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The complementarity foundations of industrial organization

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DISCUSSION PAPER

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Abstract

In this paper we review the state of the art of Games with Strategic Complementarities (GSC), which are fundamental tools in modern Industrial Organization. The originality of the paper lies in the way the material is presented. Indeed, the mathematical aspects of GSC are complex and scattered in a literature which spans a long time period and a variety of research fields such as economics, applied mathematics and operations research. We organize a large amount of material in a unified and self-contained way, and concentrate on the intuitions and conceptual points that lie in the background of the mathematical modeling, with special emphasis on the modeling of complementarity. On the technical side, we investigate in details the choice and content of the assumptions. The scope of the paper is to allow the applied researcher to understand the theory, so that she may rapidly develop her own ability to deal with concrete problems.

Keywords: strategic complementarity, oligopoly theory, supermodularity, Nash equilibria, lattices.

JEL Classification: C60, C70, C72

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1. Introduction

Industrial Organization (IO) lies at the centerpiece of economic theory since the seminal contributions of Bertrand and Cournot. Broadly speaking, IO plays two main roles in modern economics. On a theoretical side, the basic models of Bertrand and Cournot are models of price formation. In other words, they provide explanations about how prices are determined as a result of the market interaction of economic agents[1]. On the other hand, these basic models represent a cornerstone in the study of real industries both from a positive and normative point of view, and in such respect they constitute a basic tool for academics, policy makers and professionals[2].

Recently, a new and powerful set of tools have been added to those - more traditional - already available to IO scholars and practitioners. These tools are usually denoted, in a concise way, as games with strategic complementarities (GSC), also known as supermodular or quasisupermodular games.

GSC entail various methodological innovations. They are built up and structured on the basis of a fixpoint theory and of a theory of comparative statics not requiring neither continuity nor convexity assumptions, thus working for example with discrete choice sets. Furthermore, and perhaps even more importantly from a conceptual viewpoint, all the mathematical tools just mentioned rely on the modeling of and old notion in economics, that of complementarity, in terms of supermodularity and quasisupermodularity of payoffs.

As a result, the theory of GSC is complex, especially whenever full generality is at stake. Furthermore, the various aspects of the theory are presented and scattered in a literature certainly not huge, but which spans a long time period and plenty of different research fields such as applied mathematics, economics and operations research. The various writers in these fields started from different problems and developed different answers to those. Only recently all the pieces have been put together properly, setting up a general and pretty unified framework for the study of the effect of complementarities on the solution to individual optimization problems and on the solutions to games.

[1] A 'small number' of economic agents.

[2] As an example, consider the increasingly important role that IO findings have in the regulation of firms and industries and in concrete, court-based regulatory cases

In this paper, we review from scratch to the state of the art the mathematical features of GSC.

Although this paper does not contain new results, we claim some originality with respect to the way in which the theory is presented here. In particular, we plan to organize a large amount of material in a unified and self-contained frame, and to clarify the basic mathematical modeling so that the reader may rapidly develop his own ability to deal with applied research not falling entirely in the realm of well-known examples. Furthermore, we will be very focused on the intuitions and conceptual points that lie in the background of the mathematical modeling, while on the formal side we will concentrate on why the assumptions are made, and why that type of assumptions and not others.

As an example of our approach, consider that complementarity and GSC are usually treated in the literature in the context of lattices. However, lattices do not have much to do with complementarity. They are just useful when the individual choice sets are not the product of totally ordered sets. However, it is exactly in the context of choice sets which *are* the product of totally ordered sets that complementarity can be better understood.

Accordingly, the treatment in this paper does not start with lattices, but introduce them when it is needed by the natural development of the theory. Namely, we present the modeling of complementarity for choice sets which are chains at first[3], then product of chains, and finally lattices. We furthermore distinguish between cardinal and ordinal complementarity, and show why the latter should be introduced and what role it plays in the theory of GSC: namely, that of making sure that results obtained by using cardinal complementarity are indeed retained under order-preserving transformations of payoffs.

We will prove every statement with the tools developed that far, even at the obvious cost of longer proofs and more cumbersome notation. A cost due to our choice of not using lattice notions until strictly needed.

The IO literature is full of extremely relevant applications of GSC, and some books and papers are exclusively devoted to these applications, for example Vives (1999) and Amir (2005 a), just to quote some of the better known works in this field.

We will not fill this paper with applications. The interested reader is referred to the works quoted above, which contain extended references.

[3]Totally ordered sets are also called chains.

This paper is concerned with the mathematical structure of the theory of GSC and as such it complements the applied literature. However, a final section contains selected examples from IO which are constructed and presented here for the sake of illustrating particular aspects of the techniques; aspects not used extensively in the applied literature but that, we believe, can be useful and conductive of new applications.

2. Cardinal and ordinal complementarity

Complementarity is a very old notion in economics. Samuelson (1974) presents an authoritative historical perspective about it, and surveys the idea of Pareto-Edgeworth complementarity, giving it the meaning of an externality among activities. Samuelson's reconstruction lies at the basis of the current approach to complementarity. Consider an agent having preferences whose cardinality allows him to add own utility indeces. Consider a consumption bundle where tea and lemon are present. According to Samuelson, tea and lemon are Pareto-Edgeworth complements whenever, keeping all other goods fixed, a joint increase of tea and lemon gives the agent a benefit exceeding the sum of benefits he would get by increasing them separately. This means that an increase of, say, tea makes an increase of lemon more desirable: the increase of tea exerts a positive externality on increasing lemon.

Later literature (Bulow et al., 1985) independently rediscovers Samuelson's approach, calling "strategic complements" what Samuelson called Pareto-Edgeworth complements. Of course, in general there is no reason to qualify a complementarity relation as "strategic". But the terminology has spread out in economics at large. We point out, then, that strategic complementarity is exactly Pareto-Edgeworth complementarity, and call it simply "complementarity" from now on.

Samuelson heuristic description translates directly into a property of the utility function. Let $\mathbb{R}^2$ be the commodity space, with typical element (x, t), where x is the amount of lemon and t is the amount of tea. Start from a consumption bundle (x_1, t_1). Consider a consumption bundle (x_2, t_2), with $x_1 < x_2$ and $t_1 < t_2$. At (x_2, t_1) we would have increased only lemon. At (x_1, t_2) we would have increased only tea. At (x_2, t_2) we would have increased both.

Samuelson description of complementarity says that:

$$u(x_2, t_1) - u(x_1, t_1) + u(x_1, t_2) - u(x_1, t_1)$$
$$\leq$$
$$u(x_2, t_2) - u(x_1, t_1);$$

that is,

$$(1) \qquad u(x_2,t_1) - u(x_1,t_1) \leq u(x_2,t_2) - u(x_1,t_2).$$

This property has a modern name (see Topkis, 1998). Let X be a poset (a partially ordered set), T be a set, and $u : X \times T \to \mathbb{R}$. For fixed $x_1, x_2 \in X$, with $x_1 < x_2$, call

$$\Delta_{x_1 x_2}(t) := u(x_2,t) - u(x_1,t)$$

a first difference of u in x.

Definition 1. (Increasing differences). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. u has increasing differences in (x,t) if every first difference of u in x is increasing in t; that is, if for every $x_1, x_2 \in X$ with $x_1 < x_2$, for every $t_1, t_2 \in T$ with $t_1 < t_2$, inequality (1) holds.

Definition 2. (Cardinal complementarity). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. We say that activities x and t are cardinal complements if u has increasing differences in (x,t).

It is clear from the definition that $u(x,t)$ has increasing differences in (x,t) if and only if it has also increasing differences in (t,x), by which we mean that any first difference of $u(x,t)$ in t, i.e. any

$$\Delta_{t_1 t_2}(x) = u(x,t_2) - u(x,t_1)$$

with $t_1 < t_2$, is increasing in x. Hence the cardinal complementarity relation is symmetric. We will come back to this important point in the sequel.

