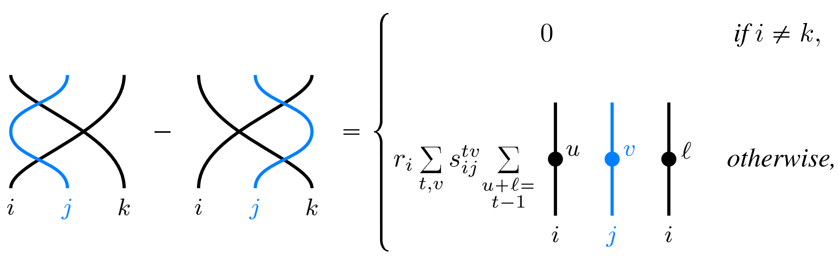
for all $i, j \in I$,

(6.2)



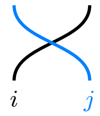
(6.3)

for all $i \neq j \in I$,



(6.4)

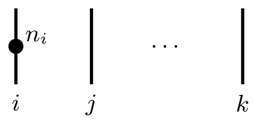
for all $i, j, k \in I$. *In addition,* $R(m)$ *is* $\mathbb{Z}$*-graded by setting*

$\deg_q$ () $:= -(\alpha_i|\alpha_j)$, $\deg_q$ () $:= (\alpha_i|\alpha_i)$.

Remark 6.0.2. *Note that for example in* (6.1)*, the sum* $\sum_{t,v} s_{ij}^{tv}$ *can be restricted to the finite number of pairs* $t, v \in \mathbb{N}$ *such that* $t(\alpha_i|\alpha_i) + v(\alpha_j|\alpha_j) = -2(\alpha_i|\alpha_j)$*. Moreover, it contains at least two non-zero elements with invertible coefficients, given by* $t = d_{ij}, v = 0$ *and* $t = 0, v = d_{ji}$*.*

As proven in [42, 44] (see also [82]), these algebras categorify the half quantum group $U_q^-(\mathfrak{g})$ associated to $(I, X, Y, (-|-))$, as a (twisted) bialgebra. The multiplication and comultiplication are categorified using respectively induction and restriction functors, obtained by putting diagrams side by side.

For each non-negative integral highest weight $N := \{n_i \in \mathbb{N} | i \in I\}$, there is a N-cyclotomic quotient $R^N(m)$ of $R(m)$ given by modding out the two-sided ideal generated by all diagrams of the form

 $= 0.$

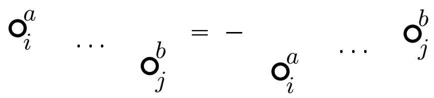
As first conjectured in [42] and proved in [33] and independently in [94], these cyclotomic quotients categorify the irreducible integrable $U_q(\mathfrak{g})$-module of highest weight N, where the action of F_i (resp. E_i) is given by induction (resp. restriction) along the map $R(m) \hookrightarrow R(m+1)$ that adds a vertical strand with label i at the right.

6.1 $\mathfrak{b}$-KLR algebra

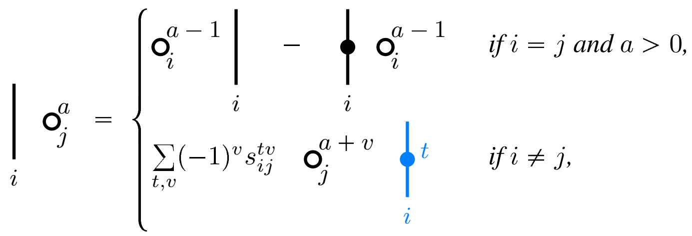
Our first goal for the chapter is to construct a dg-enhancement of the cyclotomic KLR algebras $R^N(m)$, in the same spirit as in Chapter 5.

Definition 6.1.1. *The* $\mathfrak{b}$-KLR algebra $R_{\mathfrak{b}}(m)$ *is the diagrammatic* $\Bbbk$*-algebra on* m *strands, where the strands are labeled by* I *and can carry dots. Moreover, regions in-between strands can be decorated by floating dots, which are labeled by a subscript in* I *and a superscript in* $\mathbb{N}$*. We consider these diagrams up to regular isotopy that preserves the relative height of the floating dots, and modulo all the KLR relations (6.1*

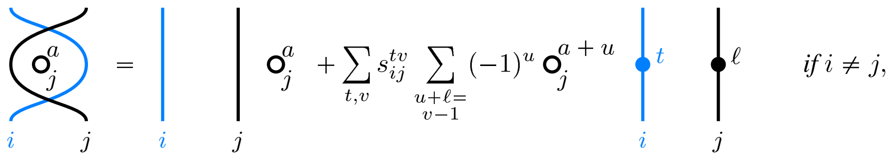
– 6.4), in addition to the following relations:



(6.5)

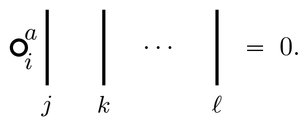


(6.6)

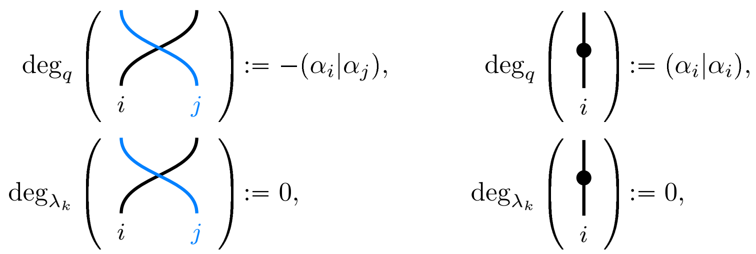


(6.7)

Moreover, a floating dot in the left-most region is zero



Given a diagram, we sometimes decorate a region with an element $K := \sum_{i \in I} k_i \cdot \alpha_i \in X^+$, where k_i denotes the number of strands with label i at the left of the region. The algebra $R_{\mathfrak{b}}$ is $\mathbb{Z}^{1+|I|}$-graded (a q-grading and a λ_k-grading for each $k \in I$) with



and

$$\deg_q\left(\begin{array}{cc} & \circ^{a}_{i} \\ K & \end{array} \right) := (1 + a - \alpha_i^\vee(K) + k_i)(\alpha_i|\alpha_i),$$

$$\deg_{\lambda_k}\left(\begin{array}{cc} & \circ^{a}_{i} \\ K & \end{array} \right) := 2\delta_{ik}.$$

This ends the definition of $R_{\mathfrak{b}}$.

6.1.1 Tightened basis

Before going any further, let us introduce some useful notation borrowed from [42]. First let $R_{\mathfrak{b}}(\nu)$ be the subalgebra of $R_{\mathfrak{b}}(m)$ given by diagrams where there is exactly ν_i strands labeled i, and this for each $i \in I$. We also denote $\mathrm{Seq}(\nu)$ the set of all ordered sequences $\boldsymbol{i} = i_1 i_2 \cdots i_m$ with $i_k \in I$ and i appearing ν_i times in the sequence. The symmetric group S_m acts on $\mathrm{Seq}(\nu)$ with a simple transposition $\sigma_k \in S_m$ acting on $\boldsymbol{i} = i_1 i_2 \cdots i_m \in \mathrm{Seq}(\nu)$ by permuting i_k and i_{k+1}. Sometimes we abuse notation by writing σ_K for $\sigma_{|K|}$ where $K = \sum_{i \in I} k_i \cdot \alpha_i \in X^+$.

For $\boldsymbol{i} = i_1 i_2 \cdots i_m \in \mathrm{Seq}(\nu)$, let $1_{\boldsymbol{i}} \in R_{\mathfrak{b}}(\nu)$ be the idempotent given by m vertical strands with labels $i_1, i_2, \ldots, i_m$, that is

$$1_{\boldsymbol{i}} := \begin{array}{cccc} \Big| & \Big| & \cdots & \Big| \\ i_1 & i_2 & & i_m \end{array}$$

By definition of the multiplication in $R_{\mathfrak{b}}(m)$, we have $1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{ij}}$ for all $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$. Hence there is a decomposition

$$R_{\mathfrak{b}}(\nu) \cong \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in \mathrm{Seq}(\nu)} 1_{\boldsymbol{j}} R_{\mathfrak{b}}(\nu) 1_{\boldsymbol{i}},$$

as $\Bbbk$-modules.

An action of $R_{\mathfrak{b}}$ on a polynomial space

We construct a polynomial representation of $R_{\mathfrak{b}}$ with a similar flavor as in [42, §2.3]. We fix $\nu \in X^+$ with $|\nu| = m$. For each $i \in I$ we define

$$Q_i := \Bbbk[x_{1,i}, \ldots, x_{\nu_i,i}] \otimes \textstyle\bigwedge^\bullet \langle \omega_{1,i}, \ldots, \omega_{\nu_i,i} \rangle.$$

We write $Q_I := \bigotimes_{i \in I} Q_i$, where $\otimes$ means the supertensor product in the sense that $\omega_{\ell,i}\omega_{\ell',j} = -\omega_{\ell',j}\omega_{\ell,i}$ for all $i, j \in I$ and $x_{i,\ell}$ commutes with everything. Thus, Q_I is a supercommutative superring. Then we construct the ring

$$Q_\nu := \bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} Q_I 1_{\boldsymbol{i}},$$

where the elements $1_{\boldsymbol{i}}$ are central idempotents.

We first construct an action of the symmetric group S^m on Q_ν by letting the simple transposition

$$\sigma_k : Q_I 1_{\boldsymbol{i}} \to Q_I 1_{s_k \boldsymbol{i}},$$

act by sending

$$x_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} x_{p+1,i} 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = i \text{ and } p = \#\{s \leqslant k | i_s = i\}, \\ x_{p-1,i} 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = i \text{ and } p = 1 + \#\{s \leqslant k | i_s = i\}, \\ x_{p,i} 1_{s_k \boldsymbol{i}}, & \text{otherwise,} \end{cases}$$

for $i \in I, p \in \{1, \dots, \nu_i\}$ and $\boldsymbol{i} = i_1 \dots i_m$, and sends

$$\omega_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} (\omega_{p,i} + (x_{p,i} - x_{p+1,i})\omega_{p+1,i})\, 1_{s_k \boldsymbol{i}}, & \text{if } i_k = i_{k+1} = i \text{ and} \\ & p = \#\{s \leqslant k | i_s = i\}, \\ \omega_{p,i} 1_{s_k \boldsymbol{i}}, & \text{otherwise,} \end{cases}$$

which we extend to Q_ν by setting $\sigma_k(fg) := \sigma_k(f)\sigma_k(g)$ for all $f, g \in Q_\nu$.

Proposition 6.1.2. *The procedure described above yields a well-defined action of S^m on Q_ν.*

Proof. The proof follows from the same reasons as Proposition 5.3.12. ∎

Then, we define inductively the element $\omega^a_{p,j} \in Q_I$ for $a \in \mathbb{N}$ as

$$\omega^0_{p,j} := \omega_{p,j}, \qquad \omega^{a+1}_{p,j} := \omega^a_{p-1,j} - x_{p,j}\omega^a_{p,j}.$$

For $K = \sum_{i \in I} k_i \cdot i \in X^+$ such that $k_i \leqslant \nu_i$, we define $\omega^a_j(K) \in Q_I$ inductively as

$$\omega^a_j(K) := \begin{cases} 0, & \text{if } k_j = 0, \\ \omega^a_{k_j,j}, & \text{if } k_i = 0 \text{ for all } i \neq j, \\ \sum\limits_{t,v} (-1)^t s^{tv}_{ij} x^t_{k_i,i} \omega^{a+v}_j(K - i), & \text{otherwise,} \end{cases}$$

where $K - i$ is a shorthand for $K - 1 \cdot \alpha_i$.

