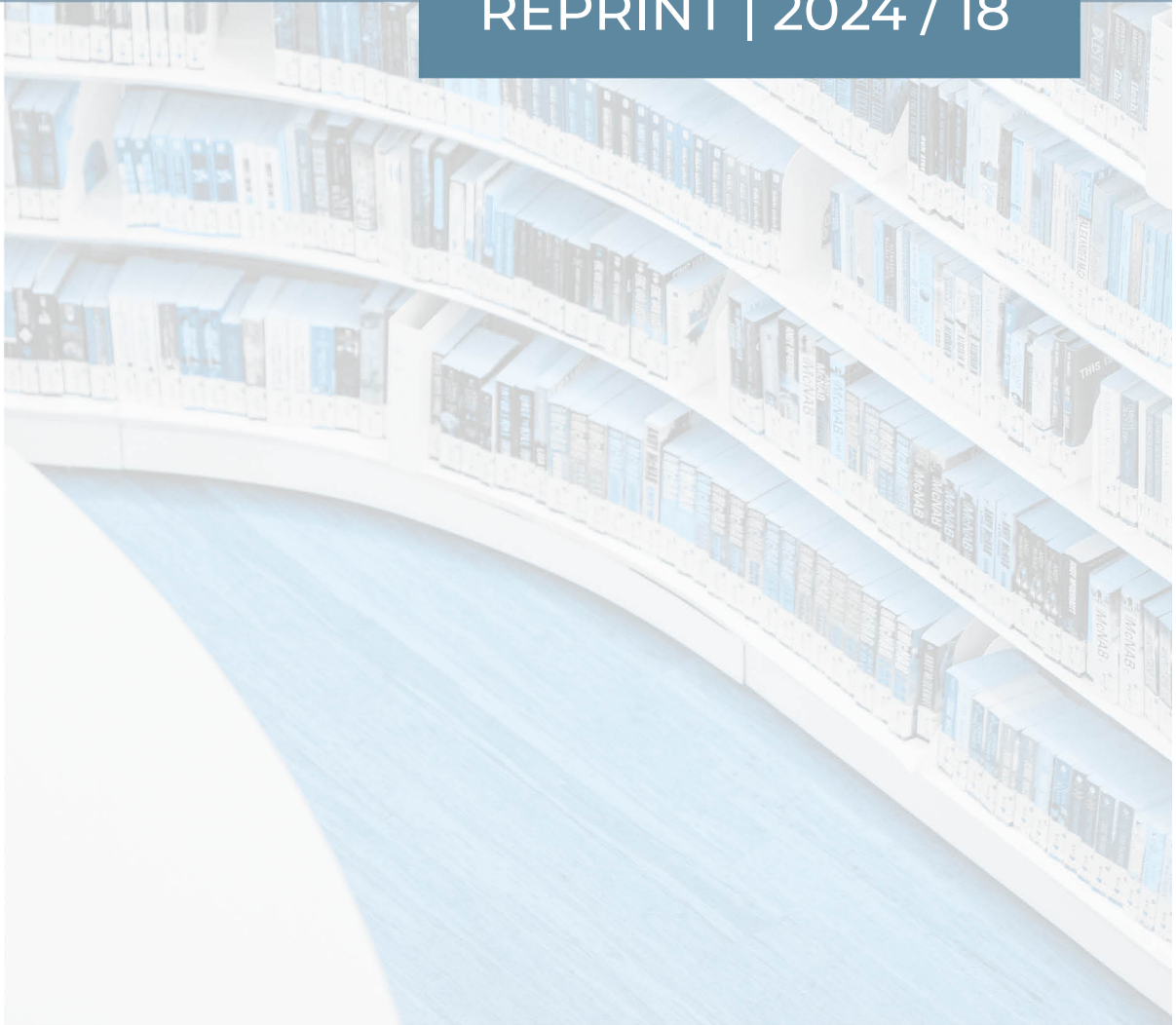


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Impact of correlation between interest rates and mortality rates on the valuation of various life insurance products

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Abstract

In this paper, we question the traditional independence assumption between mortality risk and financial risk and model the correlation between these two risks, estimating its impact on the price of different life insurance products. The interest rate and the mortality intensity are modelled as two correlated Hull and White models in an affine setup. We introduce two building blocks, namely the zero-coupon survival bond and the mortality density, calculate them in closed-form, and perform an investigation about their dependence on the correlation between mortality and financial risk, both with theoretical results and numerical analysis. We study the impact of correlation also for more structured insurance products, such as pure endowment, annuity, term insurance, whole life insurance and mixed endowment. We show that in some cases, the inclusion of correlation can lead to a severe underestimation or overestimation of the best estimate. Finally, we illustrate that the results obtained using a traditional affine diffusive setup can be generalised to affine jump diffusion by computing the price of the zero-coupon survival bond in the presence of jumps.

Keywords: stochastic mortality, correlation between mortality and financial risks, zero-coupon survival bond, mortality density, affine jump diffusion.

1 Introduction

Mortality and interest rate risks are the primary sources of risk for life insurers. On the liability side, the fair values of life insurance products depend on the future trends of interest rates and of the future trends of mortality rates of the appropriate population. On the asset side, the insurer is also exposed to interest rate risk since the investment portfolios of insurers are mainly made up of fixed-income securities. While interest rate risk has been studied for several decades, attention to mortality risk is more recent.

The mortality risk consists of two parts: the unsystematic, diversifiable mortality risk and the aggregate, systematic mortality risk. The unsystematic mortality risk takes into account random deviations from the population-wide mortality experience. According to the law of large numbers, it can be diversified away by pooling of independent contracts and can even in theory be asymptotically cancelled by increasing the size of the pool of risks. The diversifiable nature of this unsystematic mortality risk, which was for long considered as the unique type of mortality risk, explains why the traditional approach of actuaries until the early 90s, consisted in modelling mortality in a deterministic way whereas interest rates were already assumed stochastic for a long time. The systematic or aggregate mortality risk refers to risk factors (e.g. medical progress, epidemics, ...) that lead to a population-wide increase or decrease in life expectancy. This type of mortality risk cannot be eliminated since it is undiversifiable by increasing the portfolio size. This systematic risk can only be captured if mortality is modelled in a stochastic way since there is no systematic mortality risk possible when mortality is deterministic. (Milevsky and Promislow, 2001) were the first to

propose a stochastic intensity mortality model. We know from literature (see e.g. (Biffis, 2005), (Cairns and al., 2006), (Luciano and Vigna, 2008) and (Dacorogna and Cadena, 2015)) that non mean reversion models seem to be more suitable for the modelling of the mortality intensity in comparison with mean reversion models having a fixed long term target. The latter fail to capture the rectangularization phenomenon of the curve of deaths. However, (Zeddouk and Devolder, 2020) showed that incorporating a time-dependent target to the considered models improves significantly their performance.

Concerning the valuation of life insurance contracts, for which the dynamics of mortality intensities and interest rates must be concurrently modelled, the prevailing approach in the literature has been to assume independence between mortality risk and interest rate risk. Authors like (Milevsky and Promislow, 2001), (Dahl, 2004), (Biffis, 2005), (Miltersen and al., 2005) and (Cairns and al., 2006) observed and leveraged the elegant symmetry between the impact of the short rate on zero-coupon prices and that of mortality intensity on survival probabilities. The independence assumption, which permits the separate valuation of mortality risk and interest rate risk, coupled with the use of the « interest-rate machinery » for the modelling of death, yield mathematically convenient results.

However, the independence assumption between the mortality risk and the financial world is in fact more than questionable. On the short term, it is quite intuitive and also documented that catastrophic risks such as major natural disasters (e.g. earthquakes) or severe pandemics can affect human mortality and the financial markets simultaneously. Many studies attest to this dependence between short term demographic changes and financial markets. See (Brahmbhatt, 2005) and (Burns and al., 2008) for an analysis of the impact of the SARS virus on economy and (Brouwers and al., 2009) for the H1N1 virus. In (Dacorogna and Cadena, 2015) and (Dacorogna and Apicella, 2016), they study the empirical dependence between mortality and market risks and concluded that, given the absence at that time of a serious pandemic spreading rapidly around the world, the results do not allow to definitely conclude on the dependence between market and mortality risks but that they allow to give a first idea about the fact that financial markets are affected by the severe worsening of mortality. The recent COVID-19 pandemic aroused an interest of the academic research on the study of the dependence between the mortality risk and the financial world. (Barro and al., 2020) studied the relation between mortality and economic contraction during the 1918–1920 great influenza pandemic and the lessons that can be drawn from it for the coronavirus’s potential effects on mortality and economic activity. (Polyakova and al., 2020) analysed the empirical correlation between the initial economic damage and excess mortality from the COVID-19 pandemic in the U.S. based on the data at disposal at that time. (Lin and Zhang, 2022) studied the extreme co-movements between infectious disease events such as Ebola and COVID-19 outbreaks and crude oil futures prices. In (Li and al., 2023), the authors employ a bivariate affine jump-diffusion which mobilises two different kind of correlations. On the one hand the diffusive correlation between the Brownian motions and on the other hand the jump correlation between the jumps. The authors utilised the latest U.S. mortality and interest rate data to show that the estimated jump correlation remained very large in absolute value (going from -0.474 to -0.479 when the COVID-19 pandemic experience is included in the modelling) while the estimated diffusion correlation remained very low in absolute value (going from -0.039 to 0.017 when the COVID-19 pandemic experience is included in the modelling). They conclude that the COVID-19 pandemic experience greatly intensified the negative instantaneous correlation (which depends on both type of correlation and the values taken by the different model parameters) which went from -0.153 to -0.332. On the long term also, demographic changes influence the economy and the value of financial assets as shown in several works as e.g. (Nicolini, 2004), (Favero and al., 2011) and (Maurer, 2011).

The dependence between interest rates and mortality rates can become very problematic, especially for a life insurer since it links the value of its liabilities to the value of its assets which are supposed to back those liabilities, hence reducing the possibility to diversify the risks. Let us consider the case of a life insurer which has sold a large number of term life insurance contracts and invested the premiums collected in government bonds. In case of positive correlation between interest rates and mortality rates, a mortality shock will have two bad consequences for the insurer. Firstly, an increase in claims caused by the higher number of deaths. Secondly, a depreciation of the fair values of the government bonds caused by the increase of

the interest rates. (Jalen and Mamon, 2009) were among the first to relax the traditional independence assumption between mortality risk and interest rate risk and to study the resulting price of life insurance contracts. They chose two affine models, a Vasicek model for the interest rates and an Ornstein–Uhlenbeck process for the mortality rates. Same models were used in (Liu and al., 2014) for the pricing of Guaranteed Annuity Options (GAO). (Deelstra and al., 2016) also determined prices of GAO but in a Wishart affine model allowing for a non-trivial dependence between the mortality and the interest rates. In (Li and al., 2023), the authors employ a bivariate affine jump-diffusion model to describe the joint dynamics of interest rate and excess mortality, allowing for both correlated diffusions and joint jump.

In this paper, we investigate the impact on the best estimate of a large class of life insurance contracts induced by the taking into account of correlation between interest rates and mortality rates. In line with the literature, we will stay in an affine diffusion setup because of the mathematical ease of doing so. But, in accordance with (Zeddouk and Devolder, 2020), we will not opt for an Ornstein–Uhlenbeck process for the mortality rates but for a Hull and White model. We do not limit the analysis to determining the expressions of the best estimates of life insurance contracts. Indeed, we interpret the results obtained, illustrate them numerically and draw conclusions.

The general main contributions of this work are threefold. Firstly, we model the interest rate and the mortality intensity as two correlated Hull and White models in an affine setup. We introduce two building blocks, namely the zero-coupon survival bond and the mortality density, calculate them in closed-form, and perform an investigation about their dependence on the correlation between mortality and financial risk, both with theoretical results and numerical analysis. Secondly, we study the impact of correlation for more structured insurance products, such as pure endowment, annuity, term insurance, whole life insurance and mixed endowment. We show that in some cases, the inclusion of correlation can lead to a severe underestimation or overestimation of the best estimate. Our investigation further reveals that, regardless of the maturity duration, the influence of correlation on the best estimate of a pure endowment remains inescapable. Conversely, for term insurance, this impact is nullified at a specific maturity. We call that feature of a term insurance « [internal](#) hedging » of the correlation risk. Thirdly, we illustrate that the results obtained using a traditional affine diffusion setup can be generalized to affine jump diffusions by computing the price of the zero-coupon survival bond in the presence of jumps.

This paper is organised as follows. In section 2 we introduce the modelling framework. First, we study the interest rates and mortality rates dynamics separately, i.e. without taking the correlation into account. Using two separate Hull and White models, we recall the expressions of the price of a zero-coupon bond and of the survival probability. Next, we introduce the combined model with correlation that we call the Hull and White² model. We then introduce two building blocks: the price of a zero-coupon survival bond and the price of the mortality density. In section 3, we compute an explicit expression of the first building block and study the direction and the level of the impact of the taking into account of correlation on its value. That impact of correlation depends on the value taken by the different parameters. Through numerical illustrations, we capture the influence of each parameter on the correlation impact. In section 4, we compute an explicit expression of the second building block and study the impact of correlation on its value. In section 5, by suitably combining the two building blocks, we can cover quite a large class of life insurance contracts and study the change in best estimate induced by the inclusion of correlation. In section 6, we propose to improve the modelling of the interest rate and mortality intensity processes by including three jump components. In section 7, we conclude.

2 Modeling framework

2.1 Framework

Let $(\Omega, \mathcal{H}, (\mathcal{H}_t)_{t \in [0, S]}, \tilde{\mathbb{P}})$ be a filtered probability space that accommodates all sources of randomness, where $0 < S < +\infty$ denotes a finite horizon time that we choose to be larger than the maturities of all the insurance contracts considered in the following. See $(\mathcal{H}_t)_{t \in [0, S]}$ as a big « finance-insurance » filtration which

captures both the evolution of the financial and insurance quantities we are interested in.

As we want to price hybrid products and account for the dependence of financial and insurance risks, we cannot impose a product structure of the sample space Ω and the probability $\tilde{\mathbb{P}}$ because it would fully disentangle the two risks.

Following the framework developed by (Artzner and al., 2023), in the so called QP rule, we develop a valuation rule for hybrid insurance products depending on insurance and financial risks, combining the risk neutral pricing for financial risks and the real probability measure for the insurance risks.

Definition of the « extended market filtration » $(\mathcal{G}_t)_{t \in [0, S]}$ and the measure $\tilde{\mathbb{Q}}$

The zero-coupon rate at time t for maturity time s is denoted $Y(t, s)$ and is defined by

$$P(t, s) = e^{-Y(t, s)(s-t)}, \quad (1)$$

where $P(t, s)$ denotes the value of the zero-coupon bond. The short rate in t is then defined as

$$r(t) := \lim_{s \rightarrow t, s > t} Y(t, s). \quad (2)$$

Let the subfiltration $(\mathcal{F}_t)_{t \in [0, S]}$ of $(\mathcal{H}_t)_{t \in [0, S]}$, be the natural filtration generated by the evolution of the stochastic short rate process up to time t . It is supposed to satisfy the usual conditions and is defined as

$$\mathcal{F}_t := \sigma(r(u) : 0 \leq u \leq t). \quad (3)$$

For notational convenience, we will write $r_t := r(t)$.

We denote by $\tau(x)$ the positive random variable corresponding to the future lifetime, that is the number of years of life remaining, of an individual aged x at time 0.

We start from the basic idea of describing $\tau(x)$ as the first jump time of a non explosive counting process $(N_t)_{t \in [0, S]}$ recording at each time $t \geq 0$ whether the individual died ($N_t \neq 0$) or not ($N_t = 0$).