In applications, to check if a payoff function has increasing differences one often uses the following immediate result.

Lemma 1. (Differential characterization of cardinal complementarity). *Let $u : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$ be differentiable in x for every t, and let $u_x(x,t)$ be differentiable in t for every x. Function u has increasing differences in (x,t) if and only if for every $(x,t) \in \mathbb{R}^2$, $u_{xt}(x,t) \geq 0$. Furthermore, if for every $(x,t) \in \mathbb{R}^2$, $u_{xt}(x,t) > 0$, then u has strict increasing differences in (x,t).*

Proof: If u has increasing differences in (x,t), then for every $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, and for every distinct $x, x_0 \in \mathbb{R}$, we have that

$$\frac{u(x,t_2) - u(x_0,t_2) - [u(x,t_1) - u(x_0,t_1)]}{x - x_0} \geq 0.$$

Taking limit as $x \to x_0$, we get that $u_x(x_0, t_2) \geq u_x(x_0, t_1)$. Hence $u_x(x, t)$ is increasing in t, and so for every x and every distinct t, t_0 in $\mathbb{R}$,

$$\frac{u_x(x,t) - u_x(x,t_0)}{t - t_0} \geq 0.$$

Taking limit as $t \to t_0$, we get the result.

Let now $u_{xt}(x,t) \geq (>)\, 0$ for every $(x,t) \in \mathbb{R}^2$. Since for every x $u_x(x,t)$ is differentiable in t, then by the intermediate value theorem, for every t_1, t_2 with $t_1 < t_2$, there exists some $\beta \in (t_1, t_2)$ such that

$$\frac{u_x(x,t_2) - u_x(x,t_1)}{t_2 - t_1} = u_{xt}(x, \beta) \geq (>)\, 0,$$

meaning that for every x, $u_x(x,t_2) - u_x(x,t_1) \geq (>)\, 0$. Hence for $u(x,t_2) - u(x,t_1)$, which is a differentiable function of x, by the intermediate value theorem we have that, for every $x_1 < x_2$, there is some $\alpha \in (x_1, x_2)$ such that

$$\frac{u(x_2,t_2) - u(x_2,t_1) - [u(x_1,t_2) - u(x_1,t_1)]}{x_2 - x_1} =$$

$$u_x(\alpha, t_2) - u_x(\alpha, t_1) \geq (>)\, 0.$$

Hence

$$u(x_2,t_2) - u(x_2,t_1) - [u(x_1,t_2) - u(x_1,t_1)] \geq (>)\, 0,$$

and u has (strictly) increasing differences in (x,t). ■

Remark: conventional complementarity. Bulow etc. (1985) define "conventional" complementarity as a positive effects of increasing one activity, say x, on total profits. In these terms, x is a conventional complement of t if $u_x(x,t) \geq 0$, i.e. if payoff is increasing in x. On the other hand, strategic complementarity means that increasing one activity has a positive effects on the *marginal profits* associated to its complements.

Lemma 2. (Multidimensional increasing differences). *Let $u : X \times T \to \mathbb{R}$, with $X = X_1 \times \cdots \times X_n$ and $T = T_1 \times \cdots \times T_n$, each factor in the products being a poset. u has increasing differences in $((x_1, \ldots, x_m), (t_1, \ldots, t_n))$ on $X \times T$ if and only if, for every $(x'_1, \ldots, x'_m, t'_1, \ldots, t'_n)$ in $X \times T$, for every $i = 1, \ldots, m$ and every $j = 1, \ldots, n$, the function*

$$u(x'_1, \ldots, x_i, \ldots, x'_m, t'_1, \ldots, t_j, \ldots, t'_n) : X_i \times T_j \to \mathbb{R}$$

has increasing differences in (x_i, t_j) on $X_i \times T_j$.

Proof: Necessity is trivial. For sufficiency, we need two steps.

STEP 1. We first prove that u has increasing differences in $(x_i, (t_1, \ldots, t_n))$ for any $i = 1, \ldots, m$ and any fixed x_{-i}. Take $i = 1$. Fix any $(x_2, \ldots, x_n)$. Pick any $(x_i', t_1', \ldots, t_n') \leq (x_i'', t_1'', \ldots, t_n'')$. Set $(t_2, \ldots, t_n) = (t_2', \ldots, t_n')$. By assumption u has increasing differences in (x_1, t_1) for fixed $(x_2, \ldots, x_n)$ and $(t_2', \ldots, t_n')$; that is,

$$\begin{gathered} u\left(x_1'', x_2, \ldots, x_m, t_1', t_2', \ldots, t_n'\right) - u\left(x_1', x_2, \ldots, x_m, t_1', t_2', \ldots, t_n'\right) \\ \leq \\ u\left(x_1'', x_2, \ldots, x_m, t_1'', t_2', \ldots, t_n'\right) - u\left(x_1', x_2, \ldots, x_m, t_1'', t_2', \ldots, t_n'\right) \end{gathered}$$

Set now $(t_1, t_3, \ldots, t_n) = (t_1'', t_3', \ldots, t_n')$. By assumption, u has increasing differences in (x_1, t_2) for fixed $(x_2, \ldots, x_n)$ and $(t_1'', t_3', \ldots, t_n')$. Hence the inequality above can be continued into:

$$\begin{gathered} \leq \\ u\left(x_1'', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n'\right) - u\left(x_1', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n'\right) \end{gathered}$$

Set now $(t_1, t_2, t_4 \ldots, t_n) = (t_1'', t_2'', t_4' \ldots, t_n')$. By assumption, u has increasing differences in (x_1, t_3) for fixed $(x_2, \ldots, x_n)$ and $(t_1'', t_2'', t_4' \ldots, t_n')$. Continue the inequalities above into:

$$\begin{gathered} \leq \\ u\left(x_1'', x_2, \ldots, x_m, t_1'', t_2'', t_3'', t_4' \ldots, t_n'\right) - u\left(x_1', x_2, \ldots, x_m, t_1'', t_2'', t_3'', t_4' \ldots, t_n'\right) \end{gathered}$$

Proceeding this way, we get a string of inequalities ending as:

$$\begin{gathered} \cdots \\ \leq \\ u\left(x_1'', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n''\right) - u\left(x_1', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n''\right). \end{gathered}$$

Hence, for fixed $(x_2, \ldots, x_n)$,

$$\begin{gathered} u\left(x_1'', x_2, \ldots, x_m, t_1', t_2', \ldots, t_n'\right) - u\left(x_1', x_2, \ldots, x_m, t_1', t_2', \ldots, t_n'\right) \\ \leq \\ u\left(x_1'', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n''\right) - u\left(x_1', x_2, \ldots, x_m, t_1'', t_2'', \ldots, t_n''\right). \end{gathered}$$

So u has increasing differences in $(x_i, (t_1, \ldots, t_n))$ for any fixed $(x_2, \ldots, x_n)$. Redo the argument for $i = 2, \ldots, m$.