Lemma 6.1.3. *The element $\omega^a_j(K)$ is well-defined.*

Proof. Take $i \neq i' \neq j \in I$ such that $k_i > 0$ and $k_{i'} > 0$. We can suppose by induction that $\omega_j^b(K - i - i')$ is well-defined for all $b \geqslant 0$. Then we have

$$\begin{aligned}&\sum_{t,v}(-1)^t s_{ij}^{tv} x_{k_i,i}^t \sum_{t',v'}(-1)^{t'} s_{i'j}^{t'v'} x_{k_{i'},i'}^{t'} \omega_j^{a+v}(K - i - i')\\ &= \sum_{t',v'}(-1)^{t'} s_{i'j}^{t'v'} x_{k_{i'},i'}^{t'} \sum_{t,v}(-1)^t s_{ij}^{tv} x_{k_i,i} \omega_j^{a+v}(K - i' - i),\end{aligned}$$

for all $i \neq i' \neq j \in I$. This concludes the proof. ∎

It will be useful to give $\omega_j^a(K)$ a non-inductive expression. We write $K^{\setminus j} := \sum_{i \neq j} k_i \cdot \alpha_i i$. For a given non-negative integer $n_i \in \mathbb{N}$ we define

$$\varepsilon_{n_i,i}^j(\underline{x}_{k_i,i}) := \sum_{|V_i| = n_i} \left(\prod_{\ell=1}^{k_i} s_{ji}^{v_\ell t_\ell} x_{\ell,i}^{t_\ell} \right) \in P_i, \tag{6.8}$$

with the sum being over all partitions $V_i : v_1 + \cdots + v_{k_i} = v_i$ such that $(\alpha_i|\alpha_i)|v_\ell(\alpha_j|\alpha_j)$ for each ℓ, and with $t_\ell := \frac{-2(\alpha_i|\alpha_j) - v_\ell(\alpha_j|\alpha_j)}{(\alpha_i|\alpha_i)}$. This is a symmetric polynomial of q-degree $-2k_i(\alpha_i|\alpha_j) - v_i(\alpha_j|\alpha_j)$ whenever it is non zero. Clearly we can suppose $v_\ell \leqslant d_{ji}$, and therefore we can also suppose that $n_i \leqslant d_{ji} k_i$. For $n \in \mathbb{N}$ we define

$$\varepsilon_v^j(\underline{x}_K) := \sum_{|V| = n} \left(\prod_{i \neq j} \varepsilon_{n_i,i}^j(\underline{x}_{k_i,i}) \right) \in P_I, \tag{6.9}$$

with the sum being over all partitions $V : \sum_{i \neq j} n_i = n$. Notice that $\varepsilon_v^j(\underline{x}_K)$ is a polynomial of q-degree $(-\alpha_j^\vee(K^{\setminus j}) - n)(\alpha_j|\alpha_j)$.

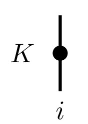
Lemma 6.1.4. *We have*

$$\omega_j^a(K) = \sum_{n=0}^{-\alpha_j^\vee(K^{\setminus j})} (-1)^n \omega_{k_j,j}^{a+n} \varepsilon_n^j(\underline{x}_K) \in P_I. \tag{6.10}$$

Proof. A straightforward computation shows that the RHS of (6.10) respects the recursive definition of $\omega_j^a(K)$, which proves the equality. ∎

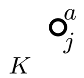
We now have all the tools we need to define an action of $R_{\mathfrak{p}}(\nu)$ on P_ν. First we choose an arbitrary orientation $i \leftarrow j$ or $i \rightarrow j$ for each pair of distinct $i, j \in I$. Then we let $a \in R_{\mathfrak{p}}(\nu)1_{\boldsymbol{j}}$ act as zero on $P_I 1_{\boldsymbol{i}}$ whenever $\boldsymbol{j} \neq \boldsymbol{i}$. Otherwise, we declare that

- the dot



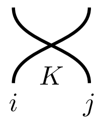
acts as multiplication by $x_{k_i+1,i}1_{\boldsymbol{i}}$;

- the floating dot



acts as multiplication by $\omega_j^a(K)1_{\boldsymbol{i}}$;

- the crossing



acts as

$$
\begin{aligned}
f1_{\boldsymbol{i}} &\mapsto r_i \frac{f1_{\boldsymbol{i}} - s_K(f1_{\boldsymbol{i}})}{x_{k_i,i} - x_{k_i+1,i}}, && \text{if } i = j,\\
f1_{\boldsymbol{i}} &\mapsto \left(\sum_{t,v} s_{ij}^{tv} x_{k_i,i}^t x_{k_j+1,j}^v\right) s_K(f1_{\boldsymbol{i}}), && \text{if } i \to j,\\
f1_{\boldsymbol{i}} &\mapsto s_K(f1_{\boldsymbol{i}}), && \text{if } i \leftarrow j.
\end{aligned}
$$

Proposition 6.1.5. *The rules above define an action of $R_{\mathfrak{b}}(\nu)$ on Q_ν.*

Proof. We have to check the validity of the KLR relations (6.1 – 6.4) and of the relations involving floating dots (6.5 – 6.7), as well as the relations coming from regular isotopies.

We start by proving the KLR relations. Clearly (6.1), (6.2) and (6.3) are satisfied. The case $i \neq k$ of (6.4) is also straightforward. For $i \leftarrow j$ and $k = i$ we compute the action of the LHS of (6.4) on $f \in Q_\nu$ as

$$
\begin{aligned}
f &\mapsto \big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\big)\sigma_1\partial_2\sigma_1(f) - \sigma_2\partial_1\big(\sum_{t,v} s_{ji}^{tv} y^t x_2^v \sigma_2(f)\big)\\
&= \big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\big)\frac{f - \sigma_1\sigma_2\sigma_1(f)}{x_1 - x_2} - \frac{\big(\sum_{t,v} s_{ji}^{tv} y^t x_2^v\big)f - \big(\sum_{t,v} s_{ji}^{tv} y^t x_1^v\big)\sigma_2\sigma_1\sigma_2(f)}{x_1 - x_2}\\
&= \sum_{t,v} s_{ji}^{tv} y^t \frac{x_1^v f - x_2^v}{x_1 - x_2} = \sum_{t,v} s_{ji}^{tv} y^t \sum_{\substack{r+s=\\ v-1}} x_1^r x_2^s,
\end{aligned}
$$

where x_1, x_2 correspond with the $x_{k_i,i}, x_{k_i+1,i}$ and y with $x_{k_j,j}$. What remains coincides with the RHS of (6.4). A similar computation applies for the case $i \to j$.

For the relations involving the floating dots, we remark that (6.5) follows from the supercommutativity of Q_ν, and $\omega_j^a(K)$ respects (6.6) by construction. For relation (6.7), we apply the action of the LHS on some $f \in Q_\nu$ and we obtain

$$f \mapsto \big(\sum_{t,v} s_{ji}^{tv} y^t x^v\big)\omega_j^a(K+j)f,$$

and on the RHS we obtain

$$\begin{aligned} f &\mapsto \big(\omega_j^a(K+i+j) + \sum_{t,v} s_{ij}^{tv} \sum_{\substack{u+\ell=\\ v-1}} (-1)^u \omega_j^{a+u}(K) x^t y^\ell\big) f \\ &= \big(\sum_{t,v} (-1)^v s_{ij}^{tv} x^t \omega_j^{a+v}(K+j) + \sum_{t,v} s_{ij}^{tv} \sum_{\substack{u+\ell=\\ v-1}} (-1)^u \omega_j^{a+u}(K) x^t y^\ell\big) f. \end{aligned}$$

Then we compute

$$\omega_j^{a+v}(K+j) = \big(\sum_{\substack{u+\ell=\\ v-1}} (-1)^{v-1-u} y^\ell \omega_j^{a+u}(K)\big) + (-1)^v y^v \omega_j^a(K+j),$$

which implies that the action of the RHS of (6.7) coincides with the one of the LHS.

The only non trivial relation coming from regular isotopies we need to verify is given by the commutation of a floating dot and a crossing at its left. This is a consequence of the fact that (6.8) is a symmetric polynomial, and divided difference operators commute with symmetric functions. ■

A generating set

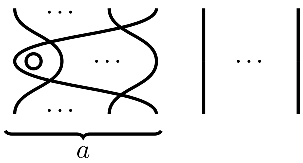
As before, we say that a floating dot is *tight* if it is placed directly at the right of the left-most strand, and has superscript 0. We can also suppose it has the same subscript as the label of the strand at its left (otherwise it would slide to the left and be zero).

Lemma 6.1.6. *The algebra $R_{\mathfrak{b}}(\nu)$ is generated by KLR elements (i.e. dots and crossings) and tight floating dots.*

Proof. This follows by observing that (5.15) holds in $R_{\mathfrak{b}}(\nu)$ when the label of both strands match with the subscript of the floating dot. This fact, together with (6.7) and (6.6) allows to bring all floating dots to the left. Then applying (6.6) recursively allows to transform all floating dots with superscript bigger than zero into dots and tight floating dots. ■

As before, we write ω for a tight floating dot, τ_a for a crossing between the ath and $(a+1)$th strands (counting from left), and x_a for a dot on the ath strand, where we suppose the label of the strands given by the context, in the form of an idempotent $1_{\boldsymbol{i}}$.

We also define the *tightened floating dot* in $R_{\mathfrak{b}}(m)$ as $\theta_a := \tau_{a-1}\cdots\tau_1\omega\tau_1\cdots\tau_{a-1}$, or diagrammatically

$\theta_a :=$ 

Also recall that we write $\theta_a^0 := \theta_a$ and $\theta_a^{-1} := 1$.

Fix $\boldsymbol{i},\boldsymbol{j} \in \mathrm{Seq}(\nu)$. Since they are both sequences of the same elements, there is a subset ${}_{\boldsymbol{j}}S_{\boldsymbol{i}} \subset S_m$ of permutations $w \in S_m$ such that $i_k = j_{w(k)}$ for all $k \in \{1,\dots,m\}$. Given such a permutation $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$, we can choose a left-adjusted reduced expression (see 5.3.4). It comes again with a partition $w^{m+1}\cdots w^2w^1 = w$ and a bijection $s : \{1,\dots,m\} \to \{1,\dots,m\}$, such that

$$w^k\cdots w^1(s(k)) = \min_w(s(k)),$$

for all $1 \leqslant k \leqslant m$. Then, consider the collection of elements

$$\begin{aligned} {}_{\boldsymbol{j}}B_{\boldsymbol{i}} = \big\{ & x_m^{k_m}\cdots x_1^{k_1}\tau_{w^{m+1}}\theta^{\ell_m}_{\min_w(s(m))}\tau_{w^m}\cdots\theta^{\ell_2}_{\min_w(s(2))}\tau_{w^2}\theta^{\ell_1}_{\min_w(s(1))}\tau_{w^1} \,\big| \\ & k_i \in \mathbb{N}, \ell_i \in \{0,-1\}, w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}\big\} \end{aligned} \tag{6.11}$$

in $1_{\boldsymbol{j}}R_{\mathfrak{b}}(m)1_{\boldsymbol{i}}$.

Proposition 6.1.7. *Elements in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ generate $1_{\boldsymbol{j}}R_{\mathfrak{b}}(m)1_{\boldsymbol{i}}$ as a $\Bbbk$-vector space.*

Proof. The proof is similar to the one of Theorem 5.3.22, relying on an induction on the number of crossings. By Lemma 6.1.6, we can already suppose all floating dots to be tight. Moreover, all braid isotopies hold up to adding terms with a lower amount of crossings thanks to (6.1) and (6.4), and the same applies for sliding dots using (6.3) and (6.2). Therefore we can apply all the arguments about the generating part of Theorem 5.3.22, with the equations of Lemma 5.3.21 also holding up to adding terms with less crossings. ∎

<u>The basis theorem</u>

Proposition 6.1.8. *The action in Proposition 6.1.5 is faithful.*

Proof. The proof is inspired by [83, Proposition 3.8] (see also [42, Theorem 2.5] for a different approach). We claim that elements in ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$ act as linearly independent endomorphisms on P_ν. The action yields morphisms

$$P_I 1_{\boldsymbol{i}} \to P_I 1_{\boldsymbol{j}},$$

that we will consider as endomorphisms of P_I.

First we extend the scalars to $\Bbbk(x_{1,i}, \dots, x_{\nu_i,i})$ in P_i for all $i \in I$. We claim that different choices of $w \in {}_{\boldsymbol{j}}S_{\boldsymbol{i}}$ and $\ell_i \in \{-1, 0\}$ give linearly independent operators. Notice that since $\boldsymbol{i}, \boldsymbol{j}$ is fixed, w is only given by choices of permutations between strands of the same label. Since crossings between strands with different labels act as multiplication by a polynomial, we can ignore them as being multiplication by a scalar. By Theorem 5.3.22, we know that different choices of permutations and tightened floating dots for strands with label i act as linearly independent operators on P_i, hence proving our claim. Finally taking into account the multiplication by the polynomial given by the choice $k_i \in \mathbb{N}$ in the first place concludes the proof. ■

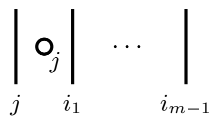
Theorem 6.1.9. *The $\Bbbk$-module $1_{\boldsymbol{j}} R_{\mathfrak{b}}(m) 1_{\boldsymbol{i}}$ is free with basis ${}_{\boldsymbol{j}}B_{\boldsymbol{i}}$.*

Proof. It follows from Proposition 6.1.7 and Proposition 6.1.8. ■

6.2 Dg-enhancement

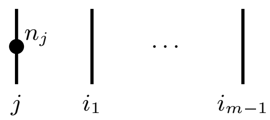
We fix a subset $I_f \subset I$ and consider the associated parabolic subalgebra $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ (see §1.2.2). For each $j \in I_f$, we also choose a weight $n_j \in \mathbb{N}$, and write $N := \{n_j\}_{j \in I_f}$.