For analytical tractability, we assume that the counting process $(N_t)_{t \in [0, S]}$ is a Cox process, driven by a subfiltration $(\mathcal{M}_t)_{t \in [0, S]}$ of $(\mathcal{H}_t)_{t \in [0, S]}$. $(\mathcal{M}_t)_{t \in [0, S]}$ is the natural filtration generated by the evolution of the systematic part of mortality up to time t . It is defined as

$$\mathcal{M}_t := \sigma(\mu_{x+u}(u) : 0 \leq u \leq t), \quad (4)$$

where $\mu_{x+t}(t)$ represents the mortality intensity of age $x + t$ at time t . It is important to note that we are adopting a cohort-based approach meaning that we are interested in the different values of mortality intensity that an individual will face during his or her life. In other words, $\mu_{x+t}(t)$ describes the future intensity of mortality for any age $x + t$ of an individual aged x at time 0. For notational convenience, we will write $\mu_{x+t} := \mu_{x+t}(t)$.

We define by

$$(\mathcal{G}_t)_{t \in [0, S]} := (\mathcal{F}_t)_{t \in [0, S]} \vee (\mathcal{M}_t)_{t \in [0, S]}, \quad (5)$$

the minimal σ -algebra containing $(\mathcal{F}_t)_{t \in [0, S]} \cup (\mathcal{M}_t)_{t \in [0, S]}$.

We will denote $\tilde{\mathbb{P}}|_{\mathcal{G}_S} = \tilde{\mathbb{Q}}$ and in particular we consider that the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_S$, namely $\tilde{\mathbb{Q}}|_{\mathcal{F}_S}$, is the risk neutral measure and the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{M}_S$, namely $\tilde{\mathbb{Q}}|_{\mathcal{M}_S}$, is the historical measure. This leads to two important remarks. Firstly, in this paper, we will compute the best estimate of liabilities and not their fair value because of the EIOPA requirement that no prudential margin should appear in the best estimate of liabilities. Secondly, this ensures that the best estimate of liabilities that we will compute will be market consistent.

Definition of the « finance-insurance filtration » $(\mathcal{H}_t)_{t \in [0, S]}$ and the measure $\tilde{\mathbb{P}}$

The filtration $(\mathcal{M}_t)_{t \in [0, S]}$ models the accumulation of information concerning the systematic mortality risk. However, it does not provide information about the actual occurrence of death (unsystematic part). That is the reason why we define the subfiltration $(\mathcal{S}_t)_{t \in [0, S]}$ of $(\mathcal{H}_t)_{t \in [0, S]}$ as the filtration generated by the evolution of the unsystematic part of mortality up to time t . It is defined as

$$\mathcal{S}_t := \sigma(1_{\{\tau(x) \leq u\}} : 0 \leq u \leq t), \quad (6)$$

which is the smallest filtration with respect to which $\tau(x)$ is a stopping time. It models whether the policyholder has died by then or not and, if so, when exactly.

We can now formally define the big « finance-insurance » filtration by

$$(\mathcal{H}_t)_{t \in [0, S]} := (\mathcal{G}_t)_{t \in [0, S]} \vee (\mathcal{S}_t)_{t \in [0, S]}, \quad (7)$$

the minimal σ -algebra containing $(\mathcal{G}_t)_{t \in [0, S]} \cup (\mathcal{S}_t)_{t \in [0, S]}$. Hence, the filtration $(\mathcal{H}_t)_{t \in [0, S]}$ models the full flow of information available as time goes by: this includes knowledge of the evolution of stochastic processes r_u and μ_{x+u} up to time t and whether the policyholder has died by then.

On one hand, it can be proven that the survival probability between time t and time s , assuming that the insured is alive at time t , conditionally on $\mathcal{G}_s$, equals

$$\tilde{\mathbb{P}}(\tau(x) > s | \tau(x) > t, \mathcal{G}_s) = \exp \left(- \int_t^s \mu_{x+u} du \right) \quad \text{with } s > t. \quad (8)$$

On the other hand, it can also be shown that the probability of dying between time $u > t$ and time $u + du$, assuming that the insured is alive at time t , conditionally on $\mathcal{G}_u$, equals

$$\tilde{\mathbb{P}}(u < \tau(x) < u + du | \tau(x) > t, \mathcal{G}_u) = \exp \left(- \int_t^u \mu_{x+v} dv \right) \mu_{x+u} du \quad \text{with } u > t. \quad (9)$$

Note in (8) the elegant symmetry between the role played by the short rate for the zero-coupon price and that played by the mortality intensity for the survival probability, which legitimises the use of « interest-rate machinery » for the modelling of death.

It is important to note that filtration $(\mathcal{G}_t)_{t \in [0, S]}$ captures whole the publicly available information (e.g. financial markets, mortality rates) and filtration $(\mathcal{H}_t)_{t \in [0, S]}$ captures on top of that, the internal additional information of the insurance company (e.g. survival times of the individual policyholders).

In the framework developed in (Artzner and al., 2023) the market-consistent valuation of an insurance liability is the expectation of the liability under an insurance-finance-consistent probability denoted $\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}$ and called « pasting of $\tilde{\mathbb{Q}}$ with $\tilde{\mathbb{P}}$ ». Concretely speaking, to compute the best estimates of an insurance liability denoted X and being an $\mathcal{H}_s$ -measurable non-negative random variable already discounted, we apply what the authors call the « QP-rule » and which implies the following:

$$\mathbb{E}_{\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}}[X | \mathcal{G}_t] = \mathbb{E}_{\tilde{\mathbb{Q}}} [\mathbb{E}_{\tilde{\mathbb{P}}} [X | \mathcal{G}_s] | \mathcal{G}_t] \quad \text{with } s > t. \quad (10)$$

2.2 Building blocks

The paper is organised around two fundamental quantities, which serve as the building blocks of more complex life insurance contingent claims. These quantities are introduced below.

The zero-coupon survival bond

A zero-coupon survival bond with maturity time s for an insured of age x at time 0, is a product which delivers 1 unit at maturity time s upon survival of the insured at that time. Hence, the payoff at time s

of a zero-coupon survival bond is equal to $1 \cdot 1_{\tau(x) > s}$ since the contract pays 1 if the insured is still alive at time s , and zero if the insured dies before. The price at time t of a zero-coupon survival bond of maturity time s , denoted $P_{r,\mu}(t, s)$, is then given by the $\mathcal{H}_t$ -conditional expectation under the $\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}$ measure of the discounted payoff, i.e.

$$P_{r,\mu}(t, s) = \mathbb{E}_{\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}} \left[1 \cdot \exp \left(- \int_t^s r_u du \right) \cdot 1_{\{\tau(x) > s\}} \middle| \mathcal{H}_t \right]. \quad (11)$$

At time t when we compute the price of the zero-coupon survival bond, we do know if the insured is dead or not. Hence, the expectation has to be taken with respect to the global filtration $\mathcal{H}_t$, which besides the evolution of the two state variables up to t (information included in $\mathcal{G}_t$), includes the death monitoring (information included in $\mathcal{S}_t$). However, if we take $1_{\{\tau(x) > t\}}$ out of the expectation (11), we can switch from a $\mathcal{H}_t$ to a $\mathcal{G}_t$ expectation which gives

$$P_{r,\mu}(t, s) = 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}} \left[\exp \left(- \int_t^s r_u du \right) \cdot 1_{\{\tau(x) > s\}} \middle| \mathcal{G}_t \right]. \quad (12)$$

The application of the QP-rule (10) and probability (8) then allows to rewrite (12) as

$$P_{r,\mu}(t, s) = 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}}} \left[\exp \left(- \int_t^s r_u du \right) \cdot \exp \left(- \int_t^s \mu_{x+u} du \right) \middle| \mathcal{G}_t \right]. \quad (13)$$

which is the formal expression of the price of a zero-coupon survival bond, first of our two building blocks. It is identical to the expression obtained in the literature (e.g. (Biffis, 2005) and (Deelstra and al., 2016)) except that the expectation here is taken under measure $\tilde{\mathbb{Q}}$. Moreover, the structure of (13) is similar as the one of a defaultable zero-coupon bond developed in the credit risk literature with the intensity of default replaced by a mortality intensity.

Mortality density

Within the same framework, we introduce a concept called the « mortality density », which is more accurately described as the « price density of a term insurance ». Despite this clarification, we deliberately choose the former terminology over the latter to explicitly acknowledge its connection to mortality.

The mortality density for an insured aged x at time 0 is a « product » which delivers 1 unit to the beneficiary of the insured at the time u when the insured dies. Hence, the payoff at time u of the mortality density is equal to $1 \cdot 1_{\tau(x)=u}$ since the contract pays 1 if the insured dies at time u . The value at time t of the mortality density of a death at time $u > t$, denoted $D_{r,\mu}(t, u)$, is given by the $\mathcal{G}_t$ -conditional expectation under the $\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}$ measure of the discounted payoff, since

$$\begin{aligned} D_{r,\mu}(t, u) &= \mathbb{E}_{\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}} \left[1 \cdot \exp \left(- \int_t^u r_v dv \right) \cdot 1_{\{\tau(x)=u\}} \middle| \mathcal{H}_t \right] \\ \Leftrightarrow D_{r,\mu}(t, u) &= 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}} \left[\exp \left(- \int_t^u r_v dv \right) \cdot 1_{\{\tau(x)=u\}} \middle| \mathcal{G}_t \right]. \end{aligned} \quad (14)$$

The application of the QP-rule (10) and probability (9) then allows to rewrite (14) as

$$D_{r,\mu}(t, u) = 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}}} \left[\exp \left(- \int_t^u r_v dv \right) \cdot \exp \left(- \int_t^u \mu_{x+v} dv \right) \mu_{x+u} \middle| \mathcal{G}_t \right] \quad (15)$$

and this is the second of our two building blocks.

By suitably combining the building blocks (13) and (15), we can cover quite a large class of insurance contracts. In practical terms, this implies that by deriving explicit expressions for the price of the building blocks, we will also obtain explicit expressions for the best estimates of the different life insurance contracts considered.

2.3 Model

As there is a natural desire to work with a computationally tractable model, we will use an affine framework for its rich structural properties and mathematical convenience. For the theory on affine diffusion processes, we refer to the very structured and accessible paper (Filipovic and Mayerhofer, 2009) which revisit the class of time-homogeneous affine diffusion processes on the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$. The key affine property is roughly that the logarithm of the $\mathcal{F}_t$ -conditional characteristic function of the state vector at maturity is affine with respect to the state vector. The coefficients appearing in this affine relationship are the solutions of generalised Riccati equations. When the latter can be solved explicitly, tractability is ensured which means that we have an explicit expression for the building blocks.

In this paper, we relax the traditional assumption of independence between interest rate and mortality intensity. The direct implication is that the expectations (13) and (15) are harder to calculate since they cannot simply be separated into the product of two expectations. To compute those expectations we rely on the affine theory. The problem is that for a lot of choices of two separate affine models, the taking into account of correlation goes hand in hand with a loss of affinity of the combined model. In those cases, the inclusion of correlation comes at the expense of tractability. As discussed below, this combined model affinity constraint restricts the range of possibilities among the pairs of univariate models considered.

Three natural candidates to model the stochastic behaviour of the interest rate are the Hull and White model (see (Hull and White, 1994)), the CIR model (see (Cox and al., 1985)) and the CIR++ model (see (Brigo and Mercurio, 2007)). The major advantage of the Hull and White model is that its Gaussian character greatly facilitates the calculations. What used to be its main drawback, namely the fact that there is no guarantee that the interest rate will remain strictly positive, is no longer a problem in a context where negative rates are observed. Therefore, it seems to be a good choice for interest rate modeling.

As mentioned in the introduction, (Zeddouk and Devolder, 2020) showed that models that incorporate a time-dependent target, namely the Hull and White model and the extended CIR model¹, are very performing to model the mortality intensity. Among those two models, the most legitimate candidate is the extended CIR model since it guarantees that the mortality intensity remains strictly positive provided that the Feller condition is met (without this condition we still have non-strict positivity).

Hence, the prevailing combined model is the Hull and White-Extended CIR model with correlation. Unfortunately, the correlation terms make it non-affine which is a considerable obstacle. Inspired by (Grzelak and Oosterlee, 2011) and (Brigo and Alfonsi, 2005), we have explored different approaches that would allow us to return, through an approximation, to an affine version of the Hull and White-Extended CIR model with correlation. The cost in terms of loss of interpretability of the results is considerable, so that we chose to fall back on a model, admittedly sub-optimal, but which has the merit of being affine, highly tractable and interpretable: the Hull and White² model with correlation between the finance world and the insurance world which is written under $\tilde{\mathbb{Q}}$ as

$$\begin{cases} \textbf{Interest rate :} & dr_t = \lambda(\bar{r}_t - r_t) dt + \eta dW_t^r & \text{Hull and White,} \\ \textbf{Mortality rate :} & d\mu_{x+t} = \omega(\bar{\mu}_{x+t} - \mu_{x+t}) dt + \varepsilon dW_t^\mu & \text{Hull and White,} \end{cases} \quad (16)$$

with

$$\mathbb{E}_{\tilde{\mathbb{Q}}} [dW_t^r dW_t^\mu] = \rho^{r,\mu} dt,$$

where $\lambda, \omega, \eta, \varepsilon$ are strictly positive constants, $\bar{r}_t$ and $\bar{\mu}_t^x$ are time dependent functions which will be explained in respectively (18) and (27) and $\rho^{r,\mu}$ is the correlation coefficient between the Brownian motions W_t^r and W_t^μ introducing therefore a dependence between financial and actuarial risks.

Before delving into the Hull and White² model, which simultaneously considers the dynamics of interest rates and mortality intensity and allows for correlation between them, it is pertinent to briefly review some key results for each standalone Hull and White model.

¹CIR model with a time-dependent target

2.3.1 Hull and White model for the interest rate

Under $\tilde{\mathbb{Q}}$, the Hull and White model for the interest rate is written as

$$dr_t = \lambda(\bar{r}_t - r_t) dt + \eta dW_t^r, \quad r_0 = r^0, \quad (17)$$

where $\bar{r}_t$ is the deterministic moving target, $\lambda > 0$ is the speed of mean reversion, $\eta > 0$ is the diffusion coefficient and W_t^r is a standard Brownian motion under $\tilde{\mathbb{Q}}$ which is assumed to be $\mathcal{F}_t$ -adapted.

Let us recall that we consider that the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{F}_S$, namely $\tilde{\mathbb{Q}}|_{\mathcal{F}_S}$, is the risk neutral measure which means that the values of λ and η can be obtained through a calibration on the market prices of derivatives (typically swaptions). In the following, we will use values extracted from the literature which are computed under the risk neutral measure.