STEP 2. We now prove the result. Pick any

$$\left(x_1', \ldots, x_m', t_1', \ldots, t_n'\right) \leq \left(x_1'', \ldots, x_m'', t_1'', \ldots, t_n''\right).$$

By the previous step, fixing $(x_2, x_3, \ldots, x_m) = (x_2', x_3', \ldots, x_m')$, u has increasing differences in $(x_1, (t_1, \ldots, t_n))$, and hence in $((t_1, \ldots, t_n), x_1)$

(here we use that increasing differences does not distinguish between the first and second variable). That is,

$$u\left(x_1', x_2', \ldots, x_m', t_1'', \ldots, t_n''\right) - u\left(x_1', x_2', \ldots, x_m', t_1', \ldots, t_n'\right)$$
$$\leq$$
$$u\left(x_1'', x_2', \ldots, x_m', t_1'', \ldots, t_n''\right) - u\left(x_1'', x_2', \ldots, x_m', t_1', \ldots, t_n'\right).$$

Fix now $(x_1, x_3, \ldots, x_m) = (x_1'', x_3', \ldots, x_m')$. By Step 1, u has increasing differences in $(x_2, (t_1, \ldots, t_n))$, and so u has increasing differences in $((t_1, \ldots, t_n), x_2)$. Thus, we can continue the inequality above into:

$$\leq$$
$$u\left(x_1'', x_2'', \ldots, x_m', t_1'', \ldots, t_n''\right) - u\left(x_1'', x_2'', \ldots, x_m', t_1', \ldots, t_n'\right).$$

Proceeding this way, we get a string of inequalities ending as:

$$\ldots$$
$$\leq$$
$$u\left(x_1'', x_2'', \ldots, x_m'', t_1'', \ldots, t_n''\right) - u\left(x_1'', x_2'', \ldots, x_m'', t_1', \ldots, t_n'\right).$$

Hence,

$$u\left(x_1', x_2', \ldots, x_m', t_1'', \ldots, t_n''\right) - u\left(x_1', x_2', \ldots, x_m', t_1', \ldots, t_n'\right)$$
$$\leq$$
$$u\left(x_1'', x_2'', \ldots, x_m'', t_1'', \ldots, t_n''\right) - u\left(x_1'', x_2'', \ldots, x_m'', t_1', \ldots, t_n'\right).$$

This prove that u has increasing differences in $((x_1, \ldots, x_m), (t_1, \ldots, t_n))$ on the product $X \times T$. ■

Corollary 1. (Differential characterization of multidimensional increasing differences). *Let $u : \mathbb{R}_x^m \times \mathbb{R}_t^n \to \mathbb{R}$ be twice differentiable on $\mathbb{R}^m \times \mathbb{R}^n$ (but we need less, see the assumptions for the differential characterization of increasing differences). Function u has increasing differences in (x, t) on $\mathbb{R}^m \times \mathbb{R}^n$ if and only if, for every $i = 1, \ldots, n$; every $j = 1, \ldots, n$; and every (x, t) in $\mathbb{R}^m \times \mathbb{R}^n$, $u_{x_i t_j}(x, t) \geq 0$.*

Proof: From Lemma 1 and Lemma 2. ■

A weaker, ordinal (preserved under increasing transformations of payoffs) notion of complementarity has been introduced by Milgrom and Shannon (1994) using the following observation: in the inequality defining increasing differences, if the left-hand-side difference is (strictly) greater than some real k, then the right-hand-side difference must also be (strictly) greater than k. For $k = 0$, this means that whenever an increase of lemon is desirable at a fixed level of tea, this increase remains desirable if tea increases too. It is a weak notion of Pareto-Edgeworth complementarity.

Definition 3. (Single crossing property). Let X and T be posets. Function $u(x,t) : X \times T \to \mathbb{R}$ has the single crossing property in (x,t) if, whenever any first difference $\Delta_{x_1 x_2}(t)$ of u in x is (strictly) positive at some t_1, it remains so at any t_2 greater than t_1. That is, for every $x_1, x_2 \in X$ with $x_1 < x_2$, for every $t_1, t_2 \in T$ with $t_1 < t_2$,

$$u(x_1,t_1) \leq (<) u(x_2,t_1) \Rightarrow u(x_1,t_2) \leq (<) u(x_2,t_2).$$

Clearly, if function $u(x,t)$ has increasing differences in (x,t), it also satisfies the single crossing property in (x,t), while the contrary does not hold necessarily, as can be shown easily.

Definition 4. (Ordinal complementarity). Let $u(x,t) : X \times T \to \mathbb{R}$, where X and T are posets. We say that activities x and t are ordinal complements if u has the single crossing property in (x,t).

The fact that $u(x,t)$ has the single crossing property in (x,t) does not imply that it has this property in (x,t), i.e. it does not imply that whenever any first difference $\Delta_{t_1 t_2}(x)$ of u in t is (strictly) positive at some x_1, it remains so at any x_2 greater than t_1. Hence, contrary to cardinal complementarity, the relation of ordinal complementarity is not symmetric. We elaborate on this in the next Section.

3. Cardinal versus ordinal complementarity

3.1. **On the lack of symmetry of ordinal complementarity.** We have seen that a function $u(x,t)$ has increasing differences in (x,t) if and only if it has increasing differences in (t,x). Hence, if increasing differences in (x,t) is meant to define the relation among variables: "x is a cardinal complement of t", this relation is symmetric.

On contrary, the single crossing condition depends on the variable with respect to whom the first difference is taken. Single crossing in (x,t) does not imply single crossing in (t,x); that is, does not imply that every first difference of u in t which is (strictly) positive at some x_1, stays (strictly) positive at any x_2 greater than x_1, as the following example shows.

Consider the function

$$u(x,t) = \frac{x}{t} + t$$

with $t \neq 0$. For any $x_1 < x_2$, the corresponding first difference of u in x is

$$\frac{1}{t}\left(x_2 - x_1\right).$$

If any such first difference is strictly positive[4], then t must be strictly positive as well, and so this first difference stays strictly positive as t increases, albeit being decreasing in t. Thus u has the single crossing property in (x,t).

On the other hand, u fails to have the single crossing property in (t,x), since for any $t_1 < t_2$ the corresponding first difference of u in t takes the form of

$$x\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + t_2 - t_1$$

which for $t_1 = 1$ and $t_2 = 2$ becomes $-\frac{x}{2} + 1$, which in turn is positive at $x_1 = 1$ but negative at $x_2 = 4$.

If the single crossing property in (x,t) is meant to define the relation among variables: "x is an ordinal complement of t", this relation is not symmetric. As we have just seen, for some u it can well be that x is an ordinal complement of t but t is not an ordinal complement of x.

Whenever we define complementarity by means of the single crossing property, we obtain a non-symmetric complementarity relation among activities. This lack of symmetry is not only a curiosity, but will require to add extra assumptions on payoffs when studying the comparative statics of the set of maximizers of $u(x,t)$ in a purely ordinal context. We will turn to this issue later on in the paper.

3.2. **On some inconsistency between cardinal and ordinal complementarity.** Consider again the function of the previous subsection. We have seen that it satisfies the single crossing property in (x,t). On the restricted domain $\{(x,t) : x \leq 1 \leq t\}$, the function satisfies also the single crossing property in (t,x). Indeed, for every $t_1 < t_2$, the expression

$$\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + t_2 - t_1$$

is (strictly) positive iff $t_1 t_2 \geq 1$ (iff $t_1 t_2 > 1$). Hence, when we rescale the negative term $\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$ multiplying it by any $x \leq 1$, so as to obtain the expression of the first difference of u in t, we see that this first difference remain positive or, respectively, strictly positive. Hence u satisfies the single crossing property in (t,x) on the defined domain.

[4] It can never be 0

However, our function u has *decreasing differences*, since

$$u_{xt}(x,t) = -\frac{1}{t^2} < 0$$

for all x and all $t \neq 0$. Hence, for this function, on the aforementioned domain, the two activities x and t are both cardinal substitutes and ordinal complements.

This ambiguity is however mitigated by the fact that, being cardinal substitutes, the activities are also ordinal substitutes. Hence they are indeed ordinally 'neutral' with each other.

It is furthermore worth remarking that, differently from the problem of the lack of symmetry of the ordinal complementarity relation, the possible inconsistency between complementarity and substitutability that we encounter when we define complementarity or substitutability in terms of properties of payoffs, does not entail operational problems for comparative statics analysis.

In fact, within the theory presented in the following sections, complementarity is used to analyze the behavior of the set of maximizers, while substitutability allows to study the set of minimizers. Hence the inconsistency has no effect on the operational aspects of the theory.