Definition 6.2.1. *The $\mathfrak{p}$-KLR algebra $R_{\mathfrak{p}}(m)$ is given by forgetting the λ_j-degree for each $j \in I_f$ in $R_{\mathfrak{b}}(m)$ and taking the quotient by the relation*



$= 0,$

*for all $j \in I_f$. The N-*cyclotomic quotient* $R_{\mathfrak{p}}^N(m)$ of $R_{\mathfrak{p}}(m)$ is given by killing all diagrams of the form*

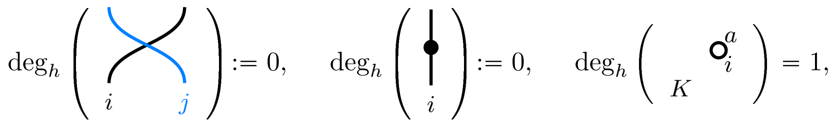


$= 0,$

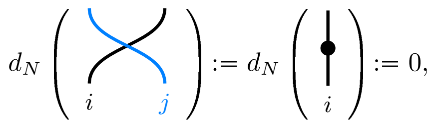
for all $j \in I_f$.

In particular, $R_{\mathfrak{g}}(m)$ is the usual KLR algebra $R(m)$ (see Definition 6.0.1) where there is no lambda grading and no floating dots. Its N-cyclotomic quotient $R_{\mathfrak{g}}^N(m)$ is also the usual cyclotomic quotient of the KLR algebra. Taking $I_f = \varnothing$ gives $\mathfrak{p} = \mathfrak{b}$ and we recover Definition 6.1.1.

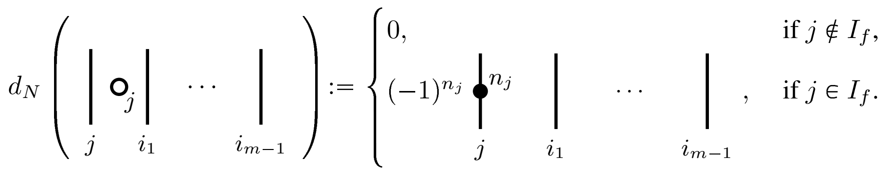
We equip $R_{\mathfrak{b}}$ with a homological $\mathbb{Z}$-grading, denoted h, by setting



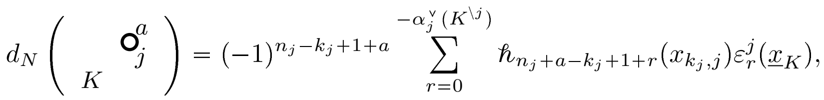
for all $i, j \in I$. Then we equip $R_{\mathfrak{b}}(m)$ with a differential d_N by setting



and



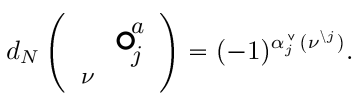
Thanks to Lemma 6.1.6 and extending by the Leibzniz rule, this is enough. From this, we derive for $j \in I_r$ that



where $x_{\ell,i}$ is a dot on the ℓth strand with label i, and $\varepsilon_r^j(\underline{x}_K)$ is defined in (6.9). We refer to the dg-algebra $(R_{\mathfrak{b}}(m), d_N)$ as *dg-enhanced KLR algebra.*

Proposition 6.2.2. *If* $n_j - \nu_j - \alpha_j^\vee(\nu^{\backslash j}) < 0$, *then* $(R_{\mathfrak{b}}(\nu), d_N)$ *is acyclic.*

Proof. Taking $a := -(n_j - \nu_j - \alpha_j^\vee(\nu^{\backslash j}) + 1)$ and considering the floating dot placed on the far right with subscript j and superscript a yields



Thus $H(R_{\mathfrak{b}}(\nu), d_N) \cong 0.$ ■

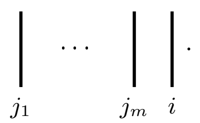
Our goal for the rest of the section will be to prove the following:

Theorem 6.2.3. *The dg-algebra* $(R_{\mathfrak{b}}(m), d_N)$ *is formal with homology*

$$H(R_{\mathfrak{b}}(m), d_N) \cong R_{\mathfrak{p}}^N(m).$$

6.2.1 Proof of Theorem 6.2.3

Denote $1_{(m,i)} := \sum_{\boldsymbol{j} \in I^m} 1_{\boldsymbol{j}i}$, or diagrammatically

$$1_{(m,i)} := \sum_{(j_1,\dots,j_m)\in I^m}$$ 

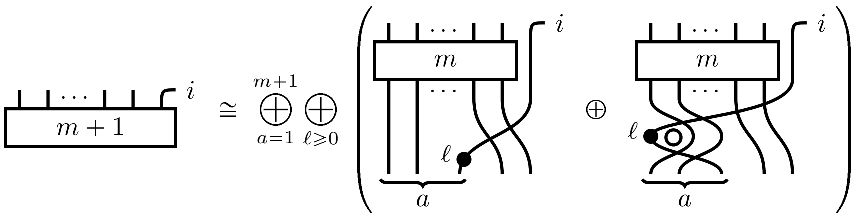
It is an idempotent of $R_{\mathfrak{b}}(m+1)$. We also define $1_{(\nu,i)} := \sum_{\boldsymbol{j}\in \mathrm{Seq}(\nu)} 1_{\boldsymbol{j}i}$ for $\nu \in X^+$. The algebra $R_{\mathfrak{b}}(m)$ acts on $1_{(m,i)}R_{\mathfrak{b}}(m+1)$ by first adding a vertical strand labeled i at the right of $D \in R_{\mathfrak{b}}(m)$ and then multiplying on the left in $R_{\mathfrak{b}}(m+1)$.

Lemma 6.2.4. *As a left* $R_{\mathfrak{b}}(m)$*-module,* $1_{(m,i)}R_{\mathfrak{b}}(m+1)1_{\boldsymbol{j}}$ *is free with decomposition*

$$\bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geqslant 0}\big(R_{\mathfrak{b}}(m)1_{(m,i)}\tau_m\cdots\tau_a x_a^\ell 1_{\boldsymbol{j}} \oplus R_{\mathfrak{b}}(m)1_{(m,i)}\tau_m\cdots\tau_a\theta_a^\ell 1_{\boldsymbol{j}}\big),$$

where $\theta_a^\ell := \tau_{a-1}\cdots\tau_1\omega x_1^\ell\tau_1\cdots\tau_{a-1}$.

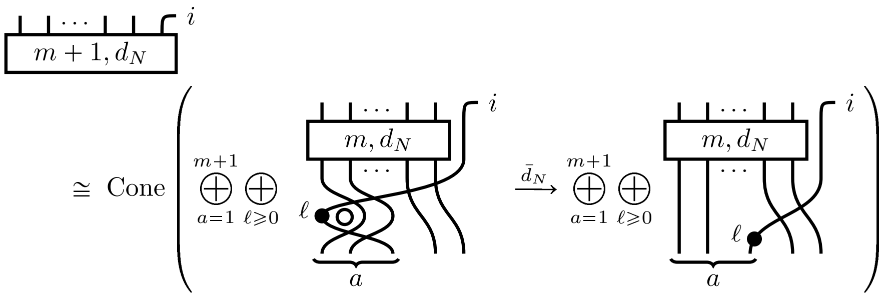
Using the same diagrammatic notations as in §5.4.1, we draw this as



where the label of the strands on the bottom is fixed by $\boldsymbol{j}$.

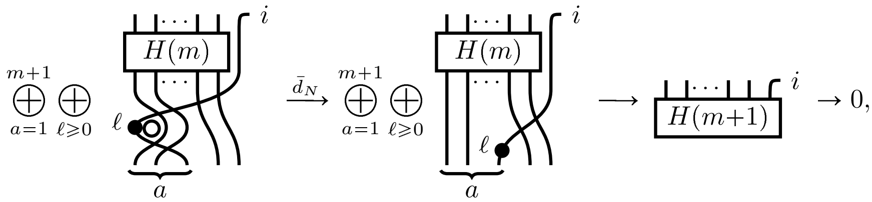
Proof. The proof is similar to the one of Lemma 5.4.3, using Theorem 6.1.9. ∎

From now on, we will draw boxes with label 'm, d_N' to denote the dg-algebra $(R_{\mathfrak{b}}(m), d_N)$. Similarly, a box with label $H(m)$ denotes its homology $H(R_{\mathfrak{b}}(m), d_N)$. Then, the decomposition in Lemma 6.2.4 directly lifts to a direct sum decomposition of dg-modules whenever $i \notin I_f$. Otherwise, for $i \in I_f$, it lifts to a mapping cone



where the map $\bar{d}_N$ is induced by the differential of $(R_{\mathfrak{b}}(m+1), d_N)$.

We will prove Theorem 6.2.3 using an induction on the number of strands m. Therefore, we can assume $(R_{\mathfrak{b}}(m), d_N)$ to be formal. We obtain an exact sequence

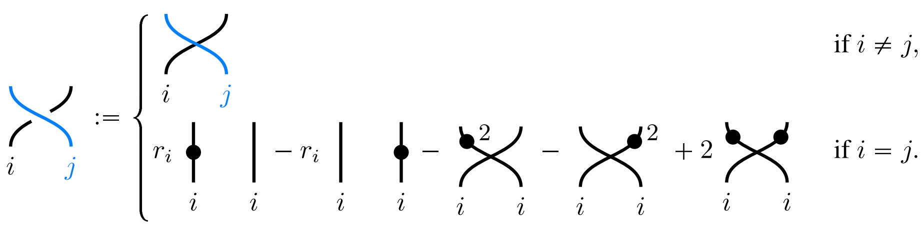


(6.12)

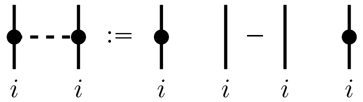
thanks to Proposition 2.2.13.

Proposition 6.2.5. *The exact sequence* (6.12) *is a short exact sequence, with* $\bar{d}_N$ *being injective.*

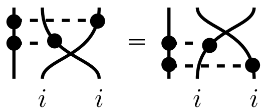
Then Theorem 6.2.3 is a direct consequence of Proposition 6.2.5. Therefore, we now focus on proving this proposition. This is in fact similar to [33, Eq. (4.13)], with morally only a change of basis, and thus we will follow the same ideas. We introduce the equivalent of g_a from the reference and draw it as an *under crossing*:



In order to shorten our diagrams, we also introduce the convenient notation

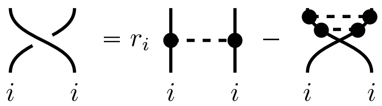


It respects the relation



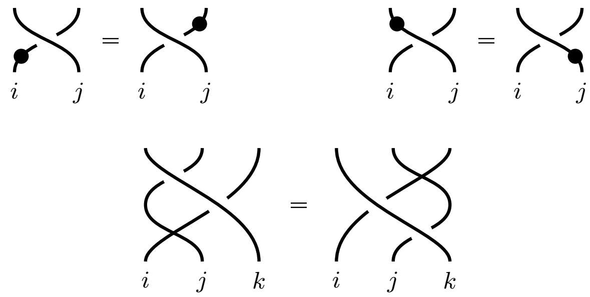
(6.13)

We also have that



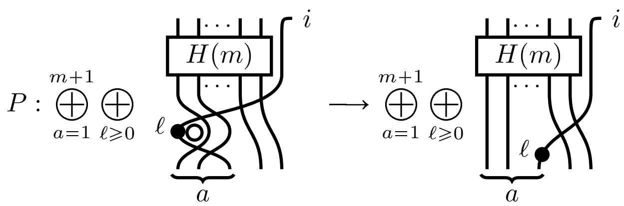
(6.14)

Lemma 6.2.6. ([33, Lemma 4.12]) *The under crossing respects the following relations:*

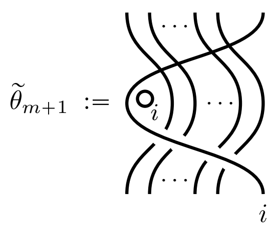


for all $i, j, k \in I$.