A judicious choice for the expression of the moving reverting target allows to incorporate the zero-coupon market curve observed on the market at $t = 0$ as an initial constraint, hence ensuring that the zero-coupon prices predicted by the model in zero coincides with the values of the zero-coupon prices observed in the market in $t = 0$. As stated by (Brigo and Mercurio, 2007), the needed expression of the moving target function $\bar{r}_t$ to ensure the exogenous constraint is the following:

$$\bar{r}_t = \frac{1}{\lambda} \frac{\partial f^M(0, t)}{\partial t} + f^M(0, t) + \frac{\eta^2}{2\lambda^2} (1 - e^{-2\lambda t}), \quad (18)$$

where $f^M(0, t)$ is the market instantaneous forward rate at time 0 for maturity t , which can be expressed from the market zero-coupon price at time 0 for maturity t , noted $P^M(0, t)$, by

$$f^M(0, t) = -\frac{\partial \ln P^M(0, t)}{\partial t}. \quad (19)$$

Applying Ito's lemma to the function $r_t e^{\lambda t}$ yields the following unique explicit solution of (17):

$$r_s = r_t e^{-\lambda(s-t)} + \lambda e^{-\lambda s} \int_t^s \bar{r}_u e^{\lambda u} du + \eta e^{-\lambda s} \int_t^s e^{\lambda u} dW_u^r. \quad (20)$$

This last expression implies that the distribution of r_s conditionally to r_t is normal with conditional moments given by

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[r_s | \mathcal{F}_t] = r_t e^{-\lambda(s-t)} + \lambda e^{-\lambda s} \int_t^s \bar{r}_u e^{\lambda u} du \quad \text{and} \quad \text{Var}_{\tilde{\mathbb{Q}}}[r_s | \mathcal{F}_t] = \frac{\eta^2}{2\lambda} \left(1 - e^{-2\lambda(s-t)}\right). \quad (21)$$

Considering the Hull and White model (17) with moving target (18), the price at time t of a zero-coupon of maturity time s , is given by

$$P(t, s) = A_r(t, s) \exp(-B(\lambda, t, s)r_t), \quad (22)$$

with

$$A_r(t, s) = \frac{P^M(0, s)}{P^M(0, t)} \exp\left(f^M(0, t)B(\lambda, t, s) - \frac{\eta^2}{4\lambda} B^2(\lambda, t, s) (1 - e^{-2\lambda t})\right), \quad (23)$$

where the function B is defined by

$$B(x, t, s) := \left(\frac{1 - e^{-x(s-t)}}{x}\right). \quad (24)$$

It is worth mentioning that, in accordance with our expectations, the prices of the zero-coupons predicted by the model in $t = 0$, coincide with those of the market, regardless of the values of the parameters λ and η , that is

$$P(0, s) = P^M(0, s), \quad (25)$$

which justifies the particular form of the moving target (18).

2.3.2 Hull and White model for the mortality intensity

Under $\tilde{\mathbb{Q}}$, the Hull and White model for the mortality intensity is written as

$$d\mu_{x+t} = \omega(\bar{\mu}_{x+t} - \mu_{x+t}) dt + \varepsilon dW_t^\mu, \quad \mu_x = \mu_x^0, \quad (26)$$

where $\bar{\mu}_{x+t}$ is the moving target, $\omega > 0$ is the speed of mean reversion, $\varepsilon > 0$ is the diffusion coefficient and W_t^μ is a standard Brownian motion under $\tilde{\mathbb{Q}}$ which is assumed to be $\mathcal{M}_t$ -adapted.

Following (Zeddouk and Devolder, 2020), we consider that the moving target $\bar{\mu}_{x+t}$ towards which the intensity of mortality tends to return with a speed of mean reversion ω is a Gompertz of the following form

$$\bar{\mu}_{x+t} = \bar{A}e^{\bar{B}t}, \quad (27)$$

with $\bar{A}$ and $\bar{B}$ two strictly positive constants. It expresses that the force of mortality increases exponentially with age, where $\bar{A}$ is the baseline mortality at age x , and $\bar{B}$ is the senescent component. It makes a lot of sense to see the stochastic mortality intensity as random deviations around that theoretical law.

Let us recall that the restriction of $\tilde{\mathbb{Q}}$ to $\mathcal{M}_S$, namely $\tilde{\mathbb{Q}}|_{\mathcal{M}_S}$, is the historical measure which means that the values of ω , ε , $\bar{A}$ and $\bar{B}$ can be obtained through a calibration on historical data. In the following, we will use values extracted from the literature which are computed under that historical measure.

Applying Ito's lemma to the function $\mu_{x+t}e^{\omega t}$ yields the following unique explicit solution of (26):

$$\mu_{x+s} = \mu_{x+t}e^{-\omega(s-t)} + \omega e^{-\omega s} \int_t^s \bar{\mu}_{x+u} e^{\omega u} du + \varepsilon e^{-\omega s} \int_t^s e^{\omega u} dW_u^\mu. \quad (28)$$

This last expression implies that the distribution of μ_{x+s} conditionally to μ_{x+t} is normal with conditional moments given by

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[\mu_{x+s}|\mathcal{F}_t] = \mu_{x+t}e^{-\omega(s-t)} + \omega e^{-\omega s} \int_t^s \bar{\mu}_{x+u} e^{\omega u} du \quad \text{and} \quad \text{Var}_{\tilde{\mathbb{Q}}}[\mu_{x+s}|\mathcal{F}_t] = \frac{\varepsilon^2}{2\omega} \left(1 - e^{-2\omega(s-t)}\right). \quad (29)$$

Considering the Hull and White model (26) with moving target (27), the survival probability of an individual initially aged x , and alive at time t to survive $s - t$ years more, is at time t given by

$$P_\mu(t, s) = A_\mu(t, s) \exp(-B(\omega, t, s)\mu_{x+t}), \quad (30)$$

with

$$A_\mu(t, s) = \exp\left(\frac{\omega \bar{A} e^{\bar{B}t}}{(\omega + \bar{B})} (B(\omega, t, s) - B(-\bar{B}, t, s)) + \frac{\varepsilon^2}{2\omega^2} \left[(s-t) - B(\omega, t, s) - \frac{\omega}{2} B^2(\omega, t, s)\right]\right) \quad (31)$$

and where the function B is defined in (24).

3 Zero-coupon survival bond

Let us compute the price of the zero-coupon survival bond (13), first of our two building blocks, whose expression is recalled below,

$$P_{r,\mu}(t, s) = 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}}} \left[\exp\left(-\int_t^s r_u du\right) \exp\left(-\int_t^s \mu_{x+u} du\right) \middle| \mathcal{G}_t \right], \quad (32)$$

knowing that the dynamics of r_t and μ_{x+t} follow the Hull and White² model (16). Using Cholesky decomposition, we can rewrite (16) with independent Brownian motions noted $\widetilde{W}_t^r$ and $\widetilde{W}_t^\mu$ as

$$\begin{cases} dr_t = \lambda(\bar{r}_t - r_t)dt + \eta d\widetilde{W}_t^r, \\ d\mu_t^x = \omega(\bar{\mu}_{x+t} - \mu_{x+t})dt + \varepsilon \left[\rho^{r,\mu} d\widetilde{W}_t^r + \sqrt{1 - (\rho^{r,\mu})^2} d\widetilde{W}_t^\mu \right]. \end{cases} \quad (33a)$$

$$\quad (33b)$$

Applying Ito's lemma respectively to the function $r_t e^{\lambda t}$ and to the function $\mu_t^x e^{\omega t}$ yields the following unique explicit solutions of equations (33a) and (33b):

$$\begin{cases} r_s = r_t e^{-\lambda(s-t)} + \lambda e^{-\lambda s} \int_t^s \bar{r}_u e^{\lambda u} du + \eta e^{-\lambda s} \int_t^s e^{\lambda u} d\widetilde{W}_u^r, \\ \mu_{x+s} = \mu_{x+t} e^{-\omega(s-t)} + \omega e^{-\omega s} \int_t^s \bar{\mu}_{x+u} e^{\omega u} du + \varepsilon \rho^{r,\mu} e^{-\omega s} \int_t^s e^{\omega u} d\widetilde{W}_u^r \\ \quad + \varepsilon \sqrt{1 - (\rho^{r,\mu})^2} e^{-\omega s} \int_t^s e^{\omega u} d\widetilde{W}_u^\mu. \end{cases} \quad (34a)$$

$$\quad (34b)$$

Let us define

$$I(t, s) = \int_t^s (r_u + \mu_{x+u}) du. \quad (35)$$

After some calculations detailed in the [online Appendix A](#), we obtain

$$\begin{aligned} I(t, s) = & r_t B(\lambda, t, s) + \int_t^s \bar{r}_u B(\lambda, u, s) du + \mu_{x+t} B(\omega, t, s) + \int_t^s \bar{\mu}_{x+u} B(\omega, u, s) du \\ & + \eta \int_t^s B(\lambda, u, s) d\widetilde{W}_u^r + \varepsilon \rho^{r,\mu} \int_t^s B(\omega, u, s) d\widetilde{W}_u^r + \varepsilon \sqrt{1 - (\rho^{r,\mu})^2} \int_t^s B(\omega, u, s) d\widetilde{W}_u^\mu, \end{aligned} \quad (36)$$

from which we can deduce

$$\exp(-I(t, s)) \sim LN(\mu, \sigma^2), \quad (37)$$

where

$$\mu = -r_t B(\lambda, t, s) - \int_t^s \bar{r}_u B(\lambda, u, s) du - \mu_{x+t} B(\omega, t, s) - \int_t^s \bar{\mu}_{x+u} B(\omega, u, s) du \quad (38)$$

and

$$\sigma^2 = \sigma_r^2 + \sigma_m^2 + \rho_{rm} \quad \text{where} \quad \begin{cases} \sigma_r^2 = \frac{\eta^2}{\lambda^2} [(s-t) - \frac{\lambda}{2} B^2(\lambda, t, s) - B(\lambda, t, s)], \\ \sigma_m^2 = \frac{\varepsilon^2}{\omega^2} [(s-t) - \frac{\omega}{2} B^2(\omega, t, s) - B(\omega, t, s)], \\ \rho_{rm} = \frac{2\eta\varepsilon\rho^{r,\mu}}{\lambda\omega} [(s-t) - B(\lambda, t, s) - B(\omega, t, s) + B(\lambda + \omega, t, s)]. \end{cases} \quad (39)$$

Given (35), (32) takes the form

$$P_{r,\mu}(t, s) = 1_{\{\tau(x) > t\}} \mathbb{E}_{\mathbb{Q}} [\exp(-I(t, s)) | \mathcal{G}_t]. \quad (40)$$

Using the formula of the expected value of a lognormal distribution, we can conclude from (38) and (39) that the price of the zero-coupon survival bond is given by the following proposition. For a detailed proof, see the [online Appendix A](#).

Proposition I

Considering the Hull and White model² (16) with explicit expressions of the moving targets (18) and (27), the price at time t of a zero-coupon survival bond of maturity time s , for an individual initially aged x at time 0, is given by

$$P_{r,\mu}(t, s) = 1_{\{\tau(x) > t\}} \cdot P_r(t, s) \cdot P_\mu(t, s) \cdot P_{\rho^{r,\mu}}(t, s), \quad (41)$$

where

- $P_r(t, s)$ is the price of a zero-coupon under the Hull and White model given by expressions (22)

and (23).

- $P_\mu(t, s)$ is the survival probability under the Hull and White model given by expressions (30) and (31).
- $P_{\rho^{r,\mu}}(t, s)$ is the « price of correlation » between interest rates and mortality rates given by

$$P_{\rho^{r,\mu}}(t, s) := \exp \left(\frac{\eta \varepsilon \rho^{r,\mu}}{\lambda \omega} [(s - t) + B(\lambda + \omega, t, s) - B(\lambda, t, s) - B(\omega, t, s)] \right), \quad (42)$$

where the function B is defined in (24).

3.1 Study of the direction of the impact of correlation

The elegant multiplicative structure (41), coupled with an analysis (see the [online Appendix B](#)) of the sign of function

$$f(T) = T + \left(\frac{1 - e^{-(\lambda+\omega)T}}{\lambda + \omega} \right) - \left(\frac{1 - e^{-\lambda T}}{\lambda} \right) - \left(\frac{1 - e^{-\omega T}}{\omega} \right), \quad (43)$$

enables to distinct three different scenarios regarding the impact of correlation on the price of the zero-coupon survival bond.

- **Positive correlation scenario:** If $\rho^{r,\mu} > 0$, $P_{\rho^{r,\mu}}(t, s) > 1$ which implies that in the case of positive correlation, the price of the zero-coupon survival bond is higher than when assuming independence, namely

$$P_{r,\mu}(t, s) > P_r(t, s) \cdot P_\mu(t, s). \quad (44)$$

This is an important result since it means that when there is a positive correlation between interest rates and mortality rates, ignoring correlation amounts to underestimate the price of the zero-coupon survival bond.

- **Zero correlation scenario:** If $\rho^{r,\mu} = 0$, $P_{\rho^{r,\mu}}(t, s) = 1$ which confirms that in absence of correlation, the price of the zero-coupon survival bond is the same as when we assume independence, namely

$$P_{r,\mu}(t, s) = P_r(t, s) \cdot P_\mu(t, s), \quad (45)$$

which is the classical multiplicative form used in life insurance.

- **Negative correlation scenario:** If $\rho^{r,\mu} < 0$, $P_{\rho^{r,\mu}}(t, s) < 1$ which implies that in the case of negative correlation, the price of the zero-coupon survival bond is lower than in the independent case,

$$P_{r,\mu}(t, s) < P_r(t, s) \cdot P_\mu(t, s). \quad (46)$$

When there is a negative correlation between interest rates and mortality rates, ignoring correlation amounts to overestimate the price of the zero-coupon survival bond.

In summary, within the framework of the Hull and White model (16), positive correlation scenarios result in an upward impact on the price of the zero-coupon survival bond, while negative correlation scenarios lead to a downward impact.

The fact that the direction (whether downward or upward) of the impact of correlation on the price of the zero-coupon survival bond is entirely driven by the sign of the correlation coefficient may not be readily apparent in terms of interpretation. To elucidate the underlying dynamics, recall that interest rates and mortality rates affect the price of the zero-coupon survival bond in a congruent manner. This coherence stems from the symmetric role they play in the formulation represented by formula (32). The upward scenario can be interpreted as follows: equation (39) indicates that considering a positive correlation leads to an increase in the overall variance, consequently resulting in a higher price for the zero-coupon survival

bond. Indeed, that higher risk translates into a higher price of the zero-coupon survival bond through the multiplication by a price of correlation strictly higher than 1, hence confirming (44). A similar analysis can be done in case of negative correlation.

3.2 Study of the level of the impact of correlation

In the preceding section, we examined the upward or downward trajectory of the zero-coupon survival bond price resulting from the incorporation of correlation. We will now shift our attention to evaluating the magnitude of this effect.

The correlation price (42) depends on the value taken by the following six distinct parameters: the correlation coefficient $\rho^{r,\mu}$, the diffusion coefficients η and ε , which symmetrically impact the correlation price (42), the speeds of mean reversion λ and ω , whose influence on the correlation price is likewise symmetrical, and lastly the time to maturity $T = s - t$. Notably, it is observed that the correlation price does not depend directly on times t and s , but solely on the difference between these two times, represented by the maturity $T = s - t$.

We start by delineating the univariate functions of each of the six aforementioned parameters on the correlation price. Subsequently, we proceed to rank these parameters based on their respective degrees of influence on the correlation price.

3.2.1 Univariate behaviours of each parameter on the correlation price

η, ε and T

The correlation price is a positive (when $\rho^{r,\mu} > 0$) or negative (when $\rho^{r,\mu} < 0$) exponential function of the parameters η, ε and of the increasing function $f(T)$. Hence, the higher the value of η, ε and T , the higher the value of $|1 - P_{\rho^{r,\mu}}(t, s)|$, that is the higher the absolute impact of correlation on the total price either upward (when $\rho^{r,\mu} > 0$) or downward (when $\rho^{r,\mu} < 0$).

The interpretation is the following. An increase of one of those parameters corresponds to a higher relative importance given to stochasticity either through a higher diffusion coefficient or through a longer time interval where stochasticity applies. That higher stochasticity translates then into:

- a higher total price reflecting a higher risk in case of positive correlation since the natural parallel influence of interest rates and mortality rates on the zero-coupon survival price is then reinforced; or
- a lower total price reflecting a lower risk in case of negative correlation since the natural parallel influence of interest rates and mortality rates on the zero-coupon survival price is then countered.