4. Monotone comparative statics on chains

Let B_t be the set of maximizers over X of function $u(x,t)$. We keep the assumption, in the comparative statics results below, that B_t is nonempty for every t in T. Conditions to assure this in the various contexts will be introduced separately. In view of these conditions, however, we introduce here an appropriate intrinsic topology for posets.

Definition 5. (Interval topology). Let X be a poset with a least and a greatest element. The interval topology of X is the topology generated by taking the closed intervals

$$[y,z] = \{x \in X : y \leq x \leq z\},$$

with $y, z \in X$, as a subbasis for closed sets.

For the sake of applications, we recall that in $\mathbb{R}^n$ the interval topology is equivalent to the standard topology (Frink, see Birkhoff, 1967, Ch. X).

The simplest case is that in which the choice set X is a chain (a totally ordered set, for example a subset of $\mathbb{R}$). We have the following comparative statics result.

Theorem 1. (Comparative statics on chains, cardinal case). *Let X be a chain, T be a poset and let $u(x,t) : X \times T \to \mathbb{R}$ have increasing differences in (x,t). For every $t_1 < t_2$, for every $a \in B_{t_1}$ and every $b \in B_{t_2}$,* $\min\{a,b\} \in B_{t_1}$ *and* $\max\{a,b\} \in B_{t_2}$.

Proof: If $a \leq b$ we are done. Let then $b < a$. We have that

$$0 \leq u(a,t_1) - u(b,t_1) \leq u(a,t_2) - u(b,t_2) \leq 0,$$

where the first and last inequalities follows from optimality, and the middle-one from increasing differences. Hence $b \in B_{t_1}$, and $a \in B_{t_2}$. ■

The generalization to the ordinal case is immediate, and the proof is a prototype of how the inequalities of the single crossing property work in monotone comparative statics theorems. Hence it is instructive to fill in all the details.

Theorem 2. (Comparative statics on chains, ordinal case). *Let X be a chain, T be a poset and let $u(x,t) : X \times T \to \mathbb{R}$ satisfy the single crossing property in (x,t). For every $t_1 < t_2$, for every $a \in B_{t_1}$ and every $b \in B_{t_2}$,* $\min\{a,b\} \in B_{t_1}$ *and* $\max\{a,b\} \in B_{t_2}$.

Proof: If $a \leq b$ we are done. Let then $b < a$. By the optimality of a at t_1

$$u(a,t_1) - u(b,t_1) \geq 0,$$

and hence by the single crossing property

$$u(a,t_2) - u(b,t_2) \geq 0,$$

which implies, by the optimality of b at t_2, that $a \in B_{t_2}$. On the other hand, by optimality of b at t_2, the following inequality fails:

$$u(a,t_2) - u(b,t_2) > 0.$$

Hence, by the single crossing property,

$$u(b,t_1) - u(a,t_1) \leq 0.$$

Thus, by the optimality of a at t_1, $b \in B_{t_1}$ and we are done. ■

Theorem 3. (Increasing extremal selections). *Let X be a chain, T be a poset and let $u : X \times T \to \mathbb{R}$ satisfy the single crossing property in (x,t). If B_t has either a least element $a_*(t)$ or a greatest element $a^*(t)$ for every $t \in T$ (or both), then this element is an increasing functions.*

Proof: Take any $t_1 < t_2$. Consider first $a_*(t)$. Take any $a \in B_{t_1}$. By the previous theorem, $c := \min\{a, a_*(t_2)\} \in B_{t_1}$. Hence, $a_*(t_1) \leq c \leq a_*(t_2)$. Analogously, for any $b \in B_{t_2}$, by the previous theorem $d := \max\{b, a^*(t_1)\} \in B_{t_2}$, and so $a^*(t_1) \leq d \leq a^*(t_2)$. ■

Conditions assuring that the argmax B_t has indeed a least element and a greatest element of every t in T are here the same as the standard ones making it nonempty. Assume X compact in its interval topology and u upper semicontinuous in x for every t. Hence every B_t is nonempty and compact and, being a chain, has a least and a greatest element.

5. Monotone comparative statics on finite products of chains

If $X = Y \times Z$, where Y and Z are chains (for example, X is a box in $\mathbb{R}^2$), then a natural way to extend to this context the definition of monotonicity of the argmax B_t of $u(y, z, t)$ over $Y \times Z$ is to use coordinate-wise minima and maxima.

Pick any (y_1, z_1),$(y_2, z_2) \in Y \times Z$. Define infima and suprema as

$$(y_1, z_1) \wedge (y_2, z_2) = (\min\{y_1, y_2\}, \min\{z_1, z_2\}),$$

$$(y_1, z_1) \vee (y_2, z_2) = (\max\{y_1, y_2\}, \max\{z_1, z_2\}).$$

Note that such infima and suprema are well defined exactly because Y and Z are chains.

We say that B_t is increasing if for every $t_1 \leq t_2$ in T, for every $(y_1, z_1) \in B_{t_1}$ and every $(y_2, z_2) \in B_{t_2}$,

$$(y_1, z_1) \wedge (y_2, z_2) \in B_{t_1},$$

$$(y_1, z_1) \vee (y_2, z_2) \in B_{t_2}.$$

In this context, increasing differences of u in (x, t), hence the single crossing property u in (x, t), where $x = (y, z)$, does no longer suffice to guarantee that B_t is increasing. The reason is examined in the following subsections.

5.1. Using cardinal complementarity for comparative statics over a product of chains.

Let

$$X = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2.$$

X is the product of chain $\{0,1\}$ by itself. Let $T = \{t_1, t_2\}$, with $t_1 < t_2$. Let $u : X \times T \to \mathbb{R}$ be defined by

$$\begin{array}{cc}
u((0,0),t_1) = 0 & u((0,0),t_2) = 0 \\
u((0,1),t_1) = 8 & u((0,1),t_2) = 16 \\
u((1,0),t_1) = 10 & u((1,0),t_2) = 15 \\
u((1,1),t_1) = 1 & u((1,1),t_2) = 10
\end{array}$$

This function has increasing differences (hence satisfies the single crossing property as well) in (x,t). Indeed, check for all $x_1 \leq x_2$, with

$$x_1, x_2 \in \{(0,0), (0,1), (1,0), (1,1)\}.$$

For $x_1 = (0,0)$:

$$\begin{array}{c}
u((1,0),t_1) - u((0,0),t_1) = 10 < 15 = u((1,0),t_2) - u((0,0),t_2) \\
u((0,1),t_1) - u((0,0),t_1) = 8 < 16 = u((0,1),t_2) - u((0,0),t_2) \\
u((1,1),t_1) - u((0,0),t_1) = 1 < 10 = u((1,1),t_2) - u((0,0),t_2)
\end{array}$$

For $x_1 = (0,1)$:

$$u((1,1),t_1) - u((0,1),t_1) = -7 < -6 = u((1,1),t_2) - u((0,1),t_2)$$

For $x_1 = (1,0)$:

$$u((1,1),t_1) - u((1,0),t_1) = -9 < -5 = u((1,1),t_2) - u((1,0),t_2)$$

For this function, $B_{t_1} = \{(1,0)\}$ and $B_{t_2} = \{(0,1)\}$. Hence monotone comparative statics of the argmax fails. What has happened here?

Starting at the optimal bundle $(1,0)$, a shift of the parameter from t_1 to t_2 makes indeed desirable to increase the amount of the second good, as increasing differences shows by taking $x_1 = (1,0) < (1,1) = x_2$; that is, the marginal utility of the second good increases with t:

$$u((1,1),t_1) - u((1,0),t_1) = -9 < -5 = u((1,1),t_2) - u((1,0),t_2).$$

Furthermore, a shift of the parameter from t_1 to t_2 makes desirable to increase the amount of the first good too (i.e. to keep it to 1, which is the maximum amount allowed by our feasible set), as the increasing differences inequality above also shows. So why $(1,1)$, or at least $(1,0)$, is not the optimal bundle at t_2?