Still as in [33], in order to construct a nearly inverse for $\bar{d}_N$, we define the map



as multiplication on the left (or diagrammatically stacking above) with the element



Lemma 6.2.7. *The map P defined above is a map of $H(R_{\mathfrak{b}}(m), d_N)$-module.*

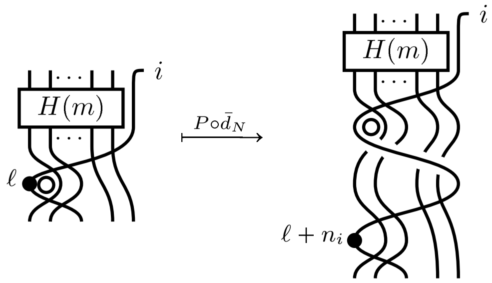
Proof. Crossings and dots slide over the upper part of the $(m+1)$th strand in $\widetilde{\theta}_{m+1}$ at the cost of adding diagrams with floating dots that are killed in $H(R_{\mathfrak{b}}(m), d_N)$, and then slide over the lower part thanks to Lemma 6.2.6. Tight floating dots with subscript $j \notin I_f$ also slide over $\widetilde{\theta}_{m+1}$ thanks to (6.7). ■

Lemma 6.2.8. *The composition $P \circ \bar{d}_N$ is given on $H(R_{\mathfrak{b}}(m), d_N) \otimes_m 1_{(\nu,i)} R_{\mathfrak{b}}(m+1)$ by multiplication by*

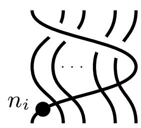
$$r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^\vee(\nu)} x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu), \tag{6.15}$$

where $\varepsilon_p^i(\underline{x}_\nu)$ is as in (6.9).

Proof. The proof is similar to [33, Theorem 4.15]. We have



We prove by induction on the number of strands m that

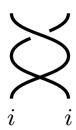


$$\equiv r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^\vee(\nu)} x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu),$$

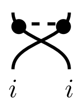
where $\equiv$ means equality up to adding elements killed in the quotient $H(R_{\mathfrak{b}}(m), d_N) \cong R_{\mathfrak{p}}^N(m)$. If $m = 0$, then it is trivial. Thus we suppose by induction that it holds for $m-1$. We fix the label of the strands on the diagram above as $i\boldsymbol{j}$ with $\boldsymbol{j} = j_1 \cdots j_m \in \mathrm{Seq}(\nu)$, and we consider the different possible cases.

If $j_m \neq i$, then the result follows by applying (6.1) with Lemma 6.2.6, and using the induction hypothesis.

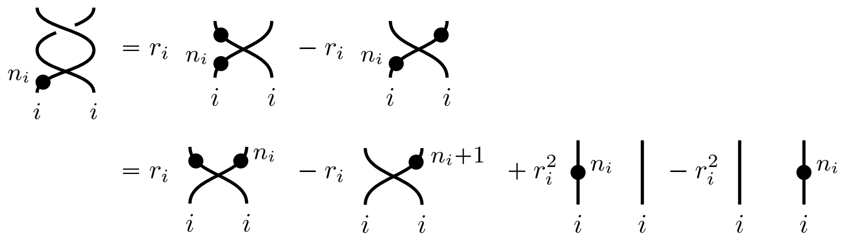
If $j_m = i$, we first observe that



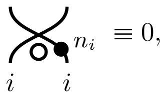
$$= r_i \tag{6.16}$$



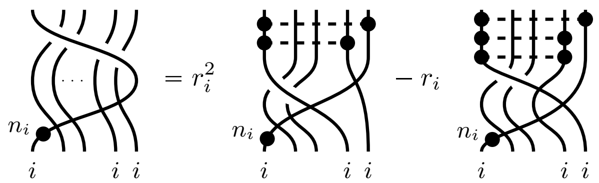
Then we need to consider j_{m-1}. If $m = 1$, we have by Lemma 5.1.5 that



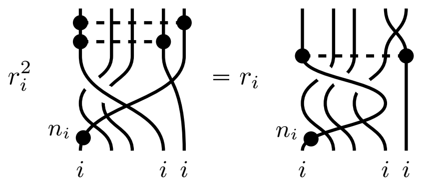
Moreover, we observe that



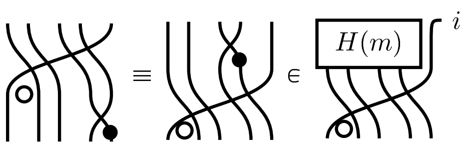
which finishes the case $m = 1$. For $j_{m-1} = i$, we have



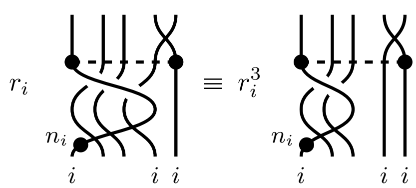
using (6.16), (6.14) and Lemma 6.2.6. Using (6.13) followed by Lemma 6.2.6 and (6.16) we obtain



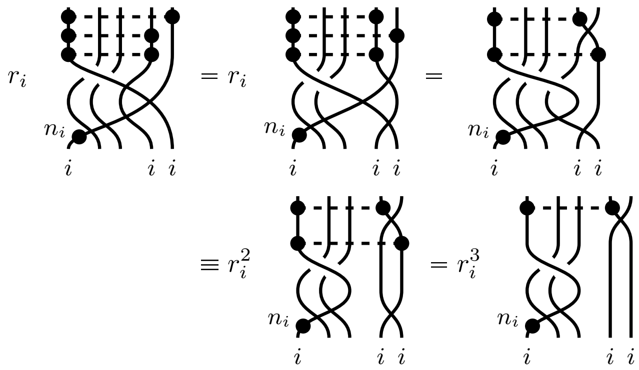
The upper part of the diagram slides to the top by (6.4) and becomes elements of $H(R_{\mathfrak{b}}(m), d_N)$,



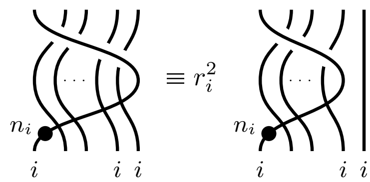
This means we can apply the induction hypothesis to get



Similarly, we have

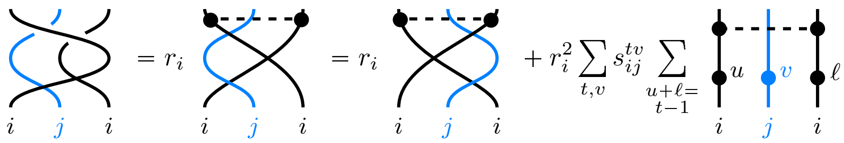


Putting these two results together and using (6.3), we obtain

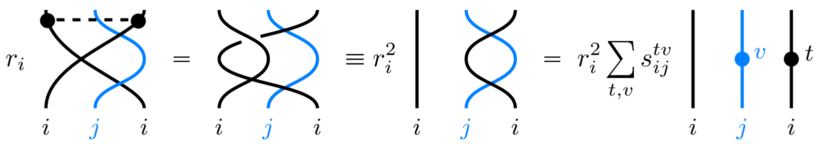


which concludes this case.

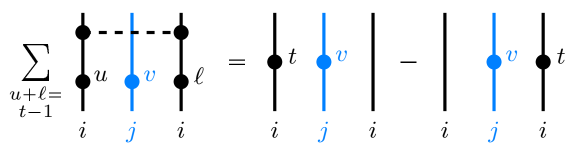
For the final case $j_{m-1} = j \neq i$, we compute



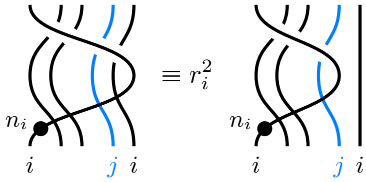
using (6.4). Then we have for the first term, putting it in the global picture and using the induction hypothesis (which again we can do because the upper crossing slide to the top inside $H(R_{\mathfrak{b}}(m), d_N)$) together with (6.1) we obtain



On the other hand, we have for all t, v that



Putting these results together with the case $j_m \neq i$ yields



which concludes the proof. ■

Proof of Proposition 6.2.5. The polynomial (6.15) is monic (up to invertible scalar) with leading terms $x_{m+1}^{n_i+2\nu_i-\alpha_i^\vee(\nu)}$. Therefore, multiplication by (6.15) yields an injective map. Thus Lemma 6.2.8 tells us that $P \circ \bar{d}_N$ is injective, and so is $\bar{d}_N$. ■

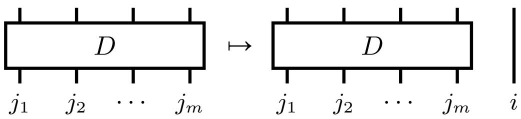
6.3 Categorical action

We write $\otimes_m := \otimes_{R_{\mathfrak{b}}(m)}$ and

$$R_{\mathfrak{b}}^{\xi_i}(\nu) := R_{\mathfrak{b}}(\nu) \otimes \Bbbk[\xi_i] \cong \bigoplus_{\ell \geqslant 0} q_i^2 R_{\mathfrak{b}}(\nu),$$

with $\deg_q(\xi_i) = (\alpha_i|\alpha_i)$.

We now generalize §5.4 to $\mathfrak{b}$-KLR algebras. For each $i \in I$ there is a (non-unital) injection $R_{\mathfrak{b}}(m) \hookrightarrow R_{\mathfrak{b}}(m+1)1_{(m,i)}$, given by adding a vertical strand with label i to the right of a diagram $D \in R_{\mathfrak{b}}(m)$:



From it we obtain induction and restriction functors

$$\begin{aligned}
\mathrm{Ind}_m^{m+i} &: R_{\mathfrak{b}}(m)\text{-mod} \to R_{\mathfrak{b}}(m+1)\text{-mod}, \\
&\quad \mathrm{Ind}_m^{m+i}(-) \cong R_{\mathfrak{b}}(m+1)1_{(m,i)} \otimes_m -, \\
\mathrm{Res}_m^{m+i} &: R_{\mathfrak{b}}(m+1)\text{-mod} \to R_{\mathfrak{b}}(m)\text{-mod}, \\
&\quad \mathrm{Res}_m^{m+i}(-) \cong 1_{(m,i)} R_{\mathfrak{b}}(m+1) \otimes_{m+1} -.
\end{aligned}$$

which are adjoint.

Theorem 6.3.1. *There is a short exact sequence*

$$\begin{aligned} 0 \to q_i^{-2} R_{\mathfrak{b}}(\nu) 1_{(m-1,i)} \otimes_{m-1} 1_{(m-1,i)} R_{\mathfrak{b}}(\nu) \to 1_{(\nu,i)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)} \\ \to R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1] \to 0 \end{aligned}$$

of $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodules for all $i \in I$. Moreover, there is an isomorphism

$$q^{-(\alpha_i|\alpha_j)} R_{\mathfrak{b}}(\nu) 1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu) \cong 1_{(\nu,j)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)}$$

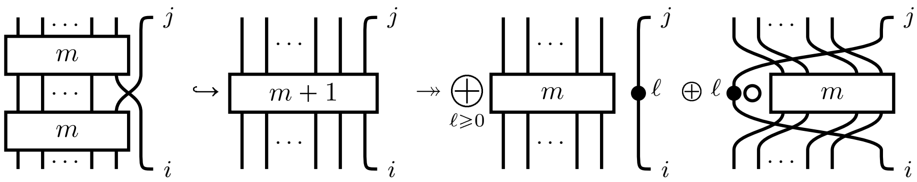
for all $i \neq j \in I$.

Proof. The proof is similar to the one of Theorem 5.4.1. The monomorphism

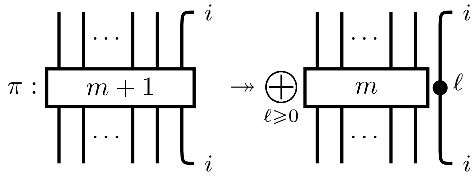
$$u_{ij} : q^{-(\alpha_i|\alpha_j)} R_{\mathfrak{b}}(\nu) 1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu) \hookrightarrow 1_{(\nu,j)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)} \quad (6.17)$$

is given by sending $x \otimes_{m-1} y$ to $x\tau_m y$ (i.e. adding a crossing). The epimorphism is given by projection on the summands $R_{\mathfrak{b}}(m)1_{(m,i)}x_{m+1}^{\ell}$ and on $R_{\mathfrak{b}}(m)1_{(m,i)}\theta_{m+1}^{\ell}$ in the decomposition of Lemma 6.2.4 (when $i = j$ and projection on 0 otherwise). For the same reasons as in Lemma 5.4.4, this defines a bimodule map, the only difference being that the Reidemeister III type-moves (i.e. (6.4)) used in the proof of Lemma 5.4.4 can add terms in the case of $R_{\mathfrak{b}}(m)$. However, these terms are killed in the quotient $1_{(\nu,j)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)} / R_{\mathfrak{b}}(\nu) 1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu)$. Finally, the exactness follows from Lemma 6.2.4. ■

We picture these facts as a short exact sequence of diagrams



where the cokernel vanishes whenever $i \neq j$. Moreover, we call π the projection



We write $\mathrm{Id}_\nu := R_{\mathfrak{b}}(\nu) \otimes_m (-)$ and we define

$$\mathcal{F}_i := \bigoplus_{m \geqslant 0} \mathrm{Ind}_m^{m+i}, \qquad \mathcal{E}_i := \bigoplus_{m \geqslant 0} \bigoplus_{|\nu| = m} \lambda_i^{-1} q_i^{\alpha_i^\vee(\nu)} \mathrm{Res}_m^{m+i} \mathrm{Id}_{\nu+i}\,.$$

These are exact functors and preserves c.b.l.f. generated projective modules, thanks to Lemma 6.2.4. Then we obtain:

Corollary 6.3.2. *There is a natural short exact sequence*

$$0 \to \mathcal{F}_i\mathcal{E}_i \,\mathrm{Id}_\nu \to \mathcal{E}_i\mathcal{F}_i \,\mathrm{Id}_\nu \to \oplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0,$$

for all $i \in I$ and there is a natural isomorphism

$$\mathcal{F}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{F}_i,$$

for all $i \neq j \in I$.