λ and ω

The correlation price is a positive exponential function of the inverse of the speeds of mean reversion in case of positive correlation and a negative exponential function of the inverse of the speeds of mean reversion in case of negative correlation. Hence, the lower the values of λ and ω , the higher the value of $|1 - P_{\rho^{r,\mu}}(t, s)|$, that is the higher the absolute impact of correlation on the total price either upward (when $\rho^{r,\mu} > 0$) or downward (when $\rho^{r,\mu} < 0$). The interpretation is the following. A lower speed of mean reversion means a higher relative importance given to stochasticity, which translates into:

- a higher total price reflecting a higher risk in case of positive correlation since the natural parallel influence of interest rates and mortality rates on the zero-coupon survival price is then reinforced by the downward shock on the speed of mean reversion.
- a lower price reflecting a lower risk in case of negative correlation since the natural parallel influence of interest rates and mortality rates on the zero-coupon survival price is then countered by the downward shock on the speed of mean reversion.

It is noteworthy that the convex nature of the exponential function results in a non-symmetric effect of correlation. Specifically, this implies that the upward impact associated with a positive correlation $|\rho^{r,\mu}|$ is relatively greater in magnitude compared to the downward impact attributed to a negative correlation $-|\rho^{r,\mu}|$.

It is important to recognise that the speed of increase or decay of the aforementioned exponential functions of one of the parameters depends on the values taken by the other parameters. The concordance of a small exponential growth/decay constant and a fairly small range of the parameter under consideration leads to an exponential behavior that progresses so gradually that it becomes hardly discernible, often appearing to be linear. The figures below illustrate this phenomenon.

3.2.2 Ranking the parameters according to the magnitude of their respective impact on the price of correlation

It appears logical to attempt to categorise the six parameters based on their varying degrees of influence on the correlation price. The objective is to ascertain whether certain factors exert a greater impact than others and thus warrant particular attention in monitoring their values.

In an attempt to elucidate the sensitivity of the correlation price to each individual parameter, we employ a first-order approximation method. This method involves computing the change in value resulting from the transition from a parameter vector denoted as $\bar{x} = \bar{x}^* = (T^*, (\rho^{r,\mu})^*, \eta^*, \epsilon^*, \lambda^*, \omega^*)$ to $\bar{x} = \bar{x}^* + \bar{\Delta x}$ where we choose $\bar{\Delta x} = \chi \bar{x}^*$ where $\chi \in \mathbb{R}$. This approach enables the decomposition of the overall change in the correlation price resulting from the shift into six distinct terms. These terms delineate the linear marginal impact, either positive or negative, of each of the six parameters on the correlation price. We obtain the following formula:

$$\begin{aligned}
P_{r,\mu}(\bar{x}^* + \bar{\Delta x}) = & P_{r,\mu}(\bar{x}) + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial \rho^{r,\mu}} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot (\rho^{r,\mu})^*}_{A} + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial \epsilon} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot \epsilon^*}_{B} + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial \eta} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot \eta^*}_{C} \\
& + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial T} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot T^*}_{D} + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial \lambda} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot \lambda^*}_{E} + \underbrace{\frac{\partial P_{r,\mu}(\bar{x})}{\partial \omega} \Big|_{\bar{x}=\bar{x}^*} \chi \cdot \omega^*}_{F} + o(\|\bar{x} - \bar{x}^*\|).
\end{aligned} \tag{47}$$

An examination of the terms A, B, C, D, E and F allows to establish a hierarchy between three groups of parameters concerning their first-order impact on the correlation price. Ranked from the most influential to the least, the classification is as follows: first, the maturity; second, the correlation coefficient and the diffusion coefficients; and third the speeds of mean reversion.

3.3 Numerical illustration

3.3.1 Calibration

We conduct separate calibrations for the Hull and White interest rate model and the Hull and White mortality rate model. Subsequently, we have the flexibility to set $\rho^{r,\mu}$ to a desired value, effectively treating the correlation coefficient as a variable parameter. We do not perform calibrations as such but refer to the literature to define a default value and a range of values for each parameter. Those values are reasonable values that can be obtained from calibration on the interest rates options market on one side and on official mortality tables on the other side.

Interest rate model: based on (Devolder and Azizieh, 2013), (Singor and al., 2013), (Casamassima and al., 2022) and (Diez, F. and Korn, R., 2020), we opt for the default values

$$\lambda = 3\% \text{ and } \eta = 1\%, \tag{48}$$

and the ranges

$$\lambda \in [0 \ 25\%] \text{ and } \eta \in [0 \ 10\%]. \quad (49)$$

We consider the following market yield rate structure at $t = 0$:

$$Y(0, i) = 1\% \quad \forall i \in \{1, 2, \dots, \tilde{\omega} - x\},$$

where $\tilde{\omega}$ is the ultimate age considered (here $\tilde{\omega} = 110$); which implies

$$P^M(0, i) = \exp(-Y(0, i) \cdot i) \quad \forall i \in \{1, 2, \dots, \tilde{\omega} - x\}.$$

Mortality rate model: in accordance with (Zeddouk and Devolder, 2020), we choose as default values for the mortality model

$$\mu_0 = 0.002600332, \omega = 0.1385505877, \varepsilon = 0.0005196101, \bar{A} = 0.002219915 \text{ and } \bar{B} = 0.100627916, \quad (50)$$

for a Belgian individual born in 1970 and initially aged 50 years old. The order of magnitude of those values is in adequacy with (Luciano and Vigna, 2008) and we consider the following ranges of values :

$$\omega \in [0 \ 25\%] \text{ and } \varepsilon \in [0 \ 1\%]. \quad (51)$$

Correlation coefficient: based on (Dacorogna and Cadena, 2015), which studies the dependence between mortality and market risks, we consider a default value $\rho^{r,\mu} = 20\%$ and the range

$$\rho^{r,\mu} \in [-100\% \ 100\%]. \quad (52)$$

3.3.2 Sensitivity analysis of the correlation price

With established default values and ranges for each of the six parameters, we proceed to evaluate their respective impacts on the correlation price. This assessment aims to discern the extent to which the inclusion of correlation influences the pricing of the zero-coupon survival bond. The four figures below illustrate the evolution of the correlation price as a function of the correlation coefficient, the maturity $T = s - t$, the diffusion coefficients and the speeds of mean reversion.

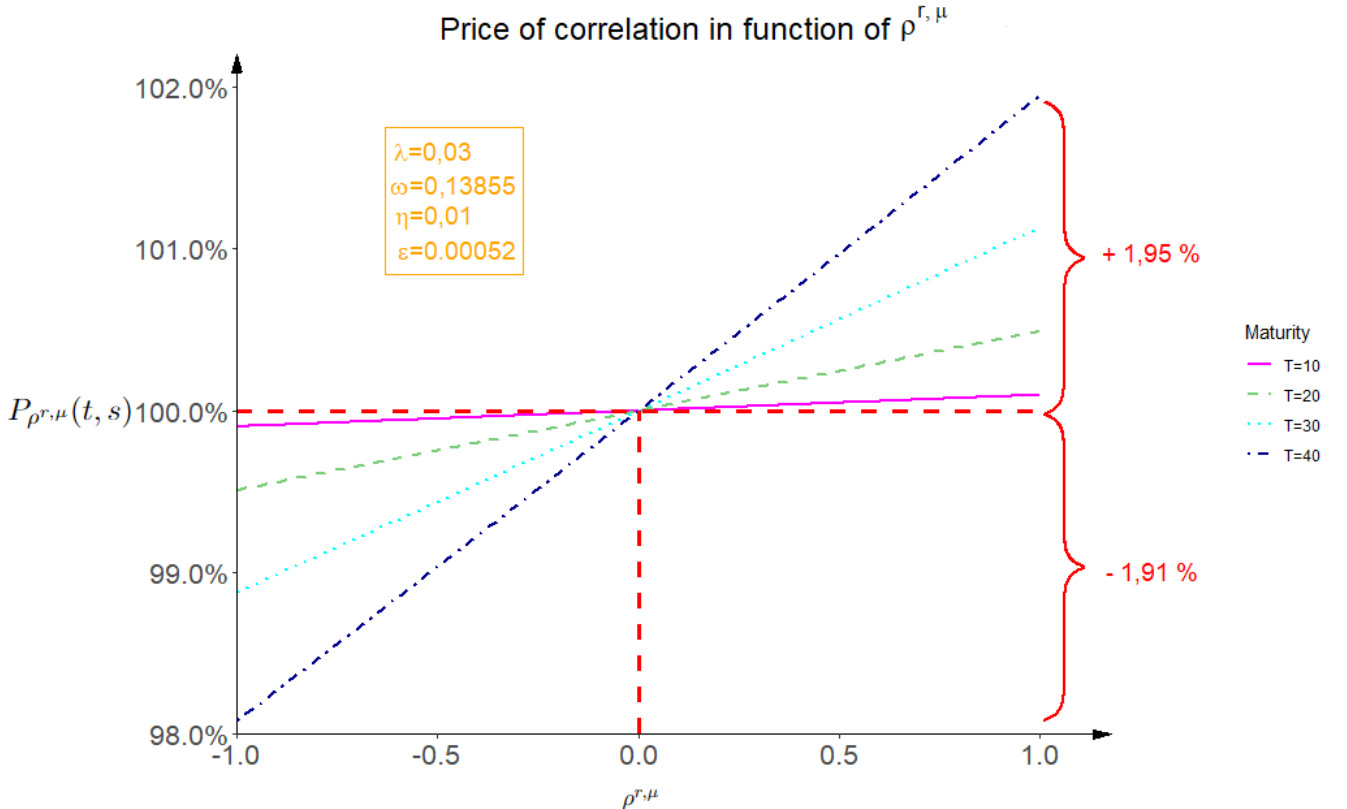


Figure 1: Price of correlation in function of the correlation coefficient for maturities 10, 20, 30 and 40.

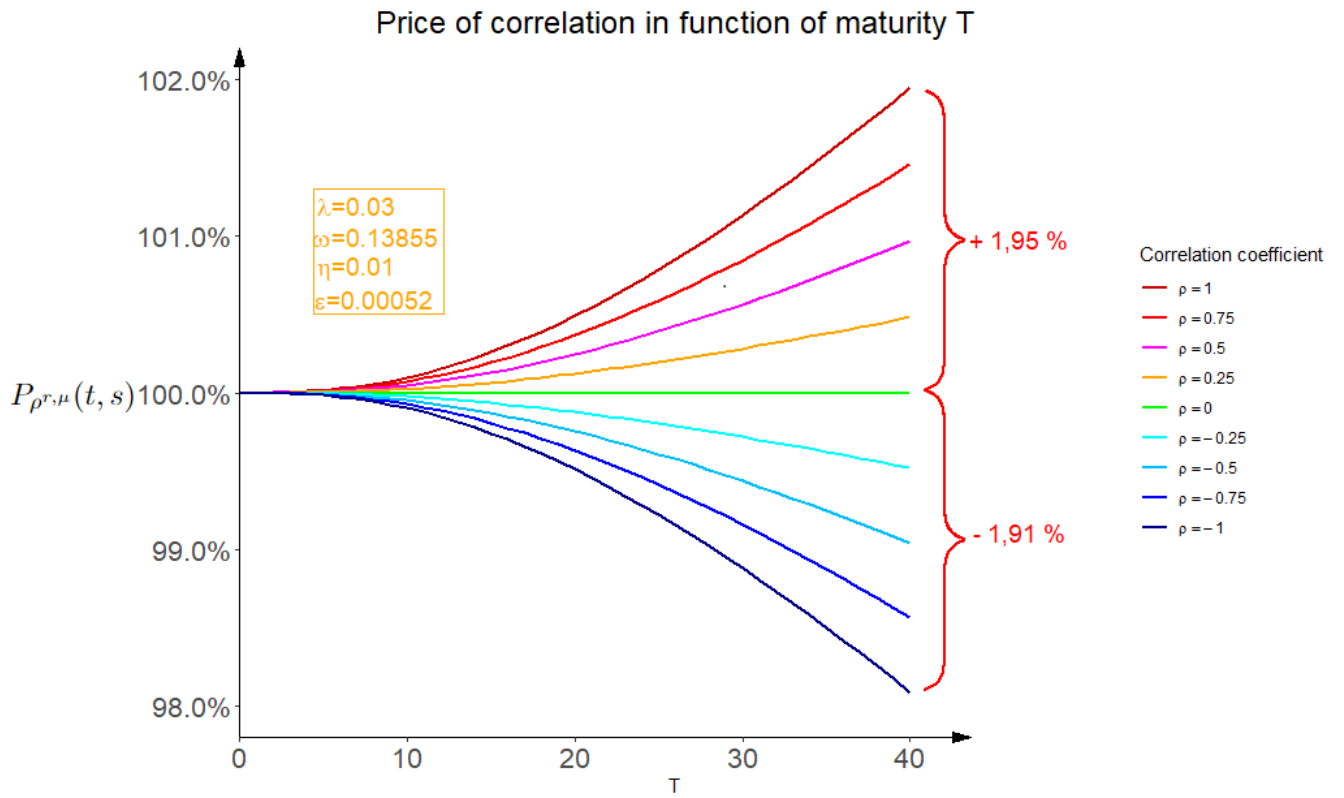


Figure 2: Price of correlation in function of maturity for different values of the correlation coefficient.

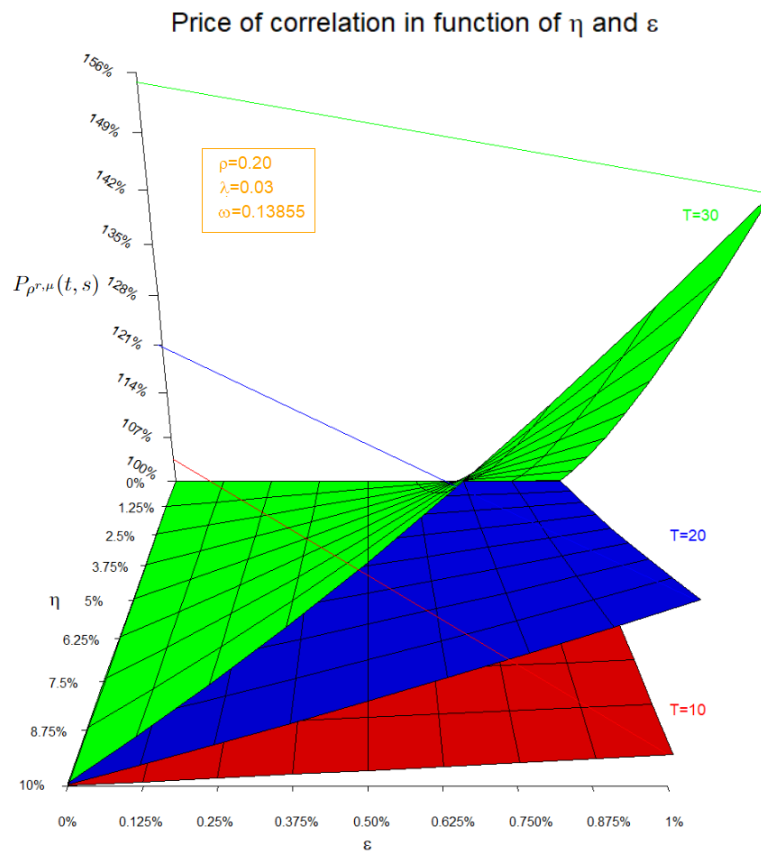


Figure 3: Price of correlation in function of η and ε for maturities 10, 20 and 30.