The problem is that, for our utility function, the marginal utility of the first good *decreases* as the amount of the second good increases, and vice versa, at any fixed level of t. Hence, the two goods are *substitutes*, and the net effect of increasing the parameter t - as this effect is determined by our utility function - is to increase the optimal consumption of the second good but to decrease that of the first good, ending up the re-optimization process at bundle $(0,1)$.

In other words, the problem in this example is that $u(y,z,t)$, albeit having increasing differences in the pair $((y,z),t)$, has *decreasing differences* in (y,z) for every t. Take in fact $y_1 = 0$ and $y_2 = 1$. The corresponding first difference of u in y is decreasing for every t as z shifts from 0 to 1. Let's check this. For $t = t_1$:

$$u((1,0),t_1) - u((0,0),t_1) = 10 > -7 = u((1,1),t_1) - u((0,1),t_1).$$

For $t = t_2$:

$$u((1,0),t_2) - u((0,0),t_2) = 15 > -6 = u((1,1),t_2) - u((0,1),t_2).$$

The conceptual point. When X is a chain, even a multidimensional chain, this cannot happen. In a chain, any reallocation following a parameter's shift needs to take the form of either an increase in the level of all goods, or a decrease. Hence the level of all goods move in the same direction. Then, complementarity between the bundle in X and the parameter suffices for monotone comparative statics to hold. On a chain, all goods can be seen as behaving as complements.

Summing up. In this example monotone comparative statics has failed because notwithstanding that each good is a complement with the parameter, i.e. that $u(y,z,t)$ has increasing difference in $((y,z),t)$, the two goods are substitutes with each other, i.e. $u(y,z,t)$ has decreasing differences in (y,z) for every t.

In order to obtain the desired monotone comparative statics, we need to assure not only that each one of the activities y and z is is a complement of the parameter t, but also that the activities y and z are complements with each other for each level of the t. Hence we need to assume both increasing differences of u in $((y,z),t)$, and increasing differences of u in (y,z) for any fixed t. The comparative statics theorem in this context is then the following:

Theorem 4. (COMP. STATICS ON A PRODUCT OF CHAINS, CARDINAL CASE). *Let Y and Z be a chains, T be a poset and $u : Y \times Z \times T \to \mathbb{R}$ have increasing differences in $((y,z),t)$, and in (y,z) for every $t \in T$.*

For every $t_1 < t_2$, for every $(y_1,z_1) \in B_{t_1}$ and every $(y_2,z_2) \in B_{t_2}$, $(y_1,z_1) \wedge (y_2,z_2) \in B_{t_1}$ and $(y_1,z_1) \vee (y_2,z_2) \in B_{t_2}$.

Proof: Apply the proof of theorem 5 below. ∎

Remark: too much complementarity in the cardinal context. Due to the symmetry of the induced cardinal complementarity relation, y and z are cardinal complements with each other, and we have seen that we need this symmetric complementarity relation for well-defined comparative statics. However, by assuming increasing differences in $((y, z), t)$, due to the symmetry of cardinal complementarity we get also that the parameter t is a complement of the activity bundle (y, z). This latter complementarity is of course completely unnecessary for comparative static analysis: the effect on the parameter t of an increase in the level of y or z is irrelevant as long as, indeed, t plays the role of a parameter. This extra assumption of complementarity is avoided in the ordinal context of the next subsection.

5.2. **Using ordinal complementarity for comparative statics over a product of chains.** If the choice space X is the product of chains Y and Z, and if we use the ordinal notion of complementarity, we need to take care of the non-symmetry of the induced complementarity relation. For $u(y, z, t)$, assuming the single crossing property of u in (y, z) for any fixed t, and of u in $((y, z), t)$, is not enough.

Indeed, we are assuming only that y is an ordinal complement of z, as well as that both y and z are ordinal complements of t. We can well get that increasing t makes in the first place the optimal level of both y and z increase, then increasing z makes y increase too, but ultimately increasing y makes z decrease. Hence we need to assume, also, that z is an ordinal complement of y. Hence, we need to assume the single crossing property in both (y, z) and (z, y), for any fixed t.

This is done in the next theorem. After reading the next section, it will be clear that the proof of theorem 2 could have been made much shorter and less pedantic. In the spirit of this paper, however, we have chosen to point out in details the role and working of ordinal complementarity, at the cost of a much longer and involved proof.

Theorem 5. (COMP. STATICS ON A PRODUCT OF CHAINS, ORDINAL CASE). *Let Y and Z be a chains, T be a poset and $u : Y \times Z \times T \to \mathbb{R}$ satisfy the single crossing property in $((y, z), t)$, and in both (y, z) and (z, y) for every $t \in T$.*

For every $t_1 < t_2$, for every $(y_1, z_1) \in B_{t_1}$ and every $(y_2, z_2) \in B_{t_2}$, $(y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$ and $(y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$.

Proof: Pick any $(y_1, z_1) \in B_{t_1}$ and any $(y_2, z_2) \in B_{t_2}$. If $(y_1, z_1) \leq (y_2, z_2)$, we are done.

Case (A): If $(y_2, z_2) \leq (y_1, z_1)$ then, by the theorem on comparative statics on chains, the single crossing property in $((y, z), t)$ suffices for the result.

Case (B): Let $y_2 < y_1$ and $z_1 < z_2$. Here we need the single crossing in $((y, z), t)$ and in (y, z). By optimality of (y_1, z_1) at $t = t_1$, the first difference

$$\Delta_{y_2 y_1}(z_1, t_1) = u(y_1, z_1, t_1) - u(y_2, z_1, t_1) \geq 0,$$

and so by the single crossing property in (y, z),

$$\Delta_{y_2 y_1}(z_2, t_1) = u(y_1, z_2, t_1) - u(y_2, z_2, t_1) \geq 0.$$

Since $(y_2, z_2) < (y_1, z_2)$, then the first difference $\Delta_{y_2 y_1}(z_2, t_1)$ coincides with the first difference $\Delta_{(y_2 z_2)(y_1 z_2)}(t_1)$ and so, because $t_1 < t_2$, by the single crossing property in $((y, z), t)$, we have that

$$\Delta_{(y_2 z_2)(y_1 z_2)}(t_2) = u(y_1, z_2, t_2) - u(y_2, z_2, t_2) \geq 0,$$

and by optimality of (y_2, z_2) at $t = t_2$, $(y_1, z_2) = (y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$. Analogously, by optimality of (y_2, z_2) at $t = t_2$,

$$u(y_2, z_2, t_2) - u(y_1, z_2, t_2) \geq 0,$$

and so by the single crossing in (y, z) and by the fact that u takes values in a chain,

$$u(y_2, z_1, t_2) - u(y_1, z_1, t_2) \geq 0.$$

Since $(y_2, z_2) < (y_1, z_2)$ and $t_1 < t_2$, then again by the single crossing in $((y, z), t)$,

$$u(y_2, z_1, t_1) - u(y_1, z_1, t_1) \geq 0.$$

Hence by optimality of (y_1, z_1) at $t = t_1$, $(y_2, z_1) = (y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$, and we are done.