Corollary 6.3.3. *There is a quasi-isomorphism*

$$\mathrm{Cone}(\mathcal{F}_i\mathcal{E}_i \,\mathrm{Id}_\nu \to \mathcal{E}_i\mathcal{F}_i \,\mathrm{Id}_\nu) \xrightarrow{\simeq} \oplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu\,.$$

Proposition 6.3.4. *For each $i, j \in I$ there is a natural isomorphism*

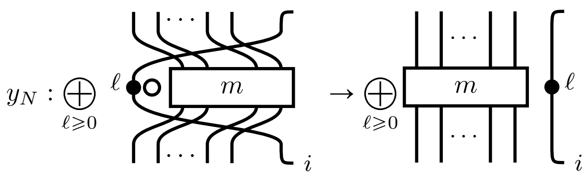
$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} \mathcal{F}_i^{2a}\mathcal{F}_j\mathcal{F}_i^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} \mathcal{F}_i^{2a+1}\mathcal{F}_j\mathcal{F}_i^{d_{ij}-2a},$$

and in particular for $(\alpha_i|\alpha_j) = 0$ we have $\mathcal{F}_i\mathcal{F}_j 1_\nu \cong \mathcal{F}_j\mathcal{F}_i 1_\nu$. By adjunction, the same isomorphism exists for the $\mathcal{E}_i, \mathcal{E}_j$.

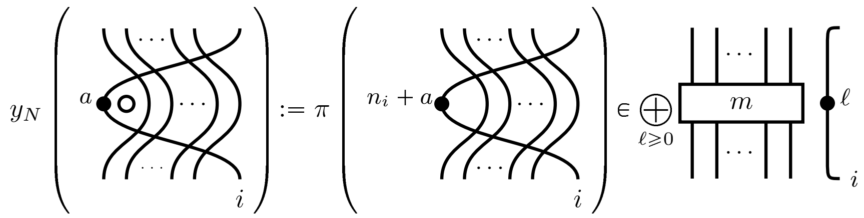
Proof. Similarly as in the case of the usual KLR algebras, it follows from (6.3) and (6.4) (we can directly apply the proof of [44, Proposition 6]). ■

6.3.1 Long exact sequence

We want to lift Theorem 6.3.1 to the dg-world of $(R_{\mathfrak{b}}(m), d_N)$, and study the long exact sequence that it induces. Therefore we define



as the $R_{\mathfrak{b}}(m)$-$R_{\mathfrak{b}}(m)$-bimodule map given by



whenever $i \in I_f$, and $y_N = 0$ for $i \notin I_f$. Then we define

$$\begin{aligned}&\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big)\\ &\qquad := \operatorname{Cone}\Big((\lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N) \xrightarrow{y_n} (R_{\mathfrak{b}}^{\xi_i}(\nu), d_N)\Big),\end{aligned}$$

and

$$\begin{aligned}&(R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m-1,i)} R_{\mathfrak{b}}(\nu), d_N)\\ &\qquad := (R_{\mathfrak{b}}(\nu)1_{(m-1,i)}, d_N) \otimes_{(R_{\mathfrak{b}}(m-1), d_N)} (1_{(m-1,i)} R_{\mathfrak{b}}(\nu), d_N).\end{aligned}$$

Proposition 6.3.5. *There is a short exact sequence of dg-bimodules*

$$\begin{aligned}&0 \to q_i^{-2}(R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m-1,i)} R_{\mathfrak{b}}(\nu), d_N) \to (1_{(\nu,i)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)}, d_N)\\ &\quad \to \big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big) \to 0\end{aligned}$$

for all $i \in I$. Moreover, there is an isomorphism

$$q^{-(\alpha_i|\alpha_j)}(R_{\mathfrak{b}}(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m_1,j)} R_{\mathfrak{b}}(\nu), d_N) \cong (1_{(\nu,j)} R_{\mathfrak{b}}(m+1) 1_{(\nu,i)}, d_N)$$

for all $i \neq j \in I$.

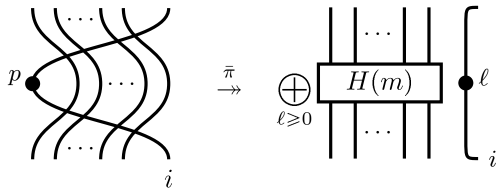
Proof. It is a straightforward consequence of Theorem 6.3.1. ■

In order to understand the consequences of this short exact sequence in homology, we need to compute the homology

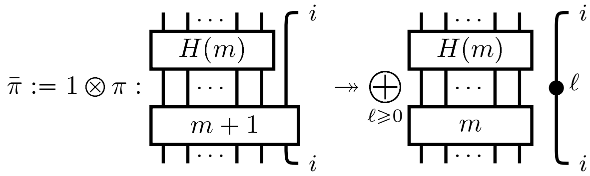
$$H\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big),$$

for all $i \in I_f$.

Therefore, we want to compute the projection of the element

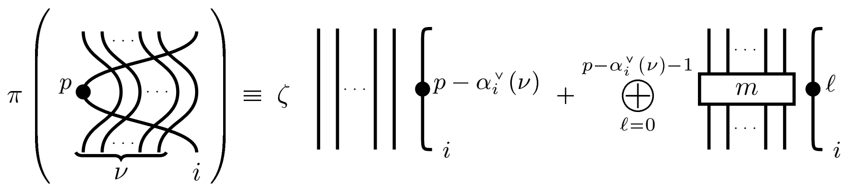


for all $p \geqslant n_i$. Note that we project on the homology of $(R_{\mathfrak{b}}(m), d_N)$. This will ease some of the computations we need to do. We write $\bar{\pi}$ when we take the composition of π with the projection on the homology of $(R_{\mathfrak{b}}(m), d_N)$. More precisely, $\bar{\pi}$ is given by

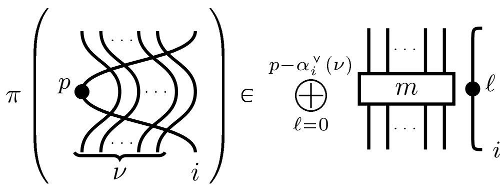


Similarly, we write $\bar{y}_N$.

Lemma 6.3.6. *If $p \geqslant 2\nu_i$, then*

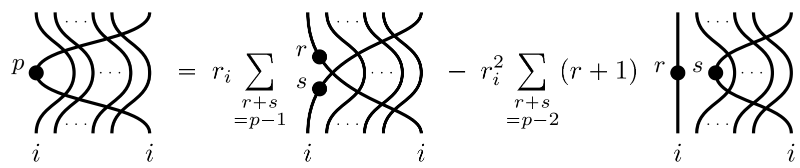


for some invertible element $\zeta \in \Bbbk^\times$. If $p < 2\nu_i$, then

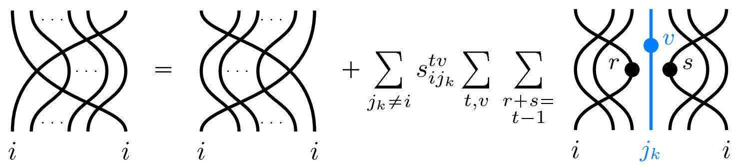


Proof. The proof is an induction on m. If $m = 0$, then it is trivial. Suppose the statement holds for $m-1$. We fix the labels of the strands as the bottom as $\boldsymbol{j} = j_1 \cdots j_m \in \mathrm{Seq}(\nu)$. If $j_1 = i$, then we apply the same computation as in Lemma 5.4.7

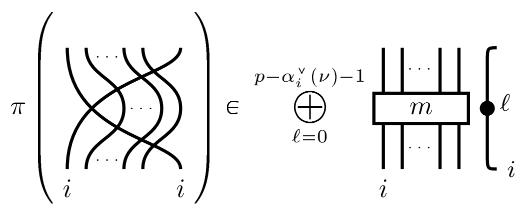
to get



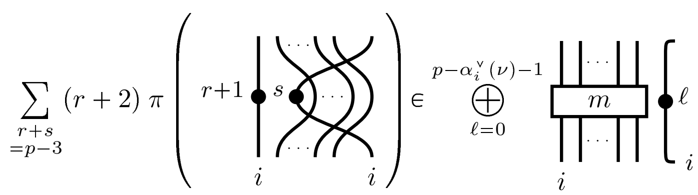
Then using (6.4) we have



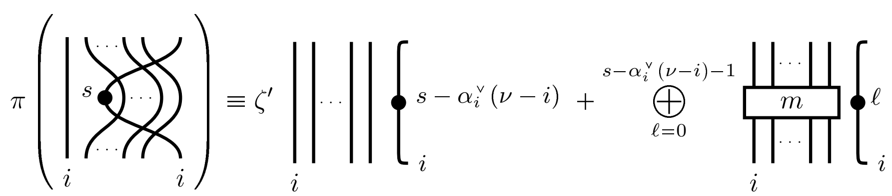
so that, since $s < d_{ij_k}$, we obtain by induction hypothesis



Moreover, still by induction hypothesis, we have

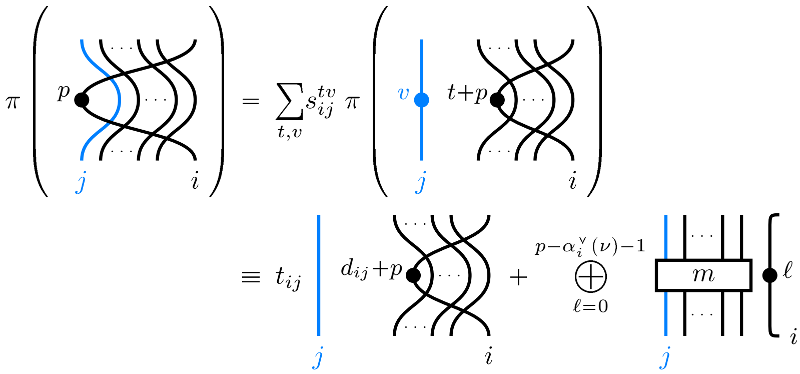


Finally, if $p \geqslant 2\nu_i$, by induction hypothesis we get for $s = p - 2$,



which concludes the case by observing that $s - \alpha_i^\vee(\nu - i) = p - \alpha_i^\vee(\nu)$, and taking $\zeta = r_i^2\zeta'$. If $p < 2\nu_i$, it is immediate by induction hypothesis.

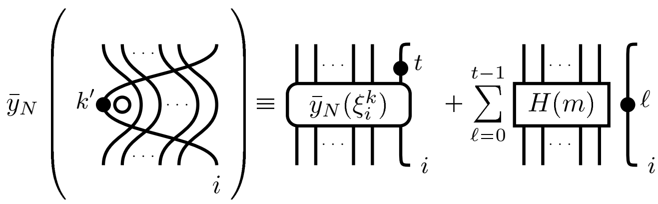
For the case $j_1 = j \neq i$, we use (6.1) and then the induction hypothesis to get



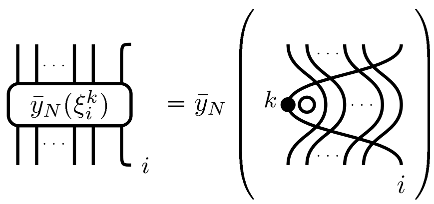
where we recall that $s_{ij}^{d_{ij}0} = t_{ij}$. We conclude by applying the induction hypothesis, observing that $d_{ij} + p - \alpha_i^\vee(\nu - j) = p - \alpha_i^\vee(\nu)$. ■

Consider also the following result, which is similar to [33, Lemma 5.4].