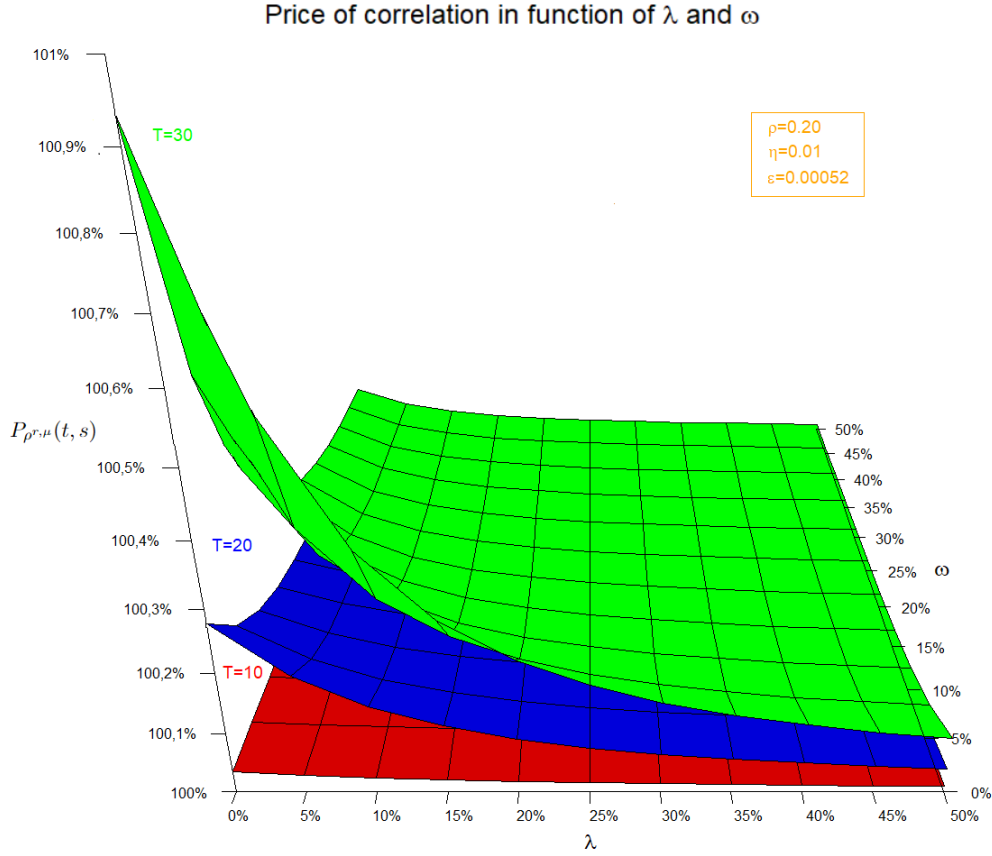


Figure 4: Price of correlation in function of λ and ω for maturities 10, 20 and 30.

A careful inspection of the four preceding figures leads to the following observations which are in line with what has been discussed above.

Figures 1 and 2

- On figure 1, we observe the positive exponential behaviour of the correlation price in function of the correlation coefficient. However, due to the very small value of the exponential constant for the default values of $\lambda, \omega, \eta, \varepsilon$ and T , the exponential behaviour is hardly perceivable. The graph also confirms that when $\rho^{r,\mu} < 0$, the correlation price is strictly below 1 (negative correlation scenario), when $\rho^{r,\mu} = 0$, it is equal to 1 (zero correlation scenario) and when $\rho^{r,\mu} > 0$, it is strictly above 1 (positive correlation scenario).
- On figure 2, we observe the positive exponential behaviour of the correlation price with maturity when $\rho^{r,\mu} > 0$ and the negative exponential behaviour of the correlation price with maturity when $\rho^{r,\mu} < 0$.
- We observe on figures 1 and 2 that for the default values of λ, ω, η and ε , the maximum upward impact is when $\rho^{r,\mu}$ and T are maximum, namely when $\rho^{r,\mu} = 1$ and $T = 40$. The corresponding value of the price of correlation is 101.95%, which corresponds to an increase of 1.95% compared to the price without correlation. We also denote that the maximum downward impact is when $\rho^{r,\mu}$ is negative and maximum in absolute value and T is maximum, namely when $\rho^{r,\mu} = -1$ and $T = 40$. The corresponding value of the price of correlation is 98.09%, which corresponds to a decrease of 1.91% compared to the price without effect of correlation. This confirms the asymmetry between the maximum upward impact and the maximum downward impact.

Figures 3 and 4

- On figure 3, we observe the positive exponential behaviour of the correlation price in function of the diffusion coefficients when $\rho^{r,\mu} > 0$. It is very clear for $T = 30$. For $T = 10$, due to the very small value of the exponential constant, the exponential behaviour is not really perceivable anymore.

- On figure 4, we observe the exponential behaviour of the correlation price with the inverse of the speeds of mean reversion when $\rho^{r,\mu} > 0$.
- We observe on figure 3 that the maximum upward impact for the chosen values of λ, ω and $\rho^{r,\mu}$, occurs when the diffusion coefficients and T are maximum, namely when $\eta = 10\%$, $\varepsilon = 1\%$ and $T = 30$. The corresponding value of the price of correlation is 155%. This means an increase of 55% compared to the price without correlation. This worst case scenario in terms of diffusion coefficients and maturity clearly underlines the absolute necessity of taking correlation into account when the diffusion coefficients are located in the highest part of their range and when maturity is high.
- In a similar way, we note on figure 4 that the maximum upward impact for the chosen values of η, ε and $\rho^{r,\mu}$, occurs when the speeds of mean reversion are minimum and T is maximum, namely when $\lambda = 0\%$, $\omega = 0\%$ and $T = 30$. The corresponding value of the price of correlation is 100.93%. This means an increase of 0.93% compared to the price without effect of correlation. The analysis highlights a significant disparity in the influence of correlation arising from adjustments in the speeds of mean reversion compared to those resulting from changes in the diffusion coefficients. Note that setting speeds of mean reversion to zero is purely theoretical. Such a scenario would fundamentally alter the nature of the models, rendering them incompatible with the Hull and White framework as the drift terms would cease to exist. This theoretical stance is especially problematic, notably for the mortality model, which operates on the foundational principle that mortality rates evolve stochastically around a Gompertz moving target, representing the average increasing mortality rate with age.

4 Mortality density

Let us compute the price of the mortality density, the second of our two building blocks, whose expression is recalled below

$$D_{r,\mu}(t, u) := 1_{\{\tau(x) > t\}} \mathbb{E}_{\tilde{\mathbb{Q}}} \left[\exp \left(- \int_t^u r_v dv \right) \cdot \exp \left(- \int_t^u \mu_{x+v} dv \right) \cdot \mu_{x+u} | \mathcal{F}_t \right]. \quad (53)$$

Let us switch from the measure $\tilde{\mathbb{Q}}$, associated to the numeraire

$$M(t) = \exp \left(\int_0^t r_v dv \right), \quad (54)$$

to a measure $\tilde{\mathbb{Q}}^{u,\mu} \sim \tilde{\mathbb{Q}}$ defined by the following Radon–Nikodym derivative of $\tilde{\mathbb{Q}}^{u,\mu}$ with respect to $\tilde{\mathbb{Q}}$,

$$\frac{d\tilde{\mathbb{Q}}^{u,\mu}}{d\tilde{\mathbb{Q}}} = \frac{\exp \left(- \int_0^u (r_v + \mu_{x+v}) dv \right)}{\mathbb{E}_{\tilde{\mathbb{Q}}} \left[\exp \left(- \int_0^u (r_v + \mu_{x+v}) dv \right) \right]}, \quad (55)$$

which given formulas (36) to (39) becomes

$$\begin{aligned} \frac{d\tilde{\mathbb{Q}}^{u,\mu}}{d\tilde{\mathbb{Q}}} = & \exp \left(- \int_0^u [\eta B(\lambda, v, s) + \varepsilon \rho^{r,\mu} B(\omega, v, s)] d\tilde{W}_v^r - \int_0^u \left[\varepsilon \sqrt{1 - (\rho^{r,\mu})^2} B(\omega, v, s) \right] d\tilde{W}_v^\mu \right. \\ & \left. - \frac{1}{2} \int_0^u [\eta B(\lambda, v, s) + \varepsilon \rho^{r,\mu} B(\omega, v, s)]^2 dv - \frac{1}{2} \int_0^u \left[\varepsilon \sqrt{1 - (\rho^{r,\mu})^2} B(\omega, v, s) \right]^2 dv \right). \end{aligned} \quad (56)$$

The change of measure allows to rewrite (53) as

$$D_{r,\mu}(t, u) = 1_{\{\tau(x) > t\}} P_{r,\mu}(t, u) \mathbb{E}_{\tilde{\mathbb{Q}}^{u,\mu}} [\mu_{x+u} | \mathcal{F}_t]. \quad (57)$$

The form of (56) implies, according to the multidimensional Girsanov theorem, that the Brownian motions $\widehat{W}_t^r$ and $\widehat{W}_t^\mu$, defined as

$$\begin{cases} \widehat{W}_t^r = \tilde{W}_t^r + \int_0^t [\eta B(\lambda, v, s) + \varepsilon \rho^{r,\mu} B(\omega, v, s)] dv, \\ \widehat{W}_t^\mu = \tilde{W}_t^\mu + \varepsilon \sqrt{1 - (\rho^{r,\mu})^2} \int_0^t B(\omega, v, s) dv, \end{cases} \quad (58)$$

are standard Brownian motions under $\tilde{\mathbb{Q}}^{u,\mu}$.

Inserting (58) in (34b) yields the following expression of μ_{x+u} under measure $\tilde{\mathbb{Q}}^{u,\mu}$:

$$\begin{aligned}\mu_{x+u} = & \mu_{x+t}e^{-\omega(u-t)} + \omega e^{-\omega u} \int_t^u \bar{\mu}_{x+v} e^{\omega v} dv + \varepsilon \rho^{r,\mu} e^{-\omega u} \int_t^u e^{\omega v} d\widehat{W}_v^r - \frac{\varepsilon \eta}{\lambda} \rho^{r,\mu} B(\omega, t, u) \\ & + \frac{\varepsilon \eta}{\lambda} \rho^{r,\mu} B(\lambda + \omega, t, u) - \frac{\varepsilon^2}{2} B^2(\omega, t, u) + \varepsilon \sqrt{1 - (\rho^{r,\mu})^2} e^{-\omega u} \int_t^u e^{\omega v} d\widehat{W}_v^\mu.\end{aligned}\quad (59)$$

Hence, in virtue of (57) and (59), we can conclude that the value of the mortality density is given by the following proposition. For more details on the calculations summarised above, please refer to the [online Appendix C](#).

Proposition II

Considering the Hull and White model² (16) with explicit expressions of the moving targets (18) and (27), the value at time t of the mortality density of maturity time u , for an individual initially aged x at time 0, is given by

$$D_{r,\mu}(t, u) = 1_{\{\tau(x) > t\}} \cdot P_{r,\mu}(t, u) \cdot M_{r,\mu}(t, u), \quad (60)$$

where

- $P_{r,\mu}(t, u)$ is the price of a zero-coupon survival bond under the Hull and White² model given by expression (41).
- $M_{r,\mu}(t, u)$ is the mortality factor given by

$$\begin{aligned}M_{r,\mu}(t, u) := & \left[\mu_{x+t} e^{-\omega(u-t)} + \frac{\omega \bar{A}}{(\bar{B} + \omega)} e^{\bar{B}u} - \frac{\omega \bar{A} e^{(\bar{B} + \omega)t}}{(\bar{B} + \omega)} e^{-\omega u} - \frac{\varepsilon^2}{2} B^2(\omega, t, u) \right. \\ & \left. + \rho^{r,\mu} \frac{\varepsilon \eta}{\lambda} [B(\lambda + \omega, t, u) - B(\omega, t, u)] \right],\end{aligned}\quad (61)$$

where the function B is defined in (24).

4.1 Interpretation of the mortality factor

The mortality density value (60) is the product of the zero-coupon price $P_{r,\mu}(t, u)$, which accounts for the survival of the insured between time t and time u plus the fact that the insured receives 1 at time u , and the mortality factor $M_{r,\mu}(t, u)$, which is linked with the death of the insured at time u . The mortality factor can be broken down in three terms denoted $I(t, u)$, $II(t, u)$ and $III(t, u)$ as following:

$$\begin{aligned}M_{r,\mu}(t, u) = & \underbrace{\mu_{x+t} e^{-\omega(u-t)} + \frac{\omega \bar{A}}{(\bar{B} + \omega)} e^{\bar{B}u} - \frac{\omega \bar{A} e^{(\bar{B} + \omega)t}}{(\bar{B} + \omega)} e^{-\omega u}}_{I(t, u)} - \overbrace{\frac{\varepsilon^2}{2} B^2(\omega, t, u)}^{II(t, u)} \\ & + \underbrace{\rho^{r,\mu} \frac{\varepsilon \eta}{\lambda} [B(\lambda + \omega, t, u) - B(\omega, t, u)]}_{III(t, u)}.\end{aligned}\quad (62)$$

- The term $I(t, u)$ is nothing else than the conditional expectation of the mortality intensity stochastic process, that is $\mathbb{E}_{\tilde{\mathbb{Q}}}[\mu_{x+u} | \mathcal{F}_t]$. By analyzing (53), we deduce that $M_{r,\mu}(t, u) = I(t, u)$ corresponds to the case where there is independence between mortality intensity and interest rates, since then the expectation appearing in (53) can be decomposed in the product of two expectations. This suggests that $II(t, u)$ and $III(t, u)$ are terms resulting from the correlation between $\exp(-\int_t^u (\mu_{x+v} + r_v) dv)$ and μ_{x+u} .

- The term $II(t, u)$ results from the correlation between the factors $\exp(-\int_t^u \mu_{x+v} dv)$ and μ_{x+u} , both driven by the Brownian motion W_t^μ .

- The term $III(t, u)$ results from the correlation between the factors $\exp(-\int_t^u r_v dv)$, driven by the Brownian motion W_t^r , and μ_{x+u} , driven by the Brownian motion W_t^μ . It can easily be proven that this term is negative in case of positive correlation and positive in case of negative correlation.

4.2 Study of the impact of correlation

Maintaining the introduced notations, in the absence of correlation, i.e., when $\rho^{r,\mu} = 0$, the value of the mortality density is expressed as follows:

$$D_{r,\mu}(t, u) = P_r(t, u) \cdot P_\mu(t, u) \cdot (I(t, u) + II(t, u)), \quad (63)$$

whereas in presence of correlation it is given by

$$D_{r,\mu}(t, u) = P_r(t, u) \cdot P_\mu(t, u) \cdot P_{\rho^{r,\mu}}(t, u) \cdot (I(t, u) + II(t, u) + \underbrace{\rho^{r,\mu} \frac{\varepsilon \eta}{\lambda} [B(\lambda + \omega, t, u) - B(\omega, t, u)]}_{III(t, u)}). \quad (64)$$

Consequently, the terms $P_{\rho^{r,\mu}}(t, u)$ and $III(t, u)$ emerge as crucial factors when considering correlation. Their respective impacts on the value of the mortality density are antagonistic. In the scenario of a positive correlation, the correlation price $P_{\rho^{r,\mu}}(t, u)$ influences the value of $D_{r,\mu}(t, u)$ upwards, while term $III(t, u)$ influences the value of $D_{r,\mu}(t, u)$ downwards. Conversely, in the case of a negative correlation, their effects are reversed. It is not possible to determine a priori which effect predominates; it depends on the parameter values. To explore this question, we will scrutinise the behavior of the ratio between the value of the mortality density with correlation (64) and its value without correlation (63). This ratio, referred to as the « correlation ratio of the mortality density » and denoted $CRD(t, u, \rho^{r,\mu})$, is defined as follows:

$$CRD(t, u, \rho^{r,\mu}) := \underbrace{P_{\rho^{r,\mu}}(t, u)}_{CRD_P(t, u, \rho^{r,\mu})} \cdot \frac{\overbrace{(I(t, u) + II(t, u) + III(t, u))}^{CRD_M(t, u, \rho^{r,\mu})}}{(I(t, u) + II(t, u))}. \quad (65)$$

Decomposing the ratio $CRD(t, u, \rho^{r,\mu})$ into the product of the zero-coupon survival price ratio $CRD_P(t, u, \rho^{r,\mu})$, which is nothing else than $P_{\rho^{r,\mu}}(t, u)$, and the mortality effect ratio $CRD_M(t, u, \rho^{r,\mu})$, provides a clearer understanding of the interplay between the upward and downward dynamics. The figures below illustrate, on the left for the default parameter values and on the right in the case of a shock on the diffusion coefficients, both correlation ratios $CRD_P(t, u, \rho^{r,\mu})$ and $CRD_M(t, u, \rho^{r,\mu})$, alongside the overall ratio $CRD(t, u, \rho^{r,\mu})$, plotted as functions of maturity $T = s - t$.