Case (C): Let now $y_1 < y_2$ and $z_2 < z_1$. Here we need the single crossing in $((y, z), t)$ and in (z, y). By optimality of (y_1, z_1) at $t = t_1$,

$$u(y_1, z_1, t_1) - u(y_1, z_2, t_1) = \Delta_{z_2 z_1}(y_1, t_1) \geq 0.$$

By the single crossing property in (z, y),

$$u(y_2, z_1, t_1) - u(y_2, z_2, t_1) = \Delta_{z_2 z_1}(y_2, t_1) \geq 0.$$

Since $(y_2, z_2) < (y_2, z_1)$ and $t_1 < t_2$, then by the single crossing in $((y, z), t)$ we have that

$$u(y_2, z_1, t_2) - u(y_2, z_2, t_2) = \Delta_{(y_2 z_1)(y_2 z_2)}(t_2) \geq 0.$$

Hence by optimality again, $(y_2, z_1) = (y_1, z_1) \vee (y_2, z_2) \in B_{t_2}$. In the same way, by optimality

$$u(y_2, z_2, t_2) - u(y_2, z_1, t_2) \geq 0.$$

Hence by the single crossing in (z, y),

$$u(y_1, z_2, t_2) - u(y_1, z_1, t_2) \geq 0.$$

Again, because $(y_2, z_2) < (y_2, z_1)$ and $t_1 < t_2$, by the single crossing in $((y, z), t)$ we have that

$$u(y_1, z_2, t_1) - u(y_1, z_1, t_1) \geq 0,$$

and so by optimality $(y_1, z_2) = (y_1, z_1) \wedge (y_2, z_2) \in B_{t_1}$. ■

Corollary 2. (Increasing extremal selections). *Let Y and Z be chains, T be a poset and $u : Y \times Z \times T \to \mathbb{R}$ satisfy the single crossing property in $((y, z)\, t)$ and in both (y, z) and (z, y) for every $t \in T$. If B_t has either a least element $a_*(t)$ or a greatest element $a^*(t)$ for every $t \in T$ (or both), then this element is an increasing function.*

Proof: Take any $t_1 < t_2$. Consider $a_*(t)$. Take any $c \in B_{t_1}$. By the previous theorem, $c \wedge a_*(t_2) \in B_{t_1}$. Hence,

$$a_*(t_1) \leq c \wedge a_*(t_2) \leq a_*(t_2).$$

Analogously, for any $b \in B_{t_2}$, by the previous theorem $a^*(t_1) \vee b \in B_{t_2}$, and so

$$a^*(t_1) \leq a^*(t_1) \vee b \leq a^*(t_2).$$

■

The conditions assuring that the argmax B_t has indeed a least and a greatest element are the same conditions assuring that it is nonempty, namely that the objective function is upper semicontinuous in the decision variables and that the choice set is compact. However, to explain why, we would need further definitions and an important topological result. We postpone all this to the next section, were lattices are introduced. See the discussion after Corollary 3.

6. Monotone comparative statics on lattices

If the choice set X is a poset which is not a finite product of chains (for example a circle in the plane), then infima and suprema of pairs of elements of X do not need to exist (or to be in X). To extend to this context the notion of increasingness of B_t that we have used in the previous section, we need to introduce lattices.

Definition 6. (LATTICE) Let X be a nonempty poset. X is a lattice if for every $x_1, x_2 \in X$, $x_1 \wedge x_2 \in X$ and $x_1 \vee x_2 \in X$.

There is an important difference with the previous sections. Albeit increasing differences and the single crossing property can still be defined on $X \times T$, these properties may be insufficient to investigate the behavior of the first differences of u.

Let us elaborate more on this. Let X be a lattice, T be a poset, and $u : X \times T \to \mathbb{R}$ have the single crossing property in (x, t). Since X is a lattice, monotonicity of the argmax B_t is well defined as in the previous section. Take unordered $x_1 \in B_{t_1}$ and $x_2 \in B_{t_2}$. By optimality of x_1 at t_1, the first differences

$$\Delta_{x_1 \wedge x_2, x_1}(t_1) = u(x_1, t_1) - u(x_1 \wedge x_2, t_1)$$

is greater than or equal to zero. By the single crossing property, the same holds at $t = t_2$. And this is all we can say by using the single crossing property[5]. In particular, we do not reach any statement about the eventual non negativity of the first difference

$$\Delta_{x_2, x_1 \vee x_2}(t_2) = u(x_1 \vee x_2, t_2) - u(x_2, t_2),$$

which is what we need to reach a comparative statics conclusion.

An obvious way to deal with the problem would be, then, to have a property of u which relate in the right way, for fixed t in T, the first difference $\Delta_{x_1 \wedge x_2, x_1}(t)$ with the first difference $\Delta_{x_2, x_1 \vee x_2}(t)$. This property is called quasisupermodularity, and has been introduced by Milgrom and Shannon (1994).

Definition 7. ((QUASI)SUPERMODULARITY). Let X be a lattice and $u : X \to \mathbb{R}$. Function u is supermodular on X if for every $x_1, x_2 \in X$,

$$u(x_1) + u(x_2) \leq u(x_1 \wedge x_2) + u(x_1 \vee x_2).$$

Function u is quasisupermodular on X if for every $x_1, x_2 \in X$,

$$u(x_1) \geq (>) u(x_1 \wedge x_2) \Rightarrow u(x_1 \vee x_2) \geq (>) u(x_2).$$

In general lattices X, (quasi)supermodularity has not a direct interpretation in terms of complementarity. It only make the choice variables in X behave "consistently" with complementarity, but it is not complementarity itself. It is essentially a useful mathematical device.

[5]An analogous argument holds were we using increasing differences.

But, as the next lemmas show, the interpretation in terms of complementarity is completely restored as soon as X takes the more familiar forms that we considered in the previous sections.

Lemma 3. (Characterization of supermodularity in terms of increasing differences). *Let $X = Y \times Z$ and consider $u : X \to \mathbb{R}$. (i) If Y and Z are chains and u has increasing differences in (y, z) on $Y \times Z$, then it is supermodular in (y, z) on $Y \times Z$. (ii) If Y and Z are lattices, and u is supermodular in (y, z) on $Y \times Z$, then it has increasing differences in (y, z) on $Y \times Z$.*

Proof: (i) Take any unordered $x' = (y', z'), x'' = (y'', z'')$ in $Y \times Z$ (if they are ordered supermodularity holds trivially). Let, without loss of generality, $x' \wedge x'' = (y', z'')$ and $x' \vee x'' = (y'', z')$ (here we are using the assumption that Y and Z are chains). By increasing differences,

$$u(y'', z'') - u(y', z'') \leq u(y'', z') - u(y', z'),$$

which is supermodularity.
(ii) Pick any $(y', z') \leq (y'', z'')$ in the lattice $Y \times Z$. For $x' := (y', z'')$ and $x'' := (y'', z')$ in $Y \times Z$, $x' \wedge x'' = (y', z')$ and $x' \vee x'' = (y'', z'')$. By supermodularity,

$$u(x'') - u(x' \wedge x'') \leq u(x' \vee x'') - u(x'),$$

which is increasing differences. ■

Lemma 4. (Vector supermodularity). *Let $u : X \to \mathbb{R}$, where $X = X_1 \times \cdots \times X_n$ and each factor is a chain. Function u is supermodular in $(x_1, \ldots, x_n)$ on X if and only if for every $(x'_1, \ldots, x'_n)$ in X, for every $h, i = 1, \ldots, n$, with $h < i$, the function*

$$u\left(x'_1, \ldots, x_h, x'_{h+1} \ldots, x_i, \ldots, x'_n\right) : X_h \times X_i \to \mathbb{R}$$

is supermodular in (x_h, x_i) on $X_h \times X_i$.

Proof: Necessity is trivial. As for sufficiency, since u is supermodular in each pair of variables (x_h, x_i), then by Lemma 3 it can be seen immediately that u has increasing differences in each (x_h, x_i) (fix the value of all the other entries x_k, $k \neq h, i$, and apply the proof of point (ii)). Then apply Lemma 2 (vector complementarity). ■

Lemma 5. (Differential characterization of supermodularity). *Let $u : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on $\mathbb{R}^n$. Function u is supermodular in x on $\mathbb{R}^n$ if and only, if for every $h, i = 1, \ldots, n$, with $h \neq i$, for every x in $\mathbb{R}^n$, $u_{x_h x_i}(x) \geq 0$.*

Proof: By Lemma 4, u has increasing difference in each pair of variables. Use then the differential characterization of increasing differences (Lemma 1). ■

difference between the following theorem and the analogous theorem which relates supermodularity with increasing difference lies in the fact that, in the implications which define quasisupermodularity, when we interchange the roles of the two initial points x_1, x_2 of lattice X we obtain different implications. This is not true for the inequalities which define supermodularity.