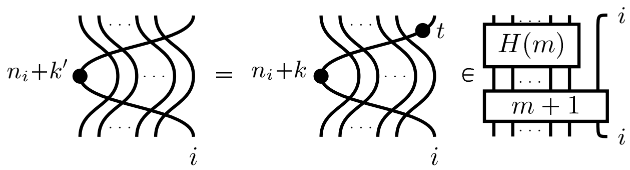
Lemma 6.3.7. *We have for $k < k'$ and $t = k' - k$,*



where

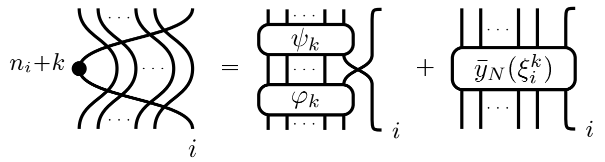


Proof. First we observe that



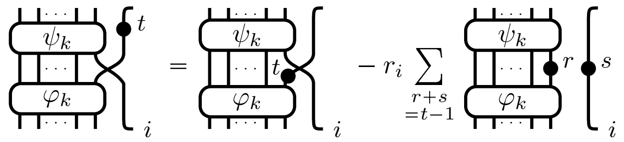
using (6.3) and (6.2), and the fact that n_i dots on the left strand is annihilated in $H(R_{\mathfrak{b}}(m), d_N)$.

Then using Lemma 6.2.4 we obtain



(6.18)

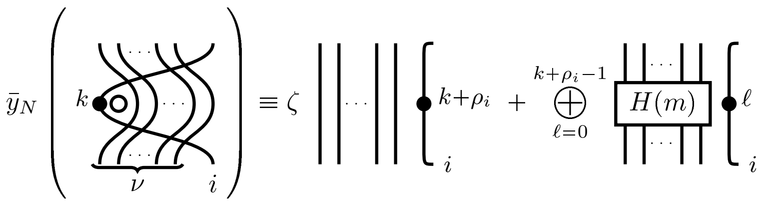
for some $\varphi_k, \psi_k \in R_{\mathfrak{b}}(m)$. We conclude by observing that



(6.19)

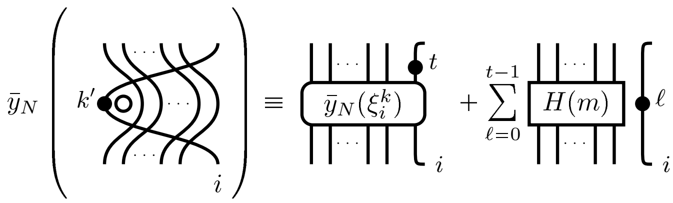
thanks to Lemma 5.1.5. ∎

Proposition 6.3.8. *Putting $\rho_i := n_i - \alpha_i^\vee(\nu)$, we have*

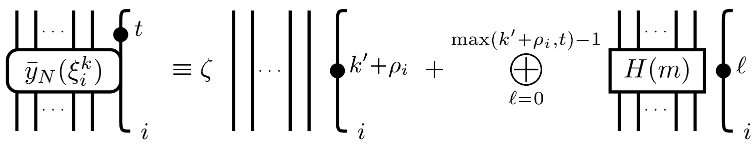


which is 0 *whenever* $k + \rho_i < 0$, *and where* $\zeta \in \Bbbk^\times$.

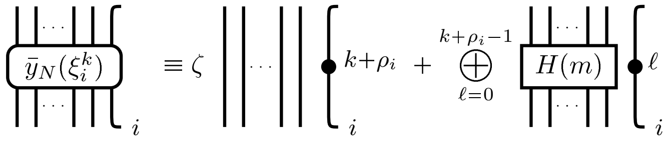
Proof. If $n_i \geqslant 2\nu_i$, then the result follows from Lemma 6.3.6. Otherwise, we take $k' = 2\nu_i - n_i$ and the result follows from Lemma 6.3.6 for $k \geqslant k'$. Suppose $k < k'$ and put $t = k' - k$. Then by Lemma 6.3.7 we obtain



Therefore, we must have



From this, we deduce



which concludes the proof. ■

We now have all the tools we need to compute the homology of the cokernel of the short exact sequence of Proposition 6.3.5.

Proposition 6.3.9. *There is an isomorphism of $R_{\mathfrak{p}}^N(\nu)$-$R_{\mathfrak{p}}^N(\nu)$-bimodules*

$$H\big(R_{\mathfrak{b}}^{\xi_i}(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_{\mathfrak{b}}^{\xi_i}(\nu)[1], d_N\big) \cong \begin{cases} \bigoplus_{\ell=0}^{\rho_i-1} q_i^{2\ell} R_{\mathfrak{p}}^N(\nu), & \text{if } \rho_i \geqslant 0, \\ \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} \bigoplus_{\ell=0}^{-\rho_i-1} q_i^{2\ell} R_{\mathfrak{p}}^N(\nu)[1], & \text{if } \rho_i \leqslant 0, \end{cases}$$

where $\rho_i = n_i - \alpha_i^\vee(\nu)$.

Proof. First suppose $\rho_i \geqslant 0$. Then Proposition 6.3.8 tells us that $\bar{y}_N(\xi_i^k)$ is a monic polynomial (up to invertible scalar) with leading terms $\xi_i^{k+\rho_i}$. This gives us the first case. If $\rho_i \leqslant 0$, then we have $\bar{y}_N(\xi_i^k) = 0$ for $k < -\rho_i$. Moreover, $\zeta^{-1}\bar{y}_N(\xi_i^{-\rho_i}) = 1$, and in general $\bar{y}_N(\xi_i^k)$ is a monic polynomial with leading term $\xi_i^{k+\rho_i}$ for $k > -\rho_i$. This concludes the proof. ■

Our next goal is to show the following:

Proposition 6.3.10. *The $(R_{\mathfrak{b}}(m), d_N)$-module $(1_{(m,i)} R_{\mathfrak{b}}(m+1), d_N)$ is strongly projective.*

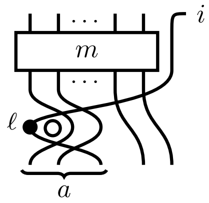
It is obvious for $i \notin I_f$ by Lemma 6.2.4, and thus we can assume $i \in I_f$. The idea of the proof is similar to the one of Proposition 5.4.10. Thus, we first construct the mapping cone

$$(Q,d_Q) := \operatorname{Cone}\Big(\bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geqslant 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^\ell \xrightarrow{d_Q} \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geqslant 0} R_{\mathfrak{g}}(m) 1_{(\nu,i)} \tau_m \cdots \tau_a x_a^\ell\Big),$$

where we think of $\tau_m \cdots \tau_a \theta_a^\ell$ as a formal symbol that represents a degree shift corresponding with the degree of the element $1_{(\nu,i)} \tau_m \cdots \tau_a \theta_a^\ell$ in $R_{\mathfrak{b}}(m+1)$. The map d_Q

is given by first embedding $R_{\mathfrak{g}}(m)$ into $R_{\mathfrak{b}}(m+1)$ through the diagrams

$$R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell \hookrightarrow$$



then applying d_N of $(R_{\mathfrak{b}}(m+1), d_N)$, then decomposing the image in the left-decomposition $\bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geqslant 0} R_{\mathfrak{b}}(m)1_{(m,i)}\tau_m\cdots\tau_a x_a^\ell$, and finally projecting unto the part in homogical degree zero of $R_{\mathfrak{b}}(m)$, which is trivially isomorphic to $R_{\mathfrak{g}}(m)$. Moreover, $(R_{\mathfrak{b}}(m), d_N)$ is a (right) module over $(R_{\mathfrak{g}}, 0)$ which act by gluing KLR diagrams on the bottom. Then we have as $(R_{\mathfrak{b}}(m), d_N)$-modules

$$(R_{\mathfrak{b}}(m+1), d_N) \cong (R_{\mathfrak{b}}(m), d_N) \otimes_{(R_{\mathfrak{g}}(m),0)} (Q, d_Q).$$

Therefore, we want to show that (Q, d_Q) is strongly projective as $(R_{\mathfrak{g}}(m), 0)$-module. We write

$$Q_1[\xi_i] := \bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geqslant 0} R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m\cdots\tau_a\theta_a^\ell,$$
$$Q_0[\xi_i] := \bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geqslant 0} R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m\cdots\tau_a x_a^\ell,$$

where we identify ξ_i with x_a in Q_0, and ξ_i^ℓ with x_1^ℓ in θ_a^ℓ. Note that d_Q is not $\Bbbk[\xi_i]$-linear.

Lemma 6.3.11. *The map*

$$d_Q : Q_1[\xi_i] \to Q_0[\xi_i]$$

defined above is injective.

Proof. Recall the map P of Lemma 6.2.7 given by multiplication with $\widetilde{\theta}_{m+1}$. Since floating dots are also annihilated in $R_{\mathfrak{g}}(m)$, it defines a map

$$P' : Q_0[\xi_i] \to Q_1[\xi_i]. \tag{6.20}$$

We reconsider the proof of Lemma 6.2.8 to show that $P' \circ d_Q$ is injective. First, we put an order on the summands of $Q_1[\xi_i] = \bigoplus_{a=1}^{m+1}\bigoplus_{\ell\geqslant 0} R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m\cdots\tau_a\theta_a^\ell$ by saying that

$$R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell} \prec R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell'},$$
$$R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell} \prec R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_{a'} \theta_{a'}^{\ell''},$$

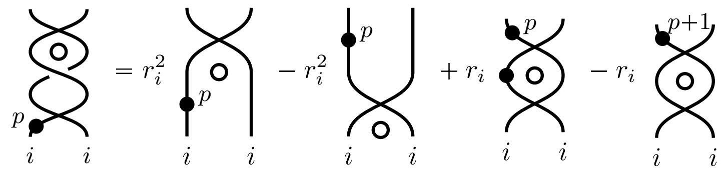
for all $a > a'$, $\ell < \ell'$, and for all ℓ''. In other words, if there are more crossings under the floating dot, then the term is smaller. If there is the same amount of crossings, then we consider the amount of dots at the left of the floating dot, and lesser dots meaning a smaller term.

We claim that if $Z \in R_{\mathfrak{g}}(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^{\ell}$ then

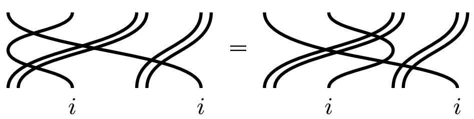
$$P' \circ d_Q(Z) = r_i^{2\nu_i} \sum_{p=0}^{2\nu_i - \alpha_i^\vee(\nu)} Z x_{m+1}^{n_i+p} \varepsilon_p^i(\underline{x}_\nu) + H,$$

where $H \prec Z x_{m+1}^{\ell+n_i+2\nu_i-\alpha_i^\vee(\nu)}$. This implies that $P' \circ d_Q$ is in echelon form (with pivot being invertible scalars), and thus is injective. By consequence, so is d_Q.

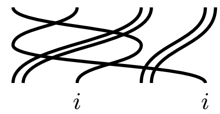
In order to prove our claim, we need to tweak the proof of Lemma 6.2.8. We need to keep track of the terms that are annihilated when working over the cyclotomic quotient, and show these appear as lower terms in the order defined above. The case $j_m \neq i$ remains the same. The case $j_m = i$ and $m = 1$ becomes



where $p = n_i + \ell$. The first term is the leading term. The second term possesses less dots on the left of the floating dot, and thus is smaller. If $a = 0$, then the last two terms possesses one more crossing at the bottom of the floating dot, thus are smaller. If $a = 1$, then they are annihilated by (6.1). Finally the two remaining cases $j_{m-1} \neq i$ and j_{m-1} follows from the same arguments as in the proof of Lemma 6.2.8, with the lower terms in the induction hypothesis only adding lower terms because:



by (6.4), and,



$$= 0,$$

by (6.4) and (6.1). This concludes the proof of the claim, and therefore of the proposition. ■

Proof of Proposition 6.3.10. The proof is a revisit of the proof of Lemma 5.4.9 (which can be found in [33, Lemma 4.18]) that applies for our particular case.

Recall the map P' from (6.20). We know that $P' \circ d_Q$ is given by multiplying by a monic polynomial with leading term $x_{m+1}^{n_i+2\nu_i-\alpha_i^\vee(\nu)}$ plus some remaining map giving lower terms. In particular, it is injective and we have a short exact sequence

$$0 \to Q_1[\xi_i] \xrightarrow{P'\circ d_Q} Q_1[\xi_i] \to \mathrm{Coker}(P' \circ d_Q) \to 0.$$

Moreover, since $P' \circ d_Q$ is in echelon form, it means that $\mathrm{Coker}(P' \circ d_Q)$ is a projective $R_{\mathfrak{g}}(m)$-module. Thus the sequence splits as $R_{\mathfrak{g}}(m)$-modules with splitting map $\sigma : Q_1[\xi_i] \to Q_1[\xi_i]$, and we get $\sigma \circ P' \circ d_Q = \mathrm{Id}_{Q_1[\xi_i]}$. Then, the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Q_1[\xi_i] \arrow[r, "d_Q"] & Q_0[\xi_i] \arrow[l, dashed, bend left, "\sigma\circ P'"] \arrow[r] & H(Q, d_Q), \arrow[r] & 0, \end{tikzcd}$$

obtained thanks to Lemma 6.3.11 splits with splitting map given by $\sigma \circ P'$. Since $Q_0[\xi_i]$ is a projective $R_g(m)$-module, so is $H(Q, d_Q)$. Finally, $d_Q(Q_1[\xi_i])$ is also projective since d_Q is injective and $Q_1[\xi_i]$ is projective. ■

6.3.2 Functors

We define for all $i \in I$ the functors

$$\mathscr{F}_i^N(-) := \bigoplus_{m\geqslant 0} R_{\mathfrak{p}}^N(m+1)1_{(m,i)} \otimes_{R_{\mathfrak{p}}^N(m)} (-),$$
$$\mathscr{E}_i^N(-) := \bigoplus_{m\geqslant 0} \bigoplus_{|\nu|=m} \lambda_i^{-1} q_i^{\alpha_i^\vee(\nu)} 1_{(\nu,i)} R_{\mathfrak{p}}^N(m+1) \otimes_{R_{\mathfrak{p}}^N(m+1)} (-),$$

where we interpret $\lambda_i = q^{n_i}$ whenever $i \in I_f$. Thanks to Proposition 6.3.10, these are exact and their derived counterpart preserve c.b.l. compact objects. Moreover, both $\mathscr{F}_i^N \,\mathrm{Id}_\nu$ and $\mathscr{E}_i^N \,\mathrm{Id}_\nu$ are of finite amplitude.