Decomposition of the correlation ratio of the mortality density

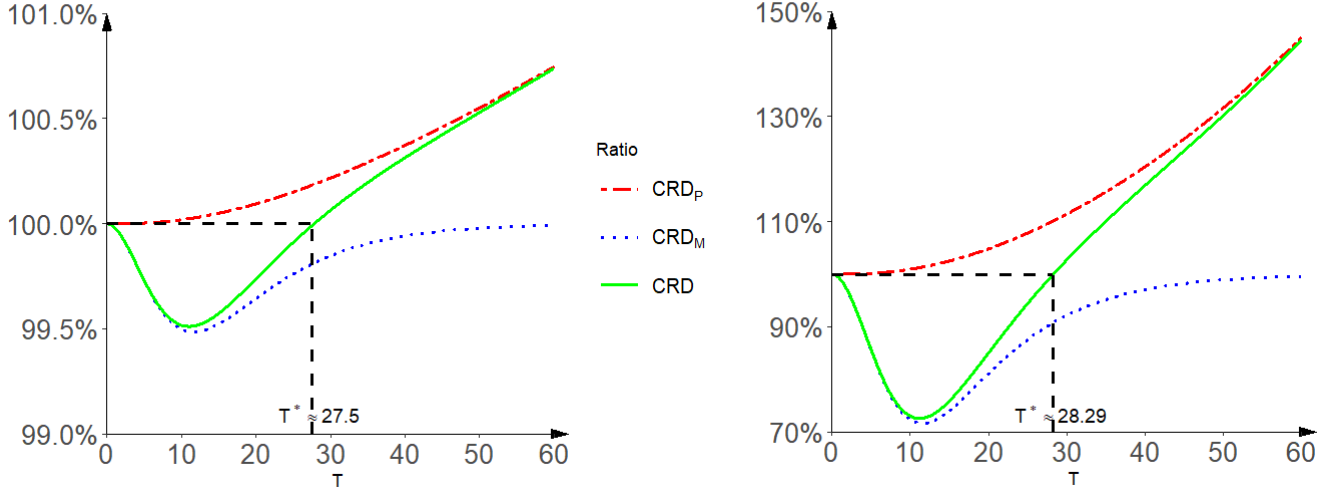


Figure 5: Decomposition of the correlation ratio of the mortality density for the default values of the parameters at the left and for default values at the right except $\eta = 0.05$ and $\varepsilon = 0.005$. Please note that the scales of the two graphs are different.

The examination of these figures yields four key observations. Firstly, it illustrates the « struggle » between the downward effect attributed to $CRD_M(t, u, \rho^{r, \mu})$ and the upward effect of $CRD_P(t, u, \rho^{r, \mu})$. It is crucial to recognise that the influence of correlation on the price of the zero-coupon survival bond was unidirectional, as the sign of the correlation coefficient dictated the direction of this impact. While the maturity value evidently influenced the magnitude of the impact, it didn't affect its direction. Conversely, regarding the value of the mortality density, there exists a specific maturity denoted as T^* , where the upward and downward effects compensate each other. Consequently, the direction of the correlation's impact hinges on whether T is less than or greater than T^* . When $T < T^*$, the downward impact induced by the mortality effect outweighs the upward effect associated with the correlation price. Conversely, when $T > T^*$, the exponential behaviour of the correlation price prevails, all the more so since the $CDR_M(t, u, \rho^{r, \mu})$ converges towards 1. This « change in direction of the impact of correlation » on the mortality density constitutes the foundation of an interesting feature inherent to a term insurance contract, which will be elaborated upon in the next section.

Secondly, we see that in the figure on the right, the scale has increased considerably. For a fixed value of correlation coefficient (here $\rho^{r, \mu} = 20\%$), increasing the values of η and ε reinforces the upward effect associated to the correlation price and the downward effect associated to the mortality factor. As observed, this scaling is anything but negligible.

Thirdly, it enables us to discern that despite the substantial differences in scale between the figures on the left and the right, the maturities T^* at which the two effects offset each other are not as disparate as one might anticipate.

Finally, it is important to underline the fact that the maturities T^* are relatively high. This means that there is a large interval of maturities ($[0, T^*]$) wherein the downward impact prevails.

5 Impact of correlation on the best estimate of typical life insurance contracts

By appropriately combining the two building blocks, we can encompass a broad range of insurance contracts. Specifically, we will consider a pure endowment, a term/life annuity, a term/whole life insurance and a term/whole life mixed endowment. The objective is to examine the change in the best estimate of these contracts resulting from the inclusion of correlation. In order to obtain these best estimates as legally imposed by the European Insurance and Occupational Pensions Authority (EIOPA), we will compute mathematical expectations under the measure $\tilde{\mathbb{Q}} \odot \tilde{\mathbb{P}}$ (see for instance (Artzner and al., 2023)).

5.1 Life pure endowment

Let us consider a pure endowment contract of maturity time s and with guarantee 1. In such a contract, the insurer pays nothing to the insured if the insured dies before time s and pays 1 if the insured lives at time s . Let us note that the best estimate at time 0 of a pure endowment of maturity time s follows immediately from our first building block and can be decomposed as

$$BE^{PE}(0, s, \rho^{r,\mu}) = P_r(0, s) \cdot P_\mu(0, s) \cdot P_{\rho^{r,\mu}}(0, s) = P^M(0, s) \cdot P_\mu(0, s) \cdot P_{\rho^{r,\mu}}(0, s), \quad (66)$$

whereas without correlation, it is given by

$$BE^{PE}(0, s, 0) = P_r(0, s) \cdot P_\mu(0, s) \cdot 1 = P^M(0, s) \cdot P_\mu(0, s). \quad (67)$$

To assess the relative impact of correlation on the best estimate, we define the ratio of the value with correlation (66) and without correlation (67) at time 0, named « correlation ratio of the pure endowment » and defined as

$$CR^{PE}(0, s, \rho^{r,\mu}) := P_{\rho^{r,\mu}}(0, s) \text{ when } P^M(0, s) \text{ and } P_\mu(0, s) \neq 0. \quad (68)$$

Firstly, it is important to note that the survival probability $P_\mu(0, s)$ converges to 0 at the ultimate age (corresponding to a maturity of 60, given the initial age is 50 and the ultimate age $\tilde{\omega}$ is 110). Consequently, the correlation ratio (68) is undefined for maturities exceeding 60. Prior to this threshold, the correlation ratio of the pure endowment coincides with the price of correlation. Based on the comprehensive investigation of the correlation price conducted above, it is established that in the event of a positive correlation, the correlation ratio (68) will exceed 1 since $P_{\rho^{r,\mu}}(0, s)$ is correspondingly higher than 1. Conversely, in the case of negative correlation, the ratio will be lower than 1. This implies that when $\rho^{r,\mu} > 0$, ignoring correlation leads to an underestimation of the best estimate of a pure endowment life insurance contract. The extent of underestimation depends on the parameter values and may, in certain scenarios, become notably pronounced, thus warranting increased attention from life insurers. This observation is confirmed in the figure below, which illustrates, for different correlation coefficient values, at the left hand side the value of the best estimate of the pure endowment (66), and at the right hand side the value of the correlation ratio (68).

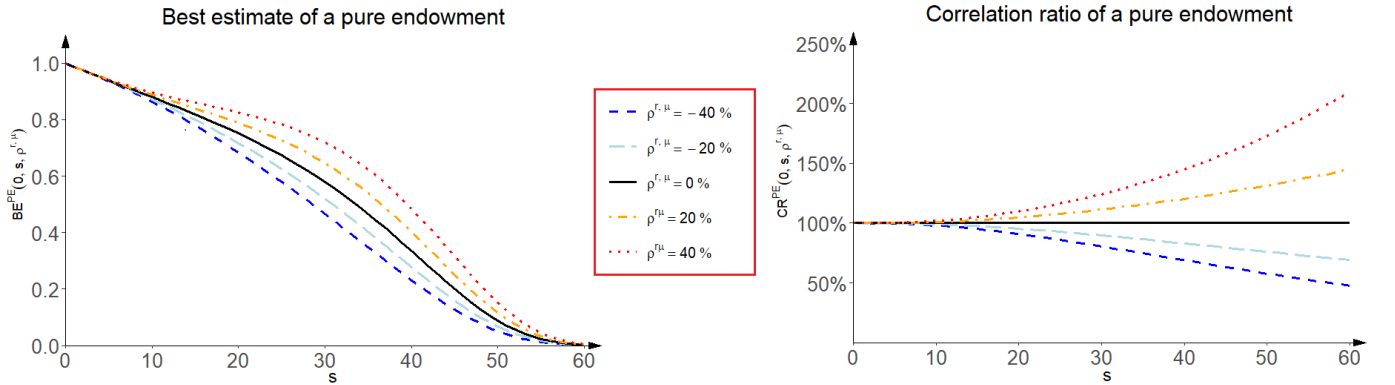


Figure 6: Best estimates and correlation ratios of a pure endowment for different values of the correlation coefficient ($\eta = 0.05$, $\varepsilon = 0.005$, $\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$ and $\bar{B} = 0.100627916$).

5.2 Term/life annuity

Let us consider, at time 0, a term/life annual annuity in arrears (paid at the end of each year) of maturity time s and payment of 1. In such a contract, the insurer pays the annuitant 1 at the end of each year until maturity time s . In case of a fixed term annuity, the maturity time s is an integer with $s < \tilde{\omega} - x$, and in case of a life annuity, $s = \tilde{\omega} - x$, since the annuitant is then paid as long as he/she is alive.

The annuity contract can be seen as a collection of pure endowment contracts. Let us note that the best estimate at time 0 of a such a contract with maturity time s is therefore equal to

$$BE^{TLA}(0, s, \rho^{r,\mu}) = \sum_{i=1}^s P_r(0, i) \cdot P_\mu(0, i) \cdot P_{\rho^{r,\mu}}(0, i) = \sum_{i=1}^s P^M(0, i) \cdot P_\mu(0, i) \cdot P_{\rho^{r,\mu}}(0, i). \quad (69)$$

Without the effect of correlation, it is given by

$$BE^{TLA}(0, s, 0) = \sum_{i=1}^s P_r(0, i) \cdot P_\mu(0, i) = \sum_{i=1}^s P^M(0, i) \cdot P_\mu(0, i). \quad (70)$$

The ratio of the best estimate with correlation (69) and without correlation (70) at time 0, named « correlation ratio of the term/life annuity » is defined as

$$CR^{TLA}(0, s, \rho^{r,\mu}) := \frac{\sum_{i=1}^s P^M(0, i) \cdot P_\mu(0, i) \cdot P_{\rho^{r,\mu}}(0, i)}{\sum_{i=1}^s P^M(0, i) \cdot P_\mu(0, i)}. \quad (71)$$

The figure below shows in function of maturity and for different values of the correlation coefficient, the best estimate (69) and the correlation ratio (71) of a term/life annuity.

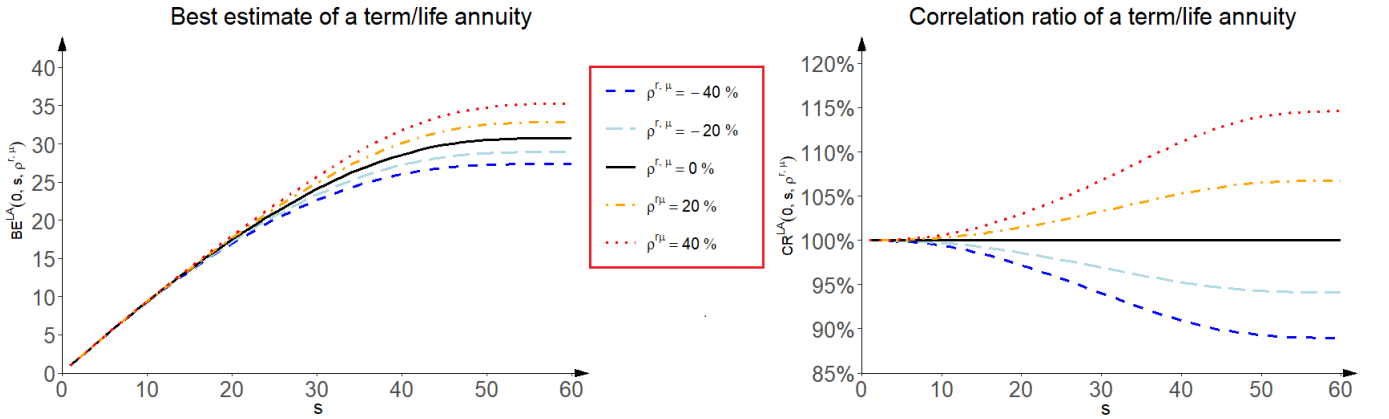


Figure 7: Best estimate and correlation ratio of a term/life annuity for different values of the correlation coefficient ($\eta = 0.05$, $\varepsilon = 0.005$, $\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$ and $\bar{B} = 0.100627916$).

Based on figures 6 and 7, we can make the following observations and comments.

- Rationally, it is discernible from the left part of figure 7 that the best estimate of the term/life annuity is strictly increasing with maturity. This observed increase can be attributed to the incremental consideration of additional positive terms within the summation (69) as the maturity period extends. Nevertheless, we observe a notable stagnation for very high maturities. This stagnation phenomenon is attributable to the fact that, as can be observed on figure 6, the latest terms of the summation (69), namely the best estimates of zero-coupon survival bonds for very high maturities, are close to zero due to the survival probabilities which are extremely low for those very high maturities.
- Given that a term/life annuity essentially represents a summation of zero-coupon survival bond prices, the influence of correlation aligns similarly to that observed for individual zero-coupon survival bonds. Specifically, positive correlation exerts an upward effect on the best estimate of the term/life annuity, while negative correlation produces an inverse impact. An examination of expression (71) reveals that under positive correlation, the correlation ratio exceeds 1. This is due to the fact that each term in the numerator's summation surpasses its corresponding term in the denominator. This disparity arises

from the observation that $P_{\rho^{r,\mu}}(0, i)$ consistently exceeds 1 for all values of i . This is confirmed when we look at the right graph of figure 7 for the cases where $\rho^{r,\mu} > 0$. Furthermore, it is noteworthy to remember that in the case of positive correlation, the impact of correlation increases with maturity. This elucidates why the correlation ratio of a term/life annuity strictly rises with maturity. However, akin to the previous point, the correlation ratio converges for higher maturities.

Let us compare, for a given maturity and under positive correlation, the extent of correlation's impact on the best estimate of pure endowment and of a term/life annuity.