Lemma 6. (Characterization of quasisupermodularity in terms of the single crossing property) *Consider $X = Y \times Z$ and $u(y,z) : X \to \mathbb{R}$. (i) If Y and Z are chains and u has the single crossing property in both (y,z) and (z,y) on $Y \times Z$, then it is quasisupermodular in (y,z) on $Y \times Z$. (ii) If Y and Z are lattices and u is quasisupermodular in (y,z) on $Y \times Z$, then it has the single crossing property in both (y,z) and (z,y) on $Y \times Z$.*

Proof:

(i) Take any unordered $(y', z'), (y'', z'')$ in $Y \times Z$ (if they are ordered quaisupermodularity holds trivially). Let, without loss of generality, $x' \wedge x'' = (y', z'')$ and $x' \vee x'' = (y'', z')$ (here we are using the assumption that Y and Z are chains). Let

$$u(y', z') - u(y', z'') \geq (>)\, 0.$$

Hence this first difference of u in z is (strictly) positive at y', and by the single crossing property in (z,y) it stays (strictly) positive at y'', meaning that

$$u(y'', z') - u(y'', z'') \geq (>)\, 0.$$

Let now

$$u(y'', z'') - u(y', z'') \geq (>)\, 0.$$

Hence this first difference of u in y is (strictly) positive at z'', and by the single crossing property in (y,z) it stays (strictly) positive at z', meaning that

$$u(y'', z') - u(y', z') \geq (>)\, 0.$$

Hence quasisupermodularity holds.

(ii) Pick any $(y', z') \leq (y'', z'')$ in the lattice $Y \times Z$. For $x' := (y', z'')$ and $x'' := (y'', z')$ in $Y \times Z$, $x' \wedge x'' = (y', z')$ and $x' \vee x'' = (y'', z'')$. By quasisupermodularity,

$$u(x') - u(x' \wedge x'') \geq (0) \Rightarrow u(x' \vee x'') - u(x'') \geq (0),$$

which is the single crossing property of u in (z, y). Again by quasisupermodularity,

$$u(x'') - u(x' \wedge x'') \geq (0) \Rightarrow u(x' \vee x'') - u(x') \geq (0),$$

which is the single crossing property of u in (y, z). ■

We now prove the monotone comparative statics theorem for this context. Theorems 1, 2, 5 are special cases of theorem 6. Also, note that the proof of theorem 5 would have been written in a much shorter and transparent way, had we used quasisupermodularity directly.

Theorem 6. (Comparative statics on lattices, Milgrom and Shannon, 1994.) *Let X be a lattice, T be a poset, and $u(x,t) : X \times T \to \mathbb{R}$ be quasisupermodular in x for every t in T, and have the single crossing property in (x, t). For every $t_1 < t_2$, for every $x_1 \in B_{t_1}$ and every $x_2 \in B_{t_2}$, $x_1 \wedge x_2 \in B_{t_1}$ and $x_1 \vee x_2 \in B_{t_2}$.*

Proof: Take any $x_1 \in B_{t_1}$ and any $x_2 \in B_{t_2}$. By optimality,

$$u(x_1 \wedge x_2, t_1) \leq u(x_1, t_1),$$

and so by quasisupermodularity

$$u(x_2, t_1) \leq u(x_1 \vee x_2, t_1).$$

Because $x_2 \leq x_1 \vee x_2$ and $t_1 < t_2$, by the single crossing property

$$u(x_2, t_2) \leq u(x_1 \vee x_2, t_2).$$

Thus, by optimality, $x_1 \vee x_2 \in B_{t_2}$. For the other part, by optimality,

$$u(x_1 \vee x_2, t_2) \leq u(x_2, t_2).$$

Hence by quasisupermodularity

$$u(x_1, t_2) \leq u(x_1 \wedge x_2, t_2),$$

and by the single crossing property,

$$u(x_1, t_1) \leq u(x_1 \wedge x_1, t_1).$$

Hence, by optimality, $x_1 \wedge x_2 \in B_{t_1}$. ■

Corollary 3. (Increasing extremal selections). *Let X be a lattice, T be a poset, and $u(x,t) : X \times T \to \mathbb{R}$ be quasisupermodular in x for every t in T, and have the single crossing property in (x, t). If B_t has either a least element $a_*(t)$ or a greatest element $a^*(t)$ for every $t \in T$ (or both), then this element in an increasing functions.*

Proof: Take any $t_1 < t_2$. Consider $a_* (t)$. Take any $c \in B_{t_1}$. By the previous theorem, $c \wedge a_* (t_2) \in B_{t_1}$. Hence,

$$a_* (t_1) \leq c \wedge a_* (t_2) \leq a_* (t_2) .$$

Analogously, for any $b \in B_{t_2}$, by the previous theorem $a^* (t_1) \vee b \in B_{t_2}$, and so

$$a^* (t_1) \leq a^* (t_1) \vee b \leq a^* (t_2) .$$

■

The conditions assuring that every B_t has a least and a greatest element are those making it nonempty. We need a further result. This discussion applies to the previous subsection too.

Definition 8. (Complete lattice). Let X be a nonempty lattice. X is a complete lattice if for every nonempty subset $S \subseteq X$, $\inf S \in X$ and $\sup S \in X$.

What we care about here is that, on setting $S = X$, a complete lattice has a least and a greatest element. We need the following important topological result.

Theorem 7. (Topological characterization of completeness). *Let X be a nonempty lattice. X is complete if and only if it is compact in its interval topology.*

Proof: Birkhoff, 1967, Ch. X, Theorem 20. ■

Assume then X compact in its interval topology and u upper semi-continuous in x for every t. Hence every B_t is nonempty and compact. Furthermore, by theorem 6, on setting $t_1 = t_2$, we see that every B_t is indeed a lattice (a sublattice of X). Thus, by theorem 7, every B_t is a complete lattice, and so it has a least and a greatest element.

For the previous subsection, assume that the lattice $X = Y \times Z$ is compact in its interval topology, and apply the same argument.

7. Applications to Cournot and Bertrand competition

7.1. **Monotone comparative statics on a chain. Bertrand competition with linear demand curves.** The firm with constant marginal cost $c > 0$ choose own scalar price $p \in [c, +\infty)$, against the pricing of its competitors $\boldsymbol{p}_{-i} \in \mathbb{R}^n_+$, to maximize:

$$\Pi \left(p, \boldsymbol{p}_{-i}\right) = (p - c) D \left(p, \boldsymbol{p}_{-i}\right) ,$$

where the demand function for firm's product is

$$D\left(p, \boldsymbol{p}_{-i}\right) = \alpha + \beta + p \sum_{j=1}^{n} \gamma_j p_{-i}^{j},$$

with $\beta, \gamma > 0$.

Direct computation shows that Π has increasing differences in $\left(p, \boldsymbol{p}_{-i}\right)$.

7.2. **Monotone comparative statics on a chain. Cournot duopoly with substitute products.** The firm choose quantity (capacity) $q \in [0, +\infty)$ to maximize

$$\Pi\left(q, q_{-i}\right) = qP\left(q, q_{-i}\right) - C\left(q\right),$$

against its opponent decision on its own capacity $q_{-i} \in [0, +\infty)$. $P\left(q, q_{-i}\right)$ is the inverse demand function, and $C\left(q\right)$ is the cost function.