Theorem 6.3.12. *For $i \notin I_f$ there is a natural short exact sequence*

$$0 \to \mathcal{F}_i^N \mathcal{E}_i^N \,\mathrm{Id}_\nu \to \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu \to \oplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \to 0, \tag{6.21}$$

and for $i \in I_f$ there is a natural isomorphism

$$\begin{aligned} \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu &\cong \mathcal{F}_i^N \mathcal{E}_i^N \,\mathrm{Id}_\nu \oplus_{[n_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu, \quad \textit{if } n_i - \alpha_i^\vee(\nu) \geqslant 0, \\ \mathcal{F}_i^N \mathcal{E}_i^N \,\mathrm{Id}_\nu &\cong \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu \oplus_{[\alpha_i^\vee(\nu) - n_i]_{q_i}} \mathrm{Id}_\nu, \quad \textit{if } n_i - \alpha_i^\vee(\nu) \leqslant 0. \end{aligned} \tag{6.22}$$

Moreover, there is a natural isomorphism

$$\mathcal{F}_i^N \mathcal{E}_j^N \cong \mathcal{E}_j^N \mathcal{F}_i^N, \tag{6.23}$$

for $i \neq j \in I$.

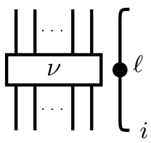
Proof. The short exact sequence (6.21) and the isomorphism (6.23) are immediate consequences of Proposition 6.3.5 and Proposition 6.3.10. For the isomorphisms (6.22), Proposition 6.3.5 and Proposition 6.3.10 gives us a long exact sequence of $R_{\mathfrak{p}}^N(\nu)$-$R_{\mathfrak{p}}^N(\nu)$-bimodules. By Proposition 6.3.9 it truncates to a short exact sequence

$$0 \to \mathcal{F}_i^N \mathcal{E}_i^N \,\mathrm{Id}_\nu \to \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu \to \oplus_{[\rho_i]} \mathrm{Id}_\nu \to 0,$$

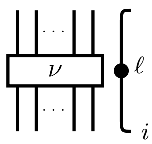
if $\rho_i = n_i - \alpha_i^\vee(\nu) \geqslant 0$, and a short exact sequence

$$0 \to \oplus_{[-\rho_i]} \mathrm{Id}_\nu \to \mathcal{F}_i^N \mathcal{E}_i^N \,\mathrm{Id}_\nu \to \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu \to 0,$$

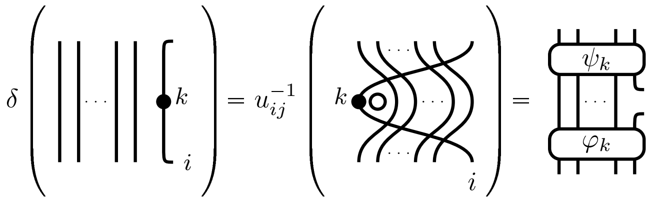
if $\rho_i = n_i - \alpha_i^\vee(\nu) \leqslant 0$. In the first case, we can identify

$$\oplus_{[\rho_i]} \mathrm{Id}_\nu \cong q_i^{1-\rho_i} \bigoplus_{\ell=0}^{\rho_i - 1}$$ 

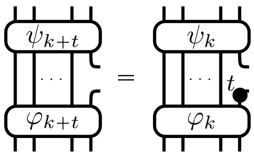
and the map $\mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu \to \oplus_{[\rho_i]_{q_i}} \mathrm{Id}_\nu$ is induced by the projection π. Thus the sequence obviously splits with the splitting map $\oplus_{[\rho_i]_{q_i}} \mathrm{Id}_\nu \to \mathcal{E}_i^N \mathcal{F}_i^N \,\mathrm{Id}_\nu$, given by the sum of maps $R_{\mathfrak{p}}^N(\nu)\xi^\ell \to R_{\mathfrak{p}}^N(\nu + i)$ that add a vertical strand labeled i carrying ℓ dots at the right of a diagram in $R_{\mathfrak{p}}^N(\nu)$. In the second case, we also identify

$$\oplus_{[-\rho_i]_{q_i}} \mathrm{Id}_\nu \cong q_i^{1+\rho_i} \bigoplus_{\ell=0}^{-\rho_i - 1}$$ 

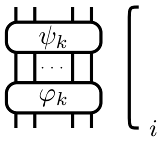
Moreover the map $\oplus_{[-\rho_i]} \mathrm{Id}_\nu \to \mathscr{F}_i^N \mathscr{E}_i^N \mathrm{Id}_\nu$ is induced by the connecting homomorphism δ. Using the notations of (6.18) it takes the form



where u_{ij} is the monomorphism defined in (6.17), and $0 \leqslant k < -\rho_i$. We also note that (6.19) tells us that

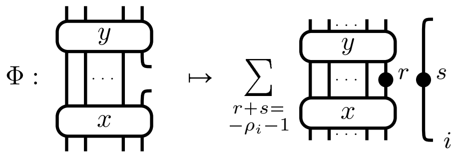

(6.24)

Moreover since $\bar{y}_N(\xi_i^{-\rho_i}) = \zeta$ and $\bar{y}_N(\xi_i^{\ell}) = 0$ for $\ell < -\rho_i$, we obtain by (6.19) again that



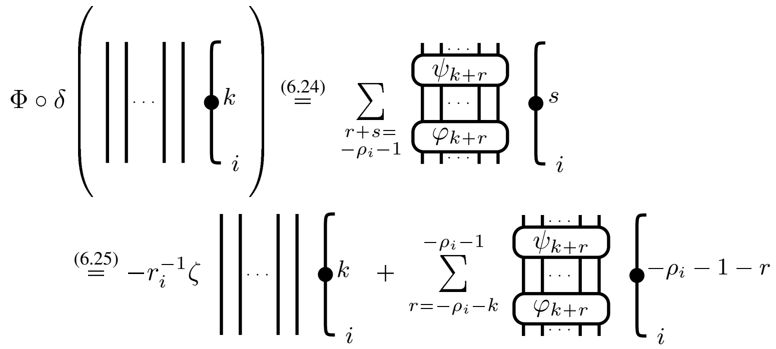
$$= \begin{cases} -r_i^{-1}\zeta, & \text{if } k = -\rho_i - 1, \\ 0, & \text{if } k < -\rho_i - 1. \end{cases} \tag{6.25}$$

As in [33, Proof of Theorem 5.2], we construct a map $\Phi : \mathscr{F}_i^N \mathscr{E}_i^N \mathrm{Id}_\nu \to \oplus_{[-\rho_i]_{q_i}} \mathrm{Id}_\nu$ induced by the morphism of bimodules



for all $x, y \in R_{\mathfrak{p}}^N(\nu)$.

Then we compute



Therefore, $\Phi \circ \delta$ is given by a triangular matrix with invertible elements on the diagonal, and thus is an isomorphism. In particular, δ is left invertible, concluding the proof. ■

Corollary 6.3.13. *There is a quasi-isomorphism*

$$\mathrm{Cone}(\mathscr{F}_i^N \mathscr{E}_i^N \,\mathrm{Id}_\nu \to \mathscr{E}_i^N \mathscr{F}_i^N \,\mathrm{Id}_\nu) \xrightarrow{\simeq} \oplus_{[\beta_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \,.$$

for $i \in I_r$ and a quasi-isomorphism

$$\mathrm{Cone}(\mathscr{F}_i^N \mathscr{E}_i^N \,\mathrm{Id}_\nu \to \mathscr{E}_i^N \mathscr{F}_i^N \,\mathrm{Id}_\nu) \xrightarrow{\simeq} \oplus_{[n_i - \alpha_i^\vee(\nu)]_{q_i}} \mathrm{Id}_\nu \,.$$

for $i \in I_f$.

Corollary 6.3.14. *For $i \in I_f$, then $1_\nu \mathscr{E}_i$ and $\mathscr{F}_i 1_\nu$ are biadjoint (up to shift).*

Proof. By the results in [13], we know the splitting map $\mathscr{E}_i^N \mathscr{F}_i^N \,\mathrm{Id}_\nu \to \mathscr{F}_i^N \mathscr{E}_i^N \,\mathrm{Id}_\nu$ of Theorem 6.3.12 together with the unit and counit of the adjunction $\mathscr{F}_i \dashv \mathscr{E}_i$ allow to construct a unit and counit for the adjunction $\mathscr{E}_i \dashv \mathscr{F}_i$. ■

Proposition 6.3.15. *For each $i, j \in I$ there is a natural isomorphism*

$$\begin{aligned}
&\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} (\mathscr{F}_i^N)^{2a} \mathscr{F}_j^N (\mathscr{F}_i^N)^{d_{ij}+1-2a} \\
&\qquad \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} (\mathscr{F}_i^N)^{2a+1} \mathscr{F}_j^N (\mathscr{F}_i^N)^{d_{ij}-2a}.
\end{aligned}$$

By adjunction, the same isomorphism exists for the $\mathscr{E}_i^N, \mathscr{E}_j^N$.

Proof. This follows from Proposition 6.3.4. ■

6.3.3 A differential on $R_{\mathfrak{p}}^N$

We fix a subset $I_f \subset I'_f \subset I$ and consider the parabolic subalgebras $U_q(\mathfrak{p}) \subset U_q(\mathfrak{p}') \subset U_q(\mathfrak{g})$. For each $j \in I'_f \backslash I_f$ we choose a weight $n'_j \in \mathbb{N}$. For $j \in I_f$ we take $n'_j := n_j \in N$, and we write $N' := \{n'_j\}_{j \in I'_f}$. Then we equip the cyclotomic $\mathfrak{p}$-KLR algebra $R_{\mathfrak{p}}^N(m)$ with a differential $d_{N'}^N$ which is zero on dots and crossings and

$$d_{N'}^N\left(\begin{array}{ccccc} \Big| & \circ_j & \Big| & \cdots & \Big| \\ j & & i_1 & & i_{m-1}\end{array}\right) := \begin{cases} 0, & \text{if } j \notin I'_f, \\ (-1)^{n_j} \begin{array}{cccc} \Big|\!\!\bullet^{n_j} & \Big| & \cdots & \Big| \\ j & i_1 & & i_{m-1}\end{array}, & \text{if } j \in I'_f \backslash I_f. \end{cases}$$

Theorem 6.3.16. *The dg-algebra $(R_{\mathfrak{p}}^N(m), d_{N'}^N)$ is formal with homology*

$$H(R_{\mathfrak{p}}^N(m), d_{N'}^N) \cong R_{\mathfrak{p}'}^{N'}(m).$$

Proof. We have $R_{\mathfrak{p}}^N(m) \cong H(R_{\mathfrak{b}}(m), d_N)$ and $R_{\mathfrak{p}'}^{N'}(m) \cong H(R_{\mathfrak{b}}(m), d_{N'})$ by Theorem 6.2.3. Moreover, $d_{N'}^N$ can be lifted to $R_{\mathfrak{b}}(m)$. We split the homological grading of $R_{\mathfrak{b}}(m)$ in three: a first one that counts the amount of floating dots with subscript in I_f, a second one for the floating dots with subscript in $I'_f \backslash I_f$, and a third one for $I \backslash I'_f$ that we ignore for the moment. Then, we have that $d_{N'}^N$ has degree $(0, -1)$ and d_N has degree $(-1, 0)$, and they commute with each other. Thus we have a (bounded) double complex $(R_{\mathfrak{b}}, d_N, d_{N'}^N)$ with total complex being $(R_{\mathfrak{b}}, d_{N'})$, since $d_{N'} = d_N + d_{N'}^N$. In particular, there is a spectral sequence from $H(R_{\mathfrak{p}}^N(m), d_{N'}^N)$ to $H(R_{\mathfrak{b}}, d_{N'}) \cong R_{\mathfrak{p}'}^{N'}(m)$. Now, Theorem 6.2.3 tells us that $H(R_{\mathfrak{b}}, d_N)$ is concentrated in homological degree zero (for the first homological grading). Thus, the spectral sequence converges at the second page, and in particular $H(R_{\mathfrak{p}}^N(m), d_{N'}^N) \cong R_{\mathfrak{p}'}^{N'}(m)$. ■

We interpret this result as a categorical version of the fact that if there is an arrow from a parabolic Verma module $M^{\mathfrak{p}}(\Lambda, N)$ to another one $M^{\mathfrak{p}'}(\Lambda', N')$ (see §1.2.2), then there is a surjection $M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow M^{\mathfrak{p}'}(\Lambda', N')$. Indeed, in that case there is a surjective quasi-isomorphism $(R_{\mathfrak{p}}^N, d_{N'}) \xrightarrow{\simeq} (R_{\mathfrak{p}'}^{N'}, 0)$, inducing equivalences of derived categories that commute up to quasi-isomorphism with the categorical actions of $U_q(\boldsymbol{g})$.