- In absolute terms, the magnitude of the impact of correlation (the difference in the best estimate with and without correlation) is greater for a term/life annuity compared to a pure endowment with the same maturity. This is because the life annuity accumulates strictly positive impacts for each year until the ultimate year (the year corresponding to the ultimate age).
- In relative terms, the magnitude of the impact of correlation (the ratio of the best estimate with and without correlation) is smaller for a term/life annuity compared to a pure endowment with same maturity. For a term/life annuity, all terms corresponding to maturities strictly lower than the final maturity contribute to reducing the value of the correlation ratio, in contrast to the pure endowment, which only captures the correlation impact for the final maturity (which typically has the highest correlation impact since the impact increases with maturity). This is confirmed when comparing the right graphs of figures 6 and 7 for a given maturity.

To further investigate the absolute and relative impact of the inclusion of correlation on the value of a life annuity, we now examine the following five stress scenarios concerning the values of the parameters $\rho^{r,\mu}$, η and ε . It is worth noting that we concentrate solely on these three parameters since, as discussed in section 3.2.2, they have - together with the maturity - the most substantial impact.

	$\rho^{r,\mu}$	η	ε	$BE^{TLA}(0, 60, 0)$	$BE^{TLA}(0, 60, \rho^{r,\mu})$	$CR^{TLA}(0, 60, \rho^{r,\mu})$
Scen. A	20%	1%	0.05196101%	30.54948	30.58924	100.130%
Scen. B	40%	1%	0.05196101%	30.54948	30.62911	100.261%
Scen. C	20%	5%	0.05196101%	30.54948	30.74941	100.654%
Scen. D	20%	1%	0.5%	30.76951	31.16260	101.278%
Scen. E	40%	5%	0.5%	30.76951	35.26051	114.596%

Table 1: Best estimate with and without correlation and corresponding correlation ratio of a life annuity ($\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$, $\bar{B} = 0.100627916$ and $T = 60$).

Firstly, it is important to observe that the value of the correlation ratio is positive across all scenarios, as each scenario involves a positive correlation. However, what is particularly notable is the exceptionally high value of the ratio in scenario E. Neglecting correlation in such a scenario results in a substantial undervaluation of the best estimate of the life annuity.

5.3 Term/whole life insurance

Let us consider, at time 0, a life insurance contract of maturity $T = s$ and with guarantee 1. In such a contract, the insurer pays 1 to the insured's beneficiary at the time of the insured's death if the death occurs before time s . In case of a term insurance, the maturity time s is an integer with $s < \tilde{\omega} - x$, and in case of a whole life insurance, $s = \tilde{\omega} - x$. Let us note the best estimate at time 0 of such a contract of maturity s as

$$BE^{TLI}(0, s, \rho^{r,\mu}) = \int_0^s D_{r,\mu}(0, u) du = \int_0^s P_r(0, u) \cdot P_\mu(0, u) \cdot P_{\rho^{r,\mu}}(0, u) \cdot (I + II + III)(0, u) du. \quad (72)$$

Without the effect of correlation, it is given by

$$BE^{TLI}(0, s, 0) = \int_0^s P_r(0, u) \cdot P_\mu(0, u) \cdot (I + II)(0, u) du. \quad (73)$$

Note that the integrals (72) and (73) cannot be solved **explicitly** and need to be computed numerically.

The ratio of the best estimate with correlation (72) and without correlation (73) at time 0, named « correlation ratio of the term/life insurance » is defined as

$$CR^{TLI}(0, s, \rho^{r,\mu}) = \frac{\int_0^s P^M(0, u) \cdot P_\mu(0, u) \cdot P_{\rho^{r,\mu}}(0, u) \cdot (I + II + III)(0, u) du}{\int_0^s P^M(0, u) \cdot P_\mu(0, u) \cdot (I + II)(0, u) du}. \quad (74)$$

The figure below shows in function of maturity and for different values of the correlation coefficient, the mortality density, the best estimate (72) and the correlation ratio (74) of a term/life insurance.

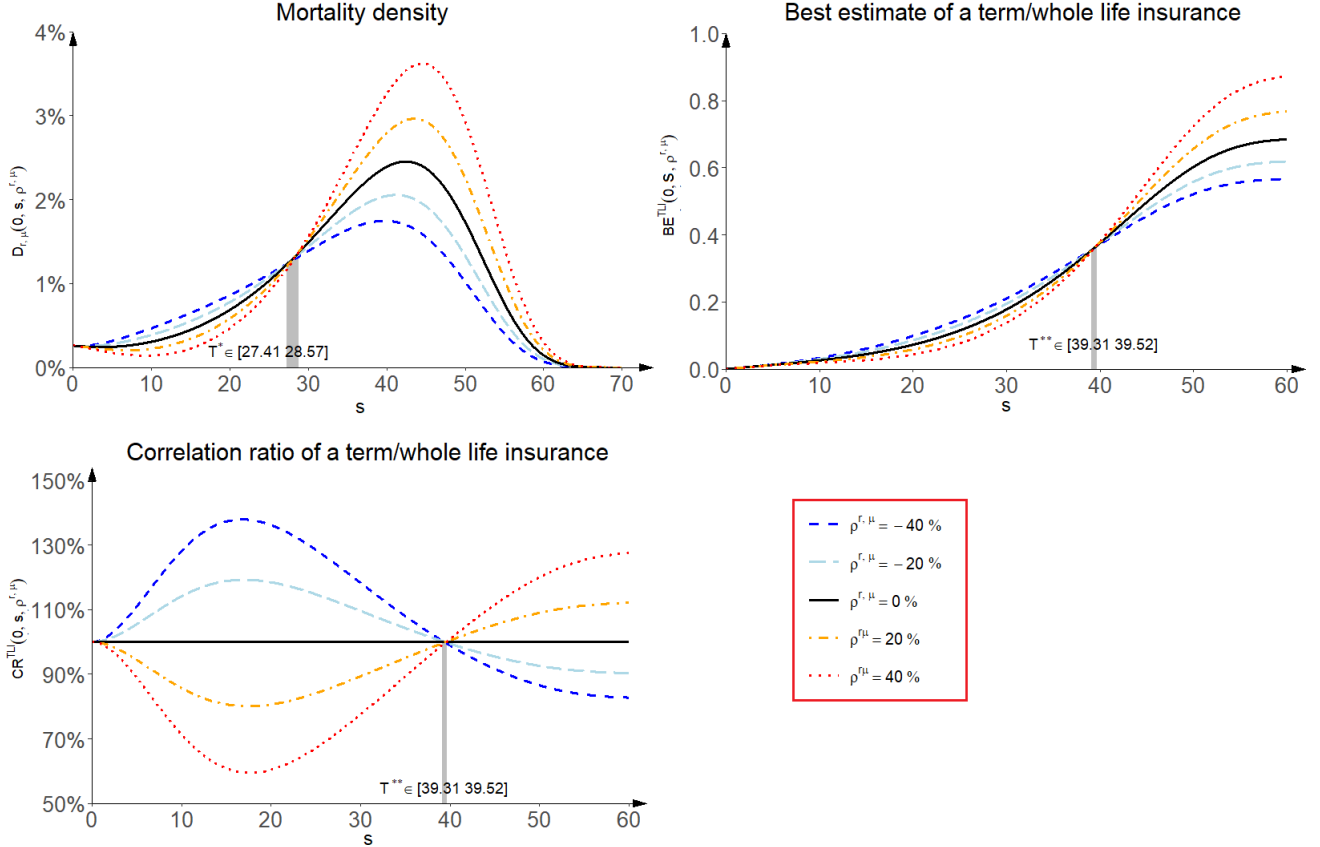


Figure 8: The mortality density value, the best estimate and the correlation ratio of a term/life insurance for different values of the correlation coefficient ($\eta = 0.05$, $\varepsilon = 0.005$, $\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$ and $\bar{B} = 0.100627916$).

The analysis of figure 8 leads to the following observations and comments.

- Let us start by recalling that in figure 5, we observed that for the selected parameter values and under positive correlation ($\rho^{r,\mu} = 0.2$), the overall impact of correlation on the mortality density value is downward when $T < T^* = 28.4$, upward when $T > T^*$, and nonexistent for $T = T^*$. It is interesting to note that for an insured individual aged 50 at $t = 0$, a maturity of 28.4 is already relatively high. On the upper left graph of figure 8, we observe that the value of T^* is not changing a lot when changing the value of the correlation coefficient. Indeed, when the parameter $\rho^{r,\mu}$ varies within the range of $[-40\% \ 40\%]$, the corresponding T^* values fluctuate within the interval of $[27.41 \ 28.57]$.
- The best estimate at time 0 of a term/life insurance with a maturity s is obtained by integrating the mortality density value between 0 and s . Essentially, each upper right curve represents the cumulative area beneath the respective upper left curve. Given that the cumulative area is strictly increasing, it follows logically that the best estimate of the term/life insurance also exhibits a strictly increasing trend. However, similar to the observation made for the term/life annuity contract, for very high

maturities, the best estimate of the term/life insurance converges. This occurs because the mortality density for such maturities gradually approaches zero.

- In the upper right graph, it is evident that regardless of the value of the correlation coefficient, there exists a specific maturity denoted as T^{**} at which there is no impact of correlation. Therefore, unlike the pure endowment and term/life annuity contracts, which are inevitably affected by correlation regardless of the maturity value, the term/life insurance exhibits a maturity for which this impact is nullified. For this specific maturity T^{**} , the impact of the correlation on the term/life insurance is internally hedged². This notion is supported by the lower left graph, which illustrates the correlation ratio's value as a function of maturity. We observe a confirmation of perfect hedging for $T = T^{**}$, as the correlation ratio equals 1.
- Examining the upper right graph, three observations can be made regarding the value of T^{**} at which the correlation impact of a term/life insurance is nullified. Firstly, it is notable that the value of T^{**} is shifted to the right compared to the value of T^* . This observation is logical since T^* represents the maturity beyond T^{**} at which the areas between the mortality density curve (for a correlation coefficient $\rho^{r,\mu} \neq 0$) and the « independent » mortality density curve (the one with $\rho^{r,\mu} = 0$) from 0 to T^* and from T^* to T^{**} are equal. Secondly, we observe that the value of T^{**} remains relatively stable when changing the value of the correlation coefficient: when $\rho^{r,\mu}$ varies within the range of $[-40\% 40\%]$, T^{**} ranges from 39.31 to 39.52. Thirdly, it is noteworthy that for the chosen parameter values, the value of T^{**} is notably high. This indicates that for a considerable range of maturities, the downward impact prevails in the case of positive correlation.

The following table presents the values of the best estimates with and without correlation, along with the correlation ratio, for a whole life insurance. These values correspond to the same five scenarios that were considered for the term/life annuity.

	$\rho^{r,\mu}$	η	ε	$BE^{TLI}(0, 60, 0)$	$BE^{TLI}(0, 60, \rho^{r,\mu})$	$CR^{TLI}(0, 60, \rho^{r,\mu})$
Scen. A	20%	1%	0.05196101%	0.6869412	0.6884605	100.221%
Scen. B	40%	1%	0.05196101%	0.6869412	0.6899871	100.443%
Scen. C	20%	5%	0.05196101%	0.6869412	0.6946117	101.117%
Scen. D	20%	1%	0.5%	0.6846576	0.6997499	102.204%
Scen. E	40%	5%	0.5%	0.6846576	0.8730708	127.519%

Table 2: Best estimate with and without correlation and corresponding correlation ratio of a whole life insurance ($\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$, $\bar{B} = 0.100627916$ and $T = 60$).

5.4 Mixed endowment

Let us consider, at time 0, a term/life mixed endowment $10/X$ of maturity time s . In such a contract, the insurer pays 10 to the insured's beneficiary at the time when the insured died if the death occurs before time s and pays X to the insured if the insured lives at time s . In case of a term mixed endowment, the maturity s is an integer with $s < \tilde{\omega} - x$, and in case of a life mixed endowment, $s = \tilde{\omega} - x$. Let us denote the best estimate at time 0 of such a contract of maturity time s as follows

$$BE^{TLM E}(0, s, \rho^{r,\mu}) = X BE^{PE}(0, s, \rho^{r,\mu}) + 10 BE^{TLI}(0, s, \rho^{r,\mu}), \quad (75)$$

where $BE^{PE}(0, s, \rho^{r,\mu})$ and $BE^{TLI}(0, s, \rho^{r,\mu})$ are respectively given by (66) and (72).

²This internal hedging should not be confused with the general concept of natural hedging in life insurance. Natural hedging refers to the partial compensation in a specific portfolio between liabilities in case of death and in case of life (for instance hedging between life time annuities and term insurance). In our model, the hedging is specific to term life and is explained by the technical presence inside its valuation formula of probabilities in case of survival and death probabilities (these two kinds of risks having opposite behaviours in terms of correlation with interest rates).

Without considering the effect of correlation, the fair value is given by

$$BE^{TLME}(0, s, 0) = X BE^{PE}(0, s, 0) + 10 BE^{TLI}(0, s, 0), \quad (76)$$

where $BE^{PE}(0, s, 0)$ and $BE^{TLI}(0, s, 0)$ are respectively given by (67) and (73). By picking some specific values for the parameter X , we deal with well known insurance contracts:

- when $X = 5$, a mixed endowment 10/5;
- when $X = 10$, a mixed endowment 10/10;
- when $X = 20$, a mixed endowment 10/20 and
- when $X = 0$, a pure term insurance.

The ratio of the best estimate with correlation (75) and without correlation (76) at time 0, named « correlation ratio of the term/life mixed insurance » is defined as

$$CR^{TLME}(0, s, \rho^{r, \mu}) = \frac{X BE^{PE}(0, s, \rho^{r, \mu}) + 10 BE^{TLI}(0, s, \rho^{r, \mu})}{X BE^{PE}(0, s, 0) + 10 BE^{TLI}(0, s, 0)}. \quad (77)$$

The figure below shows in function of maturity, the best estimate (75) and the correlation ratio (77) of different term/life mixed endowments.

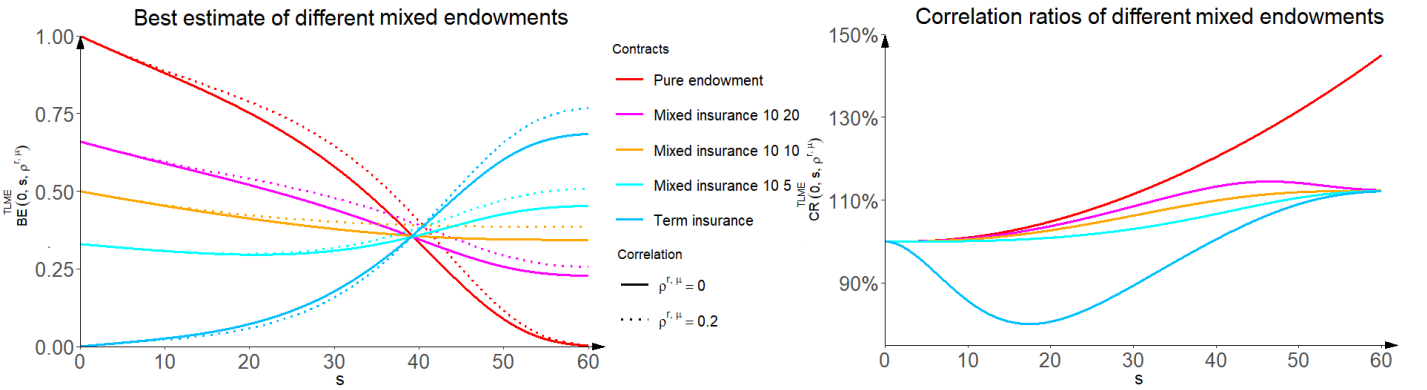


Figure 9: Best estimate and correlation ratio of different term/life mixed endowments ($\eta = 0.05$, $\varepsilon = 0.005$, $\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$ and $\bar{B} = 0.100627916$).