We make the following assumptions:
1.$P\left(q, q_{-i}\right)$ is decreasing in q_{-i}, hence the products are *gross substitutes* (we use uncompensated demand functions). This is typical in Cournot duopoly.
2.$P\left(q, q_{-i}\right)$ has decreasing differences in $\left(q, q_{-i}\right)$, meaning that in some sense own demand is less elastic when opponent's production is higher: increasing own production q leads to a possible reduction of own price which decreases, as opponent's production is higher.

As such, the problem in not supermodular. We make it supermodular by **selection of the order**. Given the total order $\left([0, +\infty), \leq\right)$, consider its **dual order** $\left([0, +\infty), \leq^d\right)$, where

$$\forall a, b \in [0, +\infty), a \leq^d b \Leftrightarrow b \leq a.$$

Consider the new decision problem where q is chosen in $[0, +\infty)$ and q_{-i} is now chosen in the dual $[0, +\infty)^d$, which is $[0, +\infty)$ endowed with $\leq^d$

Claim: $P\left(q, q_{-i}\right)$ has increasing differences in $\left(q, q_{-i}\right)$ on $[0, +\infty) \times [0, +\infty)^d$.

Proof: Pick any $\left(q', q'_{-i}\right), \left(q'', q''_{-i}\right)$ in $[0, +\infty) \times [0, +\infty)^d$ such that $q' \leq q''$ and $q'_{-i} \leq^d q''_{-i}$. Hence, $q''_{-i} \leq q'_{-i}$. So, *in the original decision problem*, by decreasing differences of P in $\left(q, q_{-i}\right)$,

$$P\left(q'', q''_{-i}\right) - P\left(q', q''_{-i}\right) \geq P\left(q'', q'_{-i}\right) - P\left(q', q'_{-i}\right),$$

which is exactly increasing difference of P in (q, q_{-i}) on $[0,+\infty) \times [0,+\infty)^d$, i.e. the first difference of P in q increases as q_{-i} increased in $[0,+\infty)^d$.

Claim: $\Pi(q, q_{-i})$ has increasing differences in (q, q_{-i}) on $[0,+\infty) \times [0,+\infty)^d$.
Proof: Pick any $\left(q', q'_{-i}\right), \left(q'', q''_{-i}\right)$ in $[0,+\infty) \times [0,+\infty)^d$ such that $q' \leq q''$ and $q'_{-i} \leq^d q''_{-i}$. By increasing differences of P on $[0,+\infty) \times [0,+\infty)^d$, rearranging terms,

$$P\left(q'', q'_{-i}\right) - P\left(q'', q''_{-i}\right) \leq P\left(q', q'_{-i}\right) - P\left(q', q''_{-i}\right).$$

Since, in the original problem, $P(q, q_{-i})$ is decreasing in q_{-i} on $[0,+\infty)$, by the same argument as in Claim 1 above $P(q, q_{-i})$ is now increasing in q_{-i} on the dual order $[0,+\infty)^d$. Hence, in the inequality above, both differences on l.h.s. and r.h.s. are negative. Since $0 \leq q' \leq q''$, can multiply as follows:

$$q''\left[P\left(q'', q'_{-i}\right) - P\left(q'', q''_{-i}\right)\right] \leq q'\left[P\left(q', q'_{-i}\right) - P\left(q', q''_{-i}\right)\right].$$

The inequality above is what we needed.

Comparative statics conclusion: The set of optimal capacities for the firm is Veinott increasing in q'_{-i} on the dual $[0,+\infty)^d$, hence it is decreasing in q'_{-i} on $[0,+\infty)$. As the opponent increases production, its optimal for the firm to decrease own production. Indeed, product substitution implies that by increasing production, the opponent lowers its price and steals consumers from the firm, which needs to lower own production to keep profits.

7.3. **Monotone comparative statics on the product of chains. Pricing and advertising in Bertrand competition with substitute products.** This example is taken from Topkis (1995). A firm with constant marginal cost $c > 0$ chooses its price $p \in [c,+\infty)$, and its advertising effort $a \in [0,+\infty)$, against the pricing of its competitors $\boldsymbol{p}_{-i} \in \mathbb{R}^n_+$. Competitors products are substitutes with that of the firm, which has cost $K(a)$ for advertising. The profit function is

$$\Pi\left(p, a, \boldsymbol{p}_{-i}\right) = (p - c)\, D\left(p, a, \boldsymbol{p}_{-i}\right) - K(a).$$

We make the following assumptions:
1. The demand function $D\left(p, a, \boldsymbol{p}_{-i}\right)$ is increasing in a
2. The demand function is increasing in $\boldsymbol{p}_{-i}$. This means that competitors' products are *gross substitutes* with that of the firm (we consider

uncompensated demand).
3. The demand function is supermodular in (p, a), meaning that an increase in own price leads to a loss of demand (if demand is decreasing) which is less, if quality is higher.
4. The demand functions has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$, meaning that (i) the loss of demand for increasing own prices is mitigated when opponents' substitute products are priced higher, since less consumers are willing to leave own market because substitute products are more expensive, and (ii) the increase in demand for increasing advertising is higher when opponents' substitute products are priced higher, since the firm has then a wider market share.

From the assumptions, profit function is supermodular in (p, a) on $[c, +\infty) \times [0, +\infty)$ and has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$ on $[c, +\infty) \times [0, +\infty) \times \mathbb{R}_{+}^{n}$. Hence, optimal prices and advertising levels are Veinott increasing in the parameters $\boldsymbol{p}_{-i}$. Note that if opponents set their advertising $\boldsymbol{a}_{-i}$ also, and D is decreasing with this, then profits do not have increasing differences in $\left((p, a), \left(\boldsymbol{p}_{-i}, \boldsymbol{a}_{-i}\right)\right)$.

To show increasing differences of Π in $\left((p, a), \boldsymbol{p}_{-i}\right)$, pick any $\left(p', a', \boldsymbol{p}'_{-i}\right) \leq \left(p'', a'', \boldsymbol{p}''_{-i}\right)$. Using the definition of increasing differences,

$$
\begin{gathered}
(p'' - c) D\left(p'', a'', \boldsymbol{p}'_{-i}\right) - K(a'') - \left[(p' - c) D\left(p', a', \boldsymbol{p}'_{-i}\right) - K(a')\right] \\
\leq \\
(p'' - c) D\left(p'', a'', \boldsymbol{p}''_{-i}\right) - K(a'') - \left[(p' - c) D\left(p', a', \boldsymbol{p}''_{-i}\right) - K(a')\right] \\
\Leftrightarrow \\
(p'' - c)\left[D\left(p'', a'', \boldsymbol{p}'_{-i}\right) - D\left(p'', a'', \boldsymbol{p}''_{-i}\right)\right] \\
\leq \\
(p' - c)\left[D\left(p', a', \boldsymbol{p}'_{-i}\right) - D\left(p', a', \boldsymbol{p}''_{-i}\right)\right]
\end{gathered}
$$

In the last inequality, the two terms in square brackets are nonpositive, due to increasingness of D in $\boldsymbol{p}_{-i}$. Furthermore, the last inequality holds without the multiplying factors $(p'' - c)$ and $(p' - c)$, by the assumption that demand has increasing differences in $\left((p, a), \boldsymbol{p}_{-i}\right)$. Since both $p'' \geq c$ and $p' \geq c$, the last inequality holds.

To show supermodularity of Π in (p, a), take any $\left(p', a', \boldsymbol{p}'_{-i}\right) \leq \left(p'', a'', \boldsymbol{p}''_{-i}\right)$ with $\boldsymbol{p}'_{-i} = \boldsymbol{p}''_{-i}$, and redo the same argument as above, by recalling that the assumption of supermodularity of demand in the vector variable (p, a) is equivalent of it having increasing differences in (p, a). This time, however, we need to use increasingness of demand in a.

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