6.4 The categorification theorems

We assume in this section that $R_{\mathfrak{b}}(\nu)$ is a $\Bbbk$-algebra over a field $\Bbbk$. We also choose an abritrary order $\prec$ for constructing $\mathbb{Z}((q, \Lambda))$ such that $0 \prec q \prec \lambda_i$ for all formal $\lambda_i = q^{\beta_i} \in \Lambda$. Every idempotent of $R_{\mathfrak{b}}(\nu)$ is the image of an idempotent of the classical KLR algebra $R_{\mathfrak{g}}(\nu)$ under the obvious inclusion $R_{\mathfrak{g}}(\nu) \hookrightarrow R_{\mathfrak{b}}(\nu)$. Thanks to [42, Section 2.5], we know all the idempotents of $R_{\mathfrak{g}}(\nu)$. As for NH_n in (5.7), we define the element

$$e_{i,n} := \tau_{\vartheta_n} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} 1_{ii\cdots i} \in R_{\mathfrak{g}}(n).$$

Let $\mathrm{Seqd}(\nu)$ be the set of expressions $i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$ for different $r \in \mathbb{N}$ and $m_\ell \in \mathbb{N}$ such that $\sum_{\ell=1}^r m_\ell \cdot \alpha_{i_\ell} = \nu$. For each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$ we define the idempotent

$$e_{\boldsymbol{i}} := e_{i_1,m_1} \otimes e_{i_2,m_2} \otimes \cdots \otimes e_{i_r,m_r} \in R_{\mathfrak{g}}(\nu),$$

where $x \otimes y$ means we put the diagram of a at the left of the one of y. Identifying $e_{\boldsymbol{i}}$ with its image in $R_{\mathfrak{b}}(\nu)$, as in [42], we define a projective left $R_{\mathfrak{b}}(\nu)$-module

$$P_{\boldsymbol{i}} := R_{\mathfrak{b}}(\nu) e_{\boldsymbol{i}},$$

Then we put

$$\langle \boldsymbol{i} \rangle := -\sum_{\ell=1}^{r} \frac{m_\ell(m_\ell - 1)}{2} d_{i_a}.$$

and we define $\tilde{P}_{\boldsymbol{i}} := q^{-\langle \boldsymbol{i} \rangle} P_{\boldsymbol{i}}$.

When writing $\dots \boldsymbol{i} \dots$ and $\dots \boldsymbol{j} \dots$ we mean we take two sequences $\boldsymbol{i}_1 \boldsymbol{i} \boldsymbol{i}_2$ and $\boldsymbol{i}_1 \boldsymbol{j} \boldsymbol{i}_2$ in $\mathrm{Seqd}(\nu)$ that are the same everywhere except on $\boldsymbol{i}$ and $\boldsymbol{j}$. From the decomposition of the nilHecke algebra (5.8), we get an isomorphism of $R_{\mathfrak{p}}^N$-modules

$$\tilde{P}_{\dots i^m \dots} \cong \oplus_{([m]_{q_i}!)} \tilde{P}_{\dots i^{(m)} \dots}.$$

Mimicking the arguments in [42, Proposition 2.13] and [44, Proposition 6] we get the following:

Proposition 6.4.1. *There are isomorphisms*

$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \tilde{P}_{\dots i^{(2a)} j i^{(d_{ij}+1-2a)} \dots} \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \tilde{P}_{\dots i^{(2a+1)} j i^{(d_{ij}-2a)} \dots}$$

for all $i \neq j \in I$.

Equipping $R_{\mathfrak{b}}(\nu)$ with d_N induces a differential on $(\tilde{P}_{\boldsymbol{i}}, d_N)$, and Proposition 6.4.1 holds for the dg-version. We put

$$\mathscr{M}(\Lambda, N) := \bigoplus_{m \geqslant 0} \mathscr{D}^{lc}(R_{\mathfrak{b}}(m), d_N)$$

with the particular case $\mathscr{M}(\Lambda)$ meaning $N = \varnothing$, and therefore $d_N = 0$.

Proposition 6.4.2. *There is an isomorphism of* $\mathbb{Q}(\!(q,\Lambda)\!)$*-modules*

$$U_q^-(\boldsymbol{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(\!(q,\Lambda)\!) \cong \boldsymbol{K}_0(\mathcal{M}(\Lambda)) \otimes_{\mathbb{Z}} \mathbb{Q},$$

and a $\mathbb{Q}(\!(q,\Lambda)\!)$*-linear epimorphism*

$$U_q^-(\boldsymbol{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(\!(q,\Lambda)\!) \twoheadrightarrow \boldsymbol{K}_0(\mathcal{M}(\Lambda, N)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

sending $F_{i_1}^{(m_1)} F_{i_2}^{(m_2)} \cdots F_{i_r}^{(m_r)}$ *to* $[(\tilde{P}_{\boldsymbol{i}}, d_N)]$ *for* $\boldsymbol{i} = i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$.

Proof. Since projective modules of $R_{\mathfrak{b}}(\nu)$ are in bijection with the ones of the classical KLR algebra $R_{\mathfrak{g}}(\nu)$ and respect the categorified Serre relations (see Proposition 6.4.1), it is a direct consequence of the main results in [42, 44], together with Theorem 3.4.26. ∎

Consider the subring $P(\nu)$ of $R_{\mathfrak{p}}^N(\nu)$ consisting of dots on vertical strands (without floating dots). It admits an action of the symmetric group permuting the strands (with labels) and dots on them (not to be confused with the action of S_m on P_ν from §6.1.1). We write $\mathrm{Sym}(\nu) := P(\nu)^{S_m}$ for the subring of invariants under this action. Since $\mathrm{Sym}(\nu)$ lies in the center of $R_{\mathfrak{p}}^N(\nu)$, any simple $R_{\mathfrak{p}}^N(\nu)$-module is annihilated by $\mathrm{Sym}^+(\nu)$, where $\mathrm{Sym}^+(\nu)$ consists of the elements in $\mathrm{Sym}(\nu)$ with non-zero degree. Then we define for each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$ the simple module

$$S_{\boldsymbol{i}} := P_{\boldsymbol{i}} / \mathrm{Sym}^+(\nu) P_{\boldsymbol{i}},$$

wich is the unique simple quotient of $P_{\boldsymbol{i}}$. We put $\tilde{S}_{\boldsymbol{i}} := q^{-\langle \boldsymbol{i} \rangle} S_{\boldsymbol{i}}$. It lifts to a dg-version $(\tilde{S}_i, d_N)$.

By Lemma 6.2.4 and Proposition 6.3.10, we know that $\mathcal{E}_i \,\mathrm{Id}_\nu$ and $\mathcal{F}_i \,\mathrm{Id}_\nu$ are exact, and induce functors that preserve c.b.l. compact modules and are of finite amplitude on the derived category $\mathcal{D}(R_{\mathfrak{b}}, d_N)$. Then Theorem 6.2.3, Theorem 6.3.12 and Proposition 6.3.15 tells us that $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\Lambda, N))$ is a $U_q(\mathfrak{g})$-weight module. By Proposition 6.4.2, we know that $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\Lambda, N))$ is cyclic as $U_q(\mathfrak{g})$-module, with highest weight generator given by the class of $(R_{\mathfrak{b}}(0), d_N) \cong (\mathbb{k}, 0)$. Thus $\boldsymbol{K}_0^{\Delta}(\mathcal{M}(\Lambda, N))$ is a highest weight module.

As before, let $\psi : R_{\mathfrak{b}}(\nu) \to R_{\mathfrak{b}}(\nu)^{\mathrm{op}}$ be the map that takes the mirror image of diagrams along the horizontal axis, and put $M^\psi := \psi(M)$. Then define the bifunctor

$$(-,-) : \mathcal{M}(\Lambda, N) \times \mathcal{M}(\Lambda, N) \to \mathcal{D}^{lc}(\mathbb{k}, 0) \quad (M, M') := M^\psi \otimes^{\mathrm{L}}_{(R_{\mathfrak{b}}, d_N)} M'.$$

Again, we have

- $((R_{\mathfrak{b}}(0), d_N), (R_{\mathfrak{b}}(0), d_N)) \cong (\mathbb{k}, 0)$;

- $(\mathrm{Ind}_m^{m+i} M, M') \cong (M, \mathrm{Res}_m^{m+i} M')$ for all $M, M' \in \mathscr{M}(\Lambda, N)$;
- $(\oplus_f M, M') \cong (M, \oplus_f M') \cong \oplus_f (M, M')$ for all $f \in \mathbb{Z}(\!(q, \Lambda)\!)$,

which implies that $(-,-)$ is a categorification of the Shapovalov form. Moreover, it turns $\tilde{S}_{\boldsymbol{i}}$ into the dual of $\tilde{P}_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in \mathrm{Seqd}(\nu)$.

Theorem 6.4.3. *The Grothendieck group $\boldsymbol{K}_0^{\Delta}(\mathscr{M}(\Lambda, N))$ is a $U_q(\mathfrak{g})$-weight module, with action of E_i, F_i given by $[\mathscr{E}_i], [\mathscr{F}_i]$. Moreover, there is an isomorphism of $U_q(\mathfrak{g})$-modules*

$$\boldsymbol{K}_0^{\Delta}(\mathscr{M}(\Lambda, N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M^{\mathfrak{p}}(\Lambda, N),$$

where $\Lambda = \{q^{\beta_i} | i \in I_r\}$ contains only formal weights.

Proof. We already proved the first claim above. Clearly, for $i \in I_f$, then $[\mathscr{F}_i], [\mathscr{E}_i]$ acts as locally nilpotent operators. In particular, the $U_q(\mathfrak{i})$-submodule of $\boldsymbol{K}_0^{\Delta}(\mathscr{M}(\Lambda, N))$ given by

$$U_q(\mathfrak{i}) \otimes_{U_q(\mathfrak{g})} [(R_{\mathfrak{b}}(0), d_N)]$$

is an integrable module over the Levi factor $U_q(\mathfrak{i})$. Since it is an integrable highest weight module, it must be isomorphic to $V(\Lambda, N)$ (see [51]). Therefore, there is a surjective $U_q(\mathfrak{g})$-module morphism

$$\gamma : M^{\mathfrak{p}}(\Lambda, N) \twoheadrightarrow \boldsymbol{K}_0^{\Delta}(\mathscr{M}(\Lambda, N)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since $M^{\mathfrak{p}}(\Lambda, N)$ is irreducible and γ is non-zero, it must be an isomorphism. ■

Let $m_r = F_{\boldsymbol{i}} v_{\Lambda,N}$ be an induced basis element of $M^{\mathfrak{p}}(\Lambda, N)$ with $\boldsymbol{i} \in \mathrm{Seq}(\nu)$. Then the isomorphism of Theorem 6.4.3 identifies m_r with the class $[(R_{\mathfrak{b}}(\nu), d_N)1_{\boldsymbol{i}}]$. Similarly, let $m'_s = F_{\boldsymbol{j}} v_{\Lambda,N}$ for $j \in \mathrm{Seqd}(\nu)$ be a canonical basis element, and let m^s be its dual in the dual canonical basis. Then the isomorphism identifies m'_s with $[(\tilde{P}_{\boldsymbol{j}}, d_N)]$ and m^s with $[(\tilde{S}_{\boldsymbol{j}}, d_N)]$. Moreover, computing the c.b.l.f. composition series of $\tilde{P}_{\boldsymbol{i}}$ or taking the c.b.l.f. cell module replacement of $\tilde{S}_{\boldsymbol{i}}$ gives a categorical version of the change of basis between canonical and dual canonical basis.

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