A mixed insurance being a combination of a pure endowment and a term insurance, both graphs of figure 9 are coherent when juxtaposed with the left graphs of figure 6 and the right graph of figure 8. Specifically, as observed on the right of figure 9, for long maturities, the correlation ratios of all mixed insurance products converge towards those of a whole life insurance. Indeed, this phenomenon is logical as the maturity lengthens, whereby the survival component of the mixed endowment (pure endowment) diminishes in significance, while the mortality component (term insurance) gains prominence. Opting for a mixed insurance product instead of a pure endowment enables partial hedging of the correlation risk through the inclusion of the term insurance component.

The following table outlines the values of the best estimates with and without correlation, as well as the correlation ratio, for a mixed endowment 10/10 with a maturity of $T = 20$. These values correspond to the same five scenarios previously considered.

	$\rho^{r,\mu}$	η	ε	$BE^{TLME}(0, 20, 0)$	$BE^{TLME}(0, 20, \rho^{r,\mu})$	$CR^{TLME}(0, 20, \rho^{r,\mu})$
Scen A	20%	1%	0.05196101%	0.3856582	0.3858577	100.052%
Scen B	40%	1%	0.05196101%	0.3856582	0.3860574	100.104%
Scen C	20%	5%	0.05196101%	0.3856582	0.3866577	100.259%
Scen D	20%	1%	0.5%	0.3875752	0.3895306	100.505%
Scen E	40%	5%	0.5%	0.3875752	0.4080213	105.275%

Table 3: Best estimate with and without correlation and corresponding correlation ratio of a term mixed endowment ($\lambda = 0.03$, $\omega = 0.1385505877$, $\mu_0 = 0.002600332$, $\bar{A} = 0.002219915$, $\bar{B} = 0.100627916$ and $T = 20$).

6 Inclusion of jumps

In this final section, we demonstrate the extension of results and analysis obtained from a conventional affine continuous diffusion framework to affine jump diffusions. This is achieved by calculating the price of the zero-coupon survival bond in the presence of jumps. For all the details, refer to the [online Appendix D](#).

We propose to improve the modelling of the state vector $\mathbf{X}_t = (r_t \ \mu_{x+t})^T$ by including three jump components respectively denoted $\mathbf{Z}_t^r$, $\mathbf{Z}_t^\mu$ and $\mathbf{Z}_t^{r,\mu}$. The model hence becomes

$$d\mathbf{X}_t = \mu(t, \mathbf{X}_t) dt + \sigma(t, \mathbf{X}_t) d\widetilde{\mathbf{W}}_t + d\mathbf{Z}_t^r + d\mathbf{Z}_t^\mu + d\mathbf{Z}_t^{r,\mu}, \quad (78)$$

with

$$\mu(t, \mathbf{X}_t) = \begin{pmatrix} \lambda(\bar{r}_t - r_t) \\ \omega(\bar{\mu}_{x+t} - \mu_{x+t}) \end{pmatrix}, \sigma(t, \mathbf{X}_t) \sigma(t, \mathbf{X}_t)^T = \begin{pmatrix} \eta^2 & \eta\varepsilon\rho^{r,\mu} \\ \eta\varepsilon\rho^{r,\mu} & \varepsilon^2 \end{pmatrix} \quad (79)$$

and

$$\mathbf{Z}_t^r := \begin{pmatrix} \sum_{i=1}^{M_t} J_{1,i}^r \\ 0 \end{pmatrix}, \mathbf{Z}_t^\mu := \begin{pmatrix} 0 \\ \sum_{i=1}^{N_t} J_{2,i}^\mu \end{pmatrix}, \mathbf{Z}_t^{r,\mu} := \begin{pmatrix} \sum_{i=1}^{O_t} J_{1,i}^{r,\mu} \\ \sum_{i=1}^{O_t} J_{2,i}^{r,\mu} \end{pmatrix}, \quad (80)$$

where

- $(M_t)_{t \in [0, S]}$ is a Poisson process with arrival intensity $\delta^r > 0$ and where $\{\mathbf{J}_i^r = (J_{1,i}^r, 0)\}_{i \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) bivariate random vectors representing the jump sizes.
- $(N_t)_{t \in [0, S]}$ is a Poisson process with arrival intensity $\delta^\mu > 0$ and where $\{\mathbf{J}_i^\mu = (0, J_{2,i}^\mu)\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. bivariate random vectors representing the jump sizes. Since the first jump component is zero, these are in fact only jumps on μ_{x+t} .
- $(O_t)_{t \in [0, S]}$ is a Poisson process with arrival intensity $\delta^{r,\mu} > 0$ and where $\{\mathbf{J}_i^{r,\mu} = (J_{1,i}^{r,\mu}, J_{2,i}^{r,\mu})\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. bivariate random vectors. This bivariate jump size distribution setup in case of simultaneous shocks allows for different jump magnitudes distribution for each process (marginals) while allowing a correlation between the two jumps sizes through a correlation coefficient ρ^j .

Each of the Poisson process mentioned above is supposed to be independent of the other Poisson processes and of the different jump size processes. The jump sizes $\mathbf{J}^r$, $\mathbf{J}^\mu$ and $\mathbf{J}^{r,\mu}$ are assumed to be independent of the Brownian process $\widetilde{\mathbf{W}}_t$.

In the literature, when a jump component is added to the Black and Scholes option-pricing framework,

two popular choices of jump size distributions are either a normal distribution, as introduced by Merton in (Merton, 1976), or a double exponential distribution, as introduced by Kou in (Kou, 2002). In (Wu and al., 2018), the authors consider mixed-exponential jumps whereas in (Li and al., 2023), in the bivariate context of an affine jump-diffusion model to describe the joint dynamics of interest rate and excess mortality, they employ a bivariate normal distribution for the jump sizes. Inspired by (Li and al., 2023), let us consider that:

- $\mathbf{J}^r$ follows a bivariate normal distribution with marginal means m_r^j and 0, standard deviations σ_r^j and 0, denoted by

$$\mathbf{J}^r \sim \mathcal{N}(m_r^j, 0; \sigma_r^j, 0); \quad (81)$$

- $\mathbf{J}^\mu$ follows a bivariate normal distribution with marginal means 0 and m_μ^j , standard deviations 0 and σ_μ^j , denoted by

$$\mathbf{J}^\mu \sim \mathcal{N}(0, m_\mu^j; 0, \sigma_\mu^j); \quad (82)$$

- $\mathbf{J}^{r,\mu}$ follows a bivariate normal distribution with marginal means $m_{r;\mu}^j$ and $m_{\mu;r}^j$, standard deviations $\sigma_{r;\mu}^j$ and $\sigma_{\mu;r}^j$, and jump size correlation coefficient ρ^j , denoted by

$$\mathbf{J}^{r,\mu} \sim N(m_{r;\mu}^j, m_{\mu;r}^j; \sigma_{r;\mu}^j, \sigma_{\mu;r}^j; \rho^j). \quad (83)$$

Before the inclusion of jumps, the only tool available to introduce dependence between the interest rates and the mortality rates was through the introduction of correlation between the two Brownian motions captured by the linear correlation coefficient $\rho^{r,\mu}$. In the model including jumps, we have on top of that, the concurrent occurrence of jumps, and the jump correlation coefficient ρ^j .

Utilising the affine jump diffusion framework outlined in the seminal work (Duffie, Pan and Singleton, 2000), we extend Proposition I to accommodate scenarios involving jumps (for the mathematical elaboration, we refer to the online Appendix D). This extension yields:

Proposition III

Considering the Hull and White² model with jumps (78) with explicit expressions of the moving targets (18) and (27) and the jump size distributions (81), (82) and (83), the price at time t of a zero-coupon survival bond of maturity time s , for an individual initially aged x at time 0, is given by

$$P_{r,\mu}(t, s) := 1_{\{\tau(x) > t\}} \cdot P_{r'}(t, s) \cdot P_{\mu'}(t, s) \cdot P_{\rho'}(t, s), \quad (84)$$

where

- $P_{r'}(t, s)$ is a term encompassing all purely interest rate impacts:

$$P_{r'}(t, s) := P_r(t, s) \cdot P_{\mathbf{J}^r}(t, s) \cdot P_{\mathbf{J}^{r,\mu}}(t, s), \quad (85)$$

where

▷ $P_r(t, s)$ accounts for the diffusion part. It is given by (22) and (23).

▷ $P_{\mathbf{J}^r}(t, s)$ accounts for the pure interest rates jumps. It is given by

$$P_{\mathbf{J}^r}(t, s) := \exp \left(\delta^r \int_t^s \exp \left(-B(\lambda, q, s) m_r^j + \frac{B^2(\lambda, q, s) (\sigma_r^j)^2}{2} \right) dq - \delta^r (s - t) \right). \quad (86)$$

▷ $P_{\mathbf{J}^{r,\mu}}(t, s)$ accounts for the interest rate component of the common jumps. It is given by

$$P_{\mathbf{J}^{r,\mu}}(t, s) := \exp \left(\delta^{r,\mu} \int_t^s \exp \left(-B(\lambda, q, s) m_{r;\mu}^j + \frac{B^2(\lambda, q, s) (\sigma_{r;\mu}^j)^2}{2} \right) dq - \delta^{r,\mu} (s - t) \right). \quad (87)$$

- $P_{\mu'}(t, s)$ is a term encompassing all purely mortality intensity impacts:

$$P_{\mu'}(t, s) := P_{\mu}(t, s) \cdot P_{\mathbf{J}^{\mu}}(t, s) \cdot P_{\mathbf{J}_{\mu}^{r, \mu}}(t, s), \quad (88)$$

where

▷ $P_{\mu}(t, s)$ accounts for the diffusion part. It is given by (30) and (31).

▷ $P_{\mathbf{J}^{\mu}}(t, s)$ accounts for the pure mortality intensity jumps. It is given by

$$P_{\mathbf{J}^{\mu}}(t, s) := \exp \left(\delta^{\mu} \int_t^s \exp \left(-B(\omega, q, s) m_{\mu}^j + \frac{B^2(\omega, q, s) (\sigma_{\mu}^j)^2}{2} \right) dq - \delta^{\mu}(s - t) \right). \quad (89)$$

▷ $P_{\mathbf{J}_{\mu}^{r, \mu}}(t, s)$ accounts for the mortality intensity component of the common jumps. It is given by

$$P_{\mathbf{J}_{\mu}^{r, \mu}}(t, s) = \exp \left(\delta^{r, \mu} \int_t^s \exp \left(-B(\omega, q, s) m_{\mu; r}^j + \frac{B^2(\omega, q, s) (\sigma_{\mu; r}^j)^2}{2} \right) dq - \delta^{r, \mu}(s - t) \right). \quad (90)$$

- $P_{\rho'}(t, s)$ is a term encompassing all the correlation impacts:

$$P_{\rho'}(t, s) := P_{\rho^{r, \mu}}(t, s) \cdot P_{\rho^j}(t, s) \cdot P_{con}(t, s), \quad (91)$$

where

▷ $P_{\rho^{r, \mu}}(t, s)$ accounts for the diffusion correlation. It is given by (42).

▷ $P_{\rho^j}(t, s)$ accounts for the jump size correlation of common jumps. It is given by

$$P_{\rho^j}(t, s) := \frac{P_{\mathbf{J}^{r, \mu}}(t, s)}{P_{\mathbf{J}_0^{r, \mu}}(t, s)}, \quad (92)$$

where

$$P_{\mathbf{J}^{r, \mu}}(t, s) := \exp \left(\delta^{r, \mu} \int_t^s \exp (I(q, s, \rho^j)) dq - \delta^{r, \mu}(s - t) \right) \quad (93)$$

and

$$P_{\mathbf{J}_0^{r, \mu}}(t, s) := \exp \left(\delta^{r, \mu} \int_t^s \exp (I(q, s, 0)) dq - \delta^{r, \mu}(s - t) \right), \quad (94)$$

with

$$\begin{aligned} I(t, s, \rho^j) := & -B(\lambda, t, s) m_{r; \mu}^j - B(\omega, t, s) m_{\mu; r}^j + \frac{B^2(\lambda, t, s) (\sigma_{r; \mu}^j)^2}{2} + \frac{B^2(\omega, t, s) (\sigma_{\mu; r}^j)^2}{2} \\ & + \rho^j B(\lambda, t, s) B(\omega, t, s) \sigma_{r; \mu}^j \sigma_{\mu; r}^j. \end{aligned} \quad (95)$$

▷ $P_{con}(t, s)$ accounts for the concurrent occurrence of the common jumps. It is given by

$$P_{con}(t, s) := \frac{P_{\mathbf{J}_0^{r, \mu}}(t, s)}{P_{\mathbf{J}^{r, \mu}}(t, s) P_{\mathbf{J}_{\mu}^{r, \mu}}(t, s)}, \quad (96)$$

where $P_{\mathbf{J}_0^{r, \mu}}(t, s)$, $P_{\mathbf{J}^{r, \mu}}(t, s)$ and $P_{\mathbf{J}_{\mu}^{r, \mu}}(t, s)$ are respectively given by (94), (87) and (90).

Notably, akin to pure diffusion models, introducing a positive correlation between jump sizes in interest rates and mortality intensities leads to an increase in the price of a pure endowment.

7 Conclusion and perspectives

This paper examines the impact of the correlation between interest rates and mortality rates on the best estimate of several life insurance contracts. Consistent with existing literature, the study employs an affine diffusion setup, namely a Hull and White² model (16), that can be seen as a nice theoretical model which leads to very elegant results whose study allows to draw interesting conclusions on the impact of correlation. By analysing this framework, the study derives expressions for the best estimates of life insurance contracts both with and without correlation, interprets the results, provides numerical illustrations, and draws pertinent conclusions.

The investigation begins by computing the explicit expression of a zero-coupon survival bond (proposition I) and a mortality density (proposition II), which are two building blocks of a wide array of life insurance contracts. The study then explores how various parameters within the correlation term impact the values of these building blocks aiming to evaluate the extent to which incorporating correlation affects their valuation. A notable finding is that, depending on specific parameter values such as maturity, speeds of mean reversion, diffusion coefficients, and correlation coefficient, ignoring correlation may result in significant overestimation or underestimation of the price of a zero-coupon survival bond. Additionally, the study reveals that while correlation inevitably influences the optimal estimate of a pure endowment across all maturities, its impact on term insurance contracts nullifies at certain maturity values, a phenomenon termed « internal hedging » of correlation risk.

The paper further proposes an enhanced model encompassing three jump components: a univariate jump component in the interest rate process, another in the mortality intensity process and a bivariate jump component accounting for simultaneous correlated shocks in both processes (proposition III). Prior to incorporating jumps, the only means to introduce dependence between the interest rates and the mortality rates was via the correlation coefficient $\rho^{r,\mu}$ of the two Brownian motions. However, with the inclusion of jumps, the model introduces the concurrent occurrence of jumps and the jump size correlation coefficient ρ^j .

In terms of perspectives, it is worth noting that alongside the consideration of jump components, the extended Cox-Ingersoll-Ross (ECIR) remains the most justifiable candidate for mortality modelling, owing to its ability to maintain positive values in mortality intensity. However, due to the non-affinity of the combined Hull and White-ECIR model, the present study opts for the Hull and White mortality model. This strategic choice is based on the resulting model's affinity, interpretability, and potential utility as a benchmark. Future research aims to explore various approximations of the Hull and White-extended CIR model to address its non-affine characteristics and compare outcomes with those discussed herein.

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Supplementary material

To view supplementary material for this article, please visit [url](#).

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Competing interest declaration